The individual time trial as an optimal control problem

De Jong, Jenny; Fokkink, Robbert; Olsder, Geert Jan; Schwab, Arend

Publication date
2017

Document Version
Final published version

Published in
Institution of Mechanical Engineers. Proceedings. Part P: Journal of Sports, Engineering and Technology

Special issue article

Institution of MECHANICAL ENGINEERS

Journal of SPORTS ENGINEERING AND TECHNOLOGY

Proc IMechE Part P: J Sports Engineering and Technology
2017, Vol. 231(3) 200–206

Reprints and permissions:
sagepub.co.uk/journalsPermissions.nav
DOI: 10.1177/1754337117705057
journals.sagepub.com/home/pip
SAGE

The individual time trial as an optimal control problem

Jenny de Jong[1], Robbert Fokkink[2], Geert Jan Olsder[2] and AL Schwab[3]

Abstract

In a cycling time trial, the rider needs to distribute his power output optimally to minimize the time between start and finish. Mathematically, this is an optimal control problem. Even for a straight and flat course, its solution is non-trivial and involves a singular control, which corresponds to a power that is slightly above the aerobic level. The rider must start at full anaerobic power to reach an optimal speed and maintain that speed for the rest of the course. If the course is flat but not straight, then the speed at which the rider can round the bends becomes crucial.

Keywords

Bicycling, individual time trial, maximum principle, optimal control, power equation

Date received: 31 August 2016; accepted: 19 March 2017

Introduction

The individual time trial is a road bicycle race, in which cyclists race alone against the clock. We use mathematical tools to determine the optimal pacing strategy of a cyclist in such an individual time trial, for which we have a relatively flat and short course in mind. The opening stage of the Giro d'Italia (Figure 1) – the prologue – through the city of Apeldoorn in 2016 is a good example. The course of a prologue can be divided into a number of relatively straight segments between bends, which the rider can round only at a limited speed. We study the optimal pacing on the straight segments as a mathematical optimal control problem. We consider the speeds at the bends as fixed external conditions, which appear in our differential equations as initial conditions. Determining the optimal speed in a bend is a challenging problem, which deserves further studies.

The problem of finding the optimal pacing strategy for a straight course has been studied before, see, for example, De Koning et al.[1] and Underwood and Jermy.[2] These studies compared a finite number of pacing strategies and selected the best strategy by numerical computation. In our considerations, we allow all possible pacing strategies and select the optimal strategy using Pontryagin's maximum principle.[3, 4] We have summarized our results previously.[5] This paper is an extended version, which contains the full analysis.

The mathematical model

We model the rider as a point mass moving on the line from start to finish in minimal time. The rider's force F counterbalances the resisting forces, which are: the air resistance F_A, slope resistance F_S, rolling resistance F_R, and bump resistance F_B. The air resistance is given by $F_A = K_A(v + v_w)^2$, where v is the velocity of the rider, v_w is the velocity of the wind, and K_A is a drag coefficient. The slope resistance is $F_S = mg\sin(\varphi)$, where g is the gravitational acceleration and φ is the angle of inclination ($\tan(\varphi)$ is the slope). The rolling resistance is $F_R = mgC_R$, where C_R is the resistance coefficient. The excess force of F minus the resisting forces will accelerate the rider, or decelerate him when the excess is negative. $F_{acc} = m_e a$, where m_e is the effective mass, which slightly exceeds the mass of the rider plus bike, m, to account for the kinetic energy of the bicycle's rotating wheels. This all adds up to

$$F = K_A(v + v_w)^2 + mg(s + C_R) + m_e a$$

The rider's power is equal to $u(t) = F(t)v(t)$, where $F(t)$ is the force and $v(t)$ is the velocity. If we substitute the expression for F into $u(t) = F(t)v(t)$, we obtain a differential equation, known as the *power equation*[6]

[1]Utrecht University, Faculty of Mathematics, The Netherlands
[2]Delft University of Technology, Faculty EEMCS, The Netherlands
[3]Delft University of Technology, Biomechanical Engineering, The Netherlands

