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Data-Driven Stability Verification of Homogeneous Nonlinear Systems with Unknown Dynamics*

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Abstract— **In this work, we propose a data-driven approach for the stability analysis of discrete-time homogeneous nonlinear systems with unknown models. The proposed framework is based on constructing Lyapunov functions via a set of data, collected from trajectories of unknown systems, while providing an a-priori guaranteed confidence on the stability of the system. In our data-driven setting, we first cast the original stability problem as a robust optimization program (ROP). Since unknown models appear in the constraint of the proposed ROP, we collect a finite number of data from trajectories of unknown systems and provide two variants of scenario optimization program (SOP) associated to the original ROP. We discuss that the proposed ROP, and its corresponding SOPs, are not convex due to having a bilinearity between decision variables. We also show that while one of the proposed SOPs is more efficient in terms of computational complexity, the other one provides Lyapunov functions with a much better performance for the original ROP. We then establish a probabilistic closeness between the optimal value of (non-convex) SOP and that of ROP, and subsequently, formally provide the stability guarantee for unknown systems with a guaranteed confidence level. We illustrate the efficacy of our proposed results by applying them to two physical case studies with unknown dynamics including (i) a DC motor and (ii) a (homogeneous) nonlinear jet engine compressor. We collect data from trajectories of unknown systems and verify their global asymptotic stability (GAS) with desirable confidence levels.**

I. Introduction

Motivations. Data-driven control approaches have received significant attentions, in the past few years, due to their broad applications in real-life safety-critical systems such as autonomous vehicles, biological networks, robotic manufacturing, (air) traffic networks, etc., to name a few. In particular, most of the existing work on the stability analysis of dynamical systems, proposed in the relevant literature, are model-based. On the downside, closed-form models for many complex and heterogeneous systems are either not available or equally complex to be of any practical use, and accordingly, one cannot employ model-based techniques to analyze them. Although there are some results in the literature to solve analysis and synthesis problems by learning approximate models utilizing *identification techniques* (*e.g.,* [1], [2], [3], [4], [5]), obtaining an accurate model for complex systems (if not impossible) can be very challenging, time-consuming and expensive. In addition, identification approaches are mainly tailored to linear or some particular classes of nonlinear systems. Hence, developing data-driven approaches is crucial to bypass the system identification phase and directly employ system's measurements for performing control and stability analyses.

Related Literature. There are already some results, proposed in the past few years, in the setting of data-driven optimization techniques. *Scenario approach* has been initially introduced in [6] to deal with semi-infinite convex programming for robust control analysis and synthesis problems. As the main benefit of the proposed approach, the robust control problem can be solved via random sampling of constraints provided that a probabilistic relaxation of the worst-case robust paradigm is established. As an extension of [6], a random convex program scheme is proposed in [7] in which an explicit bound on the upper tail probability of violation is provided. In addition, the work [7] studies the case of random convex programs with posteriori violated constraints to improve the optimal objective value while maintaining the violation probability under control. A novel scheme for constructing a probabilistic relation between the optimal value of scenario convex programs and that of robust convex programs is initially proposed in [8] in which the uncertainty takes values in a general, possibly infinite-dimensional, metric space. The proposed results are then extended to a particular class of non-convex problems by including binary decision variables.

In the past few years, there have been also some results on formal analysis of dynamical systems via data-driven approaches. A data-driven stability analysis of black-box *switched linear* systems via constructing *common* Lyapunov functions is proposed in [9] in which a stability guarantee is provided based on both the number of observations and a required level of confidence. As an extension of [9], a data-driven stability analysis of switched linear systems via sum of squares Lyapunov functions is proposed in [10]. A parameterization scheme of linear-feedback systems only using data-dependent linear matrix inequalities is proposed in [11] in which the stabilization problem is extended to the case of an output-feedback control design. A data-enabled predictive control algorithm for unknown stochastic linear systems is presented in [12] in which noise-corrupted input/output data are utilized to predict future trajectories and compute optimal control policies.

Contributions. Our main contribution is to propose a data-driven approach for the stability verification of unknown homogeneous nonlinear systems by constructing Lyapunov

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functions via a set of data, collected from trajectories of unknown systems, while providing an a-priori guaranteed confidence on the stability result. To do so, we first reformulate the original stability problem as a robust optimization program (ROP) and then provide two variants of scenario optimization program (SOP), corresponding to the original ROP. We discuss that the proposed ROP, and its corresponding SOPs, are not convex due to having a bilinearity between decision variables. We also show that while one of the proposed SOPs is more efficient in terms of computational complexity, the other one provides Lyapunov functions with a better performance for the original ROP. By establishing a probabilistic closeness between optimal values of SOP and ROP, we formally quantify the stability guarantee of unknown systems based on the number of data and a required confidence level. We verify the effectiveness of our proposed results over two physical case studies with unknown models including (i) a DC motor and (ii) a (homogeneous) nonlinear jet engine compressor.

