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RESEARCH

On the existence and approximation of localized waves in a damped Euler–Bernoulli beam on a nonlinear elastic foundation.

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Abstract Localized waves in an Euler–Bernoulli beam on a weakly nonlinear elastic foundation, which has a specific weakly nonlinear structural damping, and is under the action of a moving (constant speed) concentrated load, are studied. A structural nonlinear damping, and a damping in the elastic foundation are taking into account. New approximation formulas, which describe in one expression both linear and nonlinear damping characteristics depending on the model parameters, are found. The localized approximations of the solutions of the weakly nonlinear problem have been obtained by using the Ritz method and perturbation techniques. Relatively simple formulas obtained for the asymptotic localized solution, show that the nonlinearity in the elastic foundation leads to smaller amplitudes in the displacements than in case of a linear foundation. The nonlinearity also leads to a smaller value of the critical velocity in the considered system compared to the linear case. The coefficient introduced in the weakly nonlinear elastic foundation also leads to a larger cut-off frequency in the system as compared to the linear case. Another property and effect of the nonlinearity in the elastic foundation is that at a "velocity resonance" the amplitudes of displacement remain finite even when damping (viscosity) is absent. The critical velocity for the weakly nonlinear problem has also been obtained, and the stability of the localized solution, which depends on the system parameters, is investigated.

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1 Introduction

The dynamics of a beam, when it is under the action of a moving load, has already a rich and 120 years history of research starting in the works [1,2]. The motivation to consider such problems for beams is their importance in geotechnics, and in marine, railway track, pavement, and in aerospace engineering. Since that starting time, thousands of papers were published devoted to different aspects of the problem. The reviews on the subject of dynamics of finite and infinite beams on elastic and viscoelastic foundations under the action of different types of moving loads can be found in [3–8]. The existing literature devoted to problems of moving loads include papers which refer to all kinds of beams (Euler, Timoshenko, Rayleigh, E-B- von Karman etc., see [3,9–13,16,17]), which lie on different types of foundations (Winkler, Reissner, Filonenko-Borodich, Pasternak, non-uniform, unilateral, viscoelastic, elastic-plastic, random etc., see [3,6,7,18–21,24–27]), and under the action of various alternatives for a moving load and spring mass sys-

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tems (see[5,6,11,13,22,25,26,28,29]). Probably the first paper devoted to the steady-state solution of the problem for a physically and geometrically nonlinear beam under the action of a moving load was proposed in [14]. In [14] the first correction to the linear solutions was found by using Poincare-Lindstedt expansions for subcritical and supercritical load speeds. A simple solution for "supercritical load speeds for elastic-plastic beams and foundation materials", was also proposed. But a mathematical justification of the existence of the proposed form of the solution was not presented. In [15] "the response of infinite beams on nonlinear visco-elastic foundations subjected to harmonic moving loads" was studied. The author used a perturbation method, the Fourier transform method, and Cauchy's residue theorem to get a solution. The approximation obtained, however, gives only partial characteristics of the system. Moreover the method is inefficient for a parametric analysis, and the method might be inaccurate in some cases due to the complexity of computing residues as mentioned in [28]. In [28] a semi-analytical approach to study the dynamic response of a Timoshenko beam "resting on a nonlinear elastic foundation and subjected to a moving load represented by a series of distributed loads harmonically varying in time", was presented. The approach is based on the Adomian decomposition method combined with the wavelet-based approximation method of Coiflet type. The approximation obtained, is difficult to use for a physical interpretation as mentioned by the authors. The same approach was used in [10] to solve a similar problem for a Rayleigh beam on a nonlinear foundation. In[20] an infinite Euler–Bernoulli beam resting on a locally inhomogeneous and nonlinear Winkler foundation, subjected to a moving load, was investigated. The Winkler foundation is characterized by a piecewise-linear stiffness, and the system thus behaves piecewise linearly. The solution was obtained by applying the Laplace transform method in combination with the Finite Difference Method for the spatial discretization. As a result, the authors obtained "a computationally efficient solution method for an infinite beam system". This system exhibits nonlinear behaviour, and the influence of the foundation's nonlinear behaviour on the generated waves (i.e.,transition radiation), and on the resulting plastic deformation can be described. In [23] the authors investigate transient wave propagation in a 1-D gradient model with material nonlinearities. The study reveals that material nonlinearities lead to "energy exchange: kinetic/potential energy decreases/increases over time". Also in the presence of damping, it was found that small-amplitude waves travel opposite to the main wave direction. In [8] the oscillations of a finite beam resting on a nonlinear Kelvin–Voigt viscoelastic foundation subjected to a moving load, has been investigated. The investigation was performed by using the Galerkin method and the multiple scales perturbation method. The influence of "different parameters (such as the magnitude of the load speed, the foundation nonlinearity and the damping coefficients) on the frequency responses", were investigated as well. The Galerkin procedure was also used in [12] to investigate the dynamics of a simply-supported, finite length Euler–Bernoulli beam resting on a linear, and on a nonlinear viscoelastic foundation. In [13] the authors "analytically study the steady state responses of an infinite Euler–Bernoulli beam resting on nonlinear viscoelastic foundation under a harmonic, moving load". The governing nonlinear partial differential equation for the beam motion was "converted to two nonlinear Volterra integral equations by using the Fourier transform method, the residues theorem and the convolution theorem". The modified Adomian decomposition method was applied to obtain approximate analytical solutions for the steady state responses of the beam. In [9] the authors investigate the dynamic responses of an infinite Timoshenko beam supported by a nonlinear viscoelastic foundation subjected to a moving concentrated force. The governing differential equations are solved by using the Adomian decomposition method, a perturbation method, and the Fourier transformation method. A semi-analytical model to describe a periodically supported Euler–Bernoulli beam on a nonlinear foundation was proposed in [30]. The authors of [30] use the Fourier transform method to obtain the relation between the displacement of the beam and the reaction forces of its supports in steady state. The authors of [24] consider the problem of the wave propagation in an Euler–Bernoulli beam resting on a unilateral (tensionless) elastic foundation. Some "physically admissible" analytical solutions were found and analyzed. In[25] the authors investigate the effect of nonlinear terms, which occur due to a geometric nonlinearity, on the dynamic response of the mass-beam-foundation system. Under the assumption that the inertial effect in the axial direction is negligible, the two equations of motion reduce to one. The infinitely long beam was replaced by a sufficiently long finite beam after which a harmonic expansion method to discretize the partial differential equations of motion, was used. The nonlinear dynamics of beams on a nonlinear, fractional viscoelastic foundation subjected to a moving load with a variable speed, was considered in [26]. The geometric nonlinearity of the beam and the nonlinearity of the foundation were taken into account. To solve the problem approximately the authors applied a Galerkin procedure and a "numerical method based on the finite difference method". In [31] an incremental harmonic balance method was used to study the steady-state responses of a pinned-pinned beam on a nonlinear viscoelastic foundation due to harmonic loads. The incremental harmonic balance method is coupled to a parameter continuation technique to find the nonlinear frequency response of the system. Also numerical research results involving the dynamics of beams on nonlinear foundations subjected to moving loads,can be found in [16,18,27,32,33]. The numerical methods are very efficient tools, which are widely used in engineering practice, but are not, of course, able to provide closed form solutions. Moreover, it might be hard to find bifurcations in the problem by only using numerical methods. Closed form approximations of the solutions can be useful for numerical case studies, because they can provide a deeper and simpler engineering estimate and understanding of the problem. That is the reason why we, in this paper concentrate on analytical methods. In this paper we consider the dynamics of an infinite beam under the action of a concentrated load which is moving along the infinite beam with a constant velocity, and where the beam is connected to a viscoelastic nonlinear foundation. The main topic of this paper is the construction of a localized solution for this problem. In this paper we consider linear and weakly nonlinear models for the viscoelastic foundation, and use an analytical approach to solve the problem. We are interested in localized wave solutions mostly for two reasons. The first is that traveling non-localized solutions even can have large amplitudes, and if viscosity is taken into account, the decay can be much faster than for localized ones. Secondly, there are no known simple formulas for an approximation of a solution for such problems with weak nonlinearities, which show the influence of parameters on localized solutions. By definition a localized solution is a solution with bounded L_2 norm, i.e. $\int_{-\infty}^{\infty} u^2(x,t)dx < \infty$. We introduce a new internal beam damping model which (depending on its parameters) is capable of describing air and signum like damping models. To obtain an approximate analytical solution we use energy estimates, some a priory estimates, Ritz method, perturbation techniques, and the Contracting Map Principle. As a result, some simple formulas are obtained which allow us to draw conclusions about the influence of the nonlinearity in the beam's elastic foundation on the beam displacement and on the critical velocities of the system. Also, the influence of damping (viscosity) on the form of the localized solution is investigated as well. Special attention is paid to the behavior of the system near or in a critical velocity region. We introduce an effective coefficient for the Winkler nonlinear foundation, which makes it possible to draw some physical conclusions about waves (not only localized ones).

The paper is organized as follows. In Sect. 2 we formulate the problem. In the Sect. 3 we show that the problem is well posed. In Sect. 4 the linear problem will be considered, and in Sect. 5 we construct approximations of localized solutions for a weakly nonlinear problem, and give proofs for the existence or non-existence of localized solutions. Further, in Sect. 6, an asymptotic approximation of the solution for a weakly nonlinear problem in the region of a critical velocity, and with possible resonances in the nonlinear system, will be considered. The asymptotic formula for the critical velocities for a weakly nonlinear system will also be presented. In Sect. 7, we consider the influence of nonlinearities (both in the elastic foundation coefficients and in the structural damping coefficients) on the stability of a solution. Finally, Sect. 8 contains some concluding remarks.