Corresponding author:
Robbert Fokkink, Delft University of Technology, Faculty EEMCS, Mekelweg 4, 2628CD Delft, The Netherlands.
Email: r.j.fokkink@tudelft.nl

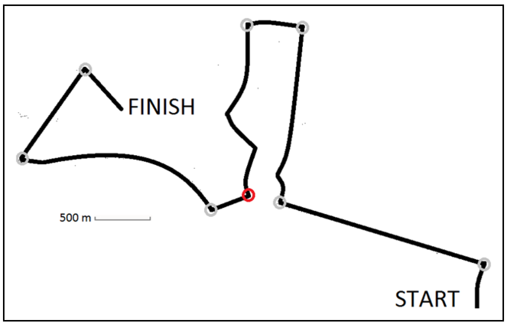

Figure 1. The course of the Giro d'Italia Prologue 2016 consisted of nine relatively straight segments, with lengths varying from 200 m to 2000 m. In a race like this, riders round the bends at high speed and are willing to take risks. Prologue specialists Jos van Emden and Stefan Küng slipped and fell in the bend marked in red. The surprise runner up, Primož Roglič, has a very good sense of balance. He used to be a ski jumper.

$$u(t) = \left[K_A(v(t) + v_w)^2 + mg(s + C_R) + m_e\frac{dv(t)}{dt}\right]v(t)$$

To simplify the mathematical details of our analysis, we assume that there is no headwind. The equation thus reduces to

$$u(t) = \left[c_1 v(t)^2 + c_2 + c_3\frac{dv(t)}{dt}\right]v(t)$$

In this equation, the constants c_1 and c_3 are positive.

Aerobic and anaerobic power

The precise measurement of a cyclist's power remains a topic of research[7] and there exist many biomechanical models to describe it. We adopt the model of Monod and Schrerrer.[8] An athlete produces power from aerobic sources, which can be sustained for a long time, and anaerobic sources, which run out quickly and are used to reach the peak power. In the model of Monod and Scherrer,[8] the power output varies between a *critical power level, CP*, which can last for a time longer than the race and a peak power level u_{max}, which can only last for a short time. In our model, we do not allow for recovery. The anaerobic reserve cannot be recharged if the rider spends some time at a power below CP. We assume that the excess power $u(t) - CP$ is limited, depending on the athlete's qualities, but we make no further assumptions. Thus, we arrive at the optimal control problem of minimizing the total time T of the time trial, under the constraints that $\int_0^T u(t) - CP\, dt$ is constant, that $CP \leqslant u(t) \leqslant u_{max}$, and that the solution $v(t)$ of the power equation satisfies $\int_0^T v(t)dt = L$, where L is the length of the circuit. This fits in the framework of Pontryagin's maximum principle.

The time trial control problem: three levels of power

To apply Pontryagin's maximum principle, it is convenient to put the problem in a different, but mathematically equivalent form. Minimizing the final time over a fixed distance is mathematically equivalent to maximizing the final distance over a fixed time T_f. This leads to the following optimal control problem, which turns out to be very similar to Goddard's problem in rocket science, as treated by Dmitruk and Samylovskiy[9]

$$\max_{CP \leqslant u(t) \leqslant u_{max}} \int_0^{T_f} v(t)\, dt$$

subject to the constraints

$$\left[\begin{array}{l} \dfrac{dx}{dt} = v(t) \\ \dfrac{dv}{dt} = \dfrac{u(t)}{c_3 v(t)} - \dfrac{c_1 v(t)^2}{c_3} - \dfrac{c_2}{c_3} \\ \dfrac{da}{dt} = u(t) - CP \end{array}\right.$$

with boundary conditions $x(0) = 0$, $v(0) = \alpha > 0$, $a(0) = 0$, $a(T_f) = W$. Note that we require that the initial velocity $v(0)$ is positive (but arbitrarily small), to avoid a singularity in the second constraint at time zero. We can now apply the maximum principle, which yields the Hamiltonian function

$$H(x, u, \lambda) = v(t) + \lambda_1(t)v(t) + \lambda_2(t) \left[\frac{u(t)}{c_3 v(t)} - \frac{c_1 v(t)^2}{c_3} - \frac{c_2}{c_3}\right] + \lambda_3(t)(u(t) - CP)$$

