CoA Report No. 136

THE COLLEGE OF AERONAUTICS

CRANFIELD

BOUNDARY LAYERS WITH SUCTION AND INJECTION

by

A. H. CRAVEN

REPORT NO. 136

September, 1960.

THE COLLEGE OF AERONAUTICS

CRANFIELD

Boundary Layers with Suction and Injection

A review of published work on skin friction

- by -

A. H. Craven, Ph.D., D.C.Ae.
(Royal Air Force Technical College, Henlow)

SUMMARY

Available data on the effects of suction and injection on skin friction are summarised and compared.

It is shown that injection into a turbulent boundary layer can produce a skin friction coefficient lower than the laminar value at the same Reynolds number on an impermeable plate.

CONTENTS

Contents (Continued)

LIST OF SYMBOLS

c	chord length
C_f	overall skin friction coefficient
c_f	local skin friction coefficient
c_p	specific heat at constant pressure
Eu	Euler number $\left[-x\, dp/dx\right] / \rho_o U_o^2$
H	shape parameter δ_1 / δ_2
k	mixing length constant
K, K_1	form parameters
l	mixing length ($l = ky$)
M	Mach number
p	pressure
Pr	Prandtl number
Pr_t	turbulent Prandtl number $\left(\epsilon_M c_p / \epsilon_H\right)$
R_x	Reynolds number based on x $(U_o x / \nu_o)$
R_{δ_2}	Reynolds number based on momentum thickness
T	temperature
u	streamwise velocity in the boundary layer
u_a	velocity at interface between laminar sublayer and turbulent outer region
U	streamwise velocity just outside the boundary layer
U_o	free stream velocity
U_τ	wall shear velocity $\left(U_\tau^2 = \frac{\tau_w}{\rho}\right)$
v	velocity normal to the wall
v_1	suction velocity
v_2	injection velocity
x	distance in stream direction
x_o	distance from leading edge to beginning of the porous surface
y	distance normal to surface
y_a	height of interface between laminar sub-layer and turbulent outer region

List of Symbols (Continued)

δ	boundary layer thickness
δ_1	displacement thickness
δ_2	momentum thickness
ϵ_D	eddy diffusion coefficient
ϵ_H	eddy thermal conductivity
ϵ_M	eddy viscosity
Θ	$c_f / c_{f(\text{Blasius})}$
μ	viscosity
ν	kinematic viscosity (μ / ρ)
ρ	density
τ	shear stress

Subscripts

o	free stream
∞	asymptotic conditions
w	wall conditions

1. Introduction

In the past most attempts to reduce skin friction have been based on the use of suction either through a porous surface or through discrete slots to maintain a laminar boundary layer and thus avoid the large values of skin friction associated with a turbulent layer. Unfortunately the application of suction raises the laminar skin friction and increases the effective heat transfer rate to the surface.

At low speeds it has been shown by theory and experiment that only moderate suction rates are required to maintain laminar flow. At higher speed (i.e. compressible flow) there is very little evidence on which to base an estimate of the suction rate necessary to prevent transition. However consideration of the change of critical Reynolds number for compressible boundary layers without suction suggest that the suction rate will be higher than for the incompressible case. It is possible that the laminar skin friction coefficient could approach the value for the turbulent layer without suction and suction does not help to solve the skin heating problem.

Injection of a cool gas into the turbulent boundary layer not only helps to keep the skin cool but also reduces the skin friction. Thus there may be considerable advantages in blowing rather than sucking, one of which could be a value of turbulent skin friction lower than the corresponding laminar value. Nowhere in this analysis has any account been taken of the pump power required and the duct losses associated with the installation.

This paper aims to summarise and review the available data on the effects of suction and injection upon the skin friction and to compare such data when comparison is possible.

2. The laminar boundary layer with suction

2.1. The incompressible laminar boundary layer with suction - zero pressure gradient

A flat plate is assumed to extend downstream from the origin of co-ordinates. The boundary conditions are :-

(i) $u(x,\infty) = U_o$

(ii) $u(o,y) = U_o$

(iii) $v(x,o) = -v_1(x)$ where $v_1(x)$ is positive

(iv) $u(x,o) = 0$

The basic laminar boundary layer equations are :-

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \qquad (2.1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.2)$$

for incompressible flow with zero pressure gradient.

(a) the solution of Griffith and Meredith[1] - constant suction

It is assumed that the final velocity profile has been reached i.e. $\partial u/\partial x = 0$

Thus (2.2) reduces to :-

$$\frac{\partial v}{\partial y} = 0 \qquad (2.3)$$

Hence it is deduced that the normal velocity everywhere is constant and equal to the suction velocity $-v_1$.

(2.1) can then be written

$$v_1\frac{\partial u}{\partial y} + \nu\frac{\partial^2 u}{\partial y^2} = 0 \qquad (2.4)$$

which has the exact solution

$$\frac{u}{U_o} = 1 - e^{-v_1 y/\nu} \tag{2.5}$$

usually known as the asymptotic solution or asymptotic velocity profile. This solution does not satisfy the fourth boundary condition above and is thus only applicable some distance downstream of the beginning of suction.

It is immediately obvious that, at distances sufficiently far from the leading edge of the plate for the asymptotic profile to hold, the momentum thickness is constant and given by

$$\delta_2 = \frac{\nu}{2 v_1} \tag{2.6}$$

the asymptotic wall shear stress is constant and of magnitude

$$\tau_\infty = \rho v_1 U_o \tag{2.7}$$

and the overall skin friction coefficient is given by

$$C_{f_\infty} = \frac{2 v_1}{U_o} \tag{2.8}$$

From wind tunnel tests Kay[2] showed that with a wind speed of 57 ft/sec. the velocity profile becomes asymptotic and momentum thickness becomes constant and equal to $\nu/2v_1$ in a length of four inches when subjected to distributed suction of constant velocity ($v_1/U_o = .0029$). The corresponding Reynolds number was $R_x = 1.2 \times 10^5$.

Equations (2.5) and (2.6) have also been obtained by Schlichting[3].

Another interesting solution of the equations (2.3) and (2.4) is obtainable in the case of Couette flow with either suction or injection applied at the stationary wall. Lilley[4] shows that the equations (2.3) and (2.4) exactly describe Couette flow with transpiration and zero pressure gradient when the boundary conditions

(i) $y = 0, \quad u = 0, \quad v = V$ positive for injection, negative for suction

(ii) $y = c, \quad u = U_o$

are applied.

Solution of the equations leads to a velocity profile

$$\frac{u}{U_o} = 1 + \frac{e^{Vc/\nu} - e^{Vy/\nu}}{1 - e^{Vc/\nu}}$$

and to a wall shear stress given by

$$\tau_w = \frac{\mu U_o}{c} - \tfrac{1}{2}\rho V U_o$$

for small values of $\frac{Vc}{\mu}$. It is seen immediately that skin friction is increased by suction and reduced by injection. Lilley's paper includes the effect of an applied pressure gradient.

(b) Suction velocity proportional to $1/x^{\frac{1}{2}}$

Putting $\eta = \frac{1}{2}(U_o/\nu x)^{\frac{1}{2}} y$ and defining a function $f(\eta)$ by

$$\psi = (\nu U_o x)^{\frac{1}{2}} f(\eta) \quad \text{where } \psi \text{ is the stream function}$$

(2.1) becomes

$$f'''(\eta) + f.f''(\eta) = 0$$

with

$$f = 2v_1 \Big/ \left(\frac{U_o \nu}{x}\right)^{\frac{1}{2}} \quad \text{at } \eta = 0 \text{ i.e. at the wall.} \qquad (2.9)$$

Thus with a suction velocity proportional to $1/x^{\frac{1}{2}}$, f is constant along the wall. It is deduced that the velocity profiles will be the same at all points of the plate. The application of suction with velocity proportional to $1/x^{\frac{1}{2}}$ implies a large normal velocity near the leading edge. Such conditions would invalidate the boundary layer assumptions.

However if the suction velocity is taken to be proportional to $1/(x + a)^{\frac{1}{2}}$, where a is a positive constant, the suction velocity is finite at the leading edge. In this relation a is related to the suction velocity at the leading edge. If, in the definition of η and ψ above, we replace x by x + a we obtain

$$u = \frac{U_o}{2} f'(\eta)$$

$$v = \tfrac{1}{2}\left(\frac{\nu U_o}{x+a}\right)^{\frac{1}{2}} (\eta f' - f)$$

and (2.1) again becomes

$$f'''(\eta) + f.\, f''(\eta) = 0$$

with $f' = 0$ at $\eta = 0$; $f' = 2$ at $\eta = \infty$

and $f = 2\, v_1 \Big/ \left(\frac{U_o \nu}{x+a}\right)^{\frac{1}{2}}$ at $\eta = 0$.

When v_1 is proportional to $1/(x+a)^{\frac{1}{2}}$, f is constant on $\eta = 0$ and again similar profiles are obtained at all points along the wall.

(c) The entry length - approximate solutions (Fig. 1)

The solution of Griffith and Meredith does not apply near the leading edge of a flat plate with distributed suction of constant velocity. It has been shown by Schlichting[3] and Thwaites[5] that, if suction begins at the leading edge of the plate, the initial velocity profile is the Blasius profile for a flat plate without suction. Thwaites defines a stream function ψ in the form

$$\psi = (2\, U_o\, \nu\, x)^{\frac{1}{2}} f(\xi, \eta)$$

where $\xi = v_1 \left(\frac{x}{2\, U_o\, \nu}\right)^{\frac{1}{2}}$ and v_1 is constant

$$\eta = \left(\frac{U_o}{2\, \nu\, x}\right)^{\frac{1}{2}} y$$

For small values of ξ (i.e. at small distances from the leading edge) the solution of the equation of motion

$$\frac{\partial^3 f}{\partial \eta^3} + f\, \frac{\partial^2 f}{\partial \eta^2} + \xi \left(\frac{\partial f}{\partial \xi}\, \frac{\partial^2 f}{\partial \eta^2} - \frac{\partial^2 f}{\partial \xi\, \partial \eta} \cdot \frac{\partial f}{\partial \eta}\right) = 0$$

with the boundary conditions

$$\text{at} \quad \eta = 0 \qquad f = \xi \,, \qquad \frac{\partial f}{\partial \eta} = \frac{u}{U_o} = 0$$

$$\eta = \infty \qquad \frac{\partial f}{\partial \eta} = 1$$

can be expressed in powers of ξ in the form

$$f(\xi, \eta) = f_o(\eta) + \xi f_1(\eta) + \xi^2 f_2(\eta) + \ldots\ldots$$

The function $f_o(\eta)$ i.e. $f(0, \eta)$ is given by

$$f_o''' + f_o f_o'' = 0$$

which is the Blasius equation and hence the existence of the Blasius profile at th e leading edge of a flat plate with uniform suction. Experimental evidence of this is given by Kay[2].

Schlichting[6] assumes a velocity profile

$$\frac{u}{U_o} = 1 - e^{-y/D} + K\frac{y}{D} e^{-y/D}$$

where D is a function of x and a measure of the boundary layer thickness, and K(x) is a form parameter equal to zero in the asymptotic state and $-\frac{1}{2}$ at the leading edge.

Thus at the leading edge Schlichting takes

$$\frac{u}{U_o} = 1 - e^{-y/D} - \frac{y}{2D} e^{-y/D}$$

to correspond to the Blasius profile at the leading edge. This is at best a doubtful approximation as it is nearer the asymptotic profile than the Blasius profile it aims to represent. Its use leads Schlichting to values of momentum thickness and shape parameter which are seriously in error.

An improved solution for the entry length is obtained by Preston[7] who takes a one-parameter family of velocity profiles having the exact Blasius profile and the asymptotic profile as limiting forms, i.e.

$$\frac{u}{U_o} = F_1 (y/\delta_1) + K_1 \left\{ F_2 (y/\delta_1) - F_1 (y/\delta_1) \right\}$$

where δ_1 is the displacement thickness

$F_1(y/\delta_1)$ is the Blasius profile

$F_2(y/\delta_1)$ is the asymptotic profile $1 - e^{-y/\delta_1}$

and $K_1(x)$ is a form parameter, zero at the leading edge and unity when asymptotic conditions apply.

This velocity profile gives the correct value of $H = \frac{\delta_1}{\delta_2} = 2.591$ at the leading edge of the plate compared with Schlichting's value of 2.182. On the plate Preston shows that

$$\frac{\delta_2}{\delta_1} = 0.38594 + 0.12800\, K_1 - 0.01394\, K_1^2$$

and $$\delta_1 = \frac{\nu}{v_1} \left\{ \frac{K_1}{a + (1 - a) K_1} \right\}$$

where $$a = \left[F_1' (y/\delta_1) \right]_{y = 0} = 0.57141$$

That Preston's solution is the more accurate is shown in Fig. 1, where the displacement and momentum thicknesses from the approximate methods are compared with the exact method due to Iglisch which is described later.