2 Formulation of the problem

The equation describing the dynamics of an infinite Euler–Bernoulli beam on a nonlinear elastic Winkler foundation, which is under the action of a concentrated moving load with a constant velocity, is given by:

$$m_0 w_{tt} + D w_{xxxx} + K(w + \alpha w^3) + \epsilon f[w(\cdot,\cdot)] = F(x - Vt), \tag{1}$$

where $w(x,t)$ is the beam transverse displacement, $x \in (-\infty,\infty)$ is the longitudinal coordinate, $t > 0$ is the time, $m_0 = S\rho$ is the mass of the beam per unit length, S is the beam's cross sectional area, ρ is the material density of the beam, $D = EI$ is the rigidity of the beam, I is the moment of the cross-section area, E is Young's modulus, $f[w(\cdot,\cdot)]$ is a functional, which defines nonlinear structural damping forces, ϵ is a damping coefficient, and $K > 0$ is the Winkler elastic foundation coefficient, and α is a nonlinear elastic foundation coefficient. We suppose that the parameters α and ϵ are small but could have different orders. Note that eq. (1) also describes the dynamics of the beam which has a structural damping and lies on a viscoelastic foundation. Therefore, both the damping force $\epsilon f[w(\cdot,\cdot)]$ (or the viscosity in the beam foundation), and the nonlinear part of the elastic foundation coefficient term are small. $F(x - Vt)$ is a moving force along the beam, where V is a constant velocity. We suppose that the external force $F(x - Vt)$ is defined by a positive function $F_r(z)$, which is even in $z = x - Vt$ such that

$$F_r(z) = 0 \quad |z| > r,$$

and we suppose that F_r is close to a δ-like function as $r \to 0$:

$$\int_{-r}^{r} F_r(z)dz = C_F > 0, \int_{-r}^{r} F_r^2(z)dz = r^{-1}C_2, F(z) = F(-z), \tag{2}$$

where $C_F >$ is a constant. Different choices for the functional $f[w(\cdot,\cdot)]$ are possible. We, follow [34], and choose the following forms for the functional f: $f[w(\cdot,\cdot)] = w_t$, or

$$f[w(\cdot,\cdot)] = \text{sign}(w_t)w^2, \tag{3}$$

where the sign function is defined as follows:

$$\text{sign}(v) = 1,\ v > 0, \quad \text{sign}(v) = -1, v < 0,$$

and we set $\text{sign}(0) = 0$. In [34] the signum type of damping model was suggested because it well describes the damping forces in steels, which are often used in the production of rails. There is a smooth version of (3), which describes (depending on the parameter choice) both cases (an air damping and a sign type of damping):

$$f_\beta[w(\cdot,\cdot)] = \tanh(\beta w_t)(w^2 + r\beta^{-1}), \tag{4}$$

where $\beta, r > 0$ are parameters. One can note that $f_\beta \to \text{sign}(w_t)w^2$ as $\beta \to +\infty$ and, on the other hand, when $\beta \to 0$, we obtain $f \to rw_t$. Another possible choice for the functional $f[w(\cdot,\cdot)]$ is the Kelvin–Voigt type of damping, when

$$f[w(\cdot,\cdot)] = w_{xxxxt}. \tag{5}$$

Initial conditions have the form

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \tag{6}$$

where w_0, w_1 are smooth functions. We consider localized solutions for the infinite beam problem. Then, the following boundary conditions hold:

$$w_x(x,t), w(x,t) \to 0, \quad x \to \pm\infty,\ t > 0. \tag{7}$$

Notice that the differential equation (1) and the given initial and boundary conditions can be transformed to a dimensionless form by introducing new variables: $x = \bar{x}L, w = \bar{w}L, L = \bar{L}\sqrt{S}, D = EI\bar{D}, t = \bar{t}\frac{L}{c_0}, c_0^2 = \frac{E}{\rho}$, $\bar{K} = \frac{KL^2}{SE}, \bar{\alpha} = L^2\alpha, \bar{\epsilon} = \frac{\epsilon L}{\sqrt{\rho E S^2}}, \bar{F} = \frac{F}{LE}$, where L is a parameter which has a length dimension, and S is the cross-sectional area of the beam. For simplification the bars are omitted.

3 An estimate of the energy

We seek solutions $w(x,t)$ of the weakly nonlinear initial-boundary value problem (IBVP) defined by (1), (6), and (7). Under certain conditions on $f_\beta[w(\cdot,\cdot)]$ the existence of solutions follows from an a priori estimate, which gives us an upper bound for energy of the beam. We use the following standard notation

$$(w, v) = \int_{-\infty}^{\infty} w(x)v(x)dx, \quad \|w\| = (w, w)^{1/2},$$

and

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(z)|^p dz\right)^{1/p}.$$

Let us introduce the dissipation functional

$$P[w_t, w] = \int_{-\infty}^{\infty} f_\beta[w(x,t)]w_t(x,t)dx.$$

Note that in the cases (3) and (4) this functional is nonnegative: $P \geq 0$ for all w_t, w. By multiplying the left and right hand sides of (1) with w_t, then by integrating with respect to x from $-\infty$ to ∞, and by taking into account the boundary conditions (7), one obtains:

$$\frac{dE(t)}{dt} = -\epsilon P[w_t, w] + \bar{F}_r(t), \tag{8}$$

where

$$E(t) = E_{\text{kin}}(t) + E_{\text{pot}}[w],$$

is the energy of the beam, which is the sum of the kinetic and potential energies:

$$E_{\text{kin}}(t) = \frac{m_0}{2}\|w_t\|^2,$$
$$E_{\text{pot}}[w] = \frac{1}{2}\Big(\int_{-\infty}^{\infty} Dw_{xx}^2 + Kw^2 + K\frac{\alpha}{2}w^4\Big)dx,$$

and

$$\bar{F}_r(t) = \int_{-\infty}^{\infty} F_r(x - Vt)w_t dx.$$

Lemma 1 *Assume that $\alpha \geq 0, \epsilon \geq 0$, and that f has the form (3) or (4). Then, for solutions of the IBVP defined by (1), (6) and the conditions (7) one has*

$$E(t) \leq \Big(\frac{bt}{2} + \sqrt{E(0)}\Big)^2, \tag{9}$$

where $E(t)$ is the value of the energy functional at the time t, and

$$b = (2(m_0 r)^{-1}C_2)^{1/2}.$$

Moreover

$$\|w_t(\cdot,t)\|^2,\ \|w(\cdot,t)\|^2, \|w_x(\cdot,t)\|^2, \|w_{xx}(\cdot,t)\|^2$$
$$< \bar{C}\left(\frac{bt}{2} + \sqrt{E(0)}\right)^2, \quad \forall t \in [0, T], \tag{10}$$

where $\bar{C}$ is a positive constant.

Proof By the Cauchy–Schwarz inequality and (2) we obtain the following estimate:

$$\bar{F}_r^2(t) \leq \int_{-\infty}^{\infty} F_r(x - Vt)^2 dx \int_{-\infty}^{\infty} w_t^2 dx$$
$$\leq C_2 r^{-1}\|w_t\|^2.$$

Therefore,

$$|\bar{F}_r(t)| \leq \big(2(m_0 r)^{-1}C_2\big)^{1/2}(E_{\text{kin}}(t))^{1/2} \leq b(E(t))^{1/2},$$

where $b = \big(2(m_0 r)^{-1}C_2\big)^{1/2} > 0$ is a constant. Moreover, one has $P[w_t, w] \geq 0$. Then (8) gives

$$dE/dt \leq bE^{1/2}.$$

This differential inequality implies (9), which in turn, gives (10). □

Note that the same estimate of E can be obtained for the Kelvin–Voigt model because the damping contribution in the energy equality (8) is given by $P = \int_{-\infty}^{\infty} w_{xxt}^2 dx$, and it also is non-negative. This result gives us a priori estimates for the L_2-norms of w and w_t, which show that generalized solutions of (1) exist for all positive times t if the initial data have bounded L_2-norms, that is,

$$\|w_0\|,\ \|w_1\| < c_0.$$

It should be noted that these energy estimates are not uniform in r as $r \to 0$. In the case of Kelvin–Voigt damping we find a similar energy estimate, which is uniform in the localization parameter $r > 0$, and therefore it holds for forces $F = C_F\delta(x - Vt)$. Thus, for the Kelvin–Voigt type of damping the energy estimate does not depend on the localization parameter. The physical reason for this effect is that the Kelvin–Voigt damping model has a stronger smoothing effect compared to the simple air or signum types of damping. Let us formulate the following lemma.

Lemma 2 *Assume that $\alpha \geq 0, \epsilon \geq 0$, and that $f = w_{xxxxt}$, and $F = C_F\delta(x - Vt)$. Then, for solutions of the IBVP defined by (1), (6), and the conditions (7) one has*

$$E(t) \leq \big(b_1 t + \sqrt{E(0)}\big)^2, \tag{11}$$

where $b_1 > 0$ is a constant.

The proof follows the same lines as for the proof of Lemma 1, but it is little bit more complicated (see the Appendix, subsection 9.1, for the details of this proof).

4 Linear problem in which F is a $\delta(z)$-like force

In this section, we let $r \to 0$ and consider the external force $F = C_F\delta(z)$ with $C_F > 0$. Furthermore, we are looking for localized solutions having the form $w(x,t) = u(z,t)$, with $z = x - Vt$. Similar solutions were obtained in some previous papers (see for example [3]), but because we will need to use these types of solutions in the next sections of the paper we will now present an alternative derivation. First, we introduce a coordinate transformation (convenient for studying special traveling wave solutions). Then, equation (1) takes the form

$$m_0 u_{tt} + D u_{zzzz} + m_0 V^2 u_{zz} - 2 m_0 V u_{zt} + K(u + \alpha u^3) + \epsilon f(u_t - V u_z, u) = F(z), \qquad (12)$$

and the boundary conditions become

$$u(z,t), u_z(z,t) \to 0, \quad z \to \pm\infty. \qquad (13)$$

4.1 Localized waves in the linear case

Let us first consider the linear case without damping, that is $\epsilon = 0, \alpha = 0$, and let us suppose that w has a traveling wave form, i.e. $w = u(z)$. We take $F = C_F \delta(z)$. Then, Eq. (12) becomes

$$\mathbf{L}u \equiv m_0 V^2 u_{zz} + D u_{zzzz} + K u = C_F \delta(z). \qquad (14)$$

It is clear that we can look for a solution $u = U$, which is even in z. In fact, an arbitrary solution can be represented as a sum of even and odd parts, $u(z) = u_{ev}(z) + u_{odd}(z)$. Then, (14) implies that $\mathbf{L}u_{\text{odd}} = 0$ and for $V \neq V_c$, where V_c is defined by

$$m_0^2 V_c^4 = 4DK,$$

this equation has no decreasing solutions for $|z| \to +\infty$. This can be seen as follows. The odd solution which tends to 0 for $|z| \to \infty$, has to satisfy $\mathbf{L}u = 0$, or equivalently

$$m_0 V^2 u_{zz} + D u_{zzzz} + K u = 0,$$

add this solution has the form

$$U_{\text{odd}}(z) = C_1 \exp(-\lambda_1 z) + C_2 \exp(-\lambda_2 z), \quad z > 0,$$

and

$$U_{\text{odd}}(z) = C_1 \exp(\lambda_1 z) + C_2 \exp(\lambda_2 z), \quad z < 0,$$

where the real parts of λ_i are negative. For odd solutions, we have $u(0) = 0$ and $d^2u/dz^2(0) = 0$. Thus $C_1 = -C_2$ where $C_1 = C_2 = 0$ or $\lambda_1^2 = \lambda_2^2$. However,$\lambda_1^2 \neq \lambda_2^2$ for $V \neq V_c$, and so we obtain that $C_1 = C_2 = 0$. Note that when $V = V_c$ the non-trivial odd localized solution does not exist because $\operatorname{Re}\lambda_1 = \operatorname{Re}\lambda_2 = 0$. Then, we have for U the following expressions. Even solutions U can be represented as

$$U(z) = U^+(z), \quad z > 0; \quad U(z) = U^-(z), \quad z < 0,$$

and for $U^\pm$ we have

$$U^+(z) = C_+ \exp(-\lambda_1 z) + C_- \exp(-\lambda_2 z), \qquad (15)$$

$$U^-(z) = C_+ \exp(\lambda_1 z) + C_- \exp(\lambda_2 z), \qquad (16)$$

where $\lambda_{1,2}$ are given by

$$\lambda_{1,2}^2 = \frac{-m_0 V^2 \pm \sqrt{(m_0 V^2)^2 - 4DK}}{2D} \qquad (17)$$

and, to have decreasing solutions $U^\pm$ for $z \to \pm\infty$, we have to take in (17) the roots $\lambda_{1,2}$ with positive real parts: $\operatorname{Re}\lambda_i > 0$ (if they exist). Indeed, the solution U is localized in z if the real parts of λ_i are positive. Three cases have to be considered:

A a low velocity force, where

$$4DK > (m_0 V^2)^2; \qquad (18)$$

B a high velocity force, where $m_0 V^2 > 2\sqrt{DK}$;

C the velocity V is equal to the critical one V_c, that is, $V = V_c$, and $m_0 V_c^2 = 2\sqrt{DK}$.

