**Delft University of Technology**
**Faculty Electrical Engineering, Mathematics and Computer Sciences**
**Delft Institute of Applied Mathematics**

# Tarski's circle squaring problem

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# "Tarski's circle squaring problem"

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# Abstract

In this thesis, we take a look Tarski's circle squaring problem: Are an open disk and an open square equidecomposable by using finitely many Borel pieces? The proof is a special case of the one given by *Marks and Unger*[1] and as such will be based on their proof. We have distilled the original proof to make it understandable for a bachelor student without requiring extra knowledge. We give an example of another case that can be solved without using the method used in the proof. We also take a look at what Borel complexity is, calculate the complexity of the sets used in the example and go over the process of calculating the complexity of the pieces used by *Marks and Unger* in broad strokes.
Since the proofs of the used lemmas are very long and/or technical, they have not been included in this report, but can be found in their respective source material.

# Contents

# 1
# Preface

As part of the Applied Mathematics bachelor programme at the TU Delft, students are required to write a thesis on a mathematical topic. While I was initially unsure about what topic to write about, I was sure that I wanted to do this at the analysis section at the TU Delft. When my supervisor, K.P. Hart, handed me the article by *Marks and Unger*, I was initially a little intimidated by the amount of unknown terms and notations, as well of the fact that it was a quite recent paper. This quickly faded however, as I started to really get into the content of the paper. The goals I set out for myself while working on this report were to be able to understand the theorems and proofs given in the paper and to be able to present these in a way such that my fellow students would be able to understand it without trouble. While many of these students dislike the dry, analytical side of mathematics and favour the more tangible subjects such as optimisation or statistics, I've tried writing this report in such a way that someone with the same education as I should be able to read it just fine.

This mentality however, did backfire a bit. Many of the proofs for the lemmas used in this report are either extremely technical or very long, which made them unfit for including them in this report. This, sadly, led to me being unable to write about one of the arguably more interesting parts of the paper by *Marks and Unger*: the Borel complexity of the decomposition used in the proof. In the end, I've managed to go over this subject in broad strokes, but it is a subject I very much would like to return to in the future.

As for any unanswered questions this report leaves: aside from the aforementioned Borel complexity, the values of the constants required to estimate the number of pieces used are still unknown and could lead to some interesting results. One can also wonder whether the method used always gives the minimal result or not and whether applying this method to other or higher dimension objects yields more favourable results.

I would like to thank my supervisor, K.P. Hart, for helping me out whenever I had any questions.

# 2

# INTRODUCTION

The oldest version of a circle-squaring problem can be found in ancient Greece. The question asked was whether given a circle, a square with the same surface area could be constructed using only compass and straightedge. This turned out to be one of three geometrical problems that the ancient Greeks weren't able to solve, and it wasn't until the 1880's that a definitive proof was given that this was impossible.

The mathematician Alfred Tarski posed a similar sounding problem in 1925: Can we take a disk in the plane, cut it into finitely many pieces and using these pieces, can we reassemble these pieces to construct a square? This problem became known as Tarski's circle-squaring problem and was proven to be possible in 1990 by Miklós Laczkovich. This proof relied heavily on the Axiom of Choice and only recently a constructive proof was found by *Marks and Unger*[1].

In their paper, Marks and Unger gave a constructive proof for any two bounded Borel sets with the same Lebesgue measure and a small enough upper Minkowski dimension of the boundary. However, this paper doesn't go deeper into the special case given by Tarski in the original problem.

Our first chapter will deal with the proof that the open disk and the open square are equidecomposable by translations using finitely many Borel pieces. It also includes an example of two equidecomposable sets that have a direct solution as opposed to the open disk and open square. This proof is largely based on the proof given by *Marks and Unger* in [1].

In the second chapter we take a look at the notion of set complexity and will calculate said complexities for the decomposition used in the example given in the first chapter. It will also give the general idea behind the calculation *Marks and Unger* gave in their paper.

# 3

# THE CIRCLE-SQUARING PROBLEM

## 3.1. PROBLEM STATING AND THE GENERAL IDEA OF THE PROOF

In this chapter we address the problem posed by Alfred Tarski, as well as look at an example of a similar problem that has a much easier solution. In the following proof we will use $A$ as an open disk in $\mathbb{R}^2$ and $B$ as an open square in $\mathbb{R}^2$ with the same surface area as $A$. To make things easier, we will work in the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, which we can identify with $[0,1)^2$. By scaling and translating, we can assume that $A, B \subseteq [0, \frac{1}{2})^2$.

The idea of the proof is as follows: first, we will construct a lattice on $\mathbb{T}^2$ using a set of integer linear independent vectors. The goal here is to take the lattice big enough that almost any point in $\mathbb{T}^2$ can be reached from any other point by walking on this lattice.

Using this lattice, we can create a graph and define a flow on the edges of this graph, which will serve as a means to keep track of the number of points that have to be transported from $A$ to $B$ over an edge. With the flow in place, we can take a look at finite rectangular subsets of the graph. Instead of monitoring the flow through every single vertex of the graph, we look at the in-and outflow of the rectangles and then move sets of points from rectangle to rectangle accordingly.