Crocco's form of the boundary layer equations

$$\mu \rho u \frac{\partial \tau}{\partial x} - \mu \frac{dp}{dx} \cdot \frac{\partial \tau}{\partial u} = \tau^2 \frac{\partial^2 \tau}{\partial u^2} \; ; \quad \tau = \mu \frac{\partial u}{\partial y}$$

is the starting point of a solution by Trilling[8]. The boundary conditions are

$$u = 0 \; ; \; \frac{\partial \tau}{\partial u} - \frac{\mu p'(x)}{\tau} = - \rho v_1 \qquad (2.10)$$

$$u = U(x) \; ; \quad \tau = 0 \qquad (2.11)$$

A series expansion for τ in powers of u is assumed in the form

$$\tau(x, u) = \sum_{0}^{\infty} \tau_n(x)\, u^n$$

$\tau_0(x)$ is the wall shear stress and, from (2.7)

$$\tau_1 = \frac{\mu\, p'(x)}{\tau_0} - \rho v_1$$

τ_2, τ_3 etc are expressed in terms of v_1, τ_0 and $p'(x)$. τ_0 can then found from the condition (2.11) using terms up to including n = 6.

A fourth approximate method is due to Ringleb[9]. In this analysis Prandtl's original boundary layer equations are taken with a velocity profile assumed to be

$$u = U(1 - e^{ay + by^2 + cy^3 + dy^4})$$

where a, b, c and d are functions of x which are supposed to have continuous derivatives of the first order.

The boundary conditions are, at y = 0,

$$\nu \frac{\partial^2 u}{\partial y^2} = - v_1 \frac{\partial u}{\partial y} - U\, U'(x)$$

$$\nu \frac{\partial^3 u}{\partial y^3} = - v_1 \frac{\partial^2 u}{\partial y^2}$$

$$\nu \frac{\partial^4 u}{\partial y^4} = \frac{\partial u}{\partial y} \cdot \frac{\partial^2 u}{\partial x\, \partial y} - v_1 \frac{\partial^3 u}{\partial y^3}$$

$$\nu \frac{\partial^5 u}{\partial y^5} = 2 \frac{\partial u}{\partial y} \cdot \frac{\partial^3 u}{\partial x\, \partial^2 z} - v_1 \frac{\partial^4 u}{\partial y^4}$$

and in addition d must be negative. b, c, d are determined in terms of a from the first three boundary conditions and a is determined from a first order differential equation obtained from the fourth condition and U and v_1. The shear stress at the wall is given by $\tau_w(x) = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} = -U(x)\ \mu a(x)$.

Ringleb's solutions for the momentum and displacement thicknesses and the wall shear stress agree very well with Iglisch's exact solution. On the scale of Fig. 1 it is not possible to differentiate between the solutions of Ringleb and Iglisch.

In a recent paper Curle[10] has used Stratford's laminar boundary layer analysis to obtain the complete skin friction distribution along a wall. In this analysis total head is taken as constant along streamlines in the outer part of the layer. Near the wall viscous forces and pressure forces must balance and the velocity profile adjusts itself accordingly. Curle includes the momentum of the fluid sucked away in the balance between viscous and pressure forces and obtains a particularly simple expression for the ratio Θ of the skin friction with uniform suction to the Blasius skin friction in the form

$$0.32867\,(\Theta - 1)\left\{\frac{1 + 2.02\,\Theta}{\Theta^2}\right\}^{\frac{1}{3}} = \left(\frac{V_1^2\,x}{U_o\,\nu}\right)^{\frac{1}{2}}\left\{1 - \frac{x_o}{x}\right\}^{\frac{1}{3}}$$

where x_o is the distance behind the leading edge at which suction starts and $x \geqslant x_o$.

This result is compared with the exact solution due to Iglisch ($x_o = 0$) in Fig. 2. It is seen that Curle's solution overestimates the skin friction by an amount increasing with suction velocity and distance from the leading edge.

(d) The exact solution due to Iglisch[11]

In this solution the basic equations (2.1 and 2.2) are taken and after several transformations of variable Iglisch obtains a second order differential equation for the velocity in terms of two space co-ordinates which is completely general and can accommodate any distribution of suction velocity. Iglisch does not solve this general problem but thereafter confines his attention to the special case of homogeneous (i.e. constant) suction. For constant suction velocity Iglisch's equation becomes the non-linear second order parabolic equation

$$V^{\frac{1}{2}}\frac{\partial^2 V}{\partial t^2} + \frac{\partial V}{\partial t}\left(2\,t\sigma + t^3 - \frac{V^{\frac{1}{2}}}{t}\right) = 2\,t^2\sigma\frac{\partial V}{\partial \sigma}$$

where V is related to the velocity u and t, σ are independent variables related to x and y. $\sigma = 0$ corresponds to the leading edge and $\sigma = \infty$ to the asymptotic state. The boundary conditions reduce to

$$V(\infty,\sigma) = 4, \qquad V(0,\sigma) = 0.$$

It is then shown that for zero suction the equation yields the Blasius solution for a flat plate. Furthermore it is shown that, at the leading edge of a plate with suction, the Blasius profile is obtained. Starting from this profile at the leading edge an iteration process is used to evaluate the velocity profile at seventeen stations between the leading edge and the point at which the profile becomes asymptotic. These profiles and corresponding values of the displacement thickness, the momentum thickness and wall shear stress are tabulated in the paper. Thus the local skin friction coefficient is immediately obtainable. The overall skin friction coefficient on a flat plate of length l can then be calculated from the relation

$$C_f = \frac{2\nu_1}{U_o} F(l) \tag{2.12}$$

where $$F(l) = \frac{1}{\left(\frac{\nu_1}{U_o}\right)^2 R_l} \int_0^{\left(\frac{\nu_1}{U_o}\right)^2 R_l} \frac{\tau_w(\xi)\, d\xi}{\rho \nu_1 U_o}$$

(see Fig. 3).

Furthermore a universal law for the skin friction coefficient on a plate of length l is obtained in the form of a unique line when $C_f \cdot \frac{U_o}{2\nu_1}$ is plotted against $\frac{\nu_1}{U_o}\sqrt{R}$. (see Fig. 4).

Iglisch's solution yields the streamline pattern for constant suction and from it the normal velocity is calculated throughout the flow field. Finally Iglisch shows that his solution gives the asymptotic profile for $\sigma = \infty$ and that, for practical purposes, all the flow characteristics lie sufficiently close to their asymptotic values at the last of the seventeen stations ($\sigma = 4$).

2.2. Prevention of laminar separation by distributed suction

Preston[7] extends his solution for the flat plate with suction to an aerofoil with a permeable surface through which suction is applied. The Blasius profile (rather than the more correct $U \alpha x$) is chosen at the leading edge. It is assumed that this will not affect the accuracy of the calculations since asymptotic conditions are assumed to exist for some distance downstream of the leading edge. Beyond this distance an adverse pressure gradient exists and Preston finds that the suction velocity required to prevent laminar separation is given by

$$\frac{v_1}{U_o} = \left[\frac{1}{\lambda R} \cdot \frac{d(U/U_o)}{d(x/c)} \right]^{\frac{1}{2}}$$

where $\lambda = \frac{\delta_1^2}{\nu} \cdot \frac{dU}{dx}$.

In the case of constant adverse velocity gradient, Preston shows that, assuming the Howarth separation profile holds in the presence of suction, the minimum suction velocity required just to prevent laminar separation is

$$\frac{v_1}{U_o} = 1.607 \left(\frac{1}{R} \cdot \left| \frac{d(U/U_o)}{d(x/c)} \right| \right)$$

Prandtl[12], using the Pohlhausen separation profile, finds that, for the same special case,

$$\frac{v_1}{U_o} = 2.18 \left(\frac{1}{R} \cdot \left| \frac{d(U/U_o)}{d(x/c)} \right| \right)$$

Other values of the numerical multiplier obtained in similar solutions are 1.55 Curle[10]

2 Head[13]

4 Thwaites[14]

2.3. Maintaining a stable laminar boundary layer with suction

From Fig. 3 it can be seen that, if a suction velocity (v/U_o) greater than 0.002 is applied to a laminar boundary layer on a flat plate the skin friction is greater than for a turbulent boundary layer without suction at the same Reynolds number. At high values of the local Reynolds number (R_x) the suction velocity ratio may be much less than 0.002. Thus there is a limit to suction velocity above which there is no advantage to be gained in maintaining a laminar layer by application of suction. It is therefore necessary to determine the suction rate needed to ensure stability of the laminar boundary layer.

The first stability analysis appears to be that of Bussmann and Munz[15] who found that the asymptotic laminar boundary layer with suction has a critical Reynolds Number ($\frac{U_o \delta_1}{\nu}$) of 7×10^4 compared with 575 for a flat plate without suction. The corresponding minimum suction velocity is $v_1 / U_o = 0.14 \times 10^{-4}$. Pretsch[16] shows that the critical Reynolds number for the asymptotic profile is raised from 680 to 5.52×10^4 before amplification of small disturbances occurs and the maximum amplification is 1/7th of that occurring without suction. According to Pretsch, the laminar boundary layer is stable for suction velocities v_1/U_o greater than 0.182×10^{-4}. Trilling[8] finds that the critical Reynolds number ($R\,\delta_1$) is raised from 511 to 41,000 by suction and that a suction rate $v_1/U_o = 0.243 \times 10^{-4}$ is required for stability.

The stability of the laminar flow before asymptotic conditions are reached is the subject of a theoretical investigation by Ulrich[17]. Eight of the nineteen exact velocity profiles calculated by Iglisch are examined. Ulrich finds that a constant suction velocity $\frac{v_1}{U_o} = 1.18 \times 10^{-4}$ is sufficient to maintain laminar conditions anywhere upon the plate. It is pointed out that this velocity is necessary only near the leading edge and at greater distances a smaller velocity will suffice, the limiting value being 0.14×10^{-4} as predicted by Bussmann and Munz. Furthermore Ulrich investigates the case when the suction is applied so that v is proportional to $1/\sqrt{x}$. Less suction is required and lower skin friction coefficients are obtained only for overall Reynolds numbers greater than 10^8.

Using the approximate formula deduced by Lin[18], Hahnemann, Freeman and Finston[19] have obtained a rather lower value than Ulrich for the critical Reynolds number and calculate that the minimum suction velocity to be $v_1/U_o = 1.5 \times 10^{-4}$. A value of $v_1/U_o = 1.7 \times 10^{-4}$ is given by Burrows, Braslow and Tetervin[20] who apply Lin's formulae to the Schlichting profiles

$$\frac{u}{U_o} = 1 - e^{-y/D} + \frac{K_y}{D} e^{-y/D}$$

Experimental investigations in a wind tunnel by Kay[2] showed that a suction rate $v_1/U_o = .0008$ (about seven times Ulrich's theoretical figure) was necessary to maintain laminar flow at a Reynolds number of approximately 0.8×10^6. Kay attributes the high rate of suction necessary to excessive free stream turbulence and the non-porous entry length on his flat plate. The value of the critical Reynolds number without suction is not given. In order to eliminate the effect of tunnel turbulence flight tests have been performed by Head[21] using a symmetrical aerofoil designed by Lighthill's method. This aerofoil, having constant velocity up to 68 per cent of its chord, is considered to have a boundary layer flow up to 0.68c similar to that on a flat plate. Head concludes that a suction rate $v_1/U_o = 1.5 \times 10^{-4}$ is sufficient to prevent transition at a Reynolds number of 3×10^6, based on the length of porous surface.

Experimental studies of uniform distributed suction applied to an isothermal laminar boundary layer by Libby, Kaufman, and Harrington[22] have shown that the layer is stabilised to an indefinitely high Reynolds number by a suction velocity (v_1/U_o) of the order of 1×10^{-4}, a value which agrees with that found experimentally by Head and theoretically by Ulrich. The critical Reynolds number (R_x) for transition without suction was found to be 1.5×10^5 in these experiments. This relatively low value of critical Reynolds number is not commented upon in the paper.

2.4. The effect of slot suction

In the previous sections the effect of distributed suction on the skin friction and stability of a laminar boundary layer has been discussed. Alternatively similar results can be obtained by applying suction at one or more discrete slots. Here the object is to position the slots so that the boundary layer thickness is kept less than that which would lead to instability and greater than that which would lead to transition caused by surface roughness[23]. Gregory and Curtis[24] have shown that, in general, minimum total drag is obtained if the minimum number of slots is used.

Experimental investigations on aerofoils by Holstein[25], Loftin and Burrows[26] and Pfenninger[27,28] have shown that the total effective drag (i.e. wake drag plus pump drag) is approximately halved by laminarisation as a result of slot suction. Furthermore by maintaining completely laminar flow the low drag range of lift coefficient could be more than doubled.

