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Document Version
Final published version

Citation (APA)
Roodenburg, F. B. (2026). *Analysis in weighted spaces: Functional calculus and parabolic boundary value problems*. [Dissertation (TU Delft), Delft University of Technology]. https://doi.org/10.4233/uuid:42bf86d7-e9ca-41bd-8eb5-f124768f2640

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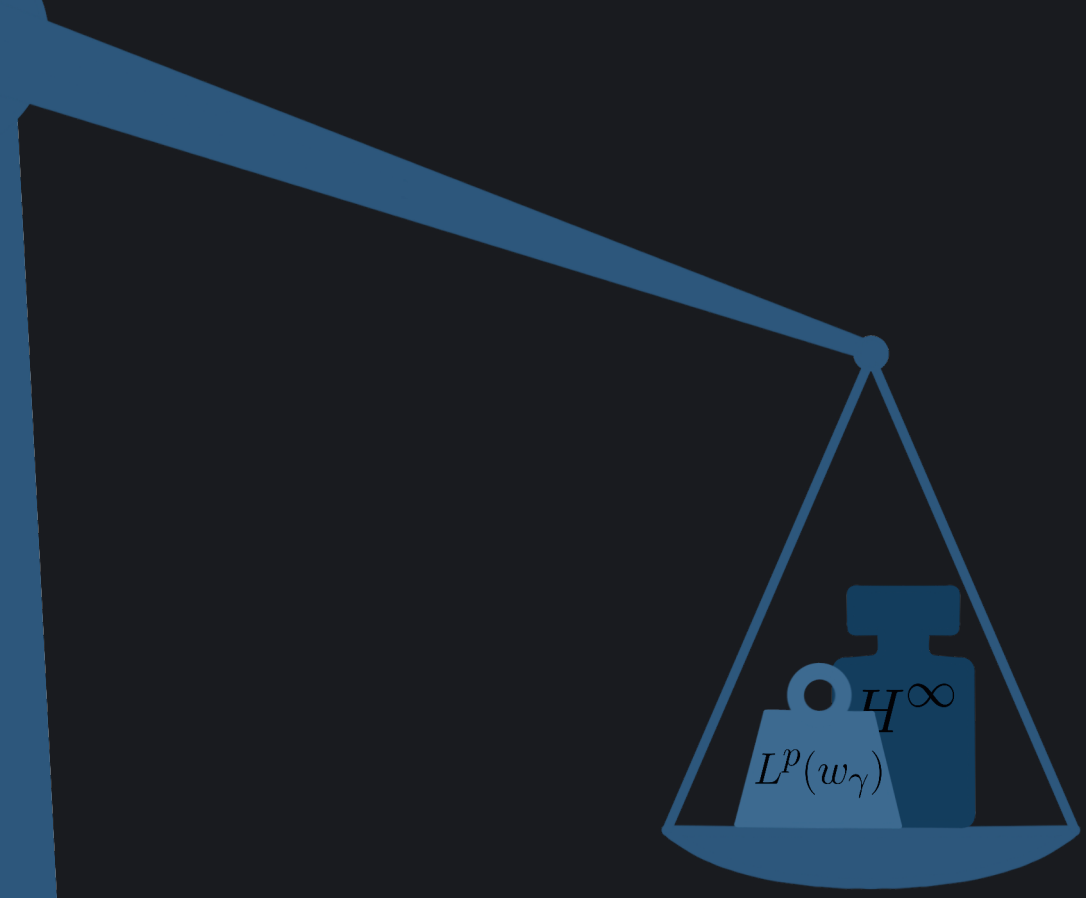
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Analysis in weighted spaces

Functional calculus and parabolic boundary value problems



Floris Roodenburg

ANALYSIS IN WEIGHTED SPACES

FUNCTIONAL CALCULUS AND PARABOLIC BOUNDARY VALUE PROBLEMS

ANALYSIS IN WEIGHTED SPACES

FUNCTIONAL CALCULUS AND PARABOLIC BOUNDARY VALUE PROBLEMS

Dissertation

for the purpose of obtaining the degree of doctor
at Delft University of Technology
by the authority of the Rector Magnificus, Prof. dr. ir. H. Bijl,
chair of the Board for Doctorates
to be defended publicly on Wednesday 7 October 2026 at 12:30

by

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The research in this dissertation is supported by the Vici grant VI.C.212.027 of the Dutch Research Council (NWO).





Keywords: damped plate equation, functional calculus, heat equation, maximal regularity, weighted function spaces

Printed by: Ipskamp Printing

Cover by: Anouk Wisse

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ISBN 978-94-6384-996-8

An electronic copy of this dissertation is available at
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1

Introduction

The primary focus of this dissertation is the development of well-posedness and maximal regularity theory for parabolic boundary value problems in the setting of weighted function spaces. The theory of maximal regularity has a long history and is well established within the setting of Sobolev spaces with respect to the Lebesgue measure. By replacing the Lebesgue measure with a weighted measure, we obtain a more flexible well-posedness and regularity theory, which avoids certain restrictive conditions required in the unweighted setting. In particular, we consider spatial power weights that allow the solution of the boundary value problem to blow up close to the boundary of the domain. In this dissertation, we develop the necessary theory to deal with boundary value problems in a weighted setting. This includes the theoretical study of weighted function spaces and their properties, the abstract study of model differential operators such as the Laplace operator, and applications to certain prototype boundary value problems.

In this chapter, we provide a self-contained introduction to the results obtained in this dissertation and an overview of related results in the existing literature. The subsequent chapters each have their own, more detailed introduction and can be read independently.

1.1. A history of maximal regularity theory

The origins of *maximal regularity* trace back to the 1960s. Since then, and particularly with the development of modern operator-theoretic methods over the last decades, its applicability to nonlinear deterministic partial differential equations (PDEs) has vastly expanded. Maximal regularity provides sharp a priori estimates for the linearised problem, which serve as an important tool for treating the original nonlinear problem afterwards. By standard tools such as the contraction principle and the implicit function theorem, one can derive well-posedness of nonlinear problems arising in, e.g., physics. Applications of maximal regularity can be found in a large variety of contexts, for example, fluid dynamics, reaction-diffusion processes, free boundary problems, and population dynamics. For a selection of applications, we refer to the monographs [14, 69, 191, 198, 230, 234, 278].

The theory of *stochastic maximal regularity* has undergone a similar evolution. While early results for stochastic equations were primarily restricted to Hilbert space settings,

over the past decades a comprehensive maximal regularity framework for stochastic partial differential equations (SPDEs) in Banach spaces has been systematically developed. In many physical phenomena noise is present, such as thermal fluctuations or turbulence. Incorporating this noise into a mathematical model means that one has to solve an SPDE. Just as in the deterministic case, stochastic maximal regularity provides sharp a priori estimates for linearised stochastic equations, which are crucial for treating nonlinear stochastic problems. For an overview of stochastic maximal regularity and its applications, we refer to, e.g., [11, 13, 58, 162, 219].

We discuss the notion of maximal regularity for both deterministic and stochastic partial differential equations below.

1.1.1. Maximal regularity for deterministic equations

Consider the linear deterministic PDE with inhomogeneous Dirichlet boundary conditions given by

$$\begin{cases} \partial_t u(t,x) + Au(t,x) = f(t,x) & (t,x) \in (0,T) \times \mathscr{O}, \\ u(t,x) = g(t,x) & (t,x) \in (0,T) \times \partial\mathscr{O}, \\ u(0,x) = u_{\mathrm{in}}(x) & x \in \mathscr{O}. \end{cases} \tag{1.1.1}$$

Here, $\mathscr{O} \subseteq \mathbb{R}^d$ is a non-empty domain and $T \in (0,\infty)$ is a time horizon. Moreover, A is a linear second-order partial differential operator (such as the Laplacian $A = -\Delta$), f is a forcing term, g is inhomogeneous boundary data, and u_{in} is the initial value. Given the inhomogeneous data f, g, and u_{in}, the question is whether we can find a unique solution u to (1.1.1). If indeed such a solution exists, we would like to know what kind of regularity properties it has. Moreover, we would like to have sharp a priori estimates on the solution, showing that the solution depends continuously on the data.

To establish well-posedness of (1.1.1), we first consider the problem with $g = 0$ and $u_{\mathrm{in}} = 0$. In this case, (1.1.1) can be considered as an *abstract Cauchy problem* given by

$$\begin{cases} \partial_t u(t) + Au(t) = f(t) & t \in (0,T), \\ u(0) = 0, \end{cases} \tag{1.1.2}$$

where now u and f are interpreted as functions of time that take values in a Banach space X_0 containing functions defined on $\mathscr{O}$. Now, $A : X_0 \supseteq D(A) \to X_0$ is an (unbounded) linear operator that models the spatial differential operator. Typically, X_0 and $D(A)$ are Lebesgue and Sobolev spaces on $\mathscr{O}$, respectively. Throughout this dissertation, we will often encounter the following prototype example.

Example 1.1.1. Let $A = -\Delta_{\mathrm{Dir}}$ be the Laplace operator with Dirichlet boundary conditions on a bounded C^2-domain $\mathscr{O} \subseteq \mathbb{R}^d$. One can consider

$$X_0 = L^p(\mathscr{O}) \quad \text{and} \quad D(A) = W^{2,p}(\mathscr{O}) \cap W_0^{1,p}(\mathscr{O}), \qquad p \in (1,\infty).$$

The corresponding equation $\partial_t u - \Delta_{\mathrm{Dir}} u = f$ is called the *heat equation*.

We do not restrict ourselves to the setting of Hilbert spaces $X_0 = L^2(\mathcal{O})$. The case of Banach spaces is particularly relevant for nonlinear problems, where one wants to use large p and q to have better Sobolev embeddings. Moreover, the case $p \neq q$ is important to exploit the scaling invariance of the problem, see, e.g., [99, 151, 235].

Definition 1.1.2. An operator A has *maximal L^q-regularity on* $(0,T)$ for some $T \in (0,\infty)$ and $q \in (1,\infty)$ if for each $f \in L^q((0,T);X_0)$ there exists a unique strong solution u to (1.1.2), which satisfies

$$\|\partial_t u\|_{L^q((0,T);X_0)} + \|u\|_{L^q((0,T);D(A))} \lesssim \|f\|_{L^q((0,T);X_0)}.$$

That is, given the regularity of f, we also obtain the maximal regularity possible for $\partial_t u$ and Au. This definition of maximal L^q-regularity stems from [273] and many standard properties of maximal regularity can be found in [82]. Early developments of maximal regularity theory for parabolic equations can be found in [55, 110, 176, 253, 256], while more modern treatments can be found in [14, 67, 126, 174, 234].

We remark that maximal regularity need not necessarily be defined in the setting of Lebesgue and Sobolev spaces. Instead, one can also consider the space of continuous or Hölder continuous functions, see, e.g., [3, 18, 49, 161, 198, 278]. The advantage of maximal L^q-regularity is that it requires the least regularity of the data.

With the above viewpoint, we have reduced the question of well-posedness to an operator-theoretical question about the differential operator A. The fundamental question is now:

> Which operators A possess the maximal L^q-regularity property? That is, can we find sufficient and/or necessary conditions for maximal L^q-regularity?

Preferably, we want to have a sufficient condition that is easy to verify for a given differential operator A on a Banach space X_0. Formally, using the variation of constants formula, the mild solution to (1.1.2) is given by

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s, \qquad t \in (0,T). \tag{1.1.3}$$

Therefore, the study of the semigroup $(e^{-tA})_{t\geq 0}$ plays an important role for Cauchy problems. By means of the Laplace transform

$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda t} e^{-tA}\,\mathrm{d}t, \qquad \lambda > 0,$$

the growth of the semigroup $\|e^{-tA}\|_{\mathscr{L}(X_0)}$ and of the resolvent $\|(\lambda+A)^{-1}\|_{\mathscr{L}(X_0)}$ are related. In particular, $-A$ is the generator of a bounded analytic semigroup if and only if A is sectorial of angle $\omega(A) < \frac{\pi}{2}$. The notion of sectoriality relates to the solvability and regularity of the elliptic equation $\lambda v + Av = h$ with $h \in X_0$.

Concerning the parabolic Cauchy problem (1.1.2), it was proved by De Simon [253] that if $-A$ generates a strongly continuous and analytic semigroup on a Hilbert space X_0, then A has maximal L^q-regularity on $(0,T)$ for all $q \in (1,\infty)$.

Unfortunately, the result of De Simon does not generalise to arbitrary Banach spaces X_0. Counterexamples were provided by Coulhon & Lamberton [51] and Kalton & Lancien [135], while Dore & Venni [83] and Prüss & Sohr [236] showed that every operator A on a UMD Banach space with bounded imaginary powers of angle $\omega_{\mathrm{BIP}}(A) < \frac{\pi}{2}$ has maximal L^q-regularity for all $q \in (1,\infty)$. Most importantly, Weis showed in [272, 273] that a sectorial operator A on a UMD Banach space X_0 has maximal L^q-regularity if and only if A is R-sectorial. Here, the UMD (unconditional martingale differences) condition is a geometric condition on the Banach space, and R-sectorial refers to a stronger notion of sectoriality requiring that the family of resolvent operators is bounded in a random sequence space defined via Rademacher variables.

The characterisation of maximal L^q-regularity requires more advanced methods based on operator-valued singular integral operators. By (1.1.3), we have that $Au(t)$ can be written as a convolution integral with kernel $K(t,s) = Ae^{-(t-s)A}\mathbf{1}_{t>s}$. Since $\|Ae^{-tA}\|_{\mathscr{L}(X_0)} \lesssim t^{-1}$, we are dealing with a singular integral operator and we enter the realm of harmonic analysis. Such integral operators are called Calderón–Zygmund operators and are thoroughly studied in the literature, see, e.g., [240]. Vector-valued harmonic analysis usually requires conditions on the geometry of the Banach space, which explains the necessity of the UMD condition.

An alternative approach to maximal regularity is via the so-called operator-sum method initiated by Da Prato & Grisvard [55] and later extended by Dore & Venni [83] and Kalton & Weis [137]. A celebrated result within this approach is the Dore–Venni theorem stating that if two sectorial operators A and B on a UMD Banach space Y have bounded imaginary powers with $\omega_{\mathrm{BIP}}(A) + \omega_{\mathrm{BIP}}(B) < \pi$ and are resolvent commuting, then

$$\|Au\|_Y + \|Bu\|_Y \lesssim \|(A+B)u\|_Y, \qquad u \in D(A) \cap D(B).$$

This result can be interpreted as a reverse triangle inequality. In particular, if the conditions of the theorem can be verified for our differential operator A and $B = \partial_t$ on $Y = L^q((0,T);X_0)$, then we obtain the required maximal regularity estimate.

Due to the characterisation of maximal regularity by Weis, it now suffices to verify the R-sectoriality condition for the differential operator A in (1.1.2). This was proved by Denk, Hieber & Prüss in [67] for a large class of uniformly elliptic operators on $\mathscr{O} = \mathbb{R}^d$, $\mathscr{O} = \mathbb{R}^d_+$, or when $\mathscr{O}$ is a smooth bounded domain. More generally, R-sectoriality remains valid for higher-order operators A and Lopatinskii–Shapiro boundary conditions. This result can be used immediately as a black box to obtain maximal L^q-regularity for A and solvability of the Cauchy problem (1.1.2). As an illustration, we consider the setting from Example 1.1.1.

Example 1.1.3. Let $p, q \in (1,\infty)$ and $T \in (0,\infty)$. Let $A = -\Delta_{\mathrm{Dir}}$ be the Dirichlet Laplacian on $X_0 = L^p(\mathscr{O})$, where $\mathscr{O}$ is a bounded C^2-domain. It follows from [67] that A has maximal L^q-regularity. In particular, for every $f \in L^q((0,T);L^p(\mathscr{O}))$ there exists a unique solution u to $\partial_t u - \Delta_{\mathrm{Dir}} u = f$ with $u|_{\partial\mathscr{O}} = 0$ and $u|_{t=0} = 0$, which satisfies

$$\|\partial_t u\|_{L^q((0,T);L^p(\mathscr{O}))} + \|u\|_{L^q((0,T);W^{2,p}(\mathscr{O}))} \lesssim \|f\|_{L^q((0,T);L^p(\mathscr{O}))}.$$

In addition, we could study higher-order spatial regularity, i.e., $f \in L^q((0,T);W^{k,p}(\mathscr{O}))$ for some $k \in \mathbb{N}_0$. However, this requires additional conditions on the smoothness of $\mathscr{O}$ and

additional unnatural boundary conditions for f. Typically, one needs to impose that $\mathcal{O}$ is a C^{k+2}-domain to apply localisation arguments. The additional boundary conditions for f are called *compatibility conditions* and are known to be necessary to obtain a sectorial operator, see [66].

Of course, this is not the end of the journey into maximal regularity, since in applications one may encounter more sophisticated operators A. For example, the Cauchy problem can be non-autonomous, meaning that $A = A(t)$ also depends on time. We refer to [14, 198, 232, 260] for some approaches to these types of problems. In addition, there is an extensive literature regarding the weakest type of regularity conditions on the space and time coefficients required for maximal regularity, see, e.g., [78, 79, 80, 95, 96, 161].

Once maximal L^q-regularity for (1.1.2) is established, the original problem (1.1.1) with nonzero initial value and inhomogeneous boundary conditions can be treated using trace theory. Nonzero initial values can be obtained using the trace method for real interpolation, see, e.g., [199, 262]. In particular, the optimal space for the initial value u_{in} is given by the real interpolation space $(X_0, D(A))_{1-\frac{1}{q},q}$, which is the temporal trace space of the solution space $W^{1,q}((0,T);X_0) \cap L^q((0,T);D(A))$. On the other hand, inhomogeneous boundary conditions require the study of the spatial trace operator as initiated by Weidemaier in [269, 270, 271] and further investigated in [68, 77, 132].

1.1.2. Maximal regularity for stochastic equations

We will now consider the maximal regularity property for stochastic equations. To be precise, consider an Itô stochastic parabolic evolution equation of the form

$$\begin{cases} \mathrm{d}u + Au\,\mathrm{d}t = f\,\mathrm{d}t + g\,\mathrm{d}W(t) & t \in (0,T), \\ u(0) = 0. \end{cases} \tag{1.1.4}$$

Again, $A : X_0 \supseteq D(A) \to X_0$ is an (unbounded) linear operator and f and g are forcing terms. Moreover, the noise in (1.1.4) is modelled by a standard one-dimensional Brownian motion W on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$. Note that for $g = 0$, (1.1.4) reduces to the deterministic Cauchy problem (1.1.2).

There is a vast literature on stochastic parabolic equations such as (1.1.4). For instance, an approach via semigroups was considered by Da Prato & Zabczyk [58], while the variational setting was investigated by Pardoux [229], Krylov & Rozovskii [168] and Liu & Röckner [193]. The setting of L^p-spaces was first studied by Krylov [162]. More recently, Agresti & Veraar [11, 12, 13] developed the theory for nonlinear stochastic equations in critical spaces using operator-theoretic properties of A for obtaining stochastic maximal regularity for the linear equation (1.1.4), see [219]. This framework is based on stochastic integration theory for UMD Banach spaces as developed by Van Neerven, Veraar & Weis [220, 221, 266].

Similar to the deterministic setting, we want the stochastic equation (1.1.4) to have a unique solution for all data f and g that depends continuously on the data. However, due

to the regularity and scaling of the Brownian motion, we need to impose more regularity on the stochastic forcing g than on the deterministic forcing f. Below, we denote by $[X_0, X_1]_\theta$ with $\theta \in (0,1)$ the complex interpolation space of two Banach spaces X_0 and X_1.

Definition 1.1.4. An operator A has *stochastic maximal L^q-regularity on* $(0,T)$ for some $q \in [2,\infty)$ if for all progressively measurable

$$f \in L^q(\Omega \times (0,T); X_0) \quad \text{and} \quad g \in L^q\big(\Omega \times (0,T); [X_0, D(A)]_{\frac{1}{2}}\big),$$

there exists a unique solution u to (1.1.4), which satisfies

$$\|u\|_{L^q(\Omega\times(0,T);D(A))} \lesssim \|f\|_{L^q(\Omega\times(0,T);X_0)} + \|g\|_{L^q(\Omega\times(0,T);[X_0,D(A)]_{\frac{1}{2}})}.$$

For deterministic equations, an estimate for $\partial_t u$ is included in Definition 1.1.2 since this can be obtained directly from the equation $\partial_t u = -Au + f$. In the stochastic case, this is not possible, and time regularity needs to be studied separately.

Analogous to the deterministic case in (1.1.3), the mild solution to (1.1.4) is given by

$$u(t) = \int_0^t e^{-(t-s)A} f(s)\,\mathrm{d}s + \int_0^t e^{-(t-s)A} g(s)\,\mathrm{d}W(s), \qquad t \in (0,T).$$

The deterministic singular integral operator can be dealt with as before. Now, stochastic maximal L^q-regularity can be established by additionally considering the stochastic singular integral operator

$$\int_0^T K(t,s)g(s)\,\mathrm{d}W(s), \qquad \text{with } K(t,s) = A^{\frac{1}{2}} e^{-(t-s)A}\mathbf{1}_{t>s}.$$

The L^q-boundedness of this stochastic singular integral operator was studied in [219], while a more general stochastic version of Calderón–Zygmund theory with operator-valued kernels was developed in [196]. Again, this requires the geometric UMD condition on the Banach space X_0.

In the special case that X_0 is a Hilbert space, it is known that if $-A$ generates a strongly continuous and analytic semigroup on X_0, then A has stochastic maximal L^q-regularity on $(0,T)$ for all $q \in [2,\infty)$. For $q = 2$ this result is due to Da Prato [54] and the general case $q > 2$ follows by extrapolation from [196], as observed in [13]. Note that this result is a stochastic version of the result by De Simon for deterministic equations as discussed before.

Similar to the deterministic setting, we want to have a sufficient condition (or even a characterisation) on the differential operator A that guarantees stochastic maximal regularity. In [219] it was proved that if A has a *bounded H^∞-functional calculus on* X_0, where X_0 is isomorphic to a closed subspace of some L^p-space with $p \in [2,\infty)$, then A has stochastic maximal L^q-regularity on $(0,T)$ for all $q \in (2,\infty)$. Nowadays, this result plays an important role in the study of SPDEs since it can be used as a black box for treating

linear stochastic equations once the H^∞-calculus condition for A is verified. This makes the H^∞-calculus an extremely powerful tool, and therefore it will also play a central role in this dissertation.

The H^∞-calculus was originally developed in [52, 204] motivated by the Kato square root problem (see [22, 205]). Later, it turned out that the H^∞-calculus yields boundedness of many singular integral operators, which also arise in the theory of (S)PDEs. Hence, there is a vast literature on the H^∞-calculus for sectorial operators, see, e.g., the monographs and lecture notes [20, 67, 118, 125, 126, 136, 174, 216, 234]. We give a formal definition of the H^∞-calculus.

Definition 1.1.5. Let A be a sectorial operator on a Banach space X_0. Let φ be a bounded and holomorphic function on a sector Σ in the right complex half-plane containing the spectrum of A. If the Cauchy integral

$$\varphi(A) := \frac{1}{2\pi \mathrm{i}} \int_{\partial\Sigma} \varphi(z) R(z, A) \, \mathrm{d}z$$

defines a bounded operator that satisfies

$$\|\varphi(A)\|_{\mathscr{L}(X_0)} \lesssim \|\varphi\|_\infty$$

for all bounded and holomorphic functions φ on Σ, then A is said to have a *bounded H^∞-calculus*.

Note that a priori it is not clear at all that $\varphi(A)$ is well defined since $\|R(z, A)\|_{\mathscr{L}(X_0)} \lesssim |z|^{-1}$, and it is a strong requirement for an operator A to have a bounded H^∞-calculus. Nonetheless, there are many known examples, including positive self-adjoint operators and generators of contraction semigroups on Hilbert spaces X_0. We also have the following result for our prototype example $A = -\Delta_{\mathrm{Dir}}$ from Example 1.1.1.

Example 1.1.6. Let $p \in (1, \infty)$. Let $A = -\Delta_{\mathrm{Dir}}$ be the Dirichlet Laplacian on $X_0 = L^p(\mathscr{O})$, where $\mathscr{O}$ is a bounded C^2-domain. Then A has a bounded H^∞-calculus, see [67]. In particular, for $p, q \in (2, \infty)$ and $T \in (0, \infty)$, it follows that $-\Delta_{\mathrm{Dir}}$ has stochastic maximal L^q-regularity. By interpolation theory with boundary conditions from [251], we have that

$$[X_0, D(A)]_{\frac{1}{2}} = \big[L^p(\mathscr{O}), W^{2,p}(\mathscr{O}) \cap W_0^{1,p}(\mathscr{O})\big]_{\frac{1}{2}} = W_0^{1,p}(\mathscr{O}).$$

Hence, our stochastic maximal regularity result means that for every progressively measurable $f \in L^q(\Omega \times (0, T); L^p(\mathscr{O}))$ and $g \in L^q(\Omega \times (0, T); W_0^{1,p}(\mathscr{O}))$ there exists a unique solution u to the stochastic heat equation

$$\mathrm{d}u - \Delta u \, \mathrm{d}t = f \, \mathrm{d}t + g \, \mathrm{d}W(t), \qquad t \in (0, T),$$

with $u|_{\partial\mathscr{O}} = 0$ and $u|_{t=0} = 0$, which satisfies

$$\|u\|_{L^q(\Omega\times(0,T);W^{2,p}(\mathscr{O}))} \lesssim \|f\|_{L^q(\Omega\times(0,T);L^p(\mathscr{O}))} + \|g\|_{L^q(\Omega\times(0,T);W^{1,p}(\mathscr{O}))}.$$

It is important to note that the data g needs to have more regularity than f, and g needs to satisfy the undesirable compatibility condition $g|_{\partial\mathscr{O}} = 0$.

1.2. A history of weighted function spaces

With the theory of maximal regularity at hand, one obtains well-posedness of many linear (stochastic) partial differential equations. However, there are classical settings where the maximal regularity theory on $X_0 = L^p(\mathcal{O})$ is insufficient. As a motivating example, we consider the heat equation with singular data f below. Another motivating example for weighted spaces in the stochastic setting can be found in [160].

Example 1.2.1. Let $T \in (0,\infty)$ and $\mathcal{O} = (0,1)$. Consider the following one-dimensional heat equation with singular data

$$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) = f(t,x) := x^{-1} & (t,x) \in (0,T)\times\mathcal{O}, \\ u(t,0) = u(t,1) = 0 & t \in (0,T), \\ u(0,x) = 0 & x \in \mathcal{O}. \end{cases} \tag{1.2.1}$$

This problem is not covered by the maximal regularity theory in Example 1.1.3, since $f \notin L^q((0,T);L^p(\mathcal{O}))$ for any $p,q \in (1,\infty)$. Nonetheless, the above problem can be solved by considering the equations

$$\begin{cases} -\partial_x^2 v_1(x) = x^{-1} & x \in \mathcal{O}, \\ v_1(0) = v_1(1) = 0, \end{cases}$$

and

$$\begin{cases} \partial_t v_2(t,x) - \partial_x^2 v_2(t,x) = 0 & (t,x) \in (0,T)\times\mathcal{O}, \\ v_2(t,0) = v_2(t,1) = 0 & t \in (0,T), \\ v_2(0,x) = -v_1(x) & x \in \mathcal{O}, \end{cases}$$

and setting $u(t,x) = v_1(x) + v_2(t,x)$. It is straightforward to check that the stationary problem has $v_1(x) = -x\log(x)$ as solution, while the problem for v_2 can be solved by classical methods such as separation of variables.

To be able to treat problems with singular data via maximal regularity theory, we have to change the underlying Banach space $X_0 = L^p(\mathcal{O})$ in such a way that it captures the singular behaviour of f close to the boundary. Therefore, we define the spatial power weight

$$w_\gamma^{\partial\mathcal{O}}(x) := \operatorname{dist}(x,\partial\mathcal{O})^\gamma, \qquad x \in \mathcal{O}, \gamma \in \mathbb{R},$$

and we let $L^p(\mathcal{O}, w_\gamma^{\partial\mathcal{O}})$ be the *weighted Lebesgue space* with measure $w_\gamma^{\partial\mathcal{O}}(x)\,\mathrm{d}x$.

In the setting of Example 1.2.1, we obtain that $f(t,x) := x^{-1}$ belongs to

$$L^q\big((0,T);L^p(\mathcal{O}, w_\gamma^{\partial\mathcal{O}})\big) \qquad \text{for all } p,q \in (1,\infty)$$

if and only if $\gamma > p-1$.

The introduction of weighted spaces can be traced back to the 1950s and 1960s, motivated by the study of PDEs on non-smooth domains. It was observed that solutions to elliptic problems on non-smooth domains blow up near the boundary, which necessitates

weighted function spaces to compensate for the singular behaviour. Many authors contributed to the early developments of this theory, amongst others Kondratiev [152, 153], Solonnikov [257] and Nečas [218]. The study of weighted spaces diverged into several directions and became a standard tool within the fields of mathematical analysis and applications to PDEs.

We mention some relevant literature about weighted function spaces. First of all, there is a vast literature containing the systematic study of the properties of weighted function spaces such as weighted Besov, Triebel–Lizorkin, Bessel potential and Sobolev spaces. The function space theory is well developed for the standard (unweighted) spaces, see, e.g., [27, 192, 262, 263]. However, weighted spaces may exhibit significantly different behaviour. We list some important properties of weighted function spaces that we want to have access to:

- density of smooth functions,
- embedding theorems and compactness of embeddings,
- extensions of functions from a domain $\mathcal{O}$ to $\mathbb{R}^d$,
- traces of functions on the boundary $\partial\mathcal{O}$,
- interpolation of weighted spaces.

Many of these elementary properties for (scalar-valued) weighted spaces can be found in [169]. In addition, compactness of embeddings is studied in, e.g., [115, 116, 117], and trace and interpolation theory can be found in, e.g., [37, 108, 252, 262].

For applications of weighted spaces to boundary value problems, we refer to the monographs [32, 154]. In particular, the application of weighted spaces plays an important role in the study of (S)PDEs on domains with conical points, see, e.g., the monographs [155, 156, 202, 217] for applications to deterministic equations. In addition, the author contributed to the study of boundary value problems on wedges with Robin-type boundary conditions [34, 35]. For applications to SPDEs, we refer to, e.g., [141, 142, 143, 167] for smooth domains and [44, 45, 46] for domains with conical points.

Given a bounded C^2-domain $\mathcal{O} \subseteq \mathbb{R}^d$, it is well known that the Laplacian with Dirichlet boundary conditions on $L^p(\mathcal{O})$ with $p \in (1,\infty)$ and the canonical domain $W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O})$ has the maximal regularity property and admits a bounded H^∞-calculus, see Example 1.1.3. However, if the regularity of $\mathcal{O}$ is too low (say Lipschitz or C^1), these properties fail and explicit counterexamples can be constructed, see [60, 203]. In such counterexamples, the derivatives of the solutions to the resolvent equation can drastically blow up near the boundary $\partial\mathcal{O}$. As a consequence, the canonical domain of the Dirichlet Laplacian on $L^p(\mathcal{O})$ is no longer a closed subspace of $W^{2,p}(\mathcal{O})$. To restore the maximal regularity with a gain of two derivatives for the solution, one has to impose additional conditions such as convexity of the domain or restrict the scale of spaces, see, e.g., the works of Dahlberg [59, 60] and Jerison & Kenig [129, 131]. In particular, Wood [277] explicitly proved that maximal L^q-regularity for the Laplacian on Lipschitz domains without

weights is only possible if the integrability exponent q is severely restricted to a narrow interval around 2.

Since the 1970s, weights also play a role in the field of harmonic analysis, see, e.g., [105, 106]. Most importantly, Muckenhoupt characterised the weights w such that the Hardy–Littlewood maximal operator on $L^p(\mathbb{R}^d, w)$ is bounded [215]. These weights are known as the *Muckenhoupt A_p weights*. In particular, if $\mathscr{O} = \mathbb{R}^d_+$, then the power weight $w_\gamma := w_\gamma^{\partial\mathbb{R}^d_+}$ belongs to the class of Muckenhoupt A_p weights if and only if $\gamma \in (-1, p-1)$. However, in applications to PDEs one typically wants to take γ large to damp the blow-up of the solution near the boundary and the range $\gamma \in (-1, p-1)$ is not sufficient, cf. Example 1.2.1. One of the main challenges in this dissertation is to lift results with Muckenhoupt power weights $\gamma \in (-1, p-1)$, which are obtained by harmonic analysis techniques, to other weighted spaces with $\gamma \geq p-1$.

The maximal regularity framework was first extended to spaces incorporating temporal weights, see e.g., [151, 179, 209, 210, 233, 235]. By introducing a time weight of the form t^μ in the definition of maximal regularity, one can allow for rough initial values. Since the real interpolation space $(X_0, D(A))_{1-\frac{\mu+1}{q},q}$ is the optimal space for the initial value, it suffices to consider temporal weights t^μ with $\mu \in [0, q-1)$. On the other hand, for spatial weights it makes sense to go beyond the case $\gamma \in (-1, p-1)$. For example, the spatial trace operator exists on $W^{2,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ with $\gamma \in (p-1, 2p-1)$ and thus it would still be interesting to establish the maximal regularity for the operator A in (1.1.1) with $X_0 = L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$ and $\gamma \in (p-1, 2p-1)$. We refer to the works of Lindemulder, Meyries & Veraar [184, 189, 190, 211, 212, 213, 214] for a systematic study of properties of weighted vector-valued spaces and applications to boundary value problems on weighted spaces.

1.3. Maximal regularity theory on weighted spaces

In the previous sections we have motivated the applications of functional and harmonic analytical tools for the study of (S)PDEs, and the application of weighted function spaces to deal with singular behaviour of solutions to (S)PDEs on domains. The main goal of this dissertation is to unite these two fields and develop a powerful toolbox for studying (S)PDEs on weighted spaces. In order to do so, we need a range of different techniques and results, including

- *Function space theory:* trace and interpolation theory for weighted Sobolev spaces,
- *Functional calculus:* we establish the H^∞-calculus for some prototype differential operators on weighted spaces,
- *Applications to (S)PDEs:* the functional calculus immediately yields maximal regularity for linear (S)PDEs on weighted spaces, allowing for a more flexible theory than in the unweighted case.

To outline the results in this dissertation, we introduce the required weighted Lebesgue and Sobolev spaces. Let $\mathscr{O}$ be the half-space $\mathbb{R}^d_+$ or a bounded domain. Recall that $w_\gamma^{\partial\mathscr{O}} =$

$\operatorname{dist}(\cdot,\partial\mathscr{O})^\gamma$ for $\gamma > -1$ denotes the spatial power weight measuring the distance to the boundary $\partial\mathscr{O}$. In particular, if $\mathscr{O} = \mathbb{R}^d_+$, we write $w_\gamma(x) := w_\gamma^{\mathbb{R}^d_+}(x) = |x_1|^\gamma$ for $x = (x_1, \widetilde{x}) \in \mathbb{R}^d_+$. Let $p \in (1,\infty)$ and let X be a Banach space. We define the weighted Lebesgue space $L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ as the space of all $f\colon \mathscr{O} \to X$ such that

$$\|f\|_{L^p(\mathscr{O},w_\gamma^{\partial\mathscr{O}};X)} := \Big(\int_{\mathscr{O}} \|f(x)\|_X^p \, w_\gamma^{\partial\mathscr{O}} \,\mathrm{d}x\Big)^{\frac{1}{p}} < \infty.$$

The k-th order weighted Sobolev space for $k \in \mathbb{N}_0$ is defined as

$$W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \big\{f \in \mathscr{D}'(\mathscr{O}; X) : \forall |\alpha| \le k, \partial^\alpha f \in L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)\big\}$$

equipped with the canonical norm. For $\gamma = 0$ and $X = \mathbb{C}$, this coincides with the standard Sobolev spaces $W^{k,p}(\mathscr{O})$.

We also refer to the spaces $W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ as *inhomogeneous weighted Sobolev spaces*, since every derivative belongs to a Lebesgue space with the same weight. On the other hand, *homogeneous weighted Sobolev spaces* (also called *Kondratiev spaces*) are often used in the literature. In this case, one imposes the condition $\partial^\alpha f \in L^p(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+|\alpha|p}; X)$. In this dissertation, we mainly focus on inhomogeneous spaces, which are natural from a functional analytic point of view since $W^{k_0,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \hookrightarrow W^{k_1,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ if $k_0 \ge k_1$.

One of the main results of this dissertation establishes the H^∞-calculus for the Laplace operator on weighted Sobolev spaces, see Chapter 6.

Theorem 1.3.1 (H^∞-calculus for the Dirichlet Laplacian)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Furthermore, suppose that*

$$\kappa > 1 - \tfrac{\gamma+1}{p} \qquad \textit{or, equivalently} \qquad \gamma > (1-\kappa)p - 1$$

and $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain. Then for all $\lambda \ge 0$ the operator

$$\lambda - \Delta_{\mathrm{Dir}} \quad \textit{on} \quad W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \quad \textit{with} \quad D(\Delta_{\mathrm{Dir}}) = \big\{u \in W^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) : \operatorname{Tr} u = 0\big\}$$

has a bounded H^∞-calculus of angle zero.

Moreover, the result also holds for $\lambda - \Delta_{\mathrm{Dir}}$ on the vector-valued space $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ if X is a UMD Banach space and $\lambda \ge 0$ is large enough. Theorem 1.3.1 generalises the result of Lindemulder & Veraar, see [190, Theorem 6.1], where the case $k = 0$ and $\mathscr{O}$ a bounded C^2-domain is treated. Theorem 1.3.1 allows for bounded C^1-domains if $\gamma \in (p-1, 2p-1)$, while for $\gamma \in (-1, p-1)$ the smoothness of the domain may depend on the weight: if the power of the weight is larger, then a rougher domain is allowed. The smoothness parameter κ is almost optimal, as is proved in [203]. We summarise the main novelties of Theorem 1.3.1.

(i) *Functional calculus.* We prove the boundedness of the H^∞-calculus, which is, in general, much harder to prove than maximal regularity and yields the boundedness of many singular integral operators. In particular, boundedness of the H^∞-calculus implies (stochastic) maximal regularity. Consequently, we can recover some maximal regularity results for the Dirichlet Laplacian from [149] (for bounded C^1-domains) and [157] (for bounded $C^{1,\kappa}$-domains and $k = 0$).

(ii) *Rough domains.* The smoothness of the domain $\mathscr{O}$ in Theorem 1.3.1 is *independent* of the smoothness k of the Sobolev space. The reason for this is that we do not use the standard localisation procedure from the half-space to domains (see, e.g., [67, 89, 166]). This standard localisation procedure typically works for C^{k+2}-domains. Instead, we apply a more sophisticated C^1-diffeomorphism suitable for the weighted setting.

(iii) *Domain characterisation.* As we have discussed in Section 1.2, if the regularity of the domain $\mathscr{O}$ is too low, then the canonical domain of the Dirichlet Laplacian on $L^p(\mathscr{O})$ is no longer a closed subspace of $W^{2,p}(\mathscr{O})$. In Theorem 1.3.1 we have an explicit characterisation of the domain of the Dirichlet Laplacian as a closed subspace of a weighted Sobolev space of order $k+2$, while only imposing minimal smoothness for the domain $\mathscr{O}$.

(iv) *Higher-order regularity without compatibility conditions.* Higher-order regularity in the unweighted setting requires additional boundary conditions for the data. Such compatibility conditions are known to be necessary, see [66]. In Theorem 1.3.1 we do not need such compatibility conditions, since due to the weight no traces exist in $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$. We will exploit this fact later on to set up a well-posedness and regularity theory for mixed-order systems without compatibility conditions.

(v) *Neumann boundary conditions.* We also prove a version of Theorem 1.3.1 for the Neumann Laplacian, see Chapters 5 and 6. The Neumann Laplacian on the half-space is studied on weighted Sobolev spaces in [80, 81] (for $k=0$), but a systematic study on bounded domains seems to be unavailable until now.

The proof of Theorem 1.3.1 consists of the following three steps. We first consider the case $\mathscr{O}=\mathbb{R}^d_+$ and $k=0$, for which the H^∞-calculus was proved by Lindemulder & Veraar in [190]. First of all, the case $\gamma\in(-1,p-1)$ can be obtained using an odd reflection and the result on the full space with weights measuring the distance to $\partial\mathbb{R}^d_+$. As we have discussed before, for this range of γ the Hardy–Littlewood maximal operator is bounded and one can employ Mihlin's multiplier theorem to obtain the H^∞-calculus via Fourier analytic methods.

The extension of the range of γ follows from a careful study of the Dirichlet heat semigroup on $\mathbb{R}^d_+$, i.e.,

$$e^{-t\Delta_{\mathrm{Dir}}}f(x)=\int_{\mathbb{R}^d_+}\big[G_t(x_1-y_1,\widetilde{x}-\widetilde{y})-G_t(x_1+y_1,\widetilde{x}-\widetilde{y})\big]f(y)\,\mathrm{d}y,$$

where

$$G_t(x):=\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}},\qquad t>0,\ x\in\mathbb{R}^d,$$

is the standard heat kernel on $\mathbb{R}^d$. Due to the cancellation in the kernel of the semigroup on $\mathbb{R}^d_+$, the range of γ can be extended to $\gamma\in(-1,2p-1)$ by a perturbation argument. This is possible since the kernel on $\mathbb{R}^d_+$ satisfies stronger estimates than the kernel G_t on $\mathbb{R}^d$. In addition, we note that for Neumann boundary conditions, the kernel on $\mathbb{R}^d_+$ is given by $G_t(x_1-y_1,\widetilde{x}-\widetilde{y})+G_t(x_1+y_1,\widetilde{x}-\widetilde{y})$ using an even reflection. Since there is no cancellation in this kernel, the range $\gamma\in(p-1,2p-1)$ is not available.