In this chapter, we will prove the following theorem:

**Theorem 3.1** *Suppose $A, B \subseteq \mathbb{R}^2$ are an open disk and an open square respectively, such that $\lambda(A) = \lambda(B) > 0$. Then $A$ and $B$ are equidecomposable by translations using finitely many Borel pieces.*

## 3.2. PROOF

The first thing we need is a systematic way of spreading points over $\mathbb{T}^2$. We do this by means of a set of integer linear independent vectors **u**; that is to say, every integer linear combinations of vectors in **u** is unique. This effectively creates a lattice in $\mathbb{T}^2$. We can choose any point $x \in \mathbb{T}^2$ as a starting point for our lattice and we let $d$ be the number of vectors in **u**. We can now introduce a group action $a_{\mathbf{u}} : \mathbb{Z}^d \times (\mathbb{T}^2)^d \to (\mathbb{T}^2)^d$, defined by:

$$(n_1, ..., n_d) \cdot_{a_{\mathbf{u}}} x = x + \sum_{p=1}^{d} n_p u_p$$

for $(n_1, ..., n_d) \in \mathbb{Z}^d$ and $x \in (\mathbb{T}^2)^d$. We also let $F_N(x, a_{\mathbf{u}}) = R_N \cdot_{a_{\mathbf{u}}} x$, with
$R_N = \{(n_1, ..., n_d) \in \mathbb{Z}^d : 0 \leq n_i \leq N \ \forall i \leq d\}$.

We note that when taking bigger $N$, the minimum distance from any point in $\mathbb{T}^2$ to the nearest point of the lattice gets smaller. If we can get this lattice dense enough to be able to cover our square and circle, we can then start looking for a way to 'transport' points over our lattice. However, before that we want to know what the minimum number of vectors in **u** is such that we can cover $A$ and $B$.

We now state the special case of a lemma by *Laczkovich*, which is exactly what we need here:

**Lemma 3.2** (Laczkovich [2]) *Let $d > 4$ be an integer. Then, for almost every $\mathbf{u} \in (\mathbb{T}^2)^d$, there is an $\varepsilon > 0$ and $M > 0$ such that for every $x \in (\mathbb{T}^2)^d$ and $N > 0$,*

$$D(F_N(x, a_{\mathbf{u}}), A) \leq \frac{M}{N^{-1-\varepsilon}}$$

Here

$$D(F, A) := \left| \frac{|F \cap A|}{|F|} - \lambda(A) \right|,$$

denotes the discrepancy of $F$ relative to $A$, for a finite $F \subseteq \mathbb{T}^k$.

So the minimum for $d$ is 5, and although the lemma still works for bigger $d$, it only makes things needlessly more complicated. We now take $d = 5$ and a $\mathbf{u} \in (\mathbb{T}^2)^5$ such that it fulfils Lemma 3.2z for both sets $A$ and $B$. Then we have some $M$ and $\varepsilon > 0$ such that

$$D(F_N(x, a_{\mathbf{u}}), A) \leq \frac{M}{N^{-1-\varepsilon}} \text{ and } D(F_N(x, a_{\mathbf{u}}), B) \leq \frac{M}{N^{-1-\varepsilon}}$$

for every $x \in \mathbb{T}^2$ and $N > 0$.

What exactly do we have now? If we visualize the result for a set point $x \in \mathbb{T}^2$, we have covered $\mathbb{T}^2$ in points, all of them translations of $x$ by linear combinations of elements of $\mathbf{u}$. What's more, the fraction of the total points that lie inside $B$ approaches $\lambda(B)$. This means we can approach any point in $B$ from any point in $A$ and vice versa if we take our $N$ big enough.

To this end, we construct a graph $G_{a_\mathbf{u}}$ using the transformations defined above: we choose as vertex set $\mathbb{T}^2$ and we create an edge from $x$ to $y$ if $y = \gamma \cdot_{a_\mathbf{u}} x$ with $\gamma = (n_1, ..., n_5)$ with $n_p \geq 0$ for $p \in \{1,2,3,4,5\}$ and $\sum_{p=1}^{5} n_p = 1$. Or, in simpler terms: we create edges from $x$ to $x+u_1$, $x+u_2$, $x+u_3$, $x+u_4$ and $x+u_5$. The idea now is to transport the points of $A$ via the edges of our newly constructed graph to $B$.

In order to do this, we need to monitor exactly how many times every edge and vertex of our graph is used (and in the case of edges, also which direction) in transporting the points from $A$ to $B$. We can define a function $\phi$ on the edges of $G_{a_\mathbf{u}}$, which we call a flow for $G_{a_\mathbf{u}}$. We also require that $\phi(x,y) = -\phi(y,x)$ for every edge $(x,y)$. Using this, we can calculate the total flow in a single vertex of $G_{a_\mathbf{u}}$: the total flow $f$ in a vertex $x$ is

$$f(x) := \sum_{y \in N(x)} \phi(x,y)$$

In order to map every single point of $A$ to a point in $B$, we want the following restrictions to this flow:

- Every $x \in A \cap B^c$ has to be moved out, so the total flow in these vertices must be $-1$.
- Every vertex in $B \cap A^c$ needs a point from $A$, so the total flow in these vertices must be 1.
- Every vertex in $A \cap B$ needs a total flow of 0, since if the point is moved, a new one needs to take its place.
- Every vertex in $A^c \cap B^c$ needs a total flow of 0, since no points need to be removed or end up here.