2.5. The compressible laminar boundary layer with suction (Fig. 5,6)

The application of suction to a compressible laminar boundary layer receives little attention in the literature. There appears to be no experimental data and theoretical treatments are restricted to two papers by Lew[29,30]. In the first paper the effect of constant distributed suction is investigated. Approximate solutions of the boundary layer equations are obtained by inserting firstly a velocity profile represented by a fourth degree polynomial and secondly an exponential profile into the von Karman momentum integral equation as simplified by the Dorodnitzyn transformation. The polynomial velocity distribution cannot be used above a certain limiting Reynolds number ($R_x = U_o x/\nu_o$) which increases with Mach number and varies inversely as the suction velocity ratio v_1/U_o. Above this limiting Reynolds number the exponential profile must be used. It is to be noted that the two profiles do not give continuous values of the overall skin friction at the limiting Reynolds number, the exponential profile leading to a somewhat higher value of the skin friction coefficient (Fig. 5).

In his second paper Lew relaxes the condition of constant suction velocity over the whole plate and investigates the effect of variable suction on the boundary layer with the polynomial velocity profile taken in the earlier paper.

Two cases are considered,

(i) suction velocity $\frac{v_1}{U_o} \propto \left[\frac{\nu_o}{U_o (x + a)}\right]^{\frac{1}{2}}$

where a is a constant to ensure a finite value of suction at the leading edge. Such a variation of suction velocity gives similar velocity profiles in incompressible flow (section 2.1b).

(ii) uniform suction starting some distance downstream of the leading edge.

The results presented in the paper are for M = 1 and zero heat transfer at the wall. It is shown (Fig. 7) that a suction velocity proportional to $1/(x + a)^{\frac{1}{2}}$ gives a lower skin friction coefficient than uniform suction for $R_x > 10^5$. The value of a in this calculation corresponds to a value of $\frac{U_o a}{\nu} = 0.552 \times 10^6$ and a suction velocity $\frac{v}{U_o} = 0.001$ at the leading edge.

2.6. The stability of the compressible laminar boundary layer with suction

As far as the author is aware no solution of the Orr-Sommerfeld equation in the case of a compressible laminar boundary layer with suction has been obtained.

In general, investigators have applied the conditions for stability of an incompressible flow on an impermeable wall to the theoretical velocity profiles obtained in compressible flow with suction. While such application of stability theory is not justified rigorously, it should give at least the qualitative influence of suction on boundary layer stability.

Following this type of argument Libby, Lew and Romano[31] have taken the critical Reynolds number $\left(\frac{U\delta_2}{\nu}\right)$ for the existence of neutral disturbances in a laminar incompressible boundary layer on an impermeable surface to be 300. It is then deduced that unstable disturbances would be likely to develop at a distance x downstream of the leading edge given by

$$R_x = \frac{Ux}{\nu} = 1.9 \times 10^5 \quad \text{at } M = 0$$
$$= 2.0 \times 10^5 \quad \text{at } M = 1$$

with suction applied at a rate $v_1/U_o = .001$, the critical Reynolds number R_x at $M = 1$ is 3.9×10^5. When $v_1/U_o = .002$ and $M = 1$, the laminar boundary layer appears to be stable to an indefinitely high Reynolds number. Repeating these tentative calculations for higher Mach numbers the following lower critical Reynolds numbers (R_x) emerge

	$v_1/U_o = 0$	.001	.002
M = 0	1.9×10^5	3.1×10^5	Indeterminate
1	2.0×10^5	3.9×10^5	"
2	2.2×10^5	5.0×10^5	"
3	2.5×10^5	Indeterminate	"
4	2.8×10^5	"	"

That these estimates of the suction quantities necessary to stabilise a compressible laminar boundary layer are large is suggested by the results of flight tests at $M = 0.70$ reported by Head, Johnson and Coxon[31a]. In these tests complete stability was achieved on a wing at a Reynolds number of 29×10^6 with a suction rate given by $v_1/U_o \approx 0.0003$.

3. The Laminar Boundary Layer with Injection

3.1. The incompressible laminar boundary layer with air injection (Fig. 8)

The basic equations for the boundary layer on a flat plate with zero pressure gradient in the presence of distributed injection are again eqns. 2.1 and 2.2. The boundary conditions are the same except that (iii) becomes

$$v(x,0) = v_2(x) \quad ; \quad v_2 \text{ being positive.}$$

Integration of (2.1) and (2.2) with respect to y gives

$$\int_0^\infty \frac{\partial}{\partial x} u^2 \, dy + \left[\overline{uv} \right]_{y=\infty} = -\nu \left[\frac{\partial \overline{u}}{\partial y} \right]_{y=0} \tag{3.1}$$

$$\int_0^\infty \frac{\partial u}{\partial x} \, dy + (v_\infty - v_2) = 0 \tag{3.2}$$

Elimination of v_∞ between (3.1) and (3.2) yields

$$\frac{d}{dx} \int_0^\infty \frac{u}{U_o} \left(1 - \frac{u}{U_o} \right) dy - \frac{v_2}{U_o} = \frac{\tau_w}{\rho U_o^2}$$

or with the usual definition of δ_2, the momentum thickness is given by

$$\delta_2 = \frac{v_2 x}{U_o} + \int_0^x \frac{\tau_w}{\rho U_o^2} \, dx \tag{3.3}$$

Schlichting[3] claims that as $x \to \infty$ and asymptotic velocity profile also exists for the case of injection, its form* being

$$\frac{u}{U_o} = 1 - e^{-2y/\delta_{1\infty}} \left(1 + 2y/\delta_{1\infty} \right) \tag{3.4}$$

where $\delta_{1\infty}$ is the asymptotic displacement thickness.

---

* The form of velocity profile appears to be empirical since it does not satisfy the equation of motion.

At all points of the plate the value of $\frac{du}{dy}$ derived from (3.4)

is zero and it is seen that the asymptotic velocity profile corresponds to a separation profile on an impermeable wall. We may infer that the wall shear stress is zero and hence at large values of x, when the first term in equ.(3.3) is large compared with the second, the momentum thickness may be written in the form given by Schlichting.

$$\delta_{2_\infty} = \frac{\nu_2 x}{U_o}$$

The independence of this relation upon viscosity implies that the equation holds for both laminar and turbulent flow.

In the entry length Schlichting[5] again takes an approximate velocity profile

$$\frac{u}{U_o} = 1 - e^{-y/D} + \frac{Ky}{D} e^{-y/D}$$

with $K = -\frac{1}{2}$ at the leading edge

$= -1$ when asymptotic conditions are reached.

Value of displacement and momentum thickness and wall shear stress are calculated at various points in the entry length.

An exact solution for the laminar boundary layer on a flat plate with air injection has been obtained by Brown and Donoughe[32] for incompressible flow (Fig. 9). The solution takes into account the pressure gradient along the plate and also any changes in fluid properties due to temperature differences between the wall and the free stream. The analysis assumes that

(i) the Mach number is small

(ii) the Euler number (i.e. the pressure gradient parameter $Eu = \frac{-x \frac{dp}{dx}}{\rho_o U_o^2}$; $U_o \propto x^{Eu}$) is constant

(iii) the wall temperature is constant

(iv) the fluid property variations are expressible as some power of the absolute temperature.

The partial differential equations of energy, momentum and continuity are transformed by the method of Falkner and Skan to two total differential equations which are solved numerically for fifty eight cases covering various pressure gradients, injection velocities and wall to free stream temperature ratios. For each case displacement, momentum and convection thicknesses, Nusselt number and local skin friction coefficient are calculated and tabulated with the corresponding velocity profile.

The only restriction on the analysis is that the injection velocity is taken to be proportional to $x^{\frac{1}{2}(Eu-1)}$. Thus, for zero pressure gradient, the injection velocity is proportional to $1/x^{\frac{1}{2}}$. Any solution for a flat plate with zero pressure gradient involving a suction velocity proportional to $1/x^{\frac{1}{2}}$ is suspect since this implies a large injection velocity near the leading edge. Such a velocity is not consistent with the assumption of zero pressure gradient. The solution gives uniform injection for the case of a pressure gradient for which the Euler number is unity.

3.2. The compressible laminar boundary layer with air injection (Fig.9)

The first paper claiming to investigate the effect of gas injection on the compressible boundary layer is apparently that on Klunker and Ivey[33]. A heat balance is set up at the surface. To solve the ensuing equation some velocity profile is needed and Klunker and Ivey take the asymptotic injection profile for uniform injection velocity first found by Schlichting. The skin friction coefficient given in this paper is in fact the same as in incompressible flow, since viscosity is assumed proportional to temperature.

Low[34] takes the compressible laminar boundary layer equations and extends the treatment of Chapman and Rubesin by including a finite normal velocity at the surface. In order to obtain similar velocity and temperature profiles Low assumes the injection velocity to be proportional to $1/\sqrt{x}$. Consistent with this assumption the temperature of the plate must be uniform. Viscosity μ and temperature T are taken to be linearly related by the equation

$$\frac{\mu}{\mu_o} = C \frac{T}{T_o}$$

A similarity variable η is defined by

$$\eta = \tfrac{1}{2} \left(\frac{U}{\nu_o x C}\right)^{\frac{1}{2}} \int_o^y \frac{T_o}{T}\, dy$$

and a stream function $f(\eta)$ is taken such that

$$\frac{u}{U_o} = f'(\eta) \qquad f'(\eta) \equiv \frac{df}{d\eta}$$

The normal velocity v is given by

$$\frac{v}{U_o} = -\tfrac{1}{2} \frac{\rho_o}{\rho} \sqrt{\frac{\nu_o C}{U_o x}}\ (f - \eta f')$$

and the momentum equation becomes

$$f f'' + f''' = 0$$

with the boundary conditions

$$f'(0) = 0 \qquad f'(\infty) = 2$$

$$f(0) = -2 \frac{\rho_2 v_2}{\rho_o U_o} \sqrt{\frac{U_o x}{\nu_o C}} \qquad (\text{\& is constant since } v_2 \propto \frac{1}{\sqrt{x}})$$

The energy equation in terms of $f(\eta)$ is

$$T'' + Pr\, fT' = \frac{1-\gamma}{4} M_o^2\, T_o\, Pr\, (f'')^2$$

with $\qquad T(0) = T_w, \quad T(\infty) = T_o.$

The momentum and energy equations are solved numerically for four values of $f(0)$ and the results for $f(\eta)$ tabulated in the paper together with its first and second derivatives. The local and overall skin friction coefficients are given by

$$c_f \equiv \frac{\tau_w}{\frac{1}{2}\rho_o U_o^2} = \tfrac{1}{2} f''(0) \sqrt{\frac{C}{R_x}}$$

$$C_f \quad \frac{1}{\frac{1}{2}\rho_o U_o^2 x} \int_o^x \tau_w dx = f''(0) \sqrt{\frac{C}{R_x}}$$

Regardless of Mach number it is shown that skin friction decreases with increase of air injection rate as a result of the decrease in $f''(0)$. Variation of Mach number is included through the constant C which depends upon Mach number.

3.3. Foreign gas injection into a compressible laminar boundary layer

In papers which the present author has been unable to obtain, Eckert and others[35, 36] have shown that injection of a light gas into a laminar boundary layer is more effective in reducing the skin friction than injection of air. Foreign gas injection reduces skin friction by thickening the boundary layer by diffusion and by altering the velocity profile at the wall.

The investigation by Smith[37] into the effect of diffusion on the compressible laminar boundary layer can be used to give a first estimate of the skin friction coefficient with foreign gas injection. Smith solves the usual boundary layer equations and the diffusion equation with the boundary conditions for an impermeable flat plate. The solution takes account of the presence of a foreign gas but considers that the injection velocity is extremely small. In other words the diffusion problem is solved without the wall boundary condition appropriate to injection. The method of solution follows that of Schuh[38] in defining a similarity variable $\eta \propto y/x^{\frac{1}{2}}$ and a stream function $f(\eta)$. The differential equations of the boundary layer are transformed into integral equations which are solved by an iterative method using the Blasius profile as the first approximation. Smith shows that four iterations are usually sufficient to obtain a velocity profile with a sufficiently small error.

Smith does not calculate skin friction coefficients but once the velocity profile is determined it is a simple matter to calculate the shear stress at the wall.

$$\tau_w = \left[\mu \frac{\partial u}{\partial y} \right]_{y=0}$$

and

$$\frac{u}{U_o} = \frac{1}{2} \frac{\rho_o}{\rho} f'(\eta) \; ; \qquad \eta = \frac{y}{2} \frac{U_o}{\nu_o x}$$

For the isothermal boundary layer Smith shows that the velocity profile is dependent only on η and not on free stream Mach number explicitly. It is readily deducable that the ratio of skin friction coefficient with the foreign gas present to the coefficient without the

foreign gas is

$$\frac{c_{f\,Hydrogen}}{c_{f\,air}} \approx 0.915$$

$$\frac{c_{f\,Helium}}{c_{f\,air}} \approx 0.698$$

for the isothermal layer.

The analysis of Smith is lacking in that the values of the injection velocity into the boundary layer is ignored and no estimate of the effect of change of injection rate can be made. Furthermore it is assumed that the gas at the wall is the foreign gas only. In considering the major processes and parameters governing gas injection and sublimation, Clarke[39] has recently investigated the effect of foreign gas injection into a simple shear layer (Couette flow) with no pressure gradient in the "mainstream" direction. In his paper Clarke points out that if the gas at the wall is to be solely injected gas the injection velocity is not negligible. Thus the application of Smith's analysis to a boundary layer with injection is not likely to lead to accurate results.