Note that the value V_c defines a velocity resonance in the linear case without damping. As $V \to V_c$ the localized solution amplitude tends to ∞. For this value $V = V_c$, the quadratic Lagrangian associated with the linear problem loses its property of positive definiteness. We will see later that in the weakly nonlinear case the amplitudes of the solutions are bounded for all V (see section 5.3). Then, we can define the critical velocity V_c^* as the velocity, which gives the largest amplitude $U(0)$. For case **A** we introduce a parameter $\kappa > 0$ defined by

$$\kappa^2 = 4DK - (m_0 V^2)^2. \qquad (19)$$

The quantity κ takes real values under the condition (18). One has

$$\lambda_1 = i k_0 - \gamma, \quad \lambda_2 = -i k_0 - \gamma$$

where $i = \sqrt{-1}$, and where k_0, and γ are constants depending on the problem parameters. Then,

$$\lambda_{1,2}^2 = (2D)^{-1}\left(-m_0 V^2 \pm i\kappa\right)$$

implying

$$|\lambda_{1,2}| = (2D)^{-1/2}\big(m_0^2V^4 + \kappa^2\big)^{1/4}, \tag{20}$$

and

$$\lambda_{1,2} = (2D)^{-1/2}\big(m_0^2V^4 + \kappa^2\big)^{1/4} \times \exp\Big(\pm\frac{i}{2}\arctan\Big(\frac{\kappa}{m_0V^2}\Big)\Big). \tag{21}$$

For small $\kappa > 0$ one has

$$k_0 = \sqrt{\frac{m_0V^2}{2D}} + O(\kappa), \quad \gamma = \frac{k_0\kappa}{2m_0V^2} + O(\kappa^2). \tag{22}$$

For not too small κ the expressions for k_0 and γ are harder to obtain and will be not given here. The unknown constants $C_\pm$ can be found by the matching conditions at $z = 0$, which can be written in the form:

$$\frac{d^pU^+(z)}{dz^p}|_{z=0} = \frac{d^pU^-(z)}{dz^p}|_{z=0} + p(p-1)(p-2)C_F/6D, \quad p = 0, 1, 2, 3. \tag{23}$$

These matching conditions guarantee a 4 times continuously differentiable solution u. For $p = 0, 2$ the conditions (23) hold automatically because the solution U is even in z. The conditions (23) for $p = 1, 3$ lead to the following system of linear algebraic equations for $C_\pm$:

$$C_+\lambda_1 + C_-\lambda_2 = 0, \quad C_+\lambda_1^3 + C_-\lambda_2^3 = C_F/D$$

which implies

$$U(z) = A_0\big(\lambda_2\exp(-\lambda_1|z|) - \lambda_1\exp(-\lambda_2|z|)\big), \tag{24}$$

where A_0 is defined by

$$A_0 = 2iW, \quad W = \frac{C_F}{4iD\lambda_1\lambda_2(\lambda_1^2 - \lambda_2^2)}. \tag{25}$$

For $z > 0$ we obtain from (24):

$$U(z) = W\exp(-\gamma z)\big(k_0\cos(k_0z) + \gamma\sin(k_0z)\big), \quad z > 0, \tag{26}$$

and

$$U'(z) = W(k_0^2 + \gamma^2)\exp(-\gamma|z|)\sin(k_0z), \tag{27}$$

where

$$W = C_F\kappa^{-1}\sqrt{D/K}.$$

Finally, for case **A** we obtain that a localized solution exists which exponentially decreases for $|z| \to \infty$. For case **C** such a localized solution does not exist since λ_i have zero real parts (according to (17)). For case **B** a localized solution exists, but it has a large amplitude when $(m_0V^2)^2 < 4DK$, and which tends to $+\infty$ as $m_0V^2 \to 2\sqrt{DK}$.

5 Localized wave solutions for a weakly non-linear problem

We consider the following boundary value problem:

$$Du_{zzzz} + m_0V^2u_{zz} + K(u + \alpha u^3) + \epsilon f(-Vu_z, u) = C_F\delta(z), \tag{28}$$

$$u(z), u_z(z) \to 0, \quad z \to \pm\infty. \tag{29}$$

5.1 Case **A**: $4DK > (m_0V^2)^2$

Let us introduce the linear operator **L** by

$$\mathbf{L}u = m_0V^2u_{zz} + Du_{zzzz} + Ku, \tag{30}$$

The corresponding Green's function $G(z-z_0)$ satisfies:

$$m_0V^2G_{zz} + DG_{zzzz} + KG = \delta(z - z_0). \tag{31}$$

The operator **L** is positively definite in case **A**. The expression for Green's function is given by the following formula(see also (24)):

$$G(z - z_0) = C_g^{-1}\big(\lambda_2\exp(-\lambda_1|z - z_0|) - \lambda_1\exp(-\lambda_2|z - z_0|)\big), \tag{32}$$

where

$$C_g = 2D\lambda_1\lambda_2(\lambda_1^2 - \lambda_2^2).$$

A similar result can be found in [13] for a case with the simple air damping. In this paper we consider a general nonlinear type of damping. The main result for case **A** is the following one:

Theorem 1 *Let $f = f(u_t, u)$ be a smooth function. Moreover, let $\kappa > 0$, $\epsilon > 0$, and $\alpha > 0$ be sufficiently small: $\epsilon < \epsilon_0(\kappa)$, $\alpha < \alpha_0(\kappa)$. Then, there exists a* *solution $u(z)$ of the boundary value problem (28), (29) such that*

$$u = U + \tilde{u}, \tag{33}$$

where U is the solution of the linearized problem, and the correction $\tilde{u}$ satisfies the estimates

$$\sup_z |\tilde{u}(z)| < C_1(\epsilon + \alpha), \quad \sup_z |\tilde{u}_z| < C_2(\epsilon + \alpha), \tag{34}$$

and

$$|\tilde{u}(z)| < C_3(\epsilon + \alpha)\exp(-a_0|z|), \tag{35}$$

where C_i and a_0 are positive constants uniform in $\epsilon > 0$, and in $\alpha > 0$.

The proof is standard and is presented in the Appendix (see subsection 9.2). It should be remarked that the proof differs from the one as given in [13]).

5.2 Local dynamic stability and global convergence to a localized solution

Let us consider the initial-boundary value problem (IBVP) defined by (1), (6) and the conditions (7). Remind that the Sobolev space $W^{1,2}(\mathbb{R})$ consists of measurable functions $f(z)$ such that this function and its weak derivative satisfy

$$\|f\|^2_{2,2} = \int_{-\infty}^{\infty}\left(f(z)^2 + \left(\frac{df}{dz}\right)^2\right)dz < \infty.$$

The following theorem can be formulated:

Theorem 2 *Let the function $f = f(u_t, u)$ be a smooth function.*

Suppose that initial data u_0, u_1 satisfy

$$\left\|U - u_0\right\|_{1,2} + \left\|\frac{dU}{dz} - u_1\right\| < \delta, \tag{36}$$

where $\delta > 0$ is a small constant. Let $|V| < V_c$. Then, for sufficiently small $\epsilon > 0$ and $\alpha > 0$ solutions of the IBVP defined by (1), (6) and the conditions (7) have the form

$$u = U + w_1, \tag{37}$$

where the correction term w_1 satisfies

$$\|w_1(\cdot, t)\|_{1,2} < C_1\delta \quad \forall t \in [0, C\epsilon^{-1}], \tag{38}$$

and where $C(\delta)$, C_1 are positive constants.

The proof is similar to the ones for Lemmas 1 and 2, and can be found in the Appendix (see subsection 9.3). Note that theorem 2 is also valid for Kelvin–Voigt damping, however, the proof for this kind of damping requires more advanced methods from Functional Analysis. The proof will be omitted in this paper.

We will investigate the spectrum of the operator $\mathbf{A}$ in section 7. As will turn out, this operator is positively definite for small $\alpha > 0$ and $\epsilon > 0$ if the localized solution of the linear problem is stable.

5.3 Approximations of solutions for a weakly non-linear problem

In this section, we propose a heuristic approach in order to find an approximation of a localized solution for a problem describing a beam on a weakly nonlinear elastic foundation ($\alpha > 0$, and small) when damping is absent ($\epsilon = 0$). The equation reads:

$$m_0V^2U_{zz} + DU_{zzzz} + K(U + \alpha U^3) = C_F\delta(z). \tag{39}$$

We apply the classical Ritz approach as follows. First we note that equation (39) subject to the boundary conditions (13) is equivalent to a minimization problem for the following Lagrangian associated with our boundary value problem:

$$\mathcal{L}[U] = C_F u(0) + \frac{1}{2}\int_{-\infty}^{\infty}\left(DU_{zz}^2 + KU^2 + \frac{K\alpha U^4}{2} - m_0V^2U_z^2\right)dz. \tag{40}$$

The minimization should be performed for the boundary conditions (13). To find an approximating minimum of the Lagrangian (40), we use the test function U defined by (26) with an unknown amplitude $W \geq 0$. This procedure leads us to the following averaged Lagrangian:

$$\bar{L}[W] = b_2\frac{W^2}{2} + \frac{\alpha b_4 W^4}{4} - C_F W, \tag{41}$$

where the coefficients b_2 and b_4 are defined by

$$b_2 = 2\int_0^\infty \left(D\hat{U}_{zz}^2 - m_0V^2\hat{U}_z^2 + K\hat{U}^2\right)dz,$$

$$b_4 = 2K\int_0^\infty \hat{U}^4 dz,$$

where

$$\hat{U}(z) = \exp(-\gamma z)\big(k_0\cos(k_0 z) + \gamma\sin(k_0 z)\big).$$

The coefficients b_2 and b_4 depend on D, K, m_0, and V in a complicated way (see in the Appendix subsection 9.4). We consider case **A** with $b_2 > 0$. By differentiating the averaged Lagrangian defined by (41) with respect to W we obtain the following equation for W:

$$b_2W + \alpha b_4W^3 = C_F. \tag{42}$$

Under the condition $b_2 > 0$ the left hand side of this equation increases in W. Therefore, this equation has a single root. This equation shows that the amplitude of a localized solution (if this solution exists) is bounded for all V. In case **A** and for small $\alpha > 0$ one obtains

$$W \approx b_2^{-1}C_F - \alpha\frac{b_4C_F^3}{b_2^4} + O(\alpha^2). \tag{43}$$

Case **C** will be considered in the next section. Eq. (42) leads to an important conclusion:

For $b_2 > 0$, and for $C_F > 0$ the amplitude of a localized solution decreases as α increases, and increases as C_F increases. The second effect is weaker for larger values of α.

