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Financial Applications of Calibrated Lévy Models

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Abstract

Exponential Lévy models are popular for option pricing, as their jump component captures empirical features of financial data that the Black-Scholes model cannot. Belomestny and Reiß [2] introduced a spectral calibration approach for a single maturity, assuming a constant Lévy triplet on the whole interval. Tendijck [27] and Koorevaar [15] extended this to a time-inhomogeneous approach estimating a distinct triplet on each maturity interval from European put and call prices. Koorevaar [15] further established asymptotic normality and confidence intervals for the each of the estimated triplets.

This thesis extends that pointwise normality result to a functional CLT in $L^2(K)$ (where $K \subset \mathbb{R}$ is compact) for the estimated Lévy density. We first derive a candidate covariance kernel of the exponentially-tilted estimation error, and identify a central structural obstruction: the rescaled kernel converges to an oscillatory kernel that is not integrable in $\mathbb{R}^2$. Hence, the associated covariance operator is not nuclear in $L^2(\mathbb{R})$, and a Giné-León [10] Hilbert space CLT cannot be obtained. We therefore restrict the domain to a compact set and modulate the error by a cosine factor to remove the oscillations, which yields a Giné-León [10] CLT for the linear part of the error. The bias and remainder terms vanish under appropriate scaling, yielding a functional Central Limit Theorem for the full cosine-modulated error.

As an application, this result is transferred through the Gil-Pelaez formula [9] to obtain a convergence-in-distribution result for the pricing error of a digital call option where the error enters through estimation of the Lévy density. This enables the computation of finite-sample confidence intervals that bridge the gap between the theory and practice. Finally, possible extensions are discussed.

Preface

This thesis marks my last chapter as a student of the Master Applied Mathematics at TU Delft. This thesis ends a period in which I have had the privilege to learn and grow as a person more than I could ever have imagined. The last two years in Delft have been incredibly formative in many ways, and that is something I will always be thankful for. Early on in my mathematical journey I discovered that I most enjoyed mathematics which could bridge the gap between theory and application, and that lead my to discover my passion for statistics. This is why I am very thankful to have conducted a project that does precisely this, and shows that behind applications there is always a natural step to captivating mathematical concepts.

I would also like to take a moment to deeply thank my supervisor Dr. Jakob Söhl for all his support and guidance throughout this thesis. I thoroughly enjoyed our bi-weekly meetings full of very interesting mathematical discussion that helped bring this project to fruition. His great advice and the freedom I was given to explore topics of my liking helped make this project an enjoyable and formative experience. Furthermore, I would also like to thank Dr. Fenghui Yu for taking the time to evaluate this project and for being on my thesis committee.

The end of this chapter in Delft also marks the end of my time as a student, a journey of five years in the Netherlands full of new and wonderful experiences, like the discovery of mathematics, the learning of a new language, and, especially, the many outstanding people I have had the privilege of getting to know. I know that without this inner circle of support I would not have made it as far as I have. The same applies to my friends from back home, who strongly encouraged me to embark on this adventure, and whose support and counsel I so deeply value. Furthermore, I want to thank my grandparents: Soledad, Juan Andrés, María Clotilde and Carlos, for all the invaluable lessons they have taught me, which have helped me countless times during my studies and I will remember all my life. Finally, I would like to especially thank my parents, Fernando and Natalia, and my brother, Pablo, for their continued love, support, encouragement and understanding, not only through my time as a student, but throughout the journey of my life, and for making every step leading up to this moment possible.

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Contents

Chapter 1
Introduction

Model calibration constitutes a fundamental pillar in the finance and banking industries, as it is the key mechanism used to price and hedge financial products such as derivatives. It ensures that the parameters of a certain model, such as the volatility or drift, are aligned with underlying market data. In turn, this allows a financial institution's prices to be market-aligned, which prevents arbitrage opportunities and is essential for risk management [13]. In option pricing, the standard model calibration approach is to take liquid, publicly traded European call and put options, and use their prices at different maturities to obtain a market-aligned model, with correct parameters. The most well-known and oldest model for option pricing is the Black-Scholes model, which provides a closed-form solution for the price of European call and put options. However, it was quickly realized that this model was insufficiently flexible to describe market behavior, as it is unable to model important empirical features of financial data such as the volatility smile, random jumps or heavy tails. This is why exponential Lévy models have become increasingly popular tools for model calibration, as they have a "jump" component that models random jumps.

In these models, an asset $(S_t)_{t\geq 0}$ follows the price

$$S_t = S_0 \, \exp\{rt + X_t\} \tag{1.1}$$

where $(X_t)_{t\geq 0}$ is a Lévy process and r is the risk-neutral interest rate (which is assumed to be constant). The study of the calibration of Lévy models has been extensive, and it has returned several relevant insights. Namely, unlike for example the Black-Scholes model, these models are able to accurately display the aforementioned jumps and volatility smile, as well as the heavy tails in parameter estimation [4].

Non-parametric estimation provides a way to reduce the risk of model misspecification. Cont and Tankov [5] provided a way to calibrate an exponential Lévy models non-parametrically using relative entropy penalization for least-squares regularization, thus making an ill-posed problem more stable. On the other hand, Belomestny and Reiß [1] provide a different methodology for solving the same problem. They proposed a spectral approach which could be used to directly extract the Lévy triplet.

It has been standard practice for non-parametric exponential Lévy model calibration from observed market data to assume that parameters are constant for the interval $[0, T]$ (homogeneity). However, as Belomestny and Reiß [1] showed, the homogeneity assumption is violated for large T. Moreover, Belomestny and Reiß [1] also highlighted the fact that calibrating these models with different maturity options can lead to conflicting information. All of this justified the introduction of a discrete time-inhomogeneous model which could allow for different parameter values within different maturity intervals $[T_{j-1}, T_j]$, where $j = 1, \ldots, n$ is the index of the maturities $T_1, \ldots, T_n$. Precisely this approach was introduced by Tendijck [27] and Koorevaar [15]. In it, the maturities are fixed but the range of strike prices grows as the mesh size for them is decreasing. This time inhomogeneous model provides an estimate for Lévy triplets $(\sigma_j^2, \gamma_j, \nu_j)$ in the different time intervals $j = 1, \ldots, n$. The calibration is based on the risk-neutral measure, given that the inference is based on market data.

For this time-inhomogeneous model, Koorevaar [15] derived a pointwise Central Limit Theorem for the error of the Lévy triplet, as well as error convergence rates and confidence intervals for the estimators of $(\sigma_j^2, \gamma_j, \nu_j)$. In previous work by Söhl [25], normality results and confidence intervals had been derived under the assumption that a Gaussian white noise model is followed. On the other hand, in Tendijck [27] and Koorevaar [15] a discrete observation model with sub-Gaussian errors is assumed.

One of the estimated parameters of the Lévy triplet is the Lévy or jump measure ν_j. As mentioned above, for the discrete-observation time-inhomogeneous model Koorevaar [15] provides a normality result for the error

$$\widetilde{\nu}_j(x) - \nu_j(x) \quad \forall x \in \mathbb{R}.$$

However, this result is for the error at every $x \in \mathbb{R}$, it is not a functional result. Nevertheless, for many applications of interest, such as the construction of simultaneous confidence bands for ν_j over a range of values, goodness-of-fit testing for parametric Lévy models, or the derivation of distributional results for pricing errors of exotic options, a functional Central Limit Theorem is required. In other words, convergence of the entire error process $\widetilde{\nu}_j(\cdot) - \nu_j(\cdot)$ as a random element in a function space (with an adequate scaling).

This thesis has two main goals. First, we establish that such a functional CLT can indeed be obtained: we show that the normalized and cosine-modulated Lévy density estimation error converges weakly to a Gaussian random element in the Hilbert space $L^2(K;\mathbb{R})$, where $K \subset \mathbb{R}$ is a compact set. The restriction to a compact domain arises from the structure of the covariance kernel of the error, which is oscillatory and has infinite $L^2(\mathbb{R}^2)$-norm, and hence a CLT in said space cannot be expected.

On $L^2(K)$ and under cosine modulation, the corresponding covariance operator is rank-one and therefore nuclear, which allows the application of the Giné-León Central Limit Theorem for Hilbert-space-valued random variables [10]. Importantly, the compact domain restriction is natural in the financial context: the Lévy measure is estimated from options with a finite range of strike prices, so the log-moneyness values at which $\nu_j(x)$ is effectively observed are bounded in practice, as liquidity is generally much lower for extremely out-the-money or very in-the-money options.

Second, we demonstrate that the functional CLT can be applied to derive new distributional results. Specifically, we consider the pricing error of a digital option, whose price is computed via the Gil-Pelaez inversion formula [9]. By expressing the normalized pricing error in terms of the Lévy density error $\widetilde{\nu}_j - \nu_j$ and as an $L^2(K)$ inner product, we can transfer the Lévy density normality result to obtain a convergence-in-distribution for the pricing error along a subsequence. This is done by applying the Extended Continuous Mapping Theorem from Van der Vaart and Wellner [28]. This analysis also reveals a structural question: the limiting variance of the pricing error depends on whether the weak limit along the subsequence is 0 or not. If it is, the CLT degenerates. Identifying an adequate scaling which can recover a non-degenerate Central Limit Theorem is left as future work.

Additionally, the convergence-in-distribution result of the error of the digital option price is applied in a finite sample setting, to derive confidence intervals. This application provides a direct bridge between the more theoretical nature of the topics this thesis explores, and the possible uses these results could find in the financial industry.

The literature relevant to this thesis can be divided in three parts. First, there is the literature related to non-parametric estimation of Lévy processes and corresponding confidence intervals or normality results. This is the component which formed the base for the time-inhomogeneous method developed in Tendijck [27] and Koorevaar [15], and upon which this research project builds. Notable contributions in the field of non-parametric Lévy estimations were provided by Figueroa-López [8], who provided sieve-based central limit theorems for the Lévy density. The main difference between this approach and one for option pricing is that in the financial setting the Lévy process is not observed directly. Furthermore, it is worth mentioning that non-parametric Lévy process estimation has been studied both from a frequentist perspective in Nickl and Reiß [19] or [18], and from a Bayesian perspective in Nickl and Söhl [20].

For non-parametric estimation based on option prices, Belomestny and Reiß [2] must be highlighted. For the building of confidence intervals and normality results, Söhl [25] and Söhl and Trabs [26] constitute significant contributions. Furthermore, the time-inhomogeneous discrete observation model and the subsequent pointwise normality results are contributions by Tendijck [27] and Koorevaar [15].

The second branch of literature relevant to this thesis is that of Central Limit Theorems in Hilbert spaces. The first relevant result in this field was provided by Kandelaki and Sozanov [14] in 1964, which proves a CLT in Hilbert spaces which requires the Lindeberg-Feller condition to be satisfied by normalizing a series of bounded linear operators. The second, and most relevant to this thesis, because it is the CLT deployed, is that of Corollary 4.3 in Giné-León [10]. To prove normality, it is required that a collection of independent random elements of the space H satisfy a positive covariance condition, an infinitesimality condition and a tail decay condition. Furthermore, Prokhorov and Statulevičius [22] explore several Hilbert Space CLTs for i.i.d random variables, and Giné and Nickl [11] provide a CLT for Banach spaces, but for which the authors note that the conditions are difficult to check. Finally, it is worth mentioning Van der Vaart and Wellner's [28] extended Continuous Mapping Theorem, which can be formulated for function spaces and forms a key link between our functional CLT and the pricing functional convergence-in-distribution result.

The third and final branch is that of option pricing of Fourier-based methods. Given that the characteristic function of Lévy processes is readily available, due to the Lévy-Khintchine representation, Fourier-based approaches to option pricing form a natural bridge between Lévy triplet estimation and applications to option pricing. Fourier-based option pricing methods are commonly based on the work of Carr and Madan [3], Lewis [16] and Raible [23]. Specifically, this thesis project explores the pricing of digital options, whose pricing formula is conveniently given by the Gil-Pelaez inversion formula [9]. This characterization of the digital option price is discussed in Kahl and Lord [17], who explore Fourier-based option pricing methods by expressing the option price in terms of the characteristic function.

The remainder of this thesis is organised as follows. Chapter 2 introduces the theoretical background on Lévy processes, exponential Lévy models, and Fourier-based option pricing. Chapter 3 describes the statistical model and the assumptions inherited from the spectral calibration framework of Tendijck [27] and Koorevaar [15]. In Chapter 4, we derive the candidate covariance kernel for the estimation error of the exponentially-tilted Lévy density and show that it is not Hilbert-Schmidt on $L^2(\mathbb{R})$, motivating the restriction to $L^2(K)$. Even on $L^2(K;\mathbb{R})$, however, the kernel's oscillations prevent it from defining a nuclear operator. Chapter 5 circumvents this by introducing a cosine modulation of the linear error term, whose covariance kernel decomposes into a rank-one limit plus oscillatory remainders that vanish by the Riemann-Lebesgue lemma. A functional CLT for the modulated linear part of the exponentially-tilted Lévy density error is then established by verifying the three conditions of the Giné-León [10] Corollary. In Chapter 6, we show that all non-linear terms in the error decomposition vanish in $L^2(K)$ under the appropriate scaling. Chapter 7 combines these results into the main theorem: the functional CLT for the full estimation error $\widetilde{\nu}_j - \nu_j$ in $L^2(K)$. Chapter 8 applies this functional CLT to derive a convergence-in-distribution result for the pricing error of a digital option via the Gil-Pelaez formula. Chapter 9 applies the main result of the previous chapter to derive confidence intervals for the error of the digital option price in a finite sample setting. Finally, Chapter 10 summarizes the main contributions and conclusions of this thesis and discusses the possible directions of future research. Lemmata required for the proofs of the main theorems are in the Appendix.

Chapter 2
Theory & Background

2.1. Time-Homogeneous Lévy Processes

Lévy processes are stochastic processes with independent and stationary increments. Named after Paul Lévy, they are effectively the continuous-time equivalent of a random walk. The best known examples of Lévy processes include Brownian Motion, Poisson processes and Gamma processes. Moreover, the theory behind Lévy processes forms the building block of several essential models in mathematical finance, such as jump diffusion models [4]. They are characterized by the following definition:

Definition 2.1 (Lévy process (Cont & Tankov [4]))**.** *An $\mathbb{R}$-valued process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is called a* Lévy process *if it is $(\mathcal{F}_t)$-adapted and satisfies:*

(i) X is continuous in probability: for every fixed $s \geq 0$ and $\varepsilon > 0$, $\mathbb{P}(|X_t - X_s| > \varepsilon) \to 0$ as $t \to s$.
(ii) $\mathbb{P}(X_0 = 0) = 1$.
(iii) For $0 \leq s \leq t$, $X_t - X_s$ is equal in distribution to X_{t-s} (stationary increments).
(iv) For $0 \leq s \leq t$, $X_t - X_s$ is independent of $\mathcal{F}_s$ (independent increments).

If a process X satisfies conditions $(i) - (iv)$, then, there exists a unique modification of the process, $\tilde{X}$, whose paths are càdlàg almost surely. It will be assumed that we are working with this càdlàg modification. The stochastic continuity condition (i) has as a consequence that only random jumps can occur in the process, as if jumps were scheduled, the probability of $|X_t - X_s| > \varepsilon$ would be greater than 0 even when $t \to s$.

Lévy processes are uniquely characterized by the Lévy triplet, a vector (σ^2, γ, ν), where:

1. $\sigma^2 \geq 0$ is the volatility or diffusion coefficient (i.e., the intensity of the process' random fluctuations).
2. $\gamma \in \mathbb{R}$ is the drift (the deterministic constant movement of the process).
3. ν is a σ-finite measure on $\mathbb{R}\backslash\{0\}$ which determines the frequency and intensity of the jumps.

More specifically, the Lévy measure is formally characterized by the following definition:

Definition 2.2 (Lévy measure)**.** *A Lévy measure on $\mathbb{R}$ is a σ-finite measure ν such that $\nu(\{0\}) = 0$ and*

$$\int_{\mathbb{R}} \left(1 \wedge |x|^2\right) \mathrm{d}\nu(x) < \infty.$$

For a Borel set $A \subset \mathbb{R} \setminus \{0\}$, $\nu(A)$ is interpreted as the expected number of jumps of X per unit time whose size belongs to A. The distribution of a Lévy process is fully characterized by its characteristic function, which, for a process of triplet (σ^2, γ, ν), is given by the Lévy-Khintchine representation:

Theorem 2.1 (Lévy–Khintchine representation (Cont & Tankov [4]))**.** *Let X be an $\mathbb{R}$-valued Lévy process. Then for each $t \geq 0$ the characteristic function of X_t admits the form*

$$\varphi_t(v) := \mathbb{E}\left[e^{ivX_t}\right] = e^{t\xi(v)}, \quad v \in \mathbb{R},$$

where the characteristic exponent ξ is given by

$$\xi(v) = -\frac{\sigma^2 v^2}{2} + i\gamma v + \int_{\mathbb{R}} \left(e^{ivx} - 1 - ivx\mathbf{1}_{|x|\leq 1}\right) \mathrm{d}\nu(x),$$

for some $\gamma \in \mathbb{R}$, $\sigma \in \mathbb{R}_{\geq 0}$, and Lévy measure ν.

In this thesis, we restrict throughout to Lévy processes where the jump component has finite activity, and where ν is absolutely continuous with respect to the Lebesgue measure. With a slight abuse of notation, we denote the corresponding density also by $\nu \in L^1(\mathbb{R})$, and write $\lambda := \|\nu\|_{L^1(\mathbb{R})} < \infty$ for the jump intensity (the expected number of jumps per unit time). Under finite activity, the compensating term $ivx\mathbf{1}_{|x|\leq 1}$ in Theorem 2.1 can be absorbed into the drift, yielding the simpler representation

$$\varphi_t(v) = \exp\left(t\left[-\frac{\sigma^2 v^2}{2} + i\gamma v + \int_{\mathbb{R}} (e^{ivx} - 1)\nu(x)\,\mathrm{d}x\right]\right), \tag{2.1}$$

which will be the working form used throughout the rest of the thesis. However, as discussed in the Introduction, option prices estimated across different maturities cannot be reproduced by a single Lévy triplet. For this reason, time-inhomogeneous Lévy processes are needed for modeling and calibration.

2.2. Additive & Time-Inhomogeneous Lévy Processes

As mentioned in section 2.1, the assumption that option prices follow a time-homogeneous Lévy process is too restrictive if an accurate calibration is to be obtained, as empirical option price surfaces have different triplets for different intervals between maturities. Indeed, Belomestny and Reiß [1] showed that for large T, the homogeneity assumption is violated. To account for maturity-dependent dynamics, the stationarity of increments assumption can be relaxed, in order to be able to model the underlying as an additive process:

Definition 2.3 (Additive process)**.** *An $\mathbb{R}$-valued process $X = (X_t)_{t\geq 0}$ defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is an* additive process *if it is $(\mathcal{F}_t)$-adapted and satisfies conditions (i), (ii), and (iv) of Definition 2.1, but not necessarily (iii); that is, the increments $X_t - X_s$ need not be stationary.*

Additive processes also have a unique càdlàg modification, and their distribution is still fully characterized by the Lévy-Khintchine representation, but with a time-dependent Lévy triplet $(\sigma_t^2, \gamma_t, \nu_t)$ varying with t . In the calibration framework of Tendijck [27] and Koorevaar [15], upon which this thesis builds, an additive Lévy process is built from a piecewise time-homogeneous structure based on the discrete grid of observed option maturities $0 = T_0 < T_1 < \cdots < T_n$. This is referred to as a time-inhomogeneous Lévy process:

Definition 2.4 (Time-inhomogeneous Lévy process, (Definition 1.4, Koorevaar [15]))**.** *An additive process X is called a* time-inhomogeneous Lévy process *on the maturity grid $\{T_0, T_1, \ldots, T_n\}$ if, on each interval $[T_{j-1}, T_j]$ with $j = 1, \ldots, n$, the increment process $(X_t - X_{T_{j-1}})_{t\in[T_{j-1},T_j]}$ is a Lévy process with characteristic triplet $(\sigma_j^2, \gamma_j, \nu_j)$.*

By independence of increments across disjoint intervals, X_{T_j} is the sum of j independent random variables, each with a known Lévy–Khintchine representation. Its characteristic function therefore factorizes as

$$\varphi_{T_j}(v) = \prod_{r=1}^{j} \exp\left((T_r - T_{r-1})\left[-\tfrac{\sigma_r^2 v^2}{2} + i\gamma_r v + \int_{\mathbb{R}} (e^{ivx} - 1)\nu_r(x)\,dx\right]\right), \tag{2.2}$$

As a result of this time-inhomogeneous characteristic function, we obtain the ratio:

$$\frac{\varphi_{T_j}(v)}{\varphi_{T_{j-1}}(v)} = \exp\left((T_j - T_{j-1})\left[-\tfrac{\sigma_j^2 v^2}{2} + i\gamma_j v + \int_{\mathbb{R}} (e^{ivx} - 1)\nu_j(x)\,dx\right]\right), \tag{2.3}$$

which directly gives the contribution of the j-th interval. This forms the basis for the spectral calibration procedure, as the j-th Lévy triplet is computed from this ratio, which is itself obtained from option prices at maturities T_{j-1} and T_j.

2.3. Hilbert-Space-Valued Random Variables

The functional Central Limit Theorem developed in this thesis is formulated in the Hilbert space $H = L^2(K; \mathbb{R})$, where $K \subset \mathbb{R}$ is compact. Before introducing the relevant limit theorems in section 2.4, we recall the basic vocabulary of probability on Hilbert spaces, with emphasis on the operator-valued analogues of mean and variance. Throughout this subsection let H denote a separable real Hilbert space with inner product $\langle\cdot,\cdot\rangle_H$ and norm $\|\cdot\|_H$. We equip H with its Borel σ-algebra $\mathcal{B}(H)$. Hence, an H-valued random element can be defined as follows:

Definition 2.5 (H-valued random element)**.** *An* H-valued random element *on a probability space* $(\Omega,\mathcal{F},\mathbb{P})$ *is a measurable map* $X:(\Omega,\mathcal{F})\to(H,\mathcal{B}(H))$.

When $\mathbb{E}\|X\|_H<\infty$, $\mathbb{E}[X]\in H$ is well-defined and satisfies $\langle\mathbb{E}[X],y\rangle_H=\mathbb{E}\langle X,y\rangle_H$ for every $y\in H$. The element $\mathbb{E}[X]$ is the mean of the random element $X\in H$. The Hilbert-space equivalent of the variance is the covariance operator:

Definition 2.6 (Covariance operator)**.** *Let X be an H-valued random element with $\mathbb{E}\|X\|_H^2<\infty$ and mean $\mathbb{E}[X]=a$. The* covariance operator *of X is the bounded linear operator $A:H\to H$ defined by*

$$\langle Ay,z\rangle_H:=\mathbb{E}\big[\langle X-a,y\rangle_H\langle X-a,z\rangle_H\big],\qquad y,z\in H.$$

The covariance operator is positive:

$$\langle Ay,y\rangle_H\geq 0\quad\forall y\in H,$$

Hermitian:

$$\langle Ay,z\rangle_H=\langle y,Az\rangle_H$$

and nuclear (trace-class), which means, for a bounded operator A on H, $\sum_n\langle|A|e_n,e_n\rangle_H<\infty$, in which case, the trace $\mathrm{Tr}(A):=\sum_n\langle Ae_n,e_n\rangle_H$ is well defined and basis-independent.

Furthermore, it has finite rank if its range $\mathrm{Range}(A)=\{Ay:y\in H\}$ is a finite-dimensional subspace of H, i.e.: $\exists\, n\in\mathbb{N}$ and elements $\phi_1,\ldots,\phi_n,\psi_1,\ldots,\psi_n\in H$ such that

$$Ay=\sum_{i=1}^{n}\langle y,\psi_i\rangle_H\cdot\phi_i\quad\forall y\in H.$$

The smallest such n is the rank of A, and $n=1$ implies a rank of one. Finally, A is Hilbert-Schmidt if $\sum_n\|Ae_n\|_H^2<\infty$ for every orthonormal basis $\{e_n\}$.

It is worth noting that finite-rank operators are nuclear; nuclear operators are Hilbert-Schmidt; Hilbert-Schmidt operators are compact and all of these are bounded [6]. An operator $A:L^2(K)\to L^2(K)$ (with kernel $\mathcal{K}:K\times K\to\mathbb{R}$) given by:

$$(Ay)(x)=\int_K\mathcal{K}(x,z)y(z)\,dz$$

is Hilbert-Schmidt if and only if $\mathcal{K}\in L^2(K\times K)$, in which case $\|A\|_{\mathrm{HS}}^2=\|\mathcal{K}\|_{L^2(K\times K)}^2$. A finite-rank kernel of the form $\mathcal{K}(x,z)=\sum_{i=1}^n\phi_i(x)\psi_i(z)$ with $\phi_i,\psi_i\in L^2(K)$ defines a nuclear operator with trace $\sum_i\langle\phi_i,\psi_i\rangle_{L^2(K)}$. Furthermore, recall that covariance operators are self-adjoint. Specifically, for the rank-one case $\mathcal{K}(x,z)=\phi(x)\phi(z)$ gives the operator $Ay=\langle y,\phi\rangle_{L^2(K)}\cdot\phi$, with trace $\|\phi\|_{L^2(K)}^2$. Operators of this form will play a central role in this thesis, as the limit of the identified covariance operator for the CLT will be rank one.

As it will be shown, the Central Limit Theorem on Hilbert spaces applied here will require nuclearity of the covariance operator. This nuclearity requirement is what forces, in Chapter 4, the restriction of the analysis from $L^2(\mathbb{R})$ to $L^2(K)$ on a compact domain. As a matter of fact, on $L^2(\mathbb{R})$, it will be shown that the operator is not even Hilbert-Schmidt (which is weaker than nuclearity). Additionally, let us recall a notion of convergence that will be useful all throughout:

Definition 2.7 (Weak convergence and Gaussian measures)**.** *Let $\{X_n\}_{n\geq1}$ and X be H-valued random elements. We say that X_n* converges weakly *(or in distribution) to X, written $X_n\Rightarrow X$, if $\mathbb{E}[f(X_n)]\to\mathbb{E}[f(X)]$ for every bounded continuous $f:H\to\mathbb{R}$. A* Gaussian measure *on H with mean $a\in H$ and covariance operator A (nuclear, positive, Hermitian) is the law of an H-valued random element G such that $\langle G,y\rangle_H\sim\mathcal{N}(\langle a,y\rangle_H,\langle Ay,y\rangle_H)$ for every $y\in H$. We write $G\sim\mathcal{N}(a,A)$.*

Finally, in Chapter 7, it will be used that bounded linear maps preserve Gaussianity: if $G\sim\mathcal{N}(a,A)$ on H and $T:H\to H$ is bounded linear with adjoint T^*, then $TG\sim\mathcal{N}(Ta,TAT^*)$.

2.4. Central Limit Theorems in Hilbert Spaces

Given that all the necessary definitions for random elements in a Hilbert space H have been established in the previous section, we can now proceed to discuss Central Limit Theorems in Hilbert Spaces. The first CLT for Hilbert-space-valued random variables was established by Kandelaki and Sozanov [14], who proved a Lindeberg-Feller-type result requiring the summands to be normalized by a sequence of bounded linear operators. While this is a key result in the field, we decided to apply another CLT in H whose conditions could be more readily verifiable in our setting. Namely, Corollary 4.3 in Giné-León [10]:

Theorem 2.2 (Corollary 4.3 in Giné-León [10]). *Given a (real) Hilbert space H, let $\{X_{nj}\}$ be an infinitesimal array of H-valued random variables and*

$$S_n = \sum_{j=1}^{k_n} X_{nj}.$$

Then there exists a sequence $\{x_n\} \subset H$ such that the laws $\{\mathcal{L}(S_n - x_n)\}$ converge weakly to a Gaussian probability measure if and only if the following conditions hold:

(i) For some (equivalently, every) $\delta > 0$ and every $y \in H$,

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \mathbb{E}\big\langle X_{nj}\mathbf{1}_{\{\|X_{nj}\|\le\delta\}} - \mathbb{E}[X_{nj}\mathbf{1}_{\{\|X_{nj}\|\le\delta\}}],\, y\big\rangle^2 = \langle Ay, y\rangle,$$

where A is a nuclear, positive, Hermitian operator on H.

(ii) For every $\varepsilon > 0$,

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \mathbb{P}\big(\|X_{nj}\| > \varepsilon\big) = 0.$$

(iii) For some (equivalently, every) $\delta > 0$ and some (equivalently, every) complete orthonormal system $\{e_i\}$ of H,

$$\lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=1}^{k_n} \sum_{i=N+1}^{\infty} \mathbb{E}\big\langle X_{nj}\mathbf{1}_{\{\|X_{nj}\|\le\delta\}} - \mathbb{E}[X_{nj}\mathbf{1}_{\{\|X_{nj}\|\le\delta\}}],\, e_i\big\rangle^2 = 0.$$

In this case, there exists $a \in H$ such that

$$\lim_{n\to\infty} \sum_{j=1}^{k_n} \mathbb{E}[X_{nj}] = a,$$

and

$$\mathcal{L}(S_n - x_n) \overset{w}{\Longrightarrow} \mathcal{N}(a, A).$$

Condition (i) identifies the limiting covariance operator A, requiring that it be nuclear. This is in turn necessary for a Gaussian measure on an infinite-dimensional Hilbert space to be tight.

Furthermore, this Hilbert Space CLT will enable us to use an additional probabilistic result in Chapter 8. If, we have a sequence of random variables X_n converging weakly to a Gaussian X in $L^2(K)$ as part of the inner product

$$\langle X_n, f_n\rangle_{L^2(K)},$$

where f_n is a deterministic sequence of functions, we effectively have a function $g_n(X_n)$ of random variables for which the classical continuous mapping theorem is insufficient, because $g_n(\cdot)$ is not fixed. Hence, the following extended version, from Van der Vaart and Wellner [28], can be used in this case:

Theorem 2.3 (Theorem 1.11.1 [28]: Extended Continuous Mapping). *Let $\mathbb{D}_n \subset \mathbb{D}$ and $f_n : \mathbb{D}_n \mapsto \mathbb{E}$, where $\mathbb{D}$ and $\mathbb{E}$ are metric spaces, satisfy the following statements: if $x_n \to x$ with $x_n \in \mathbb{D}_n$ for every n and $x \in \mathbb{D}_0$, then $f_n(x_n) \to f(x)$, where $\mathbb{D}_0 \subset \mathbb{D}$ and $f : \mathbb{D}_0 \mapsto \mathbb{E}$. Let X_n be maps with values in $\mathbb{D}_n$, let X be Borel measurable and separable, and take values in $\mathbb{D}_0$. Then*

1. $X_n \rightsquigarrow X$ *implies that* $f_n(X_n) \rightsquigarrow f(X)$;
2. $X_n \xrightarrow{\mathrm{P}^*} X$ *implies that* $f_n(X_n) \xrightarrow{\mathbb{P}} f(X)$;
3. $X_n \xrightarrow{\text{as:}} X$ *implies that* $f_n(X_n) \xrightarrow{\text{a.s.}} f(X)$.

2.5. Exponential Lévy Models & Fourier-Based Option Pricing

The time-inhomogeneous Lévy framework of Section 2.2 is connected to financial markets via the exponential Lévy model. Under the risk-neutral measure $\mathbb{Q}$, the asset price process $(S_t)_{t\geq 0}$ is modeled as in equation (1.1), where $S_0 > 0$ is the initial asset price, $r \geq 0$ is the continuously compounded risk-free interest rate (assumed constant), and $(X_t)_{t\geq 0}$ is a time-inhomogeneous Lévy process. Working under the martingale measure $\mathbb{Q}$ rather than the physical measure $\mathbb{P}$ is standard in option pricing: $\mathbb{Q}$ is inferred from observed market option prices rather than estimated from historical asset returns, so the calibrated model is automatically consistent with market data [4].

For the discounted price process $e^{-rt} S_t$ to be a Martingale with respect to $\mathbb{Q}$, the characteristic triplet $(\sigma_j^2, \gamma_j, \nu_j)$ on each interval $[T_{j-1}, T_j]$ must satisfy the Martingale Condition:

$$\frac{\sigma_j^2}{2} + \gamma_j + \int_{\mathbb{R}} (e^x - 1)\nu_j(x)\, dx = 0, \qquad j = 1, \ldots, n. \tag{2.4}$$

Under the finite-activity and finite-second-moment assumptions in this thesis, the integral in the Martingale Condition is finite.

The model is calibrated from observed prices of European call and put options $C(K_{j,k}; T_j)$ and $P(K_{j,k}; T_j)$ at maturities $T_1, \ldots, T_n$ and strike prices $K_{j,k}$, $k = 1, \ldots, m_j$. This calibration is tractable because the characteristic function of X_{T_j} is available in closed form due to to the Lévy Khintchine representation and European option prices admit Fourier representations in terms of φ_{T_j}.

The standard framework for Fourier-based option pricing, developed by Carr and Madan [3], Lewis [16], and Raible [23], expresses option prices as integrals of φ_{T_j} which turns calibration into a spectral inversion problem. The object of interest in this thesis is the Digital Call Option Π_j, whose payoff at maturity T_j is $\mathbf{1}_{\{S_{T_j} > K_{j,k}\}}$. This is a unit payment if the asset finishes above the strike $K_{j,k}$, and zero otherwise. Under $\mathbb{Q}$, the no-arbitrage price is simply the probability

$$\Pi_j := \mathbb{Q}(S_{T_j} > K_{j,k}) = \mathbb{Q}(X_{T_j} > k'),$$

where $k' := \ln(K_{j,k}/S_0) - rT_j$ is the adjusted log-moneyness. As given in Kahl and Lord [17], this tail probability admits the following Fourier representation, based on the Gil-Pelaez formula [9]:

Theorem 2.4 (Digital Call Option Price [17]). *Let Y be a real-valued random variable with characteristic function φ_Y. Then for every $k' \in \mathbb{R}$,*

$$\mathbb{Q}(Y > k') = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \Re\left(\frac{e^{-iuk'} \varphi_Y(u)}{iu} \right) du.$$

Applying Theorem 2.4 with $Y = X_{T_j}$ and $\varphi_Y = \varphi_{T_j}$ gives the digital option price as a functional of the characteristic function of X_{T_j}. This is the key structural observation upon which Chapter 8 is based on. There, φ_{T_j} is replaced by the estimated characteristic function $\widetilde{\varphi}_{T_j}$. Hence, the resulting pricing error $\widetilde{\Pi}_j - \Pi_j$ can be expressed as a functional of the estimation error $\tilde{\nu}_j - \nu_j$ in $L^2(K; \mathbb{R})$. It will be shown that, under certain conditions akin to those of Koorevaar [15] the functional CLT of Chapter 7 can be applied to obtain a convergence in distribution result for $\widetilde{\Pi}_j - \Pi_j$.

Chapter 3
Statistical Model & Assumptions

The goal of this section is to introduce the statistical framework underpinning the asymptotic results in later sections. The statistical model in this thesis is the time-inhomogeneous extension of Belomestny and Reiß [1] and Söhl and Trabs [26], which was first proposed in Tendijck [27] and further developed in Koorevaar [15]. To that end, this section aims to introduce this model and all related concepts and assumptions due to Tendijck [27] and Koorevaar [15]. Thus, most results in this section are derived from these two sources, as they contain the necessary information to carry out the objective of this thesis: the raising from pointwise to functional normality. Moreover, this section will also help to make the setup self-contained and to clearly state the elements inherited and those upon which this thesis builds.

All models in this thesis correspond to an incomplete market, which means that perfect hedges are impossible and option prices cannot be identified from the underlying. Furthermore, incompleteness also makes it impossible to identify market dynamics from historical data.

This thesis estimates processes under the risk neutral measure $\mathbb{Q}$. A crucial property of arbitrage free markets is that there exists a risk-neutral (or Martingale) measure $\mathbb{Q}$ such that the expectation of the discounted price of a security at time $t = T$ equals the price of said security at $t = 0$:

$$P_0^i = e^{-rT}\mathbb{E}_{\mathbb{Q}}(P_T^i), \tag{3.1}$$

where P_t^i is the price at time t of security $i \in I$, where I is a basket of securities. Choosing an appropriate $\mathbb{Q}$ such that the prices of securities such as options can be reproduced is what is referred to as model calibration. This is done via for example option prices, which give information about how to select the correct Martingale measure. The methods of both Belomestny and Reiß [1] and Tendijck [27] and Koorevaar [15] are calibrated via European put and call options, which are market-traded with high liquidity. By the Fundamental Theorem of Asset Pricing, it must be noted that the risk-neutral measure is only unique for complete markets. If $\mathbb{Q}$ is part of the class of time-inhomogeneous exponential Lévy models, calibrating the model from market-traded options and deriving this measure implies that the Lévy triplets $(\sigma_j^2, \gamma_j, \nu_j(x))_{j=1}^n$ are also obtained. As a result, with a full characterization of the stochastic process followed by the underlying, derivatives like exotic options can be priced in a market-consistent way.

3.1. European Options in a Time-Inhomogeneous Setting

Let us take a set of maturities $\{T_j\}_{j=1}^n$ and a set of strike prices for every maturity $\{K_{j,k}\}_{k=1}^{m_j}$, which form a grid that becomes increasingly finer as $m_j \to \infty$. Consider now European call and put options $C(T_j, K_{j,k})$ and $P(T_j, K_{j,k})$, respectively, whose underlying S_t follows an time-inhomogeneous exponential Lévy process, for which, in the interval $[T_{j-1}, T_j]$ the Lévy triplet is $(\sigma_j^2, \gamma_j, \nu_j)$.

Under a finite-variation assumption and working in the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{Q})$, it holds that

$$S_0 = \mathbb{E}(S_t|\mathcal{F}_0) \iff \mathbb{E}\left(e^{X_t}\right) = 1 \iff \frac{\sigma_j^2}{2} + \gamma_j + \int_{\mathbb{R}} (e^x - 1)\nu_j(x)\,\mathrm{d}x = 0, \tag{3.2}$$

where we have used the Lévy-Khintchine representation. Furthermore, a finite-second moment assumption is made:

$$\mathbb{E}(S_t^2) < \infty\, \forall t \geq 0, \quad \text{which implies: } \mathbb{E}(e^{2\,X_{T_j}}) < \infty.$$

Moreover, by (3.1), the prices of European options at expiry T_j can be obtained:

$$C(T_j, K_{j,k}) = e^{-rT_j}\mathbb{E}\left(S_{T_j} - K_{j,k}\right)^+; \quad P(T_j, K_{j,k}) = e^{-rT_j}\mathbb{E}\left(K_{j,k} - S_{T_j}\right)^+. \tag{3.3}$$

Let us now define the negative forward log-moneyness

$$x_{j,k} := \ln(K_{j,k}/S_0) - r\,T_j, \tag{3.4}$$

which allows us to express the put-call parity as follows:

$$C(T_j, K_{j,k}) - P(T_j, K_{j,k}) = C(x_{j,k}, T_j) - P(x_{j,k}, T_j) = S_0\mathbb{E}\left(e^{X_t} - e^{x_{j,k}}\right) = S_0(1 - e^{x_{j,k}}), \tag{3.5}$$

using the equivalence in (3.2). Note that the option prices functions $C(x_{j,k}, T_j)$ and $P(x_{j,k}, T_j)$ do not have a Fourier transform, as they do not converge to 0 as $x_{j,k}$ goes to $-\infty$ and ∞, respectively. Note furthermore that:

$$\lim_{x\to\infty} C(x, T) = \lim_{x\to-\infty} P(x, T) = 0.$$

Hence, we introduce the piecewise option price function

$$\mathcal{O}(x) := \begin{cases} S_0^{-1}C(x, T), & x \geq 0, \\ S_0^{-1}P(x, T), & x < 0. \end{cases} \tag{3.6}$$

The piecewise option price function with maturity T_j can be defined by:

$$\mathcal{O}_j(x) := \begin{cases} S_0^{-1}C(x, T_j), & x \geq 0, \\ S_0^{-1}P(x, T_j), & x < 0. \end{cases}$$

The relationship between $\mathcal{O}_j(x)$ and the characteristic function $\varphi_j(v)$ is given by the following Proposition, which is a time-inhomogeneous adaptation by Koorevaar [15] of the work of Carr and Madan [3] and Belomestny and Reiß [1]:

Proposition 3.1. *The function $\mathcal{O}_j(x)$, defined in (3.6), satisfies the following properties:*

(i) For all $x \in \mathbb{R}$, it holds that

$$\mathcal{O}_j(x) = \frac{C(x, T_j)}{S_0} - (1 - e^x)^+.$$

(ii) For all $x \in \mathbb{R}$, it holds that

$$\mathcal{O}_j(x) \in [0, 1 \wedge e^x].$$

(iii) $C_\alpha := \mathbb{E}[e^{\alpha X_{T_j}}]$ is finite for some $\alpha \geq 1 \Rightarrow$

$$\mathcal{O}_j(x) \leq C_\alpha e^{(1-\alpha)x} \quad \text{for all } x \geq 0.$$

(iv) The Fourier transform of $\mathcal{O}_j$ satisfies

$$\mathcal{F}(\mathcal{O}_j(x))(v) = \int_{-\infty}^{\infty} \mathcal{O}_j(x)e^{ivx}\,dx = \frac{1 - \varphi_{T_j}(v - i)}{v(v - i)},$$

for all $v \in \mathbb{C}$ with $\mathrm{Im}(v) \in [0, 1]$.

Our finite second moment assumptions $\mathbb{E}(e^{2X_{T_j}}) < \infty$ makes it possible, by (*ii*) and (*iii*), that $O_j(x)$ exhibits exponential decay and thus that $\mathcal{F}\mathcal{O}_j$ exists. As many different Fourier transform conventions exist, it is worth noting that the one used throughout this thesis is the following:

$$\mathcal{F}f(v) := \mathcal{F}(f(x))(v) = \int_{-\infty}^{\infty} f(x)e^{ivx}\,dx.$$

$$\mathcal{F}^{-1}F(x) := \mathcal{F}^{-1}(F(v))(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(v)e^{-ivx}\,dv.$$

Notice now that by condition (iv) in Proposition 3.1, it holds that

$$\varphi_{T_j}(v-i) = 1 - v(v-i)\mathscr{F}\mathcal{O}_j(v) = 1 + iv(1+iv)\mathscr{F}\mathcal{O}_j(v). \tag{3.7}$$

Furthermore, given that $\varphi_{T_j}(v)$ is the characteristic function in the interval $[T_{j-1}, T_j]$, given by the relation

$$\frac{\varphi_{T_j}(v)}{\varphi_{T_{j-1}}(v)} = \exp\left((T_j - T_{j-1}) \left[-\frac{\sigma_j^2 v^2}{2} + i\gamma_j v + \int_{\mathbb{R}} (e^{ivx} - 1)\nu_j(x)\, dx \right] \right),$$

the characteristic function on the previous interval, $\varphi_{T_{j-1}}(v)$, is also required. Both can be extracted from the option prices by (3.7). Once this has been established, the calibration functions $\psi_j^l(v)$, where $j = 1, \dots, n$ and $l = 0, 1$ can be defined:

$$\psi_j^l(v) := \frac{1}{T_j - T_{j-1}} \log(1 + iv(1+iv)\mathscr{F}\mathcal{O}_{j-l}(v)) = \frac{1}{T_j - T_{j-1}} \log\big(\varphi_{T_{j-l}}(v-i)\big) \tag{3.8}$$

where we take the complex logarithm such that $\psi_j^l(0) = 0$ and $\psi_j^l(v)$ is continuous on $(-v_0, v_0)$, where v_0 is the smallest positive root of $\varphi_{T_{j-l}}(v)$. The calibration procedure consists on finding the Lévy triplets $(\sigma_j^2, \gamma_j, \nu_j(x))_{j=1}^n$ which characterize the exponential Lévy process of the underlying. These are in turn, by Lévy-Khintchine, given by the characteristic function. As the characteristic functions φ_{T_j} and $\varphi_{T_{j-1}}$ are simultaneously needed, we can define the calibration function $\psi_j(v)$ as follows:

Definition 3.1 (Calibration Function $\psi_j(v)$ (equation (2.14) in Koorevaar [15])).

$$\begin{aligned}
\psi_j(v) &:= \psi_j^0(v) - \psi_j^1(v), \\
&= \frac{1}{T_j - T_{j-1}} \log\left[\frac{1 + iv(1+iv)\mathscr{F}\mathcal{O}_j(v)}{1 + iv(1+iv)\mathscr{F}\mathcal{O}_{j-1}(v)} \right], \\
&= \frac{1}{T_j - T_{j-1}} \log\left[\frac{\varphi_{T_j}(v-i)}{\varphi_{T_{j-1}}(v-i)} \right], \\
&= -\frac{\sigma_j^2 (v-i)^2}{2} + i\gamma_j(v-i) + \int_{\mathbb{R}} \big(e^{i(v-i)x} - 1\big)\nu_j(x)\, dx, \\
&= -\frac{\sigma_j^2 v^2}{2} + i(\sigma_j^2 + \gamma_j)v^2 + \left(\frac{\sigma_j^2}{2} + \gamma_j - \lambda_j \right) + \mathscr{F}\mu_j(v).
\end{aligned}$$

where $\lambda_j = \|\nu_j\|_{L^1(\mathbb{R})}$ and

$$\mu_j(x) := e^x\, \nu_j(x) \tag{3.9}$$

is the exponentially tilted Lévy density. It can be readily observed that given the option prices, the calibration formula allows us to find the Lévy triplet in a relatively straightforward way. However, the option function $\mathcal{O}_j(x)$ is not directly observable. This is because the only avaiable information is European option market prices, from which the option function has to be derived. Furthermore, because of the bid-ask spread and market inefficiencies, the quotes of option prices are often noisy and it is possible that there is not enough information available to estimate $\psi_j(v)$. This is something which must be accounted for in our modeling, as the following section will show.