In the second step, we use an induction argument on k to lift the result on $\mathbb{R}^d_+$ to higher-order weighted spaces. This lifting argument is based on Hardy's inequality

$$W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \hookrightarrow L^p(\mathbb{R}^d_+, w_\gamma; X), \qquad p \in (1,\infty),\ k \in \mathbb{N}_1,\ \gamma > -1,$$

and properties of the pointwise multiplication operator $Mu(x) := x_1 u(x)$. In particular, it holds that

$$\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \eqsim \sum_{|\alpha|\le 1} \|M\partial^\alpha f\|_{W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma+(k-1)p}; X)}, \qquad f \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X),$$

for $p \in (1,\infty)$, $k \in \mathbb{N}_1$, and $\gamma > -1$. We illustrate how this can be used to lift the H^∞-calculus for $k = 0$ to $k = 1$. If φ is a bounded and holomorphic function on a sector in the complex plane (see Definition 1.1.5) and $u \in W^{1,p}(\mathbb{R}^d_+, w_{\gamma+p}; X)$, then

$$\begin{aligned}
\|\varphi(A)u\|_{W^{1,p}(\mathbb{R}^d_+, w_{\gamma+p}; X)} &\eqsim \sum_{|\alpha|\le 1} \|M\partial^\alpha \varphi(A)u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \\
&\lesssim \sum_{|\alpha|\le 1} \|\varphi(A)M\partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} + \text{commutator term} \\
&\lesssim \sum_{|\alpha|\le 1} \|\varphi\|_\infty \|M\partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} + \text{commutator term} \\
&\lesssim \|\varphi\|_\infty \|u\|_{W^{1,p}(\mathbb{R}^d_+, w_{\gamma+p}; X)} + \text{commutator term}.
\end{aligned}$$

The core of the analysis is to calculate and estimate the commutator term. The case of arbitrary $k \ge 1$ goes similarly by induction.

Finally, to transfer the result of Theorem 1.3.1 from $\mathbb{R}^d_+$ to bounded domains, we employ the standard method of localisation and perturbation. The key ingredient is the perturbation of the H^∞-calculus on the half-space to *special domains*, i.e., domains above the graph of a function with compact support. A common method is to relate the Laplacian on the half-space and on a special domain via a diffeomorphism. However, due to the low regularity of the domain, we cannot use the standard diffeomorphism as in, e.g., [67, 89, 166, 275]. Instead, we construct a variant of the *Dahlberg–Kenig–Stein pullback*, based on ideas in [140, 149, 275]. This diffeomorphism straightens the boundary, preserves the distance to the boundary and leaves the direction of the normal vector at the boundary invariant. Moreover, higher-order derivatives exist, but blow up near the boundary of the domain. This blow-up is compensated by the weights in our spaces.

With estimates on this diffeomorphism at hand, we can employ perturbation theorems for the H^∞-calculus to extend the results to special domains. Another difficulty arising in this perturbation argument is that, if the regularity of the domain is too low, then the perturbations are of the same order as the Laplacian. It is known that the H^∞-calculus is not stable under small perturbations [207]. Additionally, we need the perturbations to be well behaved with respect to a fractional power of the original operator. Finally, by another localisation argument, based on lower-order perturbations, the H^∞-calculus on special domains is transferred to bounded domains.

To facilitate the perturbation argument in the above proof, it is important to have access to complex interpolation results of weighted Sobolev spaces with boundary conditions. If

a sectorial operator A on a Banach space X has bounded imaginary powers, then by [250] it holds that

$$D(A^\theta) = [X, D(A)]_\theta, \qquad \theta \in (0,1).$$

Hence, a characterisation of the complex interpolation space and the fractional domain is helpful to perform the perturbation argument for the H^∞-calculus. As we have seen, $D(A)$ is a Sobolev space incorporating the boundary conditions. Complex interpolation of spaces with boundary conditions was initially studied by Grisvard and Seeley in [108, 109, 250, 251] and is now widely addressed in the literature due to its numerous applications.

This dissertation contributes to the study of complex interpolation of weighted Sobolev spaces with and without boundary conditions. Let $m \in \mathbb{N}_0$ and let $\mathscr{B}$ be an *m-th order normal boundary operator* at $\partial\mathscr{O}$. We denote by $W^{k,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)$ the closed subspace of functions $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ such that $\mathscr{B} f = 0$, whenever the traces exist. One of our main results includes the following characterisation of complex interpolation spaces, which is a weighted and vector-valued extension of the interpolation results with boundary conditions obtained by Seeley [251].

Theorem 1.3.2. *Let $p \in (1,\infty)$, $k \in \{2,3,\dots\}$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and let X be a* UMD *Banach space. Let $\mathscr{O}$ be the half-space or a bounded C^∞-domain. Then for $\ell \in \{1,\dots,k-1\}$ we have*

$$\big[L^p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)\big]_{\frac{\ell}{k}} = W^{\ell,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X),$$
$$\big[L^p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), W^{k,p}_{\mathscr{B}}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)\big]_{\frac{\ell}{k}} = W^{\ell,p}_{\mathscr{B}}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X).$$

In the unweighted setting, the characterisation of complex interpolation spaces is based on harmonic analysis arguments. However, since we want to study general power weights $w^{\partial\mathscr{O}}_\gamma$ with $\gamma > -1$, we are not in the regime of Muckenhoupt A_p weights, which complicates the entire analysis. Nonetheless, with suitable adaptations of the classical arguments (as proved in, e.g., [16]), a lifting argument to increase γ and Wolff interpolation we obtain a fairly complete interpolation theory for weighted Sobolev spaces. A key ingredient for proving Theorem 1.3.2 is the characterisation of the trace spaces of weighted vector-valued Besov, Triebel–Lizorkin, Bessel potential and Sobolev spaces, which are obtained by partially extending the arguments in [16, 245] for the unweighted case.

As discussed in Section 1.1, the bounded H^∞-calculus immediately yields (stochastic) maximal regularity. So we can now use the H^∞-calculus from Theorem 1.3.1 as a black box to obtain the following results.

Example 1.3.3. Suppose that the conditions of Theorem 1.3.1 hold, and let $q \in (1,\infty)$ and $T \in (0,\infty)$. Then $-\Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ has maximal L^q-regularity. In particular, for every $f \in L^q((0,T); W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$ there exists a unique solution $u \in L^q((0,T); W^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$ to $\partial_t u - \Delta u = f$ with $u|_{\partial\mathscr{O}} = 0$ and $u|_{t=0} = 0$, which satisfies

$$\|\partial_t u\|_{L^q((0,T);W^{k,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))} + \|u\|_{L^q((0,T);W^{k+2,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))} \lesssim \|f\|_{L^q((0,T);W^{k,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))}.$$

The novelties of this result compared to Example 1.1.3 are as follows.

(i) We obtain higher-order regularity *without* compatibility conditions on the data f.

(ii) The smoothness of the domain $\mathscr{O}$ is *independent* of the order of the Sobolev space. Even for $k = 0$ and $\gamma = 0$, we can allow for a domain with less regularity than C^2, which is usually imposed in the literature.

Additionally, the above maximal regularity result allows for data with blow-up behaviour at the boundary. In particular, this means that the heat equation on $\mathscr{O} := (0,1)$ with data $f(t,x) := x^{-1}$ as studied in Example 1.2.1 can be treated with the above result, since

$$f \in L^q\big((0,T); W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big)$$

for all $p, q \in (1,\infty)$ and $\gamma \in (p-1, 2p-1)$.

As an extension of the result in Example 1.3.3, we prove maximal regularity for the heat equation with inhomogeneous boundary data, i.e.,

$$\begin{cases} \partial_t u - \Delta u = f & (t,x) \in (0,T)\times\mathscr{O}, \\ u|_{\partial\mathscr{O}} = g & t \in (0,T), \\ u|_{t=0} = 0 & x \in \mathscr{O}. \end{cases} \tag{1.3.1}$$

This requires the study of the spatial trace space of the solution space

$$\mathbb{U} := W^{1,q}\big((0,T); W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big) \cap L^q\big((0,T); W^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big).$$

For the unweighted setting with $k = 0$ and $\mathscr{O}$ a bounded C^2-domain, it is well known (see [68]) that a necessary and sufficient condition is

$$g \in F^{1-\frac{1}{2p}}_{q,p}\big((0,T); L^p(\partial\mathscr{O})\big) \cap L^q\big((0,T); B^{2-\frac{1}{p}}_{p,p}(\partial\mathscr{O})\big),$$

where B and F denote a Besov and Triebel–Lizorkin space, respectively. In this dissertation, we prove that in the setting of Example 1.3.3, the spatial trace operator Tr at $\partial\mathscr{O}$ is a continuous and surjective mapping from $\mathbb{U}$ to

$$F^{1-\frac{\gamma+1}{2p}}_{q,p}\big((0,T); L^p(\partial\mathscr{O})\big) \cap L^q\big((0,T); B^{2-\frac{\gamma+1}{p}}_{p,p}(\partial\mathscr{O})\big).$$

Hence, the regularity required for g is independent of k and for $\gamma > 0$ we can treat a larger class of data than in the unweighted setting. Moreover, this extends the weighted trace result in [190] to rougher domains and more general smoothness.

Moreover, from the H^∞-calculus in Theorem 1.3.1 we also obtain the following result for stochastic equations.

Example 1.3.4. Suppose that the conditions of Theorem 1.3.1 hold, and let $p, q \in (2,\infty)$ and $T \in (0,\infty)$. Then $-\Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ has stochastic maximal L^q-regularity. In the case that $\gamma \in (p-1, 2p-1)$, we obtain from Theorem 1.3.2 that

$$\big[W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}), W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big]_{\frac{1}{2}} = W^{k+1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}),$$

where no additional zero boundary conditions need to be specified. Hence, our stochastic maximal regularity result means that for every progressively measurable $f \in L^q(\Omega \times (0,T); W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$ and $g \in L^q(\Omega \times (0,T); W^{k+1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$ there exists a unique solution u to the stochastic heat equation

$$\mathrm{d}u - \Delta u \,\mathrm{d}t = f \,\mathrm{d}t + g \,\mathrm{d}W(t), \qquad t \in (0,T),$$

with $u|_{\partial\mathscr{O}} = 0$ and $u|_{t=0} = 0$, which satisfies

$$\|u\|_{L^q(\Omega\times(0,T);W^{k+2,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))} \lesssim \|f\|_{L^q(\Omega\times(0,T);W^{k,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))} + \|g\|_{L^q(\Omega\times(0,T);W^{k+1,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))}.$$

Hence, if $\gamma \in (p-1, 2p-1)$ we do not have to impose any compatibility conditions on g, cf. Example 1.1.6.

After studying some straightforward consequences of the H^∞-calculus, we move to maximal regularity for a more complicated equation. We consider the linear structurally damped plate equation with inhomogeneous Dirichlet–Neumann boundary conditions given by

$$\begin{cases} \partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u = f & (t,x) \in (0,T)\times\mathscr{O}, \\ u = g_0 & (t,x) \in (0,T)\times\partial\mathscr{O}, \\ \partial_{\boldsymbol{n}} u = g_1 & (t,x) \in (0,T)\times\partial\mathscr{O}, \\ u|_{t=0} = (\partial_t u)|_{t=0} = 0 & x \in \mathscr{O}. \end{cases} \tag{1.3.2}$$

Here, $\rho > 0$ is a fixed parameter and $\partial_{\boldsymbol{n}}$ denotes the inner normal derivative. Furthermore, $T \in (0,\infty]$ is a time horizon, and we consider the case of the half-space $\mathscr{O} = \mathbb{R}^d_+$ and the case of a bounded domain $\mathscr{O} \subseteq \mathbb{R}^d$. We remark that these Dirichlet–Neumann boundary conditions are mathematically more involved than the boundary conditions $u = \Delta u = 0$, which can be treated by more standard operator-theoretic methods, see, e.g., [40, 42, 246].

To study maximal regularity for (1.3.2), we rewrite it as a first-order system for $v = (u, \partial_t u)^\top$, i.e.,

$$\partial_t v + Av = \begin{pmatrix} 0 \\ f \end{pmatrix} \quad \text{with } A = \begin{pmatrix} 0 & -\mathrm{id} \\ \Delta^2 & -\rho\Delta \end{pmatrix}. \tag{1.3.3}$$

As we have discussed in Section 1.1, it suffices to establish R-sectoriality of the operator matrix A. It was shown by Denk & Schnaubelt in [74] that the operator A in (1.3.3) is R-sectorial on $W^{2,p}_0(\mathbb{R}^d_+) \times L^p(\mathbb{R}^d_+)$, but *not* on $W^{2,p}(\mathbb{R}^d_+) \times L^p(\mathbb{R}^d_+)$. Therefore, one has to include the additional boundary conditions $h_1 = \partial_{\boldsymbol{n}} h_1 = 0$ on $\partial\mathbb{R}^d_+$ for the first component of the right-hand side of the equation $(\lambda - A)v = h$. It was shown in [66] that such compatibility conditions are in general necessary for the sectoriality in unweighted spaces. As we have seen previously, we can avoid any compatibility conditions on the data by introducing weighted Sobolev spaces. We obtain R-sectoriality of A on

$$\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+) := W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}) \times W^{k-2,p}(\mathbb{R}^d_+, w_{\gamma+kp}),$$

where $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and $k \in \mathbb{N}_0$. In particular, setting $k = 2$, we can replace the space $W^{2,p}_0(\mathbb{R}^d_+) \times L^p(\mathbb{R}^d_+)$ from [74] by the weighted space $W^{2,p}(\mathbb{R}^d_+, w_{\gamma+2p}) \times L^p(\mathbb{R}^d_+, w_{\gamma+2p})$.

We obtain R-sectoriality of A on the weighted spaces $\mathbb{X}_\gamma^{k,p}(\mathbb{R}^d_+)$ by extrapolation and lifting. We start with R-sectoriality of A on the space $W_0^{2,p}(\mathbb{R}^d_+, w_\gamma) \times L^p(\mathbb{R}^d_+, w_\gamma)$ for $\gamma \in (-1, p-1)$, which can be obtained by extending the results in [74] to the weighted setting. Then we lower the smoothness of the spaces by extrapolating the operator in the sense of Amann's interpolation-extrapolation scales [14]. For the extrapolated operator on $L^p(\mathbb{R}^d_+, w_\gamma) \times W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$, due to the low regularity, the boundary conditions are not seen, so there are no compatibility conditions needed. This proves R-sectoriality on $\mathbb{X}_\gamma^{0,p}(\mathbb{R}^d_+)$.

Afterwards, with an induction argument on k, we raise the smoothness and the weight with a similar lifting technique as in the proof of Theorem 1.3.1. In this way, we can obtain R-sectoriality in weighted spaces $\mathbb{X}_\gamma^{k,p}(\mathbb{R}^d_+)$ of arbitrary smoothness $k \in \mathbb{N}_0$ without artificial compatibility conditions.

Once the R-sectoriality is obtained on $\mathbb{R}^d_+$, a localisation argument based on the *Dahlberg–Kenig–Stein pullback* allows us to treat the case of bounded domains. Therefore, we can establish the results for domains with $C^{1,\kappa}$-boundary, even for higher-order regularity. Note that even for the standard regularity $k = 2$, the classical localisation would require a domain of class C^4, as can be seen from [74, Section 5].

Finally, we obtain maximal regularity for A, which gives well-posedness of (1.3.2) with $g_0 = g_1 = 0$. Using the trace theory for intersections of weighted Sobolev spaces we can allow for rough inhomogeneous boundary conditions, see below Example 1.3.3.

To conclude this introduction, we briefly outline potential applications of the weighted setting to (stochastic) partial differential equations for future research.

- The weighted setting could be helpful in the study of stochastic equations with gradient noise, e.g.,

$$\begin{cases} \mathrm{d}u - \Delta u \,\mathrm{d}t = f \,\mathrm{d}t + (b \cdot \nabla u + g)\,\mathrm{d}W(t) & t \in (0,T), \\ u(0) = 0. \end{cases}$$

  In the unweighted setting, we have seen that the stochastic part needs to satisfy certain compatibility conditions, which in this case leads to conditions on the vector b, see [85]. In the weighted setting, any compatibility or smallness conditions on the coefficients b can be omitted, see [167]. Only in specific situations one can reduce to the case $b = 0$, see [231] and the references therein.

- Consider a problem with boundary noise, i.e.,

$$\begin{cases} \partial_t u - \Delta u = 0 & (t,x) \in (0,T) \times \mathcal{O}, \\ u = \dot{W} & (t,x) \in (0,T) \times \partial\mathcal{O}, \\ u|_{t=0} = 0 & x \in \mathcal{O}, \end{cases}$$

  as systematically studied by Da Prato & Zabczyk [56, 57]. A weighted setting is helpful to get as much regularity for the solution as possible, see [103]. Applications of the bounded H^∞-calculus and stochastic maximal regularity on unweighted spaces for the Navier–Stokes equations with boundary noise can be found in [8, 9].

- In Theorem 1.3.1 we have established a trade-off between domain regularity and the weight: lower domain regularity can be compensated by enlarging the weight exponent. A natural question is whether we can weaken the domain regularity even further to, e.g., Lipschitz domains if $\gamma \in (p-1, 2p-1)$. Within the framework of mixed weights, maximal regularity and the H^∞-calculus have been established on wedges, which are merely Lipschitz domains [47, 48].

- The operator A arising from the damped plate equation in (1.3.3) is an example of a mixed-order system, see [6, 7]. In general, semigroup theory for mixed-order boundary value problems is not available. For a general mixed-order system on the whole space, the Newton polygon theory developed in, e.g., [69, 73, 101] deals with such systems. The methods that are used to study the damped plate equation in Chapter 8 might also be applicable to more general mixed-order systems, thereby avoiding compatibility conditions.

1.4. Outline of this dissertation

The outline of this dissertation is as follows.

- In Chapter 2, we start with a discussion of some preliminary results. This includes introducing the required notation and definitions. In addition, we provide the construction of diffeomorphisms, which are required for localisation arguments. Such diffeomorphisms are standard in the literature for smooth domains, but we present a construction that is suitable for rough domains.

Subsequently, this thesis is divided into three parts.

Part I: Function space theory.

- In Chapter 3, we study trace and interpolation theory for weighted spaces. We characterise the higher-order trace spaces of weighted vector-valued Besov, Triebel–Lizorkin, Bessel potential and Sobolev spaces. This allows us to derive certain density results for weighted spaces with boundary conditions. Moreover, we characterise the complex interpolation spaces of weighted vector-valued Sobolev spaces with and without boundary conditions.

- In Chapter 4, we characterise the higher-order trace spaces of weighted vector-valued anisotropic Triebel–Lizorkin spaces, which yields the trace spaces for intersections of weighted vector-valued Sobolev spaces. This trace theory enables us to treat parabolic boundary value problems with rough boundary conditions.

Part II: Functional calculus.

- In Chapter 5, we start the investigation of functional calculus on weighted spaces. In particular, we prove that the Laplace operator on the half-space with Dirichlet or Neumann boundary conditions admits a bounded H^∞-calculus.

- In Chapter 6, we continue the study of functional calculus for the Laplace operator. Using the functional calculus for the Laplacian on $\mathbb{R}^d_+$, perturbation, and localisation arguments, we obtain that the Laplace operator on rough domains with Dirichlet or Neumann boundary conditions admits a bounded H^∞-calculus.

Part III: Applications to partial differential equations.

- In Chapter 7, we study (stochastic) maximal regularity for the Laplacian using the functional calculus developed in Chapters 5 and 6. In this way, we prove well-posedness and higher-order regularity for the (stochastic) heat equation on rough domains. In addition, we employ the trace theory from Chapter 4 to treat the heat equation with rough inhomogeneous boundary data.

- In Chapter 8, we prove well-posedness and higher-order regularity for a structurally damped plate equation with inhomogeneous Dirichlet–Neumann boundary conditions on rough domains. To this end, we establish R-sectoriality of the related first-order system on weighted spaces. The weighted spaces allow us to avoid any unnatural compatibility conditions for the data, which are necessarily present in the unweighted case.

2

Preliminaries

In this chapter, we present the background material that will be used throughout this dissertation.

2.1. Notation

We start by introducing some basic notation.

- We denote by $\mathbb{N}_0$ and $\mathbb{N}_1$ the sets of natural numbers starting at 0 and 1, respectively.
- For $a \in \mathbb{R}$, we use the notation $(a)_+ = a$ if $a \geq 0$ and $(a)_+ = 0$ otherwise.
- For $d \in \mathbb{N}_1$ the half-spaces are given by $\mathbb{R}^d_\pm = \mathbb{R}_\pm \times \mathbb{R}^{d-1}$, where $\mathbb{R}_+ = (0,\infty)$ and $\mathbb{R}_- = (-\infty, 0)$. For $x \in \mathbb{R}^d_\pm$ we write $x = (x_1, \widetilde{x})$ with $x_1 \in \mathbb{R}_\pm$ and $\widetilde{x} \in \mathbb{R}^{d-1}$.
- For two topological vector spaces X and Y, the space of continuous linear operators is denoted by $\mathscr{L}(X,Y)$ and $\mathscr{L}(X) := \mathscr{L}(X,X)$. Unless specified otherwise, X will always denote a Banach space with norm $\|\cdot\|_X$ and its dual space is $X' := \mathscr{L}(X,\mathbb{C})$.
- For a linear operator $A : X \supseteq D(A) \to X$ on a Banach space X, we denote by $\sigma(A)$ and $\rho(A)$ the spectrum and resolvent set, respectively. For $\lambda \in \rho(A)$, the resolvent operator is given by $R(\lambda, A) = (\lambda - A)^{-1} \in \mathscr{L}(X)$.
- We write $f \lesssim g$ (resp. $f \gtrsim g$) if there exists a constant $C > 0$, possibly depending on parameters which will be clear from the context or will be specified in the text, such that $f \leq Cg$ (resp. $f \geq Cg$). Furthermore, $f \eqsim g$ means $f \lesssim g$ and $g \lesssim f$.
- For an open and non-empty set $\mathscr{O} \subseteq \mathbb{R}^d$ and $\ell \in \mathbb{N}_0 \cup \{\infty\}$, the space $C^\ell(\mathscr{O}; X)$ denotes the space of ℓ-times continuously differentiable functions from $\mathscr{O}$ to the Banach space X. As usual, this space is equipped with the compact-open topology. In the case $\ell = 0$ we write $C(\mathscr{O}; X)$ for $C^0(\mathscr{O}; X)$. Furthermore, we write $C^\ell_{\mathrm{b}}(\mathscr{O}; X)$ for the space of all functions $f \in C^\ell(\mathscr{O}; X)$ such that $\partial^\alpha f$ is bounded on $\mathscr{O}$ for all multi-indices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq \ell$.

- Let $C_c^\infty(\mathcal{O};X)$ be the space of compactly supported smooth functions on $\mathcal{O}$ equipped with its usual inductive limit topology. The space of X-valued distributions is given by $\mathcal{D}'(\mathcal{O};X) := \mathcal{L}(C_c^\infty(\mathcal{O});X)$. Moreover, $C_c^\infty(\overline{\mathcal{O}};X)$ is the space of smooth functions with their support in a compact set contained in $\overline{\mathcal{O}}$.

- We denote the Schwartz space by $\mathcal{S}(\mathbb{R}^d;X)$, and $\mathcal{S}'(\mathbb{R}^d;X) := \mathcal{L}(\mathcal{S}(\mathbb{R}^d);X)$ is the space of X-valued tempered distributions. For $\mathcal{O}\subseteq\mathbb{R}^d$ we define $\mathcal{S}(\mathcal{O};X) := \{u|_{\mathcal{O}} : u\in\mathcal{S}(\mathbb{R}^d;X)\}$. For $f\in\mathcal{S}(\mathbb{R}^d;X)$ we define the d-dimensional Fourier transform and its inverse

$$(\mathcal{F}f)(\xi) := \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-\mathrm{i}x\cdot\xi}\,\mathrm{d}x, \qquad \xi\in\mathbb{R}^d,$$
$$(\mathcal{F}^{-1}f)(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} f(\xi)e^{\mathrm{i}x\cdot\xi}\,\mathrm{d}\xi, \qquad x\in\mathbb{R}^d,$$

  which extends to $\mathcal{S}'(\mathbb{R}^d;X)$ by duality.

2.2. Weighted function spaces

Let $\mathcal{O}\subseteq\mathbb{R}^d$ be open. We call a locally integrable function $w:\mathcal{O}\to(0,\infty)$ a *weight*. In particular, we will deal with the following classes of weights.

(i) *Muckenhoupt A_p weights.* Let $p\in(1,\infty)$ and let $\mathcal{O}$ be $\mathbb{R}^d$, $\mathbb{R}^d_+$ or an interval $(0,a)$ for $a>0$. A weight w on $\mathcal{O}$ belongs to the Muckenhoupt class $A_p(\mathcal{O})$ if

$$[w]_{A_p(\mathcal{O})} := \sup_B \Big(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\Big)\Big(\frac{1}{|B|}\int_B w(x)^{-\frac{1}{p-1}}\,\mathrm{d}x\Big)^{p-1} < \infty,$$

where the supremum is taken over all balls $B\subseteq\mathcal{O}$. Moreover, $A_\infty(\mathcal{O}) := \bigcup_{p>1}A_p(\mathcal{O})$. A weight is called even if $w(-x_1,\widetilde{x}) = w(x_1,\widetilde{x})$ for $(x_1,\widetilde{x})\in\mathbb{R}^d_+$. We have $w\in A_p(\mathbb{R}^d)$ and w is even if and only if $w\in A_p(\mathbb{R}^d_+)$.

The Muckenhoupt weights play an important role in harmonic analysis due to the following fact. For $p\in(1,\infty)$ it holds that $w\in A_p(\mathbb{R}^d)$ if and only if the Hardy–Littlewood maximal operator is bounded on $L^p(\mathbb{R}^d,w)$. For a detailed study of Muckenhoupt weights and their properties, we refer to [105, Chapter 7].

(ii) *Power weights.* Let $\mathcal{O}\subseteq\mathbb{R}^d$ be open such that $\mathcal{O}$ has a non-empty boundary $\partial\mathcal{O}$. For $\gamma\in\mathbb{R}$ define $w_\gamma^{\partial\mathcal{O}}$ on $\mathcal{O}$ by

$$w_\gamma^{\partial\mathcal{O}}(x) := \operatorname{dist}(x,\partial\mathcal{O})^\gamma, \qquad x\in\mathcal{O}.$$

Furthermore, we write $w_\gamma(x) := w_\gamma^{\partial\mathbb{R}^d_+}(x) = |x_1|^\gamma$ for $x=(x_1,\widetilde{x})\in\mathbb{R}^d_+$. Note that $w_\gamma(x)=|x_1|^\gamma$ also defines a weight on $\mathbb{R}^d$. Moreover, the power weight $w_\gamma(x)=|x_1|^\gamma$ is in $A_p(\mathcal{O})$ if and only if $\gamma\in(-1,p-1)$.

For $p \in [1,\infty)$, w a weight and X a Banach space, we define the weighted Lebesgue space $L^p(\mathscr{O}, w; X)$ as the Bochner space consisting of all equivalence classes of strongly measurable $f : \mathscr{O} \to X$ such that

$$\|f\|_{L^p(\mathscr{O},w;X)} := \Big(\int_{\mathscr{O}} \|f(x)\|_X^p \, w(x) \, \mathrm{d}x\Big)^{\frac{1}{p}} < \infty.$$

If $w \equiv 1$, then we simply write $L^p(\mathscr{O}; X)$. Furthermore, if $X = \mathbb{C}$, then we omit this from the notation as well.

Let $p \in (1,\infty)$ and w be such that $w^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(\mathscr{O})$. The k-th order weighted Sobolev space for $k \in \mathbb{N}_0$ is defined as

$$W^{k,p}(\mathscr{O}, w; X) := \big\{ f \in \mathscr{D}'(\mathscr{O}; X) : \forall |\alpha| \le k, \partial^\alpha f \in L^p(\mathscr{O}, w; X) \big\}$$

equipped with the canonical norm. Similar as above, if $w \equiv 1$, then we write $W^{k,p}(\mathscr{O}; X)$ and if $X = \mathbb{C}$, then we omit this from the notation.

Remark 2.2.1. The local L^1 condition for $w^{-1/(p-1)}$ ensures that all the derivatives $\partial^\alpha f$ are locally integrable in $\mathscr{O}$. If $\mathscr{O}$ is the half-space $\mathbb{R}^d_+$ or a bounded domain, then this condition for power weights $w_\gamma^{\partial\mathscr{O}}$ holds for all $\gamma \in \mathbb{R}$. On $\mathscr{O} = \mathbb{R}^d$ we consider weights $w_\gamma(x) = |x_1|^\gamma$, which are locally integrable if $\gamma > -1$. The local L^1 condition for $w_\gamma^{-1/(p-1)}$ only holds for $\gamma < p-1$ since functions might not be locally integrable near $x_1 = 0$. Hence, we can only consider $\gamma \in (-1, p-1)$ on $\mathbb{R}^d$, see also [170]. This explains why, for example, we cannot employ classical reflection arguments from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ if $\gamma \ge p-1$.

We continue with the well-known Hardy's inequality for spaces with power weights, see, e.g., [169, Section 5]. First, recall that for $p \in [1,\infty)$, $\gamma \in (-\infty, p-1)$ and X a Banach space, we have the Sobolev embedding

$$W^{1,p}(\mathbb{R}_+, w_\gamma; X) \hookrightarrow C([0,\infty); X), \tag{2.2.1}$$

so that the boundary value $u(0)$ is well defined for $u \in W^{1,p}(\mathbb{R}_+, w_\gamma; X)$, see, e.g., [190, Lemma 3.1]. We state the following version of Hardy's inequality from [190, Lemma 3.2 & Corollary 3.3], which will allow us to estimate Sobolev norms with both different weights and smoothness.

Lemma 2.2.2. *Let $p \in [1,\infty)$ and let X be a Banach space. Let $u \in W^{1,p}(\mathbb{R}_+, w_\gamma; X)$ and assume either*

(i) $\gamma < p-1$ and $u(0) = 0$, or

(ii) $\gamma > p-1$.

Then

$$\|u\|_{L^p(\mathbb{R}_+, w_{\gamma-p};X)} \le C_{p,\gamma} \|u'\|_{L^p(\mathbb{R}_+, w_\gamma;X)}.$$

In addition, if $k \in \mathbb{N}_1$ and $\gamma > p-1$, then

$$W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X).$$

2.3. Functional calculus

We introduce the notions of sectoriality and functional calculus.

2.3.1. Sectorial operators

For $\omega \in (0,\pi)$, let $\Sigma_\omega = \{z \in \mathbb{C} \setminus \{0\} : |\arg(z)| < \omega\}$ be the sector in the complex plane.

Definition 2.3.1. An injective, closed linear operator $(A, D(A))$ with dense domain and dense range on a Banach space X is called *sectorial* if there exists a $\omega \in (0,\pi)$ such that $\sigma(A) \subseteq \overline{\Sigma_\omega}$ and

$$\sup_{\lambda \in \mathbb{C}\setminus\overline{\Sigma_\omega}} \|\lambda R(\lambda, A)\| < \infty. \tag{2.3.1}$$

Furthermore, the angle of sectoriality $\omega(A)$ is defined as the infimum over all possible ω.

Remark 2.3.2. If A is a linear operator on a Banach space X satisfying (2.3.1), then A is closed. Moreover, if X is reflexive, then A has dense domain. Furthermore, if A has dense range, then A is injective. See [125, Propositions 10.1.7(3) and 10.1.9].

We state a lemma that relates the growth of the semigroup e^{-zA} to the estimate on the resolvent of A. For related results of this type, we refer to [239].

Lemma 2.3.3. *Let A be a linear operator on a Banach space X and let $\alpha \geq 1$. The following are equivalent.*

(i) There exist $\omega \in (0, \frac{\pi}{2})$ and $C_1 > 0$ such that $\sigma(A) \subseteq \overline{\Sigma_\omega}$ and

$$\|(\lambda + A)^{-1}\|_{\mathscr{L}(X)} \leq C_1\Big(\frac{1}{|\lambda|^\alpha} + \frac{1}{|\lambda|}\Big), \qquad \lambda \in \Sigma_{\pi-\omega}.$$

(ii) There exist $\eta \in (0, \frac{\pi}{2})$ and $C_2 > 0$ such that $-A$ generates an analytic C_0-semigroup on Σ_η and

$$\|e^{-zA}\|_{\mathscr{L}(X)} \leq C_2(|z|^{\alpha-1} + 1), \qquad z \in \Sigma_\eta.$$

Proof. We first prove that (i) implies (ii). Let $\mu > 0$, then there exists a constant K, independent of λ and μ, such that

$$\|(\lambda + \mu + A)^{-1}\|_{\mathscr{L}(X)} \leq C_1\Big(\frac{1}{|\lambda+\mu|^\alpha} + \frac{1}{|\lambda+\mu|}\Big) \leq \frac{K}{|\lambda|}\Big(\frac{1}{\mu^{\alpha-1}} + 1\Big),$$

since $\mu \leq k_0|\lambda+\mu|$ and $|\lambda| \leq k_1|\lambda+\mu|$ with constants $k_0, k_1 > 0$ only depending on ω. By standard theory for analytic semigroups (see [125, Theorem G.5.2]) it follows

$$\|e^{-z(\mu+A)}\|_{\mathscr{L}(X)} \leq K_1\Big(\frac{1}{\mu^{\alpha-1}} + 1\Big), \qquad z \in \Sigma_{\frac{\pi}{2}-\omega}.$$

The result follows upon taking $\mu = 1/|z|$ because

$$\|e^{-zA}\|_{\mathscr{L}(X)} = |e^{\frac{z}{|z|}}|\,\|e^{-z(\frac{1}{|z|}+A)}\|_{\mathscr{L}(X)} \leq C_2(|z|^{\alpha-1} + 1).$$

We now prove that (ii) implies (i). First, let $\lambda \in \Sigma_\omega$ for $\omega \in (0, \frac{\pi}{2})$. By the Laplace transform we obtain

$$\begin{aligned}\|(\lambda + A)^{-1}\|_{\mathscr{L}(X)} &= \Big\| \int_0^\infty e^{-\lambda t} e^{-tA} \,\mathrm{d}t \Big\|_{\mathscr{L}(X)} \leq C_2 \int_0^\infty e^{-\operatorname{Re}\lambda t}(t^{\alpha-1} + 1) \,\mathrm{d}t \\ &= K\Big(\frac{1}{(\operatorname{Re}\lambda)^\alpha} + \frac{1}{\operatorname{Re}\lambda}\Big) \leq K_1\Big(\frac{1}{|\lambda|^\alpha} + \frac{1}{|\lambda|}\Big),\end{aligned}$$

where the constants K and K_1 only depend on ω and C_2. By considering the rotated operator $e^{\mathrm{i}\theta} A$ with $|\theta| < \eta$, we obtain (i). □

2.3.2. The holomorphic functional calculus

We first introduce the following Hardy spaces. Let $\omega \in (0, \pi)$, then $H^1(\Sigma_\omega)$ is the space of all holomorphic functions $f : \Sigma_\omega \to \mathbb{C}$ such that

$$\|f\|_{H^1(\Sigma_\omega)} := \sup_{|\nu| < \omega} \|t \mapsto f(e^{\mathrm{i}\nu} t)\|_{L^1(\mathbb{R}_+, \frac{\mathrm{d}t}{t})} < \infty.$$

Moreover, let $H^\infty(\Sigma_\omega)$ be the space of all bounded holomorphic functions on the sector with norm

$$\|f\|_{H^\infty(\Sigma_\omega)} := \sup_{z \in \Sigma_\omega} |f(z)|.$$

Definition 2.3.4. Let A be a sectorial operator on a Banach space X and let $\omega \in (\omega(A), \pi)$, $\nu \in (\omega(A), \omega)$ and $f \in H^1(\Sigma_\omega)$. We define the operator

$$f(A) := \frac{1}{2\pi\mathrm{i}} \int_{\partial\Sigma_\nu} f(z) R(z, A) \,\mathrm{d}z,$$

where $\partial\Sigma_\nu$ is traversed downwards. The operator A has a *bounded $H^\infty(\Sigma_\omega)$-calculus* if there exists a $C > 0$ such that

$$\|f(A)\| \leq C \|f\|_{H^\infty(\Sigma_\omega)} \quad \text{for all } f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega).$$

Furthermore, the angle of the H^∞-calculus $\omega_{H^\infty}(A)$ is defined as the infimum over all possible $\omega > \omega(A)$.

For details on the H^∞-calculus we refer to [118] and [125, Chapter 10].

Remark 2.3.5. If the sectorial operator A is in addition invertible, then the behaviour of the function $f \in H^1(\Sigma_\omega)$ in the neighbourhood of 0 is immaterial. By Cauchy's theorem, we can equivalently define

$$f(A) = \frac{1}{2\pi\mathrm{i}} \int_{\Gamma_\nu} f(z) R(z, A) \,\mathrm{d}z,$$

where Γ_ν is the boundary of $\Sigma_\nu \setminus B(0, \delta)$ with $\delta > 0$ small enough such that $B(0, \delta) \subseteq \rho(A)$. See [118, Section 2.5.1] for more details.

2.4. The UMD property

A Banach space X satisfies the condition UMD (unconditional martingale differences) if and only if the Hilbert transform extends to a bounded operator on $L^p(\mathbb{R};X)$ for all $p\in(1,\infty)$. We list the following relevant properties of UMD spaces, see for instance [124, Chapter 4 & 5].

(i) Hilbert spaces are UMD Banach spaces. In particular, $\mathbb{C}$ is a UMD space.

(ii) If $p\in(1,\infty)$, (S,Σ,μ) is a σ-finite measure space and X is a UMD Banach space, then $L^p(S;X)$ is a UMD Banach space.

(iii) UMD Banach spaces are reflexive.

The UMD property is known to be necessary for many results on vector-valued Sobolev spaces (see [21], [124, Section 5.6] and [126, Corollary 13.3.9]). Moreover, the boundedness of the H^∞-calculus of $-\Delta$ on spaces such as $L^p(\mathbb{R}^d;X)$ is also equivalent to the UMD property (see [125, Section 10.5]).

2.5. Complex interpolation

We collect some basic properties of the complex interpolation method. For a detailed study of interpolation methods, we refer to, e.g., [27, 262].

Let $A\subseteq\mathbb{R}$ be an interval. Let $\mathbb{S}_A:=\{z\in\mathbb{C}:\operatorname{Re}z\in A\}$ be a strip in the complex plane and we write $\mathbb{S}$ if $A=(0,1)$. Let (X_0,X_1) be an interpolation couple of complex Banach spaces, i.e., a pair of Banach spaces which are continuously embedded in a linear Hausdorff space $\mathscr{X}$. We denote by $\mathscr{H}(X_0,X_1)$ the complex vector space of all continuous functions $f:\overline{\mathbb{S}}\to X_0+X_1$ such that f is holomorphic on $\mathbb{S}$ with values in X_0+X_1 and the mapping $t\mapsto f(j+\mathrm{i}t)$ belongs to $C_{\mathrm{b}}(\mathbb{R};X_j)$ for $j\in\{0,1\}$. The space $\mathscr{H}(X_0,X_1)$ endowed with the norm

$$\|f\|_{\mathscr{H}(X_0,X_1)}:=\max_{j\in\{0,1\}}\sup_{t\in\mathbb{R}}\|f(j+\mathrm{i}t)\|_{X_j}$$

is a Banach space.

Definition 2.5.1. Let (X_0,X_1) be an interpolation couple and $\theta\in(0,1)$, then the *complex interpolation space* $[X_0,X_1]_\theta$ is the complex vector space of all $x\in X_0+X_1$ such that $f(\theta)=x$ for some $f\in\mathscr{H}(X_0,X_1)$.

The space $[X_0,X_1]_\theta$ endowed with the norm

$$\|x\|_{[X_0,X_1]_\theta}:=\inf\big\{\|f\|_{\mathscr{H}(X_0,X_1)}:f(\theta)=x\big\}$$

is a Banach space and satisfies

$$X_0\cap X_1\hookrightarrow[X_0,X_1]_\theta\hookrightarrow X_0+X_1.$$

Finally, if $f\in\mathscr{H}(X_0,X_1)$ and $g_t(z):=f(z+\mathrm{i}t)$ for $z\in\overline{\mathbb{S}}$ and $t\in\mathbb{R}$, then $g_t\in\mathscr{H}(X_0,X_1)$ and

$$\|f(\theta+\mathrm{i}t)\|_{[X_0,X_1]_\theta}\le C\|f\|_{\mathscr{H}(X_0,X_1)},\qquad\theta\in(0,1),\ t\in\mathbb{R}.\tag{2.5.1}$$

2.6. Domains & localisation techniques

Let $\kappa \in (0,1]$ and let $\mathcal{O} \subseteq \mathbb{R}^{d-1}$ be open. A function $h : \mathcal{O} \to \mathbb{R}$ is called *uniformly κ-Hölder continuous on $\mathcal{O}$* if

$$[h]_{\kappa,\mathcal{O}} := \sup_{\substack{x,y\in\mathcal{O}\\ x\neq y}} \frac{|h(x)-h(y)|}{|x-y|^\kappa} < \infty.$$

In addition, for $\ell \in \mathbb{N}_0$ we define the space of κ-Hölder continuous functions by

$$C_{\mathrm{b}}^{\ell,\kappa}(\mathcal{O}) := \{h \in C_{\mathrm{b}}^{\ell}(\mathcal{O}) : [\partial^\alpha h]_{\kappa,\mathcal{O}} < \infty \text{ for all } |\alpha| = \ell\}.$$

For $\kappa = 0$ we write $C_{\mathrm{b}}^{\ell,0}(\mathcal{O}) := C_{\mathrm{b}}^{\ell}(\mathcal{O})$. By $C_{\mathrm{c}}^{\ell,\kappa}(\mathcal{O})$ we denote the subset of functions in $C^{\ell,\kappa}(\mathcal{O})$ with compact support in $\mathcal{O}$. Moreover, on $C_{\mathrm{b}}^{\ell,\kappa}(\mathcal{O})$ we define the norm

$$\|h\|_{C^{\ell,\kappa}(\mathcal{O})} := \sum_{|\alpha|\le\ell} \sup_{x\in\mathcal{O}} |\partial^\alpha h(x)| + \sum_{|\alpha|=\ell} [\partial^\alpha h]_{\kappa,\mathcal{O}}.$$

Definition 2.6.1. Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a domain, i.e., a connected open set. Let $\ell \in \mathbb{N}_0$ and $\kappa \in [0,1]$.