To show this, we apply the Implicit Function Theorem to eq. (42). Let $W(\alpha)$ be the solution of this equation for different $\alpha > 0$, and let us denote by W_α its derivative with respect to α. By differentiating (42) with respect to α one obtains

$$(b_2 + 3\alpha b_4W^2)W_\alpha = -b_4W^3, \tag{44}$$

and we see that $W_\alpha < 0$ (because for $|V| < V_c$ we have $b_2 > 0$). Note that the quantity $\bar{K} = K(1+\alpha W^2)$ can be interpreted as *an effective Winkler coefficient*, which arises as a result of linear and nonlinear effects. For $\alpha = 0$, the frequency of a harmonic perturbation $u = A\exp(ikx + i\omega t)$ satisfies the condition $-m_0\omega^2 + Dk^4 + K \le 0$ that gives a cutoff frequency $\omega_{cut} = \sqrt{K/m_0}$. If $0 < \omega < \omega_{cut}$ such harmonic waves fail to propagate. For $\alpha > 0$ we obtain that $\omega_{cut} = \sqrt{\bar{K}/m_0}$,

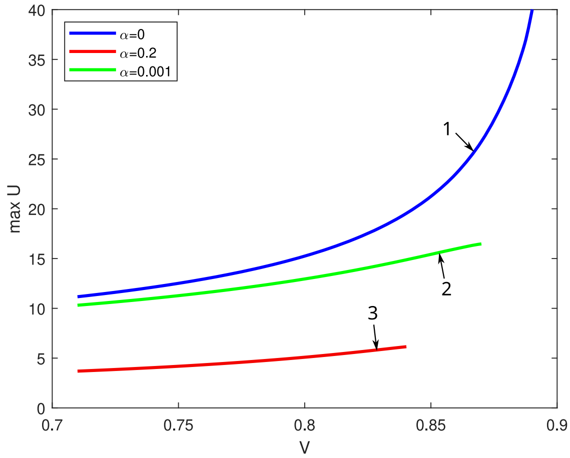

Fig. 1 The dependence of the amplitudes of localized solutions on the velocity of a concentrated load for linear and nonlinear cases. Curve 1 corresponds to the linear system, and the curves 2 and 3 to the nonlinear system. The amplitudes are found by the Ritz method (see section 5.3). The constants b_2, b_4 are computed numerically. The parameters are $D = 1$, $K = 0.16$, $m_0 = 1$, $C_F = 0.1$. No damping is present.

and this threshold depends on the amplitude W. For velocities V close to V_c this amplitude becomes large.

5.4 In case **B** localized solutions do not exist

The following theorem asserts that in case **B** localized solutions of eq. (28) do not exist.

Theorem 3 *For $|V| > V_c$ the boundary value problem (28), (29) has no solutions.*

The proof can be found in the Appendix (see subsection 9.5).

5.5 Perturbations induced by damping

The proof of Theorem 1 shows that we can find solutions by iterations. In this subsection, we will see that the first iteration allows us to identify the influence of damping. In fact, the solution without damping is even in z. Therefore, a Fourier analysis with a small odd part in the solutions permits us to see the damping influence. To show this, let us compare two cases: the simple air damping (viscosity), and the signum type of damping.

General approach. We split the solution U up into even and odd components $U^+(z)$ and $U^-(z)$, respectively. Here $F^\pm$ denotes the even and odd parts of a

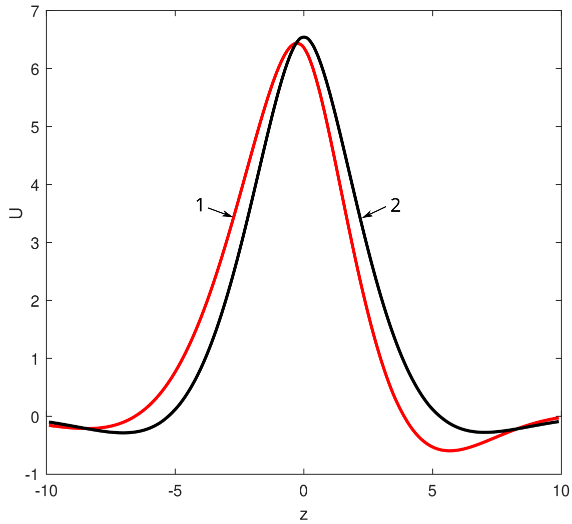

Fig. 2 Localized solutions for $\alpha = 10^{-3}$, $D = 1$, $K = 0.16$, $m_0 = 1$, $C_F = 0.1$ and velocity $V = 0.02$. The damping term is a sum of simple air damping and signum type of damping. The symmetric curve 2 corresponds to the case when damping is absent in the system. In this case the maximum is located at $z = 0$. The weakly non-symmetric curve 1 describes the solution when the damping coefficient $\epsilon = 0.1$. In this case the maximum of U shifts from position $z = 0$ to $z \approx -0.2$.

function F, that is,

$$F^{\pm}(z) = \frac{1}{2}\big(F(z) \pm F(-z)\big).$$

Then,

$$\mathbf{L}U^+ = C_F\delta(z) - \alpha K(U^+)^3 - 3\alpha K(U^-)^2U^+ \\ +\epsilon f(-V(U_z^+ + U_z^-), U^+ + U^-)^+,$$

$$\mathbf{L}U^- = -\alpha K(U^-)^3 - 3\alpha K U^-(U^+)^2 \\ +\epsilon f(-V(U_z^+ + U_z^-), U^+ + U^-)^-,$$

where $\mathbf{L}$ is the linear operator defined in (30).

Taking into account only the leading terms of orders 1, α and ϵ, we obtain the following equations:

$$\mathbf{L}U^+ = C_F\delta(z), \tag{45}$$

$$\mathbf{L}U^- = \epsilon f(-V(U_z^+ + U_z^-), U^+ + U^-)^-. \tag{46}$$

For U_z^+ we use formula (27):

$$U_z^+ = U'(z) = W(k_0^2 + \gamma^2)\exp(-\gamma|z|)\sin(k_0 z).$$

Eq. (46) is complemented by the boundary conditions

$$U^-(z) \to 0 \quad |z| \to +\infty.$$

Since eq. (46) is linear, effects of simple air damping and signum type of damping (if they act together) are additive. So, we can consider them separately. Moreover, let us note that the eqs. (45) and (46) show that the damping-induced change of the solution amplitude is of order $O(\epsilon^2)$. In fact, the maximum of the unperturbed solution is located at $z = 0$, the function U^- is odd and this perturbation does not shift this maximum because $U^-(0) = 0$, thus the shift of the z position of $\max U$ induced by damping, is of order ϵ. Localized solutions of Eq. (46) can be constructed by using some auxiliary functions $w_{k,\theta}$ and a Fourier series. In fact, we note that for small $\alpha > 0$ and $\epsilon > 0$ we can solve system (45), (46) for $U^{\pm}$ by iterations, where at the first step $U^- = 0$. At the second step, the right hand sides of eqs. (45), (46) are linear combinations of the expressions $\exp((\pm ik_0 - \gamma)z)$ and their powers, where k_0, γ are defined by (22). Let us introduce the functions $w_{k,\theta}$, which are localized solutions of the following equations:

$$\mathbf{L}w_{k,\theta} = \exp(ikz - \theta|z|), \tag{47}$$

where $\theta > 0$ is a parameter. This parameter will take values $\gamma, 2\gamma$ and 3γ. There are two cases: the non-resonant case **NR**, where $k \neq k_0$ or $\theta \neq \gamma$, and the resonant one **R**, where $k = k_0$ and $\theta = \gamma$. For the **NR** case we can construct (odd in z) localized solutions $w_{k,\theta}$ of eq. (47) in the following way. Let us define functions P defined in the complex plane $\mathbb{C}$:

$$P(y) = \left(Dy^4 + mV^2y^2 + K\right)^{-1}, \quad y \in \mathbb{C},$$

and let us define the functions $w_{k,\theta}^+$ by

$$w_{k,\theta}^{\pm}(z) = P(ik \pm \theta)\big(\exp((ik \pm \theta)z) + \tilde{w}_k^{\pm}(z)\big),$$

where the functions $\tilde{w}_k^{\pm}$ are defined by

$$\tilde{w}_k^{\pm} = a_k^{\pm}\exp(\pm(-\lambda_1 z)) + b_k^{\pm}\exp(\pm(-\lambda_2 z)).$$

Here, $a_k^{\pm}, b_k^{\pm}$ are unknown coefficients. Then, for $z > 0$ we set

$$w_{k,\theta}(z) = w_{k,\theta}^+(z),$$

and for $z < 0$

$$w_{k,\theta}(z) = w_{k,\theta}^-(z).$$

The coefficients $a_k^{\pm}, b_k^{\pm}$ can be found by the matching conditions at $z = 0$:

$$\frac{d^p w_{k,\theta}^+(z)}{dz^p}\Big|_{z=0} = \frac{d^p w_{k,\theta}^-(z)}{dz^p}\Big|_{z=0}, \quad p = 0, 1, 2, 3. \tag{48}$$

By using (48) we obtain a linear algebraic system for $a_k^{\pm}, b_k^{\pm}$:

$$\lambda_1^p a_k^+ + \lambda_2^p b_k^+ - (-\lambda_1)^p a_k^- - (-\lambda_2)^p b_k^- = c_{p,k}, \quad (49)$$

where

$$c_{p,k} = P(ik-\theta)(ik-\theta)^p - P(ik+\theta)(ik+\theta)^p.$$

In the resonant case, we modify the representation for $w_{k,\theta}^+$ as follows. For $z > 0$ we take

$$w_{k,\theta}^+ = z\Big(\frac{\exp(-\lambda_2 z)}{4D\lambda_2^3 + 2m_0V^2\lambda_2} - \frac{\exp(-\lambda_1 z)}{4D\lambda_1^3 + 2m_0V^2\lambda_1}\Big)$$

and for $z < 0$

$$w_{k,\theta}^- = z\Big(\frac{\exp(\lambda_2 z)}{4D\lambda_2^3 + 2m_0V^2\lambda_2} - \frac{\exp(\lambda_1 z)}{4D\lambda_1^3 + 2m_0V^2\lambda_1}\Big).$$

The expressions for $\tilde{w}_k$ have the same form as before, and the coefficients $a_k^{\pm}, b_k^{\pm}$ can be found by the matching relations (48).