3.2. Observation & Estimation

Since option prices are only quoted at a discrete and finite set of strikes, and are furthermore corrupted by bid-ask spreads and other market noise, the option function $\mathcal{O}_j(x)$ cannot be observed directly. To account for this, we adopt the regression model of Belomestny and Reiß [1], adapted to the time-inhomogeneous setting due to Tendijck [27] and Koorevaar [15]. For each maturity T_j, the observations $\mathcal{O}_{j,k}$ are modelled as

$$\mathcal{O}_{j,k} = \mathcal{O}_j(x_{j,k}) + \delta_{j,k}\, \varepsilon_{j,k}, \quad k = 1, \dots, m_j, \tag{3.10}$$

where $\delta_{j,k} > 0$ controls the magnitude of the noise at each design point, and $(\varepsilon_{j,k})$ are independent, centered sub-Gaussian random variables with unit variance, i.e., $\mathbb{E}[\varepsilon_{j,k}] = 0$ and $\mathrm{Var}(\varepsilon_{j,k}) = 1$. The sub-Gaussianity assumption means that the tails of the noise distribution are dominated by those of a Gaussian, which ensures that all moments are finite.

Given the noisy observations $\mathcal{O}_{j,k}$, the next step is to recover an estimate $\tilde{\mathcal{O}}_j$ of the function $\mathcal{O}_j$. In the theoretical analysis, following Tendijck [27] and Belomestny and Reiß [1], we use a linear spline interpolation scheme. Concretely, $\tilde{\mathcal{O}}_j$ is defined as

$$\tilde{\mathcal{O}}_j(x) = \beta_{0,j}(x) + \sum_{k=1}^{m_j} \mathcal{O}_{j,k}\, b_{j,k}(x), \quad x \in \mathbb{R}, \tag{3.11}$$

where $(b_{j,k})$ are triangular basis functions $b_{j,k}(x) = \Lambda\Big(\frac{x-x_{j,k}}{x_{j,k+1}-x_{j,k}}\Big)$, with $\Lambda(x) = (1-|x|)\mathbf{1}_{|x|\le 1}$, and $\beta_{0,j}$ is an auxiliary term that corrects for the jump of $\mathcal{O}_j$ at zero, such that $\beta'_{0,j}(0^+) - \beta'_{0,j}(0^-) = -1$. In practice, the estimator $\tilde{\mathcal{O}}_j$ is replaced by a natural cubic smoothing spline, which yields better finite-sample performance; however, the theoretical results derived in this thesis are established for the linear scheme (3.11), as the extension to cubic splines follows from it. From $\tilde{\mathcal{O}}_j$, an estimator of the characteristic function is obtained via (3.7):

$$\tilde{\varphi}_{T_j}(v-i) := 1 + iv(1+iv)\,\mathcal{F}\tilde{\mathcal{O}}_j(v). \tag{3.12}$$

Since the logarithm of $\tilde{\varphi}_{T_j}(v-i)$ may become numerically unstable when its argument is close to zero, which can happen due to the stochastic errors in the option prices, a trimmed complex logarithm $\log_{\ge\kappa(v,T_j)}$ is used in substitution. This trimming threshold $\kappa(v,T_j) > 0$ is chosen to prevent the argument from becoming too small, and is shown in Koorevaar [15] to satisfy

$$\kappa(v,T_j) := \tfrac{1}{2}\exp\Big(-T_j\Big(\tfrac{\sigma_{\max}^2 v^2}{2} + 2R\Big)\Big), \tag{3.13}$$

where $\sigma_{\max}$ and R are the bounding constants from the parameter class $\mathcal{G}^n_{s_j}(R,\sigma_{\max})$ introduced in the next subsection. The estimator of the calibration function $\psi_j(v)$ is then given by:

Definition 3.2 (Calibration Estimator $\tilde{\psi}_j(v)$ (Koorevaar [15]))**.**

$$\begin{aligned}\tilde{\psi}_j(v) &:= \tilde{\psi}^0_j(v) - \tilde{\psi}^1_j(v),\\ &:= \frac{1}{T_j - T_{j-1}}\log_{\ge\kappa(v,\,T_j)}\big[1 + iv(1+iv)\,\mathcal{F}\tilde{\mathcal{O}}_j(v)\big]\\ &\quad - \frac{1}{T_j - T_{j-1}}\log_{\ge\kappa(v,\,T_{j-1})}\big[1 + iv(1+iv)\,\mathcal{F}\tilde{\mathcal{O}}_{j-1}(v)\big],\end{aligned}$$

where the trimmed complex logarithm is taken to be continuous on $(-v_0, v_0)$ with $\tilde{\psi}^l_j(0) = 0$ and v_0 the smallest positive zero of $\tilde{\varphi}_{T_{j-l}}$.

Thus, $\tilde{\psi}_j(v)$ is the empirical counterpart of the calibration function $\psi_j(v)$ in Definition 3.1, replacing the unobserved option functions $\mathcal{O}_{j-l}$ with their estimated versions $\tilde{\mathcal{O}}_{j-l}$ for $l = 0, 1$. The well-definedness of this estimator is established in Koorevaar [15] under the assumptions of the following subsection. Indeed, this well-definedness means that the trimmed logarithm coincides with the ordinary logarithm with probability tending to one, and it makes it possible to obtain the pointwise asymptotic normality results in it.

3.3. Spectral Estimation of the Lévy Triplet

Given the calibration estimator $\tilde{\psi}_j(v)$, the goal is now to extract the Lévy triplets $(\tilde{\sigma}^2_j, \tilde{\gamma}_j, \tilde{\lambda}_j, \tilde{\mu}_j)^n_{j=1}$ from it. The key structural observation, already visible in Definition 3.1, is that $\psi_j(v)$ decomposes as

$$\psi_j(v) = -\frac{\sigma_j^2 v^2}{2} + i(\sigma_j^2 + \gamma_j)v + \left(\frac{\sigma_j^2}{2} + \gamma_j - \lambda_j\right) + \mathscr{F}\mu_j(v), \tag{3.14}$$

where, by the Riemann–Lebesgue lemma, $\mathscr{F}\mu_j(v) \to 0$ as $|v| \to \infty$ under the finite-activity assumption. Therefore, for sufficiently large $|v|$, the calibration function $\psi_j(v)$ behaves as a quadratic polynomial in v whose coefficients encode the parametric part $(\sigma_j^2, \gamma_j, \lambda_j)$, plus a non-parametric remainder $\mathscr{F}\mu_j(v)$ that decays to zero. As a result, the parametric and non-parametric components can be estimated separately, at different frequency regimes.

Since $\mathscr{F}\mu_j(v)$ only vanishes asymptotically, the contribution of the non-parametric part does not disappear at any fixed finite frequency. Moreover, the deviation of $\tilde{\psi}_j(v)$ from $\psi_j(v)$ grows exponentially in $|v|$ for $\sigma_j > 0$, as shown in Belomestny and Reiß [1]. It is therefore necessary to restrict estimation to a compact frequency band $[-U_j, U_j]$, for some cutoff parameter $U_j > 0$. The choice of U_j is in itself a bias-variance trade-off: a larger U_j reduces the bias from the term $\mathscr{F}\mu_j$, but makes the stochastic error larger.

Given the cutoff U_j, the parametric part is estimated by projecting $\tilde{\psi}_j$ onto the space of quadratic polynomials in a weighted L^2 sense. Concretely, we minimize the weighted residual sum of squares

$$\inf_{(\sigma_j^2, \gamma_j, \lambda_j)} \int_{-U_j}^{U_j} \tilde{w}^{U_j}(v) \left| \tilde{\psi}_j(v) + \frac{\sigma_j^2 v^2}{2} - i(\sigma_j^2 + \gamma_j)v - \left(\frac{\sigma_j^2}{2} + \gamma_j - \lambda_j \right) \right|^2 \mathrm{d}v, \tag{3.15}$$

where $\tilde{w}^{U_j}(v) := U_j^{-1} \tilde{w}^1(v/U_j)$ is a nonnegative weight function with $\tilde{w}^1$ continuous, compactly supported on $[0, 1]$, and strictly positive on $(0, 1)$. Using $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$ for $z \in \mathbb{C}$, the optimization in (3.15) separates into two independent problems via the reparametrization $(\hat{\sigma}^2, \hat{\gamma}, \hat{\lambda}) = (\tilde{\sigma}_j^2,\ \tilde{\sigma}_j^2 + \tilde{\gamma}_j,\ \tilde{\sigma}_j^2/2 + \tilde{\gamma}_j - \tilde{\lambda}_j)$:

$$\begin{aligned} (\hat{\sigma}_j^2, \hat{\lambda}_j) &:= \operatorname*{arg\,min}_{(\sigma_j^2, \lambda_j)} \int_{-U_j}^{U_j} \tilde{w}^{U_j}(v) \left(\mathrm{Re}(\tilde{\psi}_j(v)) + \frac{\sigma_j^2 v^2}{2} - \lambda_j \right)^2 \mathrm{d}v, \\ \hat{\gamma}_j &:= \operatorname*{arg\,min}_{\gamma_j} \int_{-U_j}^{U_j} \tilde{w}^{U_j}(v) \left(\mathrm{Im}(\tilde{\psi}_j(v)) - \gamma_j v \right)^2 \mathrm{d}v. \end{aligned}$$

Each of these is a simple weighted least-squares problem. Solving by setting partial derivatives to zero and inverting the resulting linear system yields explicit closed-form estimators:

$$\tilde{\sigma}_j^2 = \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\, \mathrm{Re}(\tilde{\psi}_j(v))\, \mathrm{d}v, \tag{3.16}$$

$$\tilde{\gamma}_j = -\tilde{\sigma}_j^2 + \int_{-U_j}^{U_j} w_{\gamma_j}^{U_j}(v)\, \mathrm{Im}(\tilde{\psi}_j(v))\, \mathrm{d}v, \tag{3.17}$$

$$\tilde{\lambda}_j = \frac{\tilde{\sigma}_j^2}{2} + \tilde{\gamma}_j - \int_{-U_j}^{U_j} w_{\lambda_j}^{U_j}(v)\, \mathrm{Re}(\tilde{\psi}_j(v))\, \mathrm{d}v, \tag{3.18}$$

where the derived weight functions $w_{\sigma_j}^{U_j}(v)$, $w_{\gamma_j}^{U_j}(v)$, and $w_{\lambda_j}^{U_j}(v)$ are given by $w_{\sigma_j}^{U_j}(v) = U_j^{-3} w_{\sigma_j}^1(v/U_j)$, $w_{\gamma_j}^{U_j}(v) = U_j^{-2} w_{\gamma_j}^1(v/U_j)$, and $w_{\lambda_j}^{U_j}(v) = U_j^{-1} w_{\lambda_j}^1(v/U_j)$, respectively. Here $w_{\sigma_j}^1$ is symmetric, $w_{\gamma_j}^1$ is antisymmetric, and $w_{\lambda_j}^1$ is symmetric, with all three functions bounded and supported on $[0, 1]$. Furthermore, these weight functions satisfy the normalization conditions

$$\begin{gathered} \int_{-U_j}^{U_j} -\frac{v^2}{2} w_{\sigma_j}^{U_j}(v)\, \mathrm{d}v = 1, \quad \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\, \mathrm{d}v = 0, \\ \int_{-U_j}^{U_j} v\, w_{\gamma_j}^{U_j}(v)\, \mathrm{d}v = 1, \quad \int_{-U_j}^{U_j} w_{\lambda_j}^{U_j}(v)\, \mathrm{d}v = 1, \quad \int_{-U_j}^{U_j} v^2\, w_{\lambda_j}^{U_j}(v)\, \mathrm{d}v = 0, \end{gathered}$$

which reflect the identification conditions that each estimator selects exactly one coefficient of the polynomial in (3.14).

Once $(\tilde{\sigma}_j^2, \tilde{\gamma}_j, \tilde{\lambda}_j)$ have been estimated, the non-parametric component $\tilde{\mu}_j(x)$ is recovered by subtracting the estimated polynomial from $\tilde{\psi}_j$ and applying an inverse Fourier transform, regularized by a symmetric weight function $w_{\mu_j}^{U_j}(v) = w_{\mu_j}^1(v/U_j)$ supported on $[-U_j, U_j]$:

$$\tilde{\mu}_j(x) = \mathscr{F}^{-1}\left[\left(\tilde{\psi}_j(\cdot) + \frac{\tilde{\sigma}_j^2}{2}(\cdot - i)^2 - i\tilde{\gamma}_j(\cdot - i) + \tilde{\lambda}_j\right) w_{\mu_j}^{U_j}(\cdot)\right](x), \quad x \in \mathbb{R}. \tag{3.19}$$

Since $\tilde{\mu}_j(x) = e^x \tilde{\nu}_j(x)$, the jump density itself is recovered as $\tilde{\nu}_j(x) = e^{-x}\tilde{\mu}_j(x)$. In practice, it is found that estimating $\nu_j(x)$ directly via the shifted calibration function $\tilde{\psi}_{\nu_j}(v) := \tilde{\psi}_j(v + i)$ yields more numerically stable results, avoiding the multiplication by e^{-x} which amplifies errors for large negative x. The resulting estimator is

$$\tilde{\nu}_j(x) = \mathscr{F}^{-1}\left[\left(\tilde{\psi}_{\nu_j}(\cdot) + \frac{\tilde{\sigma}_j^2}{2}(\cdot)^2 - i\tilde{\gamma}_j(\cdot) + \tilde{\lambda}_j\right) w_{\nu_j}^{U_j}(\cdot)\right](x), \quad x \in \mathbb{R}. \tag{3.20}$$

Due to the frequency truncation and the stochastic error in $\tilde{\psi}_j$, the estimated density $\tilde{\nu}_j(x)$ may take negative values. Following Belomestny and Reiß [1], we apply the correction

$$\tilde{\nu}_j^+(x; \varrho) := \max\big(0,\ \tilde{\nu}_j(x) - \varrho\big), \tag{3.21}$$

where $\varrho \geq 0$ is chosen such that $\int_{\mathbb{R}} \tilde{\nu}_j^+(x; \varrho)\, \mathrm{d}x = \tilde{\lambda}_j$. The corrected density $\tilde{\nu}_j^+$ is the unique non-negative function closest to $\tilde{\nu}_j$ in L^1 that is consistent with the estimated jump intensity.

Applying the procedure above for each $j = 1, \ldots, n$ and assembling the resulting triplets $(\tilde{\sigma}_j^2, \tilde{\gamma}_j, \tilde{\nu}_j^+)$ yields a complete, nonparametrically estimated time-inhomogeneous exponential Lévy model over the full grid of maturities $\{T_j\}_{j=1}^n$.

3.4. Assumptions

3.4.1. Lévy Triplet Assumptions

The assumption on the Lévy triplets for each time interval $[T_{j-1}, T_j]$ are given by the following definition:

Definition 3.3 (Definition 3.1 in Koorevaar [15]). *For integers $s_j \geq 0$, and $R, \sigma_{\max} > 0$, let $\mathcal{G}_{s_j}^n(R, \sigma_{\max})$ denote the set of all Lévy triplets $\tau = (\sigma_j^2, \gamma_j, \mu_j)_{j=1,\ldots,n}$ such that for all $j = 1, \ldots, n$, μ_j is s_j-times (weakly) differentiable, the martingale condition (2.4) and finite second moment assumption (2.5) are satisfied, and*

$$\sigma_j \in (0, \sigma_{\max}], \quad |\gamma_j|, \lambda_j \in [0, R], \quad \max_{k=0,\ldots,s_j} \|\mu_j^{(k)}\|_{L^2(\mathbb{R})} \leq R, \quad \textit{and} \quad \|\mu_j^{(s_j)}\|_\infty \leq R.$$

The purpose of the parameters R and $\sigma_{\max}$ is to bound the estimated parameters. This is necessary to obtain stable estimates of the calibration function, because the logarithm of $\psi_j(v)$ needs to be bounded from below by a positive number. If this is not the case, the result might be either an additional bias term or a method that is too unstable.

Furthermore, it is also assumed that $\mu_j(x)$ is s_j-times differentiable, which also means that the Lévy density $\nu_j(x)$ must also be s_j-times differentiable. The exact value of $s_j \geq 0$ is particularly important for the speed of convergence of the statistical model, as a larger s_j leads to higher convergence, since this means the Lévy measure is smoother. As shown in Koorevaar [15], the value of s_j is also important to determine the optimal U_j.

3.4.2. Weight Function Assumptions

The exact weight functions that are chosen for the estimation procedure should not affect the results or convergence of the method. The only requirement is that some basic conditions, which guarantee that the integrals over the weight functions are well-behaved, are satisfied:

Definition 3.4 (Definition 3.2 in Koorevaar [15]). *For an integer $s_j \geq 0$, let $\mathcal{W}^n_{s_j}$ denote the set of all weight functions $w^1_{\sigma_j}, w^1_{\gamma_j}, w^1_{\lambda_j}$, and $w^1_{\mu_j}$ that satisfy the implied conditions of section 2.4, and*

$$\begin{aligned} &w^1_{\sigma_j}(v)/v^{s_j}, w^1_{\gamma_j}(v)/v^{s_j}, w^1_{\lambda_j}(v)/v^{s_j}, (1 - w^1_{\mu_j}(v))/v^{s_j} \in L^2(\mathbb{R}), \\ &\mathscr{F}[w^1_{\sigma_j}(v)/v^{s_j}], \mathscr{F}[w^1_{\gamma_j}(v)/v^{s_j}], \mathscr{F}[w^1_{\lambda_j}(v)/v^{s_j}], \mathscr{F}[(1 - w^1_{\mu_j}(v))/v^{s_j}] \in L^1(\mathbb{R}). \end{aligned}$$

3.4.3. Interpolation Scheme Assumptions

The asymptotic results of this thesis rest on a set of conditions imposed on the interpolation scheme (3.11). These assumptions are made for the following: the behavior of the strike grid as $m_j \to \infty$, the regularity of the noise magnitude function δ_j, and a comparability condition between consecutive grids. Together they ensure that the Fourier transform of the interpolation error $\mathscr{F}(\tilde{\mathcal{O}}_j - \mathcal{O}_j)(v)$ is well-controlled in the asymptotic regime.

For the theoretical results, we require the strike grid $\{x_{j,k} : k = 1, \dots, m_j\}$ to become simultaneously denser and wider as $m_j \to \infty$. This is given by the conditions

$$\Delta_j := \max_{k=2,\dots,m_j} |x_{j,k} - x_{j,k-1}| \to 0 \quad \text{and} \quad H_j := \min(x_{j,m_j}, -x_{j,1}) \to \infty, \tag{3.22}$$

which respectively require the mesh size to shrink to zero and the range of observed strikes to grow to cover the entire real line. From a financial application standpoint, these conditions reflect that the number of listed options per maturity grows, with increasingly fine and wide coverage of the moneyness axis. Note that the maturities $T_1, \dots, T_n$ remain fixed throughout. In line with Koorevaar [15] and Tendijck [27], the strike grid is assumed to be equidistant, i.e.,

$$\Delta_j = |x_{j,k} - x_{j,k-1}| \quad \text{for all } k = 2, \dots, m_j. \tag{3.23}$$

This simplifies the spectral analysis considerably: the Fourier transform of each basis function $b_{j,k}$ admits the explicit closed form

$$\mathscr{F} b_{j,k}(v) = \Delta_j \, e^{ivx_{j,k}} \operatorname{sinc}^2\left(\frac{\Delta_j v}{2}\right), \tag{3.24}$$

which follows from the triangular function $b_{j,k}(x) = \Lambda\left(\frac{x - x_{j,k}}{\Delta_j}\right)$ being the convolution of two box functions, and the convolution theorem.

The noise scale $\delta_{j,k} > 0$ in the regression model (3.10) is evaluated at the design points. For the asymptotic variance of the estimators to be well-defined and finite, we require that $\delta_{j,k}$ arises from the evaluation of a function $\delta_j : \mathbb{R} \to \mathbb{R}_{>0}$ at the grid points, i.e. $\delta_{j,k} = \delta_j(x_{j,k})$, and that this function satisfies

$$\delta_{j-l} \in L^{2+\eta}(\mathbb{R}) \cap C^0(\mathbb{R}) \quad \text{for some } \eta > 0, \quad l = 0, 1. \tag{3.25}$$

The $L^{2+\eta}$-integrability ensures that the noise contribution to the asymptotic variance is finite, i.e., it guarantees that the Riemann sums $\Delta_j \sum_k \delta^2_{j,k}$ converge to $\|\delta_j\|^2_{L^2} < \infty$ as $\Delta_j \to 0$. Throughout this thesis, it suffices for the function δ_j to be continuous for the bias from the interpolation scheme to remain negligible relative to the stochastic error, i.e., no additional smoothness assumptions are necessary. Furthermore, the following integrability condition is also given:

$$\Delta_{j-l} \|\delta_{j-l}\|^2_{l^2} \leq \|\delta_{j-l}\|^2_{\infty}, \tag{3.26}$$

Moreover, we also require that:

$$\Delta_{j-1} \leq C \Delta_j \quad \text{for some constant } C > 0, \tag{3.27}$$

$$\Delta_j U_j \to 0 \quad \text{as } \Delta_j \to 0, U_j \to \infty. \tag{3.28}$$

Condition (3.26) ensures that the contributions from the j-th and $(j-1)$-th intervals to the error decomposition of $\tilde{\mu}_j(x) - \mu_j(x)$ shrink at comparable rates. Finally, condition (3.27) ensures that the cutoff parameter U_j does not go to infinity too quickly, which will help to asymptotic degeneracy issues in the upcoming chapters.

3.4.4. Well-Definedness of $\tilde{\psi}_j(v)$

Before deriving asymptotic normality of the Lévy triplet estimators, it must first be verified that the calibration estimator $\tilde{\psi}_j(v)$ is well-defined with probability tending to one. Recall from Definition 3.2 that $\tilde{\psi}_j(v)$ is constructed via a trimmed complex logarithm of $\tilde{\varphi}_{T_{j-l}}(v-i)$. The trimming is necessary because the empirical characteristic function $\tilde{\varphi}_{T_{j-l}}(v-i)$ may pass close to zero due to stochastic errors in the option prices, at which point the ordinary complex logarithm would be ill-defined or go towards infinity. The trimmed logarithm coincides with the ordinary logarithm whenever $|\tilde{\varphi}_{T_{j-l}}(v-i)| \geq \kappa(v, T_{j-l})$, so well-definedness reduces to showing that this event occurs with probability tending to one, uniformly over $v \in [0, U_j]$.

By the Lévy–Khintchine representation and the Martingale condition (3.2), one can show that

$$\left|\frac{\varphi_{T_j}(v-i)}{\varphi_{T_{j-1}}(v-i)}\right| \geq 2K(T_j - T_{j-1}, \sigma_j, R, v), \quad \text{where } K(T_j - T_{j-1}, \sigma_j, R, v) := e^{-(T_j - T_{j-1})\left(\frac{\sigma_j^2 v^2}{2} + 2R\right)},$$

and consequently, accumulating over all intervals up to T_j,

$$|\varphi_{T_j}(v-i)| \geq 2\kappa(v, T_j) := \exp\left(-T_j\left(\frac{\sigma_{\max}^2 v^2}{2} + 2R\right)\right).$$

This bound identifies the trimming threshold $\kappa(v, T_j)$ of Definition 3.2 as a genuine lower bound on the modulus of the true characteristic function.

Well-definedness then follows from showing that the empirical characteristic function $\tilde{\varphi}_{T_j}(v-i)$ is uniformly close to $\varphi_{T_j}(v-i)$ over the estimation band $[0, U_j]$. This is the content of the following proposition. Only the result is stated here, but the proof for both this proposition and the additional lemmas needed to prove are all in Koorevaar [15], where it is also shown that the decay condition $\Delta_j U_j^4 \log(U_j) \exp\left(U_j^2 \sum_{r=1}^{j}(T_r - T_{r-1})\sigma_r^2\right) \to 0$, (as $U_j \to \infty$, $\Delta_j \to 0$) is required for well-definedness to hold.

Proposition 3.2 (Asymptotic well-definedness of $\tilde{\psi}_j$, Koorevaar [15])**.** *Let $j \in \{1, \ldots, n\}$. Suppose the standing assumptions hold, and that the cutoff U_j and mesh size Δ_j satisfy the decay condition*

$$\Delta_j U_j^4 \log(U_j) \exp\left(U_j^2 \sum_{r=1}^{j}(T_r - T_{r-1})\sigma_r^2\right) \to 0 \quad \textit{as } U_j \to \infty,\ \Delta_j \to 0. \tag{3.29}$$

Then

$$\lim_{U_j \to \infty} \mathbb{P}\left(\sup_{v \in [0, U_j]} \left|\tilde{\varphi}_{T_j}(v-i) - \varphi_{T_j}(v-i)\right| > \inf_{v \in [0, U_j]} \kappa(v, T_j)\right) = 0.$$

3.5. Normality Result of $(\widetilde{\sigma}_j^2, \widetilde{\gamma}_j, \widetilde{\lambda}_j)$

This section establishes the normality results for the estimated parameters of the spectral method. Namely, the triplet $(\widetilde{\sigma}_j^2, \widetilde{\gamma}_j, \widetilde{\lambda}_j)$ for all $j = 1, \ldots, n$. Koorevaar [15] shows that the procedure to obtain an asymptotic normality is very similar for all three parameters, so the results for them are presented together in this section. On the other hand, the pointwise normality results for the estimate of the exponentially tilted Lévy measure $\widetilde{\mu}_j(x)$ are obtained via a different estimation procedure, and are thus presented in the following section.

The estimation error for each element of the triplet $(\widetilde{\sigma}_j^2, \widetilde{\gamma}_j, \widetilde{\lambda}_j)$ can be split in a linear, leading term, a remainder term and a bias term. These last two are asymptotically negligible and normality is obtained from the linear term.

3.5.1. Error Decomposition

Similarly to Koorevaar [15], let us first consider the error decomposition $\widetilde{\sigma}_j^2 - \sigma_j^2$. By Definition 3.1 and the weight function properties, the decomposition of the σ_j^2 error is the following:

$$\begin{aligned}
\tilde{\sigma}_j^2 - \sigma_j^2 &= \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\tilde{\psi}_j(v) - \psi_j(v)\big)\,dv + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\psi_j(v)\big)\,dv - \sigma_j^2 \\
&= \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\tilde{\psi}_j(v) - \psi_j(v)\big)\,dv + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\psi_j(v)\big)\,dv \\
&\quad - \sigma_j^2 \int_{-U_j}^{U_j} \left(-\frac{v^2}{2}\right) w_{\sigma_j}^{U_j}(v)\,dv - \left(\frac{\sigma_j^2}{2} + \gamma_j - \lambda_j\right) \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,dv \\
&= \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\tilde{\psi}_j(v) - \psi_j(v)\big)\,dv + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\mathcal{F}\mu_j(v)\big)\,dv \\
&= \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\tilde{\psi}_j^0(v) - \psi_j^0(v)\big)\,dv + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\tilde{\psi}_j^1(v) - \psi_j^1(v)\big)\,dv \\
&\quad + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\,\mathrm{Re}\big(\mathcal{F}\mu_j(v)\big)\,dv.
\end{aligned}$$

where in the last equality the first two summands correspond to the stochastic error and the last term to the bias. The second equality is given by the conditions of the weight functions

$$\begin{aligned}
\int_{-U_j}^{U_j} \left(-\frac{v^2}{2}\right) w_{\sigma_j}^{U_j}\,dv &= 1, \\
\int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}\,dv &= 0.
\end{aligned}$$

as well as Definition 3.1. For the pointwise normality results, Koorevaar's [15] strategy is to establish asymptotic normality of $\widetilde{\sigma}_j^2 - \sigma_j^2$ by combining two ingredients: first, the difference of the linear terms $\mathscr{L}_{\sigma_j}^0 - \mathscr{L}_{\sigma_j}^1$, suitably normalized by a deterministic standard deviation s_n, converges in distribution to a standard normal; second, the remainder terms $\mathscr{R}_{\sigma_j}^0, \mathscr{R}_{\sigma_j}^1$ and the bias term $\mathscr{B}_{\sigma_j}$ are asymptotically negligible relative to s_n. Together with Slutsky's theorem, these yield

$$\frac{\widetilde{\sigma}_j^2 - \sigma_j^2}{s_n} \xrightarrow{d} \mathcal{N}(0,1).$$

The error decompositions for $\widetilde{\gamma}_j - \gamma_j$ and $\widetilde{\lambda}_j - \lambda_j$ are not fully derived, just presented for reference. The full derivation is explicitly computed in Tendijck [27] and Koorevaar [15]. As can be seen from Definition 3.1, the spectral estimator naturally identifies the Taylor coefficients of $\psi_j(v)$ around $v = 0$, namely σ_j^2 (from the v^2 term), $\sigma_j^2 + \gamma_j$ (from the v^1 term), and $\sigma_j^2/2 + \gamma_j - \lambda_j$ (from the v^0 term). For this reason, Koorevaar [15] decomposes the combined parameterizations

$$\widehat{\gamma}_j := \widetilde{\gamma}_j + \widetilde{\sigma}_j^2, \qquad \widehat{\lambda}_j := \widetilde{\lambda}_j - \widetilde{\sigma}_j^2/2 - \widetilde{\gamma}_j,$$

instead of $\widetilde{\gamma}_j$ and $\widetilde{\lambda}_j$ themselves. The resulting decompositions have the same structure as the one obtained above for $\widetilde{\sigma}_j^2 - \sigma_j^2$, with weight functions $w_{\gamma_j}^{U_j}$ and $w_{\lambda_j}^{U_j}$ in place of $w_{\sigma_j}^{U_j}$ and with the imaginary or real parts of the integrands taken as appropriate; we refer to Koorevaar [15, eqs. (3.10), (3.11)] for the explicit expressions. Normality for $\widetilde{\gamma}_j - \gamma_j$ and $\widetilde{\lambda}_j - \lambda_j$ is then recovered at the end of the argument by subtracting the already-controlled $\widetilde{\sigma}_j^2$ (and, for $\widetilde{\lambda}_j$, also $\widetilde{\gamma}_j$) and exploiting the faster convergence rate of these terms. Hence, since the method is the same, only the error decomposition and normality results for $\widetilde{\sigma}_j^2$ are looked at in depth.

Recall now that

$$\psi_j^l(v) = \frac{1}{T_j - T_{j-1}} \log_{\geq \kappa(v, T_{j-l})}\big(\varphi_{T_{j-l}}(v-i)\big)$$

This means that the error of the calibration function can be written as

$$\widetilde{\psi}_j^l(v) - \psi_j^l(v) = \frac{1}{T_j - T_{j-1}} \Big[\log_{\geq \kappa(v, T_{j-l})}\big(\widetilde{\varphi}_{T_{j-l}}(v-i)\big) - \log\big(\varphi_{T_{j-l}}(v-i)\big)\Big].$$

which by Taylor's theorem can be split into a linearization

$$\mathcal{L}_j^l(v) := \frac{1}{T_j - T_{j-1}} \frac{\widetilde{\varphi}_{T_{j-l}}(v-i) - \varphi_{T_{j-l}}(v-i)}{\varphi_{T_{j-l}}(v-i)}$$

and remainder term $\mathcal{R}_j^l(v)$:

$$\widetilde{\psi}_j^l(v) - \psi_j^l(v) = \mathcal{L}_j^l(v) + \mathcal{R}_j^l(v).$$

where $\mathcal{R}_j^l(v) := \widetilde{\psi}_j^l(v) - \psi_j^l(v) - \mathcal{L}_j^l(v)$. As shown in Koorevaar [15], this allows for the following decomposition:

$$\begin{aligned}
\tilde{\sigma}_j^2 - \sigma_j^2 &= \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\, \Re\big(\mathcal{L}_j^0(v)\big)\, dv - \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\, \Re\big(\mathcal{L}_j^1(v)\big)\, dv \\
&\quad + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\, \Re\big(\mathcal{R}_j^0(v)\big)\, dv - \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\, \Re\big(\mathcal{R}_j^1(v)\big)\, dv \\
&\quad + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}(v)\, \Re\big(\mathscr{F}\mu_j(v)\big)\, dv \\
&=: \mathscr{L}_{\sigma_j}^0 - \mathscr{L}_{\sigma_j}^1 + \mathscr{R}_{\sigma_j}^0 - \mathscr{R}_{\sigma_j}^1 + \mathscr{B}_{\sigma_j}.
\end{aligned} \tag{3.30}$$

Furthermore, the decompositions for $\widetilde{\gamma}_j - \gamma_j$ and $\widetilde{\lambda}_j - \lambda_j$ are analogous, and use the aforementioned parametrization:

$$\begin{aligned}
(\tilde{\gamma}_j + \tilde{\sigma}_j^2) - (\gamma_j + \sigma_j^2) &= \int_{-U_j}^{U_j} w_{\gamma_j}^{U_j} \Im\big(\mathcal{L}_j^0(v)\big)\, dv - \int_{-U_j}^{U_j} w_{\gamma_j}^{U_j} \Im\big(\mathcal{L}_j^1(v)\big)\, dv \\
&\quad + \int_{-U_j}^{U_j} w_{\gamma_j}^{U_j} \Im\big(\mathcal{R}_j^0(v)\big)\, dv - \int_{-U_j}^{U_j} w_{\gamma_j}^{U_j} \Im\big(\mathcal{R}_j^1(v)\big)\, dv \\
&\quad + \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j} \Im(\mathscr{F}\mu_j(v))\, dv \\
&=: \mathscr{L}_{\gamma_j}^0 - \mathscr{L}_{\gamma_j}^1 + \mathscr{R}_{\gamma_j}^0 - \mathscr{R}_{\gamma_j}^1 + \mathscr{B}_{\gamma_j}.
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
(\tilde{\lambda}_j - \tilde{\sigma}_j^2/2 - \tilde{\gamma}_j) - (\lambda_j - \sigma_j^2/2 - \gamma_j) &= \int_{-U_j}^{U_j} w_{\lambda_j}^{U_j} \Re\big(\mathcal{L}_j^0(v)\big)\, dv - \int_{-U_j}^{U_j} w_{\lambda_j}^{U_j} \Re\big(\mathcal{L}_j^1(v)\big)\, dv \\
&\quad + \int_{-U_j}^{U_j} w_{\lambda_j}^{U_j} \Re\big(\mathcal{R}_j^0(v)\big)\, dv - \int_{-U_j}^{U_j} w_{\lambda_j}^{U_j} \Re\big(\mathcal{R}_j^1(v)\big)\, dv \\
&\quad + \int_{-U_j}^{U_j} w_{\lambda_j}^{U_j} \Re(\mathscr{F}\mu_j(v))\, dv \\
&=: \mathscr{L}_{\lambda_j}^0 - \mathscr{L}_{\lambda_j}^1 + \mathscr{R}_{\lambda_j}^0 - \mathscr{R}_{\lambda_j}^1 + \mathscr{B}_{\lambda_j}.
\end{aligned} \tag{3.32}$$

Note now that, using the interpolation scheme, we obtain that

$$\begin{aligned}
\tilde{\varphi}_{T_{j-l}}(v-i) - \varphi_{T_{j-l}}(v-i) &= iv(1+iv)\mathscr{F}\big(\tilde{\mathcal{O}}_{j-l}(x) - \mathcal{O}_{j-l}(x)\big)(v) \\
&= iv(1+iv)\mathscr{F}\left(\sum_{k=1}^{m_{j-l}} \big(\mathcal{O}_{j-l,k} - \mathcal{O}_{j-l}(x_{j,k})\big) b_{j-l,k}(x)\right)(v) \\
&= iv(1+iv) \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\, \varepsilon_{j-l,k}\, \mathscr{F} b_{j,k}(v).
\end{aligned} \tag{3.33}$$

Note furthermore that the characteristic function can be written as follows:

$$\begin{aligned}\varphi_{T_{j-l}}(v-i) &= \prod_{r=1}^{j-l} \frac{\varphi_{T_r}(v-i)}{\varphi_{T_{r-1}}(v-i)} \\ &= \exp\Bigg(-\frac{v^2}{2}\sum_{r=1}^{j-l}(T_r - T_{r-1})\sigma_r^2 + iv\sum_{r=1}^{j-l}(T_r - T_{r-1})(\sigma_r^2 + \gamma_r) \\ &\quad + \sum_{r=1}^{j-l}(T_r - T_{r-1})\left(\frac{\sigma_r^2}{2} + \gamma_r - \lambda_r\right) + \sum_{r=1}^{j-l}(T_r - T_{r-1})\mathcal{F}\mu_r(v)\Bigg) \\ &=: \exp\Bigg(-\frac{v^2}{2}A_{j-l} + ivB_{j-l} + C_{j-l} + D_{j-l}(v)\Bigg). \end{aligned} \tag{3.34}$$

This means that the linear term can be written as

$$\mathcal{L}_j^l(v) = \frac{1}{T_j - T_{j-1}} \frac{\widetilde{\varphi}_{T_{j-l}}(v-i) - \varphi_{T_{j-l}}(v-i)}{\varphi_{T_{j-l}}(v-i)} = \frac{1}{T_j - T_{j-1}} \frac{iv(1+iv)\sum_{k=1}^{m_{j-l}} \delta_{j,k}\varepsilon_{j,k}\,\mathcal{F}b_{j,k}(v)}{\varphi_{T_{j-l}}(v-i)}. \tag{3.35}$$

For $l \in \{0,1\}$, recall that the linear terms of the error decomposition $\widetilde{\sigma}_j^2 - \sigma_j^2$ are defined as follows:

$$\mathscr{L}_{\sigma_j}^l = \int_{-U_j}^{U_j} w_{\sigma_j}^{U_j}\,\mathrm{Re}\big(\mathcal{L}_j^l(v)\big)\,dv. \tag{3.36}$$

Given equation (3.35), note that $\mathscr{L}_{\sigma_j}^l$ can be written in terms of the random variables $(\varepsilon_{j,k})$ from the regression model in section (3.2) as follows:

$$\begin{aligned}\mathscr{L}_{\sigma_j}^l &= \int_{-U_j}^{U_j} \mathrm{Re}\big(\mathcal{L}_j^l(v)\big)\,w_{\sigma_j}^{U_j}(v)\,dv \\ &= U_j\int_{-1}^{1} \mathrm{Re}\big(\mathcal{L}_j^l(vU_j)\big)\,w_{\sigma_j}^{U_j}(vU_j)\,dv \\ &= U_j^{-2}\int_{-1}^{1} \mathrm{Re}\big(\mathcal{L}_j^l(vU_j)\big)\,w_{\sigma_j}^{1}(v)\,dv \\ &= U_j^{-2}\int_{0}^{1} \mathrm{Re}\big(\mathcal{L}_j^l(vU_j)\big)\,w_{\sigma_j}^{1}(v)\,dv + U_j^{-2}\int_{0}^{1} \mathrm{Re}\big(\mathcal{L}_j^l(-vU_j)\big)\,w_{\sigma_j}^{1}(-v)\,dv \\ &= 2U_j^{-2}\int_{0}^{1} \mathrm{Re}\big(\mathcal{L}_j^l(vU_j)\big)\,w_{\sigma_j}^{1}(v)\,dv \\ &= \frac{2U_j^{-2}}{T_j - T_{j-1}}\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\,\mathrm{Re}\Bigg(\int_0^1 \frac{ivU_j(1+ivU_j)\mathscr{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\,w_{\sigma_j}^{1}(v)\,dv.\Bigg) \end{aligned} \tag{3.37}$$

For $k = 1, \ldots, m_{j-l}$ we can now define the random variables

$$X_k := \frac{2U_j^{-2}}{T_j - T_{j-1}}\,\delta_{j-l,k}\,\varepsilon_{j-l,k}\,\mathrm{Re}\Bigg(\int_0^1 \frac{ivU_j(1+ivU_j)\mathscr{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\,w_{\sigma_j}^{1}(v)\,dv\Bigg), \tag{3.38}$$

where $(\varepsilon_{j-l,k})$ are centered sub-Gaussian random variables with unit variance, and, as a result X_k are as well. The linear terms of the error decomposition of γ_j and λ_j are similarly defined, and labeled $\mathscr{L}_{\gamma_j}^l, \mathscr{L}_{\lambda_j}^l$.

3.5.2. Lyapunov Condition & Asymptotic Variance

The linear term $\mathscr{L}_{\sigma_j}^l$ has been written in the previous subsubsection as a sum of m_{j-l} independent centered sub-Gaussian random variables $(X_k)_{k=1}^{m_{j-l}}$. In this setting, asymptotic normality follows from the Lyapunov Central Limit Theorem, which we recall here for completeness.

Theorem 3.1 (Lyapunov CLT)**.** *Let $X_1, \dots, X_n$ be independent random variables with $\mathbb{E}[X_k] = 0$ and $\mathbb{V}[X_k] = \sigma_k^2 < \infty$ for all $k = 1, \dots, n$. Define $T_n := \sum_{k=1}^n X_k$ and $s_n^2 := \sum_{k=1}^n \sigma_k^2$. If, for some $\eta > 0$,*

$$\lim_{n\to\infty} \frac{1}{s_n^{2+\eta}} \sum_{k=1}^n \mathbb{E}\left[|X_k|^{2+\eta}\right] = 0, \tag{3.39}$$

then $s_n^{-1} T_n \xrightarrow{d} \mathcal{N}(0,1)$.

To apply Theorem 3.1 to $\mathscr{L}^l_{\sigma_j} = \sum_{k=1}^{m_{j-l}} X_k$, two ingredients are needed: the asymptotic behavior of the variance $s_{n,l}^2 := \sum_{k=1}^{m_{j-l}} \mathbb{V}[X_k]$, and a control on the $(2+\eta)$-th moments of the X_k. We address them in turn.

The asymptotic variance of $\mathscr{L}^l_{\sigma_j}$ is identified in the following proposition, proved in Koorevaar [15, Proposition 3.2].

Proposition 3.3 (Asymptotic variance of $\mathscr{L}^l_{\sigma_j}$ (Proposition 3.2, Koorevaar [15]))**.** *Let $\delta_{j-l} \in L^\eta(\mathbb{R})$ for some $\eta \geq 2$ and $l = 0, 1$. Then, as $U_j \to \infty$,*

$$s_{n,l}^2 = |w^1_{\sigma_j}(1)|^2 \, d_{j,j-l} \, \Delta_{j-l} \, U_j^{-4} \exp(A_{j-l} U_j^2), \tag{3.40}$$

where

$$d_{j,j-l} := 2 \, \|\delta_{j-l}\|^2_{L^2} \, (T_j - T_{j-1})^{-2} \, A_{j-l}^{-2} \exp(-2C_{j-l}), \tag{3.41}$$

and A_{j-l}, C_{j-l} are the constants introduced in equation (3.34)*.*

The proof proceeds by writing $\mathrm{Re}^2(z) = \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2)$, expanding each term using (3.34) and making use of the assumed L^2-integrability of δ_{j-l} together with the limit $\mathrm{sinc}^4(U_j \Delta_{j-l}/2) \to 1$, which holds since it is assumed that $U_j \Delta_{j-l} \to 0$. The full calculation is given in Koorevaar [15, Proposition 3.2].