For the Couette flow problem with injection, Clarke uses the equations of continuity and motion with density and viscosity being appropriate to a binary gas mixture which varies in composition with distance from the wall.

It is also necessary to use the diffusion equation and the continuity equations for each of the gas species considered separately. Crocco's transformation as modified for Couette flow is applied to express derivatives with respect to distance from the wall in terms of derivatives with respect to speed in the stream direction. Expressions for the heat transfer and shear stress are determined. At the wall it is shown that the shear stress is given by

$$\tau_w = \frac{\rho_w v_2 U_o}{\exp\left(\frac{\rho_w v_2}{c\,\bar{\mu}}\right) - 1} \approx \bar{\mu}\,\frac{U_o}{c} - \frac{\rho_w v_2 U_o}{2}$$

where c is the distance between the plates and $\bar{\mu}$ is the mean viscosity. This is identical with the expression found by Lilley[4] for incompressible flow.

For the vanishingly small injection velocities envisaged by Smith the concentration of foreign gas at the wall is small and the mean viscosity will be very nearly equal to the viscosity of the main stream gas. Thus

$$\frac{c_f}{c_{f_{v_2=0}}} = 1 - \frac{\rho_w v_2 c}{c \bar{\mu}}$$

Clarke's theory shows that there is a significant reduction in skin friction when $\frac{\rho_w v_2}{\rho_o U_o}$ is of the order l/cM where l is the mean free path. Thus it may be deduced that only very small injection velocities are required to reduce the skin friction and that injection becomes more effective as the Mach number is increased.

3.4. The stability of the laminar boundary layer with injection

Here again there is an almost complete lack of precise information. The effect of injection must qualitatively be similar to the effect of roughness. Hence we may expect that transition will occur earlier with injection than without, under similar conditions. This has been shown experimentally by Libby, Kaufman and Harrington[21] when injecting air into the laminar boundary layer on a flat plate. The critical Reynolds number R_x was found to be

1.5×10^5 for zero injection

0.71×10^5 for $v_2/U_o = 10^{-3}$

0.60×10^5 for $v_2/U_o = 4 \times 10^{-3}$

0.48×10^5 for $v_2/U_o = 8 \times 10^{-3}$.

Furthermore in the case of compressible flow on an impermeable wall it is known (Lees[40], Dunn and Lin[41]) that increase of Mach number and the cooling of the wall both help to maintain the stability of the laminar boundary layer and we may infer that the figures for critical Reynolds numbers given above are underestimates for a compressible layer.

Pappas[53] finds that injection of a light gas (helium) "trips" the boundary lyer earlier than injection of air at the same rate of mass

flow per unit area while a heavy gas (freon) gives a later transition. It is immediately apparent that the injection velocity for helium is higher than for air and hence it is conjectured that the "effective roughness" of the helium jet is greater than for the air which in turn is greater than for freon injection. Furthermore it is noted that the heavier the injected gas the fuller is the velocity profile at the same wall position indicating the risk of early transition if a light gas is injected and a delayed transition associated with the injection of a heavy gas.

4. The Turbulent Boundary Layer with Suction

4.1. The incompressible turbulent boundary layer with suction

The first analysis of a turbulent boundary layer with uniform suction was due to Schlichting[5] who demonstrated that a constant momentum thickness (i.e. asymptotic conditions) could be obtained by suction, its value being given by

$$\left(\frac{U_o \delta_2}{\nu} \right)^{\frac{1}{4}} = \frac{0.01256}{v_1 / U_o}$$

The analysis stems from the momentum equation with the assumption that the skin friction on the porous surface could be obtained in terms of R_{δ_2} from existing impermeable wall data. Dutton[42] suggests that this assumption is unsatisfactory since in his experiments he found that the momentum thickness did not change over the porous surface while the skin friction (calculated from $d\delta_2/dx$) at the beginning of suction was one third of that in the asymptotic state. This follows from the momentum equation

$$\frac{\tau_w}{\rho U_o^2} = \frac{d\delta_2}{dx} + \frac{v_1}{U_o}$$

where $\frac{d\delta_2}{dx} = 0$ in the asymptotic cases.

In wind tunnel tests Kay[2] obtained velocity profiles ($u \sim y$) on the rear of a porous surface with suction which were almost identical at successive stations. It was therefore inferred that the asymptotic state could be reached with a turbulent layer but only after a considerably greater development length than for the corresponding laminar layer. In a theoretical analysis of the asymptotic state Kay notes that the wall shear stress will be the same as for the laminar asymptotic state

$$\frac{\tau_w}{\rho_o U_o^2} = \frac{v_1}{U_o} \tag{4.1}$$

Having assumed an asymptotic state $\left(\frac{\partial}{\partial x} = 0\right)$, Kay obtains the equation of motion in the form

$$v_1 \frac{du}{dy} = \frac{1}{\rho} \frac{d\tau}{dy} \tag{4.2}$$

for the outer turbulent region of the layer and, for the laminar sub-layer

$$v_1 \frac{du}{dy} = \nu \frac{d^2 u}{dy^2} \tag{4.3}$$

Using mixing length theory ($l = ky$) and the turbulent shear stress given by momentum transfer theory ($\tau = \rho l^2 \left\{\frac{du}{dy}\right\}^2$), a velocity profile

$$\frac{u}{U_o} = 1 - \frac{1}{4k^2} \frac{v_1}{U_o} \left(\log_e \frac{y}{\delta}\right)^2 \tag{4.4}$$

is obtained and is shown not to agree with measured profiles. If, however, the turbulent shear stress is taken from vorticity transfer theory

$$\frac{\partial \tau}{\partial y} = \rho l^2 \frac{du}{dy} \cdot \frac{d^2 u}{dy^2} \tag{4.5}$$

and making the approximation that $\frac{du}{dy} = 0$ at $y = \infty$ in place of $y = \delta$, the velocity profile obtained

$$\frac{u}{U_o} = 1 + \frac{1}{k^2} \frac{v_1}{U_o} \log_e \left(\frac{y}{\delta}\right) \tag{4.6}$$

is found to agree well with experiment.

In using Taylor's vorticity transfer theory it is assumed that the distribution of Reynolds stresses through the asymptotic layer is the same as in an ordinary turbulent layer without suction. Dutton's experiments show that the two distributions are quite different. Furthermore Sarnecki[43] on repeating Kay's experiments could not obtain agreement with the logarithmic law (4.6). Black and Sarnecki repeat Kay's analysis using vorticity transfer theory and obtain a velocity profile which, for an impermeable wall, reduces to a linear profile contradicting the well known logarithmic law for such cases. Using an analysis based on the momentum transfer theory similar to that of Kay and putting $\tau_w = \rho v_1 U_o$ a bilogarithmic law is obtained in the form

$$U_\tau^2 - v_1 u = \left(\frac{v_1}{2k} \log_e \frac{y}{\delta}\right)^2 \qquad (4.7)$$

A similar analysis for the impermeable wall yields the accepted

$$\frac{u}{U_\tau} = A + B \log_e \frac{U_\tau y}{\nu}.$$

The bilogarithmic law is shown to fit not only experimental measurements made by the authors but also the results of Kay and Dutton.

A section of the paper by Black and Sarnecki is devoted to the application of Coles' Wake Hypothesis to turbulent boundary layers with transpiration and it is shown that, if the local constraint shear velocity is taken to be

$$\frac{1}{k} (U_\tau^2 - v_o u_c)$$

in the case of suction, the wake hypothesis for layers on solid surfaces remains valid in layers with transpiration.

4.2. The compressible turbulent boundary layer with suction

The only analysis of the compressible turbulent boundary layer with suction is due to Dorrance and Dore[44]. The authors extend the analysis of the compressible turbulent layer on a flat plate as given by Van Driest by applying revised boundary conditions to take into account the suction at the wall. A method (based on momentum transfer theory with a linear mixing length) analogous to that used by Van Driest gives an approximate velocity distribution in the form of an integral equation for u/U_o in terms of the local skin friction coefficient, the mixing length (Von Karman) constant and an arbitrary constant.

A relation between the local skin friction, the Mach number, Reynolds number, wall temperature and suction mass flow is obtained from the momentum integral equation in the use of which turbulent conditions are assumed to extend to the wall. The mixing length constant and the arbitrary constant are determined from the local skin friction law of Von Karman and Schoenherr's average law when $M = 0$ and for zero suction. Furthermore there appears to be some confusion between local and overall skin friction coefficients.

The only experimental evidence with which the theory is compared is that of Mickley et al[45] for suction applied to a low speed turbulent layer. The agreement between theory and experiment is not good and furthermore Mickley and Davies[46] have expressed some considerable doubts on the accuracy of the experiments and suggest corrections which would worsen the comparison with the theory of Dorrance and Dore.

5. The Turbulent Boundary Layer with Injection

5.1. The incompressible turbulent boundary layer with injection

Apart from the paper by Schlichting[5] in which he points out the similarity of the asymptotic expressions for momentum thickness in the case of injection into laminar and turbulent layers, the investigations of the effect of constant uniform air injection into a turbulent boundary layer have all rested upon the momentum transfer theory with a linear mixing length to relate turbulent shear stress to the local velocity gradient $\left(\frac{\partial u}{\partial y}\right)$. Clarke, Menkes and Libby[47] simplify the equation of motion by neglecting derivatives with respect to x and obtain

$$v_2 \frac{du}{dy} = \frac{1}{\rho} \frac{d\tau}{dy} \qquad (5.1)$$

The laminar sub-layer is ignored.

Using momentum transfer theory (5.1) becomes

$$v_2 \frac{du}{dy} = k^2 \frac{d}{dy}\left[y^2 \left(\frac{du}{dy}\right)^2 \right]$$

from which, after integration, the velocity profile

$$\frac{u}{U_\tau} = a + b \log_e \left(\frac{y U_\tau}{\nu}\right) + \frac{1}{4k^2} \frac{v_2}{U_\tau} \left[\log_e \frac{y U_\tau}{\nu}\right]^2 \qquad (5.2)$$

is obtained. It can be seen that (5.2) reduces to

$$\frac{u}{U_\tau} = a + b \log_e \left(\frac{y U_\tau}{\nu}\right)$$

for the impermeable surface. The constants a, b and k are shown to be dependent upon injection velocity, but in the absence of experimental data with injection the impermeable surface values are taken.

The overall skin friction coefficient is given by

$$C_F = \frac{2}{R} \int_0^{\eta_1} \frac{F\left(\frac{v_2}{U_o}, \eta_1\right) d\eta_1}{\phi_1^2 \left[1 + \frac{v_2}{U_o} \phi_1^2\right]}$$

where

$$F\left(\frac{v_2}{U_o}, \eta_1\right) = \int_0^{\eta_1} \left(\left[1 - \frac{2\phi}{\phi_1}\right] \phi_1^2 \frac{\partial \phi}{\partial \eta_1} + \phi^2 \frac{d\phi_1}{d\eta_1} \right) d\eta$$

$$\phi = \frac{U}{U_\tau} \; ; \qquad \phi_1 = \frac{U_o}{U_\tau}$$

$$\eta = \frac{U_\tau y}{\nu} \; ; \qquad \eta_1 = \frac{U_\tau \delta}{\nu}$$

Comparison with the experimental results of Mickley and Davis[46] shows a marked discrepancy, the difference increasing with increase of injection velocity.

Mickley and Davis[46] have been able to integrate the equations for the velocity profiles obtained by Rubesin (see Sect. 5.2 below) and show that in the sub-layer

5.2. The compressible turbulent boundary layer with air injection

The two existing theoretical solutions of air injection into a turbulent compressible boundary layer were published almost simultaneously. That of Dorrance and Dore[44] could also be applied to the case of suction and the analysis has been considered previously (Sect. 4.2). Rubesin[48] also considers the compressible turbulent boundary layer on a flat plate with zero pressure gradient and assumes that the injected gas is the same as the stream and that it is at wall temperature. A relation between skin friction and injection rate is obtained for a Prandtl number of unity. The effect of changes in Prandtl number on the heat transfer coefficients is investigated separately on the assumption (taken over from the case with no injection) that the skin friction coefficient is independent of Prandtl number.

The boundary layer equations are simplified by neglecting the variation of dependent variables with respect to x when compared with their variation with respect to y. At this stage Rubesin introduces a "turbulent Prandtl number". In the definition viscosity is replaced by the eddy viscosity ϵ_M and the thermal conductivity by the eddy thermal conductivity ϵ_H. Thus the definition of turbulent Prandtl number Pr_t is

$$Pr_t = \frac{\epsilon_M c_p}{\epsilon_H}$$

A turbulent Prandtl number of unity is equivalent to Reynolds analogy.