(i) We call $\mathcal{O}$ a *special $C_{\mathrm{c}}^{\ell,\kappa}$-domain* if, after translation and rotation, it is of the form

$$\mathcal{O} = \{(x_1, \widetilde{x}) \in \mathbb{R}^d : x_1 > h(\widetilde{x})\} \tag{2.6.1}$$

for some $h \in C_{\mathrm{c}}^{\ell,\kappa}(\mathbb{R}^{d-1};\mathbb{R})$.

(ii) Given a special $C_{\mathrm{c}}^{\ell,\kappa}$-domain $\mathcal{O}$, we define

$$[\mathcal{O}]_{C^{\ell,\kappa}} := \|h\|_{C^{\ell,\kappa}(\mathbb{R}^{d-1})},$$

where $h \in C_{\mathrm{c}}^{\ell,\kappa}(\mathbb{R}^{d-1};\mathbb{R})$ is such that, after rotation and translation, (2.6.1) holds. Note that $[\mathcal{O}]_{C^{\ell,\kappa}}$ is uniquely defined due to the compact support of h.

(iii) We call $\mathcal{O}$ a *$C^{\ell,\kappa}$-domain* if every boundary point $x \in \partial\mathcal{O}$ admits an open neighbourhood V such that

$$\mathcal{O} \cap V = W \cap V \qquad \text{and} \qquad \partial\mathcal{O} \cap V = \partial W \cap V$$

for some special $C_{\mathrm{c}}^{\ell,\kappa}$-domain W.

If $\kappa = 0$, then we write C^ℓ for $C^{\ell,0}$ in the definitions above.

For any $\delta > 0$ and C^ℓ-domain $\mathcal{O}$, the special C_{c}^ℓ-domains W can always be chosen such that $[W]_{C^\ell} < \delta$. If $\kappa \in (0,1]$, $\varepsilon \in (0,\kappa)$ and $\mathcal{O}$ is a $C^{\ell,\kappa}$-domain, then for any $\delta > 0$, the special $C_{\mathrm{c}}^{\ell,\kappa}$-domains W can be chosen such that $[W]_{C^{\ell,\kappa-\varepsilon}} < \delta$. Indeed, if $h \in C_{\mathrm{c}}^{\ell,\kappa}(\mathbb{R}^{d-1};\mathbb{R})$ is associated with W, then for any $|\alpha| = \ell$, we have

$$[\partial^\alpha h]_{\kappa-\varepsilon,\mathbb{R}^{d-1}} = \sup_{\substack{x,y\in\mathbb{R}^{d-1}\\ x\neq y}} \frac{|\partial^\alpha h(x) - \partial^\alpha h(y)|}{|x-y|^\kappa} |x-y|^\varepsilon < \delta,$$

whenever $|x-y|^\varepsilon$ is small enough. Note that for $\varepsilon = 0$, the quantity $[\partial^\alpha h]_{\kappa,\mathbb{R}^{d-1}}$ cannot be made arbitrarily small.

2.6.1. Localisation

A standard technique in the theory of partial differential equations is localisation to transfer results on $\mathbb{R}^d$ to domains. This requires the construction of diffeomorphisms from special domains $\mathcal{O} := \{x \in \mathbb{R}^d : x_1 > h(\widetilde{x})\}$ to the half-space $\mathbb{R}^d_+$. In the literature, such diffeomorphisms are frequently used for smooth domains. In the rest of this section, we will construct modifications of such diffeomorphisms for rough domains.

We start with some definitions. Throughout this section, we consider $d \geq 2$.

Definition 2.6.2. Let $U, V \subseteq \mathbb{R}^d$, $\ell \in \mathbb{N}_1$ and $\kappa \in [0,1]$. A map $\Phi : U \to V$ is called a $C^{\ell,\kappa}$-*diffeomorphism* if Φ is a bijection, $\Phi \in C^{\ell,\kappa}(U;\mathbb{R}^d)$ and $\Phi^{-1} \in C^{\ell,\kappa}(V;\mathbb{R}^d)$.

To be able to deal with boundary value problems involving the normal derivative on the boundary, we need a diffeomorphism that preserves the direction of the normal vector. This is the notion of admissibility, see [275, Section 2.4].

Definition 2.6.3. Let U and V be C^1-domains, $\ell \in \mathbb{N}_1$ and $\kappa \in [0,1]$. A $C^{\ell,\kappa}$-diffeomorphism $\Phi : U \to V$ is called *admissible at the point* $x \in \partial U$ if the push-forward $D\Phi$ corresponding to Φ has the following properties:

(i) $D\Phi$ maps the tangent space $T_{\partial U,x}$ to $T_{\partial V,\Phi(x)}$;

(ii) $D\Phi$ maps the inner normal $\nu(x)$ of ∂U to the inner normal $\nu(\Phi(x))$ of ∂V.

Note that in the definition above, we do not require that the length of the normal vector is preserved. Furthermore, it holds that Φ is admissible at $x \in \partial U$ if and only if Φ^{-1} is admissible at $y = \Phi(x) \in \partial V$.

We note that for special C^1-domains of the form $\mathcal{O} := \{x \in \mathbb{R}^d : x_1 > h(\widetilde{x})\}$ for some $h \in C^1(\mathbb{R}^{d-1})$, the inner normal direction at the boundary of $\mathcal{O}$ is $\nu(x) = \nu(\widetilde{x}) = (1, -\nabla_{\widetilde{x}} h(\widetilde{x}))^\top$ and the tangent space is spanned by the vectors

$$(\partial_{x_2} h(\widetilde{x}), 1, 0, \dots, 0)^\top, \dots, (\partial_{x_d} h(\widetilde{x}), 0, 0, \dots, 1)^\top.$$

Example 2.6.4. Let $\ell \in \mathbb{N}_1$ and $\kappa \in [0,1]$. For a special domain $\mathcal{O}$ corresponding to $h \in C^{\ell,\kappa}_{\mathrm{c}}(\mathbb{R}^{d-1})$, the following two *classical* diffeomorphisms are well known in the literature.

(i) The most frequently used diffeomorphism $\Phi_{\mathrm{cl}} : \overline{\mathcal{O}} \to \overline{\mathbb{R}^d_+}$ is given by

$$\begin{aligned} \Phi_{\mathrm{cl}}(x) &= (x_1 - h(\widetilde{x}), \widetilde{x}), && x = (x_1, \widetilde{x}) \in \overline{\mathcal{O}}, \\ \Phi_{\mathrm{cl}}^{-1}(y) &= (y_1 + h(\widetilde{y}), \widetilde{y}), && y = (y_1, \widetilde{y}) \in \overline{\mathbb{R}^d_+}, \end{aligned}$$

see, e.g., [67, 89, 166]. This is a $C^{\ell,\kappa}$-diffeomorphism which, in general, is not admissible for all $x \in \partial\mathcal{O}$. This can be seen, e.g., from the push-forward of Φ_{cl}^{-1}, which is given by

$$D\Phi_{\mathrm{cl}}^{-1}(y) = \begin{pmatrix} 1 & (\nabla_{\widetilde{y}} h(\widetilde{y}))^\top \\ 0 & I_{d-1} \end{pmatrix}, \qquad y \in \overline{\mathbb{R}^d_+}.$$

(ii) Let $\ell \geq 2$. Following the construction in [275, Theorem 2.12], one can obtain a less regular diffeomorphism that is admissible for all points at the boundary. Let

$$\nu(x) = (1, \widetilde{\nu}(\widetilde{x})) = (1, -\nabla_{\widetilde{x}} h(\widetilde{x}))$$

be the inner normal direction at the boundary of $\mathscr{O}$. Consider $\overline{\Psi}_{\mathrm{cl}} : \overline{\mathbb{R}^d_+} \to \overline{\mathscr{O}}$ given by

$$\overline{\Psi}_{\mathrm{cl}}(y) = (y_1 + h(\widetilde{y}), \widetilde{y} + y_1 \widetilde{\nu}(\widetilde{y})).$$

The push-forward of $\overline{\Psi}_{\mathrm{cl}}$ at the boundary $y_1 = 0$ is

$$\begin{aligned} D\overline{\Psi}_{\mathrm{cl}}(y)|_{y_1=0} &= \begin{pmatrix} 1 & (\nabla_{\widetilde{y}} h(\widetilde{y}))^\top \\ \widetilde{\nu}(\widetilde{y}) & I_{d-1} + y_1 D_{\widetilde{y}} \widetilde{\nu}(\widetilde{y}) \end{pmatrix}\Big|_{y_1=0} \\ &= \begin{pmatrix} 1 & (\nabla_{\widetilde{y}} h(\widetilde{y}))^\top \\ -\nabla_{\widetilde{y}} h(\widetilde{y}) & I_{d-1} \end{pmatrix}, \end{aligned} \tag{2.6.2}$$

for all $\widetilde{y} \in \mathbb{R}^{d-1}$. Hence, for $y_1 = 0$ we have $|\det D\overline{\Psi}_{\mathrm{cl}}(y)| \geq 1$ and thus there exists a $\delta > 0$ such that $|\det D\overline{\Psi}_{\mathrm{cl}}(y)| \geq \frac{1}{2} > 0$ for all $y \in \overline{\mathbb{R}^d_+}$ with $0 \leq y_1 \leq \delta$. By the inverse function theorem, there exists an inverse Ψ_{cl} to $\overline{\Psi}_{\mathrm{cl}}$ on this strip. Then Ψ_{cl} is a $C^{\ell-1,\kappa}$-diffeomorphism since $\widetilde{\nu} \in C^{\ell-1,\kappa}(\mathbb{R}^{d-1})$. Moreover, (2.6.2) shows that Ψ_{cl} is an admissible $C^{\ell-1,\kappa}$-diffeomorphism for all $x \in \partial\mathscr{O}$. In the case that $\ell = 1$, we obtain that Ψ_{cl} is a homeomorphism.

If Φ_{cl} is the $C^{\ell,\kappa}$-diffeomorphism from Example 2.6.4(i), then the change of coordinates mapping $(\Phi_{\mathrm{cl}})_* f := f \circ \Phi_{\mathrm{cl}}^{-1}$ is an isomorphism between $W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$ and $W^{k,p}(\mathbb{R}^d_+, w_\gamma)$ for $\ell \geq k$. To be able to deal with rougher domains, i.e., $\ell < k$, we will mollify the diffeomorphism to make it smooth in the interior. This causes blow-up behaviour of higher-order derivatives near the boundary. Below, we present the construction of these mollified versions of Φ_{cl} and Ψ_{cl}, denoted Φ and Ψ respectively.

First, we consider a mollified version of the diffeomorphism Φ_{cl} in Example 2.6.4(i). This mollified diffeomorphism is in the literature also known as the *Dahlberg–Kenig–Stein pullback*, which dates back to [61, 62] and is, for instance, applied in [63, 90, 139]. The Dahlberg–Kenig–Stein pullback is often used for domains with low regularity (less than C^1), see the above-mentioned literature. For our purposes, we require estimates on higher-order derivatives of the pullback in the case of more regular domains (more than C^1). The following lemma is an extension of the result for C^1-domains in [149, Lemmas 2.6 & 3.8], which is based on the works [100, 182]. The result of Lemma 2.6.5 is also obtained in [140, Lemma 2.14] with a somewhat different proof than the one below. Nonetheless, we provide the proof since parts of it will be reused for constructing the diffeomorphism Ψ in Lemma 2.6.6.

Lemma 2.6.5. *Let $\mathscr{O}$ be a special C^1_{c}-domain. Then there exist continuous functions $h_1 : \overline{\mathscr{O}} \to \mathbb{R}$ and $h_2 : \overline{\mathbb{R}^d_+} \to \mathbb{R}$ which satisfy the following properties.*

(i) The map $\Phi : \overline{\mathscr{O}} \to \overline{\mathbb{R}^d_+}$ given by

$$\Phi(x) = (x_1 - h_1(x), \widetilde{x}), \qquad x = (x_1, \widetilde{x}) \in \overline{\mathscr{O}},$$

is a C^1-diffeomorphism with inverse $\Phi^{-1}:\overline{\mathbb{R}^d_+}\to\overline{\mathscr{O}}$ given by

$$\Phi^{-1}(y)=(y_1+h_2(y),\widetilde{y}),\qquad y=(y_1,\widetilde{y})\in\overline{\mathbb{R}^d_+}.$$

(ii) We have

$$\begin{aligned}\operatorname{dist}(\Phi(x),\partial\mathbb{R}^d_+)&\eqsim\operatorname{dist}(x,\partial\mathscr{O}),&&x\in\mathscr{O},\\ \operatorname{dist}(\Phi^{-1}(y),\partial\mathscr{O})&\eqsim\operatorname{dist}(y,\partial\mathbb{R}^d_+),&&y\in\mathbb{R}^d_+,\end{aligned}$$

where the implicit constants depend on $[\mathscr{O}]_{C^1}$.

(iii) We have $h_1\in C^\infty(\mathscr{O})$ and $h_2\in C^\infty(\mathbb{R}^d_+)$.

In addition, let $\ell\in\mathbb{N}_1$, $\kappa\in[0,1]$ and let $\mathscr{O}$ be a special $C^{\ell,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}}\le 1$.

(iv) The map Φ in (i) is a $C^{\ell,\kappa}$-diffeomorphism and for all $\alpha\in\mathbb{N}^d_0$, $\ell_0\in\{0,\dots,\ell\}$ and $\kappa_0\in[0,\kappa]$, we have

$$\begin{aligned}|\partial^\alpha h_1(x)|&\le C\cdot[\mathscr{O}]_{C^{\ell,\kappa}}\cdot\operatorname{dist}(x,\partial\mathscr{O})^{-(|\alpha|-\ell_0-\kappa_0)_+},&&x\in\mathscr{O},\\ |\partial^\alpha h_2(y)|&\le C\cdot[\mathscr{O}]_{C^{\ell,\kappa}}\cdot\operatorname{dist}(y,\partial\mathbb{R}^d_+)^{-(|\alpha|-\ell_0-\kappa_0)_+},&&y\in\mathbb{R}^d_+,\end{aligned}$$

where the constant $C>0$ only depends on ℓ,κ,α and d.

We have a similar result for the mollified version of the admissible diffeomorphism Ψ_{cl} in Example 2.6.4(ii).

Lemma 2.6.6. *Let $\mathscr{O}$ be a special C^1_{c}-domain. Then there exists a $\Lambda\in(0,1)$ such that if $[\mathscr{O}]_{C^1}\le\Lambda$, then there exist continuous functions $h_1:\overline{\mathscr{O}}\to\mathbb{R}$, $\widetilde{h}_1:\overline{\mathscr{O}}\to\mathbb{R}^{d-1}$, $h_2:\overline{\mathbb{R}^d_+}\to\mathbb{R}$ and $\widetilde{h}_2:\overline{\mathbb{R}^d_+}\to\mathbb{R}^{d-1}$ which satisfy the following properties.*

(i) The map $\Psi:\overline{\mathscr{O}}\to\overline{\mathbb{R}^d_+}$ given by

$$\Psi(x)=(x_1-h_1(x),\widetilde{x}-\widetilde{h}_1(x)),\qquad x=(x_1,\widetilde{x})\in\overline{\mathscr{O}},$$

is a C^1-diffeomorphism with inverse $\Psi^{-1}:\overline{\mathbb{R}^d_+}\to\overline{\mathscr{O}}$ given by

$$\Psi^{-1}(y)=(y_1+h_2(y),\widetilde{y}+\widetilde{h}_2(y)),\qquad y=(y_1,\widetilde{y})\in\overline{\mathbb{R}^d_+}.$$

(ii) We have

$$\begin{aligned}\operatorname{dist}(\Psi(x),\partial\mathbb{R}^d_+)&\eqsim\operatorname{dist}(x,\partial\mathscr{O}),&&x\in\mathscr{O},\\ \operatorname{dist}(\Psi^{-1}(y),\partial\mathscr{O})&\eqsim\operatorname{dist}(y,\partial\mathbb{R}^d_+),&&y\in\mathbb{R}^d_+.\end{aligned}$$

(iii) We have $h_1\in C^\infty(\mathscr{O})$, $\widetilde{h}_1\in C^\infty(\mathscr{O};\mathbb{R}^{d-1})$, $h_2\in C^\infty(\mathbb{R}^d_+)$ and $\widetilde{h}_2\in C^\infty(\mathbb{R}^d_+;\mathbb{R}^{d-1})$.

(iv) The diffeomorphism Ψ is admissible for all $x\in\partial\mathscr{O}$.

In addition, let $\ell\in\mathbb{N}_1$, $\kappa\in[0,1]$ and let $\mathscr{O}$ be a special $C^{\ell,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}}\le 1$.

(v) The map Ψ in (i) is a $C^{\ell,\kappa}$-diffeomorphism and for all $\alpha \in \mathbb{N}_0^d$, $\ell_0 \in \{0,\dots,\ell\}$ and $\kappa_0 \in [0,\kappa]$, we have

$$|\partial^\alpha h_1(x)| + |\partial^\alpha \widetilde{h}_1(x)| \le C \cdot [\mathscr{O}]_{C^{\ell,\kappa}} \cdot \operatorname{dist}(x,\partial\mathscr{O})^{-(|\alpha|-\ell_0-\kappa_0)_+}, \qquad x \in \mathscr{O},$$
$$|\partial^\alpha h_2(y)| + |\partial^\alpha \widetilde{h}_2(y)| \le C \cdot [\mathscr{O}]_{C^{\ell,\kappa}} \cdot \operatorname{dist}(y,\partial\mathbb{R}^d_+)^{-(|\alpha|-\ell_0-\kappa_0)_+}, \qquad y \in \mathbb{R}^d_+,$$

where the constant $C > 0$ only depends on ℓ, κ, α and d.

Remark 2.6.7. We make the following remarks about Lemmas 2.6.5 and 2.6.6.

(i) The first three statements in both lemmas are standard results for localisation. For the standard localisation procedure, one can take h_1 and h_2 equal to h, see Example 2.6.4. In our case, since h is not smooth enough, we need to use a mollifier to make h_2 smooth in Lemma 2.6.5. Afterwards, h_1 is determined using the inverse function theorem. Additionally, in Lemma 2.6.6 we need to mollify the normal vector to make $\widetilde{h}_2$ smooth.

(ii) The important part of Lemma 2.6.5 is (iv), which allows us to estimate higher-order derivatives of the diffeomorphism Φ and its inverse. If the number of derivatives exceeds the smoothness of the domain, then there is a blow-up near the boundary. We note that the construction of Φ is independent of ℓ and κ.

(iii) The important novelty of Lemma 2.6.6 is that Ψ preserves the direction of the normal and tangential vectors. This is done by choosing $\widetilde{h}_2(y) = y_1\widetilde{\mathscr{V}}(y)$, where $\widetilde{\mathscr{V}}(y)$ is a mollified version of the normal direction $-\nabla_{\widetilde{y}} h(\widetilde{y})$. While the diffeomorphism in Example 2.6.4(ii) is less regular than the domain, in Lemma 2.6.6 the regularity of the diffeomorphism is equal to the regularity of the domain. Moreover, the estimates on the derivatives in Lemma 2.6.6(v) are the same as in Lemma 2.6.5(iv).

(iv) The condition $[\mathscr{O}]_{C^{\ell,\kappa}} \le 1$ in Lemmas 2.6.5(iv) and 2.6.6(v) slightly simplifies the proofs. This condition is not necessary and can be removed. The more restrictive condition on $[\mathscr{O}]_{C^1}$ in Lemma 2.6.6 seems to be necessary to construct a global inverse. For our application in later chapters, imposing such conditions is not restrictive since $[\mathscr{O}]_{C^{\ell,\kappa}}$ can be made arbitrarily small in the localisation procedure.

(v) Our proofs would also work for Lipschitz domains if one uses an inverse function theorem for Lipschitz functions. In the setting of Lemma 2.6.5, this is done in [140]. The result in Lemma 2.6.6 appears to be new even in the case of C^1-domains.

The rest of this section is devoted to the proofs of Lemmas 2.6.5 and 2.6.6.

Proof of Lemma 2.6.5. Let $\varphi \in C_c^\infty(\mathbb{R}^{d-1})$ be a non-negative function with its support in the unit ball such that $\int_{\mathbb{R}^{d-1}} \varphi(\widetilde{x})\,\mathrm{d}\widetilde{x} = 1$. Let $h \in C_c^1(\mathbb{R}^{d-1})$ such that $\mathscr{O} = \{x \in \mathbb{R}^d : x_1 > h(\widetilde{x})\}$, see Definition 2.6.1.

Step 1: proof of (i), (ii) and (iii). Define the mapping $\overline{\Phi} : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\overline{\Phi}(y) := (y_1 + h_2(y), \widetilde{y}), \quad \text{where}$$
$$h_2(y) := \int_{\mathbb{R}^{d-1}} h(\widetilde{y} - L^{-1} y_1 \widetilde{z})\varphi(\widetilde{z})\,\mathrm{d}\widetilde{z},$$

for some suitable $L > 0$ to be chosen later. Note that $\overline{\Phi}$ maps $\partial\mathbb{R}^d_+$ to $\partial\mathscr{O}$. It holds that $h_2 \in C^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+)$. The Jacobian matrix of $\overline{\Phi}$ is given by

$$D\overline{\Phi}(y) = \begin{pmatrix} 1 + \partial_{y_1} h_2(y) & (\nabla_{\widetilde{y}} h_2(y))^\top \\ 0 & I_{d-1} \end{pmatrix}, \qquad y \in \mathbb{R}^d. \tag{2.6.3}$$

Since

$$\partial_{y_1} h_2(y) = -L^{-1} \int_{\mathbb{R}^{d-1}} (\nabla h)(\widetilde{y} - L^{-1} y_1 \widetilde{z}) \cdot \widetilde{z} \varphi(\widetilde{z}) \, \mathrm{d}\widetilde{z}, \qquad y \in \mathbb{R}^d,$$

it follows that $|\partial_{y_1} h_2(y)| \le \frac{1}{2}$ for all $y \in \mathbb{R}^d$ if $L = 2[\mathscr{O}]_{C^1}$. Hence, for all $y \in \mathbb{R}^d$, we have $1 + \partial_{y_1} h_2(y) \ge \frac{1}{2}$ and hence

$$|\det D\overline{\Phi}(y)| \ge \tfrac{1}{2} > 0, \tag{2.6.4}$$

meaning that $D\overline{\Phi}$ is invertible.

We construct an inverse of $\overline{\Phi}$ with Hadamard's inverse function theorem [159, Theorem 6.2.8]. Note that if $(y_n)_{n\ge1}$ is a sequence in $\mathbb{R}^d$ such that $|y_n| \to \infty$, then also $|\overline{\Phi}(y_n)| \to \infty$ as $n \to \infty$. Indeed, this follows from the fact that h_2 is bounded. Hence, $\overline{\Phi}$ is proper in the sense of [159, Definition 6.2.2]. This and (2.6.4) imply that all the conditions of [159, Theorem 6.2.8] are satisfied and thus that there exists a continuous inverse $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ of $\overline{\Phi}$. Since actually $\overline{\Phi} \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+;\mathbb{R}^d)$, it also follows that $\Phi \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \partial\mathscr{O};\mathbb{R}^d)$ by the inverse function theorem [159, Theorem 3.3.2]. Moreover, we obtain

$$D\Phi(x) \cdot D\overline{\Phi}(\Phi(x)) = I_d, \qquad x \in \mathbb{R}^d, \tag{2.6.5}$$

see, e.g., [159, Equation (3.37)]. From now on, we will write Φ^{-1} for $\overline{\Phi}$. Moreover, if $x \in \mathscr{O}$ and $y = \Phi(x) \in \mathbb{R}^d_+$, then by the definition of $\Phi^{-1} = \overline{\Phi}$ it holds that

$$\Phi_1(x) = y_1 = x_1 - h_2(y) = x_1 - h_2(\Phi(x)), \tag{2.6.6}$$

and hence we can write $\Phi(x) = (x_1 - h_1(x), \widetilde{x})$, where $h_1(x) = h_2(\Phi(x))$. This completes the proof of (i) and (iii).

To continue, we prove that the diffeomorphism preserves the distance to the boundary. Since both Φ and Φ^{-1} are Lipschitz on $\mathbb{R}^d$, we have

$$|y - y'| \lesssim |\Phi^{-1}(y) - \Phi^{-1}(y')| \lesssim |y - y'|, \qquad y, y' \in \mathbb{R}^d. \tag{2.6.7}$$

Let $x \in \mathscr{O}$ and set $y = \Phi(x) \in \mathbb{R}^d_+$. Then (2.6.7) gives

$$\operatorname{dist}(x, \partial\mathscr{O}) \le |x - (h(\widetilde{x}), \widetilde{x})| = |\Phi^{-1}(y) - \Phi^{-1}(0, \widetilde{y})| \lesssim y_1 = \Phi_1(x) = \operatorname{dist}(\Phi(x), \partial\mathbb{R}^d_+).$$

In addition, let $x' \in \partial\mathscr{O}$ be such that $\operatorname{dist}(x, \partial\mathscr{O}) = |x - x'|$ and set $y' = \Phi(x')$. Note that $y' = (0, \widetilde{y}')$ for some $\widetilde{y}' \in \mathbb{R}^{d-1}$, hence (2.6.7) also implies

$$\operatorname{dist}(\Phi(x), \partial\mathbb{R}^d_+) = \Phi_1(x) = y_1 \le |y - y'| \lesssim |\Phi^{-1}(y) - \Phi^{-1}(y')| = |x - x'| = \operatorname{dist}(x, \partial\mathscr{O}).$$

Thus, we have proved that $\operatorname{dist}(x, \partial\mathscr{O}) \eqsim \operatorname{dist}(\Phi(x), \partial\mathbb{R}^d_+)$, where the implicit constants depend on the Lipschitz constants of Φ and Φ^{-1} and thus on $[\mathscr{O}]_{C^1}$. The other equivalence in (ii) follows by substituting x by $\Phi^{-1}(y)$. This completes the proof of (ii).

Step 2: proof of estimates on h_2 in (iv). Let $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$ and let $\mathscr{O}$ be a special $C_{\mathrm{c}}^{\ell,\kappa}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \le 1$. For multi-indices we write $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$. By the chain rule it holds that

$$\partial_{y_1} h(\widetilde{y} - L^{-1} y_1 \widetilde{z}) = (\nabla h)(\widetilde{y} - L^{-1} y_1 \widetilde{z}) \cdot \frac{-\widetilde{z}}{L} = -L^{-1} \sum_{|\nu|=1} (\partial^\nu h)(\widetilde{y} - L^{-1} y_1 \widetilde{z}) \widetilde{z}^\nu,$$

and by iteration one can check for any $\alpha \in \mathbb{N}_0^d$ that

$$\partial^\alpha h_2(y_1, \widetilde{y}) = \frac{1}{(-L)^{\alpha_1}} \int_{\mathbb{R}^{d-1}} \sum_{|\nu|=\alpha_1} c_{\alpha,\nu} (\partial^{\nu+\widetilde{\alpha}} h)(\widetilde{y} - L^{-1} y_1 \widetilde{z}) \widetilde{z}^\nu \varphi(\widetilde{z}) \, \mathrm{d}\widetilde{z}, \tag{2.6.8}$$

for some constants $c_{\alpha,\nu}$. Take $\ell_0 \in \{0, \dots, \ell\}$. If $|\alpha| \le \ell_0$, it follows from (2.6.8) that

$$|\partial^\alpha h_2(y_1, \widetilde{y})| \le C \|h\|_{C^\ell(\mathbb{R}^{d-1})} \sum_{|\nu|=\alpha_1} \int_{\mathbb{R}^{d-1}} |\widetilde{z}^\nu \varphi(\widetilde{z})| \, \mathrm{d}\widetilde{z} \le C [\mathscr{O}]_{C^{\ell,\kappa}},$$

which proves the estimates for h_2 in (iv). Now let $|\alpha| \ge \ell_0 + 1$ and let $\beta, \overline{\beta} \in \mathbb{N}_0^d$ be such that $\beta + \overline{\beta} = \alpha$ with $|\beta| = \ell_0$ and $|\overline{\beta}| = |\alpha| - \ell_0$. From (2.6.8) and a substitution $\widetilde{z} \mapsto ((\widetilde{y} - \widetilde{z})L)/y_1$ it follows that

$$\partial^\beta h_2(y_1, \widetilde{y}) = \frac{1}{(-L)^{\beta_1}} \Big(\frac{y_1}{L}\Big)^{1-d} \int_{\mathbb{R}^{d-1}} \sum_{|\nu|=\beta_1} c_{\beta,\nu} (\partial^{\nu+\widetilde{\beta}} h)(\widetilde{z}) \Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big)^\nu \varphi\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big) \, \mathrm{d}\widetilde{z}. \tag{2.6.9}$$

By computing the $\overline{\beta}$-derivatives using (2.6.9), we claim that

$$\begin{aligned} \partial^\alpha h_2(y_1, \widetilde{y}) &= \partial^{\overline{\beta}} \partial^\beta h_2(y_1, \widetilde{y}) \\ &= C \frac{1}{y_1^{|\alpha|-\ell_0}} \Big(\frac{y_1}{L}\Big)^{1-d} \int_{\mathbb{R}^{d-1}} \sum_{|\nu|=\beta_1} c_{\beta,\nu} (\partial^{\nu+\widetilde{\beta}} h)(\widetilde{z}) \varphi_{\beta,\overline{\beta},\nu}\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big) \, \mathrm{d}\widetilde{z}, \end{aligned} \tag{2.6.10}$$

where $\varphi_{\beta,\overline{\beta},\nu} \in C_{\mathrm{c}}^\infty(\mathbb{R}^{d-1})$ and $\int \varphi_{\beta,\overline{\beta},\nu}(\widetilde{z}) \, \mathrm{d}\widetilde{z} = 0$. Indeed, if $\overline{\beta} = e_j$ is the j-th unit vector for some $j \in \{2, \dots, d\}$, then by writing $\widetilde{y} = (y_2, \dots, y_d)$ and $\widetilde{z} = (z_2, \dots, z_d)$, a calculation shows that

$$\begin{aligned} \partial_{y_j} \Big[\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big)^\nu \varphi\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big) \Big] &= \frac{L}{y_1} \Big[\nu_j \Big(\frac{(y_j - z_j)L}{y_1}\Big)^{\nu_j - 1} \prod_{\substack{n=2 \\ n \ne j}}^d \Big(\frac{(y_n - z_n)L}{y_1}\Big)^{\nu_n} \varphi\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big) \\ &\qquad + \Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big)^\nu (\partial_j \varphi)\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big) \Big] \\ &=: y_1^{-1} \varphi_{\beta,e_j,\nu}\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big). \end{aligned}$$

Moreover, note that

$$\begin{aligned} \int_{\mathbb{R}^{d-1}} \varphi_{\beta,e_j,\nu}(\widetilde{z}) \, \mathrm{d}\widetilde{z} &= \Big(\frac{y_1}{L}\Big)^{1-d} y_1 \partial_{y_j} \int_{\mathbb{R}^{d-1}} \Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big)^\nu \varphi\Big(\frac{(\widetilde{y} - \widetilde{z})L}{y_1}\Big) \, \mathrm{d}\widetilde{z} \\ &= y_1 \partial_{y_j} \int_{\mathbb{R}^{d-1}} \widetilde{z}^\nu \varphi(\widetilde{z}) \, \mathrm{d}\widetilde{z} = 0, \end{aligned} \tag{2.6.11}$$

and clearly we have $\varphi_{\beta,e_j,\nu} \in C_c^\infty(\mathbb{R}^{d-1})$. This shows (2.6.10) for $\overline{\beta} = e_j$ with $j \in \{2,\dots,d\}$. If $\overline{\beta} = e_1$, then a calculation shows that

$$\begin{aligned}
&\partial_{y_1}\Big[\Big(\frac{y_1}{L}\Big)^{1-d}\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)^{\nu}\varphi\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)\Big] \\
&\qquad = \frac{1}{y_1}\Big(\frac{y_1}{L}\Big)^{1-d}\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)^{\nu}\Big[(1-d-\beta_1)\varphi\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big) \\
&\qquad\quad - (\nabla\varphi)\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)\cdot\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)\Big] \\
&\qquad =: y_1^{-1}\Big(\frac{y_1}{L}\Big)^{1-d}\varphi_{\beta,e_1,\nu}\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big).
\end{aligned}$$

The properties of $\varphi_{\beta,e_1,\nu}$ follow similarly as in (2.6.11). Therefore, we have proved (2.6.10) for $|\overline{\beta}| = 1$. For $|\overline{\beta}| \geq 2$ we can argue by induction to show that

$$\partial_y^{\overline{\beta}}\Big[\Big(\frac{y_1}{L}\Big)^{1-d}\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)^{\nu}\varphi\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)\Big] = C\frac{1}{y_1^{|\alpha|-\ell_0}}\Big(\frac{y_1}{L}\Big)^{1-d}\varphi_{\beta,\overline{\beta},\nu}\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big).$$

This follows in the same manner as for $|\overline{\beta}| = 1$ by considering the ∂_{y_1} and ∂_{y_j} separately. Therefore, (2.6.10) follows.

Let $\kappa_0 \in [0,\kappa]$. Performing the substitution $\widetilde{z} \mapsto \widetilde{y} - L^{-1}y_1\widetilde{z}$ in (2.6.10) and using that $\varphi_{\beta,\overline{\beta},\nu}$ integrates to zero gives

$$\begin{aligned}
|\partial^\alpha h_2(y)| &\leq C\, y_1^{-(|\alpha|-\ell_0)}\int_{\mathbb{R}^{d-1}}\sum_{|\nu|=\beta_1}\big|(\partial^{\nu+\widetilde{\beta}}h)(\widetilde{y}-L^{-1}y_1\widetilde{z}) - (\partial^{\nu+\widetilde{\beta}}h)(\widetilde{y})\big|\,|\varphi_{\beta,\overline{\beta},\nu}(\widetilde{z})|\,\mathrm{d}\widetilde{z} \\
&\leq C\,\|h\|_{C^{\ell,\kappa_0}(\mathbb{R}^{d-1})}\,y_1^{-(|\alpha|-\ell_0)}\int_{\mathbb{R}^{d-1}}|L^{-1}y_1\widetilde{z}|^{\kappa_0}|\varphi_{\beta,\overline{\beta},\nu}(\widetilde{z})|\,\mathrm{d}\widetilde{z} \\
&\leq C\,[\mathscr{O}]_{C^{\ell,\kappa}}\,y_1^{-(|\alpha|-\ell_0-\kappa_0)}.
\end{aligned}$$

This completes the proof of all the estimates for h_2 in (iv).

Step 3: proof of estimates on h_1 in (iv). It remains to prove the estimates on $\partial^\alpha h_1$ for $\alpha \in \mathbb{N}_0^d$. As h_2 is bounded by Step 2, we find that $h_1(x) = h_2(\Phi(x))$ (see (2.6.6)) is bounded as well, which proves the required estimate for $\alpha = 0$. For $|\alpha| = 1$ we first show that $\|D\Phi\|$ is bounded. Note that by (2.6.5) we have

$$\|D\Phi(x)\| = \|[(D\Phi^{-1})(\Phi(x))]^{-1}\| = \frac{\|\operatorname{adj}[(D\Phi^{-1})(\Phi(x))]\|}{|\det[(D\Phi^{-1})(\Phi(x))]|}, \qquad x \in \mathscr{O},$$

where $\operatorname{adj} A$ denotes the adjugate matrix of the matrix A, i.e., the transpose of the cofactor matrix. Recall that the adjugate matrix of A only consists of polynomials of entries of A. Hence, combining this with (2.6.4) and the fact that the entries of $D\Phi^{-1}$ are bounded by Step 2, we find that

$$\|D\Phi(x)\| \leq C, \qquad x \in \mathscr{O}. \tag{2.6.12}$$

Now the estimate on $\partial^\alpha h_1$ with $|\alpha| = 1$ follows again from $h_1(x) = h_2(\Phi(x))$ together with the estimates on h_2 from Step 2, $\operatorname{dist}(\Phi(x),\partial\mathbb{R}^d_+) \eqsim \operatorname{dist}(x,\partial\mathscr{O})$ and (2.6.12).

For the general case, we proceed by induction on $|\alpha| \geq 1$. We prove that

$$|\operatorname{dist}(x,\partial\mathscr{O})|^{(|\alpha|-\ell_0-\kappa_0)_+}\|D^{|\alpha|}h_1(x)\| \leq C\,[\mathscr{O}]_{C^{\ell,\kappa}}, \qquad x\in\mathscr{O}, \tag{2.6.13}$$

for all $\ell_0 \in \{0,\dots,\ell\}$ and $\kappa_0 \in [0,\kappa]$. Recall that for $f:\mathbb{R}^d \supseteq U \to \mathbb{R}^d$, $x \in U$ and $n \in \mathbb{N}_1$, the n-th order Fréchet derivative $D^n f(x)$ is an n-linear mapping from $(\mathbb{R}^d)^n$ to $\mathbb{R}^d$, i.e., for $(\xi_1,\dots,\xi_n) \in (\mathbb{R}^d)^n$ we have $D^n f(x)(\xi_1,\dots,\xi_n) \in \mathbb{R}^d$.

The statement (2.6.13) for $|\alpha| = 1$ is proved above. Let $m \geq 1$ and assume that (2.6.13) holds for all $|\alpha| \leq m$, $\ell_0 \in \{0,\dots,\ell\}$ and $\kappa_0 \in [0,\kappa]$. It remains to prove (2.6.13) for $|\alpha| = m+1$.

By taking derivatives of the formula (2.6.5), isolating the highest-order derivatives on Φ and applying the multivariate Faà di Bruno's formula, we obtain the estimate (see [43, Lemma 4])

$$\|D^{m+1}\Phi(x)\| \lesssim \|D\Phi(x)\|\cdot\sum_{j=2}^{m+1}\|(D^j\Phi^{-1})(\Phi(x))\|\cdot\sum_{\beta}\prod_{r=1}^{m}\|D^r\Phi(x)\|^{\beta_r}, \qquad x\in\mathscr{O}, \tag{2.6.14}$$

where $\beta \in \mathbb{N}_0^m$ is such that

$$\sum_{r=1}^{m}\beta_r = j \quad \text{and} \quad \sum_{r=1}^{m} r\beta_r = m+1. \tag{2.6.15}$$

Note that $\|D\Phi(x)\|$ is uniformly bounded by (2.6.12) and that for $n \geq 2$ we have $\|D^n h_1\| = \|D^n\Phi\|$ and $\|D^n h_2\| = \|D^n\Phi^{-1}\|$. Therefore, multiplying (2.6.14) by

$$|\operatorname{dist}(x,\partial\mathscr{O})|^{(m+1-\ell_0-\kappa_0)_+},$$

it suffices to show the uniform boundedness of

$$|\operatorname{dist}(x,\partial\mathscr{O})|^{(m+1-\ell_0-\kappa_0)_+}\|(D^j h_2)(\Phi(x))\|\prod_{r=2}^{m}\|D^r h_1(x)\|^{\beta_r}, \tag{2.6.16}$$

for $j \in \{2,\dots,m+1\}$ and β such that (2.6.15) holds. Suppose that there exist $\mu,\mu_2,\dots,\mu_m \in [0,\infty)$ such that

$$\begin{aligned} &(j-\ell-\kappa)_+ \leq \mu \leq j \\ &(r-\ell-\kappa)_+ \leq \mu_r \leq r, \qquad r\in\{2,\dots,m\} \end{aligned} \tag{2.6.17}$$

and

$$\mu + \sum_{r=2}^{m}\beta_r\mu_r = (m+1-\ell_0-\kappa_0)_+. \tag{2.6.18}$$

Then, (2.6.16) can be estimated as

$$|\operatorname{dist}(x,\partial\mathscr{O})|^{\mu}\|(D^j h_2)(\Phi(x))\|\prod_{r=2}^{m}\big(|\operatorname{dist}(x,\partial\mathscr{O})|^{\mu_r}\|D^r h_1(x)\|\big)^{\beta_r} \leq C\,[\mathscr{O}]_{C^{\ell,\kappa}}, \tag{2.6.19}$$

where we have applied the induction hypothesis (2.6.13) and the estimates for h_2 from Step 2 together with $\operatorname{dist}(\Phi(x),\partial\mathbb{R}^d_+) \eqsim \operatorname{dist}(x,\partial\mathscr{O})$. It remains to show the existence of μ's satisfying (2.6.17) and (2.6.18). We distinguish several cases.