It remains to verify (3.39) with the asymptotic variance from Proposition 3.3. Since $(\varepsilon_{j-l,k})$ are sub-Gaussian, the moment condition gives, for some constant $K > 0$,

$$\mathbb{E}\left[|\varepsilon_{j-l,k}|^{2+\eta}\right] \leq K^{2+\eta} (2+\eta)^{(2+\eta)/2}. \tag{3.42}$$

Using this moment bound together with the asymptotic analysis of the integral inside X_k (analogous to the one carried out for I_v in the proof of Proposition 3.3), one obtains the bound

$$\mathbb{E}\left[|X_k|^{2+\eta}\right] \lesssim U_j^{-2\eta} \, |\delta_{j-l,k}|^{2+\eta} \exp\!\big(A_{j-l} U_j^2 \cdot \eta/2\big) \, \Delta_{j-l}^{2+\eta}. \tag{3.43}$$

Summing over $k = 1, \dots, m_{j-l}$ and dividing by $s_{n,l}^{2+\eta}$ yields, after simplification,

$$\frac{1}{s_{n,l}^{2+\eta}} \sum_{k=1}^{m_{j-l}} \mathbb{E}\left[|X_k|^{2+\eta}\right] \to 0, \tag{3.44}$$

as $U_j \to \infty$, $\Delta_{j-l} \to 0$. Hence, by Theorem 3.1,

$$\frac{\mathscr{L}^l_{\sigma_j}}{s_{n,l}} \xrightarrow{d} \mathcal{N}(0,1), \qquad l = 0, 1. \tag{3.45}$$

The corresponding normality results for $\mathscr{L}^l_{\gamma_j}$ and $\mathscr{L}^l_{\lambda_j}$ are derived analogously. For the γ_j term, a different weight function $w^{U_j}_{\gamma_j} = w^1_{\gamma_j}/U_j^2$ (which is antisymmetric) is used, and for the λ_j terms the only difference is the weight function $w^{U_j}_{\lambda_j} = w^1_{\lambda_j}/U_j$. The asymptotic normality of the linear terms $\mathscr{L}^l_{\sigma_j}$, $\mathscr{L}^l_{\gamma_j}$, and $\mathscr{L}^l_{\lambda_j}$ is summarized in the following proposition, due to Koorevaar [15, Proposition 3.3].

Proposition 3.4 (Asymptotic Normality of the Linear Terms (Proposition 3.3 in Koorevaar [15]))**.** *Let $(\varepsilon_{j-l,k})$ be independent centered sub-Gaussian random variables with $\mathbb{V}[\varepsilon_{j-l,k}] = 1$ for all $k = 1, \dots, m_{j-l}$ and $l = 0, 1$, and let $\delta_{j-l} \in L^{2+\eta}(\mathbb{R})$ for some $\eta > 0$ and $l = 0, 1$. Furthermore, let the Lévy triplets*

$(\sigma_j, \gamma_j, \lambda_j)$ belong to $\mathcal{G}^n_{s_j}$ and the weight functions $(w^1_{\sigma_j}, w^1_{\gamma_j}, w^1_{\lambda_j})$ belong to $\mathcal{W}^n_{s_j}$. Then, as $U_j \to \infty$, for $l = 0, 1$ the following asymptotic normality results hold:

$$\frac{\mathscr{L}^l_{\sigma_j}}{s_{n,l}} \xrightarrow{d} \mathcal{N}(0,1), \qquad \frac{\mathscr{L}^l_{\gamma_j}}{U_j\, s_{n,l}} \xrightarrow{d} \mathcal{N}(0,1), \qquad \frac{\mathscr{L}^l_{\lambda_j}}{U_j^2\, s_{n,l}} \xrightarrow{d} \mathcal{N}(0,1),$$

where

$$s^2_{n,l} = |w^1_{\sigma_j}(1)|^2\, d_{j,j-l}\, \Delta_{j-l}\, U_j^{-4}\, e^{A_{j-l}U_j^2},$$

with constant

$$d_{j,j-l} := 2\,\|\delta_{j-l}\|^2_{L^2}\, (T_j - T_{j-1})^{-2}\, A^{-2}_{j-l}\, e^{-2C_{j-l}},$$

and A_{j-l}, C_{j-l} as in equation (3.34).

3.5.3. Negligibility of Bias & Remainder Terms

Having established the asymptotic normality of the linear terms, it remains to show that the bias and remainder terms in the error decomposition vanish faster than the standard deviation $s_{n,l}$ of the linear part. Both results are part of Koorevaar [15] and are stated here without proof. Sections 3.3.3 and 3.3.4 there contain the full derivations.

The bias terms $\mathscr{B}_{\sigma_j}, \mathscr{B}_{\gamma_j}, \mathscr{B}_{\lambda_j}$ are deterministic and decay as a power of U_j controlled by the smoothness index s_j of the Lévy triplet.

Proposition 3.5 (Bias decay, Koorevaar [15, Proposition 3.4]). *Suppose the Lévy triplets belong to $\mathcal{G}^n_{s_j}(R, \sigma_{\max})$ and the weight functions belong to $\mathcal{W}^n_{s_j}$. Then*

$$|\mathscr{B}_{\sigma_j}| \lesssim U_j^{-(s_j+3)}, \qquad |\mathscr{B}_{\gamma_j}| \lesssim U_j^{-(s_j+2)}, \qquad |\mathscr{B}_{\lambda_j}| \lesssim U_j^{-(s_j+1)}.$$

The remainder terms $\mathscr{R}^l_{\sigma_j}, \mathscr{R}^l_{\gamma_j}, \mathscr{R}^l_{\lambda_j}$ are stochastic with mean zero. Their negligibility relative to $s_{n,l}$ holds under an additional decay condition that couples the rate $U_j \to \infty$ with $\Delta_j \to 0$.

Proposition 3.6 (Remainder negligibility, Koorevaar [15, Proposition 3.5]). *Under the assumptions of Proposition 3.4, suppose further that as $U_j \to \infty$ and $\Delta_j \to 0$,*

$$\Delta_j U_j^4 \exp\left(U_j^2 \sum_{i=1}^{j}(T_i - T_{i-1})\sigma_i^2\right) \longrightarrow 0, \tag{3.46}$$

$$\frac{\Delta^2_{j-1}}{\Delta_j} U_j^4 \exp\left(U_j^2 \left(\sum_{i=1}^{j-1}(T_i - T_{i-1})\sigma_i^2 - (T_j - T_{j-1})\sigma_j^2\right)\right) \longrightarrow 0. \tag{3.47}$$

Then, for $l = 0, 1$,

$$\frac{\mathscr{R}^l_{\sigma_j}}{s_{n,l}} \xrightarrow{P} 0, \qquad \frac{\mathscr{R}^l_{\gamma_j}}{U_j\, s_{n,l}} \xrightarrow{P} 0, \qquad \frac{\mathscr{R}^l_{\lambda_j}}{U_j^2\, s_{n,l}} \xrightarrow{P} 0.$$

The decay conditions (3.46) and (3.47) are the standing assumptions under which the pointwise normality results below are obtained. Note that the rates of convergence for the remainder terms $\mathscr{R}^l_{\gamma_j}$ and $\mathscr{R}^l_{\lambda_j}$ are scaled by U_j and U_j^2 respectively, matching the rates of the corresponding linear terms from Proposition 3.4. Similarly, the same conditions imply, with respect to the bias terms, that

$$\frac{\mathscr{B}_{\sigma_j}}{s_{n,l}} \to 0, \qquad \frac{\mathscr{B}_{\gamma_j}}{U_j\, s_{n,l}} \to 0, \qquad \frac{\mathscr{B}_{\lambda_j}}{U_j^2\, s_{n,l}} \to 0,$$

provided the additional condition

$$U_j^{2(s_j+1)}\left(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1}U_j^2}\right) \longrightarrow \infty \tag{3.48}$$

holds, which ensures that the bias term decays fast enough. Together, Propositions 3.5 and 3.6, combined with Slutsky's theorem and Proposition 3.4, yield the pointwise asymptotic normality stated in the next subsubsection.

3.5.4. Final Normality Results

Combining the asymptotic normality of the linear terms (Proposition 3.4) with the negligibility of the bias and remainder terms (Propositions 3.5 and 3.6) via Slutsky's theorem yields the pointwise asymptotic normality of the spectral triplet estimators. Before stating the result, we briefly explain how the asymptotic variances $s_{n,0}^2, s_{n,1}^2$ of the two linear terms $\mathscr{L}^0_{\sigma_j^2}, \mathscr{L}^1_{\sigma_j^2}$ combine into a single normalization for the difference. Since the regression errors $(\varepsilon_{j-l,k})$ are independent across $l = 0, 1$, the linear terms $\mathscr{L}^0_{\sigma_j^2}$ and $\mathscr{L}^1_{\sigma_j^2}$ are independent too. Their difference $\mathscr{L}^0_{\sigma_j^2} - \mathscr{L}^1_{\sigma_j^2}$ is therefore asymptotically a sum of two independent centered normals with variances $s_{n,0}^2$ and $s_{n,1}^2$, such that:

$$\frac{\mathscr{L}^0_{\sigma_j^2} - \mathscr{L}^1_{\sigma_j^2}}{\sqrt{s_{n,0}^2 + s_{n,1}^2}} \xrightarrow{d} \mathcal{N}(0,1).$$

Define now the global standard deviation

$$s_n := \sqrt{s_{n,0}^2 + s_{n,1}^2},$$

which, by Proposition 3.4 can be written explicitly as:

$$s_n = |w^1_{\sigma_j}(1)|\, U_j^{-2} \sqrt{d_{j,j}\, \Delta_j\, e^{A_j U_j^2} + d_{j,j-1}\, \Delta_{j-1}\, e^{A_{j-1} U_j^2}}.$$

Following Koorevaar [15], it is convenient to absorb the U_j^{-2} and $|w^1_{\sigma_j}(1)|$ factors into a single rate Ξ_j:

$$\Xi_j := \frac{e^{-U_j^2 \sum_{i=1}^{j}(T_i - T_{i-1})\sigma_i^2/2}}{\sqrt{d_{j,j}\, \Delta_j + d_{j,j-1}\, \Delta_{j-1}\, e^{-U_j^2 (T_j - T_{j-1})\sigma_j^2}}}, \tag{3.49}$$

where the convention $d_{j,0} := 0$ is adopted, and

$$d_{j,j-l} := 2\, \|\delta_{j-l}\|_{L^2}^2\, \frac{e^{\sum_{i=1}^{j-l}(T_i - T_{i-1})(\sigma_i^2/2 + \gamma_i - \lambda_i)}}{(T_j - T_{j-1})^2 \left(\sum_{i=1}^{j-l}(T_i - T_{i-1})\sigma_i^2\right)^2}.$$

Hence, the scaling for $\widetilde{\sigma}_j^2 - \sigma_j^2$ becomes $U_j^2 \Xi_j$. As for $\widetilde{\gamma}_j - \gamma_j$ and $\widetilde{\lambda}_j - \lambda_j$, they are recovered from the combined parameterizations $\widehat{\gamma}_j$ and $\widehat{\lambda}_j$ by subtracting the $\widetilde{\sigma}_j^2$ term, which decays more rapidly than the other two. This is explained in detail in detail in Koorevaar [15]. Their scalings are $U_j \Xi_j$ and Ξ_j, respectively.

All of these results can be summarized into the following theorem:

Theorem 3.2 (Pointwise normality of the triplet (Part of Theorem 3.2 in Koorevaar [15])). *Let $(\varepsilon_{j-l,k})$ be independent centered sub-Gaussian random variables with $\mathbb{V}[\varepsilon_{j-l,k}] = 1$ for all $k = 1, \dots, m_j$ and $l = 0, 1$, and let $\delta_{j-l} \in L^{2+\eta}(\mathbb{R}) \cap C^0(\mathbb{R})$ for some $\eta > 0$ and $l = 0, 1$ with $\Delta_j \|\delta_{j-l}\|_{L^2}^2 \le \|\delta_{j-l}\|_\infty^2$. Furthermore, let the Lévy triplets $(\sigma_j, \gamma_j, \lambda_j)$ belong to $\mathcal{G}^n_{s_j}$ and the weight functions $(w^1_{\sigma_j}, w^1_{\gamma_j}, w^1_{\lambda_j})$ belong to $\mathcal{W}^n_{s_j}$. Control $U_j \to \infty$ and $\Delta_j \to 0$ such that, for $j = 1, \dots, n$,*

$$\Delta_j U_j^4 \log U_j\, e^{U_j^2 \sum_{i=1}^{j}(T_i - T_{i-1})\sigma_i^2} \longrightarrow 0, \tag{3.50}$$

and for $j = 2, \dots, n$,

$$\frac{\Delta_{j-1}^2}{\Delta_j} U_j^4\, e^{U_j^2 \left(\sum_{i=1}^{j-1}(T_i - T_{i-1})\sigma_i^2 - (T_j - T_{j-1})\sigma_j^2\right)} \longrightarrow 0, \tag{3.51}$$

$$U_j^{2(s_j+1)} \left(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\right) \longrightarrow \infty, \quad s_j \ge 2. \tag{3.52}$$

Then, for all $j = 1, \dots, n$, *the following pointwise convergences hold:*

$$U_j^2\, \Xi_j\, (\widetilde{\sigma}_j^2 - \sigma_j^2) \xrightarrow{d} |w_{\sigma_j}^1(1)|\, Z_1,$$
$$U_j\, \Xi_j\, (\widetilde{\gamma}_j - \gamma_j) \xrightarrow{d} |w_{\gamma_j}^1(1)|\, Z_2,$$
$$\Xi_j\, (\widetilde{\lambda}_j - \lambda_j) \xrightarrow{d} |w_{\lambda_j}^1(1)|\, Z_3,$$

where $Z_1, Z_2, Z_3 \sim \mathcal{N}(0,1)$, *and* Ξ_j *is as in* (3.49).

The pointwise normality result for the spectral estimator $\widetilde{\mu}_j(x)$ of the exponentially tilted Lévy measure is obtained through a separate procedure and is treated in the next section.

3.6. Pointwise Normality Result for $\widetilde{\mu}_j(x)$

This section displays the main pointwise results for the asymptotic normality of $\widetilde{\mu}_j(x) = e^x\, \widetilde{\nu}_j(x)$, the exponentially tilted Lévy measure. As Koorevaar [15] argues, it is more straightforward to obtain normality results for this exponentially tilted version due to the nature of the estimation procedure. If $\widetilde{\nu}_j(x)$ were to be estimated, the calibration function for this Lévy density would be $\psi_{\nu_j}(v) = \psi_j(v+i)$ resulting in a Fourier transform $\mathscr{F} b_{j-l,k}$ shifted by i in its argument. The resulting e^{-x} term makes the results harder to interpret in comparison to the triplet $(\sigma_j^2, \gamma_j, \lambda_j)$, and the exponential tilting can be undone after a normality result has been obtained for $\widetilde{\mu}_j(x)$, to obtain a corresponding result for $\widetilde{\nu}_j(x)$, as multiplication by a negative exponential is a bounded, linear transformation.

3.6.1. Error Decomposition of $\widetilde{\mu}_j(x)$

Analogously to the triplet $(\sigma_j^2, \gamma_j, \lambda_j)$, the error of the exponentially tilted Lévy measure is decomposed into a leading, linear term, a bias term and a remainder. These last two are shown to decay to 0 faster than the linear term under the appropriate scaling. That term is in turn normally distributed. Recall now equation (3.19):

$$\tilde{\mu}_j(x) = \mathscr{F}^{-1}\left[\left(\tilde{\psi}_j(\cdot) + \frac{\tilde{\sigma}_j^2}{2}(\cdot - i)^2 - i\tilde{\gamma}_j(\cdot - i) + \tilde{\lambda}_j\right) w_{\mu_j}^{U_j}(\cdot)\right](x), \quad x \in \mathbb{R}.$$

This means that the following error decomposition can be carried out:

$$\begin{aligned}\tilde{\mu}_j(x) - \mu_j(x) = \mathscr{F}^{-1}&\left[\left(\tilde{\psi}_j(\cdot) + \frac{\tilde{\sigma}_j^2}{2}(\cdot - i)^2 - i\tilde{\gamma}_j(\cdot - i) + \tilde{\lambda}_j\right) w_{\mu_j}^{U_j}(\cdot)\right](x)\\ - \mathscr{F}^{-1}&\left[\left(\psi_j(\cdot) + \frac{\sigma_j^2}{2}(\cdot - i)^2 - i\gamma_j(\cdot - i) + \lambda_j\right) w_{\mu_j}^{U_j}(\cdot)\right](x). \end{aligned} \tag{3.53}$$

Recall now the definition of inverse Fourier transform, with the convention used in this thesis:

$$\mathscr{F}^{-1} f(x) = \frac{1}{2\pi}\int_{\mathbb{R}} f(v)\, e^{-ivx}\, \mathrm{d}v$$

and the fact that the weight function $w_{\mu_j}^{U_j}(v)$ is 0 outside the region $[-U_j, U_j]$. Let us furthermore assume the weight function is continuous on said interval. Then, with the linearity of $\mathscr{F}^{-1}$ the term $\tilde{\mu}_j(x) - \mu_j(x)$ can be written as

$$\begin{aligned}\tilde{\mu}_j(x) - \mu_j(x) = \frac{1}{2\pi}\Bigg[&\int_{-U_j}^{U_j} \big(\tilde{\psi}_j(v) - \psi_j(v)\big)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + \frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2}\int_{-U_j}^{U_j} (v-i)^2\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v\\ &- i(\tilde{\gamma}_j - \gamma_j)\int_{-U_j}^{U_j} (v-i)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + (\tilde{\lambda}_j - \lambda_j)\int_{-U_j}^{U_j} w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v\\ &+ \int_{\mathbb{R}\setminus[-U_j,\, U_j]} \left(\psi_j(v) + \frac{\sigma_j^2}{2}(v-i)^2 - i\gamma_j(v-i) + \lambda_j\right)\big(1 - w_{\mu_j}^{U_j}(v)\big)\, e^{-ivx}\, \mathrm{d}v\Bigg]\\ =:&\ \Psi + \Sigma + \Gamma + \Lambda + \mathscr{B}. \end{aligned} \tag{3.54}$$

The terms Σ, Γ, Λ are well-defined, and, by Section 3.5, we know they are asymptotically normal. Furthermore, it is shown in Koorevaar [15] that they converge quicker than Ψ, so we need only focus on this last term.

The Ψ term can now be decomposed in a linear and remainder parts:

$$\begin{aligned} 2\pi\Psi &= \int_{-U_j}^{U_j} \left[\mathcal{L}_j^0(v) - \mathcal{L}_j^1(v) + \mathcal{R}_j^0(v) - \mathcal{R}_j^1(v)\right] w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\ &=: \mathscr{L}_{\mu_j}^0 - \mathscr{L}_{\mu_j}^1 + \mathscr{R}_{\mu_j}^0 - \mathscr{R}_{\mu_j}^1. \end{aligned} \tag{3.55}$$

where, for $l \in \{0,1\}$ we have that:

$$\mathscr{L}_{\mu_j}^l := \frac{1}{\pi}\frac{U_j}{T_j - T_{j-1}} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\, \varepsilon_{j-l,k}\, \Re\left(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathscr{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\, e^{-ivU_jx}\, w_{\mu_j}^1(v)\right) \mathrm{d}v. \tag{3.56}$$

and, for $k = 1, \ldots, m_{j-l}$, we define the random variables

$$X_{U_j,k}(x) := \frac{1}{\pi}\frac{U_j}{T_j - T_{j-1}} \delta_{j-l,k}\, \varepsilon_{j-l,k}\, \Re\left(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathscr{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\, e^{-ivU_jx}\, w_{\mu_j}^1(v)\, \mathrm{d}v\right). \tag{3.57}$$

It must be recalled that $\varepsilon_{j,k}$ are sub-Gaussian random variables with unit variance, which means that the X_k follow this distribution too. Similarly to the $(\sigma_j^2, \gamma_j, \lambda_j)$ case, the Lyapunov CLT can be used to conclude that

$$\frac{\mathscr{L}_{\mu_j}^{\,l}}{U_j^3 s_{n,l}} \xrightarrow{d} \mathcal{N}(0,1), \qquad \text{for } l \in \{0,1\}$$

Furthermore, it is also shown in Koorevaar [15] that under the assumptions of equations (3.46), (3.47) and (3.48), the bias and remainder terms of $\widetilde{\mu}_j(x) - \mu_j(x)$ decay to 0 too. Hence, the following asymptotic normality result can be obtained:

$$\frac{1}{U_j}\,\Xi_j\Psi \xrightarrow{d} \frac{1}{2\pi}\left|w_{\mu_j}^1(1)\right| Z_4, \qquad \text{where } Z_4 \sim \mathcal{N}(0,1). \tag{3.58}$$

where Ξ_j is the same as in equation (3.49). Given that the remaining factors in the error decomposition $(\Sigma, \Gamma, \Lambda)$ vanish more quickly than the linear part of Ψ, the pointwise asymptotic normality result for $\mu_j(x)$ can be expressed in the following theorem:

Theorem 3.3 (Pointwise Normality of $\widetilde{\mu}_j(x)$ (Part of Theorem 3.2 in Koorevaar [15])**.** *Let $(\varepsilon_{j-l,k})$ be independent centered sub-Gaussian random variables with $\mathbb{V}[\varepsilon_{j-l,k}] = 1$ for all $k = 1, \ldots, m_j$ and $l = 0, 1$, and let $\delta_{j-l} \in L^{2+\eta}(\mathbb{R}) \cap C^\infty(\mathbb{R})$ for some $\eta > 0$ and $l = 0, 1$ with $\Delta_j \|\delta_{j-l}\|_{L^2}^2 \le \|\delta_{j-l}\|_\infty^2$. Furthermore, let the Lévy triplets $(\sigma_j, \gamma_j, \lambda_j)$ belong to $\mathcal{G}_{s_j}^n$ and the weight functions $(w_{\sigma_j}^1, w_{\gamma_j}^1, w_{\lambda_j}^1)$ belong to $\mathcal{W}_{s_j}^n$. Control $U_j \to \infty$ and $\Delta_j \to 0$ such that, for $j = 1, \ldots, n$,*

$$\Delta_j U_j^4 \log U_j\, e^{U_j^2 \sum_{i=1}^j (T_i - T_{i-1})\sigma_i^2} \longrightarrow 0, \tag{3.59}$$

and for $j = 2, \ldots, n$,

$$\frac{\Delta_{j-1}^2}{\Delta_j} U_j^4\, e^{U_j^2\left(\sum_{i=1}^{j-1}(T_i - T_{i-1})\sigma_i^2 - (T_j - T_{j-1})\sigma_j^2\right)} \longrightarrow 0, \tag{3.60}$$

$$U_j^{2(s_j+1)}\left(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\right) \longrightarrow \infty, \quad s_j \ge 2. \tag{3.61}$$

Then, for all $j = 1, \ldots, n$, the following pointwise convergences hold:

$$\frac{1}{U_j}\Xi_j\left(\widetilde{\mu}_j(x) - \mu_j(x)\right) \xrightarrow{d} |w_{\mu_j}^1(1)|\, Z_4,$$

where $Z_4 \sim \mathcal{N}(0,1)$, and Ξ_j is as in (3.49).

Chapter 4
Pointwise Derivation of a Candidate Covariance Kernel

As previously stated, one of the goals of this thesis is to lift Koorevaar's pointwise central limit theorem for $\widetilde{\mu}_j(x) - \mu_j(x)$ to a functional statement in the Hilbert space $L^2(K)$, in order to be able to use such a result for financial applications. As a first step, this chapter derives the covariance structure of the linear part of the estimation error and identifies a candidate covariance kernel. We show that the rescaled kernel converges pointwise to the oscillatory limit $2\mathcal{C}\cos(U_j(x-z))$, and that this limit fails to be square-integrable on $\mathbb{R}^2$. Consequently the associated covariance operator is not Hilbert-Schmidt, let alone nuclear, on $L^2(\mathbb{R})$. This is the central structural obstruction of the thesis, and it is what forces the restriction to a compact domain $K \subset \mathbb{R}$ in the chapters that follow.

4.1. Covariance Structure of the Error of $\widetilde{\mu}_j(x)$

Recall the error decomposition of $\widetilde{\mu}_j(x)$, equation (3.54) in the previous section, by which it is known that $\tilde{\mu}_j(x) - \mu_j(x)$ can be written as:

$$\begin{aligned}
\tilde{\mu}_j(x) - \mu_j(x) = \frac{1}{2\pi}\Bigg[&\int_{-U_j}^{U_j} \big(\tilde{\psi}_j(v) - \psi_j(v)\big)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + \frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2} \int_{-U_j}^{U_j} (v-i)^2\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\
&- i(\tilde{\gamma}_j - \gamma_j) \int_{-U_j}^{U_j} (v-i)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + (\tilde{\lambda}_j - \lambda_j) \int_{-U_j}^{U_j} w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\
&+ \int_{\mathbb{R}\setminus[-U_j,\,U_j]} \left(\psi_j(v) + \frac{\sigma_j^2}{2}(v-i)^2 - i\gamma_j(v-i) + \lambda_j \right) \big(1 - w_{\mu_j}^{U_j}(v)\big)\, e^{-ivx}\, \mathrm{d}v \Bigg] \\
=:\ & \Psi + \Sigma + \Gamma + \Lambda + \mathscr{B}.
\end{aligned}$$

Let us now analyze the Ψ term more closely. By Koorevaar, we know that

$$\begin{aligned}
2\pi\Psi &= \int_{-U_j}^{U_j} \left[\mathcal{L}_j^0(v) - \mathcal{L}_j^1(v) + \mathcal{R}_j^0(v) - \mathcal{R}_j^1(v)\right] w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\
&=: \mathscr{L}_{\mu_j}^0 - \mathscr{L}_{\mu_j}^1 + \mathscr{R}_{\mu_j}^0 - \mathscr{R}_{\mu_j}^1.
\end{aligned}$$

where, for $l \in \{0,1\}$ we have that:

$$\mathscr{L}_{\mu_j}^l := \frac{1}{\pi} \frac{U_j}{T_j - T_{j-1}} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\, \varepsilon_{j-l,k}\, \Re\left(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathscr{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\, e^{-ivU_jx}\, w_{\mu_j}^1(v) \right) \mathrm{d}v,$$

where, for $k = 1, \dots, m_{j-l}$, we define the random variables

$$X_{U_j,k}^l(x)' := \frac{1}{\pi} \frac{U_j}{T_j - T_{j-1}} \delta_{j-l,k}\, \varepsilon_{j-l,k}\, \mathrm{Re}\left(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathscr{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\, e^{-ivU_jx}\, w_{\mu_j}^1(v)\, \mathrm{d}v \right),$$

where it must be recalled from Koorevaar that $\varepsilon_{j,k}$ are sub-Gaussian random variables with unit variance. While Koorevaar obtained a pointwise Central Limit Theorem for the error of the Lévy measure

$$\widetilde{\mu}_j(x) - \mu_j(x)$$

for each x, our goal here is to lift this CLT to a functional form in a Hilbert Space. It will be shown that the oscillatory covariance structure does not define a trace-class operator on $L^2(\mathbb{R})$ and thus convergence in said space cannot be expected. We will therefore work in the space $L^2(K;\mathbb{R})$, with $K \subset \mathbb{R}$ compact.

To obtain a functional expression, the pointwise error must be leveraged to find the covariance structure of the error, thus retrieving a suitable covariance kernel. In Koorevaar, normality is shown for the linear parts $\mathscr{L}^l_{\mu_j}$ (for $l = 0, 1$) of the error decomposition. Thus, we will begin by analyzing the covariance structure of this object, with the objective of obtaining a $L^2(K)$ CLT for said linear parts. Later on, it will be shown that with the right normalization the rest of the terms in the decomposition are negligible and thus there is functional normality for the entire error decomposition.

To find a suitable infinite-dimensional covariance structure, one must look at the object:

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\Big(X^l_{U_j,k}(x) X^l_{U_j,k}(z)\Big).$$

4.2. Covariance Candidate Kernel

Theorem 4.1 (Candidate Covariance Kernel)**.** *Take the assumptions in section 3.4. For fixed $x, z \in \mathbb{R}$ and $l \in \{0,1\}$, as $U_j \to \infty$ and $\Delta_j \to 0$ subject to*

$$\Delta_j\, U_j^4\, \exp\Big(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2\Big) \longrightarrow 0,$$

we have

$$\lim_{U_j\to\infty}\left[\frac{1}{\pi^2(T_j - T_{j-1})^2}\,\Delta_{j-l}^{-1}\,U_j^{-2}\,e^{-A_{j-l}U_j^2}\sum_{k=1}^{m_{j-l}}\mathbb{E}\big[X^l_{U_j,k}(x)\,X^l_{U_j,k}(z)\big] - \mathcal{C}\big(e^{-iU_j(x-z)} + e^{-iU_j(z-x)}\big)\right] = 0,$$

where $\mathcal{C} := A_{j-l}^{-2}\, e^{-2C_{j-l}}\, w^1_{\mu_j}(1)^2\, \|\delta_{j-l}\|^2_{L^2} > 0$ (A_{j-l} and C_{j-l} are as in (3.14) in Koorevaar).

4.3. Proof

Note that:

- $\mathbb{E}(X^l_{U_j,k}) = 0$.
- $\mathrm{Cov}(X, Z) = \mathbb{E}(XZ) - \mathbb{E}(X)\mathbb{E}(Z)$.
- $\mathrm{Cov}(X^l_{U_j,k}(x), X^l_{U_j,k'}(z)) = 0$ for $k' \neq k$.

Define

$$\mathcal{K}_{k,j}(x,z) := \mathbb{E}\big(X^l_{U_j,k}(x)' X^l_{U_j,k}(z)'\big). \tag{4.1}$$

We will show convergence in norm of

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\big(X^l_{U_j,k}(x) X^l_{U_j,k}(z)\big).$$

To begin the proof, recall the explicit form of $X^l_{U_j,k}(x)$:

$$X^l_{U_j,k}(x)' = \frac{1}{\pi}\frac{U_j}{T_j - T_{j-1}}\,\delta_{j-l,k}\,\varepsilon_{j-l,k}\Re\left(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathscr{F}_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\,w^1_{\mu_j}(v)\,e^{-ivU_jx}\,dv\right).$$

Analogously,

$$X^l_{U_j,k}(z) = \frac{1}{\pi}\frac{U_j}{T_j - T_{j-1}}\,\delta_{j-l,k}\,\varepsilon_{j-l,k}\Re\left(\int_0^1 \frac{iwU_j(1+iwU_j)\,\mathscr{F}_{j-l,k}(wU_j)}{\varphi_{T_{j-l}}(wU_j - i)}\,w^1_{\mu_j}(w)\,e^{-iwU_jz}\,dw\right).$$

Now, compute we compute $\mathbb{E}\big(X^l_{U_j,k}(x)'\, X^l_{U_j,k}(z)'\big)$. Using the explicit representation of $X^l_{U_j,k}$, we obtain

$$\mathcal{K}_{k,j}(x,z) = M^2\,\mathbb{E}\big(\varepsilon^2_{j-l,k}\big)\,\Re\Bigg(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathscr{F}_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j-i)}\,w^1_{\mu_j}(v)\,e^{-ivU_jx}\,dv\Bigg)$$
$$\cdot\,\Re\Bigg(\int_0^1 \frac{iwU_j(1+iwU_j)\,\mathscr{F}_{j-l,k}(wU_j)}{\varphi_{T_{j-l}}(wU_j-i)}\,w^1_{\mu_j}(w)\,e^{-iwU_jz}\,dw\Bigg),$$

where

$$M := \frac{1}{\pi}\frac{U_j}{T_j-T_{j-1}}.$$

We use the identity

$$\Re(z_1)\,\Re(z_2) = \left(\frac{z_1+\overline{z}_1}{2}\right)\left(\frac{z_2+\overline{z}_2}{2}\right).$$

Hence,

$$\mathbb{E}\big(X^l_{U_j,k}(x)'\,X^l_{U_j,k}(z)'\big) = \frac{1}{4\pi^2}\left(\frac{U_j}{T_j-T_{j-1}}\right)^2 \delta_{j-l,k}\,\delta_{j-l,k}\,\big(I_v+\overline{I}_v\big)\big(I_w+\overline{I}_w\big),$$

where

$$\overline{I}_w(z) := \overline{\int_0^1 \frac{iwU_j(1+iwU_j)\,\mathscr{F}_{j-l,k}(wU_j)}{\varphi_{T_{j-l}}(wU_j-i)}\,w^1_{\mu_j}(w)\,e^{-iwU_jz}\,dw},$$

and analogously I_v is defined with x in place of z.

Summing over k, we obtain

$$\sum_{k=1}^{m}\mathbb{E}\big(X^l_{U_j,k}(x)\,X^l_{U_j,k}(z)\big) = \left(\frac{U_j}{2\pi(T_j-T_{j-1})}\right)^2\sum_{k=1}^{m}\delta_{j-l,k}\,\delta_{j-l,k}\,\big(I_v+\overline{I}_v\big)\big(I_w+\overline{I}_w\big).$$

We expand

$$(I_v+\overline{I}_v)(I_w+\overline{I}_w) = I_vI_w + I_v\overline{I}_w + \overline{I}_vI_w + \overline{I}_v\,\overline{I}_w.$$

Note that the integrals can be rewritten as:

$$I_v(x) = \int_0^1 \frac{ivU_j(1+ivU_j)\,\mathscr{F}b_{j-l,k}(vU_j)\,e^{-ivU_jx}\,w^1_{\mu_j}(v)}{\exp\{-v^2U_j^2A_{j-l}/2 + ivU_jB_{j-l} + C_{j-l} + D_{j-l}(vU_j)\}}\,dv,$$
$$\overline{I}_v(x) = \int_0^1 \frac{-ivU_j(1-ivU_j)\,\overline{\mathscr{F}}b_{j-l,k}(vU_j)\,e^{ivU_jx}\,w^1_{\mu_j}(v)}{\exp\{-v^2U_j^2A_{j-l}/2 - ivU_jB_{j-l} + C_{j-l} + D_{j-l}(vU_j)\}}\,dv.$$

Consequently,

$$I_v(x)I_w(z) = \int_0^1\int_0^1 \frac{ivU_j(1+ivU_j)\,iwU_j(1+iwU_j)\,\mathscr{F}b_{j-l,k}(vU_j)\,\mathscr{F}b_{j-l,k}(wU_j)}{\exp\{-(v^2+w^2)U_j^2A_{j-l}/2 + i(v+w)U_jB_{j-l} + 2C_{j-l} + D_{j-l}(vU_j+wU_j)\}}$$
$$\cdot\, e^{-ivU_jx}\,e^{-iwU_jz}\,w^1_{\mu_j}(v)\,w^1_{\mu_j}(w)\,dv\,dw.$$

Let us now define

$$g(v,w) := \frac{\mathscr{F}b_{j-l,k}(vU_j)\,\mathscr{F}b_{j-l,k}(wU_j)\,e^{-ivU_jx}\,e^{-iwU_jz}\,w^1_{\mu_j}(v)\,w^1_{\mu_j}(w)}{\exp\{i(v+w)U_jB_{j-l} + D_{j-l}(vU_j+wU_j)\}}.$$

Then, just like in Koorevaar,

$$I_v(x)I_w(z) = -U_j^2 \begin{pmatrix} \exp\{-2C_{j-l}\} \iint vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} g(v,w)\,dv\,dw \\ +\ iU_j \iint vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} (v+w)\,g(v,w)\,dv\,dw \\ -\ U_j^2 \iint vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} vw\,g(v,w)\,dv\,dw \end{pmatrix}.$$

Next: $I_v(x)\cdot \overline{I}_w(z)$ and $\overline{I}_v(x)\cdot I_w(z)$. We have

$$I_v(x)\,\overline{I_w(z)} = -U_j^2 \begin{pmatrix} \exp\{-2C_{j-l}\} \iint vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} g(v,-w)\,dv\,dw \\ +\ iU_j \iint vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} (v-w)\,g(v,-w)\,dv\,dw \\ -\ U_j^2 \iint vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} vw\,g(v,-w)\,dv\,dw \end{pmatrix}.$$

The term $\overline{I}_v(x)\cdot I_w(z)$ is computed symmetrically. Let us now consider:

$$\overline{I}_v(x)\cdot \overline{I}_w(z).$$

We obtain

$$\overline{I_v(x)}\,\overline{I_w(z)} = -U_j^2 \begin{pmatrix} \exp\{-2C_{j-l}\} \int_0^1\!\int_0^1 vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} g(-v,-w)\,dv\,dw \\ +\ iU_j \int_0^1\!\int_0^1 vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} (-v-w)\,g(-v,-w)\,dv\,dw \\ -\ U_j^2 \int_0^1\!\int_0^1 vw \exp\left\{A_{j-l}(v^2+w^2)\dfrac{U_j^2}{2}\right\} vw\,g(-v,-w)\,dv\,dw \end{pmatrix}.$$

Now, the following Lemma, due to Koorevaar [15] must be applied:

Lemma 4.1 (Lemma 3.3, Koorevaar [15])**.** *Let $g_U(v,w)$ be a bounded function on the unit square such that*

$$0 \le |g_U(v,w)| \le C, \qquad (v,w)\in[0,1]^2.$$

Let $h(x)$ be a function satisfying $h(x)\downarrow 0$ as $x\to\infty$, and let $f_U(v,w)$ be a positive function such that

$$\lim_{U\to\infty}\int_0^1\int_0^1 f_U(v,w)\,dv\,dw = 1$$

and

$$\lim_{U\to\infty}\int_{1-h(U)}^1\int_{1-h(U)}^1 f_U(v,w)\,dv\,dw = 1.$$

If the function $g_U(v,w)$ satisfies

$$\lim_{U\to\infty}\ \sup_{(v,w)\in[1-h(U),\,1]^2} \big|g_U(v,w)-g_U(1,1)\big| = 0,$$

then

$$\lim_{U\to\infty}\int_0^1\int_0^1 f_U(v,w)\,g_U(v,w)\,dv\,dw = \lim_{U\to\infty} g_U(1,1).$$

The Fourier transform $\mathscr{F}b_{j,k}(v)$ defined in equation 3.24 means that $\lim_{U_j\to\infty} g(v,w)$ is not necessarily finite. Hence, we need modified expressions of $g(v,w)$ to be able to apply Lemma 4.1. For $I_v(x)I_w(z)$ and $\overline{I_v(x)}\,\overline{I_w(z)}$, the modified functions $\tilde{g}(v,w)$ will be defined as

$$\tilde{g}(v,w) := vw \exp\{iU_j\,(2B_{j-l}+x+z)\}\,|\mathscr{F}b_{j,k}(U_j)|^{-2}\,g(v,w) \tag{4.2}$$

and

$$\tilde{g}(v,w) := vw \exp\{-iU_j\,(2B_{j-l}+x+z)\}\,|\mathscr{F}b_{j,k}(U_j)|^{-2}\,g(v,w) \tag{4.3}$$

respectively. For $I_v(x)\,\overline{I_w(z)}$ however, because $x \neq z$, the new expression is the following:

$$\tilde{g}(v,w) = vw \exp\{iU_j(x-z)\}\,|\mathscr{F}b_{j,k}(U_j)|^{-2}\,g(v,w) \tag{4.4}$$

The expression for $\overline{I_v(x)}\,I_w(z)$ is symmetric. With this new addition, the limit

$$\lim_{U_j\to\infty} \tilde{g}(1,1)$$

exists and is finite. Hence, it suffices to verify the conditions on $\tilde{g}$ and to find functions f_{U_j} which converge to a Dirac delta function at $(1,1)$. We will carry out the verification for $\tilde{g}(v,w)$ as in equation 4.4, as this is the only case in which our setup differs from that of Koorevaar [15]. However, note that the other cases are analogous and this fully worked out there.

Rescaling the remaining factors in the integrals so that they are of the form required in Lemma 4.1 yields

$$f_{U_j}(v,w) := A_{j-l}^2 U_j^4 \exp\{-A_{j-l}U_j^2\}\; vw\, \exp\left\{A_{j-l}\frac{v^2+w^2}{2}U_j^2\right\} =: F(v)\,F(w).$$

The function $h(x)$ in Lemma 4.1 is chosen as

$$h(x) = x^{-3/2},$$

and it is immediate that $h(x) \downarrow 0$ as $x \to \infty$. We now verify the conditions of Lemma 3.3 for this specific function $f_{U_j}(v,w)$.

First,

$$\lim_{U_j\to\infty} \int_{1-U_j^{-3/2}}^{1} \int_{1-U_j^{-3/2}}^{1} f_{U_j}(v,w)\,dv\,dw = \lim_{U_j\to\infty} \left(\int_{1-U_j^{-3/2}}^{1} F(v)\,dv\right)^2 = 1.$$

Moreover,

$$\lim_{U_j\to\infty} \int_0^1 \int_0^1 f_{U_j}(v,w)\,dv\,dw = \lim_{U_j\to\infty} \left(\int_0^1 F(v)\,dv\right)^2 = 1.$$

Here we used that, for $b(U_j) \in [0,1]$,

$$\begin{aligned}
\int_{b(U_j)}^1 F(v)\,dv &= \exp\left\{-\frac{A_{j-l}U_j^2}{2}\right\} \int_{b(U_j)}^1 vA_{j-l}U_j^2 \exp\left\{\frac{A_{j-l}v^2U_j^2}{2}\right\} dv \\
&= \exp\left\{-\frac{A_{j-l}U_j^2}{2}\right\} \left[\exp\left\{\frac{A_{j-l}v^2U_j^2}{2}\right\}\right]_{b(U_j)}^1 \\
&= 1 - \exp\left\{-\frac{A_{j-l}U_j^2}{2}\left(1-b(U_j)^2\right)\right\}
\end{aligned}$$

Thus, the function f_{U_j} satisfies the required conditions. It remains to verify the boundedness of $\tilde{g}_{U_j}$ on the unit square and to check that

$$\lim_{U_j\to\infty} \sup_{(u,v)\in[1-U_j^{-3/2},\,1]^2} \left|\tilde{g}_{U_j}(u,v) - \tilde{g}_{U_j}(1,1)\right| = 0.$$

The assumption is made that $U_j > c$ for some positive c. For ease of notation, write

$$\tilde{g}_1(v,w) = vw, \qquad \tilde{g}_2(v,w) = \frac{\mathscr{F}b_{j-l,k}(vU_j)\,\mathscr{F}b_{j-l,k}(wU_j)}{\mathscr{F}b_{j-l,k}(U_j)^2}.$$

$$\tilde{g}_3(v,w) = w_j^1(v)\,w_j^1(w), \qquad \tilde{g}_4(v,w) = \exp\{(1-v)iU_j(B_{j-l}+x)\}\cdot\exp\{(1-w)iU_j(-B_{j-l}-z)\}$$

$$\tilde{g}_5(v,w) = \exp\{-D_{j-l}(vU_j) - D_{j-l}(wU_j)\}.$$

As shown in Koorevaar [15], $\tilde{g}_1(v,w)$, $\tilde{g}_2(v,w)$, $\tilde{g}_3(v,w)$, $\tilde{g}_4(v,w)$ are uniformly bounded in the unit square. As D_{j-l} is a bounded function $\tilde{g}_5(v,w)$ is bounded uniformly. We observe that the product $\tilde{g}_1\tilde{g}_3$ is continuous at $(1,1)$. In addition, the second component of $\tilde{g}_2$ satisfies the required regularity conditions, as is shown in Appendix B.3 of Koorevaar [15]. Consequently, these terms can be taken outside the integral. Furthermore, $\tilde{g}_5$ converges uniformly to 1 as $U_j \to \infty$, which follows from the smoothness of $\mu_j(x)$. Similarly, the first part of $\tilde{g}_2(v,w)$ can be controlled, which means the only condition left to check is:

$$\begin{aligned}
&\sup_{(v,w)\in\left[1-U_j^{-3/2},\,1\right]^2} \left|\tilde{g}_4(v,w) - 1\right| = \\
&= \left|\exp\left\{i\big(1-U_j^{-3/2}\big)U_j(B_{j-l}+x)\right\}\cdot\exp\left\{i\big(1-U_j^{-3/2}\big)U_j(B_{j-l}+z)\right\} - 1\right| \\
&= \left|\exp\left\{i\,U_j^{-1/2}(B_{j-l}+x)\right\}\cdot\exp\left\{i\,U_j^{-1/2}(-B_{j-l}-z)\right\} - \exp(i\cdot 0)\right| \\
&\le \left|U_j^{-1/2}(x-z)\right| \;\longrightarrow\; 0,
\end{aligned}$$

as $U_j \to \infty$.