The equations to be solved are two in number

$$\rho_w v_2 \frac{du}{dy} = \frac{d}{dy}\left[(\mu + \epsilon_M) \frac{du}{dy} \right] \qquad (5.3)$$

$$\rho_w v_2 \frac{d}{dy}\left(c_p T + \frac{u^2}{2} \right) = \frac{d}{dy}\left[\left(\frac{\mu}{Pr} + \frac{\epsilon_M}{Pr_t} \right) \frac{d}{dy}(c_p T) + (\mu + \epsilon_M) \frac{d}{dy}\left(\frac{u^2}{2} \right) \right] \qquad (5.4)$$

where the terms are to be regarded as time-averages. The assumption that the Prandtl number and the turbulent Prandtl number are both unity leads to the deduction that

$$c_p T + \frac{u^2}{2} = a u + b$$

There is thus a direct relation between velocity and temperature and hence it is only necessary to solve the momentum equation with appropriate conditions to determine both the velocity and temperature distribution. For the laminar sub-layer the eddy transport terms are neglected and the velocity distribution is obtained in the form

$$y = \int_0^u \frac{\mu \, du}{\rho_w v_2 u + \tau_w} \quad ; \quad 0 < y < y_a$$

In the turbulent outer region the viscous terms are neglected and the eddy viscosity is determined from momentum transfer theory with a linear mixing length in the form

$$\epsilon_M = \rho l^2 \left(\frac{du}{dy}\right) \quad ; \quad l = ky$$

Since $\rho_w v_2$ is not dependent on y the parts of (5.3) appropriate to the laminar and turbulent regions respectively can be integrated separately to yield

(i) on the laminar side of the interface

$$\rho_w v_2 u = \mu \frac{du}{dy} + \text{const.}$$

which applying the boundary condition $y = 0$, $u = o$, $\mu \frac{du}{dy} = \tau_w$ leads to $\mu \frac{du}{dy} = \tau_w + \rho_w v_2 u$

(ii) on the turbulent side

$$\rho_w v_2 u = \rho k^2 y^2 \left(\frac{du}{dy}\right)^2 + \text{const} \, . \qquad (5.5)$$

Now across the interface the velocity and shear must be continuous; hence the two constants above must be the same and (5.5) can be written

$$\rho_w v_2 u + \tau_w = \rho k^2 y^2 \left(\frac{du}{dy}\right)^2$$

On integration we obtain the velocity distribution in the turbulent outer region in the form

$$y = y_a \int_{u_a}^{u} \frac{k\rho^{\frac{1}{2}}\, du}{(\rho_w v_2 u + \tau_w)^{\frac{1}{2}}} \qquad y > y_a \qquad u_a = (u)_{y=y_a}$$

where k, y_a and u_a have yet to be determined.

A relation between local skin friction coefficient and Reynolds number (R_x) is obtained by using the von Karman momentum integral. The integral cannot be expressed in closed form. However the first term of a series expansion is used giving a relation which is applicable to small values of injection velocity and skin friction coefficient. In the integration to determine the momentum thickness Rubesin uses only the velocity distribution in the turbulent outer region. The error involved in making this assumption is claimed to be less than one per cent. This means that u_a is put zero in this integration and y_a remains finite.

The determination of the constants k, y_a and u_a is now necessary to complete the solution. At low speed with no injection they can be found from velocity distribution data, from $c_f(R_{\delta_2})$ data and from $c_f(R_x)$ data, but, as Rubesin points out, the values obtained by the various methods differ markedly. There is insufficient experimental data to determine the effect of compressibility and injection on k, y_a and u_a and Rubesin assumes that the incompressible values for the impermeable surface can be applied to the case of injection. He thus uses

$$k = 0.392, \quad \frac{u_a}{U_o} = 13.1 \sqrt{\frac{\bar{c}_f}{2}}$$

and y_a calculated from u_a and the ratio of wall to free stream temperature.

While the analysis of Dorrance and Dore is essentially similar to Rubesin's treatment, the skin friction and heat transfer coefficients obtained by Dorrance and Dore are consistently lower than the values predicted by Rubesin. Rubesin's theory overestimates the skin friction coefficients obtained experimentally by Rubesin, Pappas and Okuno[49] and Tendeland and Okuno[50] on a cone at M = 2.7 and by Rubesin[54] on a flat plate by some twenty per cent. At M = 4.3 the wall temperature begins to have a marked effect on the skin friction. Rubesin's theory for wall temperature equal to stream stagnation pressure at M = 4.0 agrees well with the experimental results of Pappas and Okuno[55]. If the wall temperature is assumed to be equal to the free stream static pressure, theory underestimates experiment by some fifteen per cent.

The agreement between the above experiments and the theory of Dorrance and Dore is never satisfactory. The reason for the discrepancy between this theory and that of Rubesin is not clear. However it is noted that there are minor differences in the constants and in the former paper the velocity distribution used is more approximate than that used by Rubesin. Comparison of the theories and experiments are given in Fig. 11.

5.3. Foreign gas injection into the compressible turbulent boundary layer

In collaboration with Pappas, Rubesin[52] has extended his analysis of the effect of injection on the turbulent layer to include the case of injection of a light gas instead of air. Pappas[53] has used the same analysis to consider the effect of the injection of a heavy gas (in particular freon). The basic equations are the same as those used previously by Rubesin[48] for air injection together with an equation describing diffusion due only to concentration gradients. The other thermal and pressure dissusion processes are considered to have negligible effect. As before two regions, a laminar sub-layer and a turbulent outer region, are considered and continuity of velocity temperature, shear stress and mass and energy flux is required at the interface.

The analysis depends upon the derivation of Reynolds analogies between skin friction and diffusion and between skin friction and heat transfer to relate the local temperature and concentration to the local velocity. It is shown that there is a Reynolds analogy between heat transfer and skin friction when the turbulent Prandtl number and the turbulent Schmidt number $\left(\frac{\epsilon_M}{\rho \epsilon_D}\right)$ are unity. The Reynolds analogy between skin friction and diffusion exists for any non-zero turbulent Schmidt number. Having, by the Reynolds analogies, related local concentration and temperature to the local velocity, the velocity distributions in the laminar sublayer and the turbulent outer region can be determined by integration of the appropriate momentum equation in the form

$$\frac{\rho_o U_o y}{\mu_o} = \int_o^{u/U_o} \frac{(\mu/\mu_o)\, d\left(\frac{u}{U_o}\right)}{\frac{\rho_w v_2}{\rho_o U_o} \cdot \frac{u}{U_o} + \frac{c_f}{2}} \qquad 0 \leq y \leq y_a \qquad (5.6)$$

in the laminar sublayer and

$$y = y_a \exp\left[k \int_{\frac{u_a}{U_o}}^{u/U_o} \frac{(\rho/\rho_o)^{\frac{1}{2}} \, d\left(\frac{u}{U_o}\right)}{\left[\frac{\rho_w v_2}{\rho_o U_o} \cdot \frac{u}{U_o} + \frac{c_f}{2}\right]} \right] \quad y \geq y_a \qquad \ldots\ldots \quad (5.7)$$

in the turbulent outer region. In the integration of the turbulent momentum equation, the eddy viscosity is obtained from momentum tranfer theory with a linear mixing length.

To perform the integrations required in (5.6), it is assumed that the flow properties are constant within the laminar sub-layer so that μ / μ_o can be replaced by an average value which is constant throughout the sub-layer. The relation between density and speed required in (5.7) is obtained from the Reynolds analogy between skin friction and diffusion and Dalton's law of partial pressures applied to the isothermal boundary layer.

As in Rubesin's earlier paper the relationship between local skin friction coefficient and local Reynolds number, is obtained from the momentum equation on the assumption that the turbulent outer region extends to the surface of the plate and ignoring the laminar sub-layer. Again incompressible values of the mixing length constant and the height of and speed at the edge of the laminar sublayer on an impermeable wall are assumed.

The theoretical values of skin friction coefficient expected for the injection of hydrogen and helium into air are given in Fig. 12. Some experiments by Leadon and Scott[51] have yielded heat transfer rates at M = 3.0 when helium is injected. The theory predicts the correct trend but overestimates the measured heat transfer rate by some fifteen per cent. It may therefore be expected that the marked reduction in skin friction predicted in the theory for light gas injection will be obtainable in experiment. Such expectations have been justified in recent experiments by Pappas and Okuno[55] using helium as the injected gas. Only slight reductions are found when a heavy gas (freon 12) is used. It is also found that the effects of injection are most beneficial at the lower Mach numbers. This result, for a turbulent boundary layer, is at variance with the deduction from Clarke's solution of the effect of injection on a Couette type flow[39] that injection is most effective at the higher Mach numbers.

6. Comparison of laminar and turbulent skin friction coefficients

From the preceding section it is seen that the effect of suction is to increase skin friction and the effect of injection is to reduce skin friction. From theoretical studies (and experimental confirmation) it is known that a distributed uniform suction rate $\frac{v_1}{U_o} = .00018$ is sufficient to prevent the transition of the incompressible boundary layer on a flat plate but the skin friction coefficient is raised above the Blasius value. Injection of air at a rate $\frac{v_2}{U_o} = .007$ brings the turbulent skin friction coefficient below the laminar (Blasius) value for all values of R_x. Injection of helium at $\frac{v_2}{U_o} = .003$ or hydrogen at $\frac{v_2}{U_o} = .002$ has the same effect.

Similar effects must also be true for the compressible boundary layer. It is not known how much suction is required to stabilise the compressible boundary layer but how little the suction rate may be the skin friction cannot be less than for the layer on an impermeable surface. Thus comparing the results given in Figs. 3 and 12 it is possible to estimate injection rates which will give less turbulent skin friction than the minimum attainable laminar skin friction. The results of such an estimation are given in Table 1. It will be seen from Fig. 12 that the quoted value of injection rate is only necessary at low values of R_x. At higher values of R_x the injection rate can be reduced without the skin friction exceeding the laminar value.

	Minimum injection rate v_2/U_o		
	Air	Helium	Hydrogen
$M = 0$	.007	.003	.002
$M = 2$ $T_w = T_o$	.006	.0025	.0015
$M = 2$ $T_w = T_{stag}$	.0025	.001	.0005
$M = 4$ $T_w = T_o$	.005	.0025	.0015
$M = 4$ $T_w = T_{stag}$	.001	.0005	.0002

Table 1. Minimum injection rate to obtain turbulent skin friction lower than the impermeable value of laminar skin friction at the same R_x

7. Suggestions for future study

1. Theoretical and experimental investigations into the stability of the compressible laminar boundary layer with suction and injection and measurements of skin friction in a compressible laminar boundary layer with suction and injection.

2. Further study of the turbulent boundary layer to obtain accurate values of the mixing length constant and the height of and speed at the outer edge of the laminar sub-layer when suction or injection is applied.

3. Alternative analyses of the turbulent boundary layer to obtain confirmation or otherwise of the mixing length analysis.

8. References

1. Griffith, A.A., Meredith, F.W. — Unpublished (1936). See "Modern Developments in Fluid Dynamics" ed. Goldstein, 1938, pp 534.

2. Kay, J.M. — Boundary layer flow along a flat plate with uniform suction. A.R.C. R & M 2628, 1953.

3. Schlichting, H. — The boundary layer with suction and injection. Luftfahrtforschung, Vol.19, 1942, pp 178.

4. Lilley, G.M. — On a generalised porous-wall "Couette type" flow. Jour.Aero/Space Sci., 1959, pp 685-686.

5. Thwaites, B. — On certain types of boundary layer flow with continuous surface suction. A.R.C. R & M 2243, 1946.

6. Schlichting, H. — The boundary layer on a flat plate under conditions of suction and air injection. Luftfahrtforschung, Vol.19, 1942, pp 293.

7. Preston, J.H. — The boundary layer flow over a permeable surface through which suction is applied. A.R.C. R & M 2244, 1948.

8. Trilling, L. — The incompressible boundary layer with pressure gradient and suction. Jour. Aero.Sci., Vol.17, 1950 pp 335 - 342.

9. Ringleb, F.O. — Computation of the laminar boundary layer with suction. Jour. Aero.Sci., Vol.19, 1952, pp 48-54.

10. Curle, N. — The estimation of laminar skin friction including the effects of distributed suction. Aeronautical Quarterly, Vol.11, February 1960, pp 1 - 21.