This can be easily summed up by setting $f = \chi_A - \chi_B$, the difference between the characteristic functions of $A$ and $B$. It is easy to compute $f$ when $\phi$ is known, but can we find a $\phi$ that satisfies a given $f$? If this is the case, we call $\phi$ an $f$-flow. Let's take a look at the special case of Lemma 3.3 that is applicable to our situation:

**Lemma 3.3** (Marks and Unger [1]) *Suppose there is a function $\Phi : \mathbb{N} \to \mathbb{R}$ such that for every $y \in X$*

$$\Big| \sum_{x \in (F_{2^n}(y, a_\mathbf{u}))} f(x) \Big| < \Phi(2^n)$$

*and*

$$c = \frac{1}{16} \sum_{n=0}^{\infty} \frac{\Phi(2^n)}{2^{4n}}$$

*is finite. Then there exists an $f$−flow $\phi$ bounded by $c$*

In order to use this lemma, we only have to find a function $\Phi$ that satisfies. Because of the definition of the discrepancy and our choice of $f(x)$, we have for every $y \in \mathbb{T}^2$ and $n > 0$:

$$|\sum_{x\in R_{2^n}\cdot y} f(x)| = |\sum_{x\in F_{2^n}(y,a_{\mathbf{u}})} f(x)| = 2^{5n}|D(F_{2^n}(y,a_{\mathbf{u}}),A) - D(F_{2^n}(y,a_{\mathbf{u}}),B)|$$

Which, by the triangle inequality and Lemma 3.2 are

$$|\sum_{x\in R_{2^n}\cdot y} f(x)| \leq 2^{5n}|D(F_{2^n}(y,a_{\mathbf{u}}),A) + D(F_{2^n}(y,a_{\mathbf{u}}),B)| \leq 2\cdot 2^{5n}M2^{n(-1-\varepsilon)} = 2M2^{n(4-\varepsilon)}$$

Which gives us $\Phi(2^n) = 2M2^{(4-\varepsilon)n} = M2^{(4-\varepsilon)n+1}$ and $c = \frac{M}{8}\frac{2^\varepsilon}{2^\varepsilon-1}$ as suitable function and constant for Lemma 3.3. This means there exists an $f-$flow $\phi$ bounded by $c$.

Unfortunately, this is not enough to guarantee a perfect translation for every point in $A$ to a single point in $B$. With the current restrictions on $f$ and $\phi$, we could still end up with the following result:

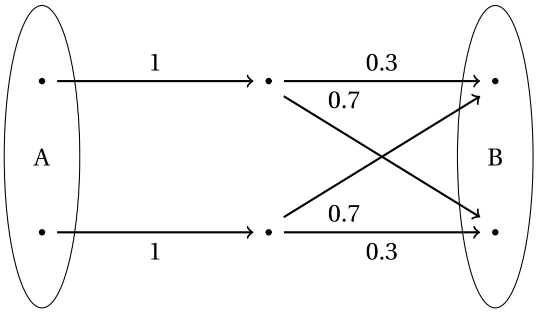


Figure 1: A possible result that can follow from the restrictions set on $f$ and $\phi$.

As we can see, all vertices in $A$ have a net outflow of exactly 1, all vertices in $B$ have a net inflow of exactly 1 and the others a net flow of 0. It follows the restrictions, but it isn't what we want, since points aren't translated but rather split up and distributed among other vertices. Which means we need to put another restriction in place: we want $\phi(x, y)$ to be an integer for every edge $(x, y)$. Hence we use the special case of a lemma by *Marks and Unger*:

**Lemma 3.4** (Marks and Unger [1]) *Suppose $a : \mathbb{Z}^5 \times X \to X$ is a free Borel action and $G_a$ is the associated graph. Then if $f : X \to \mathbb{Z}$ is a Borel function and $\phi$ is a Borel $f-$flow for $G_a$, then there is an integral Borel $f-$flow $\psi$ such that $|\phi - \psi| \leq 3^5$.*

We note that the $f$ and $\phi$ are both Borel functions, so we can apply this lemma without any more work on our part. This means we now have an integral Borel $f-$flow $\psi$ bounded by $3^5 + c$. This means we can now translate all points from $A$ to $B$. However, right now there is no guarantee that this is done using finitely many pieces. Let us look at a lemma by Gao and Jackson applied to our situation:

**Lemma 3.5** (Gao and Jackson [3]) *Let $n > 0$ be an integer. Then there is a Borel set $C \subseteq [G_{a_{\mathbf{u}}}]^{<\infty}$ such that $C$ partitions $\mathbb{T}^2$ and every $S \in C$ is a set of the form $\{(n_1, \ldots, n_5) \cdot_{a_u} x : 0 \leq n_i < N_i\}$ for some $x \in \mathbb{T}^2$ and sequence $N_1, \ldots, N_5$ where $N_i = n$ or $N_i = n+1$.*

In this lemma, $[G_{a_{\mathbf{u}}}]^{<\infty}$ denotes the set of finite subsets of $G_{a_{\mathbf{u}}}$. We note that $G_{a_{\mathbf{u}}}$ has a natural topology for the Hausdorff metric. Now, for each $n$ we can find a Borel tiling $C_n \subseteq [G_{a_{\mathbf{u}}}]^{<\infty}$ of the action $a$ by rectangles of side lengths $n$ or $n+1$. What this essentially means is that we can divide $G_{a_{\mathbf{u}}}$ into 5-hyperrectangles on our lattice, such that the union of all these rectangles is $\mathbb{T}^2$. However, due to the way our lattice lies in $\mathbb{T}^2$, these won't look anything like ordinary rectangles would:

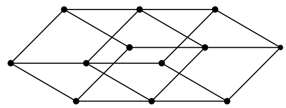

Figure 2: An example of what a rectangle with $n = 1$ could look like if $\mathbf{u}$ had only 3 elements.

Instead of looking at the total flow through a single vertex, we can now look at the total flow through a rectangle. Let us identify our rectangles by calling the unique rectangle in $C_n$ that a point $x \in \mathbb{T}^2$ is in $V_n(x)$. We call a point $x \in V_n(x)$ the starting point of the rectangle if $V_n(x) = \{(n_1, \ldots, n_5) \cdot_{a_{\mathbf{u}}} x : 0 \leq n_i < N_i\}$. If we take any point $y$ in a rectangle $V_n(y)$ that is not the starting point of this rectangle, we see that $\{(n_1, \ldots, n_5) \cdot_{a_{\mathbf{u}}} y : 0 \leq n_i < N_i\} \not\subseteq V_n(y)$. However, this goes for all points in $V_n(x)$ except the starting point. How many edges are going out of (or into) the rectangle? We can estimate this number: $|\partial V_n(x)| \leq 10 \cdot (n+1)^4 \cdot 3^5$; the number of edges times the number of points on one edge times the number of total connection in one point. This is a rough estimate, especially on the total number of connections; but this number is $O(n^4)$, so tweaking it will do little to change that.

Because a starting point exists for every $V_n(x)$, we have that:

$$-Mn^{-1-\varepsilon} \leq ||A \cap V_n(X)|/|A| - \lambda(A)| \leq Mn^{-1-\varepsilon}$$

Which means that

$$\lambda(A) - Mn^{-1-\varepsilon} \leq ||A \cap V_n(X)|/|A||$$

So

$$|A \cap V_n(X)| \geq \lambda(A)n^5 - Mn^{4-\varepsilon}, \text{ which is } O(n^5).$$

So there exists a $K$ such that

$$|A \cap V_K(X)| \geq (c+3^5)|\partial V_K(x)| \text{ and } |B \cap V_K(X)| \geq (c+3^5)|\partial V_K(x)|$$

for every $x \in \mathbb{T}^2$. We fix this $K$ and say $C = C_K$.

Now, for each rectangle $R \in C$, we let $N(R)$ be the set of rectangles with edges to $R$:

$$N(R) = \{S \in C : S \neq R \text{ and } \partial S \cap \partial R \neq \varnothing\}$$

We note that this is a finite set. We can now calculate the total flow between a rectangle $R$ and a neighbouring rectangle $S \in N(R)$:

$$\Psi(R,S) = \sum_{\{(x,y): x \in R \wedge y \in S\}} \psi(x,y),$$

where $\psi$ is our integral $f-$ flow. Also note that $\Psi(R,S)$ is an integer for all $R$ and $S$, and that $\Psi(R,S) = -\Psi(S,R)$ and

$$\sum_{S \in N(R)} \Psi(R,S) = |R \cap A| - |R \cap B|$$

Now if we look back at our choice of $K$, we can see that for every $R$

$$\sum_{S \in N(R)} \Psi(R,S) \leq |A \cap R| \text{ and } \sum_{S \in N(R)} \Psi(R,S) \leq |B \cap R|$$

Furthermore, for every $x \in R$ and $y \in S$ ($S \in N(R)$), there is a $\gamma = (\gamma_1, \ldots, \gamma_5) \in \mathbb{Z}^5$ such that $\gamma \cdot x = y$. We know that every $|\gamma_i| \leq 2(K+1)+1$, because of a maximum $K+1$ steps in the $i^{th}$ direction in $R$, $K+1$ steps in $S$ and 1 to go from $R$ to $S$.

But what we have created now is another graph $G_C$ with an integral Borel flow: if we let $C$ be our vertex set with an edge between $R$ and $S$ if $\partial R \cap \partial S \neq \varnothing$, then $\Psi$ is a flow on $G_C$ for the function $\widetilde{f}(R) = \sum_R \chi_A - \chi_B$.