Hence, Lemma 4.1 can be applied, by which all integrals in the final expressions for $I_v(x)I_w(z)$, $\overline{I_v(x)}\,\overline{I_w(z)}$, $I_v(x)\,\overline{I_w(z)}$ and $\overline{I_v(x)}\,I_w(z)$ vanish at the same asymptotic rate, with the leading contribution stemming from the final term, which is multiplied by U_j^4. Accordingly, the first two integrals for each of the terms are omitted in what follows. Using Lemma 4.1, the following limits are found:

We consider

$$I_v(x)I_w(z)\cdot\exp\{-A_{j-l}U_j^2\}\,\mathscr{F}b_{j-l,k}(U_j)^{-2}\cdot\exp\{iU_j(B_{j-l}+x)\}\cdot\exp\{iU_j(B_{j-l}+z)\}.$$

This can be written as

$$I_v(x)I_w(z) = \exp\{-2C_{j-l}\}\iint \tilde{f}\,\tilde{g}.$$

Taking limits yields

$$\lim_{U_j\to\infty}\exp\{-2C_{j-l}\}\iint \tilde{f}\,\tilde{g} = \exp\{-2C_{j-l}\}\lim_{U_j\to\infty}\tilde{g}(1,1).$$

Hence,

$$\exp\{-2C_{j-l}\}\,w_{\mu_j}^1(1)^2\lim_{U_j\to\infty}\exp\{-2D_{j-l}(U_j)\} = \exp\{-2C_{j-l}\}\,w_{\mu_j}^1(1)^2.$$

The procedure is analogous for the integral $\overline{I_v(x)}\,\overline{I_w(z)}$.

By the same procedure, one can now take:

$$\lim_{U_j\to\infty}\Big(I_v(x)\overline{I_w(z)}\,A_{j-l}^{-2}\,\exp\{-A_{j-l}U_j^2\}\,\left|\mathscr{F}b_{j-l,k}(U_j)\right|^{-2}\,\exp\{-iU_j(x-z)\}\Big) = \exp\{-2C_{j-l}\}\,w_{\mu_j}^1(1)^2.$$

The case $\overline{I_v(x)}I_w(z)$ is symmetric. Recalling the expression for $\mathcal{K}_{k,j}(x,z)$ in equation (4.1), the asymptotic covariance kernel will be found considering

$$\lim_{U_j\to\infty} \frac{\mathcal{K}_{k,j}(x,z)}{\Delta_{j-l}\, U_j^2\, A_{j-l}^{-2}\, \exp\{A_{j-l}\, U_j^2\}}$$
$$= \frac{1}{\pi\,(T_j - T_{j-1})} \lim_{U_j\to\infty} \Delta_{j-l}^{-1} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2\, A_{j-l}^2\, \exp\{-A_{j-l}\, U_j^2\}\, (I_v + \bar{I}_v)(I_w + \bar{I}_w)$$

Lemma 4.2. *Under the assumptions of section 3.4,*

$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2 A_{j-l}^2 e^{-A_{j-l}U_j^2}\, I_v(x) I_w(z)$$
$$= \lim_{U_j\to\infty} \Delta_{j-l}^{-1} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2 e^{-2C_{j-l}}\, w_{\mu_j}^1(1)^2 \left|\mathscr{F} b_{j-l,k}(U_j)\right|^2 e^{-iU_j(2B_{j-l}+x+z)}.$$

Koorevaar's [15] Lemma 3.4 handles $\mathcal{I}_v^2$ (equivalently, the $z = x$, same-index case) by: replacing I_v^2 by

$$e^{-2C_{j-l}}\, w_{\mu_j}^1(1)^2\, \mathscr{F} b_{j-l,k}(U_j)^2\, e^{-2iB_{j-l}U_j},$$

pulling k-independent factors outside the sum, and recognizing $\Delta_{j-l}^{-1}\sum_k \delta_{j-l,k}^2\, \mathscr{F} b_{j-l,k}(U_j)^2$ as a Riemann sum converging to:

$$\mathrm{sinc}^4(U_j\Delta_{j-l}/2)\, \mathscr{F}\delta_{j-l}^2(2U_j).$$

For $z \neq x$ the argument is identical for the non-cross terms $\mathcal{I}_v(x)\mathcal{I}_w(z)$ and $\overline{\mathcal{I}_v(x)}\,\overline{\mathcal{I}_w(z)}$: the leading term's phase becomes $e^{-iU_j(2B_{j-l}+x+z)}$, which is k-independent and can be pulled out, and the Riemann-sum step is unchanged. Both terms vanish by Riemann-Lebesgue applied to $\mathscr{F}\delta_{j-l}^2(2U_j)$.

$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2\, A_{j-l}^2\, e^{-A_{j-l}U_j^2}\, I_v(x) I_w(z) = 0.$$
$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2\, A_{j-l}^2\, e^{-A_{j-l}U_j^2}\, \overline{\mathcal{I}_v(x)}\,\overline{\mathcal{I}_w(z)} = 0.$$

Let us now prove an equivalent expression of Lemma 4.2 for the cross terms. For this expression, we have :

$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1}\Big(\sum_{k=1}^{m} \delta_{j-l,k}^2 A_{j-l}^2\, \exp\{-A_{j-l}\, U_j^2\}\, I_v(x)\overline{I_w(z)}$$
$$= \exp\{-2C_{j-l}\}\, w_j^1(1)^2 \lim_{U_j\to\infty} \sum_{k=1}^{m} \delta_{j-l,k}^2\, |\mathscr{F} b_{j-l,k}(U_j)^2|\, \exp\{-iU_j(x-z)\}\Big).$$

which can equivalently be written as:

$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1}\Big(\sum_{k=1}^{m} \delta_{j-l,k}^2 A_{j-l}^2\, \exp\{-A_{j-l}U_j^2\}\, I_v(x)\overline{I_w(z)}$$
$$- \sum_{k=1}^{m} \delta_{j-l,k}^2\, \exp\{-2C_{j-l}\}\, w_j^1(1)^2\, |\mathscr{F} b_{j-l,k}(U_j)^2|\, \exp\{-iU_j(x-z)\}\Big) = 0.$$

This equals

$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1}\Big(\sum_{k=1}^{m} \delta_{j-l,k}^2 A_{j-l}^2\, \exp\{-A_{j-l}^2U_j^2\}\, I_v(x)\overline{I_w(z)}$$
$$- \sum_{k=1}^{m} \delta_{j-l,k}^2\, \exp\{-2C_{j-l}\}\, w_j^1(1)^2\, \mathrm{sinc}^4(U_j\Delta_{j-l})\, \exp\{-iU_j(x-z)\}\Big) = 0$$

Now, define:

$$Q := \Delta_{j-l}^{-1} \left(\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2 A_{j-l}^2 \exp\{-A_{j-l} U_j^2\} \, I_v(x) \overline{I_w(z)} \right),$$

$$P := \Delta_{j-l}^{-1} \left(\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2 \exp\{-2C_{j-l}\} \, w_j^1(1)^2 \right),$$

$$\theta := x - z.$$

In the limiting expression we have the factor

$$\operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right),$$

whose limit equals 1 as $U_j \to \infty$.

Note:

$$\left| \exp\{-iU_j\theta\} \right| = 1.$$

Hence,

$$\lim_{U_j \to \infty} \left(Q - P \operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right) \exp\{-iU_j\theta\} \right) = 0.$$

Moreover,

$$\begin{aligned} &Q - P \exp\{-iU_j\theta\} = \\ &= \left(Q - \operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right) P \exp\{-iU_j\theta\} \right) + \left(\operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right) - 1 \right) P \exp\{-iU_j\theta\} \end{aligned}$$

Taking absolute values yields

$$\left| \left(\operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right) - 1 \right) P \exp\{-iU_j\theta\} \right| = |P| \left| 1 - \operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right) \right| \xrightarrow[U_j \to \infty]{} 0$$

Therefore,

$$\begin{aligned} \left| Q - P \exp\{-iU_j\theta\} \right| \le& \left| Q - \operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right) P \exp\{-iU_j\theta\} \right| \\ &+ \left| \left(\operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right) - 1 \right) P \exp\{-iU_j\theta\} \right| \\ &\longrightarrow 0 \quad \text{as } U_j \to \infty. \end{aligned}$$

Thus, we may ignore the factor

$$\operatorname{sinc}^4 \left(\frac{U_j \Delta_{j-l}}{2} \right)$$

in the limit. Thus

$$\begin{aligned} &\lim_{U_j \to \infty} \Delta_{j-l}^{-1} \Big(\sum_{k=1}^{m} \delta_{j-l,k}^2 A_{j-l}^2 \exp\{-A_{j-l}^2 U_j^2\} \, I_v(x) \overline{I_w(z)} \\ &- \sum_{k=1}^{m} \delta_{j-l,k}^2 \exp\{-2C_{j-l}\} \, w_j^1(1)^2 \, |\mathscr{F} b_{j-l,k}(U_j)^2| \exp\{-iU_j(x-z)\} \Big) = 0. \end{aligned}$$

And, equivalently:

$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1}\Big(\sum_{k=1}^{m} \delta_{j-l,k}^2 A_{j-l}^2 \exp\{-A_{j-l}^2 U_j^2\}\, I_v(x)\overline{I_w(z)}$$
$$-\sum_{k=1}^{m} \delta_{j-l,k}^2 \exp\{-2C_{j-l}\}\, w_j^1(1)^2 \exp\{-iU_j(x-z)\}\Big) = 0.$$

An equivalent step is carried out for the other cross term, and as the integrals not corresponding to the cross terms go to 0, we have that:

$$\lim_{U_j\to\infty} \Delta_{j-l}^{-1} A_{j-l}^2 \exp\{-A_{j-l}U_j^2\} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2 \Big(I_v(x)\overline{I_w(z)} + \overline{I_v(x)} I_w(z)\Big)$$
$$= \exp\{-2C_{j-l}\}\, w_{\mu_j}^1(1)^2\, \|\delta_{j-l}\|_{L^2}^2 \Big(\exp\{-iU_j(x-z)\} + \exp\{iU_j(x-z)\}\Big) + o(1).$$

Recalling now that

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\Big(X_{U_j,k}^l(x)' X_{U_j,k}^l(z)'\Big) = \frac{U_j^2}{4\pi^2(T_j-T_{j-1})^2} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2 \left(I_v + \overline{I_v}\right)\left(I_w + \overline{I_w}\right),$$

and that the leading contribution from each integral comes from the term multiplied by U_j^4 (by Lemma 1), while the non-cross terms vanish, we obtain

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\Big(X_{U_j,k}^l(x) X_{U_j,k}^l(z)\Big) = \frac{U_j^2}{4\pi^2(T_j-T_{j-1})^2} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2 \left(I_v(x)\overline{I_w(z)} + \overline{I_v(x)} I_w(z)\right) + o(\cdots).$$

Multiplying by $\frac{1}{\pi^2(T_j-T_{j-1})^2}\, \Delta_{j-l}^{-1}\, U_j^{-2} \exp\{-A_{j-l}U_j^2\}$ and using the asymptotic identification of the cross terms proved above, we conclude

$$\lim_{U_j\to\infty} \left(\frac{1}{\pi^2(T_j-T_{j-1})^2}\, \Delta_{j-l}^{-1}\, U_j^{-2} \exp\{-A_{j-l}U_j^2\} \sum_{k=1}^{m_{j-l}} \mathbb{E}\Big(X_{U_j,k}^l(x)' X_{U_j,k}^l(z)'\Big)\right.$$
$$\left. -\; A_{j-l}^{-2} \exp\{-2C_{j-l}\}\, w_{\mu_j}^1(1)^2\, \|\delta_{j-l}\|_{L^2}^2 \Big(\exp\{-iU_j(x-z)\} + \exp\{-iU_j(z-x)\}\Big)\right) = 0.$$

Thus, the theorem has been proved, with the normalizing prefactor

$$1/\sqrt{a_{j-l}} = \Delta_{j-l}^{-1/2}\, U_j^{-1} \exp\{-A_{j-l}U_j^2/2\}.$$

where

$$a_{j-l} := \Delta_{j-l}\, U_j^2 \exp\{A_{j-l}U_j^2\}$$

4.4. Analyzing Convergence in $L^2(K \times K)$ instead of in $L^2(\mathbb{R}^2)$

By Theorem 4.1, for every fixed $x, z \in \mathbb{R}$ the rescaled covariance kernel converges pointwise to

$$\mathcal{C}\Big(e^{-iU_j(x-z)} + e^{-iU_j(z-x)}\Big) = 2\mathcal{C}\cos\big(U_j(x-z)\big),$$

where $U_j \to \infty$ and

$$\mathcal{C} = A_{j-l}^{-2} \exp\{-2C_{j-l}\}\, w_{\mu_j}(1)^2\, \|\delta_{j-l}\|_{L^2}^2.$$

which is a finite constant. Note that A_{j-l} and C_{j-l} are defined as in equation (3.34). We show that this pointwise limit cannot belong to $L^2(\mathbb{R}^2)$, and therefore convergence in $L^2(\mathbb{R}^2)$ is impossible.

Consider the $L^2(\mathbb{R}^2)$-norm of the limiting kernel:

$$\|2\mathcal{C}\,\cos\big(U_j(x-z)\big)\|^2_{L^2(\mathbb{R}^2)} = 4\mathcal{C}^2 \int_{\mathbb{R}^2} \cos^2\big(U_j(x-z)\big)\,dx\,dz.$$

Introduce the change of variables

$$u = x - z, \qquad v = x + z,$$

whose Jacobian determinant satisfies $|J| = 1/2$. Then

$$\int_{\mathbb{R}^2} \cos^2\big(U_j(x-z)\big)\,dx\,dz = \frac{1}{2}\int_{\mathbb{R}} \cos^2(U_j u)\,du \int_{\mathbb{R}} dv.$$

Since

$$\int_{\mathbb{R}} dv = \infty,$$

it follows that

$$\|2\mathcal{C}\,\cos\big(U_j(x-z)\big)\|^2_{L^2(\mathbb{R}^2)} = \infty \qquad \text{for every } j.$$

Hence the limiting kernel does not belong to $L^2(\mathbb{R}^2)$. This, in turn, means that in this function space it is not Hilbert-Schmidt, and thus cannot be nuclear. As a result, Theorem 2.2 cannot be applied in $L^2(\mathbb{R})$, and we must look for a different space, where it is possible to achieve norm convergence. To that end, we limit the domain of the Lévy measure $\mu_j(x)$ to a compact set K s.t $x \in K \subset \mathbb{R}$, in the L^2 space. In this space,

$$\|2\mathcal{C}\,\cos\big(U_j(x-z)\big)\|^2_{L^2(K\times K)} = 4\mathcal{C}^2 \int_{K\times K} \cos^2\big(U_j(x-z)\big)\,dx\,dz \le 4\mathcal{C}^2|K|^2 < \infty$$

Restricting to K therefore resolves the integrability obstruction: the kernel is square-integrable, and the associated operator is Hilbert-Schmidt on $L^2(K)$. It does not, however, resolve the limit itself, since the kernel above still oscillates with U_j and does not converge as $U_j \to \infty$. Identifying a fixed, nuclear limiting operator requires removing these oscillations, which is achieved by cosine modulation, introduced in Chapter 5. There it will be shown that, under this modulation, all three conditions of Theorem 2.2 can be verified, including nuclearity of the covariance operator, yielding a CLT for the linear part of the Lévy density error.

Chapter 5
Central Limit Theorem of the Linear Part of $\tilde{\mu}_j - \mu_j$ in $L^2(K)$

In order to leverage the pointwise candidate covariance kernel for application in a functional setting, we need a statement that guarantees that a CLT in a functional space can be obtained. As the functional space we are working on in this thesis is $L^2(K)$, which is Hilbert, Theorem 2.2 is precisely such a statement.

This chapter will show that the linear part of the error decomposition of $\tilde{\mu}_j - \mu_j$ given by $\mathscr{L}^l_{\mu_j}$, for $l = 0, 1$, as per the error decomposition of equation (3.54) satisfies the three conditions of Theorem 2.2 when adequately scaled and modulated, and is thus asymptotically normal in the space $L^2(K)$.

In order to be able to leverage this theorem, it needs to be proved that our covariance formulation can be written in a way that is compatible with the theorem's statement:

Lemma 5.1 (Equivalence of Covariance Formulations)**.** *Let $U_j \geq 1$ and let $\{{X'_{U_j,k}}^l\}_{k=1}^{m_{j-l}}$ be random fields indexed by $x \in \mathbb{R}$. From now on, take $X^l_{U_j,k}(x) = \frac{1}{\sqrt{a_{j-l}}}{X'_{U_j,k}}^l(x) = \frac{1}{\sqrt{a_{j-l}}}\mathscr{L}^l_{\mu_j}$, where $a_{j-l} := \Delta_{j-l} \cdot U_j^2 \cdot \exp\{A_{j-l} \cdot U_j^2\}$. Fix $x, z \in \mathbb{R}$. Let $C \in \mathbb{R}$ be deterministic. The following two statements are equivalent:*

(A) *There exists a remainder function $r_{U_j} : \mathbb{R}^2 \to \mathbb{R}$ such that*

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[X^l_{U_j,k}(x)X^l_{U_j,k}(z)\Big] = \mathcal{C}\, e^{-iU_j(x-z)} + \mathcal{C}\, e^{-iU_j(z-x)} + r_{U_j}(x,z), \tag{5.1}$$

with

$$r_{U_j}(x,z) \xrightarrow[U_j\to\infty]{} 0 \quad \textit{for every fixed } (x,z) \in \mathbb{R}^2.$$

(B)

$$\lim_{U_j\to\infty}\left(\sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[X^l_{U_j,k}(x)X^l_{U_j,k}(z)\Big] - C\, e^{-iU_j(x-z)} - C\, e^{-iU_j(z-x)}\right) = 0. \tag{5.2}$$

for every fixed $(x,z) \in \mathbb{R}^2$.

Proof. We prove both implications.

(A) ⇒ (B). Assume *(A)* holds. By definition of $r_{U_j}(x,z)$ in (5.1), we have

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}[X^l_{U_j,k}(x)X^l_{U_j,k}(z)] - \mathcal{C}\, e^{-iU_j(x-z)} - \mathcal{C}\, e^{-iU_j(z-x)} = r_{U_j}(x,z).$$

Taking the limit $U_j \to \infty$ and using $r_{U_j}(x,z) \to 0$ yields (5.2).

(B) ⇒ (A). Assume *(B)* holds. Define

$$r_{U_j}(x,z) := \sum_{k=1}^{m_{j-l}} \mathbb{E}[X^l_{U_j,k}(x)X^l_{U_j,k}(z)] - \mathcal{C}\, e^{-iU_j(x-z)} - \mathcal{C}\, e^{-iU_j(z-x)}.$$

Then (5.1) holds by construction, and (5.2) implies $r_{U_j}(x,z) \to 0$ for every fixed (x,z). Thus (A) and (B) are equivalent. □

5.1. Theorem Statement

From Chapter 4 and the previous section, we have that

$$X^l_{U_j,k}(x)' = \mathscr{L}^l_{\mu_j} = \frac{1}{\pi}\frac{U_j}{T_j - T_{j-1}}\,\delta_{j-l,k}\,\varepsilon_{j-l,k}\,\Re\left(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathcal{F}_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\,w^1_{\mu_j}(v)\,e^{-ivU_jx}\,dv\right)$$
$$= \frac{1}{\pi}\frac{U_j}{T_j - T_{j-1}}\,\delta_{j-l,k}\,\varepsilon_{j-l,k}\,\Re\left(\int_0^1 h_{U_j,k}e^{-ivU_jx}\,dv\right).$$

Note that the real part of the integral is a bounded continuous function in x, so the whole random variable is bounded and continuous in x. Hence, $X^l_{U_j,k}(x)' \in L^2(K;\mathbb{R})$ for $K \subset \mathbb{R}$ compact, for every fixed k and U_j.

To prove the asymptotic normality of the linear part of the error, we will begin by showing condition (ii) in Theorem 2.2 holds. This is the only condition not subject to truncation by the indicator function $\mathbf{1}_{\{\|X_{nj}\|\leq\delta\}}$. Thereafter, it will be shown that this condition, plus the fact that the random variables $(\varepsilon_{j-l,k})$ are sub-Gaussian, means that the truncation is not necessary (and can thus be dropped) for the proofs of conditions (i) and (iii).

Furthermore, it will also be shown that the fact that the $(\varepsilon_{j-l,k})$ will also make it possible to recover a Gaussian limit for the law of the sum of the random variables, where, without loss of generality, it is obtained that $\{x_n\} \equiv 0$. Hence, given all of this, the application of Theorem 2.2 to the linear part of the error $\widetilde{\mu}_j - \mu_j$ can be stated in the following theorem:

Theorem 5.1 (Functional CLT for the Linear Terms)**.** *Given our assumptions and the decay condition*

$$\Delta_j\,U_j^4\,\exp\Big(U_j^2\sum_{i=1}^{j}(T_i - T_{i-1})\sigma_i^2\Big) \longrightarrow 0,$$

define the $L^2(K;\mathbb{R})$-valued random variables

$$\widetilde{X}^l_{U_j,k}(x) = \cos(U_j\,x)\,X^l_{U_j,k}(x) := \frac{1}{\sqrt{a_{j-l}}}\,\cos(U_j\,x)\,X^l_{U_j,k}(x)' := \frac{1}{\sqrt{a_{j-l}}}\,\cos(U_j\,x)\,\mathscr{L}^l_{\mu_j},$$

where $k = 1,\dots,m_{j-l}$, $\quad l \in \{0,1\}$,. Then the array $\{\widetilde{X}^l_{U_j,k}\}_{k=1}^{m_{j-l}}$ satisfies all three conditions of Theorem 2.2 in $H = L^2(K;\mathbb{R})$:

(i) For every $y \in L^2(K;\mathbb{R})$,

$$\lim_{U_j\to\infty}\sum_{k=1}^{m_{j-l}}\mathbb{E}\big[\langle\widetilde{X}^l_{U_j,k},y\rangle^2\big] = \langle A^l y, y\rangle,$$

where $A^l : L^2(K;\mathbb{R}) \to L^2(K;\mathbb{R})$ is the rank-one operator

$$A^l y := \tfrac{1}{2}\mathcal{C}\,\langle y, 1\rangle \cdot 1,$$

with 1 denoting the constant function 1 on K. The operator A^l is nuclear, positive, and Hermitian, with $\mathrm{Tr}(A^l) = \frac{1}{2}\mathcal{C}\,|K|$.

(ii) For every $\varepsilon > 0$,

$$\lim_{U_j\to\infty}\sum_{k=1}^{m_{j-l}}\mathbb{P}\big(\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})} > \varepsilon\big) = 0.$$

(iii) For every complete orthonormal system $\{e_i\}$ of $L^2(K;\mathbb{R})$,

$$\lim_{N\to\infty}\limsup_{U_j\to\infty}\sum_{k=1}^{m_{j-l}}\sum_{i>N}\mathbb{E}\big[\langle\widetilde{X}^l_{U_j,k},e_i\rangle^2\big] = 0.$$

Consequently,

$$\frac{1}{\sqrt{a_{j-l}}}\cos(U_j x)\,\mathscr{L}^l_{\mu_j} \Rightarrow \mathcal{N}(0, A^l) \quad \textit{in } L^2(K;\mathbb{R}), \quad \textit{for } l = 0, 1.$$

Moreover, since $a_j/(a_j + a_{j-1}) \to 1$ *and* $a_{j-1}/(a_j + a_{j-1}) \to 0$ *as* $U_j \to \infty$*, combining the two linear parts yields*

$$r_j \cos(U_j x)\big(\mathscr{L}^0_{\mu_j} - \mathscr{L}^1_{\mu_j}\big) \Rightarrow \mathcal{N}(0, A^0) \quad \textit{in } L^2(K;\mathbb{R}).$$

5.2. Cosine Modulation of $X^l_{U_j,k}$

Before starting with the proof of condition (ii), let us define a modified version of the random variables $X^l_{U_j,k}$, in which they are modulated by a cosine factor, namely, $\cos(U_j x)$. This modulation will be required to recover a non-degenerate limiting covariance operator, as the proof of condition (i) will show. Hence, as is also reflected in the statement of Theorem 5.1, the normality result will be obtained for this modulated version of the linear part of the Lévy density error. For this reason, condition (ii) will be proved for the modulated random field:

$$\widetilde{X}^l_{U_j,k}(x) = \cos(U_j x) X^l_{U_j,k}(x). \tag{5.3}$$

Notice that because $|\cos(U_j\, x)| \le 1$, this modulated random variable is also in $L^2(K)$. Thus, we now proceed to prove condition (ii) for $\widetilde{X}^l_{U_j,k}(x)$, the sum of which is the one for which asymptotic normality will be shown.

5.3. Proof of Condition (ii)

Condition (ii) requires that for every $\varepsilon > 0$,

$$\lim_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \mathbb{P}\big(\|\widetilde{X}^l_{U_j,k}\|_{L^2(K)} > \varepsilon\big) = 0. \tag{5.4}$$

Since the $\varepsilon_{j-l,k}$ are sub-Gaussian with unit variance, they have bounded fourth moments: there exists a constant $C_4 > 0$ such that $\mathbb{E}|\varepsilon_{j-l,k}|^4 \le C_4$ for all j, k. By Markov's inequality with the fourth moment,

$$\sum_{k=1}^{m_{j-l}} \mathbb{P}\big(\|\widetilde{X}^l_{U_j,k}\|_{L^2(K)} > \varepsilon\big) \le \frac{1}{\varepsilon^4} \sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^4_{L^2(K)}.$$

We bound the fourth moment sum by separating it into the maximum individual second moment and the total second moment:

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^4_{L^2(K)} \le \left(\max_{1\le k\le m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)}\right) \cdot \sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)}.$$

Take the following definition:

$$\mathcal{K}^l_j(x,z) := \sum_{k=1}^{m_{j-l}} \mathbb{E}[X^l_{U_j,k}(x) X^l_{U_j,k}(z)] \to 2\mathcal{C}\cos(U_j(x-z))$$

as given by the convergence in Theorems 4.1 and Lemma 5.1. Now, since $|\cos(U_j x)| \le 1$, the following bound can be established:

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} \le \int_K \sum_{k=1}^{m_{j-l}} \mathbb{E}|X^l_{U_j,k}(x)|^2\, dx = \int_K \mathcal{K}^l_j(x,x)\, dx.$$

By the uniform kernel bound (Lemma A.4), $|\mathcal{K}^l_j(x,x)| \le C_0$, and since K is compact,

$$\int_K \mathcal{K}^l_j(x,x)\, dx \le C_0\, |K| < \infty,$$

uniformly in U_j.

Recall now that

$$X^l_{U_j,k}(x) = (a_{j-l})^{-1/2}\, \delta_{j-l,k}\, \varepsilon_{j-l,k}\, h_{U_j,k}(x),$$

where $h_{U_j,k}(x)$ denotes the integral factor

$$h_{U_j,k}(x) := \frac{1}{\pi}\frac{U_j}{T_j - T_{j-1}} \operatorname{Re}\left(\int_0^1 \frac{ivU_j(1+ivU_j)\,\mathcal{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)}\, e^{-ivU_j x}\, w^1_{\mu_j}(v)\, dv\right).$$

Since $|\cos(U_j x)| \le 1$ and $\mathbb{E}|\varepsilon_{j-l,k}|^2 = 1$,

$$\max_{1\le k\le m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} \le (a_{j-l})^{-1}\, \delta^2_{j-l,k'} \int_K |h_{U_j,k'}(x)|^2\, dx.$$

Recall furthermore that

$$\mathscr{F}b_{j,k}(v) = \Delta_j e^{ivx_{j,k}}\, \operatorname{sinc}^2(\Delta_j v/2),$$

Hence, pulling out all the asymptotic factors outside of the integral of $h_{U_j,k}(x)$, we can obtain the following bound:

$$\begin{aligned}\max_{1\le k\le m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} &\le (a_{j-l})^{-1}\, \delta^2_{j-l,k'} \int_K |h_{k'}(x)|^2\, dx \\ &\lesssim \Delta^{-1}_{j-l}\, U_j^{-2}\, e^{-A_{j-l}\, U_j^2}\, U_j^4\, \Delta^2_{j-l}\, e^{A_{j-l}\, U_j^2} \\ &= \Delta_{j-l}\, U_j^2\end{aligned}$$

which, given that

$$\Delta_{j-l}\, U_j^4\, e^{A_{j-l}\, U_j^2} \to 0$$

the bound goes to 0 as well. Hence, condition (ii) of Theorem 2.2 is satisfied.

5.4. Reduction to Untruncated Random Variables

Conditions (i) and (iii) of Theorem 2.2 are stated in terms of the truncated, centred summands

$$\widetilde{X}^{l,\delta}_{U_j,k} := \widetilde{X}^l_{U_j,k}\, \mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\le\delta\}} - \mathbb{E}\left[\widetilde{X}^l_{U_j,k}\, \mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\le\delta\}}\right], \qquad \delta > 0.$$

Since the noise variables $\varepsilon_{j-l,k}$ are centred and enter $\widetilde{X}^l_{U_j,k}$ linearly,

$$\mathbb{E}\left[\widetilde{X}^l_{U_j,k}\right] = \cos(U_j x)\, (a_{j-l})^{-1/2}\, \delta_{j-l,k}\, h_{U_j,k}\, \mathbb{E}[\varepsilon_{j-l,k}] = 0,$$

so the untruncated summand $\widetilde{X}^l_{U_j,k}$ is already centred. The following lemma shows that, as $U_j \to \infty$, the truncated and untruncated summands yield the same limiting value of the functionals appearing in conditions (i) and (iii). The indicator $\mathbf{1}_{\{\|\cdot\|_{L^2(K;\mathbb{R})}\le\delta\}}$ may therefore be dropped, and both conditions verified directly for $\widetilde{X}^l_{U_j,k}$.

Lemma 5.2 (Reduction to untruncated summands)**.** *Fix $\delta > 0$. Under the decay condition $\Delta_{j-l}\, U_j^4\, e^{A_{j-l}U_j^2} \to 0$, the truncation tail is asymptotically negligible,*

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\left[\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\, \mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\right] \xrightarrow[U_j\to\infty]{} 0,$$

and consequently, for every $y \in L^2(K;\mathbb{R})$,

$$\lim_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \mathbb{E}\left[\langle \widetilde{X}^{l,\delta}_{U_j,k},\, y\rangle^2\right] = \lim_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \mathbb{E}\left[\langle \widetilde{X}^l_{U_j,k},\, y\rangle^2\right],$$

with the same identity holding when $\langle \cdot , y\rangle$ is replaced by the orthogonal projection Q_N onto the span of $\{e_i\}_{i>N}$, uniformly in N.

Hence conditions (i) *and* (iii) *of Theorem 2.2 may be verified for the untruncated summands $\widetilde{X}^l_{U_j,k}$.*

Proof. By Cauchy-Schwarz,

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\left[\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\right] \le \left(\sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^4_{L^2(K;\mathbb{R})}\right)^{1/2} \left(\sum_{k=1}^{m_{j-l}} \mathbb{P}\big(\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})} > \delta\big)\right)^{1/2}.$$

The second factor tends to 0 by condition (ii), established in the previous subsection, with $\varepsilon = \delta$. For the first factor, recall that

$$\widetilde{X}^l_{U_j,k}(x) = \cos(U_j x)\,(a_{j-l})^{-1/2}\,\delta_{j-l,k}\,\varepsilon_{j-l,k}\,h_{U_j,k}(x),$$

in which $\varepsilon_{j-l,k}$ is the only random factor. Hence

$$\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})} = |\varepsilon_{j-l,k}|\,\|\cos(U_j x)\,(a_{j-l})^{-1/2}\,\delta_{j-l,k}\,h_{U_j,k}\|_{L^2(K;\mathbb{R})},$$

and, since $\mathbb{E}|\varepsilon_{j-l,k}|^2 = 1$,

$$\mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^4_{L^2(K;\mathbb{R})} = \mathbb{E}|\varepsilon_{j-l,k}|^4\left(\mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\right)^2 \le C_4\left(\mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\right)^2,$$

using the bounded fourth moment $\mathbb{E}|\varepsilon_{j-l,k}|^4 \le C_4$. Factoring one power of $\mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}$ out of the square and bounding it by its maximum over k,

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^4_{L^2(K;\mathbb{R})} \le C_4\left(\max_{1\le k\le m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\right)\sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}.$$

´By the estimates of the previous subsection, $\sum_{k=1}^{m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})} \le C_0\,|K|$ uniformly in U_j, while $\max_{1\le k\le m_{j-l}} \mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})} \lesssim \Delta_{j-l}\,U_j^2$. The first factor is therefore bounded, and the truncation tail is bounded by a bounded quantity times a factor tending to 0 by condition (ii); it follows that

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\left[\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\right] \xrightarrow[U_j\to\infty]{} 0.$$

It remains to transfer this to the functionals in conditions (i) and (iii). Let us now define

$$\begin{aligned} D_{U_j,k} &:= \widetilde{X}^l_{U_j,k} - \widetilde{X}^{l,\delta}_{U_j,k} \qquad (5.5)\\ &= \widetilde{X}^l_{U_j,k} - \widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\le\delta\}} - \mathbb{E}\left[\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\le\delta\}}\right]\\ &= \widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}} - \mathbb{E}\left[\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\le\delta\}}\right] \end{aligned}$$

Given that $\mathbb{E}[\widetilde{X}^l_{U_j,k}] = \mathbb{E}\left[\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\le\delta\}}\right] + \mathbb{E}\left[\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\right] = 0$, we obtain the final expression:

$$D_{U_j,k} = \widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}} + \mathbb{E}\left[\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\right] \qquad (5.6)$$

Let us now take the norm of $D_{U_j,k}$ and apply the triangle inequality, which results in the following expression:

$$\begin{aligned} \|D_{U_j,k}\|_{L^2(K;\mathbb{R})} &\le \|\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\|_{L^2(K;\mathbb{R})} + \left\|\mathbb{E}[\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}]\right\|_{L^2(K;\mathbb{R})}\\ &\le \|\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\|_{L^2(K;\mathbb{R})} + \mathbb{E}\left[\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\right] \qquad (5.7) \end{aligned}$$

where the second inequality is obtained by applying Jensen's inequality. Note that the indicator function has been taken out of the norm because it is not a function of x. As a result,

$$\begin{aligned}
&\sum_{k=1}^{m_{j-l}} \mathbb{E}\|D_{U_j,k}\|^2_{L^2(K;\mathbb{R})} \\
&\leq \sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[\|\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\|_{L^2(K;\mathbb{R})} + \mathbb{E}\big[\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\big]\Big]^2 \\
&= \sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[\|\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\|^2_{L^2(K;\mathbb{R})} \\
&\quad + 2\,\|\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\|_{L^2(K;\mathbb{R})}\,\mathbb{E}\big[\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\big] \\
&\quad + \big(\mathbb{E}\big[\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\big]\big)^2\Big] \\
&\leq 4\sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}>\delta\}}\Big] \xrightarrow[U_j\to\infty]{} 0,
\end{aligned} \tag{5.8}$$

where the last inequality applies Jensen by the fact that the squared function is convex. Finally, for every $y \in L^2(K;\mathbb{R})$, the identity $a^2 - b^2 = (a-b)(a+b)$

$$\begin{aligned}
&\sum_{k=1}^{m_{j-l}} \big|\mathbb{E}\langle \widetilde{X}^l_{U_j,k}, y\rangle^2 - \mathbb{E}\langle \widetilde{X}^{l,\delta}_{U_j,k}, y\rangle^2\big| = \\
&= \sum_{k=1}^{m_{j-l}} \Big|\mathbb{E}\Big[\langle \widetilde{X}^l_{U_j,k}, y\rangle^2 - \langle \widetilde{X}^{l,\delta}_{U_j,k}, y\rangle^2\Big]\Big| \\
&= \sum_{k=1}^{m_{j-l}} \Big|\mathbb{E}\Big[\langle \widetilde{X}^l_{U_j,k} - \widetilde{X}^{l,\delta}_{U_j,k}, y\rangle \cdot \langle \widetilde{X}^l_{U_j,k} + \widetilde{X}^{l,\delta}_{U_j,k}, y\rangle\Big]\Big| \\
&= \sum_{k=1}^{m_{j-l}} \mathbb{E}\Big|\langle \widetilde{X}^l_{U_j,k} - \widetilde{X}^{l,\delta}_{U_j,k}, y\rangle\Big| \cdot \Big|\langle \widetilde{X}^l_{U_j,k} + \widetilde{X}^{l,\delta}_{U_j,k}, y\rangle\Big| \\
&\leq \|y\|^2 \sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[\|D_{U_j,k}\|^2_{L^2(K;\mathbb{R})} \cdot \|\widetilde{X}^l_{U_j,k} + \widetilde{X}^{l,\delta}_{U_j,k},\|^2_{L^2(K;\mathbb{R})}\Big] \\
&\leq \|y\|^2_{L^2(K;\mathbb{R})} \left(\sum_{k=1}^{m_{j-l}} \mathbb{E}\|D_{U_j,k}\|^2_{L^2(K;\mathbb{R})}\right)^{1/2} \left(\sum_{k=1}^{m_{j-l}} \mathbb{E}\big(\|\widetilde{X}^{l,\delta}_{U_j,k}\|_{L^2(K;\mathbb{R})} + \|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\big)^2\right)^{1/2} \xrightarrow[U_j\to\infty]{} 0,
\end{aligned} \tag{5.9}$$

where the last inequality due to the Cauchy-Schwarz inequality on expectations. The rightmost summand is bounded, because both $\mathbb{E}\|\widetilde{X}^{l,\delta}_{U_j,k}\|^2_{L^2(K;\mathbb{R})}$ and $\mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}$ are bounded by $\mathbb{E}\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K;\mathbb{R})}$, whose sum is at most $C_0\,|K|$, as given in the proof of condition (ii).

Equation (5.9) goes to 0 by equation (5.8), which is what forces the convergence to 0 in this case. Hence, we obtain that:

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\langle \widetilde{X}^l_{U_j,k}, y\rangle^2 = \sum_{k=1}^{m_{j-l}} \mathbb{E}\langle \widetilde{X}^{l,\delta}_{U_j,k}, y\rangle^2 \quad \forall y \in L^2(K;\mathbb{R}) \tag{5.10}$$

Hence, this means that the truncated and untruncated versions are equivalent $\forall y \in L^2(K;\mathbb{R})$, including the elements of an orthonormal system $\{e_i\}$, so we can verify conditions (i) and (iii) for $\widetilde{X}^l_{U_j,k}$ instead of $\widetilde{X}^l_{U_j,k}\,\mathbf{1}_{\{\|\widetilde{X}^l_{U_j,k}\|_{L^2(K;\mathbb{R})}\leq\delta\}}$.

□

5.5. Proof of Condition (i)

Recall from Chapter 4 and Lemma 5.1 that there exists a deterministic scalar $\mathcal{C} > 0$ such that, for every fixed $x, z \in \mathbb{R}$,

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}[X_{U_j,k}{}^l(x) X_{U_j,k}{}^l(z)] = \mathcal{C}\, e^{-iU_j(x-z)} + \mathcal{C}\, e^{-iU_j(z-x)} + r_{U_j}(x,z)$$
$$= 2\mathcal{C}\cos(U_j(x-z)) + r_{U_j}(x,z), \tag{5.11}$$

where

$$r_{U_j}(x,z) \xrightarrow[U_j\to\infty]{} 0 \quad \text{for every fixed } (x,z) \in \mathbb{R}^2.$$

5.5.1. Quadratic form representation

Fix $y \in L^2(K;\mathbb{R})$. Thus,

$$\langle X^l_{U_j,k}, y\rangle = \int_K X^l_{U_j,k}(x)\, y(x)\, dx$$

is well defined. By Fubini's theorem,

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\big[\langle X^l_{U_j,k}, y\rangle^2\big] = \sum_{k=1}^{m_{j-l}} \int_{K\times K} \mathbb{E}[X^l_{U_j,k}(x) X^l_{U_j,k}(z)]\, y(x)y(z)\, dx\, dz$$
$$= \int_{K\times K} \mathcal{K}^l_j(x,z)\, y(x)y(z)\, dx\, dz, \tag{5.12}$$

Recall now the modulated random variable in equation 5.3:

$$\widetilde{X}^l_{U_j,k}(x) = \cos(U_j x) X^l_{U_j,k}(x).$$

This is the random variable for which it will be proved that $\sum_{k=1}^{m_{j-l}} \widetilde{X}^l_{U_j,k}(x)$ (i.e., the linear part of $\widetilde{\mu}_j - \mu_j$) is asymptotically normally distributed. Let us now apply this definition to obtain the following modulated quadratic form:

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\big[\langle \widetilde{X}^l_{U_j,k}, y\rangle^2\big] = \sum_{k=1}^{m_{j-l}} \int_{K\times K} \mathbb{E}[X^l_{U_j,k}(x) X^l_{U_j,k}(z)]\, \cos(U_j x)y(x)\cos(U_j z)y(z)\, dx\, dz$$
$$= \int_{K\times K} \widetilde{\mathcal{K}^l_j}(x,z)\, y(x)y(z)\, dx\, dz. \tag{5.13}$$

5.5.2. Decomposition into leading term and remainder

As we are working in an asymptotic setting where $U_j \to \infty$, we can insert (5.11) into (5.12):

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\big[\langle \widetilde{X}^l_{U_j,k}, y\rangle^2\big] = 2\mathcal{C}\, I_{U_j}(y) + R_{U_j}(y), \tag{5.14}$$

where

$$I_{U_j}(y) = \int_{K\times K} \cos(U_j(x-z))\cos(U_j x)\cos(U_j z) y(x)y(z)\, dx\, dz,$$
$$R_{U_j} = \int_{K\times K} r_{U_j}\cos(U_j x)\cos(U_j z)(x,z)\, y(x)y(z)\, dx\, dz.$$

Using standard trigonometric identities

$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$$

we obtain:

$$\cos(U_j(x-z))\cos(U_j x)\cos(U_j z) = \cos^2(U_j x)\cos^2(U_j z) + \cos(U_j x)\cos(U_j z)\sin(U_j x)\sin(U_j z).$$

5.5.3. Exact evaluation of the main term

A direct computation yields

$$\cos(U_j x)\cos(U_j z)\sin(U_j x)\sin(U_j z) = \frac{1}{4}\sin(2U_j x)\sin(2U_j z).$$

When integrating against the test functions we obtain:

$$\int_{K\times K} \frac{1}{4}\sin(2U_j x)\sin(2U_j z)y(x)y(z)dxdz \to 0,$$

by the Riemann-Lebesgue Lemma. Furthermore, recall also the trigonometric Power Reduction identity:

$$\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta)).$$

Thus, we obtain that

$$\cos^2(U_j x)\cos^2(U_j z) = \frac{1}{4}(1+\cos(2U_j x))(1+\cos(2U_j z))$$

$$= \frac{1}{4}(1+\cos(2U_j x)+\cos(2U_j z)+\cos(2U_j x)\cos(2U_j z)).$$

When integrating against the test functions, we obtain that

$$\int \cos(2U_j x)y(x)dx \to 0,$$

once again by the Riemann-Lebesgue Lemma. Thus, the only integral left is

$$\int_{K\times K} y(x)y(z)dxdz = |\hat{y}(0)|^2.$$

Thus,

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\big[\langle X^l_{U_j,k}, y\rangle^2\big] = \frac{1}{2}\mathcal{C}\,|\widehat{y}(0)|^2 + R_{U_j}(y). \tag{5.15}$$

Hence, this means that modulation of $X^l_{U_j,k}(x)$ by $\cos(U_j x)$ makes it possible, via the application of trigonometric identities, to recover a non-oscillating element in the quadratic form which does not go to 0 by the Riemann-Lebesgue lemma. As the next subsections will show, this will make it possible to obtain a non-degenerate covariance operator.

5.5.4. Control of the remainder & Convergence of the Quadratic Form

In order to control the remainder, it will first need to be shown that $\mathcal{K}^l_j(x,z)$ is uniformly bounded in j, x, z. This is proved in Lemma A.4, in the appendix.