It is important to observe that this Hamiltonian is linear in u and therefore the optimal control u^*– the optimal power distribution of the athlete – satisfies

$$u^*(t) = \left[\begin{array}{lll} CP & \text{if} & \dfrac{\lambda_2(t)}{v(t)} < \gamma \\ u_{sing} & \text{if} & \dfrac{\lambda_2(t)}{v(t)} = \gamma \\ u_{max} & \text{if} & \dfrac{\lambda_2(t)}{v(t)} > \gamma \end{array}\right.$$

where $\gamma := -c_3\lambda_3$. The optimal power distribution has three levels: the anaerobic peak level, u_{max}, the aerobic long term level, CP, and an intermediate singular power level, u_{sing}. We will show that it is optimal to switch back in power from peak to critical power and to cross the critical level at γ only once. It does not seem possible to express γ in physical terms. The parameter c_3 is the effective mass of the rider, but λ_3 is a multiplier, which is a purely mathematical variable.

The parameter γ, which determines the switch between the power levels, needs to be computed from a system of differential equations. These equations contain the constraints on the original problem and the constraints

$$\frac{d\lambda_i}{dt} = -\frac{dH}{dx_i}$$

where $x_1 = x, x_2 = v, x_3 = a$ on the multipliers Yielding.

$$\frac{dx}{dt} = v(t) \qquad x(0) = 0 \tag{1}$$

$$\frac{dv}{dt} = \frac{u(t)}{v(t)c_3} - \frac{c_1}{c_3}(v(t))^2 - \frac{c_2}{c_3} \qquad v(0) = \alpha \tag{2}$$

$$\frac{da}{dt} = u(t) - CP \qquad a(0) = 0, a(T_f) = W \tag{3}$$

$$\frac{d\lambda_1}{dt} = 0 \qquad \lambda_1(T_f) = 0 \tag{4}$$

$$\frac{d\lambda_2}{dt} = -\left(1 + \lambda_1 - \frac{\lambda_2(t)u(t)}{c_3(v(t))^2} - 2\frac{c_1}{c_3}\lambda_2(t)v(t)\right) \qquad \lambda_2(T_f) = 0 \tag{5}$$

$$\frac{d\lambda_3}{dt} = 0 \tag{6}$$

Mathematical solution of the control problem

Our analysis will show that the rider needs to go all out at peak power at the start and aim for a velocity that can be maintained at the intermediate singular power level. Once the rider gets close to the finish, he can switch back to the critical power level and the velocity will slowly decay. Interestingly, this is entirely counter to human psychology. Any athlete will go all out once the finish line gets close. However, cold mathematical logic dictates that this is excess power, which should have been used earlier.

We need to make some straightforward assumptions to carry out our analysis. We first state them in a legible form before translating them into formulae.

(I) The trial is not too short. It is impossible to go all out and maintain peak level for the entire trial.
(II) The course is not too steep. The critical power level suffices to achieve a positive velocity.
(III) The rider does not start from a standstill. The initial velocity is positive, but small.
(IV) The rider is in shape. The anaerobic power level is sufficiently high to get to a velocity that can be maintained indefinitely at critical power.

We need to introduce some further notation to make this precise. The rider can apply CP indefinitely and, doing this, will be able to maintain a certain velocity. We denote this cruising velocity v_{CP}.

In control theory, starting at maximum power and using it all up before switching back to minimum power is called bang–bang control. In this terminology, the optimal power distribution in an individual time trial is bang–singular–bang.

If we translate our four assumptions into mathematical conditions, we get

(I) $T_f > \frac{W}{u_{max} - CP} > 0$;
(II) c_1 and c_2 are such that $v_{CP} > 0$;
(III) the initial velocity α satisfies $0 < \alpha < v_{CP}$;
(IV) the final velocity $v(T_f)$ is at least equal to v_{CP}.