References (Continued)

11.	Iglisch, R.	Exact calculation of the laminar boundary layer in longitudinal flow over a flat plate with homogeneous suction. N.A.C.A. Tech. Memo 1205, 1949.
12.	Prandtl, L.	The mechanics of viscous fluids: Aerodynamic Theory. Vol. 3, Springer, 1935.
13.	Head, M.R.	Approximate calculation of the laminar boundary layer with suction with particular reference to the suction requi rements for boundary layer stability on aerofoils of different thickness/chord ratios. A.R.C. R & M 3124, 1957.
14.	Thwaites, B.	The development of laminar boundary layers under conditions of continuous suction. A.R.C. 12,699, 1949.
15.	Bussman, K., Munz, H.	The stability of the laminar boundary layer with suction. Jahrbuch 1942 der Deutschen Luftfahrtforschung.
16.	Pretsch, J.	Transition start and suction. A.R.C. 10,291, 1942.
17.	Ulrich, A.	Theoretical investigation of the drag reduction in maintaining the laminar boundary layer by suction. N.A.C.A. Tech. Memo. 1121, 1944.
18.	Lin, C.C.	On the stability of two-dimensional parallel flows. Quarterly of Applied Mathematics, Vol. 3, 1945, pp 117 - 142 and 218 - 234

References (Continued)

19.	Hahnemann, E., Freeman, J.C., Finston, M.	Stability of boundary layers and of flow in the entrance section of a channel. Jour. Aero.Sci., Vol.15, 1948, pp 493 - 496.
20.	Burrows, D.L., Braslow, A.L., Tetervin, N.	Experimental and theoretical studies of area suction for the control of the laminar boundary layer on a porous bronze NACA 64A010 aerofoil. N.A.C.A. Tech.Note 1905, 1949.
21.	Head, M.R.	The boundary layer with distributed suction. A.R.C. R & M 2783, 1951.
22.	Libby, P.A., Kaufmann, L., Harrington, R.P.	An experimental investigation of the isothermal laminar boundary layer on a porous flat plate. Jour. Aero.Sci., Vol.19, 1952, pp 127 - 134.
23.	Lachmann, G.V.	Boundary layer control. Jour. Royal Aero.Soc., Vol.59, 1955, pp 163 - 198.
24.	Gregory, N., Curtis, A.R.	A comparison of three thick symmetrical multi-slot suction aerofoils. A.R.C. C.P. 20, 1950.
25.	Holstein, H.	The use of boundary layer suction to maintain the laminar characteristics of the friction layer. T.I.B., T2672, 1940.
26.	Loftin, L.K., Burrows, D.L.	Investigations relating to the extension of laminar flow by means of boundary layer suction through slots. N.A.C.A. Tech.Note 1961, 1949.
27.	Pfenninger, W.	Investigations on reduction of friction on wings, in particular by means of boundary layer suction. N.A.C.A. Tech.Memo.1181, 1946.

References (Continued)

28.	Pfenninger, W.	Experiments on a laminar suction aerofoil of 17 per cent thickness. Jour. Aero. Sci., Vol. 16, 1949, pp 227 - 236.
29.	Lew, H.G.	On the compressible boundary layer over a flat plate with uniform suction "Contributions to Applied Mechanics", Reissner Anniversary Volume, Ann Arbor, 1949.
30.	Lew, H.G.	The compressible laminar boundary layer over a flat plate with partial or variable suction at the wall. Polytechnic Institute of Brooklyn - Pibal report 151, 1949.
31.	Libby, P., Lew, H., Romano, F.	On the stability of a laminar compressible boundary layer over a flat plate subject to uniform suction and injection. Polytechnic Institute of Brooklyn - Pibal report 133, 1948.
31a.	Head, M.R., Johnson, D., Coxon, M.	Flight experiments on boundary layer control for low drag. A.R.C. R & M 3025, 1955.
32.	Brown, W.B., Donoughe, P.L.	Tables of exact laminar boundary layer solutions when the wall is porous and fluid properties are variable. N.A.C.A. Tech. Note 2479, 1951.
33.	Klunker, E.B.	An analysis of supersonic aerodynamic heating with continuous fluid injection. N.A.C.A. Tech. Note 1987, 1949.
34.	Low, G.M.	The compressible laminar boundary layer with fluid injection. N.A.C.A. Tech. Note 3404, 1954.
35.	Eckert, E.R.G., Schneider, P.J.	Diffusion effects in a binary isothermal boundary layer. Inst. of Technology Heat Transfer Lab., Univ. of Minnesota, Tech. report 5, 1955.

36.	Eckert, E.R.G., Schneider, P.J., Koehler, F.	Mass transfer cooling of a laminar air boundary layer by injection of a light-weight gas. Inst. of Technology Heat Transfer Lab., Univ. of Minnesota, Tech. report 8, 1956.
37.	Smith, J.W.	Effect of diffusion fields on the laminar boundary layer. Jour. Aero. Sci., Vol. 21, pp 154 - 162.
38.	Schuh, H.	The solution of the laminar boundary layer equation for the flat plate for velocity and temperature fields for variable physical properties and for diffusion fields at high concentration. N.A.C.A. Tech. Memo 1275, 1950.
39.	Clarke, J.F.	An elementary study of gas injection and sublimation into a simple shear layer. College of Aeronautics Report 126, 1960.
40.	Lees, L.	The stability of the laminar boundary layer in a compressible fluid. N.A.C.A. Report 876, 1947.
41.	Dunn, D.W., Lin, C.C.	On the stability of the laminar boundary layer in a compressible fluid. Jour. Aero. Sci. Vol. 22, 1955, pp 455 - 477.
42.	Dutton, R.A.	The effect of distributed suction on the development of turbulent boundary layers. A.R.C. 20,036, 1958.
43.	Black, T.J., Sarnecki, A.J.	The turbulent boundary layer with suction or injection. A.R.C. 20,501, 1958.
44.	Dorrance, W.H., Dore, F.J.	The effect of mass transfer on the compressible turbulent boundary layer skin friction and heat transfer. Jour. Aero. Sci. Vol. 21, 1954, pp 404-410.

References (Continued)

and Dorrance, W.H. — The effect of mass transfer on the compressible turbulent boundary layer skin friction and heat transfer - an addendum. Jour.Aero.Sci., Vol.23, 1956, pp 283-284.

45. Mickley, H.S., Ross, R.C., Squyers, A.L., Stewart, W.E. — Heat mass and momentum transfer for flow over a flat plate with blowing or suction. N.A.C.A. Tech. Note 3208, 1954.

46. Mickley, H.S., Davis, R.S. — Momentum transfer for flow over a flat plate with blowing. N.A.C.A. Tech.Note 4017, 1957.

47. Clarke, J.H., Menkes, H.R., Libby, P.A. — A provisional analysis of turbulent boundary layers with injection. Jour.Aero.Sci., Vol.22, 1955, pp 255-260.

48. Rubesin, M.W. — An analytical estimation of the effect of transpiration cooling on the heat transfer and skin friction characteristics of a compressible turbulent boundary layer. N.A.C A. Tech.Note, 3341, 1954.

49. Rubesin, M.W., Pappas, C.C., Okuno, A.F. — The effect of fluid injection on the compressible turbulent boundary layer. N.A.C.A. RM. A55I19, 1955.

50. Tendeland, T., Okuno, A.F. — The effect of fluid injection on the compressible turbulent boundary layer - the effect on skin friction of air injected into the boundary layer of a cone at M = 2.7. N.A.C.A. RM. A56D05, 1956.

51. Leadon, B.M., Scott, C.J. — Transpiration cooling experiments in a turbulent boundary layer at M = 3.0. Jour.Aero.Sci., Vol.23, 1956, pp 798-799.

and — Measurements of recovery factors and heat transfer coefficients with transpiration cooling in a turbulent boundary layer at M = 3.0 using air and helium as coolants. Univ.of Minnesota, Heat Transfer Lab. Research report 126, 1956.

52.	Rubesin, M.W., Pappas, C.C.	An analysis of the turbulent boundary layer characteristics on a flat plate with distributed light gas injection. N.A.C.A. Tech.Note 4149, 1958.
53.	Pappas, C.C.	The effect of injection of a foreign gas on the skin friction and heat transfer of the turbulent boundary layer. I.A.S. preprint 59-78, 1959.
54.	Rubesin, M.W.	The influence of surface injection on heat transfer and skin friction associated with the high speed turbulent boundary layer. N.A.C.A. RM A55L13, 1956.
55.	Pappas, C.C., Okuno, A.F.	Measurements of skin friction of the compressible turbulent boundary layer on a cone with foreign gas injection. Jour. Aerospace Sci., Vol.28, May, 1960, pp 321 - 333.
56.	Turcotte, D.L.	A sublayer theory for fluid injection into the incompressible turbulent boundary layer. Jour.Aerospace Sci. Vol.27, 1960 pp 675 - 678.
57.	Rannie, W.D.	Heat transfer in a turbulent shear flow. Jour.Aero.Sci. Vol.25, 1956, pp 485-489.