Given this fact, and since

$$r_{U_j}(x,z) = \mathcal{K}^l_j(x,z) - 2\mathcal{C}\cos(U_j(x-z)) \to 0$$

it follows that there exists a constant $C_1 > 0$ such that

$$|r_{U_j}(x,z)| \le C_1 \qquad \text{for all } U_j \ge 1,\ x,z\in\mathbb{R}.$$

As $y \in L^2(K)$, $y \in L^1(K)$ and hence:

$$\int_{K\times K} |y(x)y(z)|\,dx\,dz = \|y\|^2_{L^1(K)} < \infty,$$

Now, given that

$$r_{U_j}(x,z) \longrightarrow 0 \qquad \text{for every fixed } (x,z)\in\mathbb{R}^2.$$

we can combine this pointwise convergence with the uniform bound above, which yields:

$$|r_{U_j}(x,z)\cos(U_j x)y(x)\cos(U_j z)y(z)| \le C_1\,|y(x)y(z)| \qquad \forall \quad U_j, x, z.$$

Since $C_1|y(x)y(z)| \in L^1(K^2)$, the dominated convergence theorem applies:

$$\lim_{U_j\to\infty} R_{U_j} = \lim_{U_j\to\infty} \int_{K\times K} r_{U_j}(x,z)\,\cos(U_j x)\,\cos(U_j z)\,y(x)y(z)\,dx\,dz$$

$$= \int_{K\times K} \lim_{U_j\to\infty} r_{U_j}(x,z)\,\cos(U_j x)\,\cos(U_j z)\,y(x)y(z)\,dx\,dz = 0.$$

Combining this with (5.15), we get:

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\big[\langle \widetilde{X}^l_{U_j,k}, y\rangle^2\big] = \frac{1}{2}\mathcal{C}\,|\widehat{y}(0)|^2 + o(1), \qquad y \in L^2(K) \quad \text{for } K \text{ compact.}$$

Thus, the limiting quadratic form on the real Hilbert space $L^2(K;\mathbb{R})$ is identified:

$$y \longmapsto \frac{1}{2}\mathcal{C}\,|\widehat{y}(0)|^2.$$

5.5.5. Identification of the limiting covariance operator

By the preceding subsection, for every $y \in L^2(K)$,

$$\hat{Q}(y) = \lim_{U_j\to\infty} Q_{U_j}(y) = \tfrac{1}{2}\,\mathcal{C}\,|\hat{y}(0)|^2 = \tfrac{1}{2}\,\mathcal{C}\,\langle y, 1\rangle^2.$$

Define the operator $A^l : L^2(K;\mathbb{R}) \to L^2(K;\mathbb{R})$ by

$$A^l y := \tfrac{1}{2}\,\mathcal{C}\,\langle y, 1\rangle \cdot 1.$$

Then $\langle A^l y, y\rangle = \frac{1}{2}\,\mathcal{C}\,\langle y,1\rangle^2 = \hat{Q}(y)$ for all $y \in L^2(K)$.

Additionally, since $1 \in L^2(K)$,

$$\|A^l y\|_{L^2(K)} = \tfrac{1}{2}\,\mathcal{C}\,|\langle y,1\rangle|\,\|1\|_{L^2(K)} \le \tfrac{1}{2}\,\mathcal{C}\,\|1\|^2_{L^2(K)}\,\|y\|_{L^2(K)},$$

so A^l is bounded with $\|A^l\|_{\mathcal{L}(L^2(K))} \le \frac{1}{2}\,\mathcal{C}\,\|1\|^2_{L^2(K)}$.

For every $y \in L^2(K)$,

$$\langle A^l y, y\rangle = \tfrac{1}{2}\,\mathcal{C}\,\langle y,1\rangle^2 \ge 0,$$

since $\mathcal{C} > 0$. Finally, for every $y, z \in L^2(K)$,

$$\langle A^l y, z\rangle = \tfrac{1}{2}\,\mathcal{C}\,\langle y,1\rangle\,\langle 1,z\rangle = \tfrac{1}{2}\,\mathcal{C}\,\langle z,1\rangle\,\langle 1,y\rangle = \langle A^l z, y\rangle = \langle y, A^l z\rangle.$$

Finally, note that the operator A^l is rank-one as it has got range span$\{1\}$. Every finite-rank operator on a Hilbert space is nuclear. Explicitly, let $\{e_n\}_{n\ge1}$ be any complete orthonormal system of $L^2(K)$. Then

$$\mathrm{Tr}(A^l) = \sum_{n=1}^{\infty}\langle A^l e_n, e_n\rangle = \tfrac{1}{2}\,\mathcal{C}\sum_{n=1}^{\infty}\langle e_n, 1\rangle^2 = \tfrac{1}{2}\,\mathcal{C}\,\|1\|^2_{L^2(K)} < \infty,$$

where the last equality follows from Parseval's identity (generalization of the Pythagorean Theorem). This concludes the proof of Condition (i) from Giné-León [10].

5.6. Proof of Condition (iii)

Let $\{e_i\}_{i\geq 1}$ be a fixed complete orthonormal system of $L^2(K)$.

As mentioned previously, we have finite second moments and thus truncation is not necessary. Hence, we will drop the δ and prove the statement for $\widetilde{X}^l_{U_j,k}$. Recall that Condition (iii) requires that

$$\lim_{N\to\infty} \limsup_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \sum_{i>N} \mathbb{E}\,\langle X^l_{U_j,k}, e_i\rangle^2_{L^2(K)} = 0. \tag{5.16}$$

For $N \geq 1$, define the orthogonal projection

$$P_N x := \sum_{i=1}^{N} \langle x, e_i\rangle_{L^2(K)}\, e_i, \qquad Q_N := I - P_N.$$

By Parseval's identity,

$$\sum_{i>N} \langle x, e_i\rangle^2_{L^2(K)} = \|Q_N x\|^2_{L^2(K)} \qquad \text{for all } x \in L^2(K).$$

Therefore, for each U_j,

$$\begin{aligned} \sum_{k=1}^{m_{j-l}} \sum_{i>N} \mathbb{E}\,\langle \widetilde{X}^l_{U_j,k}, e_i\rangle^2_{L^2(K)} &= \sum_{k=1}^{m_{j-l}} \mathbb{E}\,\|Q_N \widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} \\ &= \sum_{k=1}^{m_{j-l}} \mathbb{E}\,\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} - \sum_{k=1}^{m_{j-l}} \mathbb{E}\,\|P_N \widetilde{X}^l_{U_j,k}\|^2_{L^2(K)}. \end{aligned} \tag{5.17}$$

For fixed N, Parseval's identity gives

$$\|P_N \widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} = \sum_{i=1}^{N} \langle \widetilde{X}^l_{U_j,k}, e_i\rangle^2_{L^2(K)}.$$

Summing over k and using Condition (i) of the Giné–León theorem, already proved in the previous subsection for the space $L^2(K;\mathbb{R})$, we obtain

$$\lim_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \mathbb{E}\,\|P_N \widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} = \sum_{i=1}^{N} \langle A^l e_i, e_i\rangle_{L^2(K)},$$

where A^l is the limiting covariance operator. By the uniform kernel bound (Lemma A.4), $|\widetilde{K}_{U_j}(x,z)| \leq C_0$ on $K \times K$, and hence

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}\,\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} \leq \int_K \mathcal{K}^l_j(x,x)\,dx \leq C_0\,|K| < \infty,$$

uniformly in U_j. Moreover, by Condition (i) and the identification of the covariance operator,

$$\lim_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \mathbb{E}\,\|\widetilde{X}^l_{U_j,k}\|^2_{L^2(K)} = \mathrm{Tr}(A^l) = \tfrac{1}{2}\,\mathcal{C}\,|K|.$$

Taking $\limsup_{U_j\to\infty}$ in (5.17) and using the elementary inequality $\limsup(a_j - b_j) \leq \limsup a_j - \liminf b_j$, we obtain

$$\limsup_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \sum_{i>N} \mathbb{E}\,\langle \widetilde{X}^l_{U_j,k}, e_i\rangle^2_{L^2(K)} \leq \mathrm{Tr}(A^l) - \sum_{i=1}^{N} \langle A^l e_i, e_i\rangle_{L^2(K)}.$$

Since A^l is nuclear (i.e. rank-one),

$$\sum_{i=1}^{\infty} \langle A^l e_i, e_i \rangle_{L^2(K)} = \mathrm{Tr}(A^l),$$

and hence

$$\mathrm{Tr}(A^l) - \sum_{i=1}^{N} \langle A^l e_i, e_i \rangle_{L^2(K)} \xrightarrow[N\to\infty]{} 0.$$

Each term in the sum is nonnegative, since we work in a real Hilbert space. Therefore,

$$0 \le \lim_{N\to\infty} \limsup_{U_j\to\infty} \sum_{k=1}^{m_{j-l}} \sum_{i>N} \mathbb{E}\, \langle \widetilde{X}^l_{U_j,k}, e_i \rangle^2_{L^2(K)} \le 0,$$

which proves (5.16). Hence Condition (iii) of Theorem 2.2 holds.

5.7. Conclusion

Conditions (i), (ii) and (iii) of Theorem 2.2 have now been verified for the array $\{\widetilde{X}^l_{U_j,k}\}$ in $L^2(K;\mathbb{R})$. It remains to identify the centring sequence $\{x_n\}$ and the mean of the limiting Gaussian measure. Since the noise variables $\varepsilon_{j-l,k}$ are centered, the $\widetilde{X}^l_{U_j,k}$ are also centered. I.e., $\sum_{k=1}^{m_{j-l}} \mathbb{E}[\widetilde{X}^l_{U_j,k}] = 0$ for every U_j; the array is exactly centered.

By Theorem 2.2 there exist a sequence $\{x_n\} \subset L^2(K;\mathbb{R})$ and a vector $a \in L^2(K;\mathbb{R})$ such that

$$\sum_{k=1}^{m_{j-l}} \mathbb{E}[\widetilde{X}^l_{U_j,k}] - x_n \longrightarrow a \qquad \text{and} \qquad \mathscr{L}\left(\sum_{k=1}^{m_{j-l}} \widetilde{X}^l_{U_j,k} - x_n\right) \overset{w}{\Longrightarrow} \mathcal{N}(a, A^l).$$

Substituting $\sum_{k=1}^{m_{j-l}} \mathbb{E}[\widetilde{X}^l_{U_j,k}] = 0$ into the first relation gives $x_n \to -a$. This means that, $\{x_n\}$ is a deterministic sequence converging to the constant $-a$ in $L^2(K;\mathbb{R})$. Writing

$$\sum_{k=1}^{m_{j-l}} \widetilde{X}^l_{U_j,k} = \left(\sum_{k=1}^{m_{j-l}} \widetilde{X}^l_{U_j,k} - x_n\right) + x_n,$$

the first term converges weakly to $\mathcal{N}(a, A^l)$ while the second converges deterministically to $-a$. By Slutsky's theorem, the sum converges weakly to the translate of $\mathcal{N}(a, A^l)$ by $-a$, namely $\mathcal{N}(a-a, A^l) = \mathcal{N}(0, A^l)$. Hence the centring vector is $a = 0$, and

$$\sum_{k=1}^{m_{j-l}} \widetilde{X}^l_{U_j,k} \overset{w}{\Longrightarrow} \mathcal{N}(0, A^l) \qquad \text{in } L^2(K;\mathbb{R}), \quad l = 0, 1.$$

Remember from Koorevaar that, in the decomposition of the error of $\tilde{\mu}_j(x)$

$$\mathscr{L}^l_{\mu_j} = \sum_{k=1}^{m_{j-l}} {X'_{U_j,k}}^{l}$$

Hence, the CLT can be rewritten as

$$\frac{1}{\sqrt{a_{j-l}}} \cos(U_j x)\, \mathscr{L}^l_{\mu_j} \overset{w}{\Longrightarrow} \mathcal{N}(0, A^l) \quad \text{in } L^2(K;\mathbb{R}), \quad \text{for } l = 0, 1.$$

Recall now equation (3.54), the decomposition of $\tilde{\mu}_j(x) - \mu_j(x)$:

$$\begin{aligned}
\tilde{\mu}_j(x) - \mu_j(x) &= \frac{1}{2\pi}\Bigg[\int_{-U_j}^{U_j} \big(\tilde{\psi}_j(v) - \psi_j(v)\big)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + \frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2}\int_{-U_j}^{U_j} (v-i)^2\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\
&\quad - i(\tilde{\gamma}_j - \gamma_j)\int_{-U_j}^{U_j} (v-i)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + (\tilde{\lambda}_j - \lambda_j)\int_{-U_j}^{U_j} w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\
&\quad + \int_{\mathbb{R}\setminus[-U_j,\, U_j]} \left(\psi_j(v) + \frac{\sigma_j^2}{2}(v-i)^2 - i\gamma_j(v-i) + \lambda_j\right)\big(1 - w_{\mu_j}^{U_j}(v)\big)\, e^{-ivx}\, \mathrm{d}v\Bigg] \\
&= \Psi + \Sigma + \Gamma + \Lambda + \mathscr{B}
\end{aligned}$$

where

$$\Psi = \frac{1}{2\pi}\int_{-U_j}^{U_j} \big[\mathcal{L}_j^0(v) - \mathcal{L}_j^1(v) + \mathcal{R}_j^0(v) - \mathcal{R}_j^1(v)\big]\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \tag{5.18}$$

$$= \mathscr{L}_{\mu_j}^0 - \mathscr{L}_{\mu_j}^1 + \mathscr{R}_{\mu_j}^0 - \mathscr{R}_{\mu_j}^1, \tag{5.19}$$

Hence, we will analyse the normality in the space $L^2(K;\mathbb{R})$ of the term

$$\cos(U_j x)\,(\mathscr{L}_{\mu_j}^0 - \mathscr{L}_{\mu_j}^1).$$

To account for the $\frac{1}{\sqrt{a_{j-l}}}$ prefactor, we will analyse in detail the following:

$$\frac{1}{\sqrt{a_j + a_{j-1}}}\cos(U_j x)\,(\mathscr{L}_{\mu_j}^0 - \mathscr{L}_{\mu_j}^1). \tag{5.20}$$

Note that

$$\frac{1}{\sqrt{a_j + a_{j-1}}}\cos(U_j x)\,\mathscr{L}_{\mu_j}^l = \frac{\sqrt{a_{j-l}}}{\sqrt{a_j + a_{j-1}}}\left(\frac{1}{\sqrt{a_{j-l}}}\cos(U_j x)\,\mathscr{L}_{\mu_j}^l\right) \overset{w}{\Longrightarrow} \mathcal{N}(0, A^l).$$

Hence, to obtain a CLT for (14), we need to look at the convergence of

$$\frac{a_{j-l}}{a_j + a_{j-1}} \text{ for } l = 0, 1 \tag{5.21}$$

In order to carry out this analysis, it must be remembered that Koorevaar assumes that $\Delta_{j-1} \leq C\Delta_j$ for some $C > 0$. Let us now look at when $l = 0$:

$$\begin{aligned}
\frac{a_j}{a_j + a_{j-1}} &= \frac{\Delta_j\, U_j^2\, \exp\{A_j U_j^2\}}{\Delta_j\, U_j^2\, \exp\{A_j U_j^2\} + \Delta_{j-1}\, U_j^2\, \exp\{A_{j-1} U_j^2\}} \\
&= \frac{1}{1 + \frac{\Delta_{j-1}}{\Delta_j}\exp\{-(T_j - T_{j-1})\,\sigma_j^2\, U_j^2\}} \\
&\geq \frac{1}{1 + C\exp\{-(T_j - T_{j-1})\,\sigma_j^2\, U_j^2\}} \to 1 \,\text{as}\, U_j \to \infty.
\end{aligned}$$

Given the fact that the denominator cannot vanish, as it is the sum of two positive numbers, and it is a number in $(0,1)$, we conclude by the squeeze theorem that

$$\frac{a_j}{a_j + a_{j-1}} \to 1 \,\text{as}\, U_j \to \infty. \tag{5.22}$$

Now, if $l = 1$, we have:

$$\begin{aligned}
\frac{a_{j-1}}{a_j + a_{j-1}} &= \frac{1}{1 + \frac{\Delta_j}{\Delta_{j-1}}\exp\{(T_j - T_{j-1})\,\sigma_j^2\, U_j^2\}} \\
&\leq \frac{1}{1 + \frac{1}{C}\,\exp\{(T_j - T_{j-1})\,\sigma_j^2\, U_j^2\}} \to 0 \,\text{as}\, U_j \to \infty.
\end{aligned}$$

Given that the ratio is bounded from below by 0, we can conclude by the squeeze theorem that

$$\frac{a_{j-1}}{a_j + a_{j-1}} \to 0 \,\text{as}\; U_j \to \infty. \tag{5.23}$$

Hence, in our asymptotic setup, we have the following CLT for the linear terms:

$$r_j\; \cos(U_j x)\,(\mathscr{L}^0_{\mu_j} - \mathscr{L}^1_{\mu_j}) \overset{w}{\Longrightarrow} \mathcal{N}(0, A^0), \tag{5.24}$$

where $r_j := \frac{1}{\sqrt{a_j + a_{j-1}}}$

Chapter 6
Vanishing of Non-Linear Terms of $\tilde{\mu}_j - \mu_j$ in $L^2(K)$

Recall the error decomposition for $\widetilde{\mu}_j$:

$$\begin{aligned}\tilde{\mu}_j(x) - \mu_j(x) = \frac{1}{2\pi}\Bigg[&\int_{-U_j}^{U_j} \big(\tilde{\psi}_j(v) - \psi_j(v)\big)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + \frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2}\int_{-U_j}^{U_j} (v-i)^2\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\ &- i(\tilde{\gamma}_j - \gamma_j)\int_{-U_j}^{U_j} (v-i)\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v + (\tilde{\lambda}_j - \lambda_j)\int_{-U_j}^{U_j} w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\ &+ \int_{\mathbb{R}\setminus[-U_j,\, U_j]} \left(\psi_j(v) + \frac{\sigma_j^2}{2}(v-i)^2 - i\gamma_j(v-i) + \lambda_j\right)\big(1 - w_{\mu_j}^{U_j}(v)\big)\, e^{-ivx}\, \mathrm{d}v\Bigg] \\ =:\ & \Psi + \Sigma + \Gamma + \Lambda + \mathscr{B},\end{aligned}$$

where

$$\begin{aligned}2\pi\Psi &= \int_{-U_j}^{U_j} \left[\mathcal{L}_j^0(v) - \mathcal{L}_j^1(v) + \mathcal{R}_j^0(v) - \mathcal{R}_j^1(v)\right] w_{\mu_j}^{U_j}(v)\, e^{-ivx}\, \mathrm{d}v \\ &=: \mathscr{L}_{\mu_j}^0 - \mathscr{L}_{\mu_j}^1 + \mathscr{R}_{\mu_j}^0 - \mathscr{R}_{\mu_j}^1.\end{aligned}$$

In this section, it will be proved, that under the modulation $\cos(U_j\cdot)$ and the scaling r_j, the remainder terms $\mathscr{R}_{\mu_j}^0$, the bias term $\mathscr{B}$ and the terms Σ, Γ, Λ vanish as $U_j \to \infty$.

6.1. Vanishing of the Bias Term

Recall that asymptotic normality in $L^2(K)$ has already been established for the linear term $\mathscr{L}_{\mu_j}^0 - \mathscr{L}_{\mu_j}^1$. It remains to show that the bias term $\mathscr{B}$ converges to zero in $L^2(K)$. This result is summarized in the following proposition proved in sections 6.1.1-6.1.4.

Proposition 6.1 (Vanishing of the Bias in $L^2(K;\mathbb{R})$)**.** *Take the assumptions in section 3.4, and $s_j \geq 2$. If*

$$\Delta_j\, U_j^4\, \exp\Big(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2\Big) \longrightarrow 0,$$

and

$$U_j^{2(s_j+1)}\Big(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\Big) \longrightarrow \infty,$$

then

$$r_j\, \|\cos(U_j x)\, \mathscr{B}(x)\|_{L^2(K;\mathbb{R})} \longrightarrow 0.$$

6.1.1. Pointwise bound.

It is stated section B.1 in Koorevaar [15] that there exists a constant $C > 0$ such that

$$|\mathscr{B}| \leq \frac{1}{2\pi}\, U_j^{-s_j}\, \|\mu_j^{s_j}\|_{L^2(\mathbb{R})},$$

where it must be noticed $\mathscr{B}$ is a function of x (indeed, $\mathscr{B}$ will be referred to as $\mathscr{B}(x)$ from now on) and $\|\mu_j^{s_j}\|_{L^2(\mathbb{R})}$ is finite. The function $\mathscr{B}$ is

$$\mathscr{B}(x) = \frac{1}{2\pi}\int_{\mathbb{R}\setminus[-U_j,U_j]} \left(\psi_j(v) + \frac{\sigma_j^2}{2}(v-i)^2 - i\gamma_j(v-i) + \lambda_j\right)\Big(1 - w_{\mu_j}^{U_j}(v)\Big)\, e^{-ivx}\, dv. \tag{6.1}$$

6.1.2. $L^2(K)$-norm estimate.

Since K is compact with $|K| < \infty$, the pointwise bound gives directly

$$\|\mathscr{B}\|^2_{L^2(K)} = \int_K |\mathscr{B}(x)|^2 \, dx \leq C^2 \, U_j^{-2s_j} \, |K|.$$

6.1.3. Modulation.

Since $|\cos(U_j x)| \leq 1$ for all x, we have

$$\|\tilde{\mathscr{B}}\|^2_{L^2(K)} := \|\cos(U_j x)\, \mathscr{B}(x)\|^2_{L^2(K)} = \int_K \cos^2(U_j x)\, |\mathscr{B}(x)|^2 \, dx \leq \|\mathscr{B}\|^2_{L^2(K)} \leq C^2 \, U_j^{-2s_j} \, |K|.$$

6.1.4. Prefactor.

Multiplying by $r_j^2 = (a_j + a_{j-1})^{-1}$, where $a_{j-l} = \Delta_{j-l} \, U_j^2 \, e^{A_{j-l} U_j^2}$, we obtain

$$r_j^2 \, \|\tilde{\mathscr{B}}\|^2_{L^2(K)} \lesssim (a_j + a_{j-1})^{-1} \, U_j^{-2s_j} \, |K|.$$

Hence, if:

$$U_j^{2(s_j+1)} \left(\Delta_j \, e^{A_j U_j^2} + \Delta_{j-1} \, e^{A_{j-1} U_j^2} \right) \to \infty$$

the bias goes to 0, and we have proved the proposition.

6.2. Vanishing of the Remainder Term

Control of the remainder term (also referred to as the stochastic error)

$$\mathscr{R}^l_{\mu_j}(x) = \int_{-U_j}^{U_j} \mathcal{R}^l_j(v) w^{U_j}_{\mu_j}(v) e^{-ivx} \, dv \tag{6.2}$$

requires that it converges to 0 in probability in $L^2(K; \mathbb{R})$:

$$||\mathscr{R}^l_{\mu_j}(x)||_{L^2(K)} \xrightarrow{\mathbb{P}} 0$$

Equivalently, taking into account the modulation and the prefactor needed to obtain a CLT, it must be shown that

$$\mathbb{P}\left(||r_j \, \cos(U_j x) \mathscr{R}^l_{\mu_j}(x)||_{L^2(K)} > \varepsilon \right) \to 0$$

To do so, we will first prove that, under certain model assumptions

$$\mathbb{E}\left[\left|\left| r_j \, \cos(U_j x) \mathscr{R}^l_{\mu_j}(x) \right|\right|^2_{L^2(K)} \right] \to 0$$

after which convergence to 0 in probability is guaranteed by Chebyshev's Inequality. This result is summarized in the following proposition, proved in sections 6.2.1-6.2.4:

Proposition 6.2 (Vanishing of the Remainder in $L^2(K; \mathbb{R})$)**.** *Take the assumptions in section 3.4 and*

$$\frac{\Delta^2_{j-l}}{\Delta_j} \, U_j^5 \, \exp\left(U_j^2 \left(2 \sum_{i=1}^{j-l} (T_i - T_{i-1}) \sigma_i^2 - \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 \right) \right) \longrightarrow 0, \quad \textit{for } l = 0, 1, \tag{6.3}$$

then,

$$r_j \left\| \cos(U_j x) \, \mathscr{R}^l_{\mu_j}(x) \right\|_{L^2(K; \mathbb{R})} \xrightarrow{\mathbb{P}} 0 \quad \textit{as } U_j \to \infty.$$

6.2.1. Modulation and Integral Bound

Since $|\cos(U_j x)| \le 1$, we have

$$||\cos(U_j x)\mathscr{R}^l_{\mu_j}||^2_{L^2(K)} = \int_K \cos^2(U_j x)\,|\mathscr{R}^l_{\mu_j}(x)|^2\,dx \le \int_K |\mathscr{R}^l_{\mu_j}(x)|^2\,dx = ||\mathscr{R}^l_{\mu_j}||^2_{L^2(K)}.$$

By Parseval's identity and the fact that $\mathscr{R}^l_{\mu_j}$ is band-limited to $[-U_j, U_j]$,

$$||\mathscr{R}^l_{\mu_j}||^2_{L^2(K)} \le ||\mathscr{R}^l_{\mu_j}||^2_{L^2(\mathbb{R})} = 2\pi \int_{-U_j}^{U_j} |\widehat{\mathscr{R}^l_{\mu_j}}(u)|^2\,du.$$

Hence,

$$\mathbb{E}\Big(||\cos(U_j x)\mathscr{R}^l_{\mu_j}||^2_{L^2(K)}\Big) \lesssim \mathbb{E}\Big(\int_{-U_j}^{U_j} |\widehat{\mathscr{R}^l_{\mu_j}}(u)|^2\,du\Big). \tag{6.4}$$

6.2.2. Bound of $\mathbb{E}|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathcal{O}_{j-l})|^4$

The bound of $\mathbb{E}|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathcal{O}_{j-l})|^4$ developed below largely follows the strategy of Koorevar [15, section 3.3.4], adapted to the cosine-modulated $L^2(K)$ setting. Indeed, given Koorevaar [15, eq. (3.26)], we can compute and bound the Fourier transform of the stochastic error term:

$$\begin{aligned}\widehat{\mathscr{R}^l_{u_j}}(u) &= 2\pi\,\mathcal{R}^l_j(u)\; w^{U_j}_{\mu_j}(u)\\ &\lesssim K_{j-l}(u)^{-2}\,(u^4+u^2)\,|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathcal{O}_{j-l})(u)|^2\,|w^{U_j}_{\mu_j}(u)| \qquad \forall\, u \in [-U_j, U_j],\end{aligned} \tag{6.5}$$

where

$$K_j(v) := 2^{j-1} e^{-v^2 A_j/2 + 2RT_j}.$$

Here, R is that of Definition 3.3. Thus, this means that

$$\begin{aligned}\mathbb{E}\Big(||\cos(U_j x)\mathscr{R}^l_{\mu_j}||^2_{L^2(K)}\Big) &\lesssim \int_{-U_j}^{U_j} K_{j-l}(u)^{-4}(u^4+u^2)^2\,\mathbb{E}\Big|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathcal{O}_{j-l})(u)\Big|^4\;|w^{U_j}_{\mu_j}(u)|^2\,du\\ &\lesssim \int_{-U_j}^{U_j} \mathbb{E}\Big|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathcal{O}_{j-l})(u)\Big|^4\;K_{j-l}(u)^{-4}u^8\,|w^{U_j}_{\mu_j}(u)|^2\,du.\end{aligned}$$

As per equation Koorevaar [15, eq. (3.28)], it holds that

$$\begin{aligned}&\mathbb{E}\left[|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathcal{O}_{j-l})(u)|^4\right]\\ &\le 4\mathbb{E}\left[|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathbb{E}\,\mathcal{O}_{j-l})(u)|^4\right]\\ &+ 8\left\|\mathscr{F}(\mathcal{O}_{j-l} - \mathbb{E}\tilde{\mathcal{O}}_{j-l})\right\|^2_\infty \mathbb{E}\left[|\mathscr{F}(\widetilde{\mathcal{O}}_{j-l} - \mathbb{E}\,\mathcal{O}_{j-l})(u)|^2\right]\\ &+ 4\left\|\mathscr{F}(\mathcal{O}_{j-l} - \mathbb{E}\tilde{\mathcal{O}}_{j-l})\right\|^4_\infty.\end{aligned}$$

Following Koorevaar [15, eqs. (3.29), (3.30)], we obtain

$$\begin{aligned}\mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - O_{j-l})(u)|^4\big] \lesssim\;& \mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^4\big]\\ &+ \|\delta_{j-l}\|^2_\infty \Delta^4_{j-l} + \Delta^8_{j-l}.\end{aligned} \tag{6.6}$$

Let us now bound the first summand using the interpolation scheme for the option prices:

$$\begin{aligned}\mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^4\big] &= \mathbb{E}\left[\left|\mathscr{F}\left(\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\,\varepsilon_{j-l,k}\;b_{j-l,k}\right)(u)\right|^4\right]\\ &= \mathbb{E}\left[\left|\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\,\mathscr{F}b_{j-l,k}(u)\right|^4\right].\end{aligned}$$

Write now

$$s_{j-l,k}(u) := \delta_{j-l,k}\mathscr{F}b_{j-l,k}(u).$$

Then

$$\begin{aligned}\mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^4\big] &= \mathbb{E}\left|\sum_{k=1}^{m_{j-l}} s_{j-l,k}(u)\varepsilon_{j-l,k}\right|^4 \\ &= \mathbb{E}\Big(|X_{j-l,k}(u)|^2\Big)^2,\end{aligned}$$

where

$$|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)| = \sum_{k=1}^{m_{j-l}} s_{j-l,k}(u)\varepsilon_{j-l,k}.$$

First expand the square

$$\begin{aligned}|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^2 &= \left(\sum_{i=1}^{m_{j-l}} s_{j-l,i}(u)\,\varepsilon_{j-l,i}\right)\left(\sum_{k=1}^{m_{j-l}} s_{j-l,k}(u)\,\varepsilon_{j-l,k}\right) \\ &= \sum_{i=1}^{m_{j-l}}\sum_{k=1}^{m_{j-l}} s_{j-l,i}(u)\,s_{j-l,k}(u)\varepsilon_{j-l,i}\,\varepsilon_{j-l,k}.\end{aligned}$$

Hence

$$\begin{aligned}|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^4 &= \left(\sum_{i=1}^{m_{j-l}}\sum_{k=1}^{m_{j-l}} s_{j-l,i}(u)\,s_{j-l,k}(u)\,\varepsilon_{j-l,i}\,\varepsilon_{j-l,k}\right)^2 \\ &= \sum_{i,k,r,t} s_{j-l,i}(u)\,s_{j-l,k}(u)\,s_{j-l,r}(u)\,s_{j-l,t}(u)\varepsilon_{j-l,i}\,\varepsilon_{j-l,k}\,\varepsilon_{j-l,r}\,\varepsilon_{j-l,t}.\end{aligned}$$

Taking expectations and using independence of the noise variables yields

$$\mathbb{E}|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^4 = \sum_{i,k} s_{j-l,i}(u)^2 s_{j-l,k}(u)^2\mathbb{E}[\varepsilon_{j-l,i}^2\,\varepsilon_{j-l,k}^2].$$

Since the noise variables are sub Gaussian they have finite fourth moment,

$$\mathbb{E}[\varepsilon_{j-l,i}^2\,\varepsilon_{j-l,k}^2] \le C,$$

and therefore

$$\mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^4\big] \lesssim \sum_{i,k}\delta_{j-l,i}^2\delta_{j-l,k}^2|\mathscr{F}b_{j-l,i}(u)|^2|\mathscr{F}b_{j-l,k}(u)|^2.$$

Finally, using

$$\sum_{i,k} a_i a_k = \left(\sum_k a_k\right)^2,$$

we obtain

$$\begin{aligned}\mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - \mathbb{E}O_{j-l})(u)|^4\big] &\lesssim \left(\sum_{k=1}^{m_{j-l}}\delta_{j-l,k}^2|\mathscr{F}b_{j-l,k}(u)|^2\right)^2 \\ &\lesssim \left(\sum_{k=1}^{m_{j-l}}\delta_{j-l,k}^2\|\mathscr{F}b_{j-l,k}(u)\|_\infty^2\right)^2 \\ &\lesssim \Delta_{j-l}^4\|\delta_{j-l}\|_{l^2}^4 \\ &\lesssim \Delta_{j-l}^2\,\|\delta_{j-l}\|_\infty^4\end{aligned}$$

where we used equation (3.26). Thus, we can conclude that

$$\mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - O_{j-l})(u)|^4\big] \lesssim \Delta_{j-l}^2\,\|\delta_{j-l}\|_\infty^4.$$

6.2.3. Bound of $\int_{-U_j}^{U_j} K_{j-l}(u)^{-4} u^8 \, |w_{\mu_j}^{U_j}(u)|^2 \, du$

As a U_j-independent bound for $\mathbb{E}\big[|\mathscr{F}(\tilde{O}_{j-l} - O_{j-l})(u)|^4\big]$ has been found, it holds that:

$$\begin{aligned}\mathbb{E}\Big(||\cos(U_j x)\mathscr{R}^l_{\mu_j}||^2_{L^2(K)}\Big) &\lesssim \Delta^2_{j-l}\, \|\delta_{j-l}\|^4_\infty \int_{-U_j}^{U_j} K_{j-l}(u)^{-4} u^8 \, |w_{\mu_j}^{U_j}(u)|^2 \, du \\ &\lesssim \Delta^2_{j-l} \int_{-U_j}^{U_j} K_{j-l}(u)^{-4} u^8 \, |w_{\mu_j}^{U_j}(u)|^2 \, du.\end{aligned}$$

where $\|\delta_{j-l}\|_\infty$ is dropped as it is a constant which does not contribute asymptotically. It can be readily observed that

$$\frac{1}{K_j(v)^2} = 2^{-(2j-2)} e^{v^2 A_j - 4RT_j} \lesssim e^{v^2 A_j}.$$

Moreover, we know that the weight function is bounded in the domain $[-U_j, U_j]$. Hence

$$\begin{aligned}\int_{-U_j}^{U_j} K_{j-l}(u)^{-4} u^8 \, |w_{\mu_j}^{U_j}(u)|^2 \, du &\lesssim \int_{-U_j}^{U_j} u^8 e^{2A_{j-l}u^2} \\ &\lesssim U_j^7 \int_0^{U_j} 4A_{j-l}\, u \, e^{2A_{j-l}\, u^2} \\ &\lesssim U_j^7 \, e^{2A_{j-l}\, U_j^2}.\end{aligned}$$

6.2.4. Bound of $\mathbb{E}\Big(||\cos(U_j x)\mathscr{R}^l_{\mu_j}||^2_{L^2(K)}\Big)$ & Chebyshev's Inequality

Given the preceding subsections, we can bound the expectation of the modulated remainder term as follows:

$$\mathbb{E}\Big(||\cos(U_j x)\mathscr{R}^l_{\mu_j}||^2_{L^2(K)}\Big) \lesssim \Delta^2_{j-l}\, U_j^7 \, e^{2A_j U_j^2}. \tag{6.7}$$

Furthermore, it must also be recalled that our CLT setting includes a prefactor

$$r_j = \frac{1}{\sqrt{a_j + a_{j-1}}},$$

where

$$a_{j-l} = \Delta_{j-l}\, U_j^2 \, e^{A_{j-l} U_j^2} \quad \forall \, l = 0, 1.$$

Now, applying Chebyshev's inequality to $\mathbb{P}\left(\frac{1}{\sqrt{a_j + a_{j-1}}}||\cos(U_j x)\mathscr{R}^l_{\mu_j}(x)||_{L^2(K)} > \varepsilon\right)$, we obtain

$$\begin{aligned}&\mathbb{P}\left(\frac{1}{\sqrt{a_j + a_{j-1}}}||\cos(U_j x)\mathscr{R}^l_{\mu_j}(x)||_{L^2(K)} > \varepsilon\right) \\ &\leq \frac{1}{\varepsilon^2}\mathbb{E}\left[\left|\left|\frac{1}{\sqrt{a_j + a_{j-1}}}\cos(U_j x)\mathscr{R}^l_{\mu_j}(x)\right|\right|^2_{L^2(K)}\right] \\ &\leq \frac{1}{\varepsilon^2}\left(\frac{1}{a_j + a_{j-1}}\right)\mathbb{E}\left[\left|\left|\cos(U_j x)\mathscr{R}^l_{\mu_j}(x)\right|\right|^2_{L^2(K)}\right] \\ &\lesssim \frac{1}{a_j + a_{j-1}}\Delta^4_{j-l}\, U_j^7 \, e^{2A_j U_j^2} \\ &\leq \frac{1}{a_j}\Delta^2_{j-l}\, U_j^7 \, e^{2A_{j-l} U_j^2} \\ &= \Delta^2_{j-l}\, U_j^7 \, e^{2A_{j-l} U_j^2} \cdot (\Delta_j \, U_j^2 \, e^{A_j U_j^2})^{-1} \\ &= \frac{\Delta^2_{j-l}}{\Delta_j}\, U_j^5 \, e^{(2A_{j-l} - A_j) U_j^2} \to 0.\end{aligned}$$

where the decay to 0 is given by our assumption in equation (6.3) of Proposition 6.2. Hence, the stochastic error is negligible, and the proposition is proved.

6.3. Vanishing of Additional Terms

Finally, it will be shown that the additional terms in the error decomposition $\widetilde{\mu}_j - \mu_j$ also vanish asymptotically under modulation and scaling. For that, we state and prove the following proposition, proved in sections 6.3.1-6.3.3:

Proposition 6.3 (Vanishing of Σ, Γ, Λ in $L^2(K;\mathbb{R})$)**.** *Assume the standing assumptions and the decay condition*

$$\Delta_j\, U_j^4\, \exp\Big(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2\Big) \longrightarrow 0,$$

. Then

$$r_j \,\|\cos(U_j x)\,\Sigma\|_{L^2(K;\mathbb{R})} \xrightarrow{\mathbb{P}} 0, \qquad r_j \,\|\cos(U_j x)\,\Gamma\|_{L^2(K;\mathbb{R})} \xrightarrow{\mathbb{P}} 0, \qquad r_j \,\|\cos(U_j x)\,\Lambda\|_{L^2(K;\mathbb{R})} \xrightarrow{\mathbb{P}} 0.$$

6.3.1. Proof that $r_j\|\cos(U_j x)\Sigma\|_{L^2(K)} \xrightarrow{\mathbb{P}} 0$ as $U_j \to \infty$

Since $|\cos(U_j x)| \leq 1$,

$$\|r_j \cos(U_j x)\Sigma\|^2_{L^2(K)} = r_j^2 \int_K \cos^2(U_j x)\,|\Sigma(x)|^2\,dx \leq r_j^2\,\|\Sigma\|^2_{L^2(K)}.$$

Since $L^2(K) \subset L^2(\mathbb{R})$, we can apply Parseval's identity:

$$\|\Sigma\|^2_{L^2(K)} \leq \|\Sigma\|^2_{L^2(\mathbb{R})} = \int_{-U_j}^{U_j} |\hat{\Sigma}(\xi)|^2\,d\xi = \left(\frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2}\right)^2 \int_{-U_j}^{U_j} |\xi - i|^4\,|w_{\mu_j}^{U_j}(\xi)|^2\,d\xi.$$

Since the weight function is assumed to be bounded on $[-U_j, U_j]$,

$$\|\Sigma\|^2_{L^2(K)} \lesssim \left(\frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2}\right)^2 \int_{-U_j}^{U_j} |\xi - i|^4\,d\xi \lesssim \left(\frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2}\right)^2 U_j^5.$$

Taking expectations and applying Theorem 3.2, which gives $\mathbb{E}(\tilde{\sigma}_j^2 - \sigma_j^2)^2 \lesssim \Delta_j\, e^{A_j U_j^2}\, U_j^{-4}$:

$$r_j^2\,\mathbb{E}\|\cos(U_j x)\Sigma\|^2_{L^2(K)} \lesssim r_j^2\,\Delta_j\,e^{A_j U_j^2}\,U_j^{-4}\cdot U_j^5 = r_j^2\,\Delta_j\,e^{A_j U_j^2}\,U_j.$$

Recalling that $r_j^2 = (a_j + a_{j-1})^{-1}$:

$$\begin{aligned} r_j^2\,\mathbb{E}\|\cos(U_j x)\Sigma\|^2_{L^2(K)} &\lesssim (a_j + a_{j-1})^{-1}\cdot \Delta_j\,e^{A_j U_j^2}\,U_j \\ &= U_j^{-1}\cdot \frac{\Delta_j\,e^{A_j U_j^2}\,U_j^2}{a_j + a_{j-1}} \\ &= U_j^{-1}\cdot \frac{a_j}{a_j + a_{j-1}}. \end{aligned}$$

From section 5.7 we know that

$$\frac{a_j}{a_j + a_{j-1}} \to 1$$

as $U_j \to \infty$. Hence, this means that the expression as a whole goes to 0, and, by Chebyshev's inequality, $r_j\|\cos(U_j x)\Sigma\|_{L^2(K)} \xrightarrow{\mathbb{P}} 0$.

6.3.2. Proof that $r_j\|\cos(U_j x)\Gamma\|_{L^2(K)} \xrightarrow{\mathbb{P}} 0$ as $U_j \to \infty$

The proof for the decay of this term is analogous to that of the Σ term, namely, since $|\cos(U_j x)| \leq 1$,

$$\|r_j \cos(U_j x)\Gamma\|^2_{L^2(K)} = r_j^2 \int_K \cos^2(U_j x)\,|\Gamma(x)|^2\,dx \leq r_j^2\,\|\Gamma\|^2_{L^2(K)}.$$

Since $L^2(K) \subset L^2(\mathbb{R})$, we can apply Parseval's identity:

$$\|\Gamma\|^2_{L^2(K)} \le \|\Gamma\|^2_{L^2(\mathbb{R})} = \int_{-U_j}^{U_j} |\hat{\Gamma}(\xi)|^2 \, d\xi = \left(\frac{\tilde{\gamma}_j - \gamma_j}{2}\right)^2 \int_{-U_j}^{U_j} |\xi - i|^2 \, |w^{U_j}_{\mu_j}(\xi)|^2 \, d\xi.$$

Since the weight function is bounded on $[-U_j, U_j]$,

$$\|\Gamma\|^2_{L^2(K)} \lesssim \left(\frac{\tilde{\gamma}_j - \gamma_j}{2}\right)^2 \int_{-U_j}^{U_j} |\xi - i|^2 \, d\xi \lesssim \left(\frac{\tilde{\gamma}_j - \gamma_j}{2}\right)^2 U_j^3.$$

Taking expectations and applying Theorem 3.2, which gives $\mathbb{E}(\tilde{\gamma}_j - \gamma_j)^2 \lesssim \Delta_j \, e^{A_j U_j^2} \, U_j^{-2}$:

$$r_j^2 \, \mathbb{E}\| \cos(U_j x)\Gamma\|^2_{L^2(K)} \lesssim r_j^2 \, \Delta_j \, e^{A_j U_j^2} \, U_j^{-2} \cdot U_j^3 = r_j^2 \, \Delta_j \, e^{A_j U_j^2} \, U_j.$$

Recalling that $r_j^2 = (a_j + a_{j-1})^{-1}$:

$$\begin{aligned} r_j^2 \, \mathbb{E}\| \cos(U_j x)\Gamma\|^2_{L^2(K)} &\lesssim (a_j + a_{j-1})^{-1} \cdot \Delta_j \, e^{A_j U_j^2} \, U_j \\ &= U_j^{-1} \cdot \frac{\Delta_j \, e^{A_j U_j^2} \, U_j^2}{a_j + a_{j-1}} \\ &= U_j^{-1} \cdot \frac{a_j}{a_j + a_{j-1}}. \end{aligned}$$

From section 5.7 we know that

$$\frac{a_j}{a_j + a_{j-1}} \to 1$$

as $U_j \to \infty$. Hence, this means that the expression as a whole goes to 0, and, by Chebyshev's inequality, $r_j \| \cos(U_j x)\Gamma\|_{L^2(K)} \xrightarrow{\mathbb{P}} 0$.