We will show that the three levels of power in an optimal pacing strategy correspond to three stages of the velocity v:

- initial stage of peak power, when v increases above v_{CP};
- middle stage of singular power, when v is constant;
- final stage of critical power, when v decreases but remains above v_{CP}.

The singular power level

We first consider the singular power level and assume that

$$\frac{\lambda_2}{v} = -c_3\lambda_3$$

on a certain time interval. Both c_3 and the multiplier λ_3 are constants. Differentiation gives

$$\frac{d}{dt}\left(\frac{\lambda_2}{v}\right) = \frac{v(t)\frac{d\lambda_2}{dt}(t) - \lambda_2(t)\frac{dv}{dt}(t)}{v(t)^2} = 0$$

Substituting equations (2) and (5) yields

$$\frac{d}{dt}\left(\frac{\lambda_2}{v}\right) = -\frac{1}{v(t)} + 3\frac{c_1}{c_3}\lambda_2(t) + \frac{c_2\lambda_2(t)}{c_3(v(t))^2} \tag{7}$$

On the time interval that we consider

$$\frac{\lambda_2}{v}(t) = -c_3\lambda_3 = \gamma$$

hence

$$\frac{d}{dt}\left(\frac{\lambda_2}{v}\right) = 3\frac{c_1}{c_3}\gamma v(t) + \left(\frac{c_2}{c_3}\gamma - 1\right)\frac{1}{v(t)} = 0 \tag{8}$$

It follows that the velocity $v(t)$ *remains constant under singular power, if the gradient and the wind velocity are stationary*. To be precise, the velocity is equal to

$$v(t) = \sqrt{\frac{c_3}{3c_1\gamma} - \frac{c_2}{3c_1}}$$

Using equation (2), we find that the singular power level is equal to

$$u_{\text{sing}} = \frac{(c_3 + 2c_2\gamma)\sqrt{\frac{c_3 - c_2\gamma}{c_1\gamma}}}{3\sqrt{3}\gamma} \quad (9)$$

The singular power level corresponds to a constant velocity. Now it seems clear that the rider needs to accelerate until reaching this velocity and sustain it at the singular power level. To prove that, we still need to show that the ratio

$$\frac{\lambda_2(t)}{v(t)}$$

increases monotonically, and stays fixed at $-c_3\lambda_3$ until t is close to T_f.

It is optimal to finish at minimum power level

The variable v is equal to the velocity and is positive by our assumptions. The multiplier λ_2 needs to be computed from the differential equation, and it turns out to be positive as well.

Lemma 1.

$$\frac{\lambda_2}{v} > 0$$

Proof. We prove that $\lambda_2(t) > 0$ if $t < T_f$. The boundary condition in equation (6) prescribes $\lambda_2(T_f) = 0$. We inspect the expression on the right-hand side of this equation

$$-\left(1 + \lambda_1 - \frac{\lambda_2(t)u(t)}{c_3(v(t))^2} - 2\frac{c_1}{c_3}\lambda_2(t)v(t)\right)$$

The control variate λ_1 is equal to zero by equation (5). The boundary value is $\lambda_2(T_f) = 0$ and therefore $\lambda_2'(T_f) = -1$. It follows that λ_2 is strictly positive on a final interval in $[0, T_f]$. We need to argue that, in fact, $\lambda_2(t) > 0$ for the entire time interval. If this were not the case, we would have $\lambda_2(t) = 0$ for $t < T_f$. We may take t to be the final time before T_f with this property. Since $\lambda_2'(t) = -1$ we must have that λ_2 is strictly negative in between t and T_f, which contradicts that λ_2 is strictly positive on a final interval. Therefore, this final interval is the entire time interval.

Lemma 2.

$$\lambda_3 < 0$$

Proof. This can be proved by contradiction. If $\lambda_3 \geqslant 0$, then $\gamma = -c_3\lambda_3 \leqslant 0$. We have just seen that λ_2/v is positive, so is above the switching level γ. The rider will go at peak level all the way, which contradicts our assumption I.