If $m+1 \le \ell_0$, then $(m+1-\ell_0-\kappa_0)_+ = 0$ and we can take $\mu = \mu_2 = \cdots = \mu_m = 0$. From now on, we assume that $m \ge \ell_0$. If $j \ge \ell_0 + 1$, then we can take

$$\mu = j - \ell_0 - \kappa_0 \quad \text{and} \quad \mu_r = r - 1 \text{ for } r \in \{2, \dots, m\},$$

and (2.6.15) implies that (2.6.18) is satisfied. For the remaining case $j \le \ell_0$, we will not provide the explicit values of the μ's, but only show the existence of the μ's. Taking the largest possible μ's in (2.6.17) gives

$$\mu + \sum_{r=2}^{m} \beta_r \mu_r = j + \sum_{r=2}^{m} r\beta_r = j + m + 1 - \beta_1 \ge m + 1 - \ell_0 - \kappa_0,$$

where we have used (2.6.15) and $\beta_1 \le j$. We will now take the smallest possible μ's in (2.6.17). First assume that $m \ge \ell + 1$ and that there exists an $\widetilde{r} \in \{\ell+1, \dots, m\}$ such that $\beta_{\widetilde{r}} \ge 1$. In this case, we have

$$\begin{aligned}
\mu + \sum_{r=2}^{m} \beta_r \mu_r &= (j - \ell - \kappa)_+ + \sum_{r=2}^{m} (r - \ell - \kappa)_+ \beta_r = \sum_{r=\ell+1}^{m} (r - \ell - \kappa)\beta_r \\
&\le \sum_{r=1}^{m} r\beta_r - (\ell + \kappa) \sum_{r=\ell+1}^{m} \beta_r = m + 1 - (\ell + \kappa) \sum_{r=\ell+1}^{m} \beta_r \\
&\le m + 1 - \ell - \kappa \le m + 1 - \ell_0 - \kappa_0,
\end{aligned}$$

where we have used $(j - \ell - \kappa)_+ \le (j - \ell_0 - \kappa_0)_+ = 0$ and (2.6.15). If $m \le \ell$ or $\beta_r = 0$ for all $r \in \{\ell+1, \dots, m\}$, then $\mu + \sum_{r=2}^{m} \beta_r \mu_r = 0 \le m + 1 - \ell_0 - \kappa_0$. The existence of the μ's shows that (2.6.19) holds. This finishes the induction.

Finally, we remark that the estimates on h_1 and h_2 imply that if $h \in C_{\mathrm{c}}^{\ell,\kappa}(\mathbb{R}^{d-1})$, then $\Phi \in C^{\ell,\kappa}(\overline{\mathscr{O}}; \mathbb{R}^d)$ and $\Phi^{-1} \in C^{\ell,\kappa}(\overline{\mathbb{R}^d_+}; \mathbb{R}^d)$. This proves that Φ is a $C^{\ell,\kappa}$-diffeomorphism and this finishes the proof of (iv). □

To prepare for the proof of Lemma 2.6.6, we prove the following elementary lemma.

Lemma 2.6.8. *Let A and B be $d \times d$-matrices. If $\det(A) \neq 0$ and $\|A\|^{d-1}\|B\| \cdot |\det(A)|^{-1} < 1$, then we have*

$$|\det(A+B)| \ge |\det(A)| \Big(1 - \frac{\|A\|^{d-1}\|B\|}{|\det(A)|}\Big)^d.$$

Proof. Let $0 < \sigma_1 \le \cdots \le \sigma_d$ be the singular values of A. Then we have

$$\|A^{-1}\| = \frac{1}{\sigma_1} = \frac{\prod_{j=2}^{d} \sigma_j}{|\det(A)|} \le \frac{\|A\|^{d-1}}{|\det(A)|}.$$

In particular, we have $\|A^{-1}B\| \le \|A^{-1}\|\|B\| \le \|A\|^{d-1}\|B\| \cdot |\det(A)|^{-1} < 1$. Now let $\kappa_1, \dots, \kappa_d$ be the eigenvalues of $A^{-1}B$. Then $|\kappa_j| \le \|A^{-1}B\| < 1$ for all $j \in \{1, \dots, d\}$ and

$$|\det(I_d + A^{-1}B)| = \prod_{j=1}^{d} |1 + \kappa_j| \ge \prod_{j=1}^{d} (1 - |\kappa_j|) \ge (1 - \|A^{-1}B\|)^d,$$

which, combined with the norm estimate for $\|A^{-1}\|$, yields

$$\begin{aligned}|\det(A+B)| &= |\det(A)||\det(I_d + A^{-1}B)| \\ &\geq |\det(A)|(1-\|A^{-1}B\|)^d \geq |\det(A)|\Big(1-\frac{\|A\|^{d-1}\|B\|}{|\det(A)|}\Big)^d,\end{aligned}$$

finishing the proof. □

To conclude this appendix, we give the proof of Lemma 2.6.6.

Proof of Lemma 2.6.6. Let $\varphi \in C_c^\infty(\mathbb{R}^{d-1})$ be a non-negative and radially symmetric function with its support in the unit ball such that $\int_{\mathbb{R}^{d-1}} \varphi(\widetilde{x})\,\mathrm{d}\widetilde{x} = 1$. Define $h \in C_c^1(\mathbb{R}^{d-1})$ such that $\mathscr{O} = \{x \in \mathbb{R}^d : x_1 > h(\widetilde{x})\}$, see Definition 2.6.1. Furthermore, let

$$\nu(x) = \big(1, \widetilde{\nu}(\widetilde{x})\big)^\top = \big(1, -\nabla_{\widetilde{x}} h(\widetilde{x})\big)^\top, \qquad x \in \mathbb{R}^d, \tag{2.6.20}$$

be an inward pointing normal vector at $\partial\mathscr{O}$.

Step 1: proof of (i), (ii) and (iii). Define the mapping $\overline{\Psi} : \mathbb{R}^d \to \mathbb{R}^d$ given by

$$\begin{aligned}\overline{\Psi}(y) &:= (y_1 + h_2(y), \widetilde{y} + \widetilde{h}_2(y)), \quad \text{where} \\ h_2(y) &:= \int_{\mathbb{R}^{d-1}} h(\widetilde{y} - L^{-1}y_1\widetilde{z})\varphi(\widetilde{z})\,\mathrm{d}\widetilde{z}, \\ \widetilde{h}_2(y) &:= y_1\widetilde{\mathscr{V}}(y) := y_1\int_{\mathbb{R}^{d-1}} \widetilde{\nu}(\widetilde{y} - L^{-1}y_1\widetilde{z})\varphi(\widetilde{z})\,\mathrm{d}\widetilde{z},\end{aligned}$$

for some suitable $L > 0$ to be chosen later. Note that $\overline{\Psi}$ maps $\partial\mathbb{R}^d_+$ to $\partial\mathscr{O}$. It holds that $h_2 \in C^1(\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+)$. Furthermore, we claim that $\widetilde{h}_2 \in C^1(\mathbb{R}^d;\mathbb{R}^{d-1}) \cap C^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+;\mathbb{R}^{d-1})$. Indeed, it is clear that $\widetilde{h}_2 \in C(\mathbb{R}^d;\mathbb{R}^{d-1}) \cap C^\infty(\mathbb{R}^d \setminus \partial\mathbb{R}^d_+;\mathbb{R}^{d-1})$ since $\widetilde{\nu} = -\nabla_{\widetilde{x}} h \in C(\mathbb{R}^{d-1};\mathbb{R}^{d-1})$. Note that

$$\partial_1\widetilde{h}_2(0,\widetilde{a}) = \lim_{y_1\to 0}\frac{y_1\widetilde{\mathscr{V}}((y_1,\widetilde{a}))}{y_1} = \widetilde{\mathscr{V}}(0,\widetilde{a}) = \widetilde{\nu}(\widetilde{a}), \qquad \widetilde{a} \in \mathbb{R}^{d-1},$$

and all the tangential partial derivatives of $\widetilde{h}_2$ at $\partial\mathbb{R}^d_+$ are zero. If $y \in \mathbb{R}^d \setminus \partial\mathbb{R}^d_+$ and $|\alpha| = 1$, then by a similar computation as (2.6.10) (with $|\overline{\beta}| = 1$, $\beta = 0$, $\ell_0 = 0$ and h replaced by $\widetilde{\nu}$) and a substitution $\widetilde{z} \mapsto \widetilde{y} - L^{-1}y_1\widetilde{z}$, we obtain

$$\begin{aligned}\partial^\alpha\widetilde{\mathscr{V}}(y) &= y_1^{-1}\Big(\frac{y_1}{L}\Big)^{1-d}\int_{\mathbb{R}^{d-1}} \widetilde{\nu}(\widetilde{z})\varphi_{0,\alpha,0}\Big(\frac{(\widetilde{y}-\widetilde{z})L}{y_1}\Big)\,\mathrm{d}\widetilde{z} \\ &= y_1^{-1}\int_{\mathbb{R}^{d-1}} \widetilde{\nu}(\widetilde{y} - L^{-1}y_1\widetilde{z})\varphi_{0,\alpha,0}(\widetilde{z})\,\mathrm{d}\widetilde{z},\end{aligned}$$

where $\varphi_{0,\alpha,0} \in C_c^\infty(\mathbb{R}^{d-1})$ and $\int\varphi_{0,\alpha,0}(\widetilde{z})\,\mathrm{d}\widetilde{z} = 0$. Let $y = (y_1,\widetilde{y}) \in \mathbb{R}^d \setminus \partial\mathbb{R}^d_+$, then we obtain

$$\begin{aligned}\partial_{y_1}\widetilde{h}_2(y) &= \widetilde{\mathscr{V}}(y) + y_1\partial_{y_1}\widetilde{\mathscr{V}}(y) = \int_{\mathbb{R}^{d-1}} \widetilde{\nu}(\widetilde{y} - L^{-1}y_1\widetilde{z})\big[\varphi(\widetilde{z}) + \varphi_{0,e_1,0}(\widetilde{z})\big]\,\mathrm{d}\widetilde{z}, \\ \partial_{y_j}\widetilde{h}_2(y) &= y_1\partial_{y_j}\widetilde{\mathscr{V}}(y) = \int_{\mathbb{R}^{d-1}} \widetilde{\nu}(\widetilde{y} - L^{-1}y_1\widetilde{z})\varphi_{0,e_j,0}(\widetilde{z})\,\mathrm{d}\widetilde{z}, \qquad j \in \{2,\dots,d\}.\end{aligned} \tag{2.6.21}$$

By applying the dominated convergence theorem and the properties of φ and $\varphi_{0,e_j,0}$ for $j \in \{1,\dots,d\}$, we find that for all $\widetilde{a} \in \mathbb{R}^{d-1}$

$$\lim_{y\to(0,\widetilde{a})} \partial_{y_1} \widetilde{h}_2(y) = \widetilde{\nu}(\widetilde{a}) \quad \text{and} \quad \lim_{y\to(0,\widetilde{a})} \partial_{y_j} \widetilde{h}_2(y) = 0, \quad j \in \{2,\dots,d\}.$$

Hence, all the partial derivatives of $\widetilde{h}_2$ at $\partial\mathbb{R}^d_+$ exist and are continuous. Therefore, $\widetilde{h}_2 \in C^1(\mathbb{R}^d;\mathbb{R}^{d-1})$ and we have proved the claim.

The Jacobian matrix of $\overline{\Psi}$ is given by

$$\begin{aligned} D\overline{\Psi}(y) &= \begin{pmatrix} 1+\partial_{y_1} h_2(y) & (\nabla_{\widetilde{y}} h_2(y))^\top \\ \partial_{y_1}\big(y_1 \widetilde{\mathscr{V}}(y)\big) & I_{d-1} + y_1 D_{\widetilde{y}} \widetilde{\mathscr{V}}(y) \end{pmatrix} \\ &= D\overline{\Phi}(y) + \begin{pmatrix} 0 & 0 \\ \partial_{y_1}\big(y_1 \widetilde{\mathscr{V}}(y)\big) & y_1 D_{\widetilde{y}} \widetilde{\mathscr{V}}(y) \end{pmatrix} =: D\overline{\Phi}(y) + P(y), \quad y \in \mathbb{R}^d, \end{aligned} \tag{2.6.22}$$

where $\overline{\Phi}$ is as defined in the proof of Lemma 2.6.5 (see (2.6.3)) and P is a perturbation. To show that the mapping $\overline{\Psi}$ is invertible, we make use of Lemma 2.6.8 applied to $A = D\overline{\Phi}$ and $B = P$. Note that (2.6.21) implies $\sup_{y\in\mathbb{R}^d} \|P(y)\| \le C[\mathscr{O}]_{C^1}$. Furthermore, we recall from Lemma 2.6.5(iv) and the definition of $\overline{\Phi}$ that $\sup_{y\in\mathbb{R}^d} \|D\overline{\Phi}(y)\| \le C$. From Lemma 2.6.8, (2.6.4) and the aforementioned estimates, we obtain that there exists a $\Lambda \in (0,1)$ such that if $[\mathscr{O}]_{C^1} \le \Lambda$, then

$$\begin{aligned} \inf_{y\in\mathbb{R}^d} |\det D\overline{\Psi}(y)| &\ge \inf_{y\in\mathbb{R}^d} |\det D\overline{\Phi}(y)| \Big(1 - \frac{\|D\overline{\Phi}(y)\|^{d-1}\|P(y)\|}{|\det D\overline{\Phi}(y)|}\Big)^d \\ &\ge c(1 - C[\mathscr{O}]_{C^1})^d \ge c(1-C\Lambda)^d > 0, \end{aligned}$$

for some $C, c > 0$.

Using Hadamard's inverse function theorem, we can argue similarly as in the proof of Lemma 2.6.5 to obtain an inverse $\Psi \in C^1(\mathbb{R}^d;\mathbb{R}^d) \cap C^\infty(\mathbb{R}^d \setminus \partial\mathscr{O};\mathbb{R}^d)$ to $\overline{\Psi}$. We will write Ψ^{-1} for $\overline{\Psi}$.

Moreover, if $x \in \mathscr{O}$ and $y = \Psi(x) \in \mathbb{R}^d_+$, then by the definition of $\Psi^{-1} = \overline{\Psi}$ it holds that

$$\begin{aligned} \Psi_1(x) &= y_1 = x_1 - h_2(y) = x_1 - h_2(\Psi(x)), \\ \widetilde{\Psi}(x) &= \widetilde{y} = \widetilde{x} - \widetilde{h}_2(y) = \widetilde{x} - \widetilde{h}_2(\Psi(x)), \end{aligned} \tag{2.6.23}$$

and hence we can write $\Psi(x) = (x_1 - h_1(x), \widetilde{x} - \widetilde{h}_1(x))$ with $h_1(x) = h_2(\Psi(x))$ and $\widetilde{h}_1(x) = \widetilde{h}_2(\Psi(x))$. This completes the proof of (i) and (iii). Statement (ii) follows similarly as in the proof of Lemma 2.6.5(ii).

Step 2: proof of (iv). We prove that the diffeomorphism Ψ is admissible for any $x \in \partial\mathscr{O}$. First, note that

$$(\partial_{y_1} h_2(y))|_{y_1=0} = -L^{-1}(\nabla h)(\widetilde{y}) \cdot \int_{\mathbb{R}^{d-1}} \widetilde{z}\varphi(\widetilde{z})\,\mathrm{d}\widetilde{z} = 0, \quad y \in \mathbb{R}^d,$$

where we have used the radial symmetry of φ to obtain that the latter integral vanishes, i.e., $\int_{\mathbb{R}} z_i \varphi(\widetilde{z})\,\mathrm{d}z_i = 0$ for all $i \in \{2,\dots,d\}$. Together with (2.6.20), we see that the push-forward

of $\overline{\Psi} = \Psi^{-1}$ (recall (2.6.22)) at the boundary $\partial\mathbb{R}^d_+$ is given by

$$D\overline{\Psi}(0,\widetilde{y}) = \begin{pmatrix} 1 & (\nabla_{\widetilde{y}} h(\widetilde{y}))^\top \\ \widetilde{\mathscr{V}}(0,\widetilde{y}) & I_{d-1} \end{pmatrix} = \begin{pmatrix} 1 & (\nabla_{\widetilde{y}} h(\widetilde{y}))^\top \\ -\nabla_{\widetilde{y}} h(\widetilde{y}) & I_{d-1} \end{pmatrix}, \qquad \widetilde{y} \in \mathbb{R}^{d-1}.$$

It is clear that for all $\widetilde{y} \in \mathbb{R}^{d-1}$, the push-forward $D\overline{\Psi}(0,\widetilde{y})$ maps the normal vector $(1,0)$ at $\partial\mathbb{R}^d_+$ to the normal vector $\nu(x)$ at $\partial\mathscr{O}$. Similarly, any tangent vector at $\partial\mathbb{R}^d_+$ is mapped to the tangent space at $\partial\mathscr{O}$. Moreover, note that the push-forward does not preserve the length of the vectors. This proves that $\overline{\Psi} = \Psi^{-1}$ is admissible for any $y \in \partial\mathbb{R}^d_+$, hence Ψ is admissible for any $x \in \partial\mathscr{O}$ as well.

Step 3: proof of (v). Let $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$ and let $\mathscr{O}$ be a special $C^{\ell,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \leq \Lambda$. Note that the estimates on the derivatives of h_2 follow immediately from Lemma 2.6.5(iv). Let $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$, $\ell_0 \in \{0,\dots,\ell\}$ and $\kappa_0 \in [0,\kappa]$. For the estimate on $\partial^\alpha \widetilde{h}_2$, we find with the product rule

$$|\partial^\alpha \widetilde{h}_2(y)| \lesssim |y_1 \partial^\alpha \widetilde{\mathscr{V}}(y)| + |\partial^{(\alpha_1-1,\widetilde{\alpha})} \widetilde{\mathscr{V}}(y)|, \quad y \in \mathbb{R}^d_+, \tag{2.6.24}$$

where the latter term is only present if $\alpha_1 \geq 1$. Since $\widetilde{\mathscr{V}}$ is obtained by mollifying $\widetilde{\nu} = -\nabla h$, we can redo Step 2 in the proof of Lemma 2.6.5 but with h replaced by $-\nabla h$, to obtain

$$|\partial^\beta \widetilde{\mathscr{V}}(y)| \leq C\,[\mathscr{O}]_{C^{\ell,\kappa}}\, y_1^{-(|\beta|+1-\ell_0-\kappa_0)_+}, \qquad y \in \mathbb{R}^d_+,\, \beta \in \mathbb{N}_0^d.$$

Applying this estimate to (2.6.24) gives the desired estimates for $\widetilde{h}_2$.

It remains to prove the estimates on h_1 and $\widetilde{h}_1$, which are defined as below (2.6.23). Hence, as h_2 and $\widetilde{h}_2$ are bounded, we find that h_1 and $\widetilde{h}_1$ are bounded as well. To prove the estimates on the derivatives $\partial^\alpha h_1$ and $\partial^\alpha \widetilde{h}_1$ for $|\alpha| \geq 1$, one can proceed by induction similar to Step 3 in the proof of Lemma 2.6.5. □

I

Function space theory

3

Complex interpolation of weighted Sobolev spaces

This chapter is based on:

F. Roodenburg. Complex interpolation of power-weighted Sobolev spaces with boundary conditions. To appear in Stud. Math., 2026.

Abstract. *We characterise the complex interpolation spaces of weighted vector-valued Sobolev spaces with and without boundary conditions on the half-space and on smooth bounded domains. The weights we consider are power weights that measure the distance to the boundary and do not necessarily belong to the class of Muckenhoupt A_p weights. First, we determine the higher-order trace spaces for weighted vector-valued Besov, Triebel–Lizorkin, Bessel potential and Sobolev spaces. This allows us to derive a trace theorem for boundary operators and to interpolate spaces with boundary conditions. Furthermore, we derive density results for weighted Sobolev spaces with boundary conditions.*

3.1. Introduction

In this chapter, we characterise complex interpolation spaces of weighted Sobolev spaces on a domain $\mathscr{O}$ with power weights $w_\gamma^{\partial\mathscr{O}}(x) := \operatorname{dist}(x,\partial\mathscr{O})^\gamma$ for $\gamma > -1$ measuring the distance to the boundary. The motivation for this stems from the study of partial differential equations (PDEs) and is twofold.

First, it is well known that complex interpolation serves as an essential tool for studying differential operators and evolution equations, see, e.g., the monographs [14, 16, 126, 198, 234, 278]. For example, if a sectorial operator A on a Banach space X has bounded imaginary powers, then the domain of A^θ for $\theta \in (0,1)$ is the complex interpolation space $[X, D(A)]_\theta$, see [250]. Typically, $D(A)$ is a Sobolev space on a spatial domain that incorporates the boundary conditions of the boundary value problem. Thus, with the aid of complex interpolation, we can identify domains of fractional powers, which in turn play an important role for perturbation techniques in the theory of maximal regularity, see, e.g., [65, 134, 173]. Complex interpolation of spaces with boundary conditions was studied by Grisvard and Seeley in [108, 109, 250, 251] and, due to the numerous applications, the topic is now widely addressed in the literature, see, e.g., [16, 25, 27, 107, 114, 262, 278] and the references therein.

Secondly, solutions to PDEs on a domain $\mathscr{O} \subseteq \mathbb{R}^d$ may exhibit blow-up behaviour near the boundary of $\mathscr{O}$. Using weighted spaces with weights of the form $w_\gamma^{\partial\mathscr{O}}(x) := \operatorname{dist}(x,\partial\mathscr{O})^\gamma$ for some suitable value of γ, one can nevertheless study problems which may be ill-posed in unweighted spaces. Moreover, with weighted spaces certain compatibility conditions needed in the unweighted case can be avoided. Many authors have employed weighted spaces to solve (stochastic) PDEs, see, e.g., [80, 149, 157, 164, 165, 167, 201]. In most of these works, homogeneous weighted Sobolev spaces are used, which are known to form a complex interpolation scale, see [197, Proposition 2.4]. In [190] an alternative approach to solving PDEs on the half-space is presented via the H^∞-calculus for the Laplacian on inhomogeneous Sobolev spaces with weights outside the Muckenhoupt A_p class. We will further investigate this H^∞-calculus and its consequences in later chapters. To this end, it is crucial to have access to complex interpolation for these weighted Sobolev spaces. In particular, in Chapter 6, domains of fractional powers and perturbation techniques will be used to obtain the H^∞-calculus for the Laplacian on bounded $C^{1,\kappa}$-domains for $\kappa \in [0,1]$ depending on γ.

For $m \in \mathbb{N}_0$ we denote by $\mathscr{B}$ an *m-th order normal boundary operator* at $\partial\mathbb{R}^d_+ = \{(0,\widetilde{x}) : \widetilde{x} \in \mathbb{R}^{d-1}\}$. Consider the power weight $w_\gamma(x) := \operatorname{dist}(x,\partial\mathbb{R}^d_+)^\gamma = |x_1|^\gamma$ for $x = (x_1,\widetilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$ and let $W^{k,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma)$ be the closed subspace of functions $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma)$ such that $\mathscr{B} f = 0$, whenever the traces exist. We refer to Sections 3.5 and 3.6 for the precise definitions. Our main result includes the following characterisation of complex interpolation spaces (see Section 3.6.2).

Theorem 3.1.1. *Let $p \in (1,\infty)$, $k \in \{2,3,\dots\}$ and $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$. Then for $\ell \in \{1,\dots,k-1\}$ we have*

$$\big[L^p(\mathbb{R}^d_+, w_\gamma), W^{k,p}(\mathbb{R}^d_+, w_\gamma)\big]_{\frac{\ell}{k}} = W^{\ell,p}(\mathbb{R}^d_+, w_\gamma), \tag{3.1.1a}$$

$$\big[L^p(\mathbb{R}^d_+, w_\gamma), W^{k,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma)\big]_{\frac{\ell}{k}} = W^{\ell,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma). \tag{3.1.1b}$$

Moreover, by localisation, the results also hold for smooth bounded domains.

The main novelty of Theorem 3.1.1, compared to the existing literature, is the range of weights w_γ with $\gamma > -1$. Unfortunately, the proof in [16] for the unweighted case $\gamma = 0$ does not carry over immediately to the weighted setting. In the unweighted case, the result (3.1.1a) is well known and is used to prove (3.1.1b) in [16]. In the weighted setting, we can only prove the inclusion "$\hookrightarrow$" in (3.1.1a) directly with the aid of Wolff interpolation [276] (see Proposition 3.6.3). Nevertheless, this embedding is sufficient to prove (3.1.1b) using the characterisation of the trace spaces (see Theorem 3.1.2) and similar arguments as in [16]. Then, the converse inclusion "$\hookleftarrow$" in (3.1.1a) also follows. Furthermore, we make the following remarks concerning Theorem 3.1.1.

(i) The class of normal boundary operators is considered in, e.g., [16, 98, 114, 251] and includes the important cases of Dirichlet and Neumann boundary conditions, but also mixed boundary conditions with space-dependent coefficients are allowed. Actually, we prove Theorem 3.1.1 for systems of normal boundary operators. Differential operators with these types of boundary operators are studied in, e.g., [5, 6, 7, 97, 112, 113, 198].

(ii) If $\gamma \in (-1, p-1)$, i.e., w_γ is a Muckenhoupt A_p weight, then the weighted Bessel potential and Sobolev spaces coincide: $H^{k,p}(\mathbb{R}^d_+, w_\gamma) = W^{k,p}(\mathbb{R}^d_+, w_\gamma)$ for $k \in \mathbb{N}_0$, and $H^{s,p}(\mathbb{R}^d_+, w_\gamma)$ for $s \in \mathbb{R}$ forms a complex interpolation scale, see [189, Proposition 5.5 & 5.6]. In the case $\gamma \in (-1, p-1)$ we obtain Theorem 3.1.1 for Bessel potential spaces with boundary conditions and fractional smoothness as well, see Theorem 3.6.4. This partially extends the result in [16, Theorem VIII.2.4.8].

The interpolation result in Theorem 3.1.1 without boundary conditions is new for $\gamma > p-1$. We refer to [84, 252] for results on complex interpolation of Besov and Triebel–Lizorkin spaces with weights outside the class of A_p weights.

(iii) We can allow for vector-valued weighted Sobolev spaces $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$, where X is a UMD Banach space.

A key ingredient for proving Theorem 3.1.1 is the characterisation of the trace spaces of weighted vector-valued Besov, Triebel–Lizorkin, Bessel potential and Sobolev spaces. The results are summarised in Theorem 3.1.2 (see Sections 3.3 and 3.4.1).

Theorem 3.1.2. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $m \in \mathbb{N}_0$, $\gamma > -1$, $s > m + \frac{\gamma+1}{p}$ and let X be a Banach space. Let $\mathrm{Tr}_m = \mathrm{Tr} \circ \partial_1^m$ be the m-th order trace operator at $\{(0,\widetilde{x}) : \widetilde{x} \in \mathbb{R}^{d-1}\}$. Then*

(i) $\mathrm{Tr}_m \big(B^s_{p,q}(\mathbb{R}^d, w_\gamma; X)\big) = B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$,

(ii) $\mathrm{Tr}_m \big(F^s_{p,q}(\mathbb{R}^d, w_\gamma; X)\big) = B^{s-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$,

(iii) $\mathrm{Tr}_m \big(H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)\big) = B^{s-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$ *if* $\gamma \in (-1, p-1)$,

(iv) $\mathrm{Tr}_m \big(W^{s,p}(\mathbb{R}^d_+, w_\gamma; X)\big) = B^{s-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$ *if* $s \in \mathbb{N}_1$ *and* $\gamma \notin \{jp-1 : j \in \mathbb{N}_1\}$,

where the notation $\mathrm{Tr}_m(\mathscr{A}_1) = \mathscr{A}_2$ *means that* $\mathrm{Tr}_m : \mathscr{A}_1 \to \mathscr{A}_2$ *is continuous and surjective, and that it has a continuous right inverse (the so-called extension operator* ext_m*).*

The innovative part of Theorem 3.1.2 is the extension of the range of weights for Sobolev spaces, which eventually allows us to prove Theorem 3.1.1 for these weights as well. To prove our weighted results, we partially extend the arguments in [16, 245] for the unweighted case. Moreover, we apply a Sobolev embedding for Triebel–Lizorkin spaces to deduce Theorem 3.1.2(ii). This is in contrast to [245], where an atomic approach was used.

In the unweighted scalar-valued setting, i.e., $\gamma = 0$ and $X = \mathbb{C}$, the trace spaces are well known, see [247, 262, 263]. In the unweighted vector-valued setting, the trace spaces are characterised in [16, 245].

As a consequence of Theorem 3.1.2, we can derive a trace theorem for the vector $\overline{\mathrm{Tr}}_m = (\mathrm{Tr}_0, \dots, \mathrm{Tr}_m)$ as well. For weighted scalar-valued Sobolev spaces this is proved in [262, Section 2.9.2] and [140] via different techniques.

Moreover, from Theorem 3.1.2 we derive some density results in Section 3.4.2. In particular, we show that the space of test functions

$$\{f \in C_c^\infty(\overline{\mathbb{R}^d_+}; X) : (\partial_1^m f)|_{\partial\mathbb{R}^d_+} = 0\}$$

is dense in $W^{k,p}_{\mathrm{Tr}_m}(\mathbb{R}^d_+, w_\gamma; X)$. This density result will be needed in Chapter 6 to find trace characterisations for weighted Sobolev spaces on bounded C^1-domains. Application of a standard mollification procedure would result in all traces $\overline{\mathrm{Tr}}_m$ being zero instead of only one trace. We remark that results for mollification with preservation of boundary values are contained in [39, 121].

We comment on some open and related problems. Theorem 3.1.1 only deals with interpolation of the smoothness parameter, while it might be possible to interpolate in other parameters (such as p and γ) as well. Interpolation results for Sobolev spaces with different weights (but without complicated boundary conditions) are contained in [53], [133, Section 5.3], [194] and in Proposition 5.3.8 later on in this dissertation.

For $\gamma > p-1$ we have the result of Theorem 3.1.1 only for spaces with integer smoothness. It would be interesting to see if our results carry over to spaces of fractional smoothness, for instance, by defining $H^{s,p}(\mathbb{R}^d_+, w_\gamma; X) := [L^p(\mathbb{R}^d_+, w_\gamma; X), W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)]_{s/k}$ for some $s \in \mathbb{R}_+ \setminus \mathbb{N}_1$ and $k \in \mathbb{N}_1$ such that $k > s$. It should be noted that for $\gamma > p-1$ one cannot use the standard definition of Bessel potential spaces used for Muckenhoupt weights, i.e., $\gamma \in (-1, p-1)$.

Besides complex interpolation, the real interpolation method plays an important role in the theory of evolution equations as well (see [96, Section 4.4] and [126, Section 17.2.b]). Some related results for real interpolation of weighted spaces are contained in [37, 108, 195, 262]. We note that if we have interpolation results for Besov spaces with boundary conditions, then we can obtain Theorem 3.1.1 for real interpolation, see [16, Theorem VIII.2.4.5]. We expect that our results also hold for Besov spaces with boundary conditions if $\gamma \in (-1, p-1)$. However, for $\gamma \geq p-1$ a more detailed study about interpolation of Besov

spaces and their relation to Sobolev spaces is required. A particular problem is that for $\gamma \geq p-1$ we only have the embedding

$$B^{n+\theta}_{p,q}(\mathbb{R}^d_+, w_\gamma) \hookrightarrow \big(W^{n,p}(\mathbb{R}^d_+, w_\gamma), W^{n+1,p}(\mathbb{R}^d_+, w_\gamma)\big)_{\theta,q}, \quad n \in \mathbb{N}_0, \theta \in (0,1),$$

while equality holds for $\gamma \in (-1, p-1)$, see [190, Section 7]. In particular, for $n = 0$ it is proved in [190, Remark 7.14] that the embedding "$\hookrightarrow$" fails for $\gamma \notin (-1, p-1)$.

Finally, we note that the special values $\gamma = jp-1$ with $j \in \mathbb{N}_1$ are excluded, since in these cases the study of the trace operator is more involved, see [133]. Moreover, we do not have certain trace characterisations available, which play a crucial role in the proof of the main theorem. For $\gamma \leq -1$ we have that $C^\infty_c(\mathbb{R}^d_+)$ is dense in $W^{k,p}(\mathbb{R}^d_+, w_\gamma)$ and thus all traces will be zero.

Outline

The outline of this chapter is as follows. In Section 3.2 we introduce the required function spaces and some of their properties. In Section 3.3 we determine the trace spaces for weighted Besov and Triebel–Lizorkin spaces. In Section 3.4 we extend the results to weighted Bessel potential and Sobolev spaces. As a consequence, we prove some density results. Section 3.5 deals with the characterisation of the trace space for boundary operators. Finally, in Section 3.6 we prove the main result about complex interpolation of weighted Sobolev spaces with and without boundary conditions.

3.2. Weighted function spaces

In this section, we define the required function spaces and prove certain embedding properties.

3.2.1. Weighted function spaces

We recall the following definitions from Chapter 2 about weighted Lebesgue and Sobolev spaces. For $\mathscr{O} \subseteq \mathbb{R}^d$ open, $p \in [1,\infty)$, w a weight and X a Banach space, we define the *weighted Lebesgue space* $L^p(\mathscr{O}, w; X)$, which consists of all equivalence classes of strongly measurable $f : \mathscr{O} \to X$ such that

$$\|f\|_{L^p(\mathscr{O},w;X)} := \Big(\int_{\mathscr{O}} \|f(x)\|^p_X \, w(x) \, \mathrm{d}x\Big)^{1/p} < \infty.$$

Additionally, for $k \in \mathbb{N}_0$ and w a weight such that $w^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(\mathscr{O})$, we define the *weighted Sobolev space* consisting of $f \in L^p(\mathscr{O}, w; X)$ such that

$$\|f\|_{W^{k,p}(\mathscr{O},w;X)} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_{L^p(\mathscr{O},w;X)} < \infty.$$

We continue with the definitions of weighted Besov, Triebel–Lizorkin and Bessel potential spaces. We denote by $\Phi(\mathbb{R}^d)$ the set of *inhomogeneous Littlewood–Paley sequences* consisting of sequences $(\varphi_n)_{n\geq 0} \subseteq \mathscr{S}(\mathbb{R}^d)$ such that $\widehat{\varphi}_0 := \widehat{\varphi}$ and $\widehat{\varphi}_n(\xi) := \widehat{\varphi}(2^{-n}\xi) - \widehat{\varphi}(2^{-n+1}\xi)$

for $\xi \in \mathbb{R}^d$ and $n \in \mathbb{N}_1$, where the generating function φ satisfies $0 \le \widehat{\varphi}(\xi) \le 1$ for $\xi \in \mathbb{R}^d$, $\widehat{\varphi}(\xi) = 1$ for $|\xi| \le 1$ and $\widehat{\varphi}(\xi) = 0$ for $|\xi| \ge \frac{3}{2}$. For $f \in \mathscr{S}'(\mathbb{R}^d; X)$ we use the notation

$$S_n f := \varphi_n * f \quad \text{for } n \in \mathbb{N}_0 \quad \text{and} \quad S_{-1} f = 0, \tag{3.2.1}$$

where $(\varphi_n)_{n\ge 0} \in \Phi(\mathbb{R}^d)$.

We record the following weighted and vector-valued extension of the multiplier result in [263, Section 1.6.3], see [211, Proposition 2.4].

Lemma 3.2.1. *Let $p \in [1,\infty)$, $q \in [1,\infty]$, $w \in A_\infty(\mathbb{R}^d)$ and let X be a Banach space. Let $K > 0$ be fixed and for $n \in \mathbb{N}_0$ let $f_n \in L^p(\mathbb{R}^d, w; X)$ be such that* $\operatorname{supp} \widehat{f_n} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \le 2^n K\}$. *Then*

$$\|(S_n f_n)_{n\ge 0}\|_{L^p(\mathbb{R}^d,w;\ell^q(X))} \le C \|(f_n)_{n\ge 0}\|_{L^p(\mathbb{R}^d,w;\ell^q(X))},$$

where the constant $C > 0$ is independent of $(f_n)_{n\ge 0}$.

In particular, for any fixed $n \in \mathbb{N}_0$ and $f \in L^p(\mathbb{R}^d, w; X)$ with $\operatorname{supp} \widehat{f} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \le 2^n K\}$, *it holds that*

$$\|S_n f\|_{L^p(\mathbb{R}^d,w;X)} \le C \|f\|_{L^p(\mathbb{R}^d,w;X)},$$

where the constant $C > 0$ is independent of f and n.

Remark 3.2.2. The constant C in Lemma 3.2.1 may depend on K, but later on K will always be a fixed number associated to the zeroth-order block of Littlewood–Paley decompositions.

Proof. Since $w \in A_\infty(\mathbb{R}^d)$, we can find an $r \in (0, \min\{p,q\})$ such that $w \in A_{\frac{p}{r}}(\mathbb{R}^d)$. Then [211, Proposition 2.4] implies

$$\begin{aligned}\|(S_n f_n)_{n\ge 0}\|_{L^p(\mathbb{R}^d,w;\ell^q(X))} &= \|(\mathscr{F}^{-1}(\widehat{\varphi}_n \widehat{f_n}))_{n\ge 0}\|_{L^p(\mathbb{R}^d,w;\ell^q(X))} \\ &\le C \sup_{k\ge 0} \|(1+|\cdot|^{\frac{d}{r}})\mathscr{F}^{-1}[\widehat{\varphi}_k(2^k K\cdot)]\|_{L^1(\mathbb{R}^d)} \|(f_n)_{n\ge 0}\|_{L^p(\mathbb{R}^d,w;\ell^q(X))},\end{aligned}$$

where $C > 0$ is independent of $(f_n)_{n\ge 0}$. Using the scaling property $\widehat{\varphi}_k(\cdot) = \widehat{\varphi}_1(2^{-k+1}\cdot)$ for $k \ge 1$ gives that $\widehat{\varphi}_k(2^k K\cdot)$ has its support in a ball with a radius that is independent of k. Now, [126, Lemma 14.2.12] gives the first statement. The second statement follows from applying the first statement with $(f_n)_{n\ge 0}$ replaced by $(0,\dots,0,f,0,\dots)$ with f at the n-th position. □

For $p \in (1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$, $w \in A_\infty(\mathbb{R}^d)$ and X a Banach space, we define the *weighted Besov, Triebel–Lizorkin and Bessel potential spaces* as the spaces of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ for which, respectively,

$$\begin{aligned}\|f\|_{B^s_{p,q}(\mathbb{R}^d,w;X)} &:= \|(2^{ns}\varphi_n * f)_{n\ge 0}\|_{\ell^q(L^p(\mathbb{R}^d,w;X))} < \infty, \\ \|f\|_{F^s_{p,q}(\mathbb{R}^d,w;X)} &:= \|(2^{ns}\varphi_n * f)_{n\ge 0}\|_{L^p(\mathbb{R}^d,w;\ell^q(X))} < \infty, \\ \|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} &:= \|\mathscr{F}^{-1}((1+|\cdot|^2)^{s/2}\mathscr{F}f)\|_{L^p(\mathbb{R}^d,w;X)} < \infty, \quad w \in A_p(\mathbb{R}^d).\end{aligned}$$

The definitions of the Besov and Triebel–Lizorkin spaces are independent of the chosen Littlewood–Paley sequence up to an equivalent norm, see [211, Proposition 3.4]. We note that all the spaces defined above embed continuously into $\mathscr{S}'(\mathbb{R}^d;X)$, see [183, Section 5.2.1e]. Furthermore, $\mathscr{S}(\mathbb{R}^d;X)$ embeds continuously into all the above spaces and this embedding is dense if $p,q<\infty$, see [211, Lemma 3.8].

Furthermore, we record the Fatou property for weighted Besov and Triebel–Lizorkin spaces. If $X,\mathscr{X}$ are Banach spaces such that $\mathscr{X}$ embeds continuously into $\mathscr{S}'(\mathbb{R}^d;X)$, then we say that $\mathscr{X}$ has the *Fatou property* (see [33, 94]) if for all $(f_n)_{n\geq 0}\subseteq\mathscr{X}$ such that $\lim_{n\to\infty} f_n = f$ in $\mathscr{S}'(\mathbb{R}^d;X)$ and $\liminf_{n\to\infty}\|f_n\|_{\mathscr{X}}<\infty$, we have

$$f\in\mathscr{X}\quad\text{and}\quad\|f\|_{\mathscr{X}}\leq\liminf_{n\to\infty}\|f_n\|_{\mathscr{X}}.$$

The proof of the Fatou property for Besov and Triebel–Lizorkin spaces in [94] carries over to the vector-valued case (see [245, Proposition 2.18]) and to the weighted case as well.

Proposition 3.2.3. *Let $p\in(1,\infty)$, $q\in[1,\infty]$, $s\in\mathbb{R}$, $w\in A_\infty(\mathbb{R}^d)$ and let X be a Banach space. Then $B^s_{p,q}(\mathbb{R}^d,w;X)$ and $F^s_{p,q}(\mathbb{R}^d,w;X)$ have the Fatou property.*

Let $\mathscr{O}\subseteq\mathbb{R}^d$ be open. To define the B,F and H spaces on $\mathscr{O}$, we use *restriction/factor spaces*. Let $\mathbb{F}\in\{B,F,H\}$, then we define

$$\mathbb{F}(\mathscr{O},w;X):=\big\{f\in\mathscr{D}'(\mathscr{O};X):\exists g\in\mathbb{F}(\mathbb{R}^d,w;X),g|_{\mathscr{O}}=f\big\}$$

endowed with the norm

$$\|f\|_{\mathbb{F}(\mathscr{O},w;X)}:=\inf\{\|g\|_{\mathbb{F}(\mathbb{R}^d,w;X)}:g|_{\mathscr{O}}=f\}.$$

Similarly, we can define the factor space for weighted Sobolev spaces. If $\mathscr{O}=\mathbb{R}^d_+$ and $w\in A_p(\mathbb{R}^d_+)$, then the weighted Sobolev spaces on $\mathbb{R}^d_+$ as defined above coincide with their factor spaces, see [189, Section 5]. The proof is based on the existence of an extension operator from $\mathbb{R}^d_+$ to $\mathbb{R}^d$, see, e.g., [4, Theorems 5.19 & 5.22]. Throughout the rest of the chapter, we will refer to such operators as *reflection operators*. For a discussion of reflection operators on weighted spaces, we refer to [265, Section 2.1.2].