6.3.3. Proof that $r_j \| \cos(U_j x)\Lambda\|_{L^2(K)} \xrightarrow{\mathbb{P}} 0$ as $U_j \to \infty$

The proof for the decay of the Λ term is analogous to that of the Γ term, namely, since $|\cos(U_j x)| \le 1$,

$$\|r_j \cos(U_j x)\Lambda\|^2_{L^2(K)} = r_j^2 \int_K \cos^2(U_j x) \, |\Lambda(x)|^2 \, dx \le r_j^2 \, \|\Lambda\|^2_{L^2(K)}.$$

Since $L^2(K) \subset L^2(\mathbb{R})$, we can apply Parseval's identity:

$$\begin{aligned} \|\Lambda\|^2_{L^2(K)} &\le \|\Lambda\|^2_{L^2(\mathbb{R})} \\ &= \int_{-U_j}^{U_j} |\hat{\Lambda}(\xi)|^2 \, d\xi \\ &= \left(\tilde{\lambda}_j - \lambda_j\right)^2 \int_{-U_j}^{U_j} |w^{U_j}_{\mu_j}(\xi)|^2 \, d\xi. \end{aligned}$$

Since the weight function is bounded on $[-U_j, U_j]$,

$$\|\Lambda\|^2_{L^2(K)} \lesssim \left(\tilde{\lambda}_j - \lambda_j\right)^2 \int_{-U_j}^{U_j} d\xi \lesssim \left(\tilde{\lambda}_j - \lambda_j\right)^2 U_j.$$

Taking expectations and applying Theorem 3.2, which gives $\mathbb{E}(\tilde{\lambda}_j - \lambda_j)^2 \lesssim \Delta_j \, e^{A_j U_j^2}$:

$$r_j^2 \, \mathbb{E}\| \cos(U_j x)\Lambda\|^2_{L^2(K)} \lesssim r_j^2 \, \Delta_j \, e^{A_j U_j^2} \cdot U_j.$$

Recalling that $r_j^2 = (a_j + a_{j-1})^{-1}$:

$$\begin{aligned} r_j^2 \, \mathbb{E}\| \cos(U_j x)\Lambda\|^2_{L^2(K)} &\lesssim (a_j + a_{j-1})^{-1} \cdot \Delta_j \, e^{A_j U_j^2} \, U_j \\ &= U_j^{-1} \cdot \frac{\Delta_j \, e^{A_j U_j^2} \, U_j^2}{a_j + a_{j-1}} \\ &= U_j^{-1} \cdot \frac{a_j}{a_j + a_{j-1}}. \end{aligned}$$

From section 5.7 we know that

$$\frac{a_j}{a_j + a_{j-1}} \to 1$$

as $U_j \to \infty$. Hence, this means that the expression as a whole goes to 0, and, by Chebyshev's inequality, $r_j \| \cos(U_j x) \Lambda \|_{L^2(K)} \xrightarrow{\mathbb{P}} 0$. Hence, all three additional terms go to 0 in probability, and the proposition has been proved.

Chapter 7
Central Limit Theorem for $\widetilde{\mu}_j - \mu_j$ & $\widetilde{\nu}_j - \nu_j$ in $L^2(K)$

Recall the decomposition of the error of the estimate of the Lévy measure:

$$\widetilde{\mu}_j(x) - \mu_j(x) = \Psi + \Sigma + \Gamma + \Lambda + \mathscr{B},$$

where

$$\Psi = \mathscr{L}^0_{\mu_j} - \mathscr{L}^1_{\mu_j} + \mathscr{R}^0_{\mu_j} - \mathscr{R}^1_{\mu_j}.$$

In Chapter 5 it was shown that the difference of the linear terms, multiplied by a normalization prefactor r_j and a modulation prefactor $\cos(U_j\,\cdot)$ converges weakly to a Gaussian in the space $L^2(K;\mathbb{R})$, where the decay condition needed is the same as in Koorevaar, namely:

$$\Delta_j\, U_j^4\, e^{A_j\, U_j^2} \to 0 \quad \text{as } U_j \to \infty.$$

Furthermore, it has also been shown that the rest of the terms in the error decomposition, when multiplied by $r_j\, \cos(U_j\,\cdot)$, tend to 0 in $L^2(K;\mathbb{R})$. As a result, the normality result of the linear part of the error decomposition can be extended to the whole error term $\widetilde{\mu}_j(x) - \mu_j(x)$. Hence, for the whole error term, the following CLT holds:

Theorem 7.1 (Functional CLT for the Exponentially Tilted Lévy Measure Estimator)**.** *Assume the standing assumptions, the decay conditions and bias condition*

$$\Delta_j\, U_j^4\, \exp\Big(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2\Big) \longrightarrow 0,$$

$$\frac{\Delta_{j-l}^2}{\Delta_j}\, U_j^5\, \exp\left(U_j^2\left(2\sum_{i=1}^{j-l}(T_i - T_{i-1})\sigma_i^2 - \sum_{i=1}^{j}(T_i - T_{i-1})\sigma_i^2\right)\right) \longrightarrow 0, \quad \textit{for } l = 0, 1,$$

and

$$U_j^{2(s_j+1)}\Big(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\Big) \longrightarrow \infty,$$

Then the full estimation error satisfies

$$r_j\, \cos(U_j x)\big(\widetilde{\mu}_j(x) - \mu_j(x)\big) \overset{w}{\Longrightarrow} \mathcal{N}(0, A^0) \quad \textit{in } L^2(K;\mathbb{R}),$$

where $A^0 y = \frac{1}{2}\mathcal{C}\,\langle y, 1\rangle_{L^2(K)} \cdot 1$ is the rank-one nuclear covariance operator identified in Theorem 5.1, and $r_j = (a_j + a_{j-1})^{-1/2}$.

Recall now that

$$\widetilde{\mu}_j(x) - \mu_j(x) = e^x\big(\widetilde{\nu}_j(x) - \nu_j(x)\big). \tag{7.1}$$

The goal is now to obtain an asymptotic normality result only in terms of $\widetilde{\nu}_j(x) - \nu_j(x)$, the modulation and the scaling (this will be of great help to obtain the next main result for this thesis, in Chapter 8). To that end, note that the operator

$$T : f \mapsto e^{-x}\, f$$

is a bounded linear operator in the function space $L^2(K)$ (which means it is continuous). By the Continuous Mapping Theorem, we know that for a sequence of random variables X_j and a Gaussian G, if $X_j \overset{w}{\Longrightarrow} G$, then $T\,X_j \overset{w}{\Longrightarrow} T\,G$. Moreover, given that T is linear, TG is Gaussian again with covariance

$$\operatorname{Cov}(TG) = T\operatorname{Cov}(G)\,T^* = T\operatorname{Cov}(G)\,T,$$

where the last equality is due to the fact that multiplication in a real space like $L^2(K)$ is self-adjoint, so $T = T^*$. Hence, it is possible to derive a normality result for the difference of the non-exponentially tilted Lévy measure $\widetilde{\nu}_j(x) - \nu_j(x)$, namely:

Theorem 7.2 (Functional CLT Lévy Measure Estimator)**.** *Assume the standing assumptions, the decay conditions and bias condition*

$$\Delta_j\, U_j^4\, \exp\Big(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2\Big) \longrightarrow 0,$$

$$\frac{\Delta_{j-l}^2}{\Delta_j}\, U_j^5\, \exp\left(U_j^2\left(2\sum_{i=1}^{j-l}(T_i - T_{i-1})\sigma_i^2 - \sum_{i=1}^{j}(T_i - T_{i-1})\sigma_i^2\right)\right) \longrightarrow 0, \quad \textit{for } l = 0, 1,$$

and

$$U_j^{2(s_j+1)}\Big(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\Big) \longrightarrow \infty,$$

Then the full estimation error satisfies

$$r_j \cos(U_j x)\, \big(\widetilde{\nu}_j(x) - \nu_j(x)\big) \overset{w}{\Longrightarrow} \mathcal{N}(0, T\, A^0\, T) \quad \textit{in } L^2(K; \mathbb{R}),$$

where $TA^0Ty = \frac{1}{2}\mathcal{C}\, \langle y, e^{-x}\rangle_{L^2(K)} \cdot e^{-x}$ *is the rank-one nuclear covariance operator identified and* $r_j = (a_j + a_{j-1})^{-1/2}$.

This normality result will now be leveraged to derive a convergence in distribution result for the error of the estimate of the Gil-Pelaez formula (which is also the error of the price of a Digital Option) for an exponential Lévy model. For the pricing application, the parameters $(\sigma_j^2, \gamma_j, \lambda_j)$ are assumed to be known, with only the Lévy measure estimated via the method of Tendijck [27] and Koorevaar [15]. This is justified by the fact that, as given by the convergence rates in Chapter 3 the errors of $\widetilde{\sigma}_j^2, \widetilde{\gamma}_j, \widetilde{\lambda}_j$ converge faster than that of $\widetilde{\nu}_j$, so any error resulting from treating them as known will be asymptotically negligible.

will be shown in Section 8.6, the contributions of the parametric components Σ, Γ, Λ to the pricing error vanish faster than the Lévy-measure contribution under scaling. This means that the estimation error ν_j is the dominant one and the other errors are asymptotically negligible for the option price.

Chapter 8
Digital Option Price- Convergence in Distribution

A digital or binary option is a type of option whose payoff depends on the underlying asset S being above or below a certain value K at the time of expiry T. The payoff is either a fixed monetary value or 0 [21]. For a digital call option C, the corresponding payoff function is:

$$C = \mathbf{1}_{\{S_T > K\}}. \tag{8.1}$$

Hence, under the Martingale Measure $\mathbb{P}$, the corresponding price of a digital call option at expiry is

$$\mathbb{E}_{\mathbb{P}}(C) = \mathbb{P}(S_T > K). \tag{8.2}$$

Recall that the model in Koorevaar [15] and in Belomestny and Reiß[1] is calibrated via European call and put options, $\mathcal{C}(K_{j,k};T_j)$ and $\mathcal{P}(K_{j,k};T_j)$ with maturities at T_j, $j = 1, \ldots, n$ and $k = 1, \ldots, m_j$. Let us assume that the digital option we are trying to price has the same maturity and strike price T_j and $K_{j,k}$ as one of the call or put options used to calibrate the model. It can be assumed that S_t follows an exponential Lévy model for all t:

$$S_t = S_0 e^{rt+X_t},$$

where (X_t) is a time-inhomogeneous Lévy process. This means that

$$S_t > K_{j,k} \iff \ln(S_0) + rt + X_t > \ln(K_{j,k}).$$

In order to price a digital call option with maturity T_j (which we will refer to as Π_j) based on our time-inhomogeneous Lévy model, we can leverage the characteristic function of the stochastic process. Lord and Kahl [17] do exactly this, providing the following formula, where the integral summand corresponds to the Gil-Pelaez inversion formula [9]:

$$\begin{aligned}\Pi_j &:= \mathbb{Q}\left(S_{T_j} > K_{j,k}\right) \\ &= \mathbb{Q}\left(\ln(S_0) + r\,T_j + X_{T_j} > \ln(K_{j,k})\right) \\ &= \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-iu(\ln(K_{j,k}/S_0) - r\,T_j)} \cdot \varphi_{X_t}(u)}{iu}\right) du,\end{aligned} \tag{8.3}$$

where we can take $k' := \ln(K_{j,k}/S_0) - r\,T_j$ and where $k = \ln(K)$. Now, we can assume that the parameters σ_j^2 and γ_j are available for all $j = 1, ..., n$. Since the model follows a time-inhomogeneous Lévy process, we need to look at the pricing operator at every interval between maturities $[T_{j-1}, T_j]$:

$$\varphi_{X_t} = \varphi_{T_j} := \exp\left((T_j - T_{j-1})\left(-\frac{\sigma_j^2 u^2}{2} + i\gamma_j u + \int_{\mathbb{R}} (e^{iux} - 1)\nu_j(x)\mathrm{d}x\right)\right)\varphi_{T_{j-1}}. \tag{8.4}$$

Since it is defined recursively, the characteristic function for maturity T_j depends on all the triplets of all intervals until j, i.e., $(\sigma_i^2, \gamma_i, \nu_i)_{i \le j}$. To distinguish between the characteristic function of $\widetilde{\nu}_j$ and that of ν_j, let us also define

$$\widetilde{\varphi}_{T_j} := \exp\left((T_j - T_{j-1})\left(-\frac{\sigma_j^2 u^2}{2} + i\gamma_j u + \int_{\mathbb{R}} (e^{iux} - 1)\widetilde{\nu}_j(x)\mathrm{d}x\right)\right)\widetilde{\varphi}_{T_{j-1}}.$$

Hence, with this we can also define a pricing operator for the estimated Lévy measure $\widetilde{\nu}_j$:

$$\widetilde{\Pi}_j = \mathbb{Q}(S_T > K) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-iuk'} \cdot \widetilde{\varphi}_{T_j}(u)}{iu}\right) du.$$

Although Koorevaar's procedure estimates all characteristic triplet parameters $(\sigma_j^2, \gamma_j, \nu_j)$ simultaneously, the CLT for $\tilde{\nu}_j - \nu_j$ established in Chapter 7 is driven entirely by the linear terms $\mathscr{L}^0_{\nu_j} - \mathscr{L}^1_{\nu_j}$, which depend only on the estimation error of the Lévy measure. The estimated pricing operator $\widetilde{\Pi}_j$ is obtained by replacing φ_{T_j} with $\widetilde{\varphi}_{T_j}$, the characteristic function built from the full estimated triplet $(\widetilde{\sigma}_i^2, \widetilde{\gamma}_i, \widetilde{\nu}_i)_{i\leq j}$. To isolate the contribution of the Lévy measure, we assume that σ_i^2, γ_i, and λ_i are known exactly, so that $\widetilde{\varphi}_{T_j}$ and φ_{T_j} differ only through $\widetilde{\nu}_j - \nu_j$. This is what allows the $\sigma_j^2/2 + i\gamma_j$ factor to cancel in the error term of Section 8.2. The assumption is justified by the convergence rates of Chapter 3: the parametric estimators converge strictly faster than $\widetilde{\nu}_j$, so the error incurred by treating them as exact vanishes asymptotically.

8.1. Theorem Statement

Recall Theorem 7.2, where $r_j \cos(U_j\cdot)(\widetilde{\nu}_j - \nu_j) \Rightarrow \mathcal{N}(0, TA^0T)$ in $L^2(K)$, and let $g_j \in L^2(K)$ denote the kernel given by the linearization of the Gil-Pelaez functional, which is uniformly bounded in U_j.

Theorem 8.1 (Digital option pricing error, subsequential CLT)**.** *Under the standing assumptions, the decay condition*

$$\Delta_j\, U_j^4\, \exp\Big(U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2\Big) \longrightarrow 0,$$

and the bias condition

$$U_j^{2(s_j+1)}\Big(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\Big) \longrightarrow \infty,$$

the normalized pricing error admits the linearization

$$R_j\big(\widetilde{\Pi}_j - \Pi_j\big) = \frac{2}{\pi}(T_j - T_{j-1})\, \langle R_j\, \alpha_j,\, g_j\rangle_{L^2(K)} + o_{\mathbb{P}}(1).$$

where $\alpha_j := \cos(U_j\cdot)(\widetilde{\nu}_j - \nu_j)$ $R_j = 1/\sqrt{\sum_{r=1}^{j} a_r}$, $a_r = \Delta_r\, U_j^2\, \exp\{A_r\, U_j^2\}$ *for* $r = 1, \ldots, j$, *and*

$$g_j(x) := \int_{-U_j+\varepsilon}^{\infty} \Re\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\, \varphi_{T_j}(u+U_j)\, e^{iux} \right) du.$$

Moreover, there exists a subsequence (j_k) *and a weak limit* $g \in L^2(K)$ *with* $g_{j_k} \rightharpoonup g$ *such that, along this subsequence,*

$$R_{j_k}\big(\widetilde{\Pi}_{j_k} - \Pi_{j_k}\big) \;\rightsquigarrow\; \langle G,\, g\rangle_{L^2(K)}, \qquad G \sim \mathcal{N}\left(0, TA^0T\right),$$

with limiting variance

$$\langle TA^0T\, g,\, g\rangle_{L^2(K)} = \tfrac{1}{2}\mathcal{C}\, \langle g,\, e^{-x}\rangle^2_{L^2(K)}.$$

The limiting variance is non-degenerate if and only if $\langle g, e^{-x}\rangle_{L^2(K)} \neq 0$. Whether this holds depends on the oscillatory behavior of g_j as $U_j \to \infty$ and on the structure of the Gil-Pelaez kernel.

Before starting with the proof of Theorem 8.1, we can provide a short overview of what steps will be followed to achieve the final result. First, the error $\widetilde{\Pi}_j - \Pi_j$ will be linearized. Thereafter, it will be shown that a scaling that is asymptotically equivalent to that of Theorem 7.2 eliminates all Lévy density errors for maturities $r < j$, which means for the theorem it is only necessary to focus on the error $\widetilde{\nu}_j - \nu_j$.

Afterwards, it will be shown that the remainder term decays as well, and that the principal term can be re-expressed through the change of variables $u \mapsto u + U_j$ and further simplified. Moreover, trigonometric identities will be used to recover a cosine-modulated Lévy density error as part of the pricing error formula. It will then be noted that said pricing error can be interpreted as an $L^2(K)$ duality pairing. As $L^2(K)$ is a Hilbert space, we can use the fact that every bounded sequence has a weakly convergent subsequence, plus the Extended Continuous Mapping Theorem, due to Van der Vaart and Wellner [28], to recover a convergence-in-distribution result for $\widetilde{\Pi}_j - \Pi_j$ along a subsequence.

8.2. Analysis of the Error Term $\widetilde{\Pi}_j - \Pi_j$

Looking now at the error term between the true pricing operator and the estimated pricing operator, we obtain the following expression

$$\widetilde{\Pi}_j - \Pi_j = \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-iuk'}}{iu} \left(\widetilde{\varphi}_{T_j}(u) - \varphi_{T_j}(u)\right)\right) du. \tag{8.5}$$

Note now that φ_{T_j} can be fully written out as:

$$\begin{aligned}
\varphi_{T_j}(u) = \exp&\left((T_j - T_{j-1})\left(-\frac{\sigma_j^2 u^2}{2} + i\gamma_j u + \int_{\mathbb{R}} (e^{iux} - 1)\nu_j(x)\mathrm{d}x\right)\right) \cdot \\
&\exp\left((T_{j-1} - T_{j-2})\left(-\frac{\sigma_{j-1}^2 u^2}{2} + i\gamma_{j-1} u + \int_{\mathbb{R}} (e^{iux} - 1)\nu_{j-1}(x)\mathrm{d}x\right)\right) \dots \\
&\exp\left(T_1\left(-\frac{\sigma_1^2 u^2}{2} + i\gamma_1 u + \int_{\mathbb{R}} (e^{iux} - 1)\nu_1(x)\mathrm{d}x\right)\right) = \\
&= \exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\left(-\frac{\sigma_r^2 u^2}{2} + i\gamma_r u + \int_{\mathbb{R}} (e^{iux} - 1)\nu_r(x)\mathrm{d}x\right)\right),
\end{aligned}$$

where it must be recalled that $T_0 = 0$. Thus, we can now look at the subtraction of the two terms:

$$\begin{aligned}
\widetilde{\varphi}_{T_j}(u) - \varphi_{T_j}(u) &= \exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\left(-\frac{\sigma_r^2 u^2}{2} + i\gamma_r u + \int_{\mathbb{R}} (e^{iux} - 1)\widetilde{\nu}_r(x)\mathrm{d}x\right)\right) \\
&- \exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\left(-\frac{\sigma_r^2 u^2}{2} + i\gamma_r u + \int_{\mathbb{R}} (e^{iux} - 1)\nu_r(x)\mathrm{d}x\right)\right) \\
&= \exp\left(-\sum_{r=1}^{j}(T_r - T_{r-1})\frac{\sigma_r^2 u^2}{2} + \sum_{r=1}^{j}(T_r - T_{r-1})\, i\gamma_r\right) \\
&\cdot \left(\exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\int_{\mathbb{R}} (e^{iux} - 1)\widetilde{\nu}_r(x)\mathrm{d}x\right)\right. \\
&- \left.\exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\int_{\mathbb{R}} (e^{iux} - 1)\nu_r(x)\mathrm{d}x\right)\right) \\
&= M \cdot \exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\int_{\mathbb{R}} (e^{iux} - 1)\nu_r(x)\mathrm{d}x\right) \\
&\cdot \left(\exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\int_{\mathbb{R}} (e^{iux} - 1)(\widetilde{\nu}_r(x) - \nu_r(x))\mathrm{d}x\right) - 1\right) \\
&= \varphi_{T_j}(u)\left(\exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\int_{\mathbb{R}} (e^{iux} - 1)(\widetilde{\nu}_r(x) - \nu_r(x))\mathrm{d}x\right) - 1\right).
\end{aligned}$$

Hence, the error term $\widetilde{\Pi}_j - \Pi_j$ can be rewritten as

$$\frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-iuk'}}{iu} \left(\varphi_{T_j}(u)\left(\exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\int_{\mathbb{R}} (e^{iux} - 1)(\widetilde{\nu}_r(x) - \nu_r(x))\mathrm{d}x\right) - 1\right)\right)\right) du \tag{8.6}$$

$$= \frac{1}{\pi}\int_0^{\infty} \Re\left(\frac{e^{-iuk'}}{iu} \left(\varphi_{T_j}(u)\left(\exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\psi_r(u)\right) - 1\right)\right)\right) du. \tag{8.7}$$

8.3. Linearization

Let us now carry out the following linearization:

$$\exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\psi_r(u)\right) - 1 = \sum_{r=1}^{j}(T_r - T_{r-1})\psi_r(u) + \mathcal{R}_{\Pi_j}(u), \tag{8.8}$$

where

$$|\mathcal{R}_{\Pi_j}(u)| \leq \frac{1}{2}\left(\sum_{r=1}^{j}(T_r - T_{r-1})|\psi_r(u)|\right)^2 \cdot \exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\,|\psi_r(u)|\right). \tag{8.9}$$

Hence,

$$\begin{aligned}\widetilde{\Pi}_j - \Pi_j &= \frac{1}{\pi}\int_0^\infty \Re\!\left(\frac{e^{-iuk'}}{iu}\,\varphi_{T_j}(u)\sum_{r=1}^{j}(T_r - T_{r-1})\psi_r(u)\right)du + \frac{1}{\pi}\int_0^\infty \Re\!\left(\frac{e^{-iuk'}}{iu}\,\varphi_{T_j}(u)\,\mathcal{R}_{\Pi_j}(u)\right)du \\ &:= P_j + B_j.\end{aligned}$$

8.4. Reduction to the Current-Maturity Error in P_j

Let us now recall that, after linearization, the difference of the Gil-Pelaez formulas yields a linear term of the form

$$P_j = \frac{1}{\pi}\int_0^\infty \Re\!\left(\frac{e^{-iuk'}}{iu}\,\left(\varphi_{T_j}(u)\left(\sum_{r=1}^{j}(T_r - T_{r-1})\psi_r(u)\right)\right)\right)du.$$

The analysis proceeds in two stages. First, we identify the global normalization R_j for the pricing error such that the contributions from past maturities $r < j$ vanish asymptotically, reducing the analysis to the single-maturity error $\widetilde{\nu}_j - \nu_j$. Thereafter, we decompose this surviving error into linear, remainder, bias, and Σ, Γ, Λ parts, and show that under the scaling r_j all non-linear contributions vanish. As it will also be shown that $r_j \asymp R_j$, this will be equivalent to showing that they vanishing under the new scaling.

Define now

$$a_r := \Delta_r\, U_j^2 \exp(A_r U_j^2), \qquad r = 1, \ldots, j,$$

where $A_r := \sum_{i=1}^{r}(T_i - T_{i-1})\sigma_i^2$, and let $R_j := 1/\sqrt{\sum_{r=1}^{j} a_r}$. The factor a_r corresponds to the dominant variance contribution at maturity r when the spectral inversion uses the common cut-off U_j associated with the pricing operator Π_j.

Lemma 8.1 (Cross-maturity decay). *Under the standing assumptions of Chapter 3, $a_j\, R_j^2 \to 1$ and $a_r\, R_j^2 \to 0$ for each $r \in \{1, \ldots, j-1\}$, as $U_j \to \infty$.*

Proof. By iterating Koorevaar's bound $\Delta_{i-1} \leq C\,\Delta_i$, one obtains $\Delta_r \leq C^{j-r}\,\Delta_j$. Hence

$$\frac{a_r}{a_j} = \frac{\Delta_r}{\Delta_j}\,\exp\!\big(-(A_j - A_r)\,U_j^2\big) \leq C^{j-r}\,\exp\!\big(-(A_j - A_r)\,U_j^2\big).$$

Since $\sigma_i \in (0, \sigma_{\max}]$ for all i, $A_j - A_r = \sum_{i=r+1}^{j}(T_i - T_{i-1})\sigma_i^2 > 0$ strictly. The exponential factor vanishes and dominates the bounded polynomial C^{j-r}, giving $a_r/a_j \to 0$ for each $r < j$. Using the squeeze theorem, the claims $a_r\, R_j^2 \to 0$ and $a_j\, R_j^2 \to 1$ follow from $R_j^{-2} = a_j(1 + \sum_{r<j} a_r/a_j)$. □

Consequently, for each $r < j$,

$$R_j\sqrt{a_r + a_{r-1}} \cdot \sqrt{\frac{d_{r,r}\, a_r + d_{r,r-1}\, a_{r-1}}{a_r + a_{r-1}}} \cdot \frac{1}{\sqrt{d_{r,r}\, a_r + d_{r,r-1}\, a_{r-1}}}\,(\widetilde{\nu}_r(x) - \nu_r(x)), \tag{8.10}$$

where

$$d_{r,r-l} := 2\|\delta_{r-l}\|_{L^2}^2 \frac{e^{\sum_{i=1}^{r-l}(T_i-T_{i-1})(\sigma_i^2/2+\gamma_i-\lambda_i)}}{(T_j-T_{j-1})^2(\sum_{i=1}^{r-l}(T_i-T_{i-1})\sigma_i^2}{)^2} \tag{8.11}$$

is the constant factor accompanying the asymptotic terms of the pointwise variance in Tendijck [27] and Koorevaar [15], and it is analogous to the constant factor $\mathcal{C}$ in the covariance operator of our $L^2(K)$ CLT (Theorem 7.2).

Note that the last factor in (8.10) converges pointwise to a normal distribution by Koorevaar [15], and that

$$R_j\sqrt{(a_r+a_{r-1})} = \sqrt{R_j^2\, a_r + R_j^2\, a_{r-1}} \to 0 \quad \forall r < j \tag{8.12}$$

by Lemma 8.1. Furthermore,

$$\frac{d_{r,r}\, a_r + d_{r,r-1}\, a_{r-1}}{a_r + a_{r-1}} = d_{r,r}\frac{a_r}{a_r+a_{r-1}} + d_{r,r-1}\frac{a_{r-1}}{a_r+a_{r-1}} \to d_{r,r} \tag{8.13}$$

by equations (5.22) and (5.23). For $r = j$, since $a_j + a_{j-1} \leq \sum_{r=1}^{l} a_r = 1/R_j^2$:

$$1 \geq R_j\sqrt{(a_j+a_{j-1})} \geq R_j\sqrt{a_j} \to 1$$

where we have also used Lemma 8.1. Hence, by the squeeze theorem:

$$R_j\sqrt{(a_j+a_{j-1})} = \frac{R_j}{r_j} \to 1 \tag{8.14}$$

as $U_j \to \infty$. This means that $r_j \asymp R_j$. Since $\widetilde{\nu}_r - \nu_r \in L^1(\mathbb{R})$ and the kernel $(e^{iux}-1)$ is uniformly bounded by 2, the dominated convergence theorem applies to the inner integral and allows the limit to pass inside uniformly in u:

$$\sqrt{\frac{a_r+a_{r-1}}{R_j}}\cdot\sqrt{\frac{d_{r,r}\, a_r + d_{r,r-1}\, a_{r-1}}{a_r+a_{r-1}}}\int_K (e^{iux}-1)\frac{1}{\sqrt{d_{r,r}\, a_r + d_{r,r-1}\, a_{r-1}}}(\widetilde{\nu}_r(x)-\nu_r(x))\, dx \xrightarrow{\mathbb{P}} 0, \quad \forall r < j. \tag{8.15}$$

As $R_j\sqrt{(a_j+a_{j-1})} \to 1$, by applying Slutsky's theorem to equation (8.15), we obtain convergence in probability to 0. This means that the only surviving contribution to the pricing error is in $\widetilde{\nu}_j - \nu_j$. Hence, after normalization by R_j, the pricing-error analysis reduces to the single-maturity error $\widetilde{\nu}_j - \nu_j$, and we may proceed to control its non-linear components.

Later on, in sections 8.6-8.9, it will be shown that, since the normalization R_j is asymptotically equivalent to r_j, by Slutsky's Theorem a CLT in the form of Theorem 7.2 can be recovered. This in turn will make it possible to obtain a CLT for $\widetilde{\Pi}_j - \Pi_j$.

The linear term of $\widetilde{\Pi}_j - \Pi_j$ will be further decomposed, yielding a suitable setting where a cosine prefactor $\cos(U_j\cdot)$ that multiplies $\widetilde{\nu}_j - \nu_j$ will be recovered, and where the $L^2(K)$ CLT will be leveraged by interpreting $\cos(U_j\cdot)(\widetilde{\nu}_j - \nu_j) \in L^2(K)$ as part of an $L^2(K)$ duality pairing.

8.5. Singularity Analysis of Term P_j

Given that the term P_j has a singularity at $u = 0$, it is now relevant to the remainder of the proof to show that this singularity is removable. The fact that it is removable will also help to justify that the integral around it is bounded and decays in probability, something which will simplify the analysis of P_j, and in time help retrieve a Gaussian limit. To that end, we start by proving the following Lemma:

Lemma 8.2 (Removability of the singularity in P_j)**.** *The integrand of P_j has a removable singularity at $u = 0$. Specifically, the integrand*

$$\mathcal{I}_A(u) \;:=\; \Re\!\left(\frac{e^{-iuk'}}{iu}\,\varphi_{T_j}(u)\,\sum_{r=1}^{j}(T_r - T_{r-1})\,\psi_r(u)\right)$$

extends continuously to $u = 0$, with limiting value

$$L_j \;:=\; \lim_{u\to 0}\mathcal{I}_A(u) \;=\; \sum_{r=1}^{j}(T_r - T_{r-1})\int_K x\,(\widetilde{\nu}_r(x) - \nu_r(x))\,\mathrm{d}x.$$

Proof. We Taylor-expand each factor about $u = 0$. For $\psi_r(u) = \int_{\mathbb{R}}(e^{iux} - 1)\,(\widetilde{\nu}_r - \nu_r)\,\mathrm{d}x$, the expansion $e^{iux} - 1 = iux + O(u^2x^2)$ (uniform for $x \in K$ compact) gives

$$\psi_r(u) \;=\; iu\int_K x\,(\widetilde{\nu}_r - \nu_r)\,\mathrm{d}x \;+\; O(u^2) \;=\; iu\,\mu_r^{(1)} \;+\; O(u^2),$$

where $\mu_r^{(1)} := \int_K x\,(\widetilde{\nu}_r - \nu_r)\,\mathrm{d}x$. For φ_{T_j} (a characteristic function with $\varphi_{T_j}(0) = 1$) and $e^{-iuk'}$, $\varphi_{T_j}(u) = 1 + O(u)$ and $e^{-iuk'} = 1 + O(u)$. Combining,

$$\begin{aligned}\frac{e^{-iuk'}}{iu}\,\varphi_{T_j}(u)\,\sum_{r=1}^{j}(T_r - T_{r-1})\,\psi_r(u) &= \frac{1 + O(u)}{iu}\,(1 + O(u))\,\sum_{r=1}^{j}(T_r - T_{r-1})\,\big(iu\,\mu_r^{(1)} + O(u^2)\big)\\ &= \sum_{r=1}^{j}(T_r - T_{r-1})\,\mu_r^{(1)} \;+\; O(u).\end{aligned}$$

The simple pole of $1/(iu)$ is cancelled by the simple zero of $\sum_r(T_r - T_{r-1})\psi_r(u)$ at $u = 0$. Since each $\mu_r^{(1)}$ is real, taking real parts gives

$$\mathcal{I}_A(u) \;\xrightarrow[u\to 0]{}\; \sum_{r=1}^{j}(T_r - T_{r-1})\,\mu_r^{(1)} \;=:\; L_j,$$

so $\mathcal{I}_A$ extends continuously to $u = 0$ by setting $\mathcal{I}_A(0) := L_j$. □

The continuous extension implies that on any sufficiently small neighbourhood $[0, \varepsilon]$ the integrand is bounded by some constant depending on L_j, and therefore:

$$\left|\int_0^{\varepsilon}\mathcal{I}_A(u)\,\mathrm{d}u\right| \;\le\; 2|L_j|\,\varepsilon \tag{8.16}$$

for ε sufficiently small. It remains to show that this contribution vanishes under the normalization $1/\sqrt{R_j}$.

We control the contribution through the second moment of the scalar L_j, retaining the signed structure of the error rather than passing to $\int_K |\widetilde{\mu}_j - \mu_j|\,\mathrm{d}x$. Since the noise variables $\varepsilon_{j-l,k}$ are centred, each $\mu_r^{(1)}$ is centred, and hence so is L_j; its second moment therefore equals its variance. By Chebyshev's inequality, for every $\eta > 0$,

$$\mathbb{P}\big(r_j|L_j| > \eta\big) \;\le\; \frac{\mathrm{Var}\big(r_jL_j\big)}{\eta^2}, \qquad \mathrm{Var}\big(r_jL_j\big) = \sum_{r=1}^{j}(T_r - T_{r-1})\,r_j^2\,\mathbb{E}\big[|\mu_r^{(1)}|^2\big].$$

As shown in section 8.4, $r_j \asymp R_j$. Moreover, by Lemma 8.1 only the current maturity contribution $r = j$ survives, as terms involving $r < j$ are $o_{\mathbb{P}}(1)$. Hence, it suffices to bound $r_j|L_j|$. It therefore suffices to control $r_j^2\ \mathrm{Var}\big(\mu_j^{(1)}\big)$, where

$$\begin{aligned}\mu_j^{(1)} &= \int_K x\big(\widetilde{\nu}_j(x) - \nu_j(x)\big)\mathrm{d}x \\ &= \int_K x\, e^{-x}\big(\widetilde{\mu}_j(x) - \mu_j(x)\big)\mathrm{d}x \\ &= \big\langle \widetilde{\mu}_j - \mu_j,\, y\big\rangle_{L^2(K)} \\ &= \sum_{l=0}^{1}(-1)^l\,\big\langle \mathscr{L}^l_{\mu_j},\, y\big\rangle_{L^2(K)}\end{aligned}$$

with the weight $y(x) := x\,e^{-x}$. Note that $\mu_j^{(1)}$ is a functional pairing of the linear part of the error against $y(x)$. Note that, as the bias and remainder parts of the error also vanish under scaling in $L^2(K)$ (as will be shown in section 8.6), we focus on the linear part of the error. Hence, variance is therefore a bilinear form in the rescaled covariance kernel of the linear part of the error, into which the normalization r_j is already absorbed:

$$\begin{aligned}r_j^2\ \mathrm{Var}\big(\mu_j^{(1)}\big) &= r_j^2\ \mathrm{Var}\left(\sum_{l=0}^{1}(-1)^l\,\big\langle \mathscr{L}^l_{\mu_j},\, y\big\rangle_{L^2(K)}\right) \\ &= \sum_{l=0}^{1}\int_K\!\int_K y(x)\,y(z)\ \underbrace{r_j^2\ \mathrm{Cov}\left(\sum_{k=1}^{m_{j-l}} X^l_{U_j,k}(x)',\ \sum_{k=1}^{m_{j-l}} X^l_{U_j,k}(z)'\right)}_{=:\,\mathcal{K}^l_j(x,z)}\,\mathrm{d}x\,\mathrm{d}z.\end{aligned}$$

where we have used the fact that the sub-Gaussian random variables in $\mathscr{L}^0_{\mu_j}$ and $\mathscr{L}^1_{\mu_j}$ are centered and independent.

By the covariance analysis of Chapter 4, the rescaled kernel of the linear part of the error converges, under the normalization r_j, to the oscillatory limit

$$\mathcal{K}^l_j(x,z)\ \longrightarrow\ 2\mathcal{C}\ \cos\!\big(U_j(x-z)\big),$$

and is uniformly bounded on the compact $K \times K$ by Lemma A.4, $|\mathcal{K}^l_j(x,z)| \le C_0$. It remains to evaluate the bilinear form against the fixed weight y. The leading contribution is the oscillatory double integral, which factorizes as the squared modulus of a Fourier transform:

$$\begin{aligned}2\mathcal{C}\int_K\!\int_K y(x)\,y(z)\ \cos\!\big(U_j(x-z)\big)\,\mathrm{d}x\,\mathrm{d}z &= 2\mathcal{C}\int_K\!\int_K y(x)\,y(z)\,\Re\big(e^{iU_j(x-z)}\big)\,\mathrm{d}x\,\mathrm{d}z \\ &= 2\mathcal{C}\cdot\Re\left(\int_K\!\int_K y(x)\,y(z)\,e^{iU_j(x-z)}\mathrm{d}x\,\mathrm{d}z\right) \\ &= 2\mathcal{C}\cdot\Re\left[\left(\int_K y(x)\,e^{iU_j(x)}\,\mathrm{d}x\right)\left(\int_K y(z)\,e^{-iU_j(z)}\,\mathrm{d}z\right)\right] \\ &= 2\mathcal{C}\,\big|\,\widehat{y\mathbf{1}_K}(U_j)\,\big|^2\end{aligned}$$

where

$$\widehat{y\mathbf{1}_K}(\xi) := \int_K y(x)\,e^{i\xi x}\,\mathrm{d}x.$$

Since $y\mathbf{1}_K = xe^{-x}\mathbf{1}_K \in L^1(K)$, the Riemann-Lebesgue lemma gives $\widehat{y\mathbf{1}_K}(U_j) \to 0$ as $U_j \to \infty$, hence $2\mathcal{C}\,|\widehat{y\mathbf{1}_K}(U_j)|^2 \to 0$.

Thus, it can be concluded that

$$r_j^2\ \mathrm{Var}\big(\mu_j^{(1)}\big)\ \longrightarrow\ 0, \qquad \text{hence} \qquad r_j\,|L_j|\ \xrightarrow{\ \mathbb{P}\ }\ 0\quad \text{by Chebyshev's inequality.}$$

Combining this with the patch bound (8.16), for any fixed $\varepsilon > 0$ sufficiently small,

$$R_j \left| \int_0^{\varepsilon} \mathcal{I}_A(u)\,\mathrm{d}u \right| \;\le\; 2\varepsilon\, r_j\, |L_j| \;\xrightarrow{\;\mathbb{P}\;}\; 0 \qquad \text{as } U_j \to \infty.$$

This justifies restricting the asymptotic analysis of $R_j\, P_j$ to the domain $[\varepsilon, \infty)$: the singularity-patch contribution is asymptotically negligible under the R_j normalization.

8.6. Vanishing of Non-Linear Contributions at Maturity j.

It remains to control the non-linear pieces of $\widetilde{\nu}_j - \nu_j$. Recalling that Lemma 8.2 lets us restrict the domain of integration to $[\varepsilon, \infty]$, the following bound can be obtained:

$$\begin{aligned} P_j &= \frac{1}{\pi} \int_{\varepsilon}^{\infty} \Re\!\left(\frac{e^{-iuk'}}{iu}\, \varphi_{T_j}(u)\, (T_j - T_{j-1})\, \psi_j(u) \right) du \\ &\le (T_j - T_{j-1}) \frac{2}{\pi} \int_{\varepsilon}^{\infty} \frac{e^{-A_j\, u^2/2}}{|u|} \int_K |\widetilde{\nu}_j(x) - \nu_j(x)|\, dx\, du \\ &\le M_K\, (T_j - T_{j-1}) \frac{2}{\pi} \int_{\varepsilon}^{\infty} \frac{e^{-A_j\, u^2/2}}{|u|} \int_K |\widetilde{\mu}_j(x) - \mu_j(x)|\, dx\, du, \end{aligned}$$

where $M_K := \sup_{x \in K} e^{-x}$ and we used the relation $\mu_j(x) = e^x \nu_j(x)$. If we decompose $\widetilde{\mu}_j - \mu_j = (\mathscr{L}^0_{\mu_j} - \mathscr{L}^1_{\mu_j}) + (\mathscr{R}^0_{\mu_j} - \mathscr{R}^1_{\mu_j}) + \mathscr{B}$, we show that, after multiplication by the normalization r_j, the integrals over the remainder and bias terms vanish. From Lemma A.7, under the condition:

$$\frac{\Delta^2_{j-l}}{\Delta_j}\, U_j^5\, \exp\!\left(U_j^2 \left(2 \sum_{i=1}^{j-l} (T_i - T_{i-1}) \sigma_i^2 - \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 \right) \right) \longrightarrow 0, \quad \text{for } l = 0, 1,$$

it holds that

$$r_j\, \mathscr{R}^l_{\mu_j}(x) \xrightarrow{P} 0 \quad \text{for each } x \in \mathbb{R}, \quad l = 0, 1.$$

where

$$\mathscr{R}^l_{\mu_j}(x) = \int_{-U_j}^{U_j} \mathcal{R}^l_j(v) w^{U_j}_{\mu_j}(v) e^{-ivx}\, dv.$$

Since the bound in Lemma A.7 is uniform in x, we pass to the integrated error through its first moment. By Jensen's inequality: $\mathbb{E}|X| \le (\mathbb{E}|X|^2)^{1/2}$ and hence:

$$\mathbb{E}\!\left[r_j \int_K |\mathscr{R}^l_{\mu_j}(x)|\, dx \right] = r_j \int_K \mathbb{E}|\mathscr{R}^l_{\mu_j}(x)|\, dx \le r_j \int_K \big(\mathbb{E}|\mathscr{R}^l_{\mu_j}(x)|^2 \big)^{1/2} dx \le r_j\, |K|\, \big(\Delta^2_{j-l}\, e^{2A_{j-l} U_j^2} \big)^{1/2},$$

where the last inequality uses the uniform second-moment bound of Lemma A.7. Since $r_j\, \Delta_{j-l}\, e^{A_{j-l} U_j^2} \to 0$ under the stated condition, the right-hand side vanishes, so that

$$\mathbb{E}\!\left[r_j \int_K |\mathscr{R}^l_{\mu_j}(x)|\, dx \right] \longrightarrow 0.$$

By Markov's inequality, for every $\eta > 0$,

$$\mathbb{P}\!\left(r_j \int_K |\mathscr{R}^l_{\mu_j}(x)|\, dx > \eta \right) \le \frac{1}{\eta}\, \mathbb{E}\!\left[r_j \int_K |\mathscr{R}^l_{\mu_j}(x)|\, dx \right] \longrightarrow 0,$$

hence

$$r_j \int_K |\mathscr{R}^l_{\mu_j}(x)|\, dx \xrightarrow{P} 0.$$

For the outer integral in u, the kernel $e^{-A_j (u+U_j)^2/2}$ is integrable on $(-U_j + \varepsilon, \infty)$ and serves as a dominating function. Hence, by dominated convergence,

$$(T_j - T_{j-1}) \frac{2}{\pi} \int_{\varepsilon}^{\infty} \frac{e^{-A_j\, u^2/2}}{|u|} \int_K |r_j \mathscr{R}^l_{\mu_j}(x)|\, dx\, du \xrightarrow{P} 0.$$

The bias term is deterministic. On the compact set K,

$$r_j \int_K |\mathscr{B}|\, dx \le r_j \cdot U_j^{-s_j} \cdot |K| \to 0.$$

which holds by Lemma A.6, for $s_j \ge 2$, and under the standing assumption

$$U_j^{2(s_j+1)} \left(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\right) \to \infty$$

Since this bound is independent of u, integrating against the Gaussian kernel yields

$$\begin{aligned}(T_j - T_{j-1})\frac{2}{\pi}\int_\varepsilon^\infty \frac{e^{-A_j\, u^2/2}}{|u|} \int_K r_j\, |\mathscr{B}|\, dx\, du \\ \le r_j \cdot U_j^{-s_j} \cdot |K| \cdot (T_j - T_{j-1})\frac{2}{\pi}\int_\varepsilon^\infty \frac{e^{-A_j\, u^2/2}}{|u|}\, du \to 0,\end{aligned}$$

since the Gaussian integral is bounded. For the Σ, Γ, Λ terms, we focus here on the Σ case, as the other two will be analogous. By Lemma A.8, the bound on $\mathbb{E}|\Sigma(x)|^2$ is uniform in x, and, by Jensen's Inequality, we have: $\mathbb{E}|X| \le (\mathbb{E}|X|^2)^{1/2}$. Thus:

$$\mathbb{E}\left[r_j \int_K |\Sigma(x)|\, dx\right] = r_j \int_K \mathbb{E}|\Sigma(x)|\, dx \le r_j\, |K| \left(\sup_{x\in K} \mathbb{E}|\Sigma(x)|^2\right)^{1/2} \longrightarrow 0,$$

the limit holding under the condition of Lemma A.8. By Markov's inequality, $r_j \int_K |\Sigma(x)|\, dx \xrightarrow{P} 0$. Since the u-integral $\int_\varepsilon^\infty \frac{e^{-A_j u^2/2}}{|u|}\, du$ is a finite constant, independent of the inner integral, Slutsky's theorem gives

$$(T_j - T_{j-1})\frac{2}{\pi}\int_\varepsilon^\infty \frac{e^{-A_j u^2/2}}{|u|} \int_K r_j |\Sigma(x)|\, dx\, du \xrightarrow{P} 0.$$

The same argument applies verbatim to Γ and Λ.