Simple air damping, $\alpha = 0$. Now

$$f(-V(U_z^+ + U_z^-), U^+ + U^-)^- = -VU_z^+.$$

We use formula (24) for $z > 0$ implying that

$$U_z(z) = A_0\lambda_2\lambda_1\Big(\exp(-\lambda_1 z) - \exp(-\lambda_2 z)\Big). \quad (50)$$

Note that $\lambda_1\lambda_2 = \sqrt{K/D}$. This yields

$$U^-(z) = 2\epsilon i A_0\sqrt{K/D}\,w_{k_0,\gamma}^- + O(\epsilon^2),$$

where the quantity A_0 is defined by (25) and iA_0 is a real valued number. This perturbation U^- is relatively simple and includes a single odd Fourier harmonic $\sin(k_0 z)$.

Signum type damping. In this case, the correction $\tilde{U}^-$ satisfies the equation

$$\mathbf{L}U^- = -\epsilon\,\mathrm{sign}(-VU_z^+)(U^+)^2 = \epsilon g_s(z). \quad (51)$$

By (26) we see that $U^+ = U$ and

$$g_s(z) = U^2(z)\,\mathrm{sign}_{\mathrm{per}}(z),$$

where $\mathrm{sign}_{\mathrm{per}}$ is a $2\pi/k_0$-periodic signum function, which is equal to 1 for $z \in (0, \pi/k_0)$ and -1 for $z \in (-\pi/k_0, 0)$. Therefore, one has

$$\mathbf{L}U^- = -\epsilon\,\mathrm{sign}_{\mathrm{per}}(z)U(z)^2. \quad (52)$$

We solve (52) by using Fourier's method and by using auxiliary functions $w_{k,\theta}$ as found before for the simple air damping. Let us note first that

$$\mathrm{sign}_{\mathrm{per}}(z) = 2(2\pi/k_0)^{-1/2}\sum_{n\in\mathbb{Z}}(2n+1)^{-1}\sin(k_0(2n+1)z),$$

and

$$U^2(z) = A_0^2\big(\lambda_2^2\exp(-2\lambda_1|z|) + \lambda_1^2\exp(-2\lambda_2|z|) - 2\lambda_1\lambda_2\exp(-(\lambda_1+\lambda_2)|z|)\big).$$

Then we have the following Fourier representation

$$U^- = -A_0^2\epsilon\sum_{n\in\mathbb{Z}}(2n+1)^{-1}\big(\lambda_2^2 w_{n,1,1} + \lambda_1^2 w_{n,-1,-1} - 2\lambda_1\lambda_2 w_{n,1,-1}\big). \quad (53)$$

where

$$w_{n,s_1,s_2} = (2i)^{-1}\big(w_{k_0(2n+1-s_1-s_2),-2\gamma} - w_{k_0(-2n-1-s_1-s_2),-2\gamma}\big),$$

in which s_j are spin variables: $s_j \in \{-1, 1\}$. Using these expressions, we can compute odd corrections to the main solution induced by the nonlinear signum damping term. We see opposite to the case of simple air damping, that the perturbation U^- involves a number of Fourier components.

6 An asymptotic approximation near a critical speed, and resonances

In this section, for a nonlinear case we consider the velocities V close to the critical velocity V_c. We consider the case for which $0 < V < V_c$, and we assume that the quantity $\delta^2 = m_0(V_c^2 - V^2)$ is small. Note that for the nonlinear case the amplitudes of the solutions are bounded for all V as long as $V < V_c$(see section (5.3)). Then, a nonlinear critical velocity can be defined as the V for which the amplitude $U(0)$ has a maximum. We denote this critical velocity as $V_c^*(\alpha)$.

6.1 Asymptotic approximations of solutions

Let us consider the equation

$$DU_{zzzz} + m_0V^2U_{zz} + KU = g(U, U_z), \quad (54)$$

where the nonlinear term g is given by

$$g(U, U_z) = -K\alpha U^3 - \epsilon\tanh(\beta VU_z)(U^2 + \eta\beta^{-1})$$

$$+C_F\delta(z),$$

where damping is taken into account. We approximately solve equation (54) by using the following Fourier representations:

$$g(z,U(\cdot)\ ) = \int_{-\infty}^{\infty} \hat g(k,U(\cdot))\exp(ikz)dk, \quad i=\sqrt{-1},$$

$$U(z) = \int_{-\infty}^{\infty} \hat U(k)\exp(ikz)dk$$

and

$$\hat U(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(z)\exp(-ikz)dz.$$

Then

$$\hat U(k) = R(k,V)^{-1}\hat g(k,U(\cdot)),$$

where

$$R(k,V) = Dk^4 - m_0V^2k^2 + K. \tag{55}$$

Let $k_*(V)$ be a root of $R(k,V)=0$ such that $\mathrm{Re}\,k_*>0$ and $k_c=k_*(V_c)$. Then, k_c is a root of the equation $R(k,V_c)=0$, which is given by $k_c^2=m_0V_c^2/2D$. For the root k_* we are able to find an asymptotic expression when $V\approx V_c$ with $V<V_c$, and when the parameter κ (defined by (19)) is small. This parameter measures the proximity of the system to resonance. We note that for small $\kappa>0$ one has the following asymptotic expression for $R(k,V)$:

$$R(k,V) = a_0^2(k\pm k_c)^2 + \kappa^2k_c^2 + O(k\pm k_c)^3 + O(\kappa^3),$$

for $k\approx \pm k_c$, and where $a_0^2=2m_0V_c^2$. Then, we obtain

$$U(z) \approx \mathrm{Re}\int_{-\infty}^{\infty}(a_0^2(k-k_c)^2+\kappa^2k_c^2)^{-1}\hat g(k,U(\cdot))$$
$$\times\exp(-ikz)dk.$$

implying

$$U(z)\approx \mathrm{Re}\,\frac{\pi}{a_0k_c\kappa}\hat g(k_c,U(\cdot))\exp(-ik_cz).$$

So, for small $\kappa>0$ the solution U can be approximated by

$$U(z)\approx U_b(z,W) := W\exp(-\gamma|z|)\cos(k_cz)(1+O(\kappa)), \tag{56}$$

where $\gamma=a_0\kappa$ and where W is an unknown amplitude. To find an expression for W, we use the Lagrangian defined by (40). For small $\gamma>0$ we have the following asymptotic expressions for b_2 and b_4:

$$b_2 = (\gamma^{-1}+O(\gamma))\delta^2, \quad b_4 = K\gamma^{-1}/4. \tag{57}$$

The analysis of eq. (42) shows that there are two limiting cases. If $|\kappa|^2 << \alpha$, then the nonlinear effects prevail and one obtains
$W \propto (C_F\alpha\gamma)^{1/3}$.
In the opposite case when $|\kappa|^2 \gg \alpha$, one finds that
$W\propto C_F\kappa^{-2}\gamma$.
In both cases, we observe resonances for $V\uparrow V_c$ but leading to different amplitudes. Note that in the limit $V\uparrow V_c$ the localization vanishes, and the solution transforms into a traveling harmonic wave.

6.2 Existence of resonance

The following theorem on the existence of resonance is rigorous, and does not use asymptotic representations for solutions.

Theorem 4 *(on resonance existence) The following norm of U tends to ∞ for $V\uparrow V_c$ and α fixed but small:*

$$\|U\|_1 + \|U\|_{1,2} \to \infty \quad V\uparrow V_c,\ V<V_c. \tag{58}$$

The proof can be found in the Appendix (see subsection 9.6). With some minor modifications a similar proof can be given for the following theorem which describes solutions $U(z)$ (not necessarily localized ones).

Theorem 5 *(Case **B**, $|V|>V_c$) Let $V>V_c$. Let U be a L-periodic in z solution of equation (54) for $\epsilon=0$, i.e., without damping (or viscosity). Then, the following norm of U tends to ∞ as $\alpha\downarrow 0$:*

$$\|U\|_{L,q} + \bar C_\alpha^3 > c_q\alpha^{-1} \tag{59}$$

for each $q>1$ and for some c_q uniform in α, where

$$\|U\|_{L,q} = \left(\int_{-L}^{L}|U(z)|^q dz\right)^{1/q},$$

and

$$\sup_{z\in[-L,L]}|U(z)| = \bar C_\alpha,$$

where $\bar C_\alpha$ is a positive constant.

The proof can be found in the Appendix (see subsection 9.7).

To conclude this section, let us note that these theorems 4 and 5 describe different situations. Theorem 4 describes localized solutions and shows that the solution amplitudes grow as $V \uparrow V_c$ for a fixed $\alpha > 0$. In contrast, theorem 5 describes nonlocalized solutions for a fixed $V > V_c$ and $\alpha \downarrow 0$.

The properties of localized solutions are illustrated in Fig. 1 and in Fig. 2. The system parameters for the computations were taken from [9,15]. In Fig. 1 we see that the nonlinearity weakens the speed resonance effect. Curve 1 corresponds to the linear system, and the curves 2 and 3 to the nonlinear system. Fig. 2 shows that when the damping (or viscosity) is absent (curve 2) the solution remains even in z, however, the damping leads to a weak odd perturbation (see curve 1). Therefore, we can estimate damping effects by the deformation of localized solutions. Note that the Ritz method works formally for all V. However, localized solutions exist only for values of $|V| < V_c \approx 0.87$. We thus can compute the critical V_c and V_c^*. Calculations show that $V_c^* < V_c$. Moreover, the top of the curve for the linear case should increase to $+\infty$ as $V \to V_c = 0.87$. The critical V_c values are different for linear and nonlinear cases.

6.3 Asymptotic for V_c^*

In this subsection, we compute the critical speed V_c^*, which corresponds to the largest amplitude of a localized wave in a weakly nonlinear case. This critical speed is close to V_c. Therefore we can use the Ritz method, formula (43) for the amplitude W, and the expressions (57) for b_2, and b_4. Moreover, by using expression (22) for γ and definition (19) for the parameter κ, we obtain (assuming that $V < V_c$, and $\kappa > 0$ is small) that $W(\kappa)$ satisfies the cubic equation

$$\kappa^2 W + a_1 \alpha W^3 = C_F a_2 \kappa + O(\kappa^2), \tag{60}$$

where

$$a_1 = 3KD, \quad a_2 = 16D^2.$$

Eq. (60) has the following asymptotic approximation for W:

$$W(\kappa) = c_F a_2 \kappa^{-1} - a_1 \alpha (c_F a_0)^3 \kappa^{-5} + O(1), \tag{61}$$

where α is $o(\kappa^4)$. Now we can find the value κ_* for κ, which corresponds to the maximal W. By differentiating eq. (61) we obtain

$$\kappa_* = (5\alpha C_F a_2)^{1/4}. \tag{62}$$

From the definition (19), and from (62) we obtain that:

$$4m_0^2 V_c^2 (V_c - V_c^*) = \sqrt{5\alpha} C_F a_2. \tag{63}$$

We see from (63) that V_c^* is less than V_c, and that V_c^* is of $O(\sqrt{\alpha})$. Also in this dimensionless form, the difference in magnitudes of the critical velocities looks small, however, in a dimensional form it can be large.