3.2.2. Embeddings

We continue with some standard embedding properties of vector-valued function spaces. Let $p\in(1,\infty)$, $s\in\mathbb{R}$, $k\in\mathbb{N}_0$, $w\in A_p(\mathbb{R}^d)$ and let X be a Banach space. Then

$$F^s_{p,1}(\mathbb{R}^d,w;X)\hookrightarrow H^{s,p}(\mathbb{R}^d,w;X)\hookrightarrow F^s_{p,\infty}(\mathbb{R}^d,w;X),\tag{3.2.2a}$$

$$F^k_{p,1}(\mathbb{R}^d,w;X)\hookrightarrow W^{k,p}(\mathbb{R}^d,w;X)\hookrightarrow F^k_{p,\infty}(\mathbb{R}^d,w;X),\tag{3.2.2b}$$

see [211, Proposition 3.12]. It holds that $F^s_{p,2}(\mathbb{R}^d;X)=H^{s,p}(\mathbb{R}^d;X)$ if and only if X is isomorphic to a Hilbert space, see [119] or [126, Theorem 14.7.9] for a more direct proof. For general Banach spaces X, the Bessel potential and Sobolev spaces are not equal to a Triebel–Lizorkin space, but instead, we have the embeddings (3.2.2). In particular, we note that the embedding

$$F^0_{p,1}(\mathbb{R}^d,w;X)\hookrightarrow L^p(\mathbb{R}^d,w;X),\tag{3.2.3}$$

holds for all $w \in A_\infty(\mathbb{R}^d)$. This immediately follows from taking $L^p(\mathbb{R}^d, w)$-norms in the estimate $\|f\|_X \le \sum_{n=0}^\infty \|S_n f\|_X$ with $f \in \mathscr{S}(\mathbb{R}^d; X)$ and a density argument, see [211, Lemma 3.8].

Moreover, for all $p \in [1,\infty)$, $q \in [1,\infty]$, $s \in \mathbb{R}$, $w \in A_\infty(\mathbb{R}^d)$, it holds that

$$B^s_{p,\min\{p,q\}}(\mathbb{R}^d, w; X) \hookrightarrow F^s_{p,q}(\mathbb{R}^d, w; X) \hookrightarrow B^s_{p,\max\{p,q\}}(\mathbb{R}^d, w; X), \tag{3.2.4}$$

and if, in addition, $1 \le q_0 \le q_1 \le \infty$, then

$$B^s_{p,q_0}(\mathbb{R}^d, w; X) \hookrightarrow B^s_{p,q_1}(\mathbb{R}^d, w; X) \quad \text{and} \quad F^s_{p,q_0}(\mathbb{R}^d, w; X) \hookrightarrow F^s_{p,q_1}(\mathbb{R}^d, w; X), \tag{3.2.5}$$

see [211, Proposition 3.11].

Finally, we state the following theorem on Sobolev embeddings for Besov and Triebel–Lizorkin spaces. Its proof is based on a similar result for radial weights, see [211, Theorems 1.1 & 1.2]. Moreover, in [212] general A_∞ weights are considered in the scalar-valued setting. The embedding with scalar-valued spaces can be extended to vector-valued spaces only for Muckenhoupt weights, see [212, Theorem 1.3]. However, this is not sufficient for our purposes.

Theorem 3.2.4. *Let $1 < p_0 \le p_1 < \infty$, $q_0, q_1 \in [1,\infty]$, $s_0 > s_1$, $\gamma_0, \gamma_1 > -1$ and let X be a Banach space. Suppose that*

$$\frac{\gamma_1}{p_1} \le \frac{\gamma_0}{p_0} \quad \text{and} \quad s_0 - \frac{d+\gamma_0}{p_0} = s_1 - \frac{d+\gamma_1}{p_1}.$$

Then

(i) $F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X) \hookrightarrow F^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_{\gamma_1}; X)$,

(ii) $B^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X) \hookrightarrow B^{s_1}_{p_1,q_1}(\mathbb{R}^d, w_{\gamma_1}; X)$ *if, in addition,* $q_0 \le q_1$.

Proof. Step 1: proof of (ii). By (3.2.5) it suffices to consider $q := q_0 = q_1$. Let $f \in B^{s_0}_{p_0,q}(\mathbb{R}^d, w_{\gamma_0}; X)$, $(\varphi_n)_{n\ge 0} \in \Phi(\mathbb{R}^d)$ and define $S_n f := \varphi_n * f$ as in (3.2.1). We write $s_1 + \delta = s_0$, where

$$\delta = \delta_0 + \delta_1 \quad \text{with} \quad \delta_0 := \frac{\gamma_0+1}{p_0} - \frac{\gamma_1+1}{p_1} \quad \text{and} \quad \delta_1 := \frac{d-1}{p_0} - \frac{d-1}{p_1}.$$

Let $\mathscr{F}_{x_1}$ and $\mathscr{F}_{\widetilde{x}}$ be the Fourier transform with respect to $x_1 \in \mathbb{R}$ and $\widetilde{x} \in \mathbb{R}^{d-1}$, respectively. Note that $\operatorname{supp} \mathscr{F}(S_n f) \subseteq \{\xi \in \mathbb{R}^d : |\xi| \le 3\cdot 2^{n-1}\}$, so that $\operatorname{supp} \mathscr{F}_{x_1}(S_n f(\cdot, \widetilde{x})) \subseteq \{\xi_1 \in \mathbb{R} : |\xi_1| \le 3\cdot 2^{n-1}\}$ for fixed $\widetilde{x} \in \mathbb{R}^{d-1}$ and $\operatorname{supp} \mathscr{F}_{\widetilde{x}}(S_n f(x_1, \cdot)) \subseteq \{\widetilde{\xi} \in \mathbb{R}^{d-1} : |\widetilde{\xi}| \le 3\cdot 2^{n-1}\}$ for fixed $x_1 \in \mathbb{R}$ (see [245, Theorem 4.9, Step 1 & 2]). Now, the Plancherel–Polya–Nikol'skij type inequality from [211, Proposition 4.1] twice and Minkowski's inequality yield

$$\begin{aligned}
\|S_n f\|_{L^{p_1}(\mathbb{R}^d, w_{\gamma_1}; X)} &\le C 2^{\delta_0 n} \Big(\int_{\mathbb{R}^{d-1}} \Big(\int_{\mathbb{R}} |x_1|^{\gamma_0} \|S_n f(x_1, \widetilde{x})\|_X^{p_0} \, \mathrm{d}x_1 \Big)^{\frac{p_1}{p_0}} \, \mathrm{d}\widetilde{x} \Big)^{\frac{1}{p_1}} \\
&\le C 2^{\delta_0 n} \Big(\int_{\mathbb{R}} |x_1|^{\gamma_0} \Big(\int_{\mathbb{R}^{d-1}} \|S_n f(x_1, \widetilde{x})\|_X^{p_1} \, \mathrm{d}\widetilde{x} \Big)^{\frac{p_0}{p_1}} \, \mathrm{d}x_1 \Big)^{\frac{1}{p_0}} \\
&\le C 2^{(\delta_0+\delta_1) n} \|S_n f\|_{L^{p_0}(\mathbb{R}^d, w_{\gamma_0}; X)},
\end{aligned} \tag{3.2.6}$$

where $C > 0$ is independent of f and n. Statement (ii) now follows from the definition of the Besov norms and the fact that $s_1 + \delta = s_0$.

Step 2: proof of (i). We follow the proof of [211, Theorem 1.2] for radial weights to deduce (i). By (3.2.5) it suffices to prove the required estimate for $q_1 = 1$. First, let $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X) \cap F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)$ and without loss of generality we may assume $f \neq 0$. Let $\theta_0 \in [0,1)$ be such that $\frac{1}{p_1} - \frac{1-\theta_0}{p_0} = 0$ and define the continuous function $g : (\theta_0, 1] \to \mathbb{R}$ by

$$g(\theta) = \frac{\frac{\gamma_1}{p_1} - \frac{(1-\theta)\gamma_0}{p_0}}{\frac{1}{p_1} - \frac{1-\theta}{p_0}}.$$

Note that $g(1) = \gamma_1$ and since $\gamma_1 > -1$ there exists a $\theta \in (\theta_0, 1)$ such that $\gamma := g(\theta) > -1$. Define r and t by

$$\frac{1}{p_1} = \frac{1-\theta}{p_0} + \frac{\theta}{r} \quad \text{and} \quad t - \frac{d+\gamma}{r} = s_1 - \frac{d+\gamma_1}{p_1}.$$

From the conditions on the parameters it follows that $r \in [p_1, \infty)$, and $t < s_0$ satisfies $s_1 = \theta t + (1-\theta)s_0$. Moreover, it holds that $\theta\gamma p_1/r + (1-\theta)\gamma_0 p_1/p_0 = \gamma_1$. Therefore, the Gagliardo–Nirenberg type inequality from [211, Proposition 5.1] gives

$$\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)} \leq C \|f\|^{1-\theta}_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)} \|f\|^{\theta}_{F^{t}_{r,r}(\mathbb{R}^d, w_{\gamma}; X)}. \tag{3.2.7}$$

Moreover, by (3.2.4), (ii) (using that $r \geq p_1$) and (3.2.5), we obtain

$$\|f\|_{F^{t}_{r,r}(\mathbb{R}^d, w_{\gamma}; X)} \eqsim \|f\|_{B^{t}_{r,r}(\mathbb{R}^d, w_{\gamma}; X)} \leq C \|f\|_{B^{s_1}_{p_1,p_1}(\mathbb{R}^d, w_{\gamma_1}; X)} \leq C \|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)}. \tag{3.2.8}$$

Substituting (3.2.8) in (3.2.7) yields

$$\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)} \leq C \|f\|_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)} \tag{3.2.9}$$

for $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X) \cap F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)$.

The estimate for $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)$ follows from the Fatou property of Triebel–Lizorkin spaces. We provide the argument for the convenience of the reader. For $f \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)$ the series $f_N = \sum_{n=0}^{N} S_n f$ converges to f in $\mathscr{S}'(\mathbb{R}^d; X)$ as $N \to \infty$. Note that for all $N \in \mathbb{N}_0$ we have $f_N \in F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X) \cap F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)$ by (3.2.6). From Lemma 3.2.1 we have that

$$\|f_N\|_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)} \leq C \|f\|_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)} \quad \text{for all } N \in \mathbb{N}_0,$$

and together with (3.2.9) this implies

$$\|f_N\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)} \leq C \|f\|_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)} \quad \text{for all } N \in \mathbb{N}_0.$$

Proposition 3.2.3 yields $f \in F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)$ and

$$\|f\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)} \leq \liminf_{N\to\infty} \|f_N\|_{F^{s_1}_{p_1,1}(\mathbb{R}^d, w_{\gamma_1}; X)} \leq C \|f\|_{F^{s_0}_{p_0,q_0}(\mathbb{R}^d, w_{\gamma_0}; X)}.$$

This completes the proof. □

3.3. Trace spaces of Besov and Triebel–Lizorkin spaces

In this section, we characterise the higher-order trace spaces of weighted Besov and Triebel–Lizorkin spaces on $\mathbb{R}^d$. In Section 3.3.1 we first study the zeroth-order trace operator and in Section 3.3.2 we extend the results to higher-order trace operators.

3.3.1. The zeroth-order trace operator

To define the trace operator we pursue as in [245] and use a concept due to Nikol'skij [225]. There exist several other and, in particular cases, equivalent definitions for the trace operator. We refer to [245, Remark 4.8] for an overview.

Let X be a Banach space and let $f:\mathbb{R}^d\to X$ be strongly measurable. A function $g:\mathbb{R}^{d-1}\to X$ is called the trace of f on $\{0\}\times\mathbb{R}^{d-1}$, if there are $\widetilde{f}:\mathbb{R}^d\to X$, $p\in[1,\infty]$ and $\delta>0$ such that

(i) $\widetilde{f}=f$ almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^d$,

(ii) $\widetilde{f}(x_1,\cdot)\in L^p(\mathbb{R}^{d-1};X)$ for $|x_1|<\delta$,

(iii) $\widetilde{f}(0,\cdot)=g$ almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^{d-1}$,

(iv) $\lim_{x_1\to 0}\|\widetilde{f}(x_1,\cdot)-g\|_{L^p(\mathbb{R}^{d-1};X)}=0$.

By [245, Remark 4.2 & 4.3] this definition is independent of $\widetilde{f}$, p and δ and it coincides with the restriction of f to $\{(0,\widetilde{x}):\widetilde{x}\in\mathbb{R}^{d-1}\}$ if f is continuous on $(-\delta,\delta)\times\mathbb{R}^{d-1}$. If the trace exists according to the above definition, then we write $\operatorname{Tr} f=g$, so that we obtain a linear operator Tr.

The representative $\widetilde{f}$ for f can be obtained via the formula $\widetilde{f}=\sum_{n=0}^{\infty}S_nf$, whenever this makes sense. In Proposition 3.3.2 we will use this representative to show that the trace operator is well defined if f belongs to a certain Besov space.

The following results are extensions of the arguments in [245] to the case with weights $w_\gamma(x)=|x_1|^\gamma$, which can also be found in version 1 of the arXiv preprint of [213]. We note that radial weights $|x|^\gamma$ are considered in [2, 120]. Anisotropic versions of trace theorems have appeared in, e.g., [132, 184, 185] and see the references therein.

We start with a technical lemma analogous to [245, Lemma 4.5].

Lemma 3.3.1. *Let $p\in(1,\infty)$, $\gamma>-1$ and let X be a Banach space. Let $h\in L^p(\mathbb{R}^d,w_\gamma;X)$ such that* $\operatorname{supp}\widehat{h}\subseteq\{\xi\in\mathbb{R}^d:|\xi|\le R\}$ *for some $R>0$. Then there exists a $C>0$, independent of h and R, such that for all $x_1\in\mathbb{R}$, we have*

$$\|h(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\le CR^{\frac{\gamma+1}{p}}\|h\|_{L^p(\mathbb{R}^d,w_\gamma;X)}.$$

Proof. First of all, note that h is pointwise defined by the support condition on its Fourier transform, see, e.g., [126, Lemma 14.2.9].

Step 1: the case $\gamma\in(-1,p-1)$. By scaling and translating, it suffices to consider $x_1=0$ and $R=1$. Let $\widehat{\varphi}\in C_c^\infty(\mathbb{R}^d)$ such that $\widehat{\varphi}=1$ on $B(0,1)$. Then for every $\widetilde{x}\in\mathbb{R}^{d-1}$ we have

$$h(0,\widetilde{x})=\big(\mathscr{F}^{-1}(\widehat{\varphi}\,\widehat{h})\big)(0,\widetilde{x})=\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}h(-y_1,\widetilde{x}-\widetilde{y})|y_1|^{\frac{\gamma}{p}}\varphi(y_1,\widetilde{y})|y_1|^{-\frac{\gamma}{p}}\,\mathrm{d}y_1\,\mathrm{d}\widetilde{y},$$

so that Hölder's inequality implies

$$\|h(0,\widetilde{x})\|_X \le \int_{\mathbb{R}^{d-1}} \|h(\cdot,\widetilde{x}-\widetilde{y})\|_{L^p(\mathbb{R},w_\gamma;X)} \|\varphi(\cdot,\widetilde{y})\|_{L^{p'}(\mathbb{R},w_{\gamma'})}\,\mathrm{d}\widetilde{y},$$

where $p' = \frac{p}{p-1}$ and $\gamma' = -\frac{\gamma}{p-1}$. By Minkowski's inequality we obtain

$$\|h(0,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)} \le \|h\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \int_{\mathbb{R}^{d-1}} \|\varphi(\cdot,\widetilde{y})\|_{L^{p'}(\mathbb{R},w_{\gamma'})}\,\mathrm{d}\widetilde{y} = C\|h\|_{L^p(\mathbb{R}^d,w_\gamma;X)},$$

using that $\gamma' > -1$ in the last step.

Step 2: the case $\gamma \ge p-1$. Let $\mathscr{F}_{x_1}$ be the Fourier transform with respect to $x_1 \in \mathbb{R}$. Since $\operatorname{supp}\widehat{h} \subseteq \{\xi \in \mathbb{R}^d : |\xi| \le R\}$ we have that $\operatorname{supp}\mathscr{F}_{x_1}(h(\cdot,\widetilde{x})) \subseteq \{\xi_1 \in \mathbb{R} : |\xi_1| \le R\}$ for fixed $\widetilde{x} \in \mathbb{R}^{d-1}$, see [245, Theorem 4.9, Step 1 & 2]. By the Plancherel–Polya–Nikol'skij type inequality from [211, Proposition 4.1] we obtain

$$\begin{aligned}\|h\|_{L^p(\mathbb{R}^d;X)} &= \Big(\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} \|h(x_1,\widetilde{x})\|_X^p\,\mathrm{d}x_1\,\mathrm{d}\widetilde{x}\Big)^{\frac{1}{p}} \\ &\le CR^{\frac{\gamma}{p}}\Big(\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}} |x_1|^\gamma \|h(x_1,\widetilde{x})\|_X^p\,\mathrm{d}x_1\,\mathrm{d}\widetilde{x}\Big)^{\frac{1}{p}} = CR^{\frac{\gamma}{p}}\|h\|_{L^p(\mathbb{R}^d,w_\gamma;X)}.\end{aligned} \tag{3.3.1}$$

From Step 1 applied to $\gamma = 0$ and (3.3.1), we obtain

$$\|h(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)} \le CR^{\frac{1}{p}}\|h\|_{L^p(\mathbb{R}^d;X)} \le CR^{\frac{\gamma+1}{p}}\|h\|_{L^p(\mathbb{R}^d,w_\gamma;X)}.$$

Note that this second step actually works for all $\gamma > 0$. □

Using Lemma 3.3.1 we can now prove that $\operatorname{Tr} f$ exists if $f \in B_{p,1}^{\frac{\gamma+1}{p}}(\mathbb{R}^d, w_\gamma; X)$. In particular, provided that f has this regularity, we will in the sequel always work with the identity (3.3.2). The following result is a weighted version of [245, Proposition 4.4].

Proposition 3.3.2. *Let $p \in (1,\infty)$, $\gamma > -1$ and let X be a Banach space. Then the trace of a function $f \in B_{p,1}^{\frac{\gamma+1}{p}}(\mathbb{R}^d, w_\gamma; X)$ exists and for any $(\varphi_n)_{n\ge0} \in \Phi(\mathbb{R}^d)$, we have*

$$\operatorname{Tr} f = \sum_{n=0}^{\infty} \varphi_n * f(0,\cdot) \qquad \text{in } L^p(\mathbb{R}^{d-1};X). \tag{3.3.2}$$

Proof. Let $f \in B_{p,1}^{\frac{\gamma+1}{p}}(\mathbb{R}^d, w_\gamma; X)$, $(\varphi_n)_{n\ge0} \in \Phi(\mathbb{R}^d)$ and set $\widetilde{f} = \sum_{n=0}^\infty S_n f$. We verify the conditions (i), (ii) and (iv) for the definition of the trace operator. Note that (iii) holds automatically.

Step 1: proof of (i). Since $f \in B_{p,1}^{\frac{\gamma+1}{p}}(\mathbb{R}^d, w_\gamma; X)$ we have $\sum_{n=0}^\infty \|S_n f\|_{L^p(\mathbb{R}^d,w_\gamma;X)} < \infty$ and therefore the series converges to f in $L^p(\mathbb{R}^d, w_\gamma; X)$ and thus $f = \widetilde{f}$ almost everywhere on $\mathbb{R}^d$.

Step 2: proof of (ii). By Lemma 3.3.1 (using that $\operatorname{supp}\mathscr{F}(S_n f) \subseteq \{\xi \in \mathbb{R}^d : |\xi| \leq 3\cdot 2^{n-1}\}$) we obtain for every $x_1 \in \mathbb{R}$ that

$$\begin{aligned}\|\widetilde{f}(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)} &\leq \big\|(\|S_n f(x_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)})_{n\geq 0}\big\|_{\ell^1}\\ &\leq C\big\|(2^{\frac{\gamma+1}{p}n}\|S_n f\|_{L^p(\mathbb{R}^d,w_\gamma;X)})_{n\geq 0}\big\|_{\ell^1} = C\|f\|_{B^{\frac{\gamma+1}{p}}_{p,1}(\mathbb{R}^d,w_\gamma;X)}.\end{aligned}$$

Step 3: proof of (iv). Let $\varepsilon > 0$. By applying Lemma 3.3.1, we can find an $N \geq 1$ large enough such that

$$\Big\|\sum_{n=N}^{\infty}\big(S_n f(x_1,\cdot) - S_n f(0,\cdot)\big)\Big\|_{L^p(\mathbb{R}^{d-1};X)} \leq C\sum_{n=N}^{\infty} 2^{\frac{\gamma+1}{p}n}\|S_n f\|_{L^p(\mathbb{R}^d,w_\gamma;X)} < \frac{\varepsilon}{2}. \tag{3.3.3}$$

From now on we fix such an N. Let $n < N$ and $|x_1| < 2^{-N}$, and we define $h_{x_1}(t,\widetilde{x}) := S_n f(t+x_1,\widetilde{x}) - S_n f(t,\widetilde{x})$ for $t \in \mathbb{R}$ and $\widetilde{x} \in \mathbb{R}^{d-1}$. Note that by the mean value theorem we have

$$\|h_{x_1}(t,\widetilde{x})\|_X \leq |x_1| \sup_{|y|<2^{-N}} \|\nabla S_n f(t+y,\widetilde{x})\|_X. \tag{3.3.4}$$

By Lemma 3.3.1, (3.3.4) and [245, Lemma 2.3], we obtain that

$$\begin{aligned}\|S_n f(x_1,\cdot) - S_n f(0,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)} &= \|h_{x_1}(0,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\\ &\leq C2^{\frac{\gamma+1}{p}n}\|h_{x_1}\|_{L^p(\mathbb{R}^d,w_\gamma;X)}\\ &\leq C2^{\frac{\gamma+1}{p}n}|x_1|\Big\|(t,\widetilde{x}) \mapsto \sup_{y<2^{-N}}\|\nabla S_n f(t+y,\widetilde{x})\|_X\Big\|_{L^p(\mathbb{R}^d,w_\gamma)}\\ &\leq C2^{(\frac{\gamma+1}{p}+1)n}|x_1|\big\|(\mathscr{M}\|S_n f\|_X^r)^{\frac{1}{r}}\big\|_{L^p(\mathbb{R}^d,w_\gamma)},\end{aligned}$$

for some $r \in (0,p)$, where $\mathscr{M}$ denotes the Hardy–Littlewood maximal operator. Picking r such that $w_\gamma \in A_{\frac{p}{r}}(\mathbb{R}^d)$ and using the boundedness of $\mathscr{M}$ on $L^{p/r}(\mathbb{R}^d, w_\gamma)$ gives

$$\|S_n f(x_1,\cdot) - S_n f(0,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)} \leq C2^{(\frac{\gamma+1}{p}+1)n}|x_1|\|S_n f\|_{L^p(\mathbb{R}^d,w_\gamma;X)}. \tag{3.3.5}$$

Upon choosing $|x_1|$ small enough, we now obtain by combining (3.3.3) and (3.3.5) that

$$\Big\|\sum_{n=0}^{\infty}\big(S_n f(x_1,\cdot) - S_n f(0,\cdot)\big)\Big\|_{L^p(\mathbb{R}^{d-1};X)} \leq \frac{\varepsilon}{2} + C|x_1|\sum_{n=0}^{N-1} 2^{(\frac{\gamma+1}{p}+1)n}\|S_n f\|_{L^p(\mathbb{R}^d,w_\gamma;X)} < \varepsilon.$$

For instance, it suffices to take $|x_1| < \varepsilon 2^{-N} C^{-1}\|f\|^{-1}_{B^{(\gamma+1)/p}_{p,1}(\mathbb{R}^d,w_\gamma;X)}$, where C is the constant in (3.3.5). This shows that $\lim_{x_1\to 0}\widetilde{f}(x_1,\cdot) = \widetilde{f}(0,\cdot)$ in $L^p(\mathbb{R}^{d-1};X)$. □

Remark 3.3.3. Note that for $\gamma > 0$ we have $B^{\frac{\gamma+1}{p}}_{p,1}(\mathbb{R}^d, w_\gamma; X) \hookrightarrow B^{\frac{1}{p}}_{p,1}(\mathbb{R}^d; X)$ by Theorem 3.2.4. Thus, Proposition 3.3.2 for $\gamma > 0$ can also be deduced from the unweighted result. This argument does not apply for $\gamma \in (-1,0)$.

With the above results, the trace space of a weighted Besov space can be characterised following the arguments from [245, Theorem 4.9 & Proposition 4.12] for the unweighted case. For $p = q \in (1,\infty)$ and A_p weights the result was already shown in [108, Théorème 7.1].

Theorem 3.3.4. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $\gamma > -1$, $s > \frac{\gamma+1}{p}$ and let X be a Banach space. Let* Tr *be as in Proposition 3.3.2. Then*

$$\operatorname{Tr} : B^s_{p,q}(\mathbb{R}^d, w_\gamma; X) \to B^{s-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator. Moreover, there exists a continuous right inverse ext *of* Tr *which is independent of s, p, q, γ and X.*

Proof. Let $(\varphi_n)_{n\geq 0} \in \Phi(\mathbb{R}^d)$, $(\phi_n)_{n\geq 0} \in \Phi(\mathbb{R}^{d-1})$, $f \in \mathscr{S}'(\mathbb{R}^d; X)$, $g \in \mathscr{S}'(\mathbb{R}^{d-1}; X)$ and define $S_n f := \varphi_n * f$ and $T_n g := \phi_n * g$. Moreover, set $S_{-1} f = T_{-1} g = 0$.

Step 1: trace operator. The continuity of the trace operator can be shown similarly as in Step 4 of the proof of [245, Theorem 4.9] if we apply Lemma 3.3.1 instead of [245, Equation (4.5)].

Step 2: extension operator. The extension operator is defined similarly as in [263, Section 2.7.2] and [245]. Let $g \in B^{s-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$ and take $(\rho_n)_{n\geq 0} \in \Phi(\mathbb{R})$ such that $\rho_1(0) = 2$ (and therefore $\rho_n(0) = 2^n$ for all $n \geq 0$). We set

$$\operatorname{ext} g(x_1, \widetilde{x}) = \sum_{n=0}^{\infty} 2^{-n} \rho_n(x_1) T_n g(\widetilde{x}) \quad \text{in } \mathscr{S}'(\mathbb{R}^d; X). \tag{3.3.6}$$

Note that ext is independent of s, p, q, γ and X. Moreover, since $\rho_n(x_1) = 2^{n-1}\rho_1(2^{n-1}x_1)$ it holds that

$$\|\rho_n\|_{L^p(\mathbb{R}, w_\gamma)} = 2^{(n-1)(1-\frac{\gamma+1}{p})} \|\rho_1\|_{L^p(\mathbb{R}, w_\gamma)}, \qquad n \in \mathbb{N}_0. \tag{3.3.7}$$

Therefore, using (3.3.7), Young's inequality, $s - \frac{\gamma+1}{p} > 0$ and $\gamma > -1$, we find

$$\begin{aligned}\sum_{n=0}^{\infty} 2^{-n} \|\rho_n T_n g\|_{L^p(\mathbb{R}^d, w_\gamma; X)} &= \sum_{n=0}^{\infty} 2^{-n} \|\rho_n\|_{L^p(\mathbb{R}, w_\gamma)} \|T_n g\|_{L^p(\mathbb{R}^{d-1}; X)} \\ &\leq C \|g\|_{L^p(\mathbb{R}^{d-1}; X)} \sum_{n=0}^{\infty} 2^{-n\frac{\gamma+1}{p}} \leq C \|g\|_{B^{s-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)},\end{aligned}$$

where the constant C is independent of g and n. For the last estimate we have used a geometric series ($\gamma > -1$) and that $B^{\varepsilon}_{p,q}(\mathbb{R}^{d-1}; X) \hookrightarrow B^0_{p,1}(\mathbb{R}^{d-1}; X) \hookrightarrow L^p(\mathbb{R}^{d-1}; X)$ for $\varepsilon > 0$, see [126, Theorem 14.4.19 & Proposition 14.4.18]. This shows that ext g is well defined in $L^p(\mathbb{R}^d, w_\gamma; X)$. Now, Lemma 3.2.1 and (3.3.7) imply that

$$\begin{aligned}\|S_n \operatorname{ext} g\|_{L^p(\mathbb{R}^d, w_\gamma; X)} &\leq C \sum_{j=-1}^{1} 2^{-(n+j)} \|\rho_{n+j}\|_{L^p(\mathbb{R}, w_\gamma)} \|T_{n+j} g\|_{L^p(\mathbb{R}^{d-1}; X)} \\ &\leq C \sum_{j=-1}^{1} 2^{-(n+j)\frac{\gamma+1}{p}} \|T_{n+j} g\|_{L^p(\mathbb{R}^{d-1}; X)},\end{aligned}$$

where the constant C is independent of g and n. This implies that

$$\|\operatorname{ext} g\|_{B^s_{p,q}(\mathbb{R}^d,w_\gamma;X)} \le C\sum_{j=-1}^{1}\big\|\big(2^{n(s-\frac{\gamma+1}{p})}\|T_{n+j}g\|_{L^p(\mathbb{R}^{d-1};X)}\big)_{n\ge 0}\big\|_{\ell^q} \le \|g\|_{B^{s-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1};X)}.$$

This proves the continuity of ext.

Since $\rho_n(0)=2^n$ it follows from the formulas (3.3.2) and (3.3.6) that $\operatorname{Tr}(\operatorname{ext} g)=g$ for all $g\in\mathscr{S}(\mathbb{R}^{d-1};X)$. By density (see [211, Lemma 3.8]) this extends to all $g\in B^{s-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1};X)$ with $q<\infty$. If $q=\infty$, then we have $B^{s-\frac{\gamma+1}{p}}_{p,\infty}(\mathbb{R}^{d-1};X)\hookrightarrow B^{s-\frac{\gamma+1}{p}-\varepsilon}_{p,1}(\mathbb{R}^{d-1};X)$ for all $\varepsilon>0$ (see [126, Theorem 14.4.19]). This shows that ext is also the right inverse for Tr in this case. Finally, the existence of a right inverse of Tr implies that Tr is surjective. □

Using Theorem 3.3.4 and the Sobolev embeddings for Triebel–Lizorkin spaces from Theorem 3.2.4, we characterise the trace spaces for Triebel–Lizorkin spaces. In contrast to [245, Lemma 4.15] and [120, Theorem 3.6], the following proof is Fourier analytic and does not use an atomic approach. In the unweighted scalar-valued setting, other methods are used in [263, Theorem 2.7.2] to obtain the trace space for Triebel–Lizorkin spaces.

Theorem 3.3.5. *Let $p\in(1,\infty)$, $q\in[1,\infty]$, $\gamma>-1$, $s>\frac{\gamma+1}{p}$ and let X be a Banach space. Then*

$$\operatorname{Tr}: F^s_{p,q}(\mathbb{R}^d,w_\gamma;X)\to B^{s-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)$$

is a continuous and surjective operator. Moreover, there exists a continuous right inverse ext *of* Tr *which is independent of s,p,q,γ and X. This extension operator coincides with the extension operator from Theorem 3.3.4.*

Proof. Step 1: trace operator. Let $f\in F^s_{p,q}(\mathbb{R}^d,w_\gamma;X)$ and for $\varepsilon>0$ small take $s_1:=s-\frac{\varepsilon}{p}$ and $\gamma_1:=\gamma-\varepsilon$. Then $\gamma_1/p<\gamma/p$ and $s-(d+\gamma)/p=s_1-(d+\gamma_1)/p$. So by Theorem 3.3.4, (3.2.4) and Theorem 3.2.4, we obtain

$$\begin{aligned}\|\operatorname{Tr} f\|_{B^{s-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} &= \|\operatorname{Tr} f\|_{B^{s_1-\frac{\gamma_1+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} \le C\|f\|_{B^{s_1}_{p,p}(\mathbb{R}^d,w_{\gamma_1};X)}\\ &\le C\|f\|_{F^{s_1}_{p,p}(\mathbb{R}^d,w_{\gamma_1};X)} \le C\|f\|_{F^s_{p,q}(\mathbb{R}^d,w_\gamma;X)}.\end{aligned}$$

Step 2: extension operator. Let $g\in B^{s-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)$ and for $\varepsilon>0$ small take $s_0=s+\varepsilon/p$ and $\gamma_0=\gamma+\varepsilon$. Then by Theorem 3.2.4, (3.2.4) and Theorem 3.3.4, we obtain

$$\begin{aligned}\|\operatorname{ext} g\|_{F^s_{p,q}(\mathbb{R}^d,w_\gamma;X)} &\le C\|\operatorname{ext} g\|_{F^{s_0}_{p,p}(\mathbb{R}^d,w_{\gamma_0};X)} \le C\|\operatorname{ext} g\|_{B^{s_0}_{p,p}(\mathbb{R}^d,w_{\gamma_0};X)}\\ &\le C\|g\|_{B^{s_0-\frac{\gamma_0+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} = \|g\|_{B^{s-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)}.\end{aligned}$$

The identity $\operatorname{Tr}\circ\operatorname{ext}=\mathrm{id}$ follows from Theorem 3.3.4. □

3.3.2. The higher-order trace operator

In this section, we study the m-th order trace operator $\mathrm{Tr}_m = \mathrm{Tr} \circ \partial_1^m$ for $m \in \mathbb{N}_1$. Furthermore, we will write $\mathrm{Tr}_0 = \mathrm{Tr}$.

We start with higher-order trace spaces for Besov spaces. To this end, we first define the m-th order extension operator.

Definition 3.3.6. Let $\eta_0 \in C_c^\infty((-1,1))$ and $\eta \in C_c^\infty((1,\frac{3}{2}))$ such that

$$\mathscr{F}^{-1}\eta_0(0) = \mathscr{F}^{-1}\eta(0) = 1, \tag{3.3.8}$$

where $\mathscr{F}$ is the one-dimensional Fourier transform. For $m \in \mathbb{N}_0$ and $D := -\mathrm{i}\partial_1$ we define

$$\eta_0^m := (-D)^m \eta_0 / m! \quad \text{and} \quad \eta_j^m := (-D)^m \eta(2^{-j}\cdot)/m!, \quad j \in \mathbb{N}_1.$$

Let $(\phi_j)_{j\geq 0} \in \Phi(\mathbb{R}^{d-1})$. For any $g \in \mathscr{S}(\mathbb{R}^{d-1}; X)$ we define

$$\mathrm{ext}_m g(x_1, \widetilde{x}) := \sum_{j=0}^{\infty} 2^{-j} (\mathscr{F}^{-1}\eta_j^m)(x_1)(\phi_j * g)(\widetilde{x}),$$

whenever this converges in $\mathscr{S}'(\mathbb{R}^d; X)$.

The motivation behind the definition of this extension operator ext_m is as follows. By properties of the Fourier transform, we have

$$(\mathscr{F}^{-1}\eta_j^m)(x_1) = \frac{2^j x_1^m}{m!} (\mathscr{F}^{-1}\zeta)(2^j x_1), \qquad j \in \mathbb{N}_0,\ x_1 \in \mathbb{R},$$

where $\zeta := \eta$ if $j \geq 1$ and $\zeta := \eta_0$ if $j = 0$. Using the product rule, one can calculate the derivatives ∂_1^m of ext_m to show that $\mathrm{Tr}_m \circ \mathrm{ext}_m = \mathrm{Tr} \circ \partial_1^m \circ \mathrm{ext}_m = \mathrm{id}$, i.e., ext_m is a right inverse to the m-th order trace operator. We make this rigorous in Theorems 3.3.8 and 3.3.9 for Besov and Triebel–Lizorkin spaces, respectively.

We start by showing the continuity of the extension operator ext_m on Besov spaces. The following lemma is analogous to [16, Lemma VIII.1.2.7].

Lemma 3.3.7. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $m \in \mathbb{N}_0$, $\gamma > -1$, $s > m + \frac{\gamma+1}{p}$ and let X be a Banach space. Then for $g \in B_{p,q}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$ it holds that*

(i) $\mathrm{ext}_m g$ *exists in* $B_{p,q}^s(\mathbb{R}^d, w_\gamma; X)$ *if* $q \in [1,\infty)$,

(ii) $\mathrm{ext}_m g$ *exists in* $B_{p,1}^t(\mathbb{R}^d, w_\gamma; X)$ *for any* $t < s$ *if* $q \in [1,\infty]$.

Moreover, for any $q \in [1,\infty]$, we have that

$$\mathrm{ext}_m : B_{p,q}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X) \to B_{p,q}^s(\mathbb{R}^d, w_\gamma; X)$$

is continuous.

Proof. Step 1: preparations. For $m, j \in \mathbb{N}_0$ and $(\phi_j)_{j\geq 0} \in \Phi(\mathbb{R}^{d-1})$ let η_j^m and ext_m be as defined in Definition 3.3.6. Let $\mathscr{F}_d$ be the d-dimensional Fourier transform. Moreover, let $(\rho_j)_{j\geq 0} \in \Phi(\mathbb{R})$ and set $\mathscr{F}_d \varphi_j(x) := (\mathscr{F}_1 \rho_j)(x_1)(\mathscr{F}_{d-1}\phi_j)(\tilde{x})$. For notational convenience we write $\widehat{\varphi}_j(x) = \widehat{\rho}_j(x_1)\widehat{\phi}_j(\tilde{x})$.

Let $g \in B_{p,q}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$ and for notational convenience we introduce

$$\widehat{g} := \mathscr{F}_{d-1} g \quad \text{and} \quad g_j := (\mathscr{F}_1^{-1}\eta_j^m)(\phi_j * g) \quad \text{for } j \in \mathbb{N}_0.$$

Note that for $k, j \in \mathbb{N}_1$ we have

$$\mathscr{F}_d(\varphi_k * g_j) = \widehat{\varphi}_k \mathscr{F}_d g_j = \widehat{\rho}_k \widehat{\phi}_k \mathscr{F}_d\big((\mathscr{F}_1^{-1}\eta_j^m)(\mathscr{F}_{d-1}^{-1}\widehat{\phi}_j \widehat{g})\big) = (\widehat{\rho}_k \eta_j^m)(\widehat{\phi}_k \widehat{\phi}_j \widehat{g}),$$

and thus $\varphi_k * g_j = \mathscr{F}_d^{-1}\big[(\widehat{\rho}_k \eta_j^m)(\widehat{\phi}_k \widehat{\phi}_j \widehat{g})\big]$. Since $\operatorname{supp} \eta_j^m \subseteq \{\xi_1 \in \mathbb{R} : 2^j < |\xi_1| < 3 \cdot 2^{j-1}\} \subseteq \operatorname{supp} \widehat{\rho}_j$, we have

$$\varphi_k * g_j = 0 \quad \text{if } |k-j| \geq 2. \tag{3.3.9}$$

Moreover, by properties of the Fourier transform, we obtain for $j \geq 1$

$$\mathscr{F}_1^{-1}\eta_j^m = \frac{2^{-jm}}{m!}\mathscr{F}_1^{-1}\big(((-D)^m \eta)(2^{-j}\cdot)\big) = \frac{2^{-j(m-1)}}{m!}\big(\mathscr{F}_1^{-1}((-D)^m\eta)\big)(2^j\cdot),$$

and with a substitution $x \mapsto 2^{-j}x$ we obtain that

$$\|\mathscr{F}_1^{-1}\eta_j^m\|_{L^p(\mathbb{R}, w_\gamma)} = 2^{j(1-m-\frac{\gamma+1}{p})}\|\mathscr{F}_1^{-1}\eta^m\|_{L^p(\mathbb{R}, w_\gamma)}, \tag{3.3.10}$$

where $\eta^m := (-D)^m \eta / m!$. Define $s_n g := \sum_{j=0}^n 2^{-j} g_j$ for $n \in \mathbb{N}_0$. Therefore, for $0 \leq k \leq n+1$ we obtain from (3.3.9), Lemma 3.2.1 and (3.3.10) that

$$\begin{aligned}
&2^{ks}\|\varphi_k * s_n g\|_{L^p(\mathbb{R}^d, w_\gamma; X)} \\
&\quad = 2^{ks}\Big\| \sum_{j=k-1}^{k+1} \varphi_k * 2^{-j} g_j \Big\|_{L^p(\mathbb{R}^d, w_\gamma; X)} \\
&\quad \leq C \sum_{j=-1}^{1} 2^{(k+j)s-(k+j)}\|g_{k+j}\|_{L^p(\mathbb{R}^d, w_\gamma; X)} \\
&\quad = C \sum_{j=-1}^{1} 2^{(k+j)s-(k+j)}\|\mathscr{F}_1^{-1}\eta_{k+j}^m\|_{L^p(\mathbb{R}, w_\gamma)}\|\phi_{k+j} * g\|_{L^p(\mathbb{R}^{d-1}; X)} \\
&\quad \leq C \sum_{j=-1}^{1} 2^{(k+j)(s-m-\frac{\gamma+1}{p})}\|\phi_{k+j} * g\|_{L^p(\mathbb{R}^{d-1}; X)},
\end{aligned} \tag{3.3.11}$$

and if $k \geq n+2$ we find

$$2^{ks}\|\varphi_k * s_n g\|_{L^p(\mathbb{R}^d, w_\gamma; X)} = 0. \tag{3.3.12}$$

Step 2: existence. For $g \in B_{p,q}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$ we prove that

(i) $\mathrm{ext}_m g$ exists in $B_{p,q}^s(\mathbb{R}^d, w_\gamma; X)$ if $q \in [1, \infty)$,

(ii) $\operatorname{ext}_m g$ exists in $B^t_{p,1}(\mathbb{R}^d, w_\gamma; X)$ for any $t < s$ if $q \in [1,\infty]$.