II. Problem Description

A. Notation and Preliminaries

Sets of real, positive and non-negative real numbers are denoted by $\mathbb{R}, \mathbb{R}^+$, and $\mathbb{R}_0^+$, respectively. We denote the sets of non-negative and positive integers by $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$, respectively. Given N vectors $x_i \in \mathbb{R}^{n_i}$, $x = [x_1; \dots; x_N]$ denotes the corresponding column vector of dimension $\sum_i n_i$. The maximum eigenvalue of a symmetric matrix A is denoted by $\lambda_{\max}(A)$. The absolute value of $a \in \mathbb{R}$ is denoted by $|a|$. We denote the Euclidean norm of a vector $x \in \mathbb{R}^n$ by $\|x\|$. For any matrix $P \in \mathbb{R}^{m \times n}$, we have $\|P\| := \sqrt{\lambda_{\max}(P^\top P)}$. Given a probability space $(\mathcal{D}, \mathbb{B}(\mathcal{D}), \mathbb{P})$, we denote the N-Cartesian product set of $\mathcal{D}$ by $\mathcal{D}^N$, and its corresponding product measure by $\mathbb{P}^N$. If a system Ψ fulfills a property φ, it is denoted by $\Psi \vDash \varphi$. The operator $\vDash$ is also employed to show the feasibility of a solution for an optimization problem. A function $\rho : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to be a class $\mathcal{K}$ function if it is continuous, strictly increasing, and $\rho(0) = 0$. A class $\mathcal{K}$ function $\rho(\cdot)$ is called a class $\mathcal{K}_\infty$ if $\rho(s) \to \infty$ as $s \to \infty$. A function $\rho : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\rho(r, s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the map $\rho(r, s)$ is decreasing with respect to s, and $\rho(r, s) \to 0$ as $s \to \infty$. We denote the factorial of a non-negative integer n by $n!$ as the product of all positive integers less than or equal to n, *i.e.,* $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$.

B. Discrete-Time Nonlinear Systems

In this work, we consider discrete-time nonlinear systems (dt-NS) as formalized in the next definition.

Definition 2.1 (Nonlinear Systems): A discrete-time nonlinear system (dt-NS) is characterized by the tuple

$$\Sigma = (\mathbb{R}^n, f), \tag{1}$$

where:

- $\mathbb{R}^n$ is the state space of the system;
- $f : \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function describing the state evolution of the system, which is assumed to be homogeneous of degree one, *i.e.,* for any $\lambda > 0$ and $x \in \mathbb{R}^n$, $f(\lambda x) = \lambda f(x)$ [13].

For a given initial state $x(0) \in \mathbb{R}^n$, the evolution of the state of dt-NS Σ can be characterized as

$$\Sigma\colon x(k+1) = f(x(k)), \qquad k \in \mathbb{N}. \tag{2}$$

For any initial state $x_0 = x(0) \in \mathbb{R}^n$, the sequence $x_{x_0} : \mathbb{N} \to \mathbb{R}^n$ satisfying (2) is called the *solution process* of Σ under the initial state x_0.

In this work, we are interested in global asymptotic stability of dt-NS Σ as defined below.

Definition 2.2: A dt-NS Σ in (2) is called *globally asymptotically stable (GAS)* if

$$\|x(k)\| \leq \rho(\|x(0)\|, k),$$

for all $x(0) \in \mathbb{R}^n$, and some $\rho \in \mathcal{KL}$, *i.e.,* every trajectory of Σ converges to zero as k goes to infinity.

The following theorem, borrowed from [13], provides the required conditions under which a dt-NS Σ in (2) is GAS.

Theorem 2.3 (Global Stability): Consider a dt-NS Σ as in (2). If there exist a homogeneous function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}_0^+$ of degree $\nu \in \mathbb{N}^+$, *i.e.,* for any $\lambda > 0$ and $x \in \mathbb{R}^n$, $\mathcal{V}(\lambda x) = \lambda^\nu \mathcal{V}(x)$, $\mathcal{K}_\infty$ functions $\underline{\alpha}, \overline{\alpha}$, and a constant $\gamma \in (0, 1)$ such that $\forall x \in \mathbb{R}^n$, where $\|x\| = 1$:

$$\underline{\alpha}(\|x\|) \leq \mathcal{V}(x) \leq \overline{\alpha}(\|x\|), \tag{3a}$$

$$\mathcal{V}(f(x)) - \gamma \mathcal{V}(x) \leq 0, \tag{3b}$$

then this function is called Lyapunov function with a decay rate γ, and the system Σ is globally asymptotically stable (GAS).