These two lemmas imply that an optimal pacing strategy ends at the minimum power level CP, because the switching level is positive and the ratio λ_2/v is zero at T_f because of the boundary condition on λ_2.

Switching power in optimal pacing

We are considering a time trial in which all conditions are equal along the entire course. If the rider exerts a constant power in such a stationary terrain, he will eventually reach a stationary speed that is independent of his initial velocity. Mathematically, this follows from the fact that the right-hand side of equation (3)

$$\frac{dv}{dt} = \frac{u(t)}{v(t)c_3} - \frac{c_1}{c_3}(v(t))^2 - \frac{c_2}{c_3}$$

has a unique value of $v(t)$ that makes it zero, if $u(t)$ is constant. For each of our three levels of power, there are corresponding stationary velocities $v_{CP} < v_{\text{sing}} < v_{\max}$. By our assumptions, the rider starts at a velocity below v_{CP}, so even if he would be able to apply peak power the entire time, he will never reach $v_{\max}$. Therefore, the velocity will increase whenever the rider applies peak power. We already noticed that if the rider applies the singular power level, then the velocity is constant and is necessarily equal to v_{sing}. To reach this velocity, the rider needs to apply peak power first.

Knowing all this, it may now seem obvious that the rider starts at peak power until v_{sing} is reached and then applies singular power until all the anaerobic energy runs out. However, we still need to make this mathematically precise.

Lemma 3. *Suppose that $t' < t''$ are consecutive times at which the rider switches power. In particular*

$$\frac{\lambda_2}{v}(t') = \frac{\lambda_2}{v}(t'') = \gamma$$

and

$$\frac{\lambda_2}{v}(t) \neq \gamma$$

for all $t' < t < t''$. If the rider applies peak power in the interval (t', t'') then $v(t'') \leqslant v(t')$, and if the rider applies critical power then $v(t'') \geqslant v(t')$.

Proof. We first assume that the rider applies peak power between t' and t''. In this case

$$\frac{\lambda_2}{v} > \gamma$$

between t' and t'' assumes a maximum for some value of t in this time interval. At a maximum of λ_2/v, the right-hand side of equation (8) is equal to zero. More specifically

$$3\frac{c_1}{c_3}\gamma v(t) + \left(\frac{c_2}{c_3}\gamma - 1\right)\frac{1}{v(t)} = 0$$

Since

$$3\frac{c_1}{c_3}\gamma v(t)$$

is positive, we conclude that

$$\left(\frac{c_2}{c_3}\gamma - 1\right)$$

is negative. It follows that

$$\frac{d}{dt}\left(\frac{\lambda_2}{v}\right)$$

increases with v. At time t' we have that

$$\frac{d}{dt}\left(\frac{\lambda_2}{v}\right) > 0$$

and at time t'' we have that

$$\frac{d}{dt}\left(\frac{\lambda_2}{v}\right) < 0$$

In other words, the velocity decreases at time t'' after applying peak power. Clearly, this is nonsense.

If the rider applies critical power between t' and t'' then all inequalities reverse, but the line of the argument remains the same. In this case, the velocity increases at time t'' after applying critical power. In principle, this could happen if the rider applies critical power at the start of the course, v_{CP}. This is counter intuitive, but we still need to rule it out.

Theorem 1. *In an optimal pacing strategy, the rider switches back in power.*

Proof. We already know that the rider finishes at critical power. What we need to prove now is that λ_2/v crosses the critical level γ only once. This may be a cross at a single time, in which case the rider switches back from peak power to critical power immediately, or it may be a cross in a time interval. We already know that in this case the rider maintains the constant velocity v_{sing}.

We argue by contradiction and suppose that the ratio λ_2/v crosses the critical level twice, or more. Crosses always go in opposite directions, so one of these crosses has to be from critical power to peak power. We know that, in the end, the rider switches back to critical power, so there must be a value of $t' < t''$ such that

$$\frac{\lambda_2}{v}(t') = \frac{\lambda_2}{v}(t'') = \gamma$$

and

$$\frac{\lambda_2}{v}(t) > \gamma$$

in between. By the previous lemma, the velocity would then have decreased at t'' despite having applied peak power, which is nonsense. The critical level γ can only be crossed once. By Assumption IV, the rider finishes above v_{CP} so the rider needs to apply at least singular power. However, the rider can only apply singular power at v_{sing} and to get to that velocity, he first needs to apply peak power. Therefore, the power crosses the critical level exactly once.