If $q \in [1,\infty)$ and $0 < \ell < n$, then by (3.3.9), (3.3.11) and (3.3.12), we have

$$\begin{aligned}\|s_n g - s_\ell g\|_{B^s_{p,q}(\mathbb{R}^d,w_\gamma;X)} &= \Big(\sum_{k=\ell}^{n+1}\big(2^{ks}\|\varphi_k * (s_n g - s_\ell g)\|_{L^p(\mathbb{R}^d,w_\gamma;X)}\big)^q\Big)^{\frac{1}{q}}\\ &\le C\Big(\sum_{k=\ell}^{n+1}\Big(\sum_{j=-1}^{1} 2^{(k+j)(s-m-\frac{\gamma+1}{p})}\|\phi_{k+j} * g\|_{L^p(\mathbb{R}^{d-1};X)}\Big)^q\Big)^{\frac{1}{q}} \qquad (3.3.13)\\ &\le C\Big(\sum_{k=\ell-1}^{n+2}\big(2^{k(s-m-\frac{\gamma+1}{p})}\|\phi_k * g\|_{L^p(\mathbb{R}^{d-1};X)}\big)^q\Big)^{\frac{1}{q}},\end{aligned}$$

which converges to zero as $\ell, n \to \infty$ since $g \in B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$. Thus, $(s_n g)_{n\ge 0}$ is a Cauchy sequence in $B^s_{p,q}(\mathbb{R}^d, w_\gamma; X)$. Hence, $\operatorname{ext}_m g = \sum_{j=0}^\infty 2^{-j} g_j = \lim_{n\to\infty} s_n g$ exists in this space and this proves (i).

If $q \in [1,\infty]$ and $t < s$, then by [126, Theorem 14.4.19] it follows that

$$g \in B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X) \hookrightarrow B^{t-m-\frac{\gamma+1}{p}}_{p,1}(\mathbb{R}^{d-1}; X).$$

Therefore, (i) implies that $\operatorname{ext}_m g$ exists in $B^t_{p,1}(\mathbb{R}^d, w_\gamma; X)$ and this proves (ii).

Step 3: continuity. Let $g \in B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$ with $q \in [1,\infty]$. Take $k \in \mathbb{N}_0$ and $0 < \ell < n$, then the estimate

$$2^{kt}\|\varphi_k * (s_n g - s_\ell g)\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \le C\|s_n g - s_\ell g\|_{B^t_{p,1}(\mathbb{R}^d,w_\gamma;X)},$$

together with (3.3.13) implies that $(\varphi_k * s_n g)_{n\ge 0}$ is a Cauchy sequence in $L^p(\mathbb{R}^d, w_\gamma; X)$ for all $k \in \mathbb{N}_0$. Therefore, its limit $w_k = \lim_{n\to\infty} \varphi_k * s_n g$ exists in $L^p(\mathbb{R}^d, w_\gamma; X)$ and thus in $\mathscr{S}'(\mathbb{R}^d; X)$ as well. Since $B^t_{p,1}(\mathbb{R}^d, w_\gamma; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$, Step 2(ii) implies that $s_n g \to \operatorname{ext}_m g$, and thus also $\varphi_k * s_n g \to \varphi_k * \operatorname{ext}_m g$, both in $\mathscr{S}'(\mathbb{R}; X)$ as $n \to \infty$. We conclude that $w_k = \varphi_k * \operatorname{ext}_m g$ and

$$\varphi_k * s_n g \to \varphi_k * \operatorname{ext}_m g \quad \text{in } L^p(\mathbb{R}^d, w_\gamma; X) \text{ as } n \to \infty. \qquad (3.3.14)$$

The convergence in (3.3.14) shows that we can let $n \to \infty$ in (3.3.11) to obtain for any $k \in \mathbb{N}_0$

$$2^{ks}\|\varphi_k * \operatorname{ext}_m g\|_{L^p(\mathbb{R}^d,w_\gamma;X)} \le C\sum_{j=-1}^{1} 2^{(k+j)(s-m-\frac{\gamma+1}{p})}\|\phi_{k+j} * g\|_{L^p(\mathbb{R}^{d-1};X)},$$

and therefore

$$\begin{aligned}\|\operatorname{ext}_m g\|_{B^s_{p,q}(\mathbb{R}^d,w_\gamma;X)} &\le C\Big\|\big(2^{k(s-m-\frac{\gamma+1}{p})}\|\phi_k * g\|_{L^p(\mathbb{R}^{d-1};X)}\big)_{k\ge 0}\Big\|_{\ell^q}\\ &\le C\|g\|_{B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1};X)}.\end{aligned}$$

To conclude, $\operatorname{ext}_m g \in B^s_{p,q}(\mathbb{R}^d, w_\gamma; X)$ and the lemma is proved. □

Using Lemma 3.3.7 we can now determine the higher-order trace spaces of weighted Besov spaces and show that the corresponding extension operator is indeed given as in Definition 3.3.6. We follow the arguments from [16, Theorem VIII.1.2.1].

Theorem 3.3.8. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $m \in \mathbb{N}_0$, $\gamma > -1$, $s > m + \frac{\gamma+1}{p}$ and let X be a Banach space. Then*

$$\mathrm{Tr}_m : B^s_{p,q}(\mathbb{R}^d, w_\gamma; X) \to B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator. Moreover, the extension operator ext_m *from Definition 3.3.6 defines a continuous right inverse of* Tr_m *which is independent of s, p, q, γ and X. For any $0 \le j < m$, we have* $\mathrm{Tr}_j \circ \mathrm{ext}_m = 0$.

Proof. Step 1: trace operator. Note that $\partial_1^m : B^s_{p,q}(\mathbb{R}^d, w_\gamma; X) \to B^{s-m}_{p,q}(\mathbb{R}^d, w_\gamma; X)$ is continuous, see [211, Proposition 3.10]. Therefore, since $s - m > \frac{\gamma+1}{p}$ it follows from Theorem 3.3.4 that

$$\mathrm{Tr}_m = \mathrm{Tr} \circ \partial_1^m : B^s_{p,q}(\mathbb{R}^d, w_\gamma; X) \to B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$$

is continuous.

Step 2: extension operator. Let ext_m be the extension operator from Definition 3.3.6. The continuity follows from Lemma 3.3.7. It remains to prove that ext_m is the right inverse of Tr_m and that

$$\mathrm{Tr}_j(\mathrm{ext}_m g) = \delta_{jm} g, \qquad 0 \le j \le m,\ g \in B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X), \tag{3.3.15}$$

where δ_{jm} is the Kronecker delta. Note that from Definition 3.3.6 and properties of the Fourier transform it follows that for $n \ge 0$

$$(\mathscr{F}^{-1}\eta_n^m)(x_1) = \frac{2^n x_1^m}{m!}(\mathscr{F}^{-1}\zeta)(2^n x_1), \qquad x_1 \in \mathbb{R}, \tag{3.3.16}$$

where $\mathscr{F}$ is the one-dimensional Fourier transform and $\zeta := \eta$ if $n \ge 1$ and $\zeta := \eta_0$ if $n = 0$. Therefore, if $g \in \mathscr{S}(\mathbb{R}^{d-1}; X)$, then using $\mathrm{Tr}_j = \mathrm{Tr} \circ \partial_1^j$, the product rule and (3.3.8) yields

$$\mathrm{Tr}_j\left(2^{-n}(\mathscr{F}^{-1}\eta_n^m)(\phi_n * g)\right) = \delta_{jm}(\phi_n * g), \qquad 0 \le j \le m,\ n \ge 0. \tag{3.3.17}$$

Therefore, by continuity of the trace operator (Step 1) and (3.3.17), we find for $0 \le j \le m$

$$\mathrm{Tr}_j(\mathrm{ext}_m g) = \sum_{n=0}^{\infty} \mathrm{Tr}_j\left(2^{-n}(\mathscr{F}^{-1}\eta_n^m)(\phi_n * g)\right) = \delta_{jm} \sum_{n=0}^{\infty} (\phi_n * g) = \delta_{jm} g,$$

using properties of the Littlewood–Paley sequence in the last identity. By density (see [211, Lemma 3.8]) this extends to all $g \in B^{s-m-\frac{\gamma+1}{p}}_{p,q}(\mathbb{R}^{d-1}; X)$ with $q < \infty$. If $q = \infty$, then we have $B^{s-m-\frac{\gamma+1}{p}}_{p,\infty}(\mathbb{R}^{d-1}; X) \hookrightarrow B^{s-m-\frac{\gamma+1}{p}-\varepsilon}_{p,1}(\mathbb{R}^{d-1}; X)$ for all $\varepsilon > 0$ (see [126, Theorem 14.4.19]). This proves (3.3.15). □

We continue with the characterisation of higher-order trace spaces for Triebel–Lizorkin spaces.

Theorem 3.3.9. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $m \in \mathbb{N}_0$, $\gamma > -1$, $s > m + \frac{\gamma+1}{p}$ and let X be a Banach space. Then*

$$\mathrm{Tr}_m : F^s_{p,q}(\mathbb{R}^d, w_\gamma; X) \to B^{s-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator. Moreover, the extension operator ext_m *from Definition 3.3.6 defines a continuous right inverse of* Tr_m *which is independent of s, p, q, γ and X. For any $0 \le j < m$, we have* $\mathrm{Tr}_j \circ \mathrm{ext}_m = 0$.

Proof. The proof is similar to the proof of Theorem 3.3.5 using Theorem 3.3.8 instead of Theorem 3.3.4. □

3.4. Trace spaces of Bessel potential and Sobolev spaces

Let $p \in (1,\infty)$, X a Banach space and let $\mathscr{O} \subseteq \mathbb{R}^d$ be open. Recall that the power weight $w^{\partial\mathscr{O}}_\gamma(x) := \mathrm{dist}(x,\partial\mathscr{O})^\gamma$ for $x \in \mathscr{O}$ and $\gamma > -1$. In the rest of this chapter, we mainly consider the following two weighted spaces:

(i) Bessel potential spaces $H^{s,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)$ with $s \in \mathbb{R}$ and $\gamma \in (-1, p-1)$,

(ii) Sobolev spaces $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)$ with $k \in \mathbb{N}_0$ and $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$.

To finally prove our main results in Section 3.6 about complex interpolation for these weighted spaces on bounded domains, it suffices to consider the half-space $\mathscr{O} = \mathbb{R}^d_+$ by localisation arguments. In this case, we write $w_\gamma(x) = w^{\partial\mathbb{R}^d_+}_\gamma(x) = |x_1|^\gamma$ for $x = (x_1, \widetilde{x}) \in \mathbb{R}^d$ as before.

The outline of this section is as follows. We start with the trace spaces for the above-mentioned weighted Bessel potential and Sobolev spaces on $\mathbb{R}^d$ and $\mathbb{R}^d_+$ in Section 3.4.1. Afterwards, we apply the trace theorems to prove some density results in Section 3.4.2.

3.4.1. Trace spaces of weighted spaces on the half-space

Using the results from Section 3.3.2 about weighted Besov and Triebel–Lizorkin spaces, we will now characterise higher-order trace spaces of weighted Bessel potential and Sobolev spaces. The methods we will employ depend on the weight exponent γ. For $\gamma \in (-1, p-1)$, i.e., w_γ is a Muckenhoupt weight, we can use standard techniques relying on reflection operators to transfer results from $\mathbb{R}^d$ to $\mathbb{R}^d_+$. For $\gamma > p-1$ such reflection techniques do not work since weighted Bessel potential and Sobolev spaces on $\mathbb{R}^d$ cannot be defined as usual if $\gamma > p-1$ (cf. Remark 2.2.1). Nonetheless, the definition of a Sobolev space on $\mathbb{R}^d_+$ does make sense for $\gamma > p-1$ and the main difficulty is to establish the results for this regime of weight exponents.

We first consider the case $\gamma \in (-1, p-1)$, where we can use the sandwiching embeddings (3.2.2) to obtain that, e.g.,

$$\mathrm{Tr}_m : H^{s,p}(\mathbb{R}^d, w_\gamma; X) \to B_{p,p}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X), \qquad s > m + \tfrac{\gamma+1}{p},$$

is a continuous and surjective operator, see Theorem 3.4.1 below. If $f_1, f_2 \in H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ satisfy $f_1|_{\mathbb{R}^d_+} = f_2|_{\mathbb{R}^d_+}$, then we have $\mathrm{Tr}_j f_1 = \mathrm{Tr}_j f_2$ for all $j \in \{0, \dots, m\}$. This can be proved similarly as in [189, Proposition 6.3(2)]. This implies that the trace operator Tr_m on $\mathbb{R}^d$ gives rise to a well-defined trace operator on $\mathbb{R}^d_+$, which we denote by Tr_m as well. Moreover, if f is continuous on $[0,\delta) \times \mathbb{R}^{d-1}$ for some $\delta > 0$, then the trace operator coincides with the restriction of f to $\{(0, \widetilde{x}) : \widetilde{x} \in \mathbb{R}^{d-1}\}$.

For weighted Bessel potential and Sobolev spaces on $\mathbb{R}^d$ and $\mathbb{R}^d_+$ with $\gamma \in (-1, p-1)$, we obtain the following trace results.

Theorem 3.4.1. *Let $p \in (1,\infty)$, $m \in \mathbb{N}_0$, $\gamma \in (-1, p-1)$, $\mathscr{O} \in \{\mathbb{R}^d, \mathbb{R}^d_+\}$ and let X be a Banach space.*

(i) If $s > 0$ satisfies $s > m + \frac{\gamma+1}{p}$, then

$$\mathrm{Tr}_m : H^{s,p}(\mathscr{O}, w_\gamma; X) \to B_{p,p}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator.

(ii) If $k \in \mathbb{N}_1$ satisfies $k > m + \frac{\gamma+1}{p}$, then

$$\mathrm{Tr}_m : W^{k,p}(\mathscr{O}, w_\gamma; X) \to B_{p,p}^{k-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator.

In both cases, there exists a continuous right inverse ext_m of Tr_m which is independent of k, s, p, γ and X. For any $0 \le j < m$, we have $\mathrm{Tr}_j \circ \mathrm{ext}_m = 0$.

Before we can prove Theorem 3.4.1, we need the following trace result for Besov spaces on the half-space. This proposition is used to prove Theorem 3.4.1 without a UMD condition for the Banach space X, see also Remark 3.4.4 below.

Proposition 3.4.2. *Let $p \in (1,\infty)$, $q \in [1,\infty]$, $m \in \mathbb{N}_0$, $\gamma \in (-1, p-1)$, $s > m + \frac{\gamma+1}{p}$ and let X be a Banach space. Then*

$$\mathrm{Tr}_m : B^s_{p,q}(\mathbb{R}^d_+, w_\gamma; X) \to B_{p,q}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator. Moreover, there exists a continuous right inverse ext_m of Tr_m which is independent of k, s, p, γ and X. For any $0 \le j < m$, we have $\mathrm{Tr}_j \circ \mathrm{ext}_m = 0$.

Proof. The result follows from Theorem 3.3.8 if we have a reflection operator between Besov spaces on $\mathbb{R}^d_+$ and $\mathbb{R}^d$. To obtain this reflection operator, note that for any $\widetilde{s} > 0$, $\theta \in (0,1)$, $s_0, s_1 \in \mathbb{N}_0$ such that $s_0 < s_1$ and $\widetilde{s} = (1-\theta)s_0 + \theta s_1$, we have

$$(W^{s_0,p}(\mathbb{R}^d_+, w_\gamma; X), W^{s_1,p}(\mathbb{R}^d_+, w_\gamma; X))_{\theta,q} = B^{\widetilde{s}}_{p,q}(\mathbb{R}^d_+, w_\gamma; X). \tag{3.4.1}$$

This identity follows from [211, Proposition 6.1] (using $\gamma \in (-1, p-1)$) and [189, Lemma 5.4] (which holds for real interpolation as well, see [262, Section 1.2.4]). Now, the higher-order reflection operator on Sobolev spaces from [189, Lemma 5.1] is also bounded on Besov spaces by (3.4.1). □

Remark 3.4.3. Using Theorem 3.3.9 a similar result as in Proposition 3.4.2 holds for the trace on Triebel–Lizorkin spaces on $\mathbb{R}^d_+$. For Triebel–Lizorkin spaces the construction of a reflection operator is more involved, since Triebel–Lizorkin spaces arise as interpolation spaces of Sobolev spaces using the ℓ^q-interpolation method, see [172, 186]. Alternatively, one can also redo [263, Theorem 2.9.2] using the weighted multiplier result [213, Theorem 1.3].

We will now provide the proof of Theorem 3.4.1.

Proof of Theorem 3.4.1. Step 1: $\mathscr{O} = \mathbb{R}^d$. This follows immediately from Theorem 3.3.9 and the embeddings (3.2.2a) and (3.2.2b). Note that these embeddings are only valid for $w \in A_p(\mathbb{R}^d)$, i.e., $\gamma \in (-1, p-1)$.

Step 2: $\mathscr{O} = \mathbb{R}^d_+$. Statement (ii) with $\mathscr{O} = \mathbb{R}^d_+$ follows from Step 1 and a higher-order reflection argument, see [189, Lemma 5.1].

It remains to prove statement (i) with $\mathscr{O} = \mathbb{R}^d_+$. Let $\varepsilon > 0$ be small and set $\gamma_0 := \gamma + \varepsilon$, $\gamma_1 := \gamma - \varepsilon$, $s_0 := s + \varepsilon/p$ and $s_1 := s - \varepsilon/p$. From Theorem 3.2.4, (3.2.2a) and (3.2.4) (which also hold on $\mathbb{R}^d_+$ since the spaces on $\mathbb{R}^d_+$ are defined as factor spaces), we have the chain of embeddings

$$\begin{aligned} B^{s_0}_{p,p}(\mathbb{R}^d_+, w_{\gamma_0}; X) = F^{s_0}_{p,p}(\mathbb{R}^d_+, w_{\gamma_0}; X) &\hookrightarrow F^{s}_{p,1}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow H^{s,p}(\mathbb{R}^d_+, w_\gamma; X) \\ &\hookrightarrow F^{s}_{p,\infty}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow F^{s_1}_{p,p}(\mathbb{R}^d_+, w_{\gamma_1}; X) = B^{s_1}_{p,p}(\mathbb{R}^d_+, w_{\gamma_1}; X). \end{aligned} \tag{3.4.2}$$

From Proposition 3.4.2 and (3.4.2) it follows that for $f \in H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$

$$\begin{aligned} \|\operatorname{Tr}_m f\|_{B^{s-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} &= \|\operatorname{Tr}_m f\|_{B^{s_1-m-\frac{\gamma_1+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} \\ &\leq C\|f\|_{B^{s_1}_{p,p}(\mathbb{R}^d_+, w_{\gamma_1};X)} \leq C\|f\|_{H^{s,p}(\mathbb{R}^d_+, w_\gamma;X)} \end{aligned}$$

and for $g \in B^{s-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$

$$\begin{aligned} \|\operatorname{ext}_m g\|_{H^{s,p}(\mathbb{R}^d_+, w_\gamma;X)} &\leq C\|\operatorname{ext}_m g\|_{B^{s_0}_{p,p}(\mathbb{R}^d_+, w_{\gamma_0};X)} \\ &\leq C\|g\|_{B^{s_0-m-\frac{\gamma_0+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} = \|g\|_{B^{s-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)}. \end{aligned}$$

This finishes the proof. □

Remark 3.4.4. Alternatively, in the proof of Theorem 3.4.1(i) with $\mathscr{O} = \mathbb{R}^d_+$ above, we could have used a reflection operator from $H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$ into $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$. Such a reflection operator can be obtained using [189, Lemma 5.1] and complex interpolation. However, this requires X to be UMD, see [189, Proposition 5.6]. Instead, we did the reflection argument at the level of Besov spaces in Proposition 3.4.2, where no UMD condition was required.

Remark 3.4.5. For $m \in \mathbb{N}_0$ and $\mathscr{O} \in \{\mathbb{R}^d, \mathbb{R}^d_+\}$ it is clear that the trace operator Tr_m maps $C^\infty_c(\overline{\mathscr{O}}; X)$ into $C^\infty_c(\mathbb{R}^{d-1}; X)$. Moreover, the extension operator ext_m maps $C^\infty_c(\mathbb{R}^{d-1}; X)$ into $C^\infty(\overline{\mathscr{O}}; X)$. Indeed, let $g \in C^\infty_c(\mathbb{R}^{d-1}; X)$, then $g \in B^{s-m-\frac{1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$ and $\mathrm{ext}_m g \in H^{s,p}(\mathscr{O}; X)$ for all $s > m + \frac{1}{p}$. The Sobolev embedding implies that $\mathrm{ext}_m g \in C^\infty(\overline{\mathscr{O}}; X)$.

We extend the result for Sobolev spaces in Theorem 3.4.1 from $\gamma \in (-1, p-1)$ to $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$.

Theorem 3.4.6. *Let $p \in (1,\infty)$, $m \in \mathbb{N}_0$, $k \in \mathbb{N}_1$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ such that $k > m + \frac{\gamma+1}{p}$ and let X be a Banach space. Then*

$$\mathrm{Tr}_m : W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to B^{k-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator. Moreover, there exists a continuous right inverse ext_m of Tr_m which is independent of k, p, γ and X. For any $0 \le j < m$, we have $\mathrm{Tr}_j \circ \mathrm{ext}_m = 0$.

Proof. Let $j \in \mathbb{N}_0$ be such that $\gamma \in (jp-1, (j+1)p-1)$. The case $j = 0$ follows from Theorem 3.4.1(ii), so we can assume $j \ge 1$.

Step 1: trace operator. Let $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$, then

$$\begin{aligned}\|\mathrm{Tr}_m f\|_{B^{k-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} &= \|\mathrm{Tr}_m f\|_{B^{k-j-m-\frac{\gamma-jp+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)} \\ &\le C\|f\|_{W^{k-j,p}(\mathbb{R}^d_+, w_{\gamma-jp};X)} \le C\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma;X)},\end{aligned}$$

since $\gamma - jp \in (-1, p-1)$ and $k - j > m \ge 0$ so that Theorem 3.4.1(ii) applies. The last estimate follows from Hardy's inequality, see Lemma 2.2.2.

Step 2: extension operator. Let $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ with $|\alpha| \le k$ and let $g \in B^{k-m-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X)$. Then estimating the norm on $\mathbb{R}^d_+$ by the norm on $\mathbb{R}^d$, using (3.2.3), Theorem 3.2.4 and (3.2.4), gives

$$\begin{aligned}\|\partial^\alpha \mathrm{ext}_m g\|_{L^p(\mathbb{R}^d_+, w_\gamma;X)} &\le \|\partial^\alpha \mathrm{ext}_m g\|_{L^p(\mathbb{R}^d, w_\gamma;X)} \\ &\le C\|\partial^\alpha \mathrm{ext}_m g\|_{F^0_{p,1}(\mathbb{R}^d, w_\gamma;X)} \\ &\le C\|\partial^\alpha \mathrm{ext}_m g\|_{F^{s_0}_{p,p}(\mathbb{R}^d, w_{\gamma_0};X)} \\ &\le C\|\partial^\alpha \mathrm{ext}_m g\|_{B^{s_0}_{p,p}(\mathbb{R}^d, w_{\gamma_0};X)},\end{aligned} \tag{3.4.3}$$

where $\gamma_0 > \gamma$ and $s_0 > 0$ are such that $s_0 - \frac{\gamma_0}{p} = -\frac{\gamma}{p}$. With a similar computation as in the

proof of Lemma 3.3.7 (see (3.3.11)), it follows that for $\ell \in \mathbb{N}_0$, we have

$$\begin{aligned} &2^{\ell s_0}\|\varphi_\ell * \partial^\alpha \operatorname{ext}_m g\|_{L^p(\mathbb{R}^d, w_{\gamma_0}; X)} \\ &\le C \sum_{i=-1}^{1} 2^{\ell(s_0-1)} \|\partial_1^{\alpha_1}(\mathscr{F}^{-1}\eta_{\ell+i}^m)\|_{L^p(\mathbb{R}, w_{\gamma_0})} \|\partial^{\widetilde{\alpha}}(\phi_{\ell+i} * g)\|_{L^p(\mathbb{R}^{d-1}; X)}, \end{aligned} \tag{3.4.4}$$

where $\mathscr{F}$ is the one-dimensional Fourier transform and $(\phi_\ell)_{\ell\ge 0} \in \Phi(\mathbb{R}^{d-1})$. Taking the ℓ^p-norm of (3.4.4) and combining this with (3.4.3) gives

$$\begin{aligned} &\|\partial^\alpha \operatorname{ext}_m g\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \\ &\le C\Big(\sum_{\ell=0}^{\infty} \sum_{i=-1}^{1} 2^{\ell p(s_0-1)} \|\partial_1^{\alpha_1}(\mathscr{F}^{-1}\eta_{\ell+i}^m)\|_{L^p(\mathbb{R}, w_{\gamma_0})}^p \|\partial^{\widetilde{\alpha}}(\phi_{\ell+i} * g)\|_{L^p(\mathbb{R}^{d-1}; X)}^p \Big)^{\frac{1}{p}}. \end{aligned} \tag{3.4.5}$$

We estimate the two L^p-norms in the summation above separately. We start with estimating the $L^p(\mathbb{R}, w_{\gamma_0})$-norm. Let $\ell \in \mathbb{N}_0$ and $i \in \{-1, 0, 1\}$. By (3.3.16) and the product rule, we compute

$$\begin{aligned} \partial_{x_1}^{\alpha_1}\big(\mathscr{F}^{-1}\eta_{\ell+i}^m\big)(x_1) &= \frac{2^{\ell+i}}{m!}\, \partial_{x_1}^{\alpha_1}\big(x_1^m (\mathscr{F}^{-1}\zeta)(2^{\ell+i}x_1)\big) \\ &= \frac{2^{\ell+i}}{m!} \sum_{n=0}^{\alpha_1} c_n\, x_1^{m-n}\, 2^{(\ell+i)(\alpha_1-n)} (\mathscr{F}^{-1}\zeta)^{(\alpha_1-n)}(2^{\ell+i}x_1), \end{aligned}$$

where $c_n = \binom{\alpha_1}{n} \frac{m!}{(m-n)!}$ if $n \le m$ and zero otherwise. Taking norms and applying Hardy's inequality (Lemma 2.2.2), yields

$$\begin{aligned} &\|\partial_1^{\alpha_1}\big(\mathscr{F}^{-1}\eta_{\ell+i}^m\big)\|_{L^p(\mathbb{R}, w_{\gamma_0})}^p \\ &\quad\le C\, 2^{\ell p} \sum_{n=0}^{\alpha_1} 2^{(\ell+i)(\alpha_1-n)p} \big\| x_1 \mapsto x_1^{m-n} (\mathscr{F}^{-1}\zeta)^{(\alpha_1-n)}(2^{\ell+i}x_1)\big\|_{L^p(\mathbb{R}, w_{\gamma_0})}^p \\ &\quad\le C\, 2^{\ell p(1+\alpha_1)} \big\| x_1 \mapsto x_1^m (\mathscr{F}^{-1}\zeta)^{(\alpha_1)}(2^{\ell+i}x_1)\big\|_{L^p(\mathbb{R}, w_{\gamma_0})}^p \\ &\quad\le C\, 2^{\ell p(1+\alpha_1-m-\frac{\gamma_0+1}{p})}. \end{aligned} \tag{3.4.6}$$

For the $L^p(\mathbb{R}^{d-1}; X)$-norm we estimate with the notation $T_n g := \phi_n * g$

$$\begin{aligned} \|\partial^{\widetilde{\alpha}}(\phi_{\ell+i} * g)\|_{L^p(\mathbb{R}^{d-1}; X)}^p &\le \sum_{n=-1}^{1} \|\partial^{\widetilde{\alpha}}(\phi_{\ell+i} * T_{\ell+i+n} g)\|_{L^p(\mathbb{R}^{d-1}; X)}^p \\ &\le C\, 2^{\ell p|\widetilde{\alpha}|} \sum_{n=-1}^{1} \|T_{\ell+i+n} g\|_{L^p(\mathbb{R}^{d-1}; X)}^p. \end{aligned} \tag{3.4.7}$$

Combining the estimates (3.4.5), (3.4.6) and (3.4.7) yields

$$\begin{aligned} \|\partial^\alpha \operatorname{ext}_m g\|_{L^p(\mathbb{R}^d, w_\gamma; X)} &\le C\Big(\sum_{\ell=0}^{\infty} 2^{\ell p(s_0+|\alpha|-m-\frac{\gamma_0+1}{p})} \|T_\ell g\|_{L^p(\mathbb{R}^{d-1}; X)}^p \Big)^{\frac{1}{p}} \\ &\le C\Big(\sum_{\ell=0}^{\infty} 2^{\ell p(k-m-\frac{\gamma+1}{p})} \|\phi_\ell * g\|_{L^p(\mathbb{R}^{d-1}; X)}^p \Big)^{\frac{1}{p}} \\ &= C\|g\|_{B_{p,p}^{k-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)}, \end{aligned}$$

where the constants are independent of g and ℓ and we have used that

$$s_0 + |\alpha| - m - \frac{\gamma_0 + 1}{p} = |\alpha| - m - \frac{\gamma + 1}{p} \leq k - m - \frac{\gamma + 1}{p}.$$

This proves the continuity for the extension operator. □

To close this section we state a related result for the vector of traces given by

$$\overline{\mathrm{Tr}}_m := (\mathrm{Tr}_0, \ldots, \mathrm{Tr}_m), \qquad m \in \mathbb{N}_0.$$

We follow the proof of [16, Theorem VIII.1.3.2] to determine the trace space for $\overline{\mathrm{Tr}}_m$ from the trace space for Tr_m. A similar result is contained in [262, Section 2.9.2] for weighted scalar-valued Sobolev spaces.

Proposition 3.4.7. *Let $p \in (1,\infty)$, $m \in \mathbb{N}_0$ and let X be a Banach space.*

(i) If $s > 0$, $\gamma \in (-1, p-1)$, $s > m + \frac{\gamma+1}{p}$ and $\mathscr{O} \in \{\mathbb{R}^d, \mathbb{R}^d_+\}$, then

$$\overline{\mathrm{Tr}}_m : H^{s,p}(\mathscr{O}, w_\gamma; X) \to \prod_{j=0}^{m} B_{p,p}^{s-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator.

(ii) If $k \in \mathbb{N}_1$, $\gamma \in (-1,\infty) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$ and $k > m + \frac{\gamma+1}{p}$, then

$$\overline{\mathrm{Tr}}_m : W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to \prod_{j=0}^{m} B_{p,p}^{k-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

is a continuous and surjective operator.

In both cases, there exists a continuous right inverse $\overline{\mathrm{ext}}_m$ of $\overline{\mathrm{Tr}}_m$ which is independent of k, s, p, γ and X.

Proof. The proof is similar to [16, Theorem VIII.1.3.2] and for the convenience of the reader, we sketch the proof.

Step 1: proof of (i). For $j \in \{0, \ldots, m\}$ let Tr_j and ext_j be as in Theorem 3.4.1(i). The continuity of $\overline{\mathrm{Tr}}_m$ follows from Theorem 3.4.1(i). We continue with constructing an extension operator $\overline{\mathrm{ext}}_m$. Let

$$(g_0, \ldots, g_m) \in \prod_{j=0}^{m} B_{p,p}^{s-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

and define $f_0 := \mathrm{ext}_0 g_0$. For $j \in \{1, \ldots, m\}$ we recursively define

$$f_j := f_{j-1} + \mathrm{ext}_j(g_j - \mathrm{Tr}_j f_{j-1}) = H_j f_{j-1} + \mathrm{ext}_j g_j,$$

where $H_j := 1 - \mathrm{ext}_j \circ \mathrm{Tr}_j$ is a bounded operator on $H^{s,p}(\mathscr{O}, w_\gamma; X)$ by Theorem 3.4.1(i). It is straightforward to verify that

$$f_m = \mathrm{ext}_m g_m + \sum_{j=0}^{m-1} \Big(\prod_{n=j+1}^{m} H_n \Big) \mathrm{ext}_j g_j. \tag{3.4.8}$$

Moreover, note that from Theorem 3.4.1(i) we have for $j \in \{0,\dots,m\}$

$$\begin{aligned} \operatorname{Tr}_j \circ H_j &= \operatorname{Tr}_j \circ (1 - \operatorname{ext}_j \circ \operatorname{Tr}_j) = \operatorname{Tr}_j - \operatorname{Tr}_j = 0 \quad \text{and} \\ \operatorname{Tr}_{j_1} \circ H_j &= \operatorname{Tr}_{j_1} \circ (1 - \operatorname{ext}_j \circ \operatorname{Tr}_j) = \operatorname{Tr}_{j_1} \qquad \text{for } j_1 \in \{0,\dots,j-1\}. \end{aligned} \tag{3.4.9}$$

From (3.4.8) and (3.4.9) it follows that $\operatorname{Tr}_j f_m = g_j$ for $j \in \{0,\dots,m\}$. Hence, we can define $\overline{\operatorname{ext}}_m(g_0,\dots,g_m) := f_m$ and it is straightforward to verify that this operator satisfies the desired properties.

Step 2: proof of (ii). This case is similar to Step 1 using Theorem 3.4.6 instead of Theorem 3.4.1(i). □

3.4.2. Density results

As an application of the trace theorems in the previous section, we prove density of certain classes of test functions in weighted spaces with zero boundary conditions.

Bessel potential spaces

Let $p \in (1,\infty)$, $s \in \mathbb{R}$, $\gamma \in (-1,p-1)$, $\mathscr{O} \in \{\mathbb{R}^d, \mathbb{R}^d_+\}$ and let X be a Banach space. Then we define

$$H^{s,p}_0(\mathscr{O}, w_\gamma; X) := \Big\{ f \in H^{s,p}(\mathscr{O}, w_\gamma; X) : \operatorname{Tr}(\partial^\alpha f) = 0 \text{ if } s - |\alpha| > \tfrac{\gamma+1}{p} \Big\}.$$

All the traces in the above definition are well defined by Theorem 3.4.1. We note that if $s \le \frac{\gamma+1}{p}$, then

$$H^{s,p}_0(\mathscr{O}, w_\gamma; X) = H^{s,p}(\mathscr{O}, w_\gamma; X). \tag{3.4.10}$$

Moreover, we set $\mathbb{R}^d_\bullet := \mathbb{R}^d \setminus \{(0,\widetilde{x}) : \widetilde{x} \in \mathbb{R}^{d-1}\}$.

To prove a density result for $H^{s,p}_0$, we first record the following proposition about the boundedness of the pointwise multiplication operator $\mathbf{1}_{\mathbb{R}^d_+}$. The following proposition is an extension of [189, Proposition 6.2].

Proposition 3.4.8. *Let $p \in (1,\infty)$, $\gamma \in (-1,p-1)$ and let X be a Banach space. Let $s > -1 + \frac{\gamma+1}{p}$ and $m \in \mathbb{N}_0 \cup \{-1\}$ be such that $m + \frac{\gamma+1}{p} < s < m + 1 + \frac{\gamma+1}{p}$. Then for all $f \in H^{s,p}(\mathbb{R}^d, w_\gamma; X) \cap \{g \in W^{m+1,1}_{\mathrm{loc}}(\mathbb{R}^d; X) : g(0,\cdot) = \cdots = (\partial_1^m g)(0,\cdot) = 0\}$ we have*

$$\big\| \mathbf{1}_{\mathbb{R}^d_+} f \big\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)} \le C \| f \|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)},$$

where the constant $C > 0$ only depends on p, γ, s and X.

As a consequence, $\mathbf{1}_{\mathbb{R}^d_+}$ is a pointwise multiplier on the closure of $C^\infty_{\mathrm{c}}(\mathbb{R}^d_\bullet; X)$ with respect to the norm of $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$. Moreover, it holds that

$$\partial^\alpha(\mathbf{1}_{\mathbb{R}^d_+} f) = \mathbf{1}_{\mathbb{R}^d_+}(\partial^\alpha f) \qquad \text{for } |\alpha| \le m \text{ and } f \in \overline{C^\infty_{\mathrm{c}}(\mathbb{R}^d_\bullet; X)}^{H^{s,p}(\mathbb{R}^d, w_\gamma; X)}.$$

The proof of Proposition 3.4.8 relies on an improvement of [189, Lemma 4.2], where the UMD condition on the Banach space can be omitted.

Lemma 3.4.9. *Let $p \in (1,\infty)$, $s \in \mathbb{R}$, $\sigma \geq 0$, $w \in A_p(\mathbb{R}^d)$ and let X be a Banach space. Then $f \in H^{s+\sigma,p}(\mathbb{R}^d, w; X)$ if and only if $f, (-\Delta)^{\sigma/2} \in H^{s,p}(\mathbb{R}^d, w; X)$. In this case, it holds that*

$$\|f\|_{H^{s+\sigma,p}(\mathbb{R}^d,w;X)} \eqsim \|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} + \|(-\Delta)^{\frac{\sigma}{2}} f\|_{H^{s,p}(\mathbb{R}^d,w;X)},$$

where the constant only depends on p, s, σ, w and X.

Proof. The Laplacian $-\Delta$ is a sectorial operator on $L^p(\mathbb{R}^d, w; X)$ with domain $D(\Delta) = H^{2,p}(\mathbb{R}^d, w; X)$. By lifting we obtain that $-\Delta$ is a sectorial operator on $H^{s,p}(\mathbb{R}^d, w; X)$ with $D(\Delta) = H^{s+2,p}(\mathbb{R}^d, w; X)$ for any $s \in \mathbb{R}$. Now, [126, Proposition 15.2.12] yields for any $s \in \mathbb{R}$ and $\sigma \geq 0$ that

$$\begin{aligned}\|f\|_{H^{s+\sigma,p}(\mathbb{R}^d,w;X)} &= \left\|(1-\Delta)^{\frac{\sigma}{2}} f\right\|_{H^{s,p}(\mathbb{R}^d,w;X)} \\ &\eqsim \|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} + \left\|(-\Delta)^{\frac{\sigma}{2}} f\right\|_{H^{s,p}(\mathbb{R}^d,w;X)}.\end{aligned}$$

This finishes the proof. □

As a consequence of Lemma 3.4.9, we can answer the question for which Banach spaces X the pointwise multiplication operator $\mathbf{1}_{\mathbb{R}^d_+}$ is a bounded operator on $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$. In particular, this solves [126, Problem Q.12].

Theorem 3.4.10. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, $-1 + \frac{\gamma+1}{p} < s < \frac{\gamma+1}{p}$ and let X be a Banach space. Then the pointwise multiplication operator $\mathbf{1}_{\mathbb{R}^d_+}$ extends to a bounded operator on $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$.*

Proof. The proof is precisely the same as [189, Theorem 4.1], where this theorem is proved for UMD Banach spaces X. The only place where their proof uses the UMD condition is through the norm equivalence in [189, Lemma 4.2]. Lemma 3.4.9 gives precisely this norm equivalence without the UMD condition. □

We now turn to the proof of Proposition 3.4.8. Comparing the statements of Proposition 3.4.8 and Theorem 3.4.10, we note that in Theorem 3.4.10 the additional condition $s < (\gamma+1)/p$ is required. This is precisely the setting where no traces at $\partial\mathbb{R}^d_+$ exist. In the setting of Proposition 3.4.8, certain traces exist if $s \geq (\gamma+1)/p$, but all these traces are zero. This implies that derivatives and $\mathbf{1}_{\mathbb{R}^d_+}$ commute, which allows us to remove the condition $s < (\gamma+1)/p$.

Proof of Proposition 3.4.8. The proof is completely analogous to [189, Lemma 6.1 & Proposition 6.2]. First, [189, Lemma 6.1] can be extended from $d = 1$ to $d \geq 1$. Afterwards, redoing the proof of [189, Proposition 6.2] using Theorem 3.4.10 instead of [189, Theorem 4.1], gives the desired result. □

We continue with a density result for $H^{s,p}_0(\mathscr{O}, w_\gamma; X)$, which will be needed in Section 3.6 to characterise complex interpolation spaces of Bessel potential spaces with vanishing boundary conditions. The proposition below is an extension of the results in [189, Propositions 6.4 & 6.6].

Proposition 3.4.11. *Let $p \in (1,\infty)$, $s > -1 + \frac{\gamma+1}{p}$ such that $s \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$, $\gamma \in (-1, p-1)$ and let X be a Banach space. Then*

(i) $H_0^{s,p}(\mathbb{R}^d, w_\gamma; X) = \overline{C_c^\infty(\mathbb{R}^d_\bullet; X)}^{H^{s,p}(\mathbb{R}^d, w_\gamma; X)}$,

(ii) $H_0^{s,p}(\mathbb{R}^d_+, w_\gamma; X) = \overline{C_c^\infty(\mathbb{R}^d_+; X)}^{H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)}$.