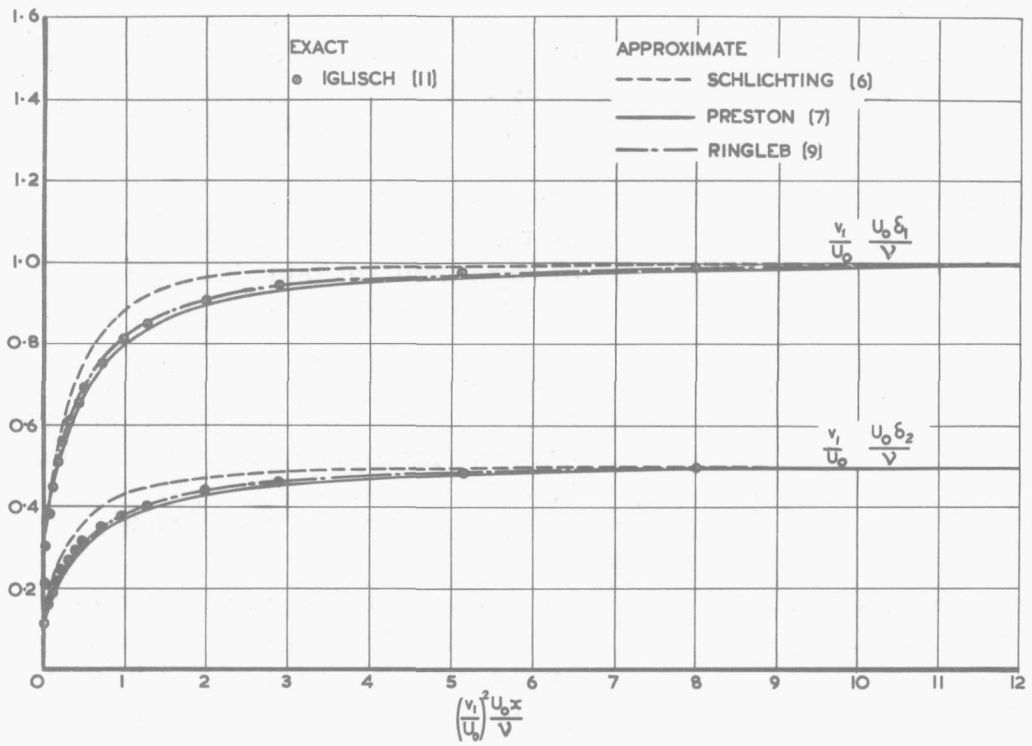

FIG. 1. COMPARISON OF APPROXIMATE SOLUTIONS WITH THE EXACT METHOD OF IGLISCH FOR LAMINAR BOUNDARY LAYER WITH SUCTION

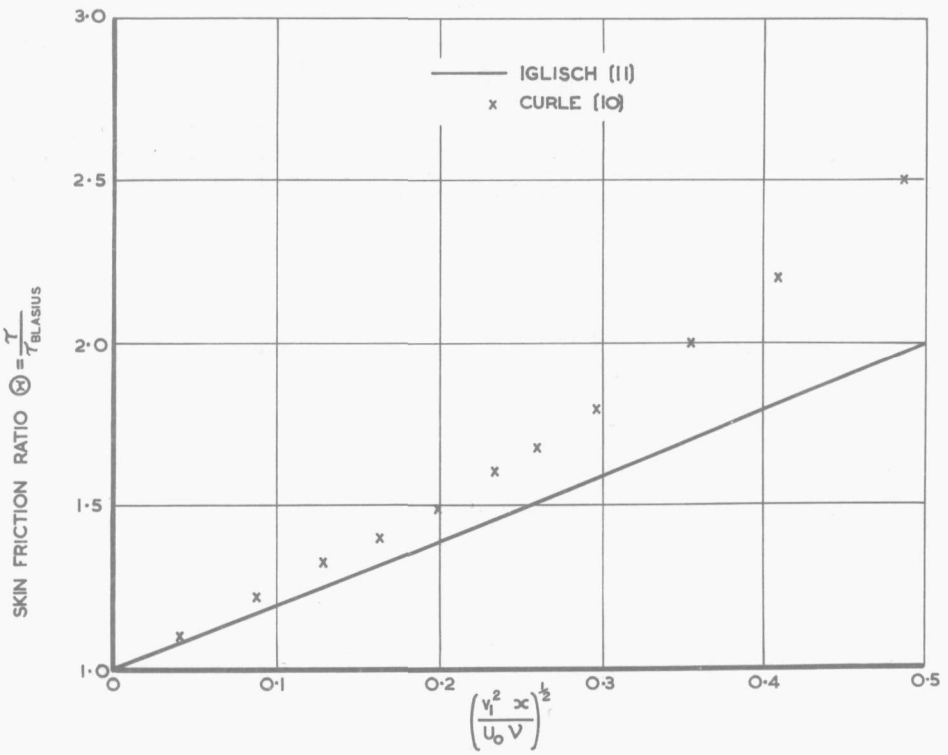

FIG. 2. COMPARISON OF EXACT AND APPROXIMATE SOLUTIONS FOR THE LAMINAR BOUNDARY WITH SUCTION

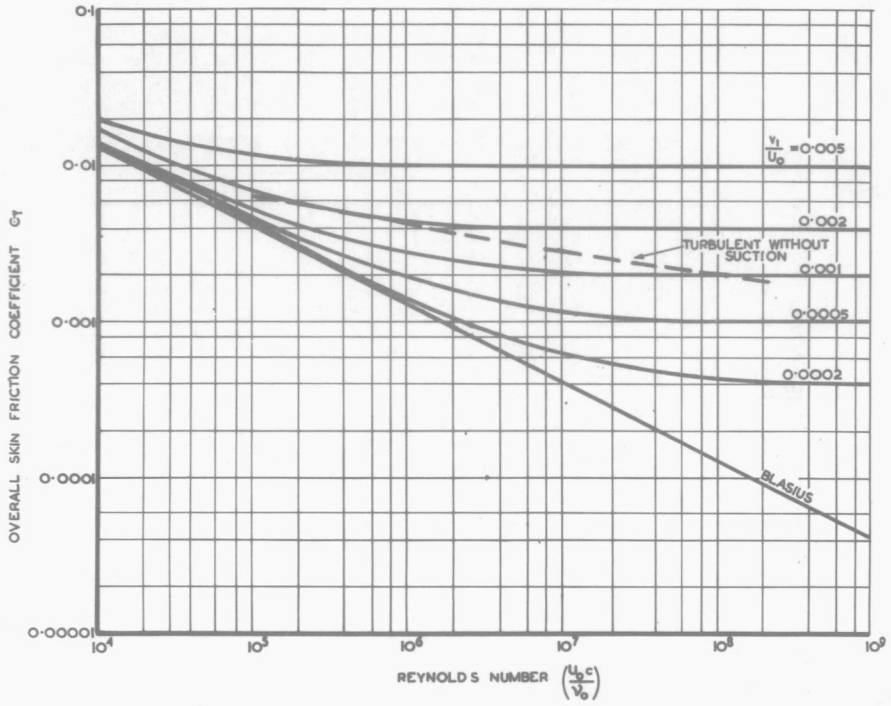

FIG. 3. EXACT SOLUTION FOR THE INCOMPRESSIBLE LAMINAR BOUNDARY LAYER WITH SUCTION – IGLISCH (11)

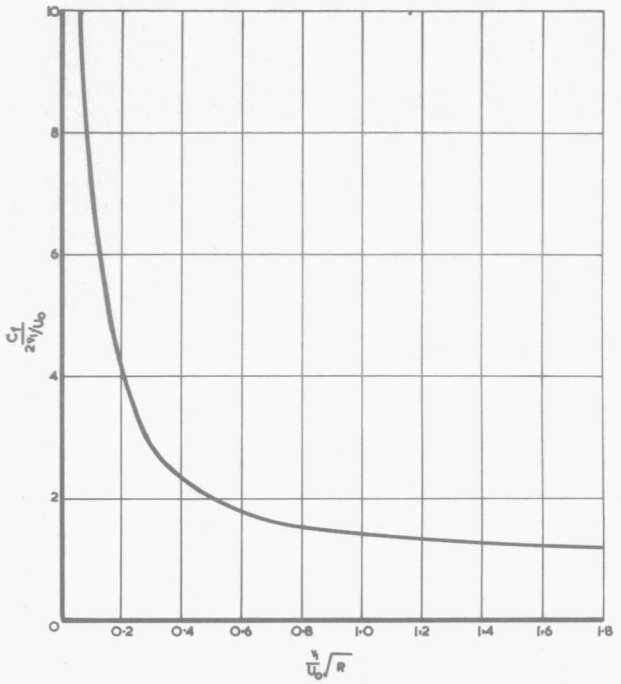

FIG. 4. UNIVERSAL SKIN FRICTION LAW – IGLISCH (11) LAMINAR BOUNDARY LAYER WITH SUCTION

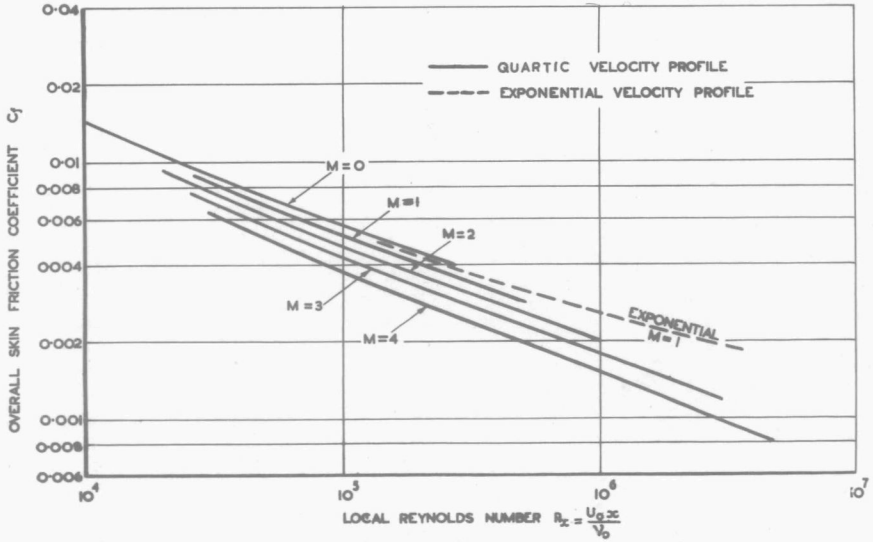

FIG. 5. THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH SUCTION; EFFECT OF MACH NUMBER, $\frac{v_1}{U_0}$ =·001 LEW (29)

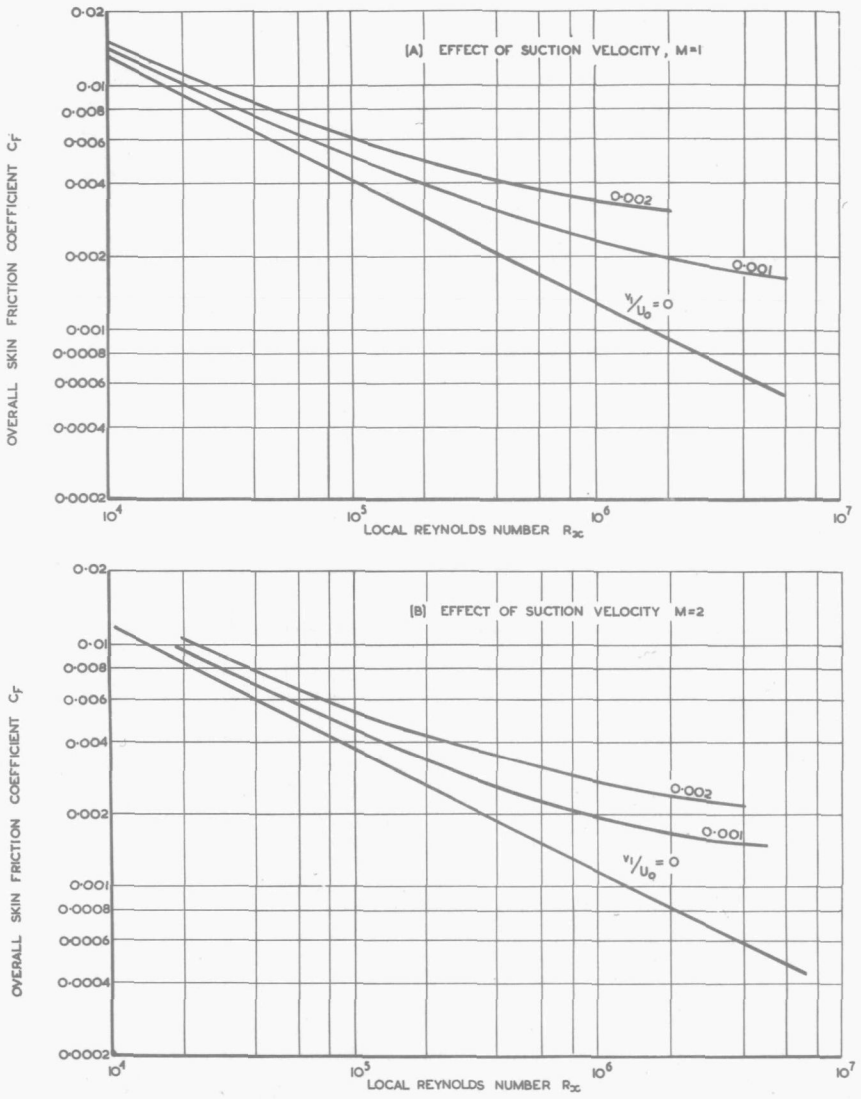

FIG.6. THE COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH SUCTION, LEW (29)