Example

As an example, we consider a 5 km time trial with the following parameters: initial velocity $\alpha = 1$ m/s, total energy $W = 20,000$ J, maximum power $u_{max} = 800$ W and critical power $CP = 300$ W, which is comparable to the values of Atkinson and Brunskill.[10] The constants in the power equation are $c_1 = 0.128$, $c_2 = 3.924$ and $c_3 = 78$. These parameters were computed from $c_1 = 0.5C_dA\rho$, where we set the product of the drag coefficient and the frontal area equal to $C_dA = 0.217$ and ρ is air density; $c_2 = mg(s + C_R)$, where we take slope $s = 0$, $C_R = 0.005$ and $c_3 = m = 78$. These parameters are comparable to those of Wilson and Papadopoulus,[6] who recommend $C_R = 0.002 - 0.008$ and $C_dA = 0.32$. The optimal pacing strategy depends on the choice of the parameters, but the overall qualitative picture remains the same. More results are contained in De Jong.[11]

If the rider goes all out at maximum power, then W is depleted after roughly 40 s and the rider has covered approximately 1 km. For this relatively short time trial, it is optimal to use up all anaerobic energy at peak level before switching back to critical power. For a longer trial of 5 km (Figure 2), it is optimal to switch back through the singular power level. The rider sustains the maximum power level for 10 s, reaching v_{sing} of 13 m/s and switches back to the singular power level. In the final minute, he switches back to critical power and the velocity decreases to v_{CP} of 12 m/s. This result is very similar to the results of De Koning et al. [1] for short trials, who found similar optimal velocity curves.

In Figure 3, the switching function λ_2/v, the value γ and the optimal control u^* can be seen for this example.

The singular power cannot be computed in a straightforward way. It can only be determined

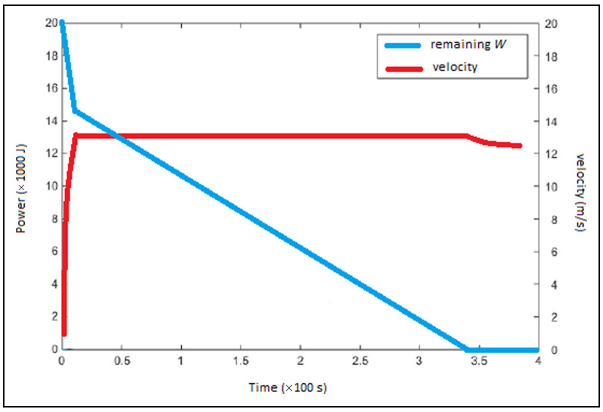

Figure 2. Example of optimal power output and energy distribution in a 5 km time trial. The blue line represents the remaining rider's energy and the red line the rider's velocity as a function of time.