Now we can finally start creating a bijection that will show that $A$ and $B$ are $a_{\mathbf{u}}$- equidecomposable using finitely many Borel pieces. What we have to construct is a Borel bijection $g : A \to B$ such that for all $x \in A$, $g(x) = \gamma_x \cdot x$, for some $\gamma_x = (\gamma_1, \ldots, \gamma_5)$, with $|\gamma_i| \leq 2K+3$,. What does $g$ look like? The idea is as follows. First of all, if $\Psi(R,S) > 0$ we move $\Psi(R,S)$ points from $A \cap R$ to $B \cap S$, for every $R$ and every $S \in N(R)$. After doing this, there should be an equal number of points in $A \cap R$ and $B \cap R$ for every $R$, so we move these points from $A \cap R$ to $B \cap R$.

First, we fix a Borel linear ordering $<_C$ of $C$ and a Borel linear ordering $<_{\mathbb{T}}$ of $\mathbb{T}^2$. For every $R \in C$, we can order the elements of $N(R)$ using our newly introduced $<_C$.

Now we inductively let $A_i(R) \subseteq A \cap R$ be the $\Psi(R,S)$ first elements in $A \cap R$ according to $<_{\mathbb{T}}$ that are not in any $A_j(R)$ with $j < i$. Here, $S$ is the $i^{th}$ element of our sorted $N(R)$ and if $\Psi(R,S) \leq 0$ for some $S \in N(R)$, we say the corresponding $A_i(R) = \varnothing$. Also, let $A'(R) = (A \cap R) \setminus \bigcup_i A_i(R)$.

Similarly, let $B_i(R) \subseteq B \cap R$ be the $-\Psi(R,S)$ first elements in $B \cap R$ according to $<_{\mathbb{T}}$ that are not in any $B_j(R)$ with $j < i$. Here $S$ is the $i^{th}$ element of our sorted $N(R)$ and if $\Psi(R,S) \geq 0$ for some $S \in N(R)$, we say the corresponding $B_i(R) = \varnothing$. Also, let $B'(R) = (B \cap R) \setminus \bigcup_i B_i(R)$.

By the properties mentioned earlier, we have that $|A'(R)| = |B'(R)|$ for every $R \in C$. Now we define $g : A \to B$ as follows: given $x \in A$, let $R \in C$ be the unique element containing $x$. If there is an $i$ such that $x \in A_i(R)$ and $x$ is the $j^{th}$ element (according to $<_{\mathbb{T}}$) of $A_i(R)$, then we let $g(x)$ be the $j^{th}$ element of $B_m(S)$, in which $S$ is the $i^{th}$ element of $N(R)$ and $m$ is the index for $R \in N(S)$. If we cannot find such an $i$, then $x \in A'(R)$. If $x$ is the $j^{th}$ element of $A'(R)$, then we let $g(x)$ be the $j^{th}$ element of $B'(R)$. □

This means that a single 'piece' that is moved from $A$ to $B$ can be identified by the number of steps in the 5 directions; an explicit definition of a piece $P(n_1, n_2, n_3, n_4, n_5)$ would hence be: $P(n_1, n_2, n_3, n_4, n_5) = \{x \in A : g(x) = x + n_1 u_1 + n_2 u_2 + n_3 u_3 + n_4 u_4 + n_5 u_5\}$.

## 3.3. AN EASIER EXAMPLE

The process used in the previous section isn't just for a square and a circle; this method can be used on any two sets with a 'clean' boundary and an equal surface area. Using this method doesn't automatically yield the least number of pieces however. So how many pieces would we be using exactly? An estimation will be hard to find, since we don't know the exact values of the chosen variables $\varepsilon$, $M$ and $K$. What we do know is that every point takes a maximum of $2K+3$ steps in every direction, which means the maximum number of possible different translations is $O(K^5)$.

In this section we take a look at an example that has an explicit solution with only 3 pieces: an open disk and an open disk united with a single point on the boundary. The solution can seem counter-intuitive since what has to be done essentially boils down to 'letting a point disappear'. What in reality happens is that the point is translated so that it is now in the interior of the circle. However, the new location the point is in was already occupied, so this point has to be moved. Moving the second point displaces a third, the third displaces a fourth, etc. This proof will show that this is possible with no displaced points remaining.

**Theorem 3.6** *Let* $c = (\frac{1}{2}, \frac{1}{2}) \in \mathbb{T}^2$ *and* $r = \frac{1}{4}$. *We consider the following sets:*
$C = \{x \in \mathbb{T}^2 : |x-c| < r\}$, *the open disk in* $\mathbb{T}^2$ *and*
$C^* = \{x \in \mathbb{T}^2 : |x-c| < r\} \cup \{(\frac{3}{4}, 0)\}$, *the open disk in* $\mathbb{T}^2$ *together with one point on its boundary.*
*These two sets are equidecomposable by translation using only three Borel pieces.*

*Proof of Theorem 3.6* Let us first construct the following sequence $(x_n)_{n\geq 0}$ defined by:
$x_0 = (\frac{3}{4}, 0)$ and

$$x_n = \begin{cases} x_{n-1} - \frac{1}{\pi} & \text{if } x_{n-1} - \frac{1}{\pi} \in C \\ x_{n-1} + 2r - \frac{1}{\pi} & \text{if } x_{n-1} - \frac{1}{\pi} \notin C \end{cases} \quad \text{for } n > 0$$

One thing to note is that this is basically a translation of $\frac{1}{\pi}$ to the left, modulo the diameter of the circle.