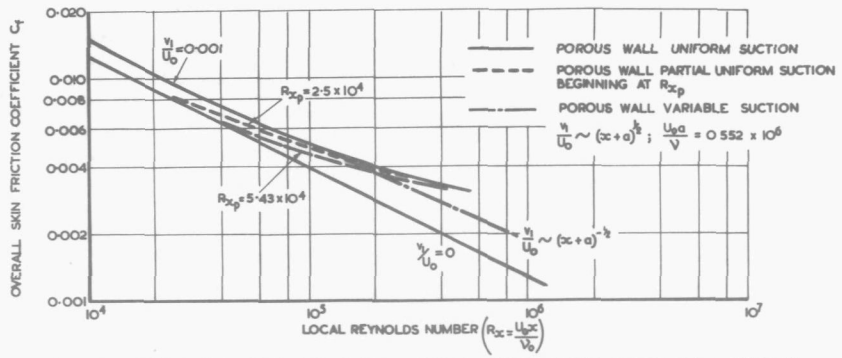

FIG.7. LAMINAR COMPRESSIBLE BOUNDARY LAYER WITH SUCTION, LEW (30)
EFFECT OF PARTIAL AND VARIABLE SUCTION ; M = 1

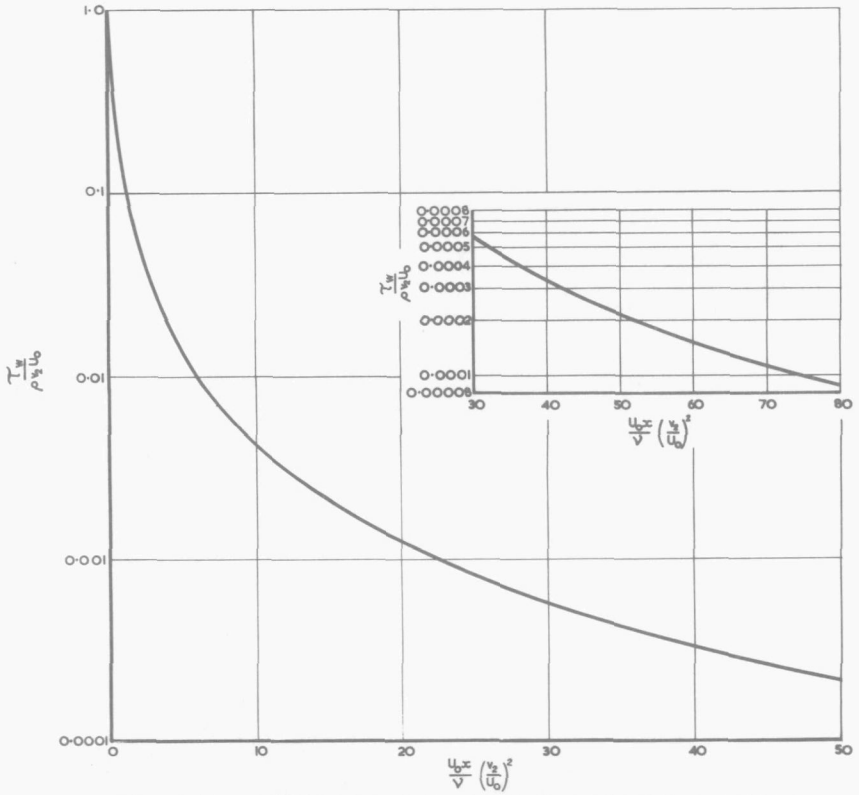

FIG.8. INCOMPRESSIBLE LAMINAR BOUNDARY LAYER WITH INJECTION
WALL SHEAR STRESS IN THE ENTRY LENGTH

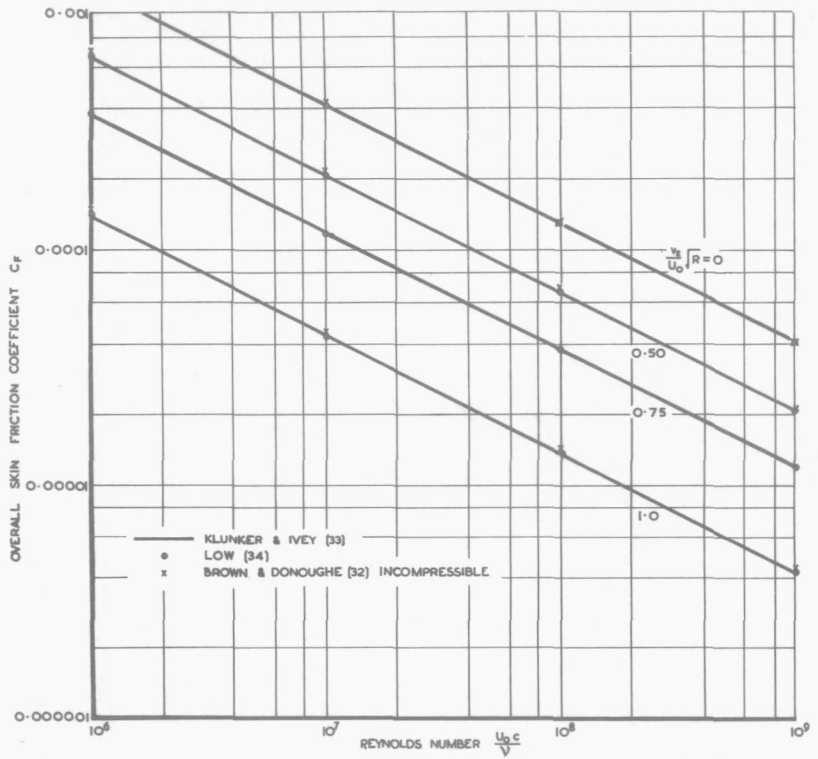

FIG.9. COMPRESSIBLE LAMINAR BOUNDARY LAYER WITH AIR INJECTION

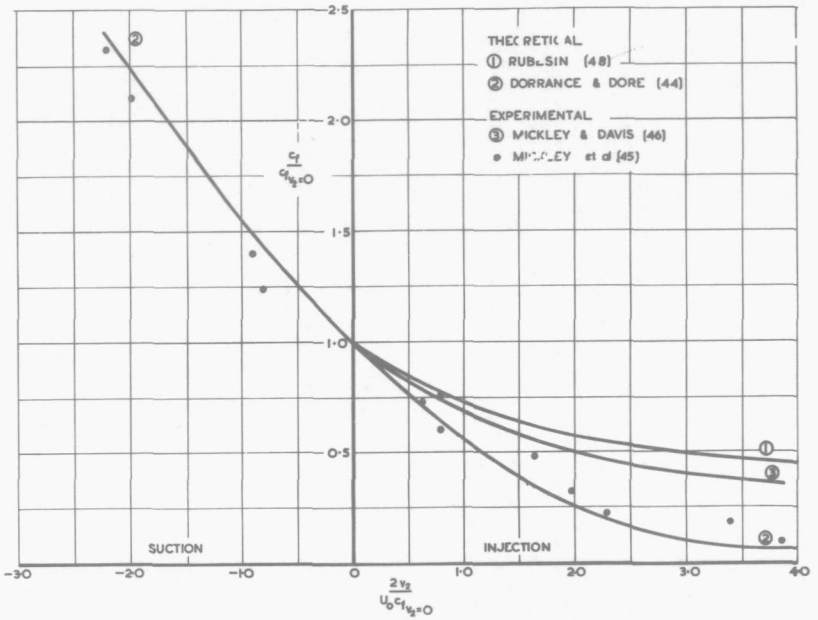

FIG.10. EFFECT OF SUCTION AND AIR INJECTION ON THE INCOMPRESSIBLE TURBULENT BOUNDARY LAYER

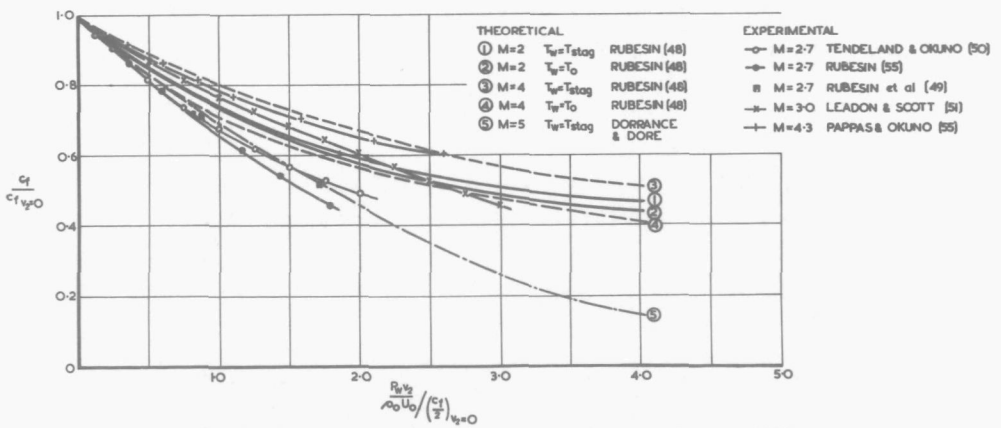

FIG.11. AIR INJECTION INTO A COMPRESSIBLE TURBULENT BOUNDARY LAYER

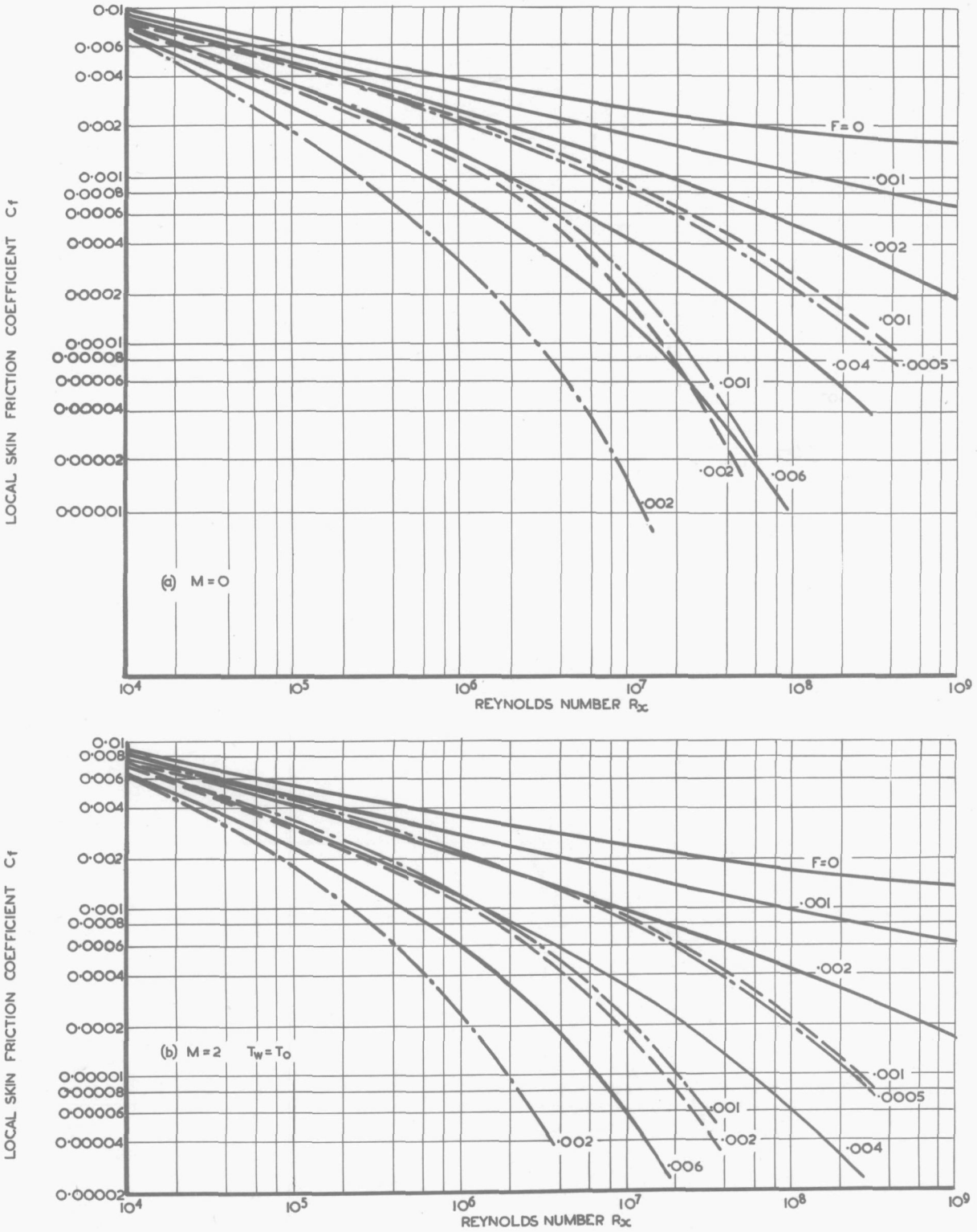

FIG. 12. INJECTION OF FOREIGN GAS INTO A COMPRESSIBLE TURBULENT BOUNDARY LAYER

$$F = \frac{\rho_w v_2}{\rho_o U_o}$$

——— AIR
– – – – HELIUM
——·—— HYDROGEN

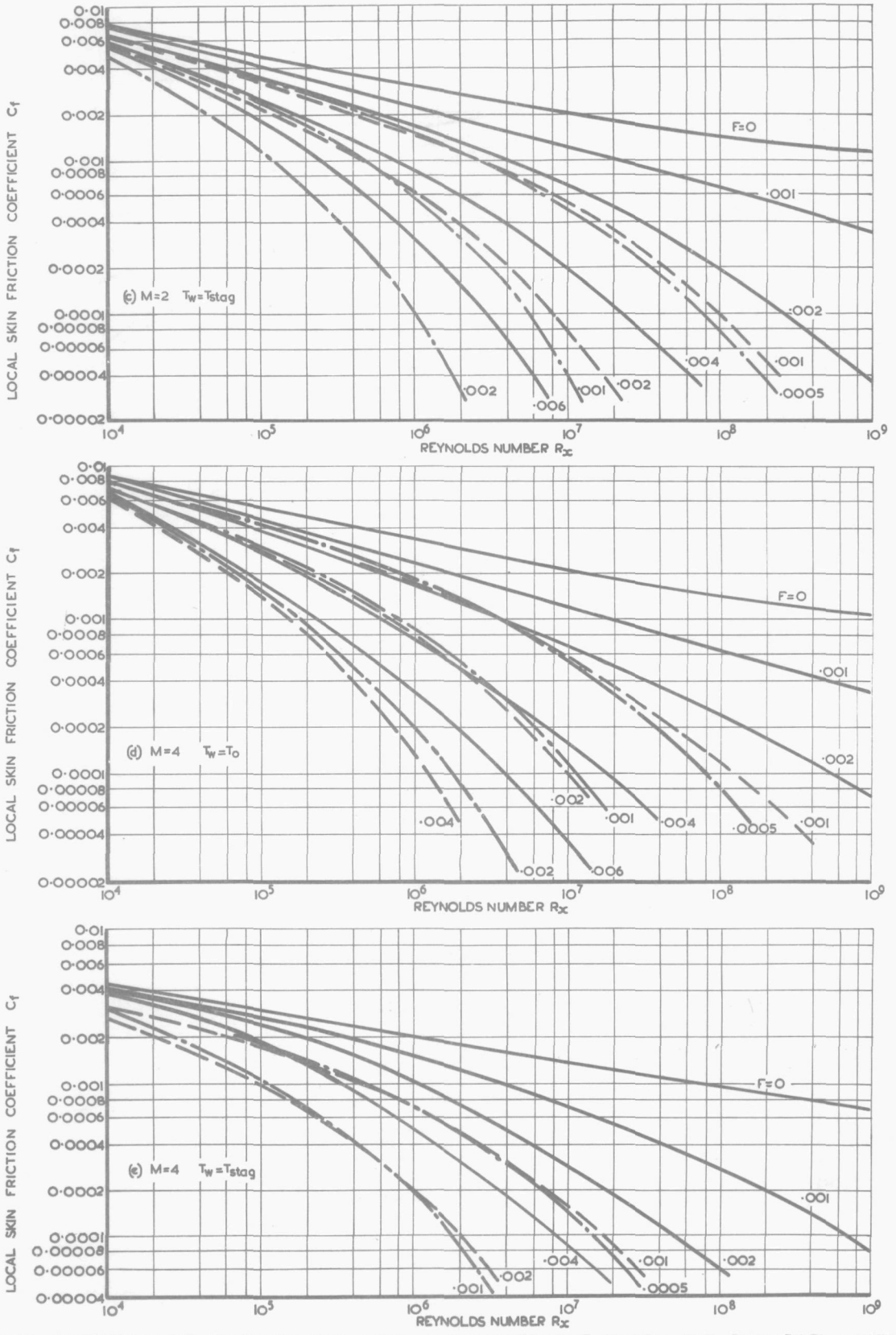

FIG.12. INJECTION OF FOREIGN GAS INTO A COMPRESSIBLE TURBULENT BOUNDARY LAYER

$$F = \frac{\rho_w v_s}{\rho_o U_o}$$