Proof. Let $m \in \mathbb{N}_0 \cup \{-1\}$ be such that $m + \frac{\gamma+1}{p} < s < m+1+\frac{\gamma+1}{p}$. For $m = -1$ no traces exist (see also (3.4.10)) and the arguments below can easily be adapted to this case by ignoring any trace conditions. Moreover, we note that derivatives in the directions $\widetilde{x} = (x_2, \dots, x_d)$ commute with the trace operator (which follows with a density argument), and therefore for $m \in \mathbb{N}_0$ and $\mathscr{O} \in \{\mathbb{R}^d, \mathbb{R}^d_+\}$ it holds that

$$H_0^{s,p}(\mathscr{O}, w_\gamma; X) = \{f \in H^{s,p}(\mathscr{O}, w_\gamma; X) : \overline{\mathrm{Tr}}_m f = 0\}.$$

Step 1: proof of (i). For $f \in C_c^\infty(\mathbb{R}^d_\bullet; X)$ it is clear that $\overline{\mathrm{Tr}}_m f = 0$ and by continuity (see Proposition 3.4.7(i)) this extends to the closure of $C_c^\infty(\mathbb{R}^d_\bullet; X)$. Conversely, let $f \in H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ such that $\overline{\mathrm{Tr}}_m f = 0$. By [189, Lemma 3.4] there exists a sequence $(f_n^{(1)})_{n\geq 1} \subseteq C_c^\infty(\mathbb{R}^d; X)$ such that

$$f_n^{(1)} \to f \quad \text{in } H^{s,p}(\mathbb{R}^d, w_\gamma; X) \text{ as } n \to \infty. \tag{3.4.11}$$

Let $\overline{\mathrm{ext}}_m$ be as in Proposition 3.4.7(i) and define $f_n^{(2)} := f_n^{(1)} - \overline{\mathrm{ext}}_m(\overline{\mathrm{Tr}}_m f_n^{(1)})$ for $n \in \mathbb{N}_1$. Then by Remark 3.4.5 we have $f_n^{(2)} \in C^\infty(\mathbb{R}^d; X)$ and from Proposition 3.4.7(i) it follows that $\overline{\mathrm{Tr}}_m f_n^{(2)} = 0$ for all $n \in \mathbb{N}_1$. Moreover, using $\overline{\mathrm{Tr}}_m f = 0$, Proposition 3.4.7(i) twice and (3.4.11), it follows

$$\begin{aligned}\|\overline{\mathrm{ext}}_m(\overline{\mathrm{Tr}}_m f_n^{(1)})\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)} &\leq C \sum_{j=0}^m \|\overline{\mathrm{Tr}}_m f - \overline{\mathrm{Tr}}_m f_n^{(1)}\|_{B_{p,p}^{s-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)} \\ &\leq C\|f - f_n^{(1)}\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)} \to 0 \quad \text{as } n \to \infty\end{aligned}$$

and therefore

$$\|f - f_n^{(2)}\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)} \leq \|f - f_n^{(1)}\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)} + \|\overline{\mathrm{ext}}_m(\overline{\mathrm{Tr}}_m f_n^{(1)})\|_{H^{s,p}(\mathbb{R}^d, w_\gamma; X)} \to 0$$

as $n \to \infty$. By a standard cut-off argument we find a sequence $(f_n^{(3)})_{n\geq 1} \subseteq C_c^\infty(\mathbb{R}^d; X)$ such that $\overline{\mathrm{Tr}}_m f_n^{(3)} = 0$ for all $n \in \mathbb{N}_1$ and $f_n^{(3)} \to f$ in $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ as $n \to \infty$.

It remains to show that any $g \in C_c^\infty(\mathbb{R}^d; X)$ with $\overline{\mathrm{Tr}}_m g = 0$ can be approximated by functions in $C_c^\infty(\mathbb{R}^d_\bullet; X)$. By writing $g = \mathbf{1}_{\mathbb{R}^d_+} g + \mathbf{1}_{\mathbb{R}^d_-} g =: g_+ + g_-$, we see that $g_\pm \in H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ by Proposition 3.4.8 and thus it suffices to approximate g_+ and g_- separately. Let $\phi^\pm \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} \phi^\pm \, \mathrm{d}x = 1$ and $\operatorname{supp}\phi^+ \subseteq [1,\infty) \times \mathbb{R}^{d-1}$ and $\operatorname{supp}\phi^- \subseteq (-\infty, -1] \times \mathbb{R}^{d-1}$. Define $\phi_n^\pm(x) := n^d \phi^\pm(nx)$ for $n \in \mathbb{N}_1$ and $x \in \mathbb{R}^d$. Then $\phi_n^\pm * g_\pm \in C_c^\infty(\mathbb{R}^d_\bullet; X)$ and $\phi_n^\pm * g_\pm \to g_\pm$ in $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ as $n \to \infty$ by [189, Lemma 3.6].

Step 2: proof of (ii). Again, the embedding "$\hookleftarrow$" is straightforward to prove. For the other embedding, let $f \in H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$ such that $\overline{\mathrm{Tr}}_m f = 0$. Then there exists an $\widetilde{f} \in H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ such that $\widetilde{f}|_{\mathbb{R}^d_+} = f$ and $\overline{\mathrm{Tr}}_m \widetilde{f} = \overline{\mathrm{Tr}}_m f = 0$ (see the discussion about traces on the half-space at the beginning of Section 3.4.1). By Step 1 there exists a sequence $\widetilde{f}_n \in C_c^\infty(\mathbb{R}^d_\bullet; X)$ such that $\widetilde{f}_n \to \widetilde{f}$ in $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ as $n \to \infty$. It follows that $f_n := \widetilde{f}_n|_{\mathbb{R}^d_+} \in C_c^\infty(\mathbb{R}^d_+; X)$ and $f_n \to f$ in $H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$ as $n \to \infty$. □

Sobolev spaces

Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and X a Banach space. Then we define

$$W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X) := \left\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \operatorname{Tr}(\partial^\alpha f) = 0 \text{ if } k - |\alpha| > \tfrac{\gamma+1}{p} \right\}.$$

Moreover, if $m \in \mathbb{N}_0$, then we define

$$W^{k,p}_{\operatorname{Tr}_m}(\mathbb{R}^d_+, w_\gamma; X) := \left\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \operatorname{Tr}_m f = 0 \text{ if } k - m > \tfrac{\gamma+1}{p} \right\}.$$

All the traces in the above definitions are well defined by Theorem 3.4.6. Note that for $m = 0$ and $m = 1$ the above Sobolev spaces with zero boundary conditions coincide with the Dirichlet and Neumann spaces as we will define later on in Chapter 5. Similar to the Bessel potential spaces, it holds that $C_c^\infty(\mathbb{R}^d_+; X)$ is dense in $W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$, see, e.g., Lemma 5.3.3 or [190, Proposition 3.8].

To deal with higher-order boundary conditions we need spaces of smooth functions where only higher-order derivatives are compactly supported. For $m \in \mathbb{N}_0$ we define

$$C^\infty_{c,m}(\overline{\mathbb{R}^d_+}; X) := \{ f \in C_c^\infty(\overline{\mathbb{R}^d_+}; X) : \partial_1^m f \in C_c^\infty(\mathbb{R}^d_+; X) \}.$$

In particular, note that $f \in C^\infty_{c,m}(\overline{\mathbb{R}^d_+}; X)$ satisfies $\partial^\alpha f \in C_c^\infty(\mathbb{R}^d_+; X)$ for all $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ with $\alpha_1 \geq m$.

The following two density results will be required in Chapter 6 to find trace characterisations for weighted Sobolev spaces on domains with, e.g., C^1-boundary. We first prove a density result for homogeneous boundary conditions of order m.

Proposition 3.4.12. *Let $p \in (1,\infty)$, $k, m \in \mathbb{N}_0$ such that $k \geq m$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \{1,\dots,k-m\}\}$ and let X be a Banach space. Then*

$$W^{k,p}_{\operatorname{Tr}_m}(\mathbb{R}^d_+, w_\gamma; X) = \overline{\left\{ f \in C_c^\infty(\overline{\mathbb{R}^d_+}; X) : (\partial_1^m f)|_{\partial\mathbb{R}^d_+} = 0 \right\}}^{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)}.$$

Proof. Step 1: the case $\gamma > (k-m)p - 1$. Note that for $\gamma > (k-m)p-1$ it holds that $W^{k,p}_{\operatorname{Tr}_m}(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$, so that the result follows from

$$\begin{aligned} &C^\infty_{c,m}(\overline{\mathbb{R}^d_+}; X) \subseteq \left\{ f \in C_c^\infty(\overline{\mathbb{R}^d_+}; X) : (\partial_1^m f)|_{\partial\mathbb{R}^d_+} = 0 \right\} \quad \text{and} \\ &C^\infty_{c,m}(\overline{\mathbb{R}^d_+}; X) \overset{\text{dense}}{\hookrightarrow} W^{k,p}(\mathbb{R}^d_+, w_\gamma; X), \end{aligned}$$

see Lemma 5.3.3.

Step 2: the case $\gamma < (k-m)p - 1$. Let $\varepsilon > 0$ and take $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ such that $\operatorname{Tr}_m f = 0$. By [169, Theorem 7.2 & Remark 11.12(iii)], which also holds in the vector-valued case, and a standard cut-off argument, there exists an $f^{(1)} \in C_c^\infty(\overline{\mathbb{R}^d_+}; X)$ such that

$$\| f - f^{(1)} \|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} < \varepsilon. \tag{3.4.12}$$

Let ext_m be as in Theorem 3.4.6 and define $f^{(2)} := f^{(1)} - \operatorname{ext}_m(\operatorname{Tr}_m f^{(1)})$. Then by Remark 3.4.5 we have $f^{(2)} \in C^\infty(\overline{\mathbb{R}^d_+}; X)$ and from Theorem 3.4.6 it follows that $(\partial_1^m f^{(2)})|_{\partial\mathbb{R}^d_+} = 0$ and

$$\|f - f^{(2)}\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} \le \|f - f^{(1)}\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} + \|\operatorname{ext}_m(\operatorname{Tr}_m f^{(1)})\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)}.$$

From $\operatorname{Tr}_m f = 0$, Theorem 3.4.6 twice and (3.4.12), it follows

$$\begin{aligned}\|\operatorname{ext}_m(\operatorname{Tr}_m f^{(1)})\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} &\le C\|\operatorname{Tr}_m f - \operatorname{Tr}_m f^{(1)}\|_{B_{p,p}^{k-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)} \\ &\le C\|f - f^{(1)}\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} < C\varepsilon.\end{aligned}$$

This proves that $\|f - f^{(2)}\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} < (C+1)\varepsilon$. Finally, a standard cut-off argument gives an approximating sequence in $C_c^\infty(\overline{\mathbb{R}^d_+}; X)$ satisfying the boundary condition. □

Under suitable restrictions on γ, such that traces of order $\ge m+1$ do not exist, we can prove density of $C^\infty_{c,m}(\overline{\mathbb{R}^d_+}; X)$ in $W^{k,p}_{\operatorname{Tr}_m}(\mathbb{R}^d_+, w_\gamma; X)$.

Proposition 3.4.13. *Let $p \in (1,\infty)$, $k, m \in \mathbb{N}_0$ such that $k \ge m+1$, $(k-m-1)p - 1 < \gamma < (k-m)p - 1$ and let X be a Banach space. Then*

$$W^{k,p}_{\operatorname{Tr}_m}(\mathbb{R}^d_+, w_\gamma; X) = \overline{C^\infty_{c,m}(\overline{\mathbb{R}^d_+}; X)}^{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)}.$$

Proof. Let $\varepsilon > 0$ and take $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ such that $\operatorname{Tr}_m f = 0$. The conditions on γ guarantee that $\operatorname{Tr}_m f$ exists and traces of order $\ge m+1$ do not exist. By Proposition 3.4.12 there exists a $g \in C_c^\infty(\overline{\mathbb{R}^d_+}; X)$ with its support in $[0, R] \times [-R, R]^{d-1}$ for some $R > 0$ such that $(\partial_1^m g)|_{\partial\mathbb{R}^d_+} = 0$ and

$$\|f - g\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} < \varepsilon. \tag{3.4.13}$$

Let $\phi \in C^\infty(\mathbb{R}_+)$ be such that $\phi = 0$ on $[0, \frac{1}{2}]$ and $\phi = 1$ on $[1, \infty)$ and set $\phi_n(x_1) := \phi(nx_1)$. We construct a sequence $(g_n)_{n\ge1}$ as follows:

- If $m = 0$, define $g_n(x) := \phi_n(x_1) g(x)$.
- If $m \ge 1$, define

$$g_n(x) := \sum_{j=0}^{m-1} (\partial_1^j g)(1, \widetilde{x}) \frac{(x_1 - 1)^j}{j!} + \frac{1}{(m-1)!} \int_1^{x_1} (x_1 - t)^{m-1} \phi_n(t) \partial_1^m g(t, \widetilde{x})\, \mathrm{d}t.$$

Note that by integration by parts $g_n(x) = g(x)$ for all $x \in \mathbb{R}^d_+$ with $x_1 \ge \frac{1}{n}$. For $|\alpha| \le k$ with $\alpha_1 \le m$ and $x \in \mathbb{R}^d_+$ with $x_1 < 1$ we have

$$\|\partial^\alpha g_n(x)\|_X \le C_m \|g\|_{C_b^{k+m}(\mathbb{R}^d_+; X)} \|\phi\|_{L^\infty(\mathbb{R}_+)}.$$

Moreover, we have

$$\partial_1^m g_n(x) = \phi_n(x_1) \partial_1^m g(x), \qquad x \in \mathbb{R}^d_+, \tag{3.4.14}$$

so that in particular $\partial_1^m g_n \in C_c^\infty(\mathbb{R}^d_+;X)$. We write $\alpha=(\alpha_1,\widetilde{\alpha})\in\mathbb{N}_0\times\mathbb{N}_0^{d-1}$. If $\alpha_1\in\{m+1,\dots,k\}$, then it follows from (3.4.14) and the product rule that

$$\begin{aligned}\|\partial_1^{\alpha_1} g_n(x)\|_X &= \big\|\partial_1^{\alpha_1-m}\big(\phi_n(x_1)\partial_1^m g(x)\big)\big\|_X\\ &\le n^{\alpha_1-m}\big\|\phi^{(\alpha_1-m)}(nx_1)\partial_1^m g(x)\big\|_X\\ &\quad + C_{k,m}\sum_{j=0}^{\alpha_1-m-1} n^j\big\|\phi^{(j)}(nx_1)\partial_1^{\alpha_1-j} g(x)\big\|_X\\ &\le n^{k-m}\|\phi\|_{C_b^{k-m}(\mathbb{R}_+)}\big\|\partial_1^m g(x)\big\|_X\\ &\quad + C_{k,m}\, n^{k-m-1}\|\phi\|_{C_b^{k-m-1}(\mathbb{R}_+)}\|g\|_{C_b^k(\mathbb{R}^d_+;X)}.\end{aligned}\tag{3.4.15}$$

Let $K_R:=[-R,R]^{d-1}$. The properties of g_n and (3.4.15) imply that

$$\begin{aligned}\|g-g_n\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)} &\le \sum_{|\alpha|\le k}\Big(\int_{K_R}\int_0^{\frac{1}{n}} x_1^\gamma\|\partial^\alpha g(x)\|_X^p\,\mathrm{d}x_1\,\mathrm{d}\widetilde{x}\Big)^{\frac{1}{p}}\\ &\quad+\sum_{\substack{|\alpha|\le k\\ \alpha_1\le m}}\Big(\int_{K_R}\int_0^{\frac{1}{n}} x_1^\gamma\|\partial^\alpha g_n(x)\|_X^p\,\mathrm{d}x_1\,\mathrm{d}\widetilde{x}\Big)^{\frac{1}{p}}\\ &\quad+n^{k-m}\|\phi\|_{C_b^{k-m}(\mathbb{R}_+)}\Big(\int_{K_R}\int_0^{\frac{1}{n}} x_1^\gamma\|\partial^{\widetilde{\alpha}}\partial_1^m g(x)\|_X^p\,\mathrm{d}x_1\,\mathrm{d}\widetilde{x}\Big)^{\frac{1}{p}}\\ &\quad+C_{k,m}\,n^{k-m-1}\|\phi\|_{C_b^{k-m-1}(\mathbb{R}_+)}\|g\|_{C_b^k(\mathbb{R}^d_+;X)}\Big(\int_{K_R}\int_0^{\frac{1}{n}} x_1^\gamma\,\mathrm{d}x_1\,\mathrm{d}\widetilde{x}\Big)^{\frac{1}{p}}\\ &\le C_{k,m}\,n^{\frac{1}{p}((k-m-1)p-1-\gamma)}\cdot\big(\tfrac{(2R)^{d-1}}{\gamma+1}\big)^{\frac{1}{p}}\|g\|_{C_b^{k+m}(\mathbb{R}^d_+;X)}\|\phi\|_{C_b^{k-m-1}(\mathbb{R}_+)}\\ &\quad+n^{k-m-1}\|\phi\|_{C_b^{k-m}(\mathbb{R}_+)}\Big(\int_{K_R}\int_0^{\frac{1}{n}} x_1^{\gamma-p}\|\partial^{\widetilde{\alpha}}\partial_1^m g(x)\|_X^p\,\mathrm{d}x_1\,\mathrm{d}\widetilde{x}\Big)^{\frac{1}{p}}\\ &\le C_{k,m,p}\,n^{\frac{1}{p}((k-m-1)p-1-\gamma)}\cdot\big(\tfrac{(2R)^{d-1}}{\gamma+1}\big)^{\frac{1}{p}}\|g\|_{C_b^{k+m}(\mathbb{R}^d_+;X)}\|\phi\|_{C_b^{k-m}(\mathbb{R}_+)},\end{aligned}$$

where in the last step we first applied Hardy's inequality (using condition (i) in Lemma 2.2.2 since $\gamma-p>-p-1$ and $(\partial_1^m g)|_{\partial\mathbb{R}^d_+}=0$), then estimated g and finally calculated the remaining integrals over x_1 and $\widetilde{x}$. Hence, as $\gamma>(k-m-1)p-1$, taking n large enough gives with (3.4.13) that

$$\|f-g_n\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)}\le\|f-g\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)}+\|g-g_n\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)}<2\varepsilon.$$

This completes the proof. □

3.5. Trace theorem for boundary operators

In this section, we study differential operators on the boundary of the half-space and we prove a trace theorem for systems of normal boundary operators on weighted Bessel potential and Sobolev spaces.

We start with the definition of a boundary operator which is in the same spirit as [16, Section VIII.2].

Definition 3.5.1 (Boundary operators)**.** Let $p \in (1,\infty)$, $s>0$, $\gamma \in (-1,\infty)\setminus\{jp-1 : j\in\mathbb{N}_1\}$, $m\in\mathbb{N}_0$ and let X, Y be Banach spaces. Then the *boundary operator of order m*, given by

$$\mathscr{B} := \sum_{|\alpha|\le m} b_\alpha \operatorname{Tr}\partial^\alpha \qquad \text{with } b_\alpha \neq 0 \text{ for some } |\alpha| = m,$$

is of *type* (p,s,γ,m,Y) if

(i) $s > m + \frac{\gamma+1}{p}$,

(ii) for all $|\alpha|\le m$ we have $b_\alpha \in C_{\mathrm{b}}^{\ell}(\mathbb{R}^{d-1};\mathscr{L}(X,Y))$ for some $\ell\in\mathbb{N}_0$ such that $\ell > s - m - \frac{\gamma+1}{p}$.

Note that for $\alpha=(\alpha_1,\widetilde{\alpha})\in\mathbb{N}_0\times\mathbb{N}_0^{d-1}$, we have $\operatorname{Tr}\partial^\alpha = \partial^{\widetilde{\alpha}}\operatorname{Tr}_{\alpha_1}$ and therefore the boundary operator $\mathscr{B}$ can be rewritten as

$$\mathscr{B} = \sum_{j=0}^{m} b_j \operatorname{Tr}_j, \quad \text{where} \quad b_j(\cdot,\nabla_{\widetilde{x}}) := \sum_{|\widetilde{\alpha}|\le m-j} b_{(j,\widetilde{\alpha})}\partial^{\widetilde{\alpha}}. \tag{3.5.1}$$

Note that the leading order coefficient b_m is independent of $\nabla_{\widetilde{x}}$. In the following lemma, we prove that $\mathscr{B}$ is a bounded operator from a weighted Bessel potential or Sobolev space to its trace space.

Lemma 3.5.2. *Let $p\in(1,\infty)$, $m\in\mathbb{N}_0$ and let X, Y be Banach spaces.*

(i) If $s>0$, $\gamma\in(-1,p-1)$ and $\mathscr{B}$ is of type (p,s,γ,m,Y), then

$$\mathscr{B} : H^{s,p}(\mathbb{R}^d_+, w_\gamma; X) \to B_{p,p}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};Y) \quad \textit{is bounded.}$$

(ii) If $k\in\mathbb{N}_1$, $\gamma\in(-1,\infty)\setminus\{jp-1 : j\in\mathbb{N}_1\}$ and $\mathscr{B}$ is of type (p,k,γ,m,Y), then

$$\mathscr{B} : W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to B_{p,p}^{k-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};Y) \quad \textit{is bounded.}$$

Proof. Step 1: proof of (i). By Theorem 3.4.1(i) we have that

$$\operatorname{Tr}_j : H^{s,p}(\mathbb{R}^d_+, w_\gamma; X) \to B_{p,p}^{k-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X) \quad \text{for all } j\in\{0,\dots,m\}, \tag{3.5.2}$$

is bounded and by [211, Proposition 3.10] it follows that

$$\partial^{\widetilde{\alpha}} : B_{p,p}^{s-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X) \to B_{p,p}^{s-j-|\widetilde{\alpha}|-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X) \quad \text{for all } |\widetilde{\alpha}|\le m-j, \tag{3.5.3}$$

is bounded. Note that (3.5.3) and the embedding $B_{p,p}^{s_1}(\mathbb{R}^{d-1};X)\hookrightarrow B_{p,p}^{s_2}(\mathbb{R}^{d-1};X)$ for $s_1\ge s_2$, imply that

$$\partial^{\widetilde{\alpha}} : B_{p,p}^{s-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X) \to B_{p,p}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X) \quad \text{for all } |\widetilde{\alpha}|\le m-j, \tag{3.5.4}$$

is bounded as well. Moreover, by Definition 3.5.1(ii) and [126, Example 14.4.33], the pointwise multiplication operator

$$M_{b_\alpha} : B_{p,p}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X) \to B_{p,p}^{s-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};Y) \tag{3.5.5}$$

is bounded. Combining (3.5.1), (3.5.2), (3.5.4) and (3.5.5), yields the result.

Step 2: proof of (ii). This case is analogous to Step 1 using Theorem 3.4.6 to obtain (3.5.2) for weighted Sobolev spaces with $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$. □

We make the following definitions about (systems of) normal boundary operators.

Definition 3.5.3 (Normal boundary operators)**.** An operator $\mathscr{B}$ of type (p,s,γ,m,Y) is a *normal boundary operator of type* (p,s,γ,m,Y) if the leading order coefficient b_m in (3.5.1) admits a continuous right inverse $b_m^{\mathrm{c}} \in C_{\mathrm{b}}^{\ell}(\mathbb{R}^{d-1};\mathscr{L}(Y,X))$, where ℓ is as in Definition 3.5.1.

In the above definition, $b_m(\widetilde{x})$ is a retraction for any $\widetilde{x} \in \mathbb{R}^{d-1}$ with coretraction $b_m^{\mathrm{c}}(\widetilde{x})$, see [262, Section 1.2.4].

Definition 3.5.4 (System of normal boundary operators)**.** Let $n \in \mathbb{N}_0$ and

(i) $0 \le m_0 < m_1 < \cdots < m_n$ be integers,

(ii) $Y_0,\dots,Y_n$ be Banach spaces,

(iii) $\mathscr{B}^{m_i}$ be a normal boundary operator of type (p,s,γ,m_i,Y_i).

Let $\overline{m} := (m_0,\dots,m_n)$ and $\overline{Y} := (Y_0,\dots,Y_n)$. Then the operator

$$\mathscr{B} := (\mathscr{B}^{m_0},\dots,\mathscr{B}^{m_n})$$

is a *normal boundary operator of type* $(p,s,\gamma,\overline{m},\overline{Y})$.

Note that a normal boundary operator of type $(p,s,\gamma,\overline{m},\overline{Y})$ satisfies $s > m_n + \frac{\gamma+1}{p}$. In particular, $(\mathrm{Tr}_{m_i})_{i=0}^n$ is a normal boundary operator of type $(p,s,\gamma,\overline{m},\overline{X})$ for $s > m_n + \frac{\gamma+1}{p}$. This includes the important cases for Dirichlet ($n=0$ and $m_0=0$) and Neumann ($n=0$ and $m_0=1$) boundary conditions.

Remark 3.5.5. In the case that $X = \mathbb{C}^q$ for some $q \in \mathbb{N}_1$, the definition of a normal boundary system as given in [251, Definition 3.1] coincides with our definition above. The definition in [251] requires that the coefficient matrix for the highest-order derivative in $\mathscr{B}^{m_i}$ is surjective, which implies that the boundary operator can be replaced by a $q \times q$ matrix operator (leaving the kernel of $\mathscr{B}^{m_i}$ invariant) such that the coefficient for the highest-order derivative is a projection-valued matrix, see [251, Section 3]. This is a key ingredient for characterising the complex interpolation spaces in [251]. For general Banach spaces X we replace these conditions by the retraction and coretraction in Definition 3.5.3. In the case that X and Y_i are Hilbert spaces we refer to [16, Lemma VIII.2.1.3] for a sufficient condition for $\mathscr{B}$ to be normal.

We note that [251, Definition 3.1] arises from Agmon's condition [5] about minimal growth of the resolvent of an elliptic operator.

We can now prove the following trace theorem for normal boundary operators analogous to [16, Theorem VIII.2.2.1].

Theorem 3.5.6. *Let $p \in (1,\infty)$ and let X be a Banach space. Assume that the conditions (i)-(ii) of Definition 3.5.4 hold.*

(i) If $s > 0$, $\gamma \in (-1, p-1)$ and $\mathscr{B}$ is of type $(p, s, \gamma, \overline{m}, \overline{Y})$, then

$$\mathscr{B} : H^{s,p}(\mathbb{R}^d_+, w_\gamma; X) \to \prod_{j=0}^{n} B_{p,p}^{s-m_j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; Y_j)$$

is a continuous and surjective operator.

(ii) If $k \in \mathbb{N}_1$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and $\mathscr{B}$ is of type $(p, k, \gamma, \overline{m}, \overline{Y})$, then

$$\mathscr{B} : W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to \prod_{j=0}^{n} B_{p,p}^{k-m_j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; Y_j)$$

is a continuous and surjective operator.

In both cases, there exists a continuous right inverse $\operatorname{ext}_{\mathscr{B}}$ of $\mathscr{B}$ which is independent of k, s, p and γ. For any $0 \le j < m_n$ such that $j \notin \{m_0, \dots, m_{n-1}\}$, we have $\operatorname{Tr}_j \circ \operatorname{ext}_{\mathscr{B}} = 0$.

Proof. The proof is similar to [16, Theorem VIII.2.2.1] and for the convenience of the reader, we provide the proof.

The continuity of $\mathscr{B}$ follows from Lemma 3.5.2. We construct the right inverse $\operatorname{ext}_{\mathscr{B}}$ in the case of Bessel potential spaces. The case of Sobolev spaces is the same if one invokes Theorem 3.4.6 instead of Theorem 3.4.1(i) in the construction below. Let

$$\mathscr{B}^{m_i} = \sum_{j=0}^{m_i} b_{i,j} \operatorname{Tr}_j, \quad b_{i,j} = \sum_{|\widetilde{\alpha}| \le m_i - j} b_{(j,\widetilde{\alpha})}^{m_i} \partial^{\widetilde{\alpha}} \quad \text{for } i \in \{0, \dots, n\},$$

where $b_\alpha^{m_i}$ are the coefficients of the operator $\mathscr{B}^{m_i}$. According to Definition 3.5.3 there exists a right inverse $b_{i,m_i}^{\mathrm{c}} \in C_{\mathrm{b}}^{\ell}(\mathbb{R}^{d-1}; \mathscr{L}(Y_i, X))$ of b_{i,m_i}. Define the operator

$$\widetilde{\mathscr{C}}^{m_i} = -\sum_{j=0}^{m_i-1} b_{i,m_i}^{\mathrm{c}} b_{i,j} \operatorname{Tr}_j, \quad \widetilde{\mathscr{C}}^{0} := 0. \tag{3.5.6}$$

Reasoning as in the proof of Lemma 3.5.2 gives that

$$\widetilde{\mathscr{C}}^{m_i} : H^{s,p}(\mathbb{R}^d_+, w_\gamma; X) \to B_{p,p}^{s-m_i-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

is a bounded linear operator. Now, define the operator $\widetilde{\mathscr{C}} := (\widetilde{\mathscr{C}}^j)_{j=0}^{m_n}$ by setting $\widetilde{\mathscr{C}}^j = \widetilde{\mathscr{C}}^{m_i}$ if $j = m_i$ for some $i \in \{0, \dots, n\}$, and $\widetilde{\mathscr{C}}^j = 0$ if $j \notin \{m_0, \dots, m_n\}$. Then

$$\widetilde{\mathscr{C}} : H^{s,p}(\mathbb{R}^d_+, w_\gamma; X) \to \prod_{j=0}^{m_n} B_{p,p}^{s-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$$

is linear and bounded as well. Let $g = (g_{m_0}, \dots, g_{m_n}) \in \prod_{i=0}^{n} B_{p,p}^{s-m_i-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; Y_i)$ and define $h = (h_j)_{j=0}^{m_n} \in \prod_{j=0}^{m_n} B_{p,p}^{s-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)$ by setting $h_j = h_{m_i} := b^{\mathrm{c}}_{i,m_i} g_{m_i}$ if $j = m_i$ for some $i \in \{0, \dots, n\}$, and $h_j = 0$ if $j \notin \{m_0, \dots, m_n\}$. Furthermore, let $f_0 := \operatorname{ext}_0 h_0$ and for $j \in \{1, \dots, m_n\}$ we recursively define

$$f_j := f_{j-1} + \operatorname{ext}_j \big(h_j + \widetilde{\mathscr{C}}^j f_{j-1} - \operatorname{Tr}_j f_{j-1} \big).$$

Note that $f_j \in H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$ for all $j \in \{0, \dots, m_n\}$ by Theorem 3.4.1(i). We will prove that $\operatorname{ext}_{\mathscr{B}} g := f_{m_n}$ defines the right inverse we are looking for.

By Theorem 3.4.1(i) we have that $\operatorname{Tr}_\ell \circ \operatorname{ext}_j = \delta_{j\ell}$ for $\ell \le j$ and therefore

$$\begin{aligned} \operatorname{Tr}_j f_j &= h_j + \widetilde{\mathscr{C}}^j f_{j-1} && \text{for } 0 \le j \le m_n, \\ \operatorname{Tr}_\ell f_j &= \operatorname{Tr}_\ell f_{j-1} && \text{for } 0 \le \ell < j \le m_n. \end{aligned} \tag{3.5.7}$$

Suppose that $0 \le \ell \le j \le m_n$, then by (3.5.7) we have

$$\operatorname{Tr}_\ell f_j = \operatorname{Tr}_\ell f_{j-1} = \cdots = \operatorname{Tr}_\ell f_\ell = h_\ell + \widetilde{\mathscr{C}}^\ell f_{\ell-1}. \tag{3.5.8}$$

Moreover, if $\ell = m_i \in \{m_0, \dots, m_n\}$, then by (3.5.6) and (3.5.7)

$$\widetilde{\mathscr{C}}^\ell f_{\ell-1} = \widetilde{\mathscr{C}}^\ell f_\ell = \cdots = \widetilde{\mathscr{C}}^\ell f_j. \tag{3.5.9}$$

Hence, (3.5.6) and (3.5.9) imply

$$\operatorname{Tr}_{m_i} f_j = h_{m_i} + \widetilde{\mathscr{C}}^{m_i} f_j \quad \text{for } m_i \le j \le m_n, \quad 0 \le i \le n.$$

Multiplying from the left with b_{i,m_i} gives

$$\mathscr{B}^{m_i} f_j = g_{m_i} \qquad \text{for } m_i \le j \le m_n, \quad 0 \le i \le n. \tag{3.5.10}$$

Define

$$\mathscr{C}^{m_i} : \prod_{j=0}^{i} B_{p,p}^{s-m_j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; Y_j) \to H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$$

by $\mathscr{C}^{m_i}(g_{m_0}, \dots, g_{m_i}) := f_{m_i}$ for $i \in \{0, \dots, n\}$. Then from (3.5.10) we obtain for $0 \le \ell \le i \le n$ that

$$\mathscr{B}^{m_\ell} \mathscr{C}^{m_i}(g_{m_0}, \dots, g_{m_i}) = \mathscr{B}^{m_\ell} f_{m_i} = g_{m_\ell}.$$

So indeed, $\operatorname{ext}_{\mathscr{B}} := \mathscr{C}^{m_n}$ is the right inverse for $\mathscr{B}$.

Finally, if $\ell \notin \{m_0, \dots, m_n\}$, then $h_\ell = 0$ and $\widetilde{\mathscr{C}}^\ell = 0$, so that (3.5.8) implies

$$\operatorname{Tr}_j(\operatorname{ext}_{\mathscr{B}} g) = \operatorname{Tr}_j f_{m_n} = 0.$$

This completes the proof. □

3.6. Complex interpolation of Bessel potential and Sobolev spaces

In this final section, we prove the characterisations for the complex interpolation spaces of weighted Bessel potential and Sobolev spaces. In Section 3.6.1 we start with some preliminary interpolation results on the half-space required for our main results which are stated in Section 3.6.2. The proofs of the main results are given in Section 3.6.3.

3.6.1. Interpolation results for weighted spaces

We start with an interpolation result for the weighted Bessel potential spaces $H_0^{s,p}(\mathscr{O}, w_\gamma; X)$, see Section 3.4.2. Interpolation theory for vector-valued Bessel potential (and Sobolev) spaces relies on the vector-valued Mihlin multiplier theorem, which requires the UMD condition, see [214, Section 3]. Consequently, unlike in previous sections where this condition was unnecessary, we will require the UMD condition from now on.

Proposition 3.6.1. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, $\mathscr{O} \in \{\mathbb{R}^d, \mathbb{R}^d_+\}$ and let X be a* UMD *Banach space. Assume that $\theta \in (0,1)$, $-1+\frac{\gamma+1}{p} < s_0 < s_\theta < s_1$ with $s_\theta = (1-\theta)s_0+\theta s_1$ satisfy $s_0, s_\theta, s_1 \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$. Then*

$$H_0^{s_\theta,p}(\mathscr{O}, w_\gamma; X) = \big[H_0^{s_0,p}(\mathscr{O}, w_\gamma; X), H_0^{s_1,p}(\mathscr{O}, w_\gamma; X)\big]_\theta.$$

Proof. Step 1: the case $\mathscr{O} = \mathbb{R}^d$. We define $H^{s,p}_{\mathbb{R}^d_\pm}(\mathbb{R}^d, w_\gamma; X)$ as the closed subspace of $H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ consisting of all functions with their support in $\mathbb{R}^d_\pm$. These spaces form a complex interpolation scale (see [189, Proposition 5.7]) and below we construct an isomorphism to reduce to this known interpolation result.

Let $s > -1 + \frac{\gamma+1}{p}$ be such that $s \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$. Then we claim that $\overline{\mathrm{Tr}}_m f = 0$ for integers $m \in [0, s - \frac{\gamma+1}{p})$ and $f \in H^{s,p}_{\mathbb{R}^d_\pm}(\mathbb{R}^d, w_\gamma; X)$. Indeed, this follows from the fact that if $f \in H^{s,p}(\mathbb{R}^d, w_\gamma; X)$ satisfies $f|_{(0,\delta)\times\mathbb{R}^{d-1}} = 0$ or $f|_{(-\delta,0)\times\mathbb{R}^{d-1}} = 0$ for some $\delta > 0$, then $\overline{\mathrm{Tr}}_m f = 0$. This observation can be proved similarly as in [189, Proposition 6.3(2)] using the continuity of $\overline{\mathrm{Tr}}_m$ from Proposition 3.4.7(i).

The claim proves that the mapping

$$R : H^{s,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w_\gamma; X) \times H^{s,p}_{\mathbb{R}^d_-}(\mathbb{R}^d, w_\gamma; X) \to H_0^{s,p}(\mathbb{R}^d, w_\gamma; X), \qquad R(g,h) := g + h,$$

is well defined and continuous. Furthermore, by Propositions 3.4.8 and 3.4.11 the mapping

$$S : H_0^{s,p}(\mathbb{R}^d, w_\gamma; X) \to H^{s,p}_{\mathbb{R}^d_+}(\mathbb{R}^d, w_\gamma; X) \times H^{s,p}_{\mathbb{R}^d_-}(\mathbb{R}^d, w_\gamma; X), \qquad Sf := (\mathbf{1}_{\mathbb{R}^d_+} f, \mathbf{1}_{\mathbb{R}^d_-} f),$$

is well defined and continuous. Since $R^{-1} = S$ the result now follows from [189, Proposition 5.7].

Step 2: the case $\mathscr{O} = \mathbb{R}^d_+$. Let $m \in \mathbb{N}_0$ be the smallest integer such that $m \geq \max\{|s_0|, |s_1|\}$ and let $s > -1 + \frac{\gamma+1}{p}$ such that $|s| \leq m$ and $s \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$. Moreover, let

$$\mathscr{E}_+^m \in \mathscr{L}(H^{s,p}(\mathbb{R}^d_+, w_\gamma; X), H^{s,p}(\mathbb{R}^d, w_\gamma; X))$$

be the reflection operator from [189, Proposition 5.6]. Then by [189, Proposition 5.6] the mapping

$$S : H_0^{s,p}(\mathbb{R}^d_+, w_\gamma; X) \to H_0^{s,p}(\mathbb{R}^d, w_\gamma; X), \qquad Sf := \mathscr{E}_+^m f,$$

is well defined and continuous. Furthermore, let $R : H_0^{s,p}(\mathbb{R}^d, w_\gamma; X) \to H_0^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$ be the restriction operator. The result now follows from Step 1 and [189, Lemma 5.3]. □

We continue with interpolation results for Sobolev spaces. First, we introduce the following multiplication operator. Properties of this multiplication operator are studied in more detail in Section 5.3.2.

For $\kappa \in \mathbb{R}$ we define the pointwise multiplication operator

$$M^\kappa : C_c^\infty(\mathbb{R}^d_+; X) \to C_c^\infty(\mathbb{R}^d_+; X) \quad \text{by} \quad M^\kappa u(x) = x_1^\kappa u(x), \qquad x \in \mathbb{R}^d_+,$$

which extends to an operator on $\mathscr{D}'(\mathbb{R}^d_+; X)$ by duality and $M^{-\kappa}$ acts as inverse for M^κ. Moreover, we write $M := M^1$.

The first interpolation result is for the Sobolev spaces $W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ with $\gamma > -1$, see Section 3.4.2. In this case, due to the zero boundary conditions, the operator $M^j : W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to W_0^{k,p}(\mathbb{R}^d_+, w_{\gamma - jp}; X)$ is an isomorphism for all $j \in \mathbb{N}_1$. This allows us to lift the interpolation result for $\gamma \in (-1, p-1)$ to $\gamma \in (jp-1, (j+1)p-1)$ for any $j \in \mathbb{N}_1$.

Proposition 3.6.2. *Let $p \in (1,\infty)$, $k_0 \in \mathbb{N}_0$, $k_1 \in \mathbb{N}_1 \setminus \{1\}$, $\ell \in \{1, \dots, k_1 - 1\}$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and let X be a* UMD *Banach space. Then*

$$\big[W_0^{k_0,p}(\mathbb{R}^d_+, w_\gamma; X), W_0^{k_0+k_1,p}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}} = W_0^{k_0+\ell,p}(\mathbb{R}^d_+, w_\gamma; X).$$

Proof. We first consider the case $k_0 = 0$. If $\gamma \in (-1, p-1)$, then the statement follows from [190, Proposition 3.15]. For $\gamma \in (jp-1, (j+1)p-1)$ with $j \in \mathbb{N}_1$ we obtain, by applying Lemma 5.3.5, properties of the complex interpolation method and the case $j = 0$, that

$$\begin{aligned}
&\big[L^p(\mathbb{R}^d_+, w_\gamma; X), W_0^{k_1,p}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}} \\
&\quad = \big[M^{-j} L^p(\mathbb{R}^d_+, w_{\gamma - jp}; X), M^{-j} W_0^{k_1,p}(\mathbb{R}^d_+, w_{\gamma - jp}; X)\big]_{\frac{\ell}{k_1}} \\
&\quad = M^{-j}\big[L^p(\mathbb{R}^d_+, w_{\gamma - jp}; X), W_0^{k_1,p}(\mathbb{R}^d_+, w_{\gamma - jp}; X)\big]_{\frac{\ell}{k_1}} \\
&\quad = M^{-j} W_0^{\ell,p}(\mathbb{R}^d_+, w_{\gamma - jp}; X) = W_0^{\ell,p}(\mathbb{R}^d_+, w_\gamma; X).
\end{aligned}$$

The case $k_0 \in \mathbb{N}_1$ follows from the case $k_0 = 0$ and reiteration for the complex interpolation method (see [27, Theorem 4.6.1]):

$$\begin{aligned}
&\big[W_0^{k_0,p}(\mathbb{R}^d_+, w_\gamma; X), W_0^{k_0+k_1,p}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}} \\
&\quad = \big[[L^p(\mathbb{R}^d_+, w_\gamma; X), W_0^{k_0+k_1,p}(\mathbb{R}^d_+, w_\gamma; X)]_{\frac{k_0}{k_0+k_1}}, W_0^{k_0+k_1,p}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}} \\
&\quad = \big[L^p(\mathbb{R}^d_+, w_\gamma; X), W_0^{k_0+k_1,p}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{k_0+\ell}{k_0+k_1}} = W_0^{k_0+\ell,p}(\mathbb{R}^d_+, w_\gamma; X).
\end{aligned}$$

This completes the proof. □

We proceed with the interpolation of weighted Sobolev spaces without boundary conditions. As a consequence of Proposition 3.6.2, we can derive such a result if γ is large enough so that certain traces do not exist. On the other hand, for arbitrary $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ we can only prove one of the embeddings for the complex interpolation space using Wolff interpolation. The converse embedding will be proved in Proposition 3.6.6.