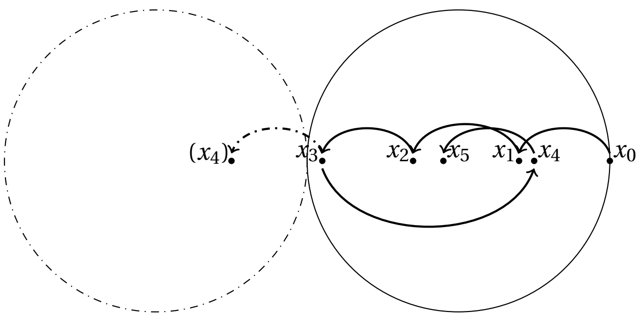


Figure 3: The first 5 steps of $x_n$ worked out.

**Lemma 3.7** *The series $(x_n)_{n\geq 0}$ has no two terms that are equal.*

*Proof of Lemma 3.7.* We look at the terms of $(x_n)$ and translate them $\frac{c}{2}$ to the left, so that all terms are in $[0,2r) \mod 2r$. Now all of our terms are of the form $2r - m\frac{1}{\pi}$. If two terms were to be equal, say terms $k$ and $l$, then we would have:

$$\begin{aligned} 2r - \tfrac{k}{\pi} \mod 2r &= 2r - \tfrac{l}{\pi} \mod 2r \\ \Leftrightarrow \tfrac{l-k}{\pi} &= 0 \mod 2r \\ \Leftrightarrow l - k &= q2r\pi = \tfrac{q}{2}\pi \end{aligned}$$

which is impossible, since $k, l, q \in \mathbb{Z}$.
Hence all terms are different. □

We now see that there are no points that map to $x_0$, because if there was a point $x_m$ such that $x_{m+1} = x_0$, our sequence would be periodical, which contradicts the above lemma. Now we have three subsets of $C^*$, namely:

- $C_0 = \{x_n : x_n - \frac{1}{\pi} \in C\}$
- $C_1 = \{x_n : x_n - \frac{1}{\pi} \notin C\}$
- $C_2 = C^* \setminus (C_0 \cup C_1)$

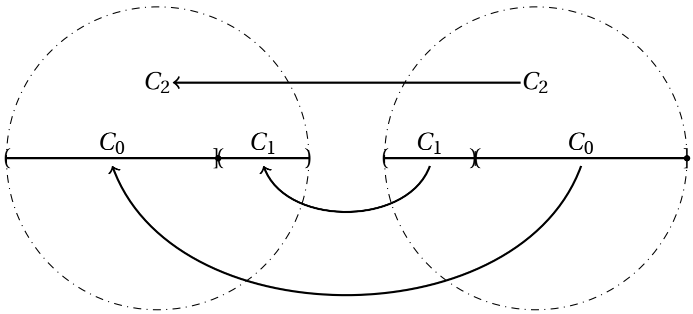


Figure 4: A visualisation of our decomposition.

We now move all elements of $C_0$ $\frac{1}{\pi}$ to the left, we move $C_1$ $2r - \frac{1}{\pi}$ to the right, and keep $C_2$ where it is. These translations essentially boil down to the map $x_n \mapsto x_{n+1}$. We now see that $(C_0 - \frac{1}{\pi}) \cup (C_1 + 2r - \frac{1}{\pi}) \cup C_2 = C$ □

The method used in this section doesn't just work for the particular case of $C$ and $C^*$, it can easily be extended to work with intervals as well. If one were to use this method on for example an open disk and a closed disk, the process would be as follows:

1. Divide the boundary of the closed disk into four quarters $Q_1, ..., Q_4$.
2. Construct a series similar to $(x_n)_{n\geq 0}$ that moves $Q_1$ into the interior of $C$.
3. Repeat step 2 for $Q_2, Q_3$ and $Q_4$.
4. Move the quarters inside one by one.

Special care has to be taken at step 2: to make sure that the quarter stays inside $C$ in its entirety, where the original series was taken modulo $2r$ a smaller number has to be taken. If one were to try this method to sets with a different shape, care has to be taken to only take a series in a direction that does not have a parallel with a tangent of the boundary, since this risks creating an interval in the direction of the tangent, which doesn't allow the series to continue infinitely.

# 4

# THE COMPLEXITY OF THE DECOMPOSITIONS

## 4.1. THE BOREL HIERARCHY

When mathematicians talk about sets and specifically Borel sets, some are often called 'beautiful' sets, meaning they have many useful properties and thus are easy to work with. But how do descriptive set theorists determine how complex a set exactly is? The complexity of a set can be described using a pointclass. Pointclasses are essentially families of sets with specific properties. For example, all sets that can be written as a countable union of closed sets, known as the $F_\sigma$ sets don't include all possible countable intersections of open sets.