In the setting of our work, we fix the structure of Lyapunov functions as

$$\mathcal{V}(q, x) = \sum_{j=1}^{r} q_j p_j(x), \tag{4}$$

with unknown coefficients $q = [q_1; \dots; q_z] \in \mathbb{R}^z$ and user-defined basis functions $p_j(x)$ which are homogeneous of degree ν. We also fix the structure of functions $\underline{\alpha}, \overline{\alpha}$ as

$$\underline{\alpha}(s) = \sum_{j=1}^{\nu} \underline{\alpha}_j s^j,\ \overline{\alpha}(s) = \sum_{j=1}^{\nu} \overline{\alpha}_j s^j,\ \forall s \in \mathbb{R}_0^+. \tag{5}$$

In order to ensure that $\underline{\alpha}, \overline{\alpha}$ in (5) are $\mathcal{K}_\infty$ functions, the derivatives of $\underline{\alpha}, \overline{\alpha}$ with respect to s should be positive, *i.e.,* strictly increasing. Then one has

$$\sum_{j=1}^{\nu} j \underline{\alpha}_j > 0, \quad \sum_{j=1}^{\nu} j \overline{\alpha}_j > 0, \tag{6}$$

with $s = \|x\| = 1$. To enforce the required conditions in Theorem 2.3, we cast our stability problem as the following robust optimization program (ROP):

$$\text{ROP}:\begin{cases}\min\limits_{[\Phi;\mu]} & \mu,\\ & \Phi=[\gamma;\underline{\alpha}_1;\ldots;\underline{\alpha}_\nu;\overline{\alpha}_1;\ldots;\overline{\alpha}_\nu;q_1;\ldots;q_z],\\ & \mu\in\mathbb{R},\ \gamma\in(0,1),\\ \text{s.t.} & \max\big\{h_1(x,\Phi),h_2(x,\Phi,\gamma)\big\}\leq\mu,\\ & \forall x\in\mathbb{R}^n\colon \|x\|=1,\end{cases}\tag{7}$$

where:

$$h_1(x,\Phi)=\max\Big\{\sum_{j=1}^{\nu}\underline{\alpha}_j-\mathcal{V}(q,x),\mathcal{V}(q,x)-\sum_{j=1}^{\nu}\overline{\alpha}_j,$$
$$-\sum_{j=1}^{\nu}j\underline{\alpha}_j,-\sum_{j=1}^{\nu}j\overline{\alpha}_j\Big\},$$
$$h_2(x,\Phi,\gamma)=\mathcal{V}(q,f(x))-\gamma\mathcal{V}(q,x).\tag{8}$$

Note that the ROP in (7) is always feasible by its definition since for a given Φ, there always exists a $\mu\in\mathbb{R}$ satisfying ROP in (7). If $\mu\leq 0$, a solution to the ROP implies the satisfaction of conditions in Theorem 2.3, and consequently, the dt-NS Σ in (2) is GAS. We denote the optimal value of ROP by $\mu^*_{\mathcal{R}}$.

To solve the proposed ROP in (7), one faces two main difficulties. First, the proposed ROP has infinitely many constraints given that the state of the system lives in a continuous set (*i.e.,* $\{x\in\mathbb{R}^n:\|x\|=1\}$). Second and more challenging, one needs to know the precise model f to solve the ROP in (7). These challenges motivated us to develop a data-driven approach, in the next section, for the stability analysis of unknown dt-NS without directly solving the ROP in (7).

III. Data-Driven Construction of Lyapunov Functions

Here, we assume that the map f in (2) is unknown, and we employ the term *unknown model* to refer to this type of systems. In our data-driven setting, we take two consecutive sampled data-points from trajectories of unknown systems as the pair of $(x(k),x(k+1))$ and denote it by $(\hat{x},f(\hat{x}))$. The main goal is to verify the stability of unknown dt-NS in (2) via a Lyapunov function constructed from data with some guaranteed confidence. Let $(\hat{x}_i,f(\hat{x}_i))_{i=1}^{\mathcal{N}}$ be $\mathcal{N}$ independent-and-identically distributed (i.i.d.) sampled data. Instead of solving the ROP in (7), we rather solve the following scenario optimization program (SOP):

$$\text{SOP}_1:\begin{cases}\min\limits_{[\Phi;\mu]} & \mu,\\ & \Phi=[\gamma;\underline{\alpha}_1;\ldots;\underline{\alpha}_\nu;\overline{\alpha}_1;\ldots;\overline{\alpha}_\nu;q_1;\ldots;q_z],\\ & \mu\in\mathbb{R},\ \gamma\in(0,1),\\ \text{s.t.} & \max\big\{h_1(x,\Phi),h_2(\hat{x}_i,\Phi,\gamma)\big\}\leq\mu,\\ & \forall x\in\mathbb{R}^n\colon \|x\|=1,\ \forall\hat{x}_i\in\mathbb{R}^n\colon\|\hat{x}_i\|=1,\\ & \forall i\in\{1,\ldots,\mathcal{N}\},\end{cases}\tag{9a}$$

where h_1,h_2 are the same functions as in (8). We denote the optimal value of SOP_1 by $\mu^*_{\mathcal{N}_1}$. As it can be observed, $f(\hat{x}_i)$ in h_2 can be substituted by measurements of unknown dt-NS after one-step evolution starting from $\hat{x}_i$. Remark that condition h_1 in (9a) should be still satisfied for any $x\in\mathbb{R}^n$: $\|x\|=1$.