The prefactor r_j used here coincides, up to constants, with the pointwise CLT prefactor of Koorevaar:

$$R_j \asymp r_j \asymp \frac{1}{U_j}\, \Xi_j, \qquad \Xi_j := \frac{\exp\left(-U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2/2\right)}{\sqrt{d_{j,j}\Delta_j + d_{j,j-1}\Delta_{j-1}\, e^{-U_j^2 (T_j - T_{j-1})\sigma_j^2}}}.$$

Hence, after normalization by r_j, or, equivalently, by R_j, only the linear terms $\mathscr{L}^0_{\nu_j} - \mathscr{L}^1_{\nu_j}$ survive inside the Gil–Pelaez integral. These are the terms to which the $L^2(K)$ functional CLT will be applied in the following section.

8.7. Decay of the term B_j

Let us now proceed to bound B_j. Recall that $\widetilde{\nu}_r - \nu_r$ is supported on a compact set $K \subset \mathbb{R}$. For any $\varepsilon > 0$, write

$$\begin{aligned}B_j &= \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-iuk'}}{iu}\varphi_{T_j}(u)\mathcal{R}_{\Pi_j}(u)\right) du \\ &= \frac{1}{\pi}\int_0^\varepsilon \Re\left(\frac{e^{-iuk'}}{iu}\varphi_{T_j}(u)\mathcal{R}_{\Pi_j}(u)\right) du \\ &+ \frac{1}{\pi}\int_\varepsilon^\infty \Re\left(\frac{e^{-iuk'}}{iu}\varphi_{T_j}(u)\mathcal{R}_{\Pi_j}(u)\right) du.\end{aligned}$$

where

$$|\mathcal{R}_{\Pi_j}(u)| \leq \frac{1}{2}\left(\sum_{r=1}^{j}(T_r - T_{r-1})|\psi_r(u)|\right)^2 \cdot \exp\left(\sum_{r=1}^{j}(T_r - T_{r-1})\,|\psi_r(u)|\right).$$

For the first integral, since $\psi_r(u) = \int_{\mathbb{R}}(e^{iux}-1)(\tilde{\nu}_r - \nu_r)dx$ satisfies $\psi_r(0) = 0$ and $|\psi_r(u)| \leq |u| \int_{\mathbb{R}} |x||\tilde{\nu}_r - \nu_r|dx$, we have $R_j(u) = O(u^2)$ as $u \to 0$. For $u \in (0, \varepsilon]$,

$$\left|\frac{1}{u}\mathcal{R}_{\Pi_j}(u)\right| \;\leq\; C|u| \;\leq\; C\varepsilon,$$

where the $|u|$ from the bound on $|R_j(u)|$ cancels with the $|u|$ in the denominator, leaving a quantity that vanishes as $u \to 0$. Hence the integrand of the first integral is bounded near $u = 0$, and

$$\left|\frac{1}{\pi}\int_0^{\varepsilon} \Re\left(\frac{e^{-iuk'}}{iu}\,\varphi_{T_j}(u)\,\mathcal{R}_{\Pi_j}(u)\right)\mathrm{d}u\right| \;\leq\; \frac{C}{\pi}\int_0^{\varepsilon}|u|\,\mathrm{d}u \;=\; \frac{C\,\varepsilon^2}{2\pi} \;\xrightarrow[\varepsilon\to 0]{}\; 0,$$

so the contribution from $[0, \varepsilon]$ vanishes, leaving no contribution from the apparent pole.

8.7.1. Bounding $|B_j|$

$$\begin{aligned} |B_j| &\leq \left|\frac{1}{2\pi}\int_{\varepsilon}^{\infty}\frac{1}{u}\,|\varphi_{T_j}(u)|\left|\sum_{r=1}^{j}(T_r - T_{r-1})\psi_r(u)\right|^2 du\right| \\ &\lesssim \sum_{r=1}^{j}\int_{\varepsilon}^{\infty}\frac{1}{u}\left|\varphi_{T_j}(u)\right|\,|\psi_r(u)|^2\,du. \end{aligned}$$

Writing the error in terms of μ via $\widetilde{\nu}_r - \nu_r = e^{-x}(\widetilde{\mu}_r - \mu_r)$,

$$\psi_r(u) = \int_K e^{-x}\big(e^{iux} - 1\big)\,\big(\widetilde{\mu}_r(x) - \mu_r(x)\big)\,\mathrm{d}x = \big\langle \widetilde{\mu}_r - \mu_r,\; h_u\big\rangle_{L^2(K)},$$

a functional pairing of the error against the deterministic weight $h_u(x) := e^{-x}\big(e^{-iux} - 1\big)$ (the conjugate placing the pairing in $L^2(K)$). Since the noise variables are centred, so is $\psi_r(u)$, and its second moment equals its variance.

As shown in section 8.4, $r_j \asymp R_j$. Moreover, by Lemma 8.1 only the current maturity contribution $r = j$ survives, as terms involving $r < j$ are $o_{\mathbb{P}}(1)$. Hence, it suffices to bound $r_j \int_{\varepsilon}^{\infty} \frac{1}{u}\left|\varphi_{T_j}(u)\right|\,|\psi_j(u)|^2\,du$.

For each fixed u, the variance of the rescaled pairing is a bilinear form in the rescaled covariance kernel, into which the normalization r_j is already absorbed:

$$r_j^2\,\mathbb{E}\big[|\psi_j(u)|^2\big] = r_j^2\,\mathbb{E}\big[|\sum_{l=0}^{1}\big\langle \mathscr{L}^l_{\mu_j},\; h_u\big\rangle_{L^2(K)}|^2\big] = \sum_{l=0}^{1}\big\langle \mathcal{K}^l_j\,h_u,\; h_u\big\rangle_{L^2(K)}$$

where we have used the independence of the sub-Gaussian random variable, and where

$$\mathcal{K}^l_j(x, z) := r_j^2\;\mathrm{Cov}\left(\sum_{k=1}^{m_{j-l}} X^l_{U_j,k}(x)',\; \sum_{k=1}^{m_{j-l}} X^l_{U_j,k}(z)'\right).$$

By the covariance analysis of Chapter 4, the rescaled kernel converges to $2\mathcal{C}\cos(U_j(x-z))$ and is uniformly bounded on the compact $K \times K$ by Lemma A.4, $|\mathcal{K}^l_j(x, z)| \leq C_0$. The leading contribution factorises through the Fourier transform of the fixed integrable weight $h_u\mathbf{1}_K \in L^1(K)$. Writing the covariance bilinear form with the conjugate in the second slot in $\langle\cdot,\cdot\rangle_{L^2(K)}$, and expanding $\cos(U_j(x-z)) = \frac{1}{2}\big(e^{iU_j(x-z)} + e^{-iU_j(x-z)}\big)$, the two halves separate into a product of single integrals:

$$\begin{aligned}\big\langle \mathcal{K}_j^l\, h_u, h_u\big\rangle_{L^2(K)} &\longrightarrow\ 2\mathcal{C}\int_K\!\!\int_K h_u(x)\,\overline{h_u(z)}\,\cos\!\big(U_j(x-z)\big)\,\mathrm{d}x\,\mathrm{d}z\\ &=\ \mathcal{C}\Big(\widehat{h_u\mathbf{1}_K}(U_j)\,\overline{\widehat{h_u\mathbf{1}_K}(U_j)}\ +\ \widehat{h_u\mathbf{1}_K}(-U_j)\,\overline{\widehat{h_u\mathbf{1}_K}(-U_j)}\Big)\\ &=\ \mathcal{C}\,\Big(\big|\,\widehat{h_u\mathbf{1}_K}(U_j)\,\big|^2\ +\ \big|\,\widehat{h_u\mathbf{1}_K}(-U_j)\,\big|^2\Big),\end{aligned}$$

where $\widehat{h_u\mathbf{1}_K}(\xi) := \int_K h_u(x)\,e^{i\xi x}\,\mathrm{d}x$. Since $h_u\mathbf{1}_K \in L^1(K)$, the Riemann–Lebesgue lemma applies to each term separately, giving

$$\big|\,\widehat{h_u\mathbf{1}_K}(U_j)\,\big|^2 \to 0 \qquad \text{and} \qquad \big|\,\widehat{h_u\mathbf{1}_K}(-U_j)\,\big|^2 \to 0 \qquad \text{as } U_j\to\infty,$$

and hence

$$\sum_{l=0}^{1}\big\langle \mathcal{K}_j^l\, h_u, h_u\big\rangle_{L^2(K)} \xrightarrow[U_j\to\infty]{} 0 \qquad \text{for each fixed } u.$$

It remains to integrate over $u\in[\varepsilon,\infty)$. The uniform kernel bound gives, for every u,

$$r_j^2\,\mathbb{E}\big[|\psi_j(u)|^2\big] \le C_0\,\|h_u\|^2_{L^2(K)} \le 4\,C_0\,|K|\,\sup_{x\in K} e^{-2x}\ =:\ C_1 < \infty,$$

a bound independent of u. Since $u\mapsto \frac{1}{u}\,|\varphi_{T_j}(u)|$ is integrable on $[\varepsilon,\infty)$ (as $\frac{1}{u}\le\frac{1}{\varepsilon}$ there and $\varphi_{T_j}\in L^1[\varepsilon,\infty)$), the dominated convergence theorem yields

$$r_j^2\,\mathbb{E}\big[|B_j|^2\big]\ \lesssim\ \int_\varepsilon^\infty \frac{1}{u}\,|\varphi_{T_j}(u)|\,\Big(r_j^2\,\mathbb{E}\big[|\psi_j(u)|^2\big]\Big)\,\mathrm{d}u \xrightarrow[U_j\to\infty]{} 0.$$

By Markov's inequality, for every $\eta>0$,

$$\mathbb{P}\big(r_j|B_j|>\eta\big) \le \frac{1}{\eta^2}\,r_j^2\,\mathbb{E}\big[|B_j|^2\big]\ \longrightarrow\ 0,$$

so $r_j|B_j| \xrightarrow{\mathbb{P}} 0$, i.e.

$$|B_j| = o_{\mathbb{P}}\big(1/R_j\big).$$

8.8. Variable Change: $u\mapsto u+U_j$

Hence, now we will look at the term

$$\begin{aligned}P_j &= \frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-iuk'}}{iu}\,\varphi_{T_j}(u)\sum_{r=1}^{j}(T_r-T_{r-1})\,\psi_r(u)\right)du\\ &= \sum_{r=1}^{j}(T_r-T_{r-1})\frac{1}{\pi}\int_0^\infty \Re\left(\frac{e^{-iuk'}}{iu}\,\varphi_{T_j}(u)\,\psi_r(u)\right)du.\end{aligned}$$

Note that

$$P_j = \sum_{r=1}^{j}(T_r-T_{r-1})\frac{1}{\pi}\int_{-U_j}^\infty \Re\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\,\psi_r(u+U_j)\right)du$$

8.8.1. Intermezzo

Recall that

$$\psi_r(u+U_j) = \int_K (e^{i(u+U_j)x}-1)(\widetilde{\nu}_r(x)-\nu_r(x))\mathrm{d}x.$$

Moreover, it holds that

$$\cos(U_j x) = \frac{1}{2}\left(e^{iU_j x} + e^{-iU_j x}\right).$$

Hence, by Lemma A.5 this means that $\psi_r(u+U_j)$ can be rewritten as:

$$\psi_r(u+U_j) = 2\psi_{\nu_r}(u) - \psi_r(u-U_j) - 2$$

where

$$\psi_{\nu_r}(u) := \int_K e^{iux}\,\cos(U_j x)\,(\widetilde{\nu}_r(x) - \nu_r(x))\mathrm{d}x.$$

This rewriting means that P_j can also be rewritten, as follows:

$$P_j = P_j' - D_j + E_j,$$

where:

$$P_j' := 2\sum_{r=1}^{j}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\,\psi_{\nu_r}(u)\right)du \tag{8.17}$$

$$D_j := \sum_{r=1}^{j}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\,\psi_r(u-U_j)\right)du \tag{8.18}$$

$$E_j := \sum_{r=1}^{j}(T_r - T_{r-1})\frac{2}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\,\int_K(\widetilde{\nu}_r(x) - \nu_r(x))dx\right)du. \tag{8.19}$$

8.9. Term P_j'

It is proved in Lemma A.9 that both D_j and E_j are $o_{\mathbb{P}}(1/R_j)$, so the asymptotic behavior of P_j is determined entirely by

$$P_j' = 2\sum_{r=1}^{j}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\,\psi_{\nu_r}(u)\right)du.$$

We can now define the term

$$M_j(u) := \frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j), \tag{8.20}$$

via which we can rewrite P_j' as:

$$\begin{aligned}
P_j' &= 2\sum_{r=1}^{j}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re(M_j(u)\,\psi_{\nu_r}(u))\,du \\
&= 2\sum_{r=1}^{j}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(M_j(u)\,\int_K e^{iux}\,\cos(U_j x)\,(\widetilde{\nu}_r(x) - \nu_r(x))\mathrm{d}x\right)du \\
&= 2\,(T_j - T_{j-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(M_j(u)\,\int_K e^{iux}\,\cos(U_j x)\,(\widetilde{\nu}_j(x) - \nu_j(x))\mathrm{d}x\right)du \\
&\quad + 2\sum_{r=1}^{j-1}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(M_j(u)\,\int_K e^{iux}\,\cos(U_j x)\,(\widetilde{\nu}_r(x) - \nu_r(x))\mathrm{d}x\right)du \\
&=: 2\,(T_j - T_{j-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\left(M_j(u)\,\int_K e^{iux}\,\cos(U_j x)\,(\widetilde{\nu}_j(x) - \nu_j(x))\mathrm{d}x\right)du + P_{j-1}'.
\end{aligned} \tag{8.21}$$

If we now multiply P_j' by the prefactor R_j, by Lemma 8.1 we can ignore the term P_{j-1}', as under this normalization the terms with index $r < j$ decay asymptotically. Hence, we get the expression:

$$P_j' \asymp 2(T_j - T_{j-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty} \Re\bigg(M_j(u)\int_K e^{iux}\, \cos(U_j x)\,(\widetilde{\nu}_j(x) - \nu_j(x))\mathrm{d}x\bigg)\, du.$$

Since $\alpha_j(x) := \cos(U_j x)\,(\widetilde{\nu}_j(x) - \nu_j(x))$ does not depend on u, by Fubini we can interchange the integrals:

$$P_j' \asymp 2\,(T_j - T_{j-1})\frac{1}{\pi}\int_K g_j(x)\,\alpha_j(x)\,dx,$$

where

$$g_j(x) := \int_{-U_j+\varepsilon}^{\infty} \Re\big(M_j(u)\,e^{iux}\big)\,du$$

We now treat the integral

$$P_j' := \int_K g_j(x)\,\alpha_j(x)\,dx$$

as an $L^2(K)$ inner product $\langle \alpha_j, g_j\rangle_{L^2(K)}$. Note that, for every $x \in K$,

$$|g_j(x)| \le \int_{-U_j+\varepsilon}^{\infty} |M_j(u)|\,du \le \int_{-U_j+\varepsilon}^{\infty} \frac{e^{-A_j(u+U_j)^2/2}}{|u+U_j|}\,du.$$

Substituting $v = u + U_j$:

$$= \int_{\varepsilon}^{\infty} \frac{e^{-A_j v^2/2}}{v}\,dv \le \frac{1}{\varepsilon}\int_{\varepsilon}^{\infty} e^{-A_j v^2/2}\,dv \le \frac{1}{\varepsilon}\sqrt{\frac{\pi}{2A_j}} < \infty,$$

uniformly in U_j (since P_j is bounded away from 0). Hence $|g_j(x)| \le M$ for all $x \in K$ and all U_j, and therefore

$$\|g_j\|_{L^2(K)}^2 = \int_K |g_j(x)|^2\,dx \le M^2\,|K| < \infty.$$

8.9.1. Verifying Convergence in Distribution for the Extended CMT

Since $\|g_j\|_{L^2(K)} \le C$ uniformly in U_j, the sequence (g_j) is bounded in $L^2(K)$. Since $L^2(K)$ is a Hilbert space, we know that every bounded sequence has a weakly convergent subsequence (as is proved in Theorem 16.6 in Fitzpatrick and Royden [24]). Hence, this guarantees that there exists a subsequence (U_{j_k}) and a limit $g \in L^2(K)$ such that

$$g_{j_k} \rightharpoonup g \quad \text{weakly in } L^2(K).$$

Now, to obtain a convergence-in-distribution result, recall the Extended Continuous Mapping Theorem, i.e., Theorem 2.3.

Recall furthermore that our functional $r_j \cos(U_j\cdot)(\widetilde{\nu}_j - \nu_j)$ converges weakly in $L^2(K)$ to a Gaussian $G \sim \mathcal{N}(0, T\,A^0\,T)$, by the $L^2(K)$ CLT established in Chapter 7. Thus, we can now verify that the extended CMT applies along the subsequence (U_{j_k}) for the $L^2(K)$ inner product. By Lemma 8.1, we know that

$$R_j \cos(U_j\cdot)(\widetilde{\nu}_j - \nu_j) = \frac{R_j}{r_j}\cdot r_j \cos(U_j\cdot)(\widetilde{\nu}_j - \nu_j)$$

where

$$\begin{aligned}\left(\frac{R_j}{r_j}\right)^2 &= R_j^2\,(a_j + a_{j-1})\\ &\le R_j^2\,a_j\,(1 + C^{j-1}\exp\{(A_{j-1} - A_j)\,U_j^2\})\\ &= \frac{a_j\,(1 + C^{j-1}\exp\{(A_{j-1} - A_j)\,U_j^2\})}{a_j(1 + \sum_{r<j} a_r/a_j)} \to 1\end{aligned}$$

and

$$\left(\frac{R_j}{r_j}\right)^2 = R_j^2(a_j + a_{j-1}) \geq R_j^2\, a_j = \frac{a_j}{a_j(1 + \sum_{r<j} a_r/a_j)} \to 1.$$

Hence, by the squeeze theorem, $\frac{R_j}{r_j} \to 1$, and this allows us to recover a Gaussian limit under this scaling as well. Thus, $R_{j_k}\alpha_{j_k} \rightsquigarrow G$ in $L^2(K)$, where $\alpha_{j_k} = \cos(U_{j_k}\cdot)(\widetilde{\nu}_{j_k} - \nu_{j_k})$ and G is the Gaussian from Theorem 7.2. We verify the hypothesis of the extended continuous mapping theorem (Theorem 2.3) for the deterministic functionals

$$f_{j_k} : L^2(K) \to \mathbb{R}, \qquad f_{j_k}(h) := \langle h,\, g_{j_k}\rangle_{L^2(K)}, \qquad f_\infty(h) := \langle h,\, g\rangle_{L^2(K)}.$$

Since the g_{j_k} are deterministic, the maps f_{j_k} are non-random. We use two facts established above: the uniform bound $\sup_k \|g_{j_k}\|_{L^2(K)} \leq C < \infty$, and the weak convergence $g_{j_k} \rightharpoonup g$ in $L^2(K)$ established above.

Hence, we can now proceed to verify that the conditions making the application of Theorem 2.3 hold, and hence that it can be applied. To do so, let $h_{j_k} \to h$ in norm in $L^2(K)$, with $h \in L^2(K)$ arbitrary. Then

$$f_{j_k}(h_{j_k}) - f_\infty(h) = \underbrace{\langle h_{j_k} - h,\, g_{j_k}\rangle_{L^2(K)}}_{\text{(I)}} + \underbrace{\langle h,\, g_{j_k} - g\rangle_{L^2(K)}}_{\text{(II)}}.$$

For (I), by Cauchy–Schwarz and the uniform bound,

$$|\langle h_{j_k} - h,\, g_{j_k}\rangle_{L^2(K)}| \leq \|h_{j_k} - h\|_{L^2(K)}\, \|g_{j_k}\|_{L^2(K)} \leq C\, \|h_{j_k} - h\|_{L^2(K)} \longrightarrow 0,$$

since $h_{j_k} \to h$ in norm. For (II), the weak convergence $g_{j_k} \rightharpoonup g$ means precisely that $\langle t,\, g_{j_k} - g\rangle_{L^2(K)} \to 0$ for every fixed $t \in L^2(K)$. Applying this with the fixed vector $t = h$ gives (II) $\to 0$. Hence $f_{j_k}(h_{j_k}) \to f_\infty(h)$ for every norm-convergent sequence $h_{j_k} \to h$ in $L^2(K)$, which is the hypothesis of Theorem 2.3.

As

$$R_{j_k}\, \alpha_{j_k} \;\rightsquigarrow\; G \qquad \text{in } L^2(K), \text{ s.t.} \quad G \sim \mathcal{N}(0,\, TA^0T).$$

and since the f_{j_k} satisfy the convergence hypothesis, the extended CMT yields

$$\left\langle R_{j_k}\, \alpha_{j_k},\, g_{j_k} \right\rangle_{L^2(K)} \;\rightsquigarrow\; \langle G,\, g\rangle_{L^2(K)},$$

where the variance of is

$$\langle TA^0T\, g, g\rangle_{L^2(K)} = \tfrac{1}{2}\mathcal{C}\, \langle g, e^{-x}\rangle^2_{L^2(K)},$$

which is positive as long as $\int_K g(x)\, e^{-x}\, dx \neq 0$.

Chapter 9
Applications

This section will focus on how the convergence-in-distribution result obtained for $\widetilde{\Pi}_j$ has tangible financial applications in the form of finite-sample confidence intervals for the pricing error $\widetilde{\Pi}_j - \Pi_j$, which are derived akin to Söhl and Trabs [26] and Koorevaar [15]. This application makes it possible to move from the abstract setting of an $L^2(K)$ CLT to an application in a finite sample setting which is directly related to the needs of the financial sector.

9.1. Finite Sample Variance

A Central Limit Theorem obtained along a subsequence for the price of a digital call option $\widetilde{\Pi}_{j_k}$, where j_k is the index of the maturity T_{j_k}, opens the door to several financial applications. The most immediate is uncertainty quantification. Traders do not observe the true price Π_{j_k} directly, but the estimated version $\widetilde{\Pi}_{j_k}$ obtained via calibration. It is essential that the pricing models used by financial institutions are aligned with the market, as acting on a position based on a wrong price carries significant risk for such an institution. Hence, it is key to know how far the price estimate deviates from the real price Π_{j_k}, to have an adequate picture of the risk profile. The Central Limit Theorem answers this question by providing an asymptotic distribution of the error $\widetilde{\Pi}_{j_k} - \Pi_{j_k}$ which can in turn be leveraged to provide $(100 - \alpha)\%$ confidence intervals.

However, when trying to apply our asymptotic result obtained in Chapter 8 for uncertainty quantification, there are several issues that come up, and that justify applying the result in a finite sample setting. Firstly, the fact that convergence in distribution is obtained along a subsequence means that, to derive confidence intervals, the right subsequence which results in convergence to a fixed covariance operator would have to be found. In practice, this can turn out to be a challenging and time consuming task, something which financial institutions can hardly afford given the fast-paced and quickly changing nature of the financial industry. Secondly, as argued in Tendijck [27] and Koorevaar [15] for the time-inhomogeneous case and shown in Söhl and Trabs [26] for the time-homogeneous case, the asymptotic confidence intervals for the Lévy triplet obtained from our spectral calibration method need not be satisfactory in terms of coverage probabilities. This occurs because in real-life applications, only a finite sample of grid points will be available for calibration, and a finite value for U_j will have to be picked for the spectral method.

Hence, given that in this thesis, the asymptotic normality results of $\widetilde{\Pi}_{j_k}$ are derived by "transferring" the uncertainty of the spectral estimate of the Lévy measure $\widetilde{\nu}_{j_k}$, which will have been computed in a finite-sample setting, it makes sense to focus on a finite-sample approach to derive confidence intervals for the estimated price. Indeed, as it will be shown in the following section, under a finite sample setting it is even possible to obtain confidence intervals for the digital call price Π_j for all $j \in \{1, \dots, n\}$.

Similarly to Söhl and Trabs [26] and Koorevaar [15], we compute the finite sample variance for the error of the Lévy triplet, the idea behind the method is to decompose the pricing error $\widetilde{\Pi}_j - \Pi_j$ in a "principal" part, which is normally distributed in our asymptotic setting, and the asymptotically negligible remainder terms. Thereafter, the assumption will be made that these remainder terms are also negligible in a finite sample setting, as it has been shown that under the appropriate scaling, they converge to 0 faster than the main term. It must also be noted that a CLT along a subsequence in Chapter 8 is only guaranteed if the covariance operator is not zero, something which need not be the case under the current scaling. However, for the purposes of our finite sample setting, those considerations can be deferred, and the study of a scaling that will guarantee a CLT for $\widetilde{\Pi}_j - \Pi_j$ is left as an open question.

9.1.1. Finite Sample Confidence Interval

Recall from Chapter 8 the definitions:

$$\alpha_j(x) = \cos(U_j x)(\widetilde{\nu}_j(x) - \nu_j(x)); \quad g_j(x) = \int_{-U_j+\varepsilon}^{\infty} \Re(M_j(u)\, e^{iux})\, du$$

and

$$M_j(u) = \frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\, \varphi_{T_j}(u+U_j).$$

To construct $(100-\alpha)\%$ finite sample confidence intervals for $\widetilde{\Pi}_j$, the variance of the estimator, $s^2_{\Pi_j} \approx \mathbb{V}(\widetilde{\Pi}_j)$ has to be computed. For this, let us once again look at the error decomposition of the pricing formula:

$$\begin{aligned}\widetilde{\Pi}_j - \Pi_j &= P_j + B_j \\ &= P'_j - D_j + E_j + B_j \\ &= 2\,(T_j - T_{j-1})\frac{1}{\pi}\int_K g(x)\,\alpha_j(x)\,dx + P_j j - 1' - D_j + E_j + B_j \\ &= P'_j + P'_{j-1} - D_j + E_j + B_j.\end{aligned}$$

Recall that, in Chapter 8, all factors in the error decomposition but for P'_j vanish asymptotically under the scaling R_j. D_j, E_j and B_j converge to 0 in probability. On the other hand, as shown in Chapter 8, the scaling R_j makes the terms corresponding to maturities T_r where $r < j$ negligible. These terms are the ones contained in P'_{j-1}.

Hence, analogously to Söhl and Trabs' [26] and Koorevaar's [15] assumption for the error of the Lévy triplet, we make the assumption that, in a finite sample setting, the non-P'_j terms can be neglected. Hence, we obtain

$$\widetilde{\Pi}_j - \Pi_j \approx P'_j = \frac{2(T_j - T_{j-1})}{\pi}\int_K g_j(x)\,\alpha_j(x)\,dx. \tag{9.1}$$

This is justified by the aforementioned asymptotic vanishing under scaling. Koorevaar [15] makes the same assumption that the bias and remainder terms can also be dropped in the finite sample setting for the error $\widetilde{\nu}_j(x) - \nu_j(x)$. Making the same linearity assumption here for the Lévy density error, it holds that $\mathbb{E}(P'_j) = 0$. As a result, for the finite sample variance $s^2_{\Pi_j} \approx \mathbb{V}(\widetilde{\Pi}_j)$, we obtain:

$$s^2_{\Pi_j} \approx \mathbb{V}(\widetilde{\Pi}_j) = \mathbb{E}\left(\widetilde{\Pi}_j - \mathbb{E}\left(\widetilde{\Pi}_j\right)\right)^2 \approx \mathbb{E}\left(\widetilde{\Pi}_j - \Pi_j\right)^2 \approx \mathbb{E}(P'_j)^2 = \mathbb{V}(P'_j). \tag{9.2}$$

Take now $h_j(x) := g_j(x)\,\cos(U_j x)$, such that $g_j(x)\,\alpha_j(x) = h_j(x)\,(\widetilde{\nu}_j(x) - \nu_j(x))$. Thus, we obtain

$$P'_j = \frac{2(T_j - T_{j-1})}{\pi}\int_K h_j(x)\,(\widetilde{\nu}_j(x) - \nu_j(x))\,dx. \tag{9.3}$$

Hence, we can now further look at the term $\mathbb{V}(P'_j)$ more closely:

$$\mathbb{V}(P'_j) = \left(\frac{2(T_j - T_{j-1})}{\pi}\right)^2 \int_{K\times K} h_j(x)\,h_j(z)\,\mathrm{Cov}[(\widetilde{\nu}_j(x) - \nu_j(x)); (\widetilde{\nu}_j(z) - \nu_j(z))]\,dx\,dz. \tag{9.4}$$

Similarly to the $\widetilde{\mu}_j(x)$, the error decomposition for $\widetilde{\nu}_j(x)$ is as follows:

$$\begin{aligned}\widetilde{\nu}_j(x)-\nu_j(x) &= \frac{1}{2\pi}\Bigg[\int_{-U_j}^{U_j}\big(\widetilde{\psi}_{\nu_j}(v)-\psi_{\nu_j}(v)\big)w_{\nu_j}^{U_j}(v)e^{-ivx}\,dv \\ &\quad+\frac{\widetilde{\sigma}_j^2-\sigma_j^2}{2}\int_{-U_j}^{U_j}v^2w_{\nu_j}^{U_j}(v)e^{-ivx}\,dv \\ &\quad-i(\widetilde{\gamma}_j-\gamma_j)\int_{-U_j}^{U_j}v\,w_{\nu_j}^{U_j}(v)e^{-ivx}\,dv+(\widetilde{\lambda}_j-\lambda_j)\int_{-U_j}^{U_j}w_{\nu_j}^{U_j}(v)e^{-ivx}\,dv \\ &\quad+\int_{\mathbb{R}\setminus[-U_j,U_j]}\left(\psi_{\nu_j}(v)+\frac{\sigma_j^2}{2}v^2-i\gamma_jv+\lambda_j\right)(1-w_{\nu_j}^{U_j}(v))e^{-ivx}\,dv\Bigg] \\ &=:\Psi+\Sigma+\Gamma+\Lambda+\mathscr{B},\end{aligned}\tag{9.5}$$

where Ψ is further decomposed as

$$\begin{aligned}\Psi &= \frac{1}{2\pi}\int_{-U_j}^{U_j}\big[\mathcal{L}_j^0(v)-\mathcal{L}_j^1(v)+\mathcal{R}_j^0(v)-\mathcal{R}_j^1(v)\big]w_{\nu_j}^{U_j}(v)e^{-ivx}\,dv \\ &=:\mathscr{L}_{\nu_j}^0-\mathscr{L}_{\nu_j}^1+\mathscr{R}_{\nu_j}^0-\mathscr{R}_{\nu_j}^1.\end{aligned}\tag{9.6}$$

In the decomposition of the error of the Lévy measure, the errors of the other parameters are also present. All of these errors are multiplied by integrals which are inverse Fourier transforms. These are well-defined due to the conditions imposed on $w_{\nu_j}^{U_j}\in\mathcal{W}_{s_j}^n$. These integrals will be labeled, for ease of notation, as

$$t_{U_j}^{(k)}(x):=\mathscr{F}^{-1}\Big[v^kw_{\nu_j}^{U_j}(v)\Big](x)\quad\forall\,k\in\{0,1,2\}.\tag{9.7}$$

Furthermore, let us approximate the errors $\widetilde{\sigma}_j^2-\sigma_j^2$, $\widetilde{\gamma}_j-\gamma_j$ and $\widetilde{\lambda}_j-\lambda_j$ by their linear parts, given their decomposition in equations (3.30), (3.31) and (3.32):

$$\widetilde{\sigma}_j^2-\sigma_j^2\approx\mathscr{L}_{\sigma_j^2}^0-\mathscr{L}_{\sigma_j^2}^1\tag{9.8}$$

$$\widetilde{\gamma}_j-\gamma_j\approx\mathscr{L}_{\gamma_j}^0-\mathscr{L}_{\sigma_j^2}^0+\mathscr{L}_{\sigma_j^2}^1-\mathscr{L}_{\gamma_j}^1\tag{9.9}$$

$$\widetilde{\lambda}_j-\lambda_j\approx\mathscr{L}_{\lambda_j}^0+\mathscr{L}_{\gamma_j}^0-\frac{1}{2}\mathscr{L}_{\sigma_j^2}^0-\mathscr{L}_{\lambda_j}^1-\mathscr{L}_{\gamma_j}^1+\frac{1}{2}\mathscr{L}_{\sigma_j^2}^1.\tag{9.10}$$

Hence, similarly to Koorevaar [15], $\widetilde{\nu}_j(x)-\nu_j(x)$ can be written as

$$\begin{aligned}&\widetilde{\nu}_j(x)-\nu_j(x)= \\ &\left\{\mathscr{L}_{\nu_j}^0+\mathscr{L}_{\sigma_j^2}^0\left(\frac{1}{2}t_{U_j}^{(2)}(x)+it_{U_j}^{(1)}(x)-\frac{1}{2}t_{U_j}^{(0)}(x)\right)+\mathscr{L}_{\gamma_j}^0\left(-it_{U_j}^{(1)}(x)+t_{U_j}^{(0)}(x)\right)+\mathscr{L}_{\lambda_j}^0t_{U_j}^{(0)}(x)\right\} \\ &-\left\{\mathscr{L}_{\nu_j}^1+\mathscr{L}_{\sigma_j^2}^1\left(\frac{1}{2}t_{U_j}^{(2)}(x)+it_{U_j}^{(1)}(x)-\frac{1}{2}t_{U_j}^{(0)}(x)\right)+\mathscr{L}_{\gamma_j}^1\left(-it_{U_j}^{(1)}(x)+t_{U_j}^{(0)}(x)\right)+\mathscr{L}_{\lambda_j}^1t_{U_j}^{(0)}(x)\right\}.\end{aligned}\tag{9.11}$$

Recall now that

$$\mathscr{L}_{\nu_j}^l=\frac{1}{2\pi}\int_{-U_j}^{U_j}\mathcal{L}_{\nu_j}^l(v)\,w_{\nu_j}^{U_j}(v)e^{-ivx}\,dv,$$

where

$$\begin{aligned}\mathcal{L}_{\nu_j}^l(v) &= \frac{1}{T_j-T_{j-1}}\frac{\widetilde{\varphi}_{T_{j-l}}(v)-\varphi_{T_{j-l}}(v)}{\varphi_{T_{j-l}}(v)} \\ &= -\frac{1}{T_j-T_{j-1}}\frac{v(v+i)\sum_{k=1}^{m_j}\delta_{j-l,k}\varepsilon_{j-l,k}\,\mathscr{F}b_{j-l,k}(v+i)}{\varphi_{T_{j-l}}(v)},\end{aligned}$$

where the last equality is given by equation (3.35). Similarly to the approach laid out in Chapter 3 we can now approximate $\mathscr{F} b_{j-l,k}(v+i)$ using the interpolation scheme, with the difference that this time the evaluation is at $v+i$ instead of at v:

$$\begin{aligned}\mathscr{F} b_{j-l,k}(v+i) &= \int_{\mathbb{R}} b_{j-l,k}(x) e^{i(v+i)x}\,dx = \int_{\mathbb{R}} b_{j-l,k}(x) e^{-x} e^{ivx}\,dx \\ &\approx \frac{x_{j-l,k+1} - x_{j-l,k-1}}{2}\, e^{-x_{j-l,k}} e^{ivx_{j-l,k}} = \Delta_{j-l} e^{-x_{j-l,k}} e^{ivx_{j-l,k}},\end{aligned} \tag{9.12}$$

where the approximation is due to the fact that the grid is equidistant, i.e.:

$$\Delta_{j-l} = |x_{j-l,k+1} - x_{j-l,k}| = |x_{j-l,k} - x_{j-l,k-1}|. \tag{9.13}$$

Let us now define the following function:

$$f^l_{\nu_j}(v) := -w^{U_j}_{\nu_j}(v)\,\frac{iv(1+iv)}{(T_j - T_{j-1})\,\varphi_{T_{j-l}}(v-i)}. \tag{9.14}$$

We can now plug (9.12) and (9.14) into the linear term $\mathscr{L}^l_{\nu_j}$, obtaining the following:

$$\begin{aligned}\mathscr{L}^l_{\nu_j} &= \frac{1}{2\pi}\int_{-U_j}^{U_j} \mathcal{L}^l_{\nu_j}(v)\, w^{U_j}_{\nu_j}(v) e^{-ivx}\,dv \\ &= \frac{1}{2\pi}\int_{-U_j}^{U_j} w^{U_j}_{\nu_j}(v)\,\frac{-v(v+i)\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\,\mathscr{F} b_{j-l,k}(v+i)}{\varphi_{T_{j-l}}(v)}\, e^{-ivx}\,dv \\ &= \frac{1}{2\pi}\int_{-U_j}^{U_j} f^l_{\nu_j}(v) \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\,\mathscr{F} b_{j-l,k}(v+i) e^{-ivx}\,dv \\ &\approx \frac{1}{2\pi}\int_{-U_j}^{U_j} f^l_{\nu_j}(v) \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\Delta_j e^{-x_{j-l,k}} e^{ivx_{j-l,k}} e^{-ivx}\,dv \\ &= \frac{1}{2\pi}\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\Delta_j e^{-x_{j-l,k}} \int_{-U_j}^{U_j} f^l_{\nu_j}(v) e^{-iv(x-x_{j-l,k})}\,dv \\ &= \frac{1}{2\pi}\sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\Delta_j e^{-x_{j-l,k}} 2\pi \frac{1}{2\pi}\int_{\mathbb{R}} f^l_{\nu_j}(v) e^{-iv(x-x_{j-l,k})}\,dv \\ &= 2\pi \sum_{k=1}^{m_{j-l}} \frac{e^{-x_{j-l,k}}}{2\pi}\,\delta_{j-l,k}\,\varepsilon_{j-l,k}\,\Delta_j\,\mathscr{F}^{-1} f^l_{\nu_j}(x - x_{j-l,k}),\end{aligned} \tag{9.15}$$

where in the penultimate equality we have used the fact that $w^{U_j}_{\nu_j}$ is compactly supported on $[-U_j, U_j]$. The expressions for $\mathscr{L}^l_{\sigma^2_j}, \mathscr{L}^l_{\gamma_j}, \mathscr{L}^l_{\lambda_j}$ are approximated similarly to that of $\mathscr{L}^l_{\nu_j}$, resulting in:

$$\mathscr{L}^l_{\sigma^2_j} \approx 2\pi \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\Delta_{j-l}\Re\Big(\mathscr{F}^{-1} f^l_{\gamma_j}(-x_{j-l,k})\Big) \tag{9.16}$$

$$\mathscr{L}^l_{\gamma_j} \approx 2\pi \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\Delta_{j-l}\Im\Big(\mathscr{F}^{-1} f^l_{\gamma_j}(-x_{j-l,k})\Big) \tag{9.17}$$

$$\mathscr{L}^l_{\lambda_j} \approx 2\pi \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}\varepsilon_{j-l,k}\Delta_{j-l}\Re\Big(\mathscr{F}^{-1} f^l_{\sigma_j}(-x_{j-l,k})\Big), \tag{9.18}$$

where

$$f^l_{\xi_j}(v) := w^{U_j}_{\xi_j}(v)\,\frac{iv(1+iv)}{(T_j - T_{j-1})\,\varphi_{T_{j-l}}(v-i)} \quad \forall \xi_j \in \{\sigma^2_j, \gamma_j, \lambda_j\}. \tag{9.19}$$

The derivation of these approximations, just like that of the linear part of the Lévy density error, follows Koorevaar [15], where they are computed in detail in sections 4.1-4.4.

The approximations conducted above result in the fact that equation (9.11) can be approximated as follows:

$$\begin{aligned}
\widetilde{\nu}_j(x) - \nu_j(x) = 2\pi \sum_{k=1}^{m_j} &\delta_{j,k}\,\varepsilon_{j,k}\,\Delta_j \\
&\cdot \Bigg(\frac{e^{-x_{j,k}}}{2\pi} \mathscr{F}^{-1} f^0_{\nu_j}(x - x_{j,k}) + \Re\Big(\mathscr{F}^{-1} f^0_{\sigma_j}(-x_{j,k})\Big) \left(\frac{1}{2} t^{(2)}_{U_j}(x) + i t^{(1)}_{U_j}(x) - \frac{1}{2} t^{(0)}_{U_j}(x)\right) \\
&\qquad + \Im\Big(\mathscr{F}^{-1} f^0_{\gamma_j}(-x_{j,k})\Big) \Big(-i t^{(1)}_{U_j}(x) + t^{(0)}_{U_j}(x)\Big) + \Re\Big(\mathscr{F}^{-1} f^0_{\lambda_j}(-x_{j,k})\Big)\, t^{(0)}_{U_j}(x) \Bigg) \\
&- 2\pi \sum_{k=1}^{m_{j-1}} \delta_{j-1,k}\,\varepsilon_{j-1,k}\,\Delta_{j-1} \\
&\cdot \Bigg(\frac{e^{-x_{j,k}}}{2\pi} \mathscr{F}^{-1} f^1_{\nu_j}(x - x_{j,k}) + \Re\Big(\mathscr{F}^{-1} f^1_{\sigma_j}(-x_{j,k})\Big) \left(\frac{1}{2} t^{(2)}_{U_j}(x) + i t^{(1)}_{U_j}(x) - \frac{1}{2} t^{(0)}_{U_j}(x)\right) \\
&\qquad + \Im\Big(\mathscr{F}^{-1} f^1_{\gamma_j}(-x_{j,k})\Big) \Big(-i t^{(1)}_{U_j}(x) + t^{(0)}_{U_j}(x)\Big) + \Re\Big(\mathscr{F}^{-1} f^1_{\lambda_j}(-x_{j,k})\Big)\, t^{(0)}_{U_j}(x) \Bigg) \\
&= 2\pi \sum_{l=0}^{1} \sum_{k=1}^{m_{j-l}} (-1)^l\, \delta_{j-l,k}\,\varepsilon_{j-l,k}\,\Delta_{j-l} \\
&\cdot \Bigg(\frac{e^{-x_{j,k}}}{2\pi} \mathscr{F}^{-1} f^l_{\nu_j}(x - x_{j,k}) + \Re\Big(\mathscr{F}^{-1} f^l_{\sigma_j}(-x_{j,k})\Big) \left(\frac{1}{2} t^{(2)}_{U_j}(x) + i t^{(1)}_{U_j}(x) - \frac{1}{2} t^{(0)}_{U_j}(x)\right) \\
&\qquad + \Im\Big(\mathscr{F}^{-1} f^l_{\gamma_j}(-x_{j,k})\Big) \Big(-i t^{(1)}_{U_j}(x) + t^{(0)}_{U_j}(x)\Big) + \Re\Big(\mathscr{F}^{-1} f^l_{\lambda_j}(-x_{j,k})\Big)\, t^{(0)}_{U_j}(x) \Bigg).
\end{aligned}$$

Hence, the final expression we obtain for the error is:

$$\widetilde{\nu}_j(x) - \nu_j(x) = 2\pi \sum_{l=0}^{1} \sum_{k=1}^{m_{j-l}} (-1)^l\, \delta_{j-l,k}\,\varepsilon_{j-l,k}\,\Delta_{j-l}\,\Phi^l_{j,k}(x), \tag{9.20}$$

where

$$\begin{aligned}
\Phi^l_{j,k}(x) := \Bigg(&\frac{e^{-x_{j,k}}}{2\pi} \mathscr{F}^{-1} f^l_{\nu_j}(x - x_{j,k}) + \Re\Big(\mathscr{F}^{-1} f^l_{\sigma_j}(-x_{j,k})\Big) \left(\frac{1}{2} t^{(2)}_{U_j}(x) + i t^{(1)}_{U_j}(x) - \frac{1}{2} t^{(0)}_{U_j}(x)\right) \\
&+ \Im\Big(\mathscr{F}^{-1} f^l_{\gamma_j}(-x_{j,k})\Big) \Big(-i t^{(1)}_{U_j}(x) + t^{(0)}_{U_j}(x)\Big) + \Re\Big(\mathscr{F}^{-1} f^l_{\lambda_j}(-x_{j,k})\Big)\, t^{(0)}_{U_j}(x). \Bigg)
\end{aligned} \tag{9.21}$$

Now, note that the covariance term from equation (9.4) can be expanded as follows:

$$\begin{aligned}
\mathrm{Cov}[(\widetilde{\nu}_j(x) - \nu_j(x)); (\widetilde{\nu}_j(z) - \nu_j(z))] &= \mathbb{E}\left[(\widetilde{\nu}_j(x) - \nu_j(x)) \cdot (\widetilde{\nu}_j(z) - \nu_j(z))\right] \\
&\quad - \mathbb{E}\left[(\widetilde{\nu}_j(x) - \nu_j(x))\right] \mathbb{E}\left[(\widetilde{\nu}_j(z) - \nu_j(z))\right] \\
&\approx \mathbb{E}\left[(\widetilde{\nu}_j(x) - \nu_j(x)) \cdot (\widetilde{\nu}_j(z) - \nu_j(z))\right],
\end{aligned}$$

because the linearity assumption in Koorevaar results in $\mathbb{E}\left[(\widetilde{\nu}_j(x) - \nu_j(x))\right] \approx 0$. This means that

$$\begin{aligned}&\operatorname{Cov}[(\widetilde{\nu}_j(x) - \nu_j(x)); (\widetilde{\nu}_j(z) - \nu_j(z))] = \\ &= \mathbb{E}\left[\left(2\pi \sum_{l=0}^{1} \sum_{k=1}^{m_{j-l}} (-1)^l \, \delta_{j-l,k} \, \varepsilon_{j-l,k} \, \Delta_{j-l} \, \Phi^l_{j,k}(x)\right); \left(2\pi \sum_{l'=0}^{1} \sum_{k'=1}^{m_{j-l'}} (-1)^{l'} \, \delta_{j-l',k'} \, \varepsilon_{j-l',k'} \, \Delta_{j-l'} \, \Phi^{l'}_{j}(z)\right)\right].\end{aligned} \tag{9.22}$$

Given that the random variables $(\varepsilon_{j,k})$ are independent and centered, the cross terms of the product can be dropped, and we only focus on the terms where $l = l'$ and $k = k'$:

$$\begin{aligned}&\operatorname{Cov}[(\widetilde{\nu}_j(x) - \nu_j(x)); (\widetilde{\nu}_j(z) - \nu_j(z))] = \\ &= 4\pi^2 \sum_{l=0}^{1} \sum_{k=1}^{m_{j-l}} \delta^2_{j-l,k} \, \mathbb{E}(\varepsilon^2_{j-l,k}) \, \Delta^2_{j-l} \, \Phi^l_{j,k}(x) \, \Phi^l_{j,k}(z) \\ &= 4\pi^2 \sum_{l=0}^{1} \sum_{k=1}^{m_{j-l}} \delta^2_{j-l,k} \, \Delta^2_{j-l} \, \Phi^l_{j,k}(x) \, \Phi^l_{j,k}(z).\end{aligned} \tag{9.23}$$

Hence, this means that we can now obtain an expression for the variance of the leading term $\mathbb{V}(P^l_j)$, and, as a result, a finite sample approximation of the variance of the pricing error $\widetilde{\Pi}_j - \Pi_j$, under the linearity assumption:

$$\begin{aligned}s^2_{\Pi_j} &\approx \mathbb{V}(\widetilde{\Pi}_j) \\ &\approx \mathbb{V}(P'_j) = \left(\frac{2(T_j - T_{j-1})}{\pi}\right)^2 \int_{K \times K} h_j(x) \, h_j(z) \operatorname{Cov}[(\widetilde{\nu}_j(x) - \nu_j(x)); (\widetilde{\nu}_j(z) - \nu_j(z))] \, dx \, dz \\ &\approx 16\pi(T_j - T_{j-1}) \sum_{l=0}^{1} \sum_{k=1}^{m_{j-l}} \int_{K \times K} h_j(x) \, h_j(z) \, \delta^2_{j-l,k} \, \Delta^2_{j-l} \, \Phi^l_{j,k}(x) \, \Phi^l_{j,k}(z) \, dx \, dz.\end{aligned} \tag{9.24}$$

It must be noted that the characteristic function $\varphi_{T_{j-l}}$ in the $\Phi^l_{j,k}(\cdot)$ term is not directly observable, and, as a result, the version estimated via the spectral method, $\widetilde{\varphi}_{T_{j-l}}$ must be plugged in to obtain a finite sample variance when conducting simulations. Once this has been taken into account, we are able to use the Continuous Mapping & Slutsky's Theorem to construct $(100 - \alpha)\%$ confidence intervals for the estimated price of a digital call option with maturity T_j, Π_j:

$$\left[\widetilde{\Pi}_j + z_{\alpha/2} \, s_{\Pi_j}; \widetilde{\Pi}_j - z_{100-\alpha/2} \, s_{\Pi_j}\right]. \tag{9.25}$$

Chapter 10
Conclusion

10.1. Summary

The purpose of this thesis has been twofold. First, to extend the pointwise asymptotic normality of the Lévy density estimate $\widetilde{\nu}_j$ in the spectral calibration framework for time-inhomogeneous exponential Lévy models to asymptotic normality in the functional space $L^2(K)$. Second, to leverage said functional Central Limit Theorem for the uncertainty quantification of the price of a digital call option.