All the pointclasses together form the Borel hierarchy. The lowest pointclasses in the hierarchy are the family of open sets, known as $G$ (from the German *Gebiet*), and the family of closed sets, known as $F$ (from the French *Fermé*). Going higher up gives us $F_\sigma$ ($\sigma$ for the French *Somme*) and the countable intersections of closed sets, $G_\delta$ ($\delta$ for the German *Durchschnitt*). Going up even further gives us the $F_{\sigma\delta}$, $G_{\delta\sigma}$,$F_{\sigma\delta\sigma}$, etc. However, just adding alternating sigmas and deltas doesn't make for a very readable text. So, another notation is used.

| $\mathbf{\Delta}^0_1$ (Clopen sets) | |
|---|---|
| $\mathbf{\Sigma}^0_1 = G$ (Open sets) | $\mathbf{\Pi}^0_1 = F$ (Closed sets) |
| $\mathbf{\Delta}^0_2$ | |
| $\mathbf{\Sigma}^0_2 = G_\delta$ | $\mathbf{\Pi}^0_2 = F_\sigma$ |
| ... | |
| $\mathbf{\Sigma}_{\omega_1} = \mathbf{\Pi}_{\omega_1} = \mathbf{\Delta}_{\omega_1} = \mathbf{\Delta}^1_0 = \boldsymbol{B}$ (Borel) | |
| $\mathbf{\Sigma}^1_0$ | $\mathbf{\Pi}^1_0$ |
| ... | |

Table 4.1: A small part of the (boldface) Borel hierarchy.

In Table 4.1, every level includes the sets included in the level above. For $G$ and $F$ this is easy to see, since a clopen set is automatically closed and open, so they are included in both. The classes are derived inductively from each other:

- A set is in $\mathbf{\Sigma}^0_1$ iff it is open.
- A set is in $\mathbf{\Pi}^0_n$ iff its complement is in $\mathbf{\Sigma}^0_n$.
- A set $A$ is in $\mathbf{\Sigma}^0_n$ for $n > 1$ iff there is a sequence of sets $A_1, A_2, ...$ such that each $A_i$ is in $\mathbf{\Pi}^0_{a_i}$ for some $a_i < n$ and $A = \bigcup A_i$.
- A set is in $\mathbf{\Delta}^0_n$ iff it is in both $\mathbf{\Sigma}^0_n$ and $\mathbf{\Pi}^0_n$

From this we easily see that taking the countable intersection of sets in $\mathbf{\Sigma}^0_n$ gives us a set in $\mathbf{\Pi}^0_{n+1}$ and taking the countable union of sets in $\mathbf{\Pi}^0_n$ gives us a set in $\mathbf{\Sigma}^0_{n+1}$. We also see that we can only go up in the hierarchy by alternating countable unions and countable intersections. We can do this as many times as we want, creeping ever closer to $\boldsymbol{B}$. Only by taking a countable number of these alternating unions and intersections will we reach $\boldsymbol{B}$, after exactly $\omega_1$ times. Here, $\omega_1$ is the smallest uncountable ordinal.

In the boldface Borel hierarchy, sets are assigned to a pointclass depending on how little of the alternating union and intersection signs one can describe the set with. It is entirely possible that a set once though to belong to a certain class can be described using far fewer signs. There is generally no easy way to see these kinds of things and thus there is almost no way to know for certain that the lowest class has been reached for a set.

A more in-depth explanation of the basics of descriptive set theory (and more) can be found in [4].

## 4.2. CALCULATING THE COMPLEXITY OF DECOMPOSITIONS

Calculating the complexity of a decomposition is rather straightforward work. In this section we will calculate the complexity used in the case of the open disk and the open disk with an additional point. As noted in chapter 3.3, the pieces used in the decomposition are:

- $C_0 = \{x_n : x_n - \frac{1}{\pi} \in C\}$
- $C_1 = \{x_n : x_n - \frac{1}{\pi} \notin C\}$
- $C_2 = C^* \setminus (C_0 \cup C_1)$

The first thing of importance is the series $(x_n)_{\geq 0}$ used in the proof. The set $X = \{x : x \in (x_n)_{\geq 0}\}$ can also be written as $X = \bigcup_n \{x : x = x_0 - \frac{n}{\pi} \mod 2r\}$ which is a countable union of points, and since points are closed and therefore in $\mathbf{\Pi^0_1}$, we can conclude that $X$ is $\mathbf{\Sigma^0_2}$.

We can now calculate the complexities of $C_0$ and $C_1$. We define $I_0$ as the open rectangle from $(\frac{1}{4} + \frac{1}{\pi}, \frac{1}{4})$ to $(\frac{7}{8}, \frac{3}{4})$ and $I_1$ as the open rectangle from $(\frac{1}{4}, \frac{1}{4})$ to $(\frac{1}{4} + \frac{1}{\pi}, \frac{3}{4})$. Since these are open sets, these are obviously sets in $\mathbf{\Sigma^0_1}$ and because of the properties mentioned in 4.1, also sets in $\mathbf{\Sigma^0_2}$. But then $C_0 = I_0 \cap X$ and $C_1 = I_1 \cap X$ are both also in $\mathbf{\Sigma^0_2}$.