We now propose another variant of scenario optimization program, denoted by SOP_2, which is more relaxed compared to (9a) in the sense that condition h_1 should be fulfilled only over sampled data:

$$\text{SOP}_2:\begin{cases}\min\limits_{[\Phi;\mu]} & \mu,\\ & \Phi=[\gamma;\underline{\alpha}_1;\ldots;\underline{\alpha}_\nu;\overline{\alpha}_1;\ldots;\overline{\alpha}_\nu;q_1;\ldots;q_z],\\ & \mu\in\mathbb{R},\ \gamma\in(0,1),\\ \text{s.t.} & \max\big\{h_1(\hat{x}_i,\Phi),h_2(\hat{x}_i,\Phi,\gamma)\big\}\leq\mu,\\ & \forall\hat{x}_i\in\mathbb{R}^n\colon\|\hat{x}_i\|=1,\ \forall i\in\{1,\ldots,\mathcal{N}\}.\end{cases}\tag{9b}$$

We denote the optimal value of SOP_2 by $\mu^*_{\mathcal{N}_2}$. In the case study section, we show that solving (9b) is more efficient than (9a) since one needs to only solve a linear programing in (9b) rather than, for instance, a semi-definite programing using sum-of-squares (SOS) optimization problem (*i.e.,* equivalently a semi-infinite linear programming) for h_1 in (9a). On the other hand, the acquired Lyapunov function via solving (9a) provides a better performance with a smaller optimal value $\mu^*_{\mathcal{R}}$ for the original ROP compared to the Lyapunov function obtained from (9b) (cf. DC motor case study).

Remark 3.1: Note that function h_2 in (9a)-(9b) is not convex due to a bilinearity between decision variables q and decay rate γ. To deal with this non-convexity, we assume that γ lies within a finite set with the cardinality m, *i.e.,* $\gamma\in\{\gamma_1,\ldots,\gamma_m\}$, and convert SOP_1 and SOP_2 to, respectively, mixed-integer semi-definite and linear programming. We then leverage the cardinality m in computing the minimum number of data required for solving SOP_1 and SOP_2 (cf. Theorem 4.3).

In the next section, we establish a probabilistic relation between the optimal value of SOP_1 (respectively SOP_2) and that of ROP, and consequently, verify the stability of unknown dt-NS with an a-priori guaranteed confidence level.

IV. Stability Guarantee over Unknown Systems

In this section, inspired by the fundamental results of [8], we aim at establishing a formal relation between the optimal value of SOP_1 (respectively SOP_2) and that of ROP in (7). Accordingly, we formally provide the stability guarantee of unknown dt-NS based on the number of data and a required confidence level. We now state the main problem that we aim to solve in this work.

Problem 4.1: Consider a dt-NS in (2) with an unknown map f. Construct a Lyapunov function via solving (non-convex) SOP_1 in (9a) (respectively SOP_2 in (9b)) and provide a formal guarantee on GAS of unknown dt-NS with an a-priori confidence level $1-\beta$, $\beta\in[0,1)$, as

$$\mathbb{P}^{\mathcal{N}}\Big\{\Sigma\models\text{GAS}\Big\}\geq 1-\beta.$$

To address Problem 4.1, we first propose the following assumption.

Assumption 4.2: Suppose h_2 (respectively h_1) are Lipschitz continuous with respect to x with Lipschitz constants $\mathscr{L}_{2_k}$ (respectively $\mathscr{L}_1$), for given γ_k where $k \in \{1, \ldots, m\}$.

Under Assumption 4.2, we verify the stability of unknown dt-NS with an a-priori confidence level via the next theorem.

Theorem 4.3: Consider an unknown dt-NS as in (2) and let Assumption 4.2 hold. Consider the SOP_1 in (9a) (respectively SOP_2 in (9b)) with its associated optimal value $\mu^*_{\mathcal{N}_1}$ (respectively $\mu^*_{\mathcal{N}_2}$) and solution $\Phi^* = [\underline{\alpha}^*_1; \ldots; \underline{\alpha}^*_\nu; \overline{\alpha}^*_1; \ldots; \overline{\alpha}^*_\nu; q^*_1; \ldots; q^*_z]$, with $\mathcal{N} \geq \bar{\mathcal{N}}(\bar{\varepsilon}, \beta)$, $\bar{\varepsilon} := (\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_m)$, where

$$\bar{\mathcal{N}}(\bar{\varepsilon}, \beta) := \min\Big\{\mathcal{N} \in \mathbb{N} \,|\, \sum_{k=1}^{m}\sum_{i=0}^{c-1}\binom{\mathcal{N}}{i}\bar{\varepsilon}_k^i(1-\bar{\varepsilon}_k)^{\mathcal{N}-i} \leq \beta\Big\}, \tag{10}$$

$\beta \in [0, 1)$, $\bar{\varepsilon}_k = w(\frac{\varepsilon_k}{\mathscr{L}_{h_k}})$, with $\varepsilon_k \in [0, 1), \mathscr{L}_{h_k} := \mathscr{L}_{2_k}$ (respectively $\mathscr{L}_{h_k} := \max\{\mathscr{L}_1, \mathscr{L}_{2_k}\}$), and $w(r) : \mathbb{R}^+_0 \to [0, 1]$ which depends on the sampling distribution and the geometry of the uncertainty set X. Then the following statement holds with a confidence of at least $1 - \beta$: if

$$\mu^*_{\mathcal{N}_1} + \max_k \varepsilon_k \leq 0, \quad (\text{respectively } \mu^*_{\mathcal{N}_2} + \max_k \varepsilon_k \leq 0),$$

then the unknown dt-NS is GAS according to Theorem 2.3.