7 Influence of a non-linear Winkler foundation and an arbitrary damping on stability of the system

In this Section, we consider stability and instability effects induced by nonlinearities. These effects can be studied and described by making use of perturbation theory. We consider three cases: (**a**) a simple type of damping (or viscosity) $f = u_t$, (**b**) Kelvin–Voigt type of damping $f = u_{txxxx}$, and (**c**) smoothed signum type of damping defined by $f = \tanh(\beta u_t) u^2$. We linearize the main equation (1) around $u = U$. Then, we obtain

$$m_0 \tilde{u}_{tt} + D\tilde{u}_{zzzz} + m_0 V^2 \tilde{u}_{zz} - 2m_0 V \tilde{u}_{zt} + K\tilde{u} = G(z, \tilde{u}, \tilde{u}_t) \tag{64}$$

where

$$G = \epsilon\big(g_0(z) + \tilde{g}_{fr}(z, \alpha, \epsilon, \tilde{u}(\cdot))\big) + k\alpha g_{nl}(z, \tilde{u}),$$

and where

$$g_{nl} = 3U^2\tilde{u} + 3U\tilde{u}^2 + \tilde{u}^3,$$

in case (**a**)

$$g_0 = -VU_z, \quad \tilde{g}_{fr} = -V\tilde{u}_z + \tilde{u}_t, \tag{65}$$

in case (**b**)

$$g_0 = -VU_{zzzz}, \quad \tilde{g}_{fr} = -V\tilde{u}_{zzzz} + \tilde{u}_{zzzzt}, \tag{66}$$

and in case (**c**)

$$g_0 = U^2 \tanh(\beta U_z),$$
$$\tilde{g}_{fr} = (2\tilde{u}U + \tilde{u}^2)\tanh(\beta V U_z) - \beta \cosh^{-2}(\beta V U_z) U^2(-V\tilde{u}_z + \tilde{u}_t) + O(\tilde{u}_t^2).$$

(67)

Again we use the Fourier transformation method:

$$u(z,t) = \int_{-\infty}^{+\infty} \hat{u}_k(t)\exp(ikz)dk,$$

where

$$\hat{u}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} u(z,t)\exp(-ikz)dk.$$

Then, for $\hat{u}_k$ we obtain

$$\begin{aligned} &m_0\hat{u}_{tt}(k,t) + R(k,V)\hat{u}(k,t) - 2m_0 Vik\hat{u}_t(k,t) \\ &= \alpha\hat{g}_{nl}(k,\tilde{u}(\cdot,\cdot)) + \epsilon\hat{g}_0(k) \\ &+\epsilon\hat{g}_{fr}(k,\tilde{u}(\cdot,\cdot),\tilde{u}_t(\cdot,\cdot)), \end{aligned} \tag{68}$$

where

$$\hat{g}_0(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_0(z)\exp(-ikz)dz,$$

$$\hat{g}_{nl}(k,\tilde{u}(\cdot,\cdot)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g_{nl}(z,\tilde{u}(\cdot,\cdot))\exp(-ikz)dz,$$

$$\hat{g}_{fr}(k,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}_{fr}(z,\tilde{u}(\cdot,\cdot),\tilde{u}_t(\cdot,\cdot)) \exp(-ikz)dz.$$

Our strategy is as follows. We solve eq. (68) approximately by using the method of iterations making use of the fact that $\tilde{u}$ is small for small and positive ϵ and α. In the right hand side of eq. (68) we can thus remove the terms which are nonlinear in $\tilde{u}$ and $\tilde{u}_t$. Since the left hand side of eq. (68) is linear in $\tilde{u}$, we can decompose $\hat{u}$ as follows:

$$\hat{u}(k,t) = \hat{u}^{(0)}(k,t) + \hat{u}^{(1)}(k,t),$$

where $\hat{u}^{(0)}(k,t)$ satisfies the equation

$$\begin{aligned} &m_0\hat{u}_{tt}^{(0)}(k,t) + R(k,V)\hat{u}^{(0)}(k,t) - 2m_0 Vik\hat{u}_t^{(0)}(k,t) \\ &= \epsilon\hat{g}_0(k), \end{aligned} \tag{69}$$

where the right hand side does not depend on $\tilde{u}$, and where $\hat{u}^{(1)}(k,t)$ satisfies the equation

$$\begin{aligned} &m_0\hat{u}_{tt}^{(1)}(k,t) + R(k,V)\hat{u}^{(1)}(k,t) - 2m_0 Vik\hat{u^{(1)}}_t(k,t) \\ &= \alpha\hat{g}_{nl}(k,\tilde{u}(\cdot,\cdot)) + \epsilon\hat{g}_{fr}(k,\tilde{u}(\cdot,\cdot),\tilde{u}_t(\cdot,\cdot)). \end{aligned} \tag{70}$$

A particular solution of (69) is:

$$\hat{u}^{(0)}(k,t) = \epsilon\frac{\hat{g}^{(0)}(k)}{R(k,V)}. \tag{71}$$

For small $\alpha, \epsilon > 0$ an approximating solution of (70) can be found by using standard asymptotic methods. It is convenient to introduce a parameter $\bar{\alpha}$ by the relation

$$\alpha = \bar{\alpha}\epsilon$$

and let us define $\tau = \epsilon t$. To find $\hat{u}^{(1)}(k,t)$, we use the following asymptotic representation (multiple time-scales approach):

$$\begin{aligned} \hat{u}^{(1)}(k,t) &= C_1(k,\tau)\exp(\theta_1(k)t) \\ &+C_2(k,\tau)\exp(\theta_2(k)t), \end{aligned} \tag{72}$$

where C_j are unknown functions of the slow time $\tau = \epsilon t$, and

$$\theta_{1,2}(k) = Vik \pm im_0^{-1/2}\sqrt{Dk^4 + K}, \quad i = \sqrt{-1}. \tag{73}$$

We substitute the expression (72) into eq. (70), and we take into account that the terms in the left hand side are of order ϵ. Then, we obtain the following equations

$$2m_0(\theta_1(k) - Vik)\frac{\partial C_1(k,\tau)}{\partial\tau} = f_1(k,\tilde{u},\tilde{u}_t), \tag{74}$$

$$2m_0(\theta_2(k) - Vik)\frac{\partial C_2(k,\tau)}{\partial\tau} = f_2(k,\tilde{u},\tilde{u}_t), \tag{75}$$

where

$$\begin{aligned} f_{1,2}(k,\tilde{u},\tilde{u}_t) = \exp(-\theta_{1,2}(k)t)\big(&\bar{\alpha}\hat{g}_{nl}(k,\tilde{u}) + \hat{g}_0(k) \\ &+ \hat{g}_{fr}(k,\tilde{u}(\cdot,\cdot),\tilde{u}_t(\cdot,\cdot))\big). \end{aligned}$$

To exclude the fast time t, we average the right hand side over a large time interval $[0,T]$, and replace the right hand sides by their averages $\bar{f}_{1,2}(k,C)$:

$$\bar{f}_{1,2}(k,C) = \lim_{T\to+\infty} T^{-1}\int_0^T f_{1,2}(k,\tilde{u},\tilde{u}_t)dt.$$

Then, we obtain

$$\begin{aligned} 2m_0(\theta_{1,2}(k) - Vik)\frac{\partial C_{1,2}(k,\tau)}{\partial\tau} &= f_{1,2}(k,\tilde{u}(\cdot,\cdot), \\ &\tilde{u}_t(\cdot,\cdot)). \end{aligned} \tag{76}$$

Now let us consider the three different cases. In case (**a**) we have

$$\bar{f}_{1,2}(k,C(\cdot,t)) = \big(\mu - \theta_{1,2}(k)\big)C_{1,2}(k,t), \tag{77}$$

where

$$\mu = -3(2\pi)^{-1}\bar{\alpha}K\int_{-\infty}^{+\infty} U^2(z)dz.$$

Then, we find that

$$\hat{u}(k,t) = \bar{C}_1 \exp\big((\theta_1+\tilde{\theta}_1)t\big)+\bar{C}_2 \exp\big((\theta_2+\tilde{\theta}_2)t\big), \quad (78)$$

where

$$\tilde{\theta}_{1,2} = \tilde{\theta}_{1,2}^{(nl)} + \tilde{\theta}_{1,2}^{(fr)}, \quad (79)$$

and

$$\tilde{\theta}_{1,2}^{(nl)} = \frac{\mu\epsilon}{2m_0(\theta_{1,2} - Vik)},$$
$$\tilde{\theta}_{1,2}^{(fr)} = -\frac{\epsilon\theta_{1,2}(k)}{2m_0(\theta_{1,2} - Vik)}. \quad (80)$$

The first contribution $\tilde{\theta}_{1,2}^{(nl)}$ is a purely imaginary number, which determines a small frequency shift as a result of the nonlinear effects. The second term $\tilde{\theta}_{1,2}^{(fr)}$ is a real number, and it describes how damping influences the amplitudes of the solution. For the Kelvin–Voigt type of damping, case (**b**), in a similar way, we obtain expression (78) with the same $\tilde{\theta}_{1,2}^{(nl)}$ and

$$\tilde{\theta}_{1,2}^{(fr)} = -\frac{\epsilon k^4\theta_{1,2}(k)}{2m_0(\theta_{1,2} - Vik)}. \quad (81)$$

We see that the second damping term involves an additional factor k^4. Finally, in case (**c**) , one has

$$\tilde{\theta}_{1,2}^{(fr)} = \frac{1}{2m_0(Vik - \theta_{1,2})} \int_{-\infty}^{+\infty} \big(2U \tanh(\beta V U_z)\big)$$
$$+ (-Vik + \theta_{1,2})U^2\beta \cosh^{-2}(\beta V U_z)\big)dz. \quad (82)$$

Recall that U is even in z. It shows that the contribution of the term with the hyperbolic tangent is equal to zero. So, we have

$$\tilde{\theta}_{1,2}^{(fr)} = -\frac{\int_{-\infty}^{+\infty} U^2\beta \cosh^{-2}(\beta V U_z)dz}{2m_0}, \quad (83)$$

and we can conclude that the localized solution is stable.