Now we can also calculate the complexity of $C_2$. Since $C^*$ is neither open nor closed, but can be written as $C \cup (\frac{3}{4}, \frac{1}{2})$, it is a set in $\mathbf{\Delta^0_2}$. Then:

$$C^* \setminus (C_0 \cup C_1) = C^* \cap (C_0 \cup C_1)^c = C^* \cap (C_0^c \cap C_1^c)$$

Since the complements of $C_0$ and $C_1$ reside in $\mathbf{\Pi^0_2}$, we now know that $C_2$ is also a set in $\mathbf{\Pi^0_2}$.

## 4.3. SOME COMMENTS ON THE COMPLEXITY OF THE DECOMPOSITIONS IN 3.2

As stated earlier, it is generally not possible to see whether a given set has a better description, which means that in order to get a meaningful result one has to optimize the functions and algorithms to use sets that have an as low complexity as possible. However, this optimization was not the goal of this project and as such, there is little reason to estimate the complexity of the pieces used in 3.2 ourselves. In their article [1], *Marks and Unger* have done this and in this section we take a look at what choices they have made and the results that followed. *Dubins, Hirsch and Karush*[5] showed that $A$ and $B$ are not scissor congruent, which means that it is impossible to cut a circle into pieces with a scissor and create a square using these pieces. These scissor-pieces are all closed sets, so we can say that the pieces used in the decomposition are more complex than $\mathbf{\Pi^0_1}$.

Instead of following the 'traditional' way of determining the complexity by going back and forth between $\mathbf{\Pi}$ and $\mathbf{\Sigma}$, they start by introducing the following:

- $\mathbf{\Sigma}_1^{A,B}$ is the collection of all open balls in $\mathbb{R}^2$, translates of $A$ and translates of $B$.
- Inductively, $\mathbf{B}_n^{A,B}$ is the collection of all finite Boolean combinations of $\mathbf{\Sigma}_n^{A,B}$ sets and $\mathbf{\Sigma}_{n+1}^{A,B}$ is the collection of all countable unions of sets in $\mathbf{B}_n^{A,B}$.

Since we are working with the open square and the open disk, it is obvious that these are in $\mathbf{\Sigma}_1^0$. This means every set in $\mathbf{\Sigma}_1^{A,B}$ is $\mathbf{\Sigma}_1^0$ and because of how $\mathbf{B}_n^{A,B}$ and $\mathbf{\Sigma}_n^{A,B}$ were defined, we can inductively conclude that every set in $\mathbf{\Sigma}_n^{A,B}$ is $\mathbf{\Sigma}_n^0$.

Using this, they note the following:
**Remark 4.1** (Marks and Unger [1]) *Suppose $C \subseteq \mathbb{T}^2$ is defined in terms of some sets $D_1, ..., D_n \subseteq \mathbb{T}^2$. If there is some $m$ and a deterministic algorithm which decides if $x \in C$ based on inspecting what vertices of the $m-$ball around $x$ in the graph $G_{a_u}$ lie in $D_1, ..., D_n$, then $C$ is a finite boolean combination of the sets $g \cdot D_i$, where $|g|_\infty \leq 2$ and $1 \leq i \leq n$. Hence, if $D_1, ..., D_n \in \mathbf{B}_m^{A,B}$, then also $C \in \mathbf{B}_m^{A,B}$.*

Three choices matter in finding the minimal complexity: the choice of $\phi$, the choice of the ordering on $\mathbb{T}^2$ and the choice of $\psi$.

The choice of $\phi$ follows naturally from the proof of Lemma 3.3. By taking $\phi$ as a sum that can only take on a finite amount of rational values in its first $k$ terms, $\{x : \phi(x, \gamma \cdot x) < a\}$ is only a set in $\mathbf{\Sigma}_2^{A,B}$.

The choice of $<_\mathbb{T}$ can be done in such a way that it combines well with Remark 4.1.

The choice of $\psi$ follows from the proof of Lemma 3.4 and the complexity of the set $\{x \in \mathbb{T}^2 : \psi(x, \gamma \cdot x) = m\}$ depends on the complexity of the $D_i$s, since Remark 4.1 will be used, as well as the complexity of the set that followed from the choice of $\phi$.

They arrive at the conclusion that $\{x \in \mathbb{T}^2 : \psi(x, \gamma \cdot x) = m\} \in \mathbf{\Sigma}_4^{A,B}$, which means that the resulting pieces are sets in $\mathbf{B}_4^{A,B}$. As mentioned earlier, it is difficult to see whether a set's currently found minimal complexity is its minimal complexity, but considering all the sets involved in the creation of the pieces, we believe the chances of finding less complex pieces for the decomposition in this proof to be nigh zero.

# 5

# References


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