Proof: We first establish a probabilistic relation between optimal values of ROP and SOP_1, and then provide stability guarantees over the unknown dt-NS via the established relation. Based on [8, Theorems 4.1, 4.3], the probabilistic distance between optimal values of ROP and SOP_1 can be formally lower bounded as

$$\mathbb{P}^{\mathcal{N}}\Big\{0 \leq \mu^*_{\mathcal{R}} - \mu^*_{\mathcal{N}_1} \leq \max_k \varepsilon_k\Big\} \geq 1 - \beta,$$

provided that

$$\mathcal{N} \geq \bar{\mathcal{N}}\big(w(\frac{\varepsilon_k}{\mathrm{L_{SP}}\mathscr{L}_{h_k}}), \beta\big),$$

where $w(s) : \mathbb{R}^+_0 \to [0, 1]$ depends on the sampling distribution and the geometry of the uncertainty set X, and $\mathrm{L_{SP}}$ is a Slater constant as defined in [8, equation (5)]. Based on [8, Remark 3.5], since the original ROP in (7) can be cast as a min-max optimization problem, the Slater constant $\mathrm{L_{SP}}$ can be selected as 1. We refer the interested reader to [8, equation (5)] for more details on the formal definition of Slater point.

One can readily conclude that $\mu^*_{\mathcal{N}_1} \leq \mu^*_{\mathcal{R}} \leq \mu^*_{\mathcal{N}_1} + \max_k \varepsilon_k$ with a confidence of at least $1 - \beta$. If $\mu^*_{\mathcal{N}_1} + \max_k \varepsilon_k \leq 0$ (as the main condition of the theorem), then $\mu^*_{\mathcal{R}} \leq 0$, implying the satisfaction of conditions in Theorem 2.3 and ensuring the global asymptotic stability of unknown dt-NS with a confidence of at least $1 - \beta$. One can readily utilize the same reasoning to establish a probabilistic relation between optimal values of ROP and SOP_2, and then provide stability guarantees over unknown dt-NS via the established relation, which completes the proof. ■

Remark 4.4: As discussed in [8, Proposition 3.8], the function w in (10) satisfies the following inequity:

$$w(r) \leq \mathbb{P}\big[\mathbb{B}_r(x)\big], \qquad \forall r \in \mathbb{R}^+_0, \forall x \in X,$$

where $\mathbb{B}_r(x) \subset X$ is an open ball centered at x with radius r. In the case of collecting samples from a *unit* n-dimensional sphere with a *uniform* distribution, *i.e.,* $\frac{X}{\|X\|}$ where X is distributed with a normal distribution with zero mean and identity covariance, the function w in (10) is computed as

$$w(r) = \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi}\,\Gamma(\frac{n-1}{2})}\int_0^{\theta 0}(\sin\theta)^{n-2}\mathrm{d}\theta, \tag{11}$$

where $\theta_0 = 2\arcsin\frac{r}{2}$, $\pi \approx 3.14$, and Γ is the Gamma function defined as $\Gamma(n) = (n-1)!$ for any positive integer n and $\Gamma(n + \frac{1}{2}) = (n - \frac{1}{2}) \times (n - \frac{3}{2}) \times \cdots \times \frac{1}{2} \times \pi^{\frac{1}{2}}$ for any non-negative integer n. In the case of collecting samples with a *uniform* distribution from a *unit circle*, the function w in (11) is reduced to $w(r) = \frac{2}{\pi}\arcsin\frac{r}{2}$ (cf. case studies).

We propose Algorithm 1 to describe the required procedure in Theorem 4.3.