8 Conclusions

The results obtained in this paper (in particular, Theorem 2) allow us to draw some conclusions for large time behaviour of the solutions and their dependence on the speed V. Consider the initial boundary value problem (IBVP) for (1) in which the initial data are sufficiently close to the traveling wave $U(x - Vt)$ as studied in previous sections. This implies that the initial displacement $u_0(x) = u(x,0)$ is close to $U(x)$ in the L_2-norm, and that the initial speed $u_1(x) = u_t(x,t)|_{t=0}$ is close to $-VU(x)$ in the same norm. Then, the solution can be represented as the sum of a localized wave solution U and a Fourier integral $\tilde{u}(z,t)$:

$$\tilde{u}(z,t) = \int_{-\infty}^{\infty} \hat{u}(k,t)\exp(ikz)\,dk,$$

where $z = x - Vt$, and the time evolution of $\tilde{u}(z,t)$ can be described by the asymptotic formulas as presented in the previous section. For small $\epsilon > 0$ in case (**a**) it is shown in the previous section that the coefficients $\hat{u}(k,t)$ consist of two terms. The first term does not depend on t, and the second one decreases in t as $\exp(-\epsilon \operatorname{Re}\tilde{\theta}(k)t)$, where $\operatorname{Re}\tilde{\theta}(k) > 0$. Therefore, for $V < V_c$ the solution is the sum of the three terms. The first one is the localized wave $U(z)$, the second term is a small perturbation of this wave, and the third term is slowly damped in t. For V close to V_c we obtain resonance and for $V > V_c$ the localized solution breaks up and vanishes. So, the existence of localized waves in Euler–Bernoulli beams on a weakly nonlinear elastic foundation, which are under the action of moving concentrated loads have been investigated. The case for a beam with a structural damping or viscosity in the elastic foundation is considered as well. To simplify the analysis of the problem, we introduced a new damping model, which depending on its parameters, is capable to describe the simple air damping (viscosity) and the signum type of damping. Also the problem with a Kelvin–Voigt type of damping in the system is considered. The localized wave solution for a weakly nonlinear problem has been obtained by using the Ritz method and perturbation techniques. For velocities $|V| > V_c$ localized wave solutions do not exist. Using a simple analytical expression obtained from a simple cubic equation for the solution amplitudes for a weakly nonlinear case it was shown that the nonlinearity of hardening type in an elastic foundation leads to smaller displacement amplitude than in a case of a linear elastic foundation. This conclusion is supported by results obtained numerically in [30]. We introduced for a weakly nonlinear system an effective coefficient for the elastic foundation, which is larger than the elastic coefficient for a linear beam foundation. This causes a shift of the system cut-off frequency, and led us to the conclusion that some of the traveling harmonic waves, which propagate in a linear system, will not propagate in a weakly nonlinear system. Rigorous and a priori estimates show the existence of a velocity resonance for values V close to a critical value V_c for small nonlinearities. Other estimates show that for non-localized solutions their magnitudes tend to ∞ as the coefficient of the nonlinearity in the elastic foundation goes to zero. Another effect of the nonlinearity in the elastic foundation is that for a "velocity resonance" the amplitude of the displacement remains finite even when damping (viscosity) is absent. It is shown that the nonlinearity in the elastic foundation leads to a new critical velocity which is smaller, than the critical velocity in the linear case. The stability of the obtained localized solution is also investigated. Asymptotic formulas for velocities below the critical value, and in the presence of damping, show that all small disturbances decay slowly and exponentially. Theorem 5.2 gives an energy estimate for the disturbances. Explicit asymptotic expressions are obtained for the frequency shift and amplitude change when damping (viscosity) is present in the system. It also turned out that the nonlinear damping influences the wave profile shift weaker than linear damping. The following rigorous statements are proved (Theorem 6.1 and 6.2). From Theorem 6.1 it follows that in the absence of damping (viscosity), the norm of the solution increases without limit as the nonlinearity parameter decreases and the velocity approaches the resonant (critical) value. From Theorem 6.2 it can concluded that if the velocity is higher than the critical value, then the norm of the solution increases without limit as the nonlinearity parameter decreases (for periodic solutions). The norm can be large due to the following factors — either the solution spreads out and damps weakly, or the amplitude is large, or both. For these cases, energy estimates are obtained which bound the growth of the energy for any values of the velocity (that is, both for above and below the critical value). The energy grows no faster than the root of t. For simple air and signum types of damping, the estimates depend on the force localization parameter. Then, strong localization can lead to a larger energy growth. The Kelvin–Voigt type of damping bounds the growth of energy stronger than simple air type of damping and signum type of damping. The asymptotic behavior of the solutions for velocities close to the critical one were also studied. It turned out that the nonlinear equation for the amplitude has a simple form, and damping (viscosity) makes a small contribution to the displacement of the beam.

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Declarations

Conflicts of Interest The authors declare that they do not have any commercial or associative interest that represents a conflict of interest in connection with the work submitted. The authors declare that they have no conflict of interest.

9 Appendix

9.1 Proof of Lemma 2

Proof From (8) it follows that:

$$\frac{dE}{dt} = -\epsilon \|u_{xxt}\|^2 + c_F u_t(Vt). \tag{84}$$

Let us estimate $|u_t(x - Vt)|$ by using Sobolev norms. We observe that

$$u_t(Vt)^2 = \frac{1}{2}\int_{-\infty}^{Vt} u_t u_{xt} dx$$

due to the boundary conditions (7). Thus, by using the Schwarz inequality

$$u_t(Vt)^2 \le \frac{1}{2}\|u_t\|\|u_{xt}\|$$

it follows that for each $a > 0$:

$$u_t(Vt)^2 \le \frac{1}{4}\left(a\|u_t\|^2 + a^{-1}\|u_{xt}\|^2\right).$$

Now let us estimate $\|u_{xt}\|^2$ by using the Parseval identity, and let us denote the Fourier coefficients of $u(x,t)$ by $\hat{u}(k,t)$. Then,

$$\|u_t\|^2 = \int_{-\infty}^{\infty} |\hat{u}_t(k,t)|^2 dk,$$
$$\|u_{xt}\|^2 = \int_{-\infty}^{\infty} k^2 |\hat{u}_t(k,t)|^2 dk,$$

$$\|u_{xxt}\|^2 = \int_{-\infty}^{\infty} k^4 |\hat{u}_t(k,t)|^2 dk.$$

We observe that for each $b > 0$ one has $2k^2 \leq bk^4 + b^{-1}$. Therefore,

$$\|u_{xt}\|^2 \leq \frac{1}{2}\big(b\|u_{xxt}\|^2 + b^{-1}\|u_t\|^2\big).$$

Using this estimate and (84) we obtain

$$\frac{dE}{dt} \leq -\epsilon\|u_{xxt}\|^2 + c_F\Big(\frac{a}{4}\|u_t\|^2 + \frac{1}{8a}\big(b\|u_{xxt}\|^2 + b^{-1}\|u_t\|^2\big)\Big)^{1/2}, \tag{85}$$

where a, b are arbitrary positive coefficients, which can depend on t. Consider the function

$$Y(v, u_t) = -\epsilon v^2 + c_F\Big(\frac{a}{4}\|u_t\|^2 + \frac{1}{8a}\big(bv^2 + b^{-1}\|u_t\|^2\big)\Big)^{1/2}.$$

If b is small enough, this function is decreasing in v. We take such a $b = b_0$. Then $Y(v, u_t) \leq Y(0, u_t)$, and (85) reduces to

$$\frac{dE}{dt} \leq c_F\Big(\frac{a}{4} + \frac{1}{8ab_0}\Big)^{1/2}\|u_t\| \tag{86}$$

which can be simplified to

$$\frac{dE}{dt} \leq b_1 E^{1/2}.$$

This differential inequality completes the proof of Lemma 2. □

9.2 Proof of Theorem 1

Proof We use the representation (33). For the unknown function $\tilde{u}$ one has

$$\mathbf{L}\tilde{u} = m_0 V^2 \tilde{u}_{zz} + D\tilde{u}_{zzzz} + K\tilde{u} = g(\tilde{u}, \tilde{u}_z), \tag{87}$$

where

$$g(\tilde{u}, \tilde{u}_z) = -\epsilon f(-V(U + \tilde{u}_z), U + \tilde{u}) - K\alpha(U^3 + 3U^2\tilde{u} + 3U\tilde{u}^2 + \tilde{u}^3).$$

We will show that $\tilde{u}$ is uniformly bounded in z. Using the Green's function G, we rewrite (87) into an integral equation:

$$\tilde{u}(z) = \int_{-\infty}^{\infty} G(z - z_0) g(\tilde{u}, \tilde{u}_z)(z_0) dz_0. \tag{88}$$

We consider this equation in the Banach space $\mathbb{B}_1$ consisting of differentiable functions w bounded in the C^1-norm:

$$|w|_1 = \sup_{z\in\mathbb{R}}(|w(z)| + |w_z(z)|).$$

Consider the nonlinear operator defined by the right hand side of (88):

$$\mathbf{T}(\tilde{u}(\cdot))(z) = \int_{-\infty}^{\infty} G(z - z_0) g(\tilde{u}, \tilde{u}_z)(z_0) dz_0$$

and consider balls B_R in $\mathbb{B}_1$

$$B_R = \{w \in \mathbb{B}_1 : |w|_1 \leq R\}.$$

Let us estimate $|\mathbf{T}(\tilde{u}(\cdot))|_1$ for $\tilde{u} \in B_R$, and let us observe that if $\tilde{u} \in B_R$, then

$$\sup|G|, \quad \sup|G_z| < cC_g\kappa^{-1},$$

where $c > 0$ is a constant uniform in $\kappa > 0$. Moreover,

$$\sup_z |g(\tilde{u}, \tilde{u}_z)| < c_2\kappa^{-2}(\epsilon R^2 + K\alpha R),$$

where $c_2 > 0$ is a constant uniform in $\epsilon, \alpha > 0$. Here, we took into account that U is proportional to κ^{-1} for small $\kappa > 0$. The integral over z_0 gives us an additional factor κ^{-1} because the space damping coefficient in G is γ, which is proportional to κ. Therefore,

$$\sup_{\tilde{u}\in B_R} |\mathbf{T}(\tilde{u})|_1 < c_2\kappa^{-3}(\epsilon R^2 + K\alpha R).$$

For sufficiently small $\epsilon, \alpha > 0$ there exists a $R > 0$ such that

$$R > c_2\kappa^{-3}(\epsilon R^2 + K\alpha R). \tag{89}$$

Under the condition (89) the nonlinear operator $\mathbf{T}$ maps the ball B_R into itself, and so we can use Schauder's fixed point Theorem. This proves the existence of a solution and validity of the estimates (34). Estimate (35) can be obtained in a similar way by using Green's function. To conclude the proof, we can use a norm with a parameter $a > 0$

$$|\tilde{u}|_a = \sup_{z\in\mathbb{R}} \exp(a|z|)|\tilde{u}(z)|,$$

and we choose an appropriate value for a such that $0 < a < \gamma$. And so, it can be shown easily that the operator $\mathbf{T}$ is a contraction, leading to the results as described in Theorem 1. □