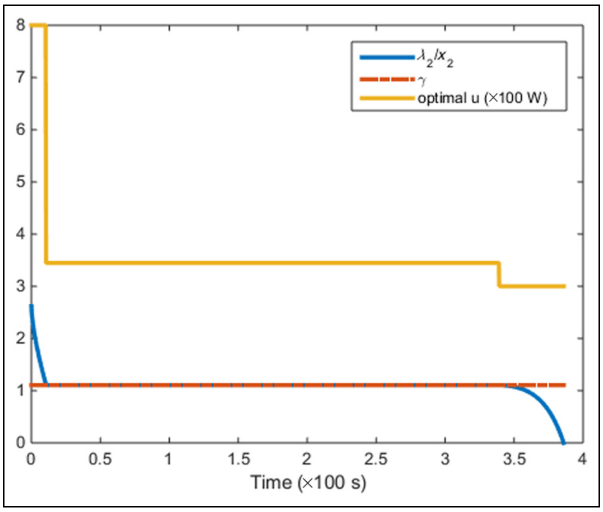

Figure 3. Optimal power output and energy distribution in a 5 km time trial. The yellow line represents the power output, the blue line the switching function λ_2/v and the red dashed line the value γ. Note that the rider starts with maximum power output u_{max} and after 10 s switches to a singular power level of u_{sing} to sustain a velocity of 13 m/s. Approximately 1 minute before the finish, his energy has run out and he coasts to the finish. Mathematically, it is optimal to use up the additional power before and reach the end at level *CP*. In reality, a rider will of course never do this, but will speed up when approaching the finish. The psychological effect of reaching the finish is not included in our power model.

numerically. Our computations show that u_{sing} approaches u_{max} in short trials and it approaches CP in long trials.

Conclusions

Using only minimal assumptions, we have shown that the optimal pacing strategy in an individual time trial involves three levels of power. In an optimal pacing strategy, the rider needs to go all out at the beginning until he reaches a velocity that can be maintained for almost the entire course. The peak power and the critical power are invariant, and only depend on the athlete. The intermediate singular power level depends on the terrain of the time trial, but can be computed numerically.

In our computations, external variables such as wind velocity and slope were constants. We have chosen stationary parameters to keep our computations simple and transparent. It is possible to use variable wind velocity and slope. The computational effort remains the same.

In our model of the rider's power model, the anaerobic reserve cannot recharge. It is not straightforward to extend our analysis to a power model that does allow such a recharge. Our analysis of the three levels of power has to be adapted; settling the mathematical details will require further study. Our analysis only applies to relatively short time trials.

In a short and flat time trial, it is crucial to round the bends at the highest possible velocity. The optimal way to round a bend in an individual time trial is important and deserves further study.

Acknowledgements

We would like to thank Teun van Erp and other members of the Giant-Sunweb cycling team for supplying data, selecting an appropriate power model and tuning the coefficients.

Declaration of conflicting interest

The authors declare that there is no conflict of interest.

Funding

This research received no specific grant from any funding agency in the public, commercial or not-for-profit sectors.

References

1. De Koning JJ, Bobbert MF and Foster C. Determination of optimal pacing strategy in track cycling with an energy flow model. *J Sci Med Sport* 1999; 2(3): 266–277.
2. Underwood I and Jermy M. Determining optimal pacing strategy for the track cycling individual pursuit event with a fixed energy mathematical model. *Sports Eng* 2014; 17: 183–196.
3. Başar T and Olsder GJ. *Dynamic noncooperative game theory*. Philadelphia: Society for Industrial and Applied Mathematics, 1999.
4. Bryson AE and Ho Y. *Applied optimal control: optimization, estimation and control*. New York: Taylor & Francis, 1975.
5. De Jong JW, Fokkink RJ, Olsder GJ, et al. A variational approach to determine the optimal power distribution for cycling in a time trial. *Procedia Eng* 2016; 147: 907–911.

6. Wilson DG and Papadopoulus J. *Bicycling science*. 3rd ed. Cambridge, MA: MIT Press, 2004, p.188.
7. Driss T and Vandewalle H. The measurement of maximal (anaerobic) power output on a cycle ergometer: a critical review. *Biomed Res Int* 2013; 2013: 589361.
8. Monod H and Schrerrer J. The work capacity of a synergic muscular group. *Ergonomics* 1981; 8: 329–338.
9. Dmitruk A and Samylovskiy I. A simplified Goddard problem in the presence of a nonlinear media resistance and a bounded thrust. In: *2013 European control conference*, Zurich, Switzerland, 17–19 July 2013, pp.341–346. Piscataway, NJ: IEEE.
10. Atkinson G and Brunskill A. Pacing strategies during a cycling time trial with simulated headwinds and tailwinds. *Ergonomics* 2000; 43(10): 1449–1460.
11. De Jong JW. *The optimal power distribution for cycling a time trial*. MSc Thesis, Utrecht University, The Netherlands, 2015.