Algorithm 1 Ensuring GAS guarantee for unknown dt-NS

Require: $\mathscr{L}_1, \mathscr{L}_{2_k}, \varepsilon_k, \beta \in [0, 1)$ as desired

1: Compute the required number of samples as $\mathcal{N} \geq \bar{\mathcal{N}}\big(w(\frac{\varepsilon_k}{\mathscr{L}_{h_k}}), \beta\big)$ in (10)

2: Solve SOP_1 in (9a) (respectively SOP_2 in (9b)) with the acquired data and obtain $\mu^*_{\mathcal{N}_1}$ (respectively $\mu^*_{\mathcal{N}_2}$)

3: If $\mu^*_{\mathcal{N}_1} + \max_k \varepsilon_k \leq 0$ (respectively $\mu^*_{\mathcal{N}_2} + \max_k \varepsilon_k \leq 0$), then unknown dt-NS is GAS with a confidence of at least $1 - \beta$, *i.e.,* $\mathbb{P}^{\mathcal{N}}\big\{\Sigma \models \mathrm{GAS}\big\} \geq 1 - \beta$, otherwise inconclusive

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In order to compute the required number of samples in Theorem 4.3, one needs to first compute $\mathscr{L}_{h_k}$ which is based on $\mathscr{L}_1, \mathscr{L}_{2_k}$. We propose in the next lemmas an explicit way to compute $\mathscr{L}_1, \mathscr{L}_{2_k}$ for both (unknown) nonlinear and linear systems.

Lemma 4.5: For a dt-NS as in (2), let $\|f(x)\| \leq \mathcal{M}_f \in \mathbb{R}_{\geq 0}$, and $\|\frac{\partial f(x)}{\partial x}\| \leq \mathscr{L}_f \in \mathbb{R}_{\geq 0}$ for any $x \in X$. Then $\mathscr{L}_1, \mathscr{L}_{2_k}$ for a quadratic Lyapunov function of the form $x^\top Px$, with a positive-definite matrix $P \in \mathbb{R}^{n \times n}$, is quantified as $\mathscr{L}_1 = 2\,\lambda_{\max}(P), \mathscr{L}_{2_k} = 2\lambda_{\max}(P)(\mathcal{M}_f\mathscr{L}_f + \gamma_k)$. The proof is straightforward and is omitted here.

Remark 4.6: Note that one needs to know an upper bound for $\lambda_{\max}(P)$ in order to compute $\mathscr{L}_{h_k}$ and the required number of samples in Step 1 in Algorithm 1. We compute the required upper bound for $\lambda_{\max}(P)$ a-priori using Gershgorin circle theorem [14] and then enforce it as an additional condition while solving the SOP_1 in (9a) (respectively SOP_2 in (9b)) as mentioned in Step 2 of Algorithm 1 (cf. case studies).

Similarly, we provide another lemma for the computation of $\mathscr{L}_{h_k}$ but for linear dynamical systems.

Lemma 4.7: For a linear system $x(k + 1) = Ax(k)$ with $A \in \mathbb{R}^{n \times n}$, let $\|A\| \leq \mathscr{L}_f \in \mathbb{R}^+_0$. Then $\mathscr{L}_1, \mathscr{L}_{2_k}$ for a quadratic Lyapunov function of the form $x^\top Px$, with a positive-definite matrix $P \in \mathbb{R}^{n\times n}$, is quantified as $\mathscr{L}_1 = 2\,\lambda_{\max}(P), \mathscr{L}_{2_k} = 2\,\lambda_{\max}(P)(\mathscr{L}_f^2 + \gamma_k)$.

Lemmas 4.5 and 4.7 provide a systematic approach for computing $\mathscr{L}_{h_k}$ tailored to quadratic Lyapunov functions of the form $x^\top Px$. Nevertheless, one can still utilize these results for polynomial-type Lyapunov functions as in (4) given that they can be reformulated as a quadratic function of monomials of the form $x^\top Px$.

V. Case Studies

DC Motor. We first apply our data-driven approaches to a DC motor borrowed from [15]:

$$x_1(k+1) = x_1(k) + \tau\big(\frac{-R}{L}x_1(k) - \frac{k_{dc}}{L}x_2(k)\big),$$
$$x_2(k+1) = x_2(k) + \tau\big(\frac{k_{dc}}{J}x_1(k) - \frac{b}{J}x_2(k)\big),$$

where x_1, x_2, $R = 1$, $L = 0.01$, and $J = 0.01$ are the armature current, the rotational speed of the shaft, the electric resistance, the electric inductance, and the moment of inertia of the rotor, respectively. In addition, $\tau = 0.01$, $b = 0.1$, and $K_{dc} = 0.01$ represent, respectively, the sampling time, the motor torque and the back electromotive force. We assume that the model is unknown. We construct a Lyapunov function via data collected from trajectories of unknown system by solving SOP$_1$ in (9a), and accordingly, verify that the DC motor is GAS with respect to its given equilibrium point $(0,0)$.