Chapters 1 and 2 introduced the motivation for the time-inhomogeneous Lévy framework and reviewed the background on (time-inhomogeneous) Lévy processes, exponential Lévy models, probability on Hilbert spaces and option pricing via Fourier methods. Chapter 3 explained the statistical model and the assumptions inherited from the spectral calibration procedure of Tendijck [27] and Koorevaar [15], and recalled how the Lévy process parameters $(\sigma_j^2, \gamma_j, \lambda_j, \nu_j)$ are extracted from observed plain European call and put option prices via the calibration estimator $\widetilde{\psi}_j$, as well as the pointwise normality results for the errors $\widetilde{\sigma}_j^2 - \sigma_j^2$, $\widetilde{\gamma}_j - \gamma_j$, $\widetilde{\lambda}_j - \lambda_j$ and $\widetilde{\nu}_j - \nu_j$.

Chapter 4 derived the candidate covariance kernel for the exponentially-tilted estimation error $\widetilde{\mu}_j(x) - \mu_j(x)$ and established the central structural obstruction of this thesis: the rescaled kernel converges pointwise to an oscillatory limit $2\mathcal{C}\cos(U_j(x-z))$ which is not square-integrable on $\mathbb{R}^2$. The candidate covariance operator is therefore not nuclear on $L^2(\mathbb{R})$, so the Hilbert-space CLT of Giné–León cannot be applied in that space. This motivated restricting the domain of the Lévy measure to a compact set $K \subset \mathbb{R}$ and working in $L^2(K;\mathbb{R})$. Because the variable $x \in \mathbb{R}$ is the negative forward log-moneyness, and the grid points for the model, it makes sense to restrict its domain to a compact set. The reason for this is that there are very few vanilla options quoted on the market very far OTM or ITM.

Furthermore, Chapter 5 established the functional CLT for the linear part of the exponentially-tilted error decomposition by verifying the three conditions of Giné–León [10] Corollary 4.3, namely the positive covariance, infinitesimality, and tail decay conditions. To obtain this functional CLT, it was necessary to modulate the linear part of the error decomposition by the cosine factor $\cos(U_j\cdot)$, in order to eliminate oscillations and obtain convergence to a fixed covariance object. In Chapter 6 it was shown that, under the appropriate normalization $r_j\,\cos(U_j\,\cdot)$, all non-linear terms of the error decomposition, the bias, the stochastic remainder, and the contributions from $\widetilde{\sigma}_j^2 - \sigma_j^2$, $\widetilde{\gamma}_j - \gamma_j$, and $\widetilde{\lambda}_j - \lambda_j$, vanish in $L^2(K)$. Chapter 7 combined these results into the main theorem of the thesis: a functional CLT for the cosine modulated full estimation error: $r_j\,\cos(U_j x)\,(\widetilde{\nu}_j(x) - \nu_j(x))$ in $L^2(K)$. It must be noted that the decay conditions required to obtain an $L^2(K)$ CLT are almost the same as those in Tendijck [27] and Koorevaar [15], with a small adjustment in the decay condition which makes the remainder term converge to 0 in probability. Here, the fact the object that must converge to 0 is the expectation of the $L^2(K)$-norm of the remainder means that this object is raised to a higher power than in the scalar case. As a result, a mildly stronger condition is required.

In Chapter 8, this functional CLT is applied to the error of the price of a digital call option. The digital call price Π_j (where j indicates that the maturity is T_j, for $j \in \{1,\dots,n\}$) can be written as a linear functional of the characteristic function via the Gil-Pelaez formula [9]. This, combined with the $L^2(K)$ CLT and the extended Continuous Mapping Theorem of Van der Vaart and Wellner [28] yields a convergence in distribution result for the pricing error $\widetilde{\Pi}_{j_k} - \Pi_{j_k}$ along a subsequence j_k. The restriction to a subsequence arises from the weak convergence argument used to find a limit for the oscillatory function g_j, and the limit is non-degenerate only when the associated covariance operator does not vanish.

Finally, in Chapter 9 this asymptotic result was applied, by developing a finite-sample variance approximation for the pricing error, similar to that of Söhl and Trabs [26] and Koorevaar [15], in order to obtain $(100-\alpha)\%$ finite-sample confidence intervals for the estimated digital option price for all T_j where $j \in \{1, \dots, n\}$. Taken together, these results move the spectral calibration framework from an abstract $L^2(K)$ CLT to a concrete instrument for quantifying the uncertainty of calibrated option prices, something that is directly relevant to the risk management needs of financial institutions.

10.2. Discussion of Results

Having summarized the results of the thesis, it is now worth taking a step back to assess their meaning in a broader context, as well as some of the modeling choices made throughout.

First, one of the central structural decisions made was the choice to restrict analysis from $L^2(\mathbb{R})$ to $L^2(K)$, where $K \subset \mathbb{R}$ is a compact set. It is worth exploring how limiting this restriction is. This restriction is forced, rather than a modeling choice. This is because the pointwise limit of the covariance kernel for the linear part of the error $\widetilde{\mu}_j - \mu_j$ is $2\mathcal{C}\cos(U_j(x-z))$, where $\mathcal{C} > 0$ is a constant and $U_j \to \infty$. This object has infinite $L^2(\mathbb{R}^2)$ norm. Hence, the corresponding operator is not Hilbert-Schmidt in $L^2(\mathbb{R})$. This means it is also not nuclear in that space, and hence, it is not possible to recover a Central Limit Theorem in the form of Theorem 2.2 in the space $L^2(\mathbb{R})$. Hence, this justifies the restriction to $K \subset \mathbb{R}$, under which, in Chapter 5, it is shown that, after cosine modulation, it is possible to recover a Giné-León CLT for the linear part of the exponentially-tilted Lévy density error.

It is also worth analyzing the consequences of such a restriction from a financial application perspective. It can be argued that restricting to a compact domain is a natural modeling choice that reflects market structure. Recall that the variable x in $\nu_j(x)$ is the negative forward log-moneyness, and that the Lévy density is calibrated over a finite, liquid range of European option strikes. In the market, there are very few strikes quoted deep in-the-money or out-the-money, so, de facto, ν_j is observed only over a finite range of values. Thus, the cost of the restriction has limited practical impact.

From a theoretical standpoint, it is also worth noting that a Giné-León CLT might be recovered for an entire space if we look at a space different to $L^2(\mathbb{R})$. This is something that remains unexplored in this thesis, but possible natural alternatives could be a weighted $L^2(\mathbb{R}, w)$ space, (with some weight w which can recover a finite norm for the covariance kernel) or a weaker topology like a Sobolev space. For the first alternative, a caveat worth noting is that a weighted-space CLT might be less easily transferrable to pricing applications like the one in Chapter 8. For the second one, it is worth noting that the strength of the conclusion is traded for having a larger domain, and that in an asymptotic setting where $U_j \to \infty$ it can introduce additional complications that might make convergence unattainable.

Furthermore, another point worth discussing is the slight strengthening of the decay condition for the remainder to decay in probability, given by Proposition 6.2, to:

$$\frac{\Delta_{j-l}^2}{\Delta_j} U_j^5 \exp\left(U_j^2 \left(2\sum_{i=1}^{j-l} (T_i - T_{i-1})\sigma_i^2 - \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2 \right) \right) \longrightarrow 0, \quad \text{for } l = 0, 1,$$

where the difference is that we have U_j^5 instead of U_j^4. It is worth noting that this is the only decay condition that changes with respect to Tendijck [27] and Koorevaar [15]. The reason for this change is that in the pointwise setting, the object required to decay in probability is the remainder itself, whereas in $L^2(K)$ it is the integral of the square over K of this object. This produces an additional factor of U_j, requiring a stronger decay condition from U_j^4 to U_j^5. However, for practical purposes, the strengthening itself is relatively benign. The admissible classes of Lévy models remains intact, as does the smoothness index s_j. Indeed, any model satisfying the original conditions with some margin can satisfy the new one too.

Finally, it is worth reflecting on where the results of this thesis sit in the literature, and what gaps are filled. The results sit at the intersection of three branches of literature. For non-parametric estimation of Lévy models from option prices, it builds on the work of Belomestny and Reiß [2], Söhl [25], Söhl and Trabs [26], and the time-inhomogeneous framework of Tendijck [27] and Koorevaar [15]. The main contribution is the raising the pointwise asymptotic normality result of $\widetilde{\nu}_j$ to a functional statement in $L^2(K)$. Within literature of Central Limit Theorems in Hilbert Spaces, such as Kandelaki-Sozanov [14], Giné-León [10] and Prokhorov-Statulevičius [22], this thesis is a non-trivial application, as it shows that the Giné-León CLT can be applied to a statistical spectral-calibration problem. Additionally, this thesis links Fourier-based option pricing methods in the line of Carr and Madan [3] or Lord and Kahl [17] to the uncertainty quantification resulting from the spectral calibration method.

10.3. Possible Extensions

Having obtained a natural financial application from the theoretical results of this thesis in Chapter 9, the question arises of how the conclusions obtained here could be extended and which related topics are worth exploring further.

Firstly, it is worth recalling that a CLT along a subsequence for $\widetilde{\Pi}_{j_k}$ is only obtained when the covariance operator

$$\frac{1}{2}\mathcal{C}\langle g, e^{-x}\rangle^2_{L^2(K)} \neq 0,$$

which happens if and only if $\langle g, e^{-x}\rangle_{L^2(K)} \neq 0$. This depends on the explicit oscillatory behavior of g_{j_k}. Hence, a possible line of exploration would be to find an adjusted scaling for the pricing error, which could (possibly only for a subsequence) recover a fixed non-zero limit. To recover this covariance, it would not only be necessary to counteract the magnitude of the estimation error (which R_j achieves already) but also the oscillations that occur in g_{j_k}. Proving the existence and computing such a scaling would lead to a definitive CLT result and asymptotic confidence intervals, whose coverage could be compared to that of the finite sample ones.

Additionally, another possible line of future exploration is a financial one, i.e., using the Lévy density CLT to recover the pricing error of other exotic options. For that, it is necessary that the pricing formula of said options can be expressed in terms of the characteristic function. Indeed, it is because the price of a digital call option can be expressed in terms of the Gil-Pelaez formula that it is possible to use the Extended Continuous Mapping Theorem to "transfer" this normality, while not needing to use the delta method. Examples of other types of options (such as those closely resembling digital or European options) to which this methodology could apply include gap or power options, as shown by the formulas in Carr and Madan [3] or Fang [7]. Moreover, another application worth mentioning concerns linear combinations of the above, such as straddles or risk reversals. These types of option contracts are extensively used in the financial industry. On the other hand, an application to options whose payoff cannot be shown to depend directly on the characteristic function would be out of reach for this method. A good example of this would be path-dependent options, whose price depends on the law of the path $(X_t)_{t\leq T_j}$ [12].

An additional relevant financial extension would be that of hedging. Calibrated models and the corresponding pricing errors are commonly used for constructing and rebalancing hedging portfolios. Indeed, via Carr and Madan [3], option Greeks can be computed in terms of the characteristic function φ_{T_j}, and therefore of the calibrated Lévy triplet. Hence, the estimation errors propagate to the hedging ratios, which means that the natural question arises of whether a normality or convergence in distribution result can be obtained for the error of said ratios. This would make it possible to extend the finite sample uncertainty quantification of section 9.1.1 to the cost and risk of hedging a trade, something which could also be directly applicable in a financial industry setting.

Finally, a more practical extension concerns implementation. The convergence results of this thesis are stated for the estimation error of the time-inhomogeneous Lévy triplet $(\widetilde{\sigma}^2_j, \widetilde{\gamma}_j, \widetilde{\lambda}_j, \widetilde{\nu}_j)$, and the spectral calibration procedure that produces these estimates has already been implemented and tested by Koorevaar [15], including Monte Carlo simulations and an empirical calibration to S&P 500 option data. An implementation of the digital option pricing error and the finite-sample confidence intervals of Section 9.1.1 could hence be built directly on top of the existing calibration code.

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Chapter A
Lemmata

Lemma A.4 (Uniform Covariance Kernel Bound)**.** *Let*

$$\mathcal{K}_j^l(x,z) := \sum_{k=1}^{m_{j-l}} \mathbb{E}[X^l_{U_j,k}(x) X^l_{U_j,k}(z)], \qquad x,z \in \mathbb{R},$$

where $X^l_{U_j,k}(x) = \frac{1}{\sqrt{a_j^l}} X_{U_j,k}{}^l(x)'$*, where* $a_j^l := \Delta_{j-l} \cdot U_j^2 \cdot \exp\{A_{j-l} \cdot U_j^2\}$*, given explicitly by*

$$X_{U_j,k}{}^l(x)' = \frac{1}{\pi} \frac{U_j}{T_j - T_{j-1}} \, \delta_{j-l,k} \varepsilon_{j-l,k} \, \Re\left(\int_0^1 \frac{ivU_j(1+ivU_j)\, \mathcal{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)} e^{-ivU_j x} w^1_{\mu_j}(v)\, dv \right).$$

Assume the assumptions in section 3.4 and that, as $U_j \to \infty$*,* $\Delta_j \to 0$*:*

$$\Delta_j U_j^4 \exp\left\{ U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1}) \sigma_i^2 \right\} \longrightarrow 0.$$

Then there exists a finite constant $C_0 > 0$ *such that*

$$\sup_{U_j \geq 1} \sup_{x,z \in \mathbb{R}} |\mathcal{K}_j^l(x,z)| \leq C_0.$$

Proof. Fix $U_j \geq 1$ and $x, z \in \mathbb{R}$. Recall that

$$X^l_{U_j,k}(x) = \frac{1}{\sqrt{a_j^l}} \, X'_{U_j,k}{}^l(x), \qquad a_j^l = \Delta_{j-l}\, U_j^2 \, e^{A_{j-l} U_j^2}.$$

Using the identity

$$\Re(a)\Re(b) = \tfrac{1}{4}\big(a\bar{b} + \bar{a}b + ab + \bar{a}\bar{b}\big),$$

we can write

$$\begin{aligned} \mathcal{K}_j^l(x,z) = \frac{(a_j^l)^{-1}}{4} \sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[& I_v(x)\, \overline{I_w(z)} \\ & + \overline{I_v(x)}\, I_w(z) \\ & + I_v(x)\, I_w(z) \\ & + \overline{I_v(x)}\, \overline{I_w(z)} \Big]. \end{aligned}$$

where

$$I_{U_j,k}(x) = \int_0^1 h_k(v)\, e^{-ivU_j x}\, dv.$$

and

$$h_{U_j,k}(x) := \frac{1}{\pi} \frac{U_j}{T_j - T_{j-1}} \, \mathrm{Re}\left(\int_0^1 \frac{ivU_j(1+ivU_j)\, \mathcal{F}b_{j-l,k}(vU_j)}{\varphi_{T_{j-l}}(vU_j - i)} \, e^{-ivU_j x}\, w^1_{\mu_j}(v)\, dv \right). \tag{A.1}$$

As shown in the proof of Theorem 4.1 in Chapter 4 the non-cross terms vanish, and only the cross terms give rise to the non-zero part of the limit. Their contribution gives rise to the oscillatory kernel identified previously and is uniformly bounded by $2|\mathcal{C}|$. It therefore suffices to show that the remaining non-cross terms go to 0. These are encompassed in the remainder:

$$r_{U_j}(x,z) = \frac{(a_j^l)^{-1}}{4} \sum_{k=1}^{m_{j-l}} \mathbb{E}\Big[I_v(x)I_w(z) + \overline{I_v(x)}\,\overline{I_w(z)}\Big]$$

are uniformly bounded in (x, z) and U_j.

By the triangle inequality and the fact that $|\mathbb{E}[XY]| \leq \mathbb{E}[|X||Y|]$, we obtain

$$|r_{U_j}(x,z)| \leq C \sum_{k=1}^{m_{j-l}} (a_j^l)^{-1}\, \mathbb{E}\big[|I_v(x)|\,|I_w(z)|\big].$$

Furthermore, we estimate

$$|I_v(x)| \leq \int_0^1 |h_{U_j,k}(v)|\, dv, \qquad |I_w(z)| \leq \int_0^1 |h_{U_j,k}(w)|\, dw,$$

and, using $\mathbb{E}[\varepsilon_{j-l,k}^2] = 1$ and $|e^{-ivU_jx}| \leq 1$, we obtain the bound

$$\mathbb{E}\big[|I_v(x)|\,|I_w(z)|\big] \leq \left(\int_0^1 |h_{U_j,k}(v)|\, dv\right)^2.$$

The conjugate case is analogous. Now, using the explicit form of $h_{U_j,k}$ and the Fourier transform of the triangular spline:

$$\mathscr{F} b_{j,k}(v) = \Delta_j e^{ivx_{j,k}} \operatorname{sinc}^2(\Delta_j v/2),$$

we obtain

$$\int_0^1 |h_{U_j,k}(v)|\, dv \leq C\, \delta_{j-l,k}\, \Delta_{j-l}\, U_j\, e^{A_{j-l}U_j^2/2}.$$

Consequently,

$$|r_{U_j}(x,z)| \leq C \sum_{k=1}^{m_{j-l}} (a_j^l)^{-1}\, \delta_{j-l,k}^2\, \Delta_{j-l}\, U_j^2\, e^{A_{j-l}U_j^2}.$$

Substituting the definition of a_j^l yields

$$\begin{aligned} |r_{U_j}(x,z)| &\leq C\Delta_{j-l}^2\, U_j^2\, e^{A_{j-l}U_j^2} \sum_{k=1}^{m_{j-l}} \delta_{j-l,k}^2. \\ &\leq \Delta_{j-l}\, U_j^2\, e^{A_{j-l}U_j^2}\, \|\delta_{j-l}\|_{L^2}^2 \to 0 \end{aligned}$$

By the condition

$$\Delta_j U_j^4 \exp\left\{ U_j^2 \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2 \right\} \longrightarrow 0.$$

we conclude that

$$\sup_{U_j \geq 1} \sup_{x,z \in \mathbb{R}} |r_{U_j}(x,z)| \to 0$$

Thus, as the cross-terms are uniformly bounded in U_j, x, z, because they result in a factor $\cos(U_j(x-z))$ as given by Theorem 4.1, and r_{U_j} is uniformly bounded, we obtain

$$\sup_{U_j \geq 1} \sup_{x,z \in \mathbb{R}} |\mathcal{K}_j^l(x,z)| \leq C_0$$

for some finite constant $C_0 > 0$. □

Lemma A.5 (Cosine Modulation Expansion)**.** *For any $x \in \mathbb{R}$ and $U_j > 0$, it holds that:*

$$e^{i(u+U_j)x} - 1 = 2\,\cos(U_j x) e^{iux} - (e^{i(u-U_j)x} - 1) - 2$$

Proof. Note that the right hand side can be rewritten as follows:

$$\begin{aligned}
e^{i(u+U_j)x} - 1 &= e^{i(u+U_j)x} - e^{iU_j x} + e^{i(u-U_j)x} - e^{-iU_j x} - e^{i(u-U_j)x} + 1 + e^{iU_j x} - 1 + e^{-iU_j x} - 1 \\
&= \left(e^{iU_j x} + e^{-iU_j x}\right)(e^{iux} - 1) - (e^{i(u-U_j)x} - 1) + (e^{iU_j x} - 1) + (e^{-iU_j x} - 1) \\
&= 2\,\cos(U_j x)(e^{iux} - 1) - (e^{i(u-U_j)x} - 1) + (e^{iU_j x} - 1) + (e^{-iU_j x} - 1) \\
&= 2\,\cos(U_j x)\, e^{iux} - 2\,\cos(U_j x) - (e^{i(u-U_j)x} - 1) + (e^{iU_j x} + e^{-iU_j x}) - 2 \\
&= 2\,\cos(U_j x)\, e^{iux} - 2\,\cos(U_j x) - (e^{i(u-U_j)x} - 1) + 2\,\cos(U_j x) - 2 \\
&= 2\,\cos(U_j x)\, e^{iux} - (e^{i(u-U_j)x} - 1) - 2
\end{aligned}$$

□

Lemma A.6 (Bias Decay)**.** *Under the condition*

$$U_j^{2(s_j+1)} \left(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\right) \to \infty$$

, and assuming $s_j \geq 2$, we have:

$$r_j |\mathscr{B}| \to 0 \quad \textit{as } U_j \to \infty. \tag{A.2}$$

Proof. By Section B.1 in Koorevaar, $|\mathscr{B}|^2 \lesssim U_j^{-2s_j}$, and by definition $r_j^{-2} = a_j + a_{j-1}$, so that:

$$r_j^2\, |\mathscr{B}|^2 \lesssim (a_j + a_{j-1})^{-1}\, U_j^{-2s_j}$$

which will go to 0 given the condition

$$U_j^{2(s_j+1)} \left(\Delta_j\, e^{A_j U_j^2} + \Delta_{j-1}\, e^{A_{j-1} U_j^2}\right) \to \infty$$

□

Lemma A.7 (Remainder Decay)**.** *Under the condition*

$$\frac{\Delta_{j-l}^2}{\Delta_j}\, U_j^5\, \exp\!\left(U_j^2 \left(2 \sum_{i=1}^{j-l} (T_i - T_{i-1})\sigma_i^2 - \sum_{i=1}^{j} (T_i - T_{i-1})\sigma_i^2 \right)\right) \longrightarrow 0, \quad \textit{for } l = 0, 1,$$

, it holds that

$$r_j\, \mathscr{R}^l_{\mu_j} \to 0 \tag{A.3}$$

in probability, i.e.:

$$\mathscr{R}^l_{\mu_j} \lesssim o_{\mathbb{P}}(1/r_j) \tag{A.4}$$

Proof. In order to show that

$$\mathbb{P}\left(|r_j\, \mathscr{R}^l_{\mu_j}| > \varepsilon\right) \to 0$$

Markov's inequality will be used. In Koorevaar [15] (page 45, equation (3.33)), it is known that

$$\mathbb{E}\left(|\mathscr{R}^l_{\mu_j}|^2\right) \lesssim \Delta_{j-l}^2\, e^{2\, A_{j-l}\, U_j^2}$$

Multiplying this bound by the prefactor $r_j^2 \leq 1/a_j$, the bound obtained is

$$\begin{aligned}
r_j^2\, \mathbb{E}\left(|\mathscr{R}^l_{\mu_j}|^2\right) &\lesssim \Delta_j^{-1}\, U_j^{-2}\, e^{-A_j\, U_j^2} \cdot \Delta_{j-l}^2\, e^{2\, A_{j-l}\, U_j^2} \\
&= \frac{\Delta_{j-l}^2}{\Delta_j}\, U_j^{-2}\, e^{(2A_{j-l} - A_j)\, U_j^2} \to 0 \quad \text{as } U_j \to \infty
\end{aligned}$$

Markov's inequality then gives, for every $\varepsilon > 0$,

$$\mathbb{P}\big(|r_j\, \mathscr{R}^l_{\mu_j}| > \varepsilon\big) \leq \frac{r_j^2\, \mathbb{E}(|\mathscr{R}^l_{\mu_j}|^2)}{\varepsilon^2} \longrightarrow 0,$$

so $r_j\, \mathscr{R}^l_{\mu_j} \xrightarrow{P} 0$. □

Lemma A.8 (Decay of the Σ, Γ and Λ terms.)**.** *Under the standing assumptions, it holds that*

$$r_j\,\Sigma, \quad r_j\,\Gamma, \quad r_j\,\Lambda \to 0$$

in probability as $U_j \to \infty$.

Proof. It will be shown that $r_j\,\Sigma \to 0$ in probability. The proofs for the other two terms are analogous. Recall that:

$$\Sigma = \frac{\tilde{\sigma}_j^2 - \sigma_j^2}{2} \int_{-U_j}^{U_j} (v-i)^2\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\,\mathrm{d}v$$

The aim is to show that

$$\mathbb{P}\left(|r_j\,\Sigma| > \varepsilon\right) \to 0$$

In order to do this, recall Chebyshev's Inequality:

$$\begin{aligned}\mathbb{P}\left(|r_j\,\Sigma| > \varepsilon\right) &\leq \frac{1}{\varepsilon}\mathbb{E}\left(|r_j\,\Sigma|^2\right)\\ &\lesssim r_j^2\,\mathbb{E}(\widetilde{\sigma}_j^2 - \sigma_j^2)^2 \left|\int_{-U_j}^{U_j} (v-i)^2\, w_{\mu_j}^{U_j}(v)\, e^{-ivx}\,\mathrm{d}v\right|^2\end{aligned}$$

Now, by Cauchy-Schwarz, we can further bound the expression as follows:

$$\begin{aligned}\mathbb{P}\left(|r_j\,\Sigma| > \varepsilon\right) &\leq r_j^2\,\mathbb{E}(\widetilde{\sigma}_j^2 - \sigma_j^2)^2 \int_{-U_j}^{U_j} |v-i|^4\,\mathrm{d}v\\ &\lesssim \Delta_j^{-1}\,U_j^{-2}\,e^{-A_j\,U_j^2}\,\Delta_j\,U_j^{-4}\,e^{A_j\,U_j^2}\,U_j^5\\ &= U_j^{-1} \to 0\end{aligned}$$

The proof for the other terms is analogous, and thus the conclusion reached is the same. ☐

Lemma A.9 (Decay of D_j and E_j)**.** *Assume the standing assumptions in section 3.4. Then*

$$|D_j| = o_{\mathbb{P}}\left(1/R_j\right) \quad \text{and} \quad |E_j| = o_{\mathbb{P}}\left(1/R_j\right) \quad \text{as } U_j \to \infty.$$

Proof. **Bound for D_j.** Recall that:

$$D_j := \sum_{r=1}^{j}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty} \Re\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\,\psi_r(u-U_j)\right) du$$

Note that, using the bounds for the characteristic function the following inequality can be established:

$$\begin{aligned}|D_j| &\leq \sum_{r=1}^{j}(T_r - T_{r-1})\frac{1}{\pi}\int_{-U_j+\varepsilon}^{\infty} \frac{|\varphi_j(u+U_j)|}{|u+U_j|}\,|\psi_r(u-U_j)|\,du\\ &\leq \sum_{r=1}^{j}(T_r - T_{r-1})\frac{2}{\pi}\int_{-U_j+\varepsilon}^{\infty} \frac{e^{-A_j\,(u+U_j)^2/2}}{|\varepsilon|}\left|\int_K (e^{i(u-U_j)x} - 1)(\widetilde{\nu}_r(x) - \nu_r(x))\mathrm{d}x\right| du\end{aligned}$$

Writing $\widetilde{\nu}_r - \nu_r = e^{-x}(\widetilde{\mu}_r - \mu_r)$,

$$\psi_r(u-U_j) = \int_K \left(e^{i(u-U_j)x} - 1\right) e^{-x}\left(\widetilde{\mu}_r(x) - \mu_r(x)\right)\mathrm{d}x = \left\langle \widetilde{\mu}_r - \mu_r,\ h_u\right\rangle_{L^2(K)},$$

a fixed-functional pairing of the error against the deterministic weight $h_u(x) := e^{-x}\left(e^{-i(u-U_j)x} - 1\right)$ (the conjugate placing the pairing in $L^2(K)$). Since the noise variables are centred, so is $\psi_r(u-U_j)$, and its second moment equals its variance.

As shown in section 8.4, $r_j \asymp R_j$. Moreover, by Lemma 8.1 only the current maturity contribution $r = j$ survives, as terms involving $r < j$ are $o_{\mathbb{P}}(1)$. Hence, it suffices to bound $r_j|D_j|$. Bounding the deterministic factor and applying the Cauchy–Schwarz inequality to the outer u-integral,

$$|D_j|^2 \lesssim \left(\frac{1}{|\varepsilon|}\int_{-U_j+\varepsilon}^{\infty} e^{-A_j(u+U_j)^2/2}\,\mathrm{d}u\right) \cdot \frac{1}{|\varepsilon|}\int_{-U_j+\varepsilon}^{\infty} e^{-A_j(u+U_j)^2/2}\,\big|\psi_j(u-U_j)\big|^2\,\mathrm{d}u.$$

The first factor is the deterministic Gaussian integral evaluated above; by the substitution $v = \sqrt{A_j}(u + U_j)$ and the bound $\operatorname{erfc}(z) \le 1$,

$$\frac{1}{|\varepsilon|}\int_{-U_j+\varepsilon}^{\infty} e^{-A_j(u+U_j)^2/2}\,\mathrm{d}u \;\le\; \frac{1}{|\varepsilon|}\,\frac{1}{\sqrt{A_j}}\sqrt{\frac{\pi}{2}} \;=:\; G_j.$$

Taking expectations and using Tonelli to interchange expectation and the outer integral,

$$r_j^2\,\mathbb{E}\big[|D_j|^2\big] \;\lesssim\; G_j\,\frac{1}{|\varepsilon|}\int_{-U_j+\varepsilon}^{\infty} e^{-A_j(u+U_j)^2/2}\left(r_j^2\,\mathbb{E}\big[|\psi_j(u-U_j)|^2\big]\right)\mathrm{d}u.$$

For each fixed u, the variance of the rescaled pairing is a bilinear form in the rescaled covariance kernel, into which the normalization r_j is already absorbed:

$$r_j^2\,\mathbb{E}\big[|\psi_j(u-U_j)|^2\big] = r_j^2\,\mathbb{E}\Big[\big|\textstyle\sum_{l=0}^{1}\langle \mathscr{L}_{\mu_j}^l,\ h_u\rangle_{L^2(K)}\big|^2\Big] = \textstyle\sum_{l=0}^{1}\langle \mathcal{K}_j^l\,h_u,\ h_u\rangle_{L^2(K)},$$

where we have used that the sub-Gaussian noise variables are centred and mutually independent across the maturity index l, so the cross terms $l \neq l'$ vanish in expectation, and where

$$\mathcal{K}_j^l(x,z) := r_j^2\ \operatorname{Cov}\!\left(\sum_{k=1}^{m_{j-l}} X_{U_j,k}^l(x)',\ \sum_{k=1}^{m_{j-l}} X_{U_j,k}^l(z)'\right).$$

By the covariance analysis of Chapter 4, the rescaled kernel converges to $2\mathcal{C}\cos(U_j(x-z))$ and is uniformly bounded on the compact $K \times K$ by Lemma A.4, $|\mathcal{K}_j^l(x,z)| \le C_0$. Writing the covariance bilinear form with the conjugate in the second slot and expanding $\cos(U_j(x-z)) = \frac{1}{2}\big(e^{iU_j(x-z)} + e^{-iU_j(x-z)}\big)$, the two halves separate into a product of single integrals,

$$\big\langle \mathcal{K}_j^l\,h_u, h_u\big\rangle_{L^2(K)} \;\longrightarrow\; \mathcal{C}\left(\big|\,\widehat{h_u\mathbf{1}_K}(U_j)\,\big|^2 + \big|\,\widehat{h_u\mathbf{1}_K}(-U_j)\,\big|^2\right), \qquad \widehat{h_u\mathbf{1}_K}(\xi) := \int_K h_u(x)\,e^{i\xi x}\,\mathrm{d}x.$$

The weight $h_u(x) = e^{-x}\big(e^{-i(u-U_j)x} - 1\big)$ itself depends on U_j, so we expand each transform into transforms of the fixed function $e^{-x}\mathbf{1}_K$. Distributing the bracket,

$$\begin{aligned}\widehat{h_u\mathbf{1}_K}(U_j) &= \widehat{e^{-x}\mathbf{1}_K}(2U_j-u) \;-\; \widehat{e^{-x}\mathbf{1}_K}(U_j),\\ \widehat{h_u\mathbf{1}_K}(-U_j) &= \widehat{e^{-x}\mathbf{1}_K}(-u) \;-\; \widehat{e^{-x}\mathbf{1}_K}(-U_j).\end{aligned}$$

For the $+U_j$ term, both frequencies $2U_j - u$ and U_j tend to infinity as $U_j \to \infty$ (for fixed u), so by the Riemann-Lebesgue lemma applied to the fixed integrable function $e^{-x}\mathbf{1}_K$, $\widehat{h_u\mathbf{1}_K}(U_j) \to 0$. For the $-U_j$ term, however, the first piece $\widehat{e^{-x}\mathbf{1}_K}(-u)$ sits at the fixed frequency $-u$ and does not vanish pointwise, only the second piece $\widehat{e^{-x}\mathbf{1}_K}(-U_j) \to 0$. Hence

$$\big\langle \mathcal{K}_j^l\,h_u, h_u\big\rangle_{L^2(K)} \;\longrightarrow\; \mathcal{C}\,\big|\,\widehat{e^{-x}\mathbf{1}_K}(-u)\,\big|^2 =: \Phi(u),$$

a bounded, U_j-independent function of u which does not vanish pointwise. The pointwise limit is therefore not zero and the required decay instead arises after integration against the Gaussian weight, which localizes u near $-U_j$.

Indeed, the uniform kernel bound gives $r_j^2\,\mathbb{E}[|\psi_j(u-U_j)|^2] \le C_0\|h_u\|^2_{L^2(K)} \le C_1$, a constant independent of u, so by dominated convergence the surviving contribution to $r_j^2\,\mathbb{E}[|D_j|^2]$ is controlled by

$$G_j\,\frac{1}{|\varepsilon|}\int_{-U_j+\varepsilon}^{\infty} e^{-A_j(u+U_j)^2/2}\,\Phi(u)\,\mathrm{d}u \;+\; o(1).$$

Substituting $v = u+U_j$ (so $u = v-U_j$ and the domain becomes $[\varepsilon,\infty)$), the Gaussian becomes stationary while the argument of Φ shifts:

$$\frac{1}{|\varepsilon|}\int_{\varepsilon}^{\infty} e^{-A_jv^2/2}\,\big|\,\widehat{e^{-x}\mathbf{1}_K}(U_j-v)\,\big|^2\,\mathrm{d}v,$$

using $\Phi(u) = |\widehat{e^{-x}\mathbf{1}_K}(-u)|^2$ with $-u = U_j - v$. For each fixed v, the frequency $U_j - v \to \infty$ as $U_j \to \infty$, so $\widehat{e^{-x}\mathbf{1}_K}(U_j - v) \to 0$ by the Riemann-Lebesgue lemma. Thus, the integrand is dominated by the fixed, integrable function $\|e^{-x}\mathbf{1}_K\|^2_{L^1(K)}\,e^{-A_jv^2/2}$. Dominated convergence then gives

$$\frac{1}{|\varepsilon|}\int_{\varepsilon}^{\infty} e^{-A_jv^2/2}\,\big|\,\widehat{e^{-x}\mathbf{1}_K}(U_j-v)\,\big|^2\,\mathrm{d}v \;\xrightarrow[U_j\to\infty]{}\; 0.$$

Combining with the vanishing of the $+U_j$ term, we conclude

$$r_j^2\,\mathbb{E}\big[|D_j|^2\big] \;\longrightarrow\; 0.$$

By Markov's inequality, for every $\eta > 0$,

$$\mathbb{P}\big(r_j|D_j| > \eta\big) \le \frac{1}{\eta^2}\,r_j^2\,\mathbb{E}\big[|D_j|^2\big] \;\longrightarrow\; 0,$$

hence $r_j|D_j| \xrightarrow{\mathbb{P}} 0$, i.e.

$$|D_j| \;=\; o_{\mathbb{P}}\left(1/R_j\right) \qquad \text{as } U_j \to \infty.$$

Bound for E_j.

Note that the error error can be written in terms of μ_r via $\widetilde{\nu}_r - \nu_r = e^{-x}(\widetilde{\mu}_r - \mu_r)$,

$$\int_K \big(\widetilde{\nu}_r(x) - \nu_r(x)\big)\,\mathrm{d}x = \int_K e^{-x}\big(\widetilde{\mu}_r(x) - \mu_r(x)\big)\,\mathrm{d}x = \big\langle \widetilde{\mu}_r - \mu_r,\, e^{-x}\big\rangle_{L^2(K)},$$

a fixed-functional pairing against the real, u-independent weight e^{-x}. Since this factor does not depend on u, it factors out of the outer integral, leaving

$$E_j = \sum_{r=1}^{j}(T_r - T_{r-1})\,\big\langle \widetilde{\mu}_r - \mu_r,\, e^{-x}\big\rangle_{L^2(K)}\;\cdot\;\frac{2}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\!\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\right)\mathrm{d}u.$$

The deterministic outer integral is bounded exactly as for D_j: with $|u+U_j| \ge \varepsilon$ and the Gaussian bound on φ_{T_j}, the substitution $v = u + U_j$ and $\mathrm{erfc}(z) \le 1$ give

$$\left|\frac{2}{\pi}\int_{-U_j+\varepsilon}^{\infty}\Re\!\left(\frac{e^{-i(u+U_j)k'}}{i(u+U_j)}\,\varphi_{T_j}(u+U_j)\right)\mathrm{d}u\right| \;\le\; \frac{2}{\pi}\,\frac{1}{\varepsilon}\,\frac{1}{\sqrt{A_j}}\sqrt{\frac{\pi}{2}} \;=:\; G_j.$$

As shown in section 8.4, $r_j \asymp R_j$. Moreover, by Lemma 8.1 only the current maturity contribution $r = j$ survives, as terms involving $r < j$ are $o_{\mathbb{P}}(1)$. Hence, it suffices to bound $r_j|E_j|$. Furthermore, since the deterministic factor G_j separates we obtain

$$r_j^2\,\mathbb{E}\big[|E_j|^2\big] \;\le\; G_j^2\,\cdot\,r_j^2\,\mathbb{E}\Big[\,\big|\big\langle \widetilde{\mu}_j - \mu_j,\, e^{-x}\big\rangle_{L^2(K)}\big|^2\,\Big].$$

The pairing $\langle \widetilde{\mu}_j - \mu_j,\, e^{-x} \rangle$ is of exactly the form treated for L_j, against the fixed real weight e^{-x}, so its rescaled variance is the bilinear form

$$r_j^2\, \mathbb{E}\Big[\, \big|\big\langle \widetilde{\mu}_j - \mu_j,\, e^{-x} \big\rangle\big|^2 \,\Big] = \sum_{l=0}^{1} \big\langle \mathcal{K}_j^l\, e^{-x},\, e^{-x} \big\rangle_{L^2(K)},$$

where we have used that the sub-Gaussian noise variables are centered and independent across the maturity index l. By Lemma A.4 each $\mathcal{K}_j^l$ converges to $2\mathcal{C}\cos(U_j(x-z))$ and is bounded by C_0; since e^{-x} is real, the bilinear form factorizes as a single squared modulus,

$$\big\langle \mathcal{K}_j^l\, e^{-x}, e^{-x} \big\rangle_{L^2(K)} \;\longrightarrow\; 2\mathcal{C} \int_K\!\int_K e^{-x} e^{-z} \cos\!\big(U_j(x-z)\big)\, \mathrm{d}x\, \mathrm{d}z = 2\mathcal{C}\, \big|\, \widehat{e^{-x}\mathbf{1}_K}(U_j)\, \big|^2,$$

with $\widehat{e^{-x}\mathbf{1}_K}(\xi) = \int_K e^{-x} e^{i\xi x}\, \mathrm{d}x$. The weight $e^{-x}\mathbf{1}_K \in L^1(K)$ is fixed and integrable, so by the Riemann-Lebesgue lemma $\widehat{e^{-x}\mathbf{1}_K}(U_j) \to 0$ as $U_j \to \infty$, whence

$$r_j^2\, \mathbb{E}\Big[\, \big|\big\langle \widetilde{\mu}_j - \mu_j,\, e^{-x} \big\rangle\big|^2 \,\Big] \;\longrightarrow\; 0.$$

Combining with the bounded factor G_j, $r_j^2\, \mathbb{E}[|E_j|^2] \to 0$, and by Markov's inequality $r_j |E_j| \xrightarrow{\mathbb{P}} 0$, i.e.

$$|E_j| = o_{\mathbb{P}}\left(1/R_j\right) \qquad \text{as } U_j \to \infty.$$

□