9.3 Proof of Theorem 2

Proof Let us introduce the energy functional $\tilde{E}[w]$ associated with the perturbation or correction term w:

$$E_\alpha[w] = \frac{1}{2}\Big(m_0\|w_t(\cdot,t)\|^2 + D\|w_{zz}(\cdot,t)\|^2 + K\|w\|^2 + \alpha K\int_{-\infty}^{+\infty}(U^3 + 3U^2(z)w(z,t)^2 + 2U(z)w(z,t)^3 + \frac{1}{2}w(z,t)^4)dz\Big).$$

Note that the term with U^3 does not depend on w, and so it gives a constant contribution. Note that E_0 is a perturbed energy for the linear case and

$$\big|E_\alpha[w] - E_0[w]\big| < \alpha c_1\|w\|^2\Big(\sup_{z\in(-\infty,\infty)}|w| + \sup_{z\in(-\infty,\infty)}|w|^2\Big),$$

where $c_1 > 0$ is uniform in $\alpha > 0$. We can estimate $\sup|w|^2$ in the following way:

$$w^2(z) = \left|\frac{1}{2}\int_{-\infty}^{z} w_x w dx\right| \le \frac{1}{2}\|w\|\|w_z\| \le \|w\|^2 + \|w_z\|^2.$$

By using Fourier integrals it can be shown for $|V| < V_c$ that

$$\|w\|^2 + \|w_z\|^2 \le c_2 E_0[w],$$

where $c_2 > 0$ is uniform in w. The last inequalities imply

$$\big|E_\alpha[w] - E_0[w]\big| \le c_3\alpha(E_0^3/2 + E_0^2), \tag{90}$$

where $c_3 > 0$ is uniform in w and $\alpha > 0$. Eq. (1) implies

$$m_0 w_{tt} + D w_{zzzz} - 2m_0 V w_{zt} + K w + h(w,z) = \epsilon q(w, w_t, z) \tag{91}$$

where

$$h = \alpha K(U^3 + 3Uw^2 + w^3),$$
$$q = -f(-VU_z - Vw_z - w_t, U + w).$$

We multiply both sides of (91) with w_t, and integrate over the entire z axis, yielding

$$\frac{dE_\alpha}{dt} = \epsilon S(t), \tag{92}$$

where

$$S(t) = \int_{-\infty}^{+\infty} q(w(z,t), w_t(z,t), z)w_t dz.$$

Let us estimate $|S(t)|$. One has

$$\sup_{z\in\mathbb{R}} |q(z, w(\cdot,\cdot), w_t(\cdot,\cdot)) - q_0(z)| \le c_5(|w_t| + |w|),$$

where $q_0 = -f(-VU_z, U)$ is a bounded function, and c_5 is a positive constant. The last estimate implies

$$|S(t)| < c_6\|w_t\| + c_5(\|w_t\|^2 + \|w\|\|w_t\|) < c_7(E_0[w]^{1/2} + E_0[w]),$$

where c_6, c_7 are positive constants uniform in α, ϵ. And so it follows from (92) that

$$\frac{dE_\alpha}{dt} \le \epsilon c_7\Big(E_0[w]^{1/2} + E_0[w]\Big). \tag{93}$$

Then it is follows from (90) under the condition $E_0 < C$ (where C is a constant, uniform in α, ϵ and where $\alpha > 0$ is small) that inequality (93) can be replaced by

$$\frac{dE_\alpha}{dt} \le \epsilon c_8\Big(E_\alpha[w]^{1/2} + E_\alpha[w]\Big). \tag{94}$$

This differential inequality shows that $E_\alpha[w(\cdot,t)] < c_9\delta$ for $t \in [0, c_{10}\epsilon^{-1}]$, and completes the proof of the Theorem 2.

□

9.4 Computation of b_2 and b_4

In this subsection, we derive expressions for b_2 and b_4. Consider the integrals

$$J_{p,q} = \int_0^\infty \Big(\frac{d^p\hat{U}}{dz^p}\Big)^q dz.$$

We note that

$$b_4 = 2KJ_{0,4}, \quad b_2 = 2(DJ_{2,2} - m_0V^2J_{1,2} + KJ_{0,2}).$$

For $\hat{U}$ we use the formula (41) which for $z > 0$ gives

$$\hat{U}(z) = 4\big(\lambda_1\exp(-\lambda_2 z) - \lambda_2\exp(-\lambda_1 z)\big).$$

We obtain the following results:

$$J_{0,2} = 16\frac{(\lambda_1^2 - \lambda_2^2)^2 + \lambda_1\lambda_2(\lambda_1 - \lambda_2)^2}{\lambda_1\lambda_2((\lambda_1 + \lambda_2))},$$

$$J_{1,2} = 16\frac{\lambda_1\lambda_2(\lambda_1 - \lambda_2)^2}{\lambda_1 + \lambda_2},$$

$$J_{2,2} = 16\frac{\lambda_1^2\lambda_2^2(\lambda_1-\lambda_2)^2}{\lambda_1+\lambda_2},$$

and

$$J_{0,4} = 256\Big(\frac{\lambda_1^4}{4\lambda_2} - 4\frac{\lambda_1^3\lambda_2}{3\lambda_2+\lambda_1} + 3\frac{\lambda_1^2\lambda_2^2}{\lambda_2+\lambda_1} - 4\frac{\lambda_1\lambda_2^3}{3\lambda_1+\lambda_2} + \frac{\lambda_2^4}{4\lambda_1}\Big).$$

These expressions are complicated, but for $V \approx V_c$ we can use the asymptotics

$$\hat{U}(z) \approx 4\exp(-\gamma z)(\cos(k_0 z) + O(\gamma))$$

for small $\gamma > 0$. By using (22) we then obtain

$$b_4 = 24(\gamma^{-1} + O(1)),$$
$$b_2 = 4(Dk_0^4 - m_0V^2k_0^2 + K)((\gamma^{-1} + O(1)).$$

9.5 Proof of Theorem 3

If condition (29) holds, then for any sufficiently small $\varepsilon_0 > 0$ there exists a z_0 such that

$$|u(z)| < \varepsilon_0, \quad z \in (z_0, \infty). \tag{95}$$

Therefore, for $z > z_0$ and small $\varepsilon > 0$ eq. (28) can be rewritten as

$$Du_{zzzz} + m_0V^2u_{zz} + Ku + \varepsilon_0\phi(z, \varepsilon_0)u = 0, \tag{96}$$

where ϕ is a smooth bounded function. Let us construct asymptotic solutions of this linear equation by the method of variation of constants. Let k satisfy the equation

$$Dk^4 - m_0V^2k^2 + K = 0.$$

Let us consider asymptotic solutions of (96) of the following form

$$u_{as,k} = (C_0(Z) + \kappa C_1(Z) + \dots)\exp(ikz),$$

where $Z = \varepsilon_0 z$ is a slow variable. The unknown coefficients C_j can be found step by step. For C_0 one obtains

$$ik\frac{dC_0}{dZ}(-4Dk^2 - 2m_0V^2) = -\phi(z, \varepsilon_0),$$

and the equations for higher order C_j have the following form

$$ik\frac{dC_j}{dZ}(-4Dk^2 - 2m_0V^2) = F_j(z),$$

where F_j can be expressed in ϕ and $C_0, C_1, \dots, C_{j-1}$. One can check that $u_{as,k}$ are solutions oscillating in z and they do not decrease as $z \to \infty$. This proves the Theorem.

9.6 Proof of Theorem 4

For V close to V_c there exists a $k = k_*(V)$ such that

$$R(k_*, V) = Dk_*^4 - m_0V^2k_*^2 + K = -\mu^2,$$

where $c_1 > 0$ is a constant uniform in the parameter $\mu = \sqrt{|V - V_c|} > 0$. Let us multiply both sides of (28) with $\psi = \exp(ik_*z - bz^2/2)$ where $b > 0$ is small. Then,we integrate the so-obtained expression with respect to z over $(-\infty, +\infty)$, yielding

$$\int_{-\infty}^{\infty} U\phi dz + K\alpha\int_{-\infty}^{\infty} U^3\psi dz = C_F, \tag{97}$$

where

$$\phi = D\psi_{zzzz} + m_0V^2\psi_{zz} + K\psi.$$

From the definition of ψ, it follows that:

$$\psi_{zz} = ((ik - bz)^2 - b)\psi,$$
$$\psi_{zzzz} = P(z, b)\psi,$$

where for small $b > 0$ the polynomial P satisfies the estimate

$$|P(z, b) - k^4| < c_8\big(b(|z| + 1) + (b|z|)^2 + (b|z|)^3 + (b|z|)^4\big),$$

in which $c_8 > 0$ is a positive constant uniform in $b > 0$. For the first integral in(97) it now follows that:

$$I_0 = \int_{-\infty}^{\infty} U\phi dz = \kappa^2\int_{-\infty}^{\infty} U\psi dz + \tilde{I}_0(b), \tag{98}$$

where

$$|\tilde{I}_0(b)| < c_9 b^{1/2},$$

where c_9 is a positive constant. We assume that $b \to 0$ as $\kappa \to 0$, where κ is defined by (19). Furthermore, we use the following estimates:

$$\left|\int_{-\infty}^{\infty} U\psi dz\right| \le \|U\|_1,$$

$$\sup|U(z)|^2 \le 2\left|\int_{-\infty}^{z} U_s U(s)ds\right| \le 2\|U_z\|_2\|U\|_2.$$

The last estimate gives

$$\left|\int_{-\infty}^{\infty} U^3\psi dz\right| \le 2\|U_z\|_2\|U\|_2^3.$$

By using these estimates and (97) we obtain that

$$\kappa^2\|U\|_1 + C_0 b^{1/2}\|U\|_2 + 2K\alpha\|U_z\|_2\|U\|_2^3 \ge C_F,$$

where C_0 is a constant, uniform in $\kappa > 0$. The last inequality implies (58), and so, Theorem 6.1 is proved.

9.7 Proof of Theorem 5

For $V > V_c$ there exists a $k = k_*(V)$ such that

$$R(k_*, V) = Dk_*^4 - m_0 V^2 k_*^2 + K = 0.$$

Let us multiply both sides of (28) with $\psi = \exp(ik_* z)$, and let us integrate the so-obtained equation with respect to z from $z = -L$ to $z = L$, where L is the period of the periodic solution. Then, we obtain

$$K\alpha \int_{-L}^{L} U^3 \psi dz = C_F. \quad (99)$$

By using use the following estimate:

$$\left| \int_{-L}^{L} U^3 \psi dz \right| \le \bar{C}_\alpha^3 L,$$

and (99), we obtain that

$$K\alpha \bar{C}_\alpha^3 \ge C_F.$$

This last inequality proves inequality (5), and so, we have

$$\|U\|_{L,q} + \bar{C}_\alpha^3 > C_8 \alpha^{-1}$$

as $\alpha \downarrow 0$.

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