We fix the structure of our homogeneous Lyapunov function as $\mathcal{V}(q,x) = q_1x_1^4 + q_2x_1^2x_2^2 + q_3x_2^4$ with degree $\nu = 4$. We now follow Algorithm 1 in order to utilize the results of Theorem 4.3. We first fix threshold $\varepsilon_k = 0.0028, \forall k \in \{1,\dots,m\}$ and the confidence $\beta = 10^{-7}$ a-priori. Now we need to compute $\mathscr{L}_{h_k}$ which is required in computing the minimum number of data. We construct matrix P based on coefficients of the nominated Lyapunov function. By enforcing each coefficient of the Lyapunov between $[-0.01, 0.01]$, we ensure that $\lambda_{\max}(P) \leq 0.02$ as discussed in Remark 4.6. We assume that $\gamma \in \{0.35, 0.65, 0.95\}$ with the cardinality $m = 3$. Then according to Lemma 4.7, we compute $\mathscr{L}_{h_1} = 0.046$, $\mathscr{L}_{h_2} = 0.058$, and $\mathscr{L}_{h_3} = 0.07$. By collecting samples from a *unit circle* with a *uniform* distribution, we compute $\bar{\varepsilon}_k = w(\frac{\varepsilon_k}{\mathscr{L}_{h_k}}) = \frac{2}{\pi}\arcsin(\frac{\varepsilon_k}{2\mathscr{L}_{h_k}})$ (cf. Remark 4.4). Now we have all the required ingredients to compute $\mathcal{N}$. The minimum number of data required for solving SOP$_2$ in (9b) is computed as $\mathcal{N} = 3147$.

We now solve the SOP$_2$ (9b) with the acquired $\mathcal{N}$, and the additional conditions on the coefficient of the nominated Lyapunov function. The solution Φ^* together with the optimal objective value of SOP$_2$ are computed as

$$\mathcal{V}(q,x) = 0.01x_1^4 + 0.01x_1^2x_2^2 + 0.01x_2^4,\ \mu^*_{\mathcal{N}_2} = -0.0029,$$
$$\underline{\alpha}_1 = 0.5, \underline{\alpha}_2 = -0.4985, \underline{\alpha}_3 = -0.5, \underline{\alpha}_4 = 0.5,$$
$$\overline{\alpha}_1 = \overline{\alpha}_2 = \overline{\alpha}_3 = \overline{\alpha}_4 = 0.5.$$

Since $\mu^*_{\mathcal{N}_2} + \max_k \varepsilon_k = -10^{-4} \leq 0$, according to Theorem 4.3 and via the constructed Lyapunov function from data, one can verify that the unknown DC motor is GAS with a confidence of at least $1-10^{-7}$. To illustrate the effectiveness of our results, we assume that we have access to the model of DC motor and verify our data-driven Lyapunov function via SOSTOOLS [16].

In order to have a practical analysis on the required number of collected data in Theorem 4.3, we plotted in Fig. 1 the required number of data in terms of the threshold ε_k and the confidence parameter β based on (10) for the DC motor. As it can be observed, the required number of data decreases by increasing either the threshold ε_k or β. However, in practice, one needs to select β as small as possible to provide a reasonable confidence (*i.e.,* $1-\beta$) over the stability of the original unknown dt-NS. Besides, in order to ensure the stability of unknown dt-NS with some desirable confidence, condition $\mu^*_{\mathcal{N}_1} + \max_k \varepsilon_k \leq 0$ (respectively $\mu^*_{\mathcal{N}_2} + \max_k \varepsilon_k \leq 0$) needs to hold.

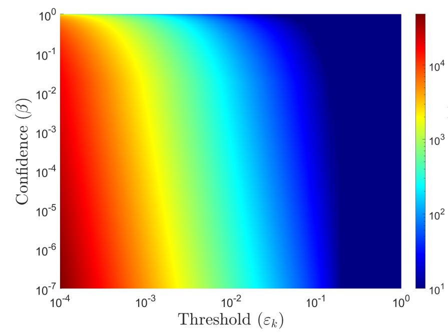

Fig. 1. Required number of data, represented by 'colour bar', in terms of the threshold ε_k and the confidence parameter β. Plot is in the logarithmic scale. The required number of data decreases by increasing either the threshold ε_k or the confidence parameter β.

In order to have a comparison between the performance of Lyapunov functions acquired from SOP$_1$ and SOP$_2$, we also solve SOP$_1$ in (9a). We now assume we know the model and compute the optimal value $\mu^*_{\mathcal{R}}$ in ROP (7) via Lyapunov functions obtained from SOP$_1$ and SOP$_2$. The corresponding results are plotted in Fig. 2 for different data sets. As it can be observed, optimal values $\mu^*_{\mathcal{R}}$ acquired from Lyapunov functions via solving SOP$_1$ in (9a) (*i.e.,* purple curve) is always less than the optimal values $\mu^*_{\mathcal{R}}$ obtained from Lyapunov functions via solving SOP$_2$ in (9b) (*i.e.,* green curve). This it expected and the reason is that Lyapunov functions obtaining from SOP$_1$ are positive for all range of $\{x \in \mathbb{R}^n : \|x\| = 1\}$, and not only over finite data points which is the case in SOP$_2$. This implies that the Lyapunov function via solving SOP$_1$ in (9a) has a better performance compared to the one acquired from SOP$_2$ in (9b). On the other hand, solving (9b) is more efficient than (9a) since one needs to only solve a mixed-integer linear programming in (9b) rather than a mixed-integer semi-definite programming using sum-of-squares (SOS) in (9a).

Jet Engine Compressor. In order to show the applicability of our results to *nonlinear* systems, we apply our approaches to the following (homogeneous) nonlinear jet engine compressor [17]:

$$x_1(k+1) = x_1(k) + \tau\big(-x_2(k) - \frac{3}{2}x_1^2(k) - \frac{1}{2}x_1^3(k)\big),$$
$$x_2(k+1) = x_2(k) + \tau\big(x_1(k) - u(k)\big), \quad (12)$$

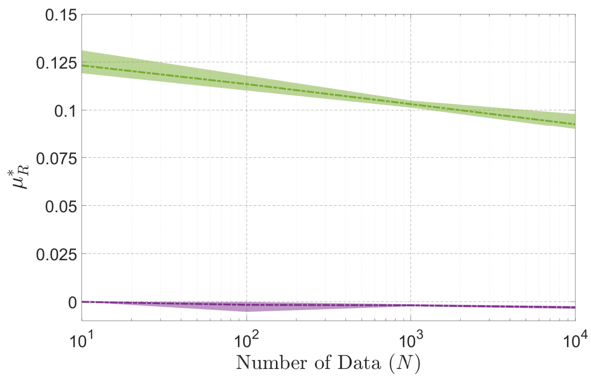

Fig. 2. Optimal values $\mu^*_{\mathcal{R}}$ acquired from Lyapunov functions via solving SOP_1 in (9a) (*i.e.,* purple curve) and SOP_2 in (9b) (*i.e.,* green curve).

where $x_1 = \Psi - 1$, $x_2 = \hat{\Psi} - \bar{\Psi} - 2$, with Ψ, $\hat{\Psi}$, and $\bar{\Psi}$ being, respectively, the mass flow, the pressure rise, and a constant, u is the control input, and $\tau = 0.01$. As shown in [17], the closed-loop jet engine in (12) is homogeneous with a controller

$$u = 3x_1 - (x_1^2+1)(y + x_1^2 y + x_1 y^2), \text{ where } y = 2\frac{x_1^2 + x_2}{x_1^2+1}.$$

We assume that the model is unknown to us. The main goal is to construct a Lyapunov function via data collected from trajectories of unknown system to show that the system is GAS with respect to the equilibrium point $(0,0)$.

We fix the structure of our Lyapunov function as $\mathcal{V}(q,x) = q_1 x_1^2 + q_2 x_1 x_2 + q_3 x_2^2$ with degree $\nu = 2$. We also fix the threshold $\varepsilon_k = 0.07, \forall k \in \{1,\dots,m\}$ and the confidence parameter $\beta = 10^{-7}$ a-priori. By enforcing each coefficient of the Lyapunov function within $[-0.5, 0.5]$, we ensure that $\lambda_{\max}(P) \leq 1$ as discussed in Remark 4.6. We assume $\gamma \in \{0.5, 0.7, 0.9\}$ with the cardinality $m = 3$. Then according to Lemma 4.7, we compute $\mathscr{L}_{h_1} = 5.15$, $\mathscr{L}_{h_2} = 5.55$, and $\mathscr{L}_{h_3} = 5.95$. By collecting samples from a *unit circle* with a *uniform* distribution, we compute $\bar{\varepsilon}_k = w(\frac{\varepsilon_k}{\mathscr{L}_{h_k}}) = \frac{2}{\pi}\arcsin(\frac{\varepsilon_k}{2\mathscr{L}_{h_k}})$ (cf. Remark 4.4). The minimum number of data required for solving SOP_2 in (9b) is computed as $\mathcal{N} = 8792$.

After solving SOP_2 (9b) with the acquired number of data, the solution Φ^* together with the optimal objective value of SOP_2 are computed as

$$\mathcal{V}(q,x) = 0.09x_1^2 + 0.03x_1x_2 + 0.1x_2^2, \ \mu^*_{\mathcal{N}_2} = -0.0872,$$
$$\underline{\alpha}_1 = -0.5, \underline{\alpha}_2 = 0.2936, \ \overline{\alpha}_1 = \overline{\alpha}_2 = 0.5.$$

Since $\mu^*_{\mathcal{N}_2} + \max_k \varepsilon_k = -0.0172 \leq 0$, according to Theorem 4.3 and via the constructed Lyapunov function from data, one can verify that the unknown jet engine compressor is GAS with a confidence of at least $1 - 10^{-7}$.

VI. Discussion

In this work, we proposed a data-driven approach for the stability analysis of discrete-time homogeneous nonlinear systems with unknown models. We first reformulated the original stability problem as a robust optimization program (ROP) and provided two variants of scenario optimization program (SOP) corresponding to the original ROP by collecting a finite number of data from trajectories of the unknown system. We then established a probabilistic closeness between optimal values of SOP and ROP, and accordingly, formally provided the stability guarantee of unknown systems based on the number of data and a guaranteed confidence level. We illustrated our data-driven results over two physical case studies with unknown dynamics. Developing a data-driven approach for synthesizing *stabilizing controllers* for discrete-time homogeneous nonlinear systems is under investigation as a future work.

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