Proposition 3.6.3. *Let $p \in (1,\infty)$, $k_0 \in \mathbb{N}_0$, $k_1 \in \mathbb{N}_1 \setminus \{1\}$, $\ell \in \{1,\dots,k_1-1\}$ and let X be a* UMD *Banach space.*

(i) If $\gamma \in ((k_0+\ell)p-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$, then

$$\big[W^{k_0,p}(\mathbb{R}^d_+,w_\gamma;X), W^{k_0+k_1,p}(\mathbb{R}^d_+,w_\gamma;X)\big]_{\frac{\ell}{k_1}} = W^{k_0+\ell,p}(\mathbb{R}^d_+,w_\gamma;X).$$

(ii) If $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$, then

$$\big[W^{k_0,p}(\mathbb{R}^d_+,w_\gamma;X), W^{k_0+k_1,p}(\mathbb{R}^d_+,w_\gamma;X)\big]_{\frac{\ell}{k_1}} \hookrightarrow W^{k_0+\ell,p}(\mathbb{R}^d_+,w_\gamma;X).$$

Proof. Step 1: the proof of (i). We first consider the case $k_0 = 0$. It follows from Proposition 3.6.2 that

$$\begin{aligned} W^{\ell,p}(\mathbb{R}^d_+,w_\gamma;X) = W^{\ell,p}_0(\mathbb{R}^d_+,w_\gamma;X) &= \big[L^p(\mathbb{R}^d_+,w_\gamma;X), W^{k_1,p}_0(\mathbb{R}^d_+,w_\gamma;X)\big]_{\frac{\ell}{k_1}} \\ &\hookrightarrow \big[L^p(\mathbb{R}^d_+,w_\gamma;X), W^{k_1,p}(\mathbb{R}^d_+,w_\gamma;X)\big]_{\frac{\ell}{k_1}}. \end{aligned}$$

To prove the other embedding, it suffices to show that

$$\|u\|_{W^{\ell,p}(\mathbb{R}^d_+,w_\gamma;X)} \le C\|u\|_{[L^p(\mathbb{R}^d_+,w_\gamma;X),W^{k_1,p}(\mathbb{R}^d_+,w_\gamma;X)]_{\frac{\ell}{k_1}}}, \qquad u \in C^\infty_c(\mathbb{R}^d_+;X),$$

by [190, Lemma 3.9]. Let $j \in \mathbb{N}_0$ be such that $\gamma \in (jp-1,(j+1)p-1)$. The case $j=0$ follows from [189, Proposition 5.5 & 5.6]. For $j \in \mathbb{N}_1$ we obtain, by applying Lemma 5.3.5, the case $j=0$ and properties of the complex interpolation method, that

$$\begin{aligned} \|u\|_{W^{\ell,p}(\mathbb{R}^d_+,w_\gamma;X)} &= \|M^{-j}M^j u\|_{W^{\ell,p}(\mathbb{R}^d_+,w_\gamma;X)} \\ &\le C\|M^j u\|_{W^{\ell,p}(\mathbb{R}^d_+,w_{\gamma-jp};X)} \\ &\le C\|M^j u\|_{[L^p(\mathbb{R}^d_+,w_{\gamma-jp};X),W^{k_1,p}(\mathbb{R}^d_+,w_{\gamma-jp};X)]_{\frac{\ell}{k_1}}} \\ &\le C\|u\|_{[L^p(\mathbb{R}^d_+,w_\gamma;X),W^{k_1,p}(\mathbb{R}^d_+,w_\gamma;X)]_{\frac{\ell}{k_1}}}. \end{aligned}$$

Similar to the proof of Proposition 3.6.2, the case $k_0 \in \mathbb{N}_1$ follows from the case $k_0 = 0$ and reiteration. This completes the proof of (i).

Step 2: the proof of (ii). From now on, we abbreviate $W^{k,p}(w_\gamma) := W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)$ for notational convenience. Again, let $j \in \mathbb{N}_0$ be such that $\gamma \in (jp-1,(j+1)p-1)$. As also noted in Step 1, for $j=0$ the result (ii) follows from [189, Proposition 5.5 & 5.6], so we may assume that $j \ge 1$.

Step 2a: the case $j \in \{1,\dots,k_0\}$. We first prove the embedding in (ii) for $j \in \{1,\dots,k_0\}$. Note that $k_0 - j \ge 0$ and $\gamma - jp \in (-1,p-1)$, so [189, Proposition 5.5 & 5.6] imply that

$$\big[W^{k_0-j,p}(w_{\gamma-jp}), W^{k_0+k_1-j,p}(w_{\gamma-jp})\big]_{\frac{\ell}{k_1}} \hookrightarrow W^{k_0+\ell-j,p}(w_{\gamma-jp}).$$

We prove by induction on $j_1 \ge 0$ that

$$\big[W^{k_0+j_1-j,p}(w_{\gamma+(j_1-j)p}), W^{k_0+k_1+j_1-j,p}(w_{\gamma+(j_1-j)p})\big]_{\frac{\ell}{k_1}} \hookrightarrow W^{k_0+\ell+j_1-j,p}(w_{\gamma+(j_1-j)p}), \tag{3.6.1}$$

so that (ii) for $\gamma < (k_0+1)p-1$ follows from (3.6.1) with $j_1 = j$. Assume that (3.6.1) holds for some $j_1 \geq 0$. Let

$$\widetilde{k} := k_0 + j_1 - j \quad \text{and} \quad \widetilde{\gamma} := \gamma + (j_1 - j)p.$$

Let $u \in W^{\widetilde{k}+k_1+1,p}(w_{\widetilde{\gamma}+p})$, then

$$\|u\|_{W^{\widetilde{k}+\ell+1,p}(w_{\widetilde{\gamma}+p})} = \|u\|_{W^{\widetilde{k}+\ell,p}(w_{\widetilde{\gamma}+p})} + \sum_{|\alpha|=1} \sum_{|\beta|=\widetilde{k}+\ell} \|M\partial^{\alpha+\beta}u\|_{L^p(w_{\widetilde{\gamma}})}.$$

Using $\partial_1^n M = M\partial_1^n + n\partial_1^{n-1}$ for $n \in \mathbb{N}_1$ and the induction hypothesis (3.6.1), we obtain

$$\begin{aligned}
&\sum_{|\alpha|=1} \sum_{|\beta|=\widetilde{k}+\ell} \|M\partial^{\alpha+\beta}u\|_{L^p(w_{\widetilde{\gamma}})} \\
&\quad \leq \sum_{|\alpha|=1} \Big(\|M\partial^{\alpha}u\|_{W^{\widetilde{k}+\ell,p}(w_{\widetilde{\gamma}})} + \sum_{\substack{|\beta|=\widetilde{k}+\ell \\ \beta_1>0}} \|\partial^{\beta}M\partial^{\alpha}u - \beta_1\partial_1^{\beta_1-1}\partial^{\widetilde{\beta}+\alpha}u\|_{L^p(w_{\widetilde{\gamma}})} \Big) \\
&\quad \leq C\Big(\sum_{|\alpha|=1} \|M\partial^{\alpha}u\|_{W^{\widetilde{k}+\ell,p}(w_{\widetilde{\gamma}})} + \|u\|_{W^{\widetilde{k}+\ell,p}(w_{\widetilde{\gamma}})} \Big) \\
&\quad \leq C\Big(\sum_{|\alpha|=1} \|M\partial^{\alpha}u\|_{[W^{\widetilde{k},p}(w_{\widetilde{\gamma}}),W^{\widetilde{k}+k_1,p}(w_{\widetilde{\gamma}})]_{\frac{\ell}{k_1}}} + \|u\|_{[W^{\widetilde{k},p}(w_{\widetilde{\gamma}}),W^{\widetilde{k}+k_1,p}(w_{\widetilde{\gamma}})]_{\frac{\ell}{k_1}}} \Big) \\
&\quad \leq C\|u\|_{[W^{\widetilde{k}+1,p}(w_{\widetilde{\gamma}+p}),W^{\widetilde{k}+k_1+1,p}(w_{\widetilde{\gamma}+p})]_{\frac{\ell}{k_1}}},
\end{aligned}$$

where the last estimate follows from the fact that for $|\alpha| = 1$ the operators

$$\begin{aligned}
&M\partial^{\alpha}, \mathrm{id} : W^{\widetilde{k}+1,p}(w_{\widetilde{\gamma}+p}) \to W^{\widetilde{k},p}(w_{\widetilde{\gamma}}) \quad \text{and} \\
&M\partial^{\alpha}, \mathrm{id} : W^{\widetilde{k}+k_1+1,p}(w_{\widetilde{\gamma}+p}) \to W^{\widetilde{k}+k_1,p}(w_{\widetilde{\gamma}})
\end{aligned} \tag{3.6.2}$$

are bounded. Indeed, from Hardy's inequality (see Lemma 2.2.2 using $\widetilde{\gamma} > j_1 p - 1 \geq -1$) it follows that the operator id in (3.6.2) is bounded. Furthermore, by Lemma 5.3.5 it follows that the operator $M\partial^{\alpha}$ with $|\alpha| = 1$ in (3.6.2) is bounded.

Thus, we have proved

$$\|u\|_{W^{\widetilde{k}+\ell+1,p}(w_{\widetilde{\gamma}+p})} \leq C\|u\|_{[W^{\widetilde{k}+1,p}(w_{\widetilde{\gamma}+p}),W^{\widetilde{k}+k_1+1,p}(w_{\widetilde{\gamma}+p})]_{\frac{\ell}{k_1}}}, \quad u \in W^{\widetilde{k}+k_1+1,p}(w_{\widetilde{\gamma}+p}),$$

and the induction is completed by noting that $W^{\widetilde{k}+k_1+1,p}(w_{\widetilde{\gamma}+p})$ is dense in the interpolation space (see [262, Theorem 1.9.3]).

Step 2b: the case $j \in \{k_0+1, \dots, k_0+\ell-1\}$. Assume that $\ell \in \{2, \dots, k_1-1\}$ (the case $\ell = 1$ is covered by Step 1 and 2c) and $j \in \{k_0+1, \dots, k_0+\ell-1\}$. We apply Step 1 and 2a together with Wolff interpolation to prove (ii). Since $\gamma > jp-1$ and $1 \leq j-k_0 \leq \ell-1$, Step 1 implies

$$\big[W^{k_0,p}(w_{\gamma}), W^{k_0+\ell,p}(w_{\gamma})\big]_{\lambda} = W^{j,p}(w_{\gamma}), \quad \text{with } \lambda := \frac{j-k_0}{\ell} \in [0,1],$$

and since $\gamma < (j+1)p-1$, Step 2a implies

$$\big[W^{j,p}(w_{\gamma}), W^{k_0+k_1,p}(w_{\gamma})\big]_{\mu} \hookrightarrow W^{k_0+\ell,p}(w_{\gamma}), \quad \text{with } \mu := \frac{k_0+\ell-j}{k_0+k_1-j} \in [0,1].$$

Wolff interpolation [127, Corollary 3] gives that

$$\big[W^{k_0,p}(w_\gamma), W^{k_0+k_1,p}(w_\gamma)\big]_\eta \hookrightarrow W^{k_0+\ell,p}(w_\gamma), \quad \text{with } \eta := \frac{\mu}{1+\lambda(\mu-1)}.$$

A straightforward calculation shows that $\eta = \frac{\ell}{k_1}$.

Step 2c: the case $j \geq k_0+\ell$. If $j \geq k_0+\ell$, then $\gamma > jp-1 \geq (k_0+\ell)p-1$, so (ii) follows from Step 1. This completes the proof. □

3.6.2. Main results about complex interpolation of weighted spaces

We start with the definition of weighted spaces with vanishing boundary conditions on the half-space. Let $p \in (1,\infty)$ and let X be a Banach space. Assume that the conditions (i)-(ii) of Definition 3.5.4 hold.

(i) If $s>0$, $\gamma \in (-1,p-1)$ and $\mathscr{B}$ is of type $(p,s,\gamma,\overline{m},\overline{Y})$, then we define for $t \leq s$

$$H^{t,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X) = \Big\{ f \in H^{t,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathscr{B}^{m_i} f = 0 \text{ if } m_i + \frac{\gamma+1}{p} < t \Big\}.$$

(ii) If $k \in \mathbb{N}_1$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and $\mathscr{B}$ is of type $(p,k,\gamma,\overline{m},\overline{Y})$, then we define for $\ell \leq k$

$$W^{\ell,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X) = \Big\{ f \in W^{\ell,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathscr{B}^{m_i} f = 0 \text{ if } m_i + \frac{\gamma+1}{p} < \ell \Big\}.$$

All the traces in the above definition are well defined by Theorem 3.5.6. We note that if $t, \ell < m_0 + \frac{\gamma+1}{p}$, then the condition $\mathscr{B}^{m_i} f = 0$ is empty for all $i \in \{0,\dots,n\}$, so that the spaces with and without boundary conditions coincide.

The following two theorems characterise the complex interpolation spaces for weighted spaces with boundary conditions on the half-space. The proofs of these theorems are given in Section 3.6.3.

Theorem 3.6.4 (Complex interpolation of weighted Bessel potential spaces)**.** *Let $p \in (1,\infty)$, $\gamma \in (-1,p-1)$ and let X be a* UMD *Banach space. Let $\mathscr{B}$ be a normal boundary operator of type $(p,s,\gamma,\overline{m},\overline{Y})$ as in Definition 3.5.4. Assume that $\theta \in (0,1)$, $s_0 < s_\theta < s_1 \leq s$ with $s_\theta = (1-\theta)s_0 + \theta s_1$ satisfy $s_0, s_\theta, s_1 \notin \{m_i + \frac{\gamma+1}{p} : 0 \leq i \leq n\}$ and $s_0 > -1 + \frac{\gamma+1}{p}$. Then*

$$\begin{aligned} H^{s_\theta,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X) &= \big[H^{s_0,p}(\mathbb{R}^d_+, w_\gamma; X), H^{s_1,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)\big]_\theta \\ &= \big[H^{s_0,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X), H^{s_1,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)\big]_\theta. \end{aligned}$$

Theorem 3.6.5 (Complex interpolation of weighted Sobolev spaces)**.** *Let $p \in (1,\infty)$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and let X be a* UMD *Banach space. Let $\mathscr{B}$ be a normal boundary operator of type $(p,k,\gamma,\overline{m},\overline{Y})$ as in Definition 3.5.4. Assume that $k_0 \in \mathbb{N}_0$ and $k_1 \in \mathbb{N}_1 \setminus \{1\}$ are such that $k_0 + k_1 \leq k$. Then for $\ell \in \{1,\dots,k_1-1\}$ we have*

$$\begin{aligned} W^{k_0+\ell,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X) &= \big[W^{k_0,p}(\mathbb{R}^d_+, w_\gamma; X), W^{k_0+k_1,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}} \\ &= \big[W^{k_0,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X), W^{k_0+k_1,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}}. \end{aligned}$$

As a consequence of Theorem 3.6.5 we can characterise the complex interpolation space of weighted Sobolev spaces without boundary conditions. One of the embeddings, which is needed in the proof of Theorem 3.6.5, was already proved in Proposition 3.6.3.

Proposition 3.6.6. *Let $p \in (1,\infty)$, $k_0 \in \mathbb{N}_0$, $k_1 \in \mathbb{N}_1 \setminus \{1\}$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$, $\ell \in \{1,\dots,k_1-1\}$ and let X be a* UMD *Banach space. Then*

$$W^{k_0+\ell,p}(\mathbb{R}^d_+, w_\gamma; X) = \big[W^{k_0,p}(\mathbb{R}^d_+, w_\gamma; X), W^{k_0+k_1,p}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}}.$$

Proof. Note that the embedding "$\hookleftarrow$" has been proved in Proposition 3.6.3(ii). For the other embedding, let $j \in \mathbb{N}_0$ be the smallest integer such that $k_0 + \ell < j + \frac{\gamma+1}{p}$. Then by Theorem 3.6.5 we obtain

$$\begin{aligned} W^{k_0+\ell,p}(\mathbb{R}^d_+, w_\gamma; X) &= W^{k_0+\ell,p}_{\mathrm{Tr}_j}(\mathbb{R}^d_+, w_\gamma; X) \\ &= \big[W^{k_0,p}(\mathbb{R}^d_+, w_\gamma; X), W^{k_0+k_1,p}_{\mathrm{Tr}_j}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}} \\ &\hookrightarrow \big[W^{k_0,p}(\mathbb{R}^d_+, w_\gamma; X), W^{k_0+k_1,p}(\mathbb{R}^d_+, w_\gamma; X)\big]_{\frac{\ell}{k_1}}. \end{aligned}$$

This completes the proof. □

By standard localisation techniques, the complex interpolation results above can be extended to smooth bounded domains. Using localisation the definition of the traces on the half-space and the function space $W^{\ell,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)$ can be extended to smooth bounded domains $\mathscr{O} \subseteq \mathbb{R}^d$, we refer to [190, Section 5.2] for details. Recall that $w^{\partial\mathscr{O}}_\gamma(x) := \mathrm{dist}(x,\partial\mathscr{O})^\gamma$ for $x \in \mathscr{O}$. From Theorem 3.6.5 and Proposition 3.6.6 we obtain the following result. Similar results for Bessel potential spaces on domains with $\gamma \in (-1, p-1)$ can be obtained using Theorem 3.6.4.

Proposition 3.6.7. *Assume that the conditions of Theorem 3.6.5 hold and let $\mathscr{O}$ be a bounded C^∞-domain. Then for $\ell \in \{1,\dots,k_1-1\}$ we have*

$$\begin{aligned} W^{k_0+\ell,p}_{\mathscr{B}}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X) &= \big[W^{k_0,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), W^{k_0+k_1,p}_{\mathscr{B}}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)\big]_{\frac{\ell}{k_1}} \\ &= \big[W^{k_0,p}_{\mathscr{B}}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), W^{k_0+k_1,p}_{\mathscr{B}}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)\big]_{\frac{\ell}{k_1}}, \end{aligned}$$

and

$$W^{k_0+\ell,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X) = \big[W^{k_0,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X), W^{k_0+k_1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)\big]_{\frac{\ell}{k_1}}.$$

For simplicity, we only deal with smooth domains to avoid problems with the space-dependent coefficients in the boundary operators, although this assumption can be weakened. In Chapter 6, we prove an interpolation result for Dirichlet and Neumann boundary conditions, but with minimal smoothness assumptions on the domain.

We close this section with an application of the complex interpolation result in Proposition 3.6.6 to derive Fubini's property (mixed derivative theorem) for weighted Sobolev spaces on the half-space. It extends the range of weights obtained in [190, Corollary 3.19]. The Fubini property for unweighted spaces is studied in [264, Section I.4] and some results for weighted spaces are contained in [185, Section 5].

Proposition 3.6.8. *Let $p \in (1,\infty)$, $d \geq 2$, $k \in \mathbb{N}_0$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and let X be a* UMD *Banach space. Then*

$$L^p(\mathbb{R}^{d-1}; W^{k,p}(\mathbb{R}_+, w_\gamma; X)) \cap W^{k,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X)) = W^{k,p}(\mathbb{R}^d_+, w_\gamma; X).$$

Proof. The embedding "$\hookleftarrow$" is clear. For the other embedding, let $k_1, k_2 \in \{0,\dots,k\}$ such that $k_1 + k_2 = k$. By [190, Theorem 3.18] and Proposition 3.6.6, we have

$$\begin{aligned}
&L^p(\mathbb{R}^{d-1}; W^{k,p}(\mathbb{R}_+, w_\gamma; X)) \cap W^{k,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X)) \\
&\qquad = H^{0,p}(\mathbb{R}^{d-1}; W^{k,p}(\mathbb{R}_+, w_\gamma; X)) \cap H^{k,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X)) \\
&\qquad \hookrightarrow \big[H^{0,p}(\mathbb{R}^{d-1}; W^{k,p}(\mathbb{R}_+, w_\gamma; X)), H^{k,p}(\mathbb{R}^{d-1}; L^p(\mathbb{R}_+, w_\gamma; X))\big]_{\frac{k_1}{k}} \\
&\qquad = H^{k_1,p}(\mathbb{R}^{d-1}; [W^{k,p}(\mathbb{R}_+, w_\gamma; X), L^p(\mathbb{R}_+, w_\gamma; X)]_{\frac{k_1}{k}}) \\
&\qquad = W^{k_1,p}(\mathbb{R}^{d-1}; W^{k_2,p}(\mathbb{R}_+, w_\gamma; X)),
\end{aligned}$$

where it was used that $1 - \frac{k_1}{k} = \frac{k_2}{k}$. This proves the result. □

3.6.3. The proofs of Theorems 3.6.4 and 3.6.5

Using the trace theorem for $\mathscr{B}$ from Section 3.5, we will prove Theorems 3.6.4 and 3.6.5. The proofs follow the arguments in [16, Theorems VIII.2.4.3, VIII.2.4.4 & VIII.2.4.8] for unweighted Bessel potential spaces. We will provide the proof of Theorem 3.6.5 in full detail and afterwards indicate how the arguments should be adapted to prove Theorem 3.6.4.

Proof of Theorem 3.6.5. First, we prove in Steps 1 and 2 that for $\theta := \frac{\ell}{k_1}$ we have

$$W^{k_0+\ell,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X) = \big[W^{k_0,p}(\mathbb{R}^d_+, w_\gamma; X), W^{k_0+k_1,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)\big]_\theta. \tag{3.6.3}$$

For notational convenience we write $W^{k,p} := W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ for $k \in \mathbb{N}_0$.

Step 1: the embedding "$\hookleftarrow$". Note that by Proposition 3.6.3(ii) it holds that

$$\big[W^{k_0,p}, W^{k_0+k_1,p}_{\mathscr{B}}\big]_\theta \hookrightarrow \big[W^{k_0,p}, W^{k_0+k_1,p}\big]_\theta \hookrightarrow W^{k_0+\ell,p}. \tag{3.6.4}$$

Now, suppose that $u \in [W^{k_0,p}, W^{k_0+k_1,p}_{\mathscr{B}}]_\theta$. We prove that $u \in W^{k_0+\ell,p}_{\mathscr{B}}$ in Step 1a and 1b, i.e., u also satisfies the required boundary conditions. We distinguish between the case where the boundary traces do not exist (Step 1a), and the case where they exist and must be shown to vanish (Step 1b).

Step 1a. Assume that $k_0 + \ell < m_0 + \frac{\gamma+1}{p}$. Then it holds that $W^{k_0+\ell,p}_{\mathscr{B}} = W^{k_0+\ell,p}$. Therefore, the embedding

$$\big[W^{k_0,p}, W^{k_0+k_1,p}_{\mathscr{B}}\big]_\theta \hookrightarrow W^{k_0+\ell,p}_{\mathscr{B}}$$

follows from (3.6.4).

Step 1b. Assume that $k_0 + \ell > m_0 + \frac{\gamma+1}{p}$ and let $i \leq n$ be the largest integer such that $m_i + \frac{\gamma+1}{p} < k_0 + \ell$. By definition of the complex interpolation method (see Section 2.5),

there exists an $f \in \mathscr{H}(W^{k_0,p}, W^{k_0+k_1,p}_{\mathscr{B}})$ such that $f(\theta) = u$. By (2.5.1) and (3.6.4) we have that the restriction $f|_{\mathbb{S}_{[\theta,1]}}$ is bounded and continuous with values in $W^{k_0+\ell,p}$ and is holomorphic on $\mathbb{S}_{(\theta,1)}$. Therefore, if $0 \le j \le i$, then $\mathscr{B}^{m_j} f$ is bounded and continuous on $\mathbb{S}_{[\theta,1]}$ with values in $B^{k_0+\ell-m_j-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; Y_j)$ (see Theorem 3.5.6) and is holomorphic on $\mathbb{S}_{(\theta,1)}$. Moreover, for $\operatorname{Re}(z) = 1$ we have $\mathscr{B}^{m_j} f(z) = 0$. By the three lines lemma (see [27, Lemma 1.1.2]), $\mathscr{B}^{m_j} f(z)$ vanishes identically and thus

$$\mathscr{B}^{m_j} u = \mathscr{B}^{m_j} f(\theta) = 0 \qquad \text{for } 0 \le j \le i.$$

This proves $u \in W^{k_0+\ell,p}_{\mathscr{B}}$ and thus $[W^{k_0,p}, W^{k_0+k_1,p}_{\mathscr{B}}]_\theta \hookrightarrow W^{k_0+\ell,p}_{\mathscr{B}}$.

Step 2: the embedding "$\hookrightarrow$". We continue with the proof of the embedding "$\hookrightarrow$" in (3.6.3). Again, we consider two cases depending on which traces exist. In Step 2a, we consider the case where all traces exist. For $u \in W^{k_0+\ell,p}_{\mathscr{B}}$ we explicitly construct a decomposition $u = v + (u - v)$, where v satisfies the boundary conditions, and $u - v$ has vanishing traces. In this setting, we construct a right inverse for an extended system of boundary operators $\mathscr{C}$ to find such a v. In Step 2b, we consider the case where certain traces do not exist and the result can be deduced by reiteration.

Before proceeding with the main argument, we construct a modified boundary operator such that its leading order coefficient is a projection. This is a necessary preliminary step for the construction of the extended system of boundary operators $\mathscr{C}$ in Step 2a.

Let $0 \le i \le n$ and let b_{i,m_i} be the leading order coefficient of $\mathscr{B}^{m_i}$ for which b^{c}_{i,m_i} is the right inverse from Definition 3.5.3. Define the boundary operator $\widetilde{\mathscr{B}}^{m_i} := b^{\mathrm{c}}_{i,m_i}\mathscr{B}$. Then it is straightforward to verify that

$$\ker(\widetilde{\mathscr{B}}^{m_i}) = \ker(\mathscr{B}^{m_i}) \qquad \text{for } 0 \le i \le n. \tag{3.6.5}$$

Similar to (3.5.1) we write

$$\widetilde{\mathscr{B}}^{m_i} = \sum_{j=0}^{m_i} \widetilde{b}_{i,j}\operatorname{Tr}_j \qquad \text{where } \widetilde{b}_{i,j} := b^{\mathrm{c}}_{i,m_i} b_{i,j}.$$

We claim that $\pi_{m_i} := \widetilde{b}_{i,m_i} : \mathbb{R}^{d-1} \to \mathscr{L}(X)$ is a projection. Indeed, for $0 \le j \le m_i$, we have

$$\pi_{m_i}\widetilde{b}_{i,j} = b^{\mathrm{c}}_{i,m_i} b_{i,m_i} b^{\mathrm{c}}_{i,m_i} b_{i,j} = b^{\mathrm{c}}_{i,m_i} b_{i,j} = \widetilde{b}_{i,j}. \tag{3.6.6}$$

Moreover, (3.6.6) implies that for $i \in \{0,\dots,n\}$ we have

$$\pi_{m_i}\widetilde{\mathscr{B}}^{m_i} = \widetilde{\mathscr{B}}^{m_i}. \tag{3.6.7}$$

Step 2a. Assume that $k_0 + k_1 > m_n + \frac{\gamma+1}{p}$. We start with the construction of an extended system of normal boundary operators $\mathscr{C}$ whose kernel coincides with $W^{k_0+k_1,p}_0$.

Let a be the largest integer such that $a + \frac{\gamma+1}{p} < k_0 + k_1$ and define $\overline{a} = (0,1,\dots,a)$. Moreover, by setting $\overline{X} = (X,\dots,X)$ with length $a+1$, we can define a normal bound-

ary operator of type $(p, k_0+k_1, \gamma, \overline{a}, \overline{X})$ by

$$\begin{aligned} \mathscr{C} &= (\mathscr{C}^0, \dots, \mathscr{C}^a) : W^{k_0+k_1,p} \to \prod_{j=0}^{a} B_{p,p}^{k_0+k_1-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X), \\ \mathscr{C}^j &:= \begin{cases} \operatorname{Tr}_j & \text{if } j \notin \{m_0, \dots, m_n\}, \\ (1-\pi_j)\operatorname{Tr}_j + \widetilde{\mathscr{B}}^j & \text{if } j \in \{m_0, \dots, m_n\}. \end{cases} \end{aligned} \tag{3.6.8}$$

Note that by (3.6.7) we have

$$\pi_j \mathscr{C}^j = \pi_j \widetilde{\mathscr{B}}^j = \widetilde{\mathscr{B}}^j \qquad \text{for } j \in \{m_0, \dots, m_n\}. \tag{3.6.9}$$

We show that

$$\mathscr{C} v = 0 \quad \text{if and only if} \quad \overline{\operatorname{Tr}}_a v = 0 \qquad \text{for } v \in W^{k_0+k_1,p}. \tag{3.6.10}$$

Indeed, let $v \in W^{k_0+k_1,p}$ and $\mathscr{C} v = 0$, then for $0 \le \widetilde{a} \le a$ with $\widetilde{a} \notin \{m_0, \dots, m_n\}$, we have

$$\operatorname{Tr}_{\widetilde{a}} v = \mathscr{C}^a v = 0. \tag{3.6.11}$$

If $\widetilde{a} = m_i \in \{m_0, \dots, m_n\}$ for some $0 \le i \le n$, then by (3.6.7) and (3.6.11) it holds that

$$\begin{aligned} 0 = \mathscr{C}^{m_i} v &= (1-\pi_{m_i})\operatorname{Tr}_{m_i} v + \pi_{m_i} \widetilde{\mathscr{B}}^{m_i} v \\ &= \operatorname{Tr}_{m_i} v + \sum_{j=0}^{m_i-1} \widetilde{b}_{i,j} \operatorname{Tr}_j v \\ &= \operatorname{Tr}_{m_i} v + \sum_{j \in \{m_0, \dots, m_{i-1}\}} \widetilde{b}_{i,j} \operatorname{Tr}_j v. \end{aligned}$$

It follows that $\operatorname{Tr}_{m_0} v = \mathscr{C}^{m_0} v = 0$ and successively we obtain $\operatorname{Tr}_{m_i} v = \mathscr{C}^{m_i} v = 0$ for $0 \le i \le n$ as well. The converse statement is trivial, so we have proved (3.6.10).

From (3.6.10) and the definition of $W_0^{k_0+k_1,p}$, it follows that

$$W_{\mathscr{C}}^{k_0+k_1,p} = W_{\overline{\operatorname{Tr}}_a}^{k_0+k_1,p} = W_0^{k_0+k_1,p}.$$

By Theorem 3.5.6 there exists a right inverse

$$\operatorname{ext}_{\mathscr{C}} : \prod_{j=0}^{a} B_{p,p}^{k_0+k_1-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X) \to W^{k_0+k_1,p}$$

for $\mathscr{C}$, which is universal with respect to $k_0 + k_1 \in (a + \frac{\gamma+1}{p}, k]$.

Let $u \in W_{\mathscr{B}}^{k_0+\ell,p} = W_{\widetilde{\mathscr{B}}}^{k_0+\ell,p}$ (recall (3.6.5)) and we will now prove that $u \in [W^{k_0,p}, W_{\mathscr{B}}^{k_0+k_1,p}]_\theta$ with $\theta = \frac{\ell}{k_1}$. Using the extension operator $\operatorname{ext}_{\mathscr{C}}$ from above, we construct a suitable $v \in [W^{k_0,p}, W_{\mathscr{B}}^{k_0+k_1,p}]_\theta$ such that $u = v + (u - v)$ and all the traces of $u - v$ vanish.

Let a_1 be the largest integer such that $a_1 + \frac{\gamma+1}{p} < k_0 + \ell$ and define for $j \le a_1$

$$g_j := \begin{cases} \mathscr{C}^j u & \text{if } j \notin \{m_0, \dots, m_n\}, \\ (1-\pi_j)\mathscr{C}^j u & \text{if } j \in \{m_0, \dots, m_n\}. \end{cases}$$

By Lemma 3.5.2 and [126, Theorem 14.4.30], we obtain

$$g_j \in B_{p,p}^{k_0+\ell-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X) = \big[B_{p,p}^{k_0-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X), B_{p,p}^{k_0+k_1-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X)\big]_\theta.$$

Therefore, by definition of the complex interpolation method, there exists a

$$\widetilde{f}_j \in \mathscr{H}\big(B_{p,p}^{k_0-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X), B_{p,p}^{k_0+k_1-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X)\big)$$

such that $\widetilde{f}_j(\theta) = g_j$ for $j \le a_1$. In addition, for $j \le a_1$ we define

$$f_j := \begin{cases} \widetilde{f}_j & \text{if } j \notin \{m_0,\dots,m_n\}, \\ (1-\pi_j)\widetilde{f}_j & \text{if } j \in \{m_0,\dots,m_n\}, \end{cases}$$

which satisfies the same properties as $\widetilde{f}_j$. Then

$$F := \operatorname{ext}_{\mathscr{C}}(f_0,\dots,f_{a_1},0,\dots,0) \in \mathscr{H}(W^{k_0,p}, W^{k_0+k_1,p})$$

and

$$v := F(\theta) = \operatorname{ext}_{\mathscr{C}}\big(f_0(\theta),\dots,f_{a_1}(\theta),0,\dots,0\big) = \operatorname{ext}_{\mathscr{C}}(g_0,\dots,g_{a_1},0,\dots,0).$$

Moreover, F is bounded and continuous on $\{z \in \mathbb{C} : \operatorname{Re} z = 1\}$ with values in $W^{k_0+k_1,p}$ and by (3.6.9), properties of the right inverse $\operatorname{ext}_{\mathscr{C}}$ and the definition of f_j, we have for $j \in \{m_0,\dots,m_n\}$

$$\begin{aligned} \widetilde{\mathscr{B}}^j F(1) &= \pi_j \mathscr{C}^j \operatorname{ext}_{\mathscr{C}}(f_0(1),\dots,f_{a_1}(1),0,\dots,0) \\ &= \begin{cases} \pi_j f_j(1) = 0 & \text{if } j \in \{m_0,\dots,m_n\},\ j \le a_1, \\ 0 & \text{if } j \in \{m_0,\dots,m_n\},\ a_1 < j \le a. \end{cases} \end{aligned}$$

This implies $F \in \mathscr{H}(W^{k_0,p}, W_{\widetilde{\mathscr{B}}}^{k_0+k_1,p})$ and thus by (3.6.5) we have

$$v = F(\theta) \in \big[W^{k_0,p}, W_{\mathscr{B}}^{k_0+k_1,p}\big]_\theta. \tag{3.6.12}$$

To continue, we prove that $u - v \in W_{\mathscr{C}}^{k_0+\ell,p}$. Indeed, by definition of g_j, (3.6.9) and the fact that $\widetilde{\mathscr{B}}^j u = 0$ for $j \in \{m_0,\dots,m_n\}$, we obtain

$$\begin{aligned} \mathscr{C}^j(u-v) &= \mathscr{C}^j u - \mathscr{C}^j \operatorname{ext}_{\mathscr{C}}(g_0,\dots,g_{a_1},0,\dots,0) \\ &= \mathscr{C}^j u - g_j = \begin{cases} g_j - g_j = 0 & \text{if } j \notin \{m_0,\dots,m_n\},\ j \le a_1, \\ \mathscr{C}^j u - (1-\pi_j)\mathscr{C}^j u = 0 & \text{if } j \in \{m_0,\dots,m_n\},\ j \le a_1. \end{cases} \end{aligned}$$

It follows from Proposition 3.6.2 that

$$u - v \in W_{\mathscr{C}}^{k_0+\ell,p} = W_0^{k_0+\ell,p} = \big[W_0^{k_0,p}, W_0^{k_0+k_1,p}\big]_\theta \hookrightarrow \big[W^{k_0,p}, W_{\mathscr{B}}^{k_0+k_1,p}\big]_\theta. \tag{3.6.13}$$

Then combining (3.6.12) and (3.6.13) implies $u = v + (u-v) \in [W^{k_0,p}, W_{\mathscr{B}}^{k_0+k_1,p}]_\theta$ with $\theta = \frac{\ell}{k_1}$. This completes the proof of the embedding "$\hookrightarrow$" in (3.6.3) if $k_0 + k_1 > m_n + \frac{\gamma+1}{p}$.

Step 2b. We prove the embedding "$\hookrightarrow$" in (3.6.3) if $k_0+k_1<m_n+\frac{\gamma+1}{p}$. Fix $a\in\mathbb{N}_0$ such that $k_0+a\in(m_n+\frac{\gamma+1}{p},k]$. Then by reiteration for the complex interpolation method (see [27, Theorem 4.6.1]) and Step 2a twice, we obtain

$$\big[W^{k_0,p},W_{\mathscr{B}}^{k_0+k_1,p}\big]_{\frac{\ell}{k_1}}=\big[W^{k_0,p},[W^{k_0,p},W_{\mathscr{B}}^{k_0+a,p}]_{\frac{k_1}{a}}\big]_{\frac{\ell}{k_1}}=\big[W^{k_0,p},W_{\mathscr{B}}^{k_0+a,p}\big]_{\frac{\ell}{a}}=W_{\mathscr{B}}^{k_0+\ell,p}.$$

Combining Steps 1 and 2 completes the proof of (3.6.3).

Step 3. It remains to prove for $k_0\in\mathbb{N}_1$ that

$$W_{\mathscr{B}}^{k_0+\ell,p}(\mathbb{R}^d_+,w_\gamma;X)=\big[W_{\mathscr{B}}^{k_0,p}(\mathbb{R}^d_+,w_\gamma;X),W_{\mathscr{B}}^{k_0+k_1,p}(\mathbb{R}^d_+,w_\gamma;X)\big]_{\frac{\ell}{k_1}}.$$

Fix $a\in\mathbb{N}_0$ such that $a<k_0$ and define $\theta_1:=(k_0-a)/(k_0+k_1-a)$. Then by reiteration and (3.6.3) twice, we obtain

$$\begin{aligned}\big[W_{\mathscr{B}}^{k_0,p},W_{\mathscr{B}}^{k_0+k_1,p}\big]_{\frac{\ell}{k_1}}&=\big[[W^{a,p},W_{\mathscr{B}}^{k_0+k_1,p}]_{\theta_1},W_{\mathscr{B}}^{k_0+k_1,p}\big]_{\frac{\ell}{k_1}}\\&=\big[W^{a,p},W_{\mathscr{B}}^{k_0+k_1,p}\big]_{(1-\frac{\ell}{k_1})\theta_1+\frac{\ell}{k_1}}=W_{\mathscr{B}}^{k_0+\ell,p}.\end{aligned}$$

This completes the proof of the theorem. □

We conclude with the proof of Theorem 3.6.4.

Proof of Theorem 3.6.4. We first prove in Steps 1 and 2 that

$$H_{\mathscr{B}}^{s_\theta,p}(\mathbb{R}^d_+,w_\gamma;X)=\big[H^{s_0,p}(\mathbb{R}^d_+,w_\gamma;X),H_{\mathscr{B}}^{s_1,p}(\mathbb{R}^d_+,w_\gamma;X)\big]_\theta. \tag{3.6.14}$$

For notational convenience we write $H^{s,p}:=H^{s,p}(\mathbb{R}^d_+,w_\gamma;X)$.

Step 1: the embedding "$\hookleftarrow$". By [189, Proposition 5.6] it holds that

$$\big[H^{s_0,p},H_{\mathscr{B}}^{s_1,p}\big]_\theta\hookrightarrow\big[H^{s_0,p},H^{s_1,p}\big]_\theta=H^{s_\theta,p}.$$

Thus the embedding $[H^{s_0,p},H_{\mathscr{B}}^{s_1,p}]_\theta\hookrightarrow H_{\mathscr{B}}^{s_\theta,p}$ can be proved similarly as in Step 1 of the proof of Theorem 3.6.5.

Step 2: the embedding "$\hookrightarrow$". We adopt the same notation as in Step 2 of the proof of Theorem 3.6.5.

Step 2a. Assume that $s_1>m_n+\frac{\gamma+1}{p}$ and $s_1\notin\mathbb{N}_0+\frac{\gamma+1}{p}$. Note that the second condition on s_1 is new compared to the proof of Theorem 3.6.5. This condition will be removed in Step 2b.

Let a be the largest integer such that $a+\frac{\gamma+1}{p}<s_1$. The operator

$$\mathscr{C}=(\mathscr{C}^0,\dots,\mathscr{C}^a):H^{s_1,p}\to\prod_{j=0}^{a}B_{p,p}^{s_1-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X),$$

as defined in (3.6.8) satisfies (3.6.10) for $v\in H^{s_1,p}$. Therefore, we have

$$H_{\mathscr{C}}^{s_1,p}=H_{\overline{\mathrm{Tr}_a}}^{s_1,p}=H_0^{s_1,p}.$$

We can now proceed similarly as in Step 2a of the proof of Theorem 3.6.5 using Proposition 3.6.1 to obtain (cf. (3.6.13))

$$H^{s_\theta,p}_{\mathscr{C}} = H^{s_\theta,p}_0 = \big[H^{s_0,p}_0, H^{s_1,p}_0\big]_\theta \hookrightarrow \big[H^{s_0,p}, H^{s_1,p}_{\mathscr{B}}\big]_\theta.$$

Step 2b. Assume that $s_1 > m_n + \frac{\gamma+1}{p}$ and there exists an $\ell \in \mathbb{N}_0$ such that $s_1 = \ell + \frac{\gamma+1}{p}$. Fix $t \in (s_1, s]$ such that $t \notin \mathbb{N}_0 + \frac{\gamma+1}{p}$. Furthermore, let $\theta_1 := (s_1 - s_0)/(t - s_0)$. Then by reiteration for the complex interpolation method (see [27, Theorem 4.6.1]) and Step 2a twice, we obtain

$$\big[H^{s_0,p}, H^{s_1,p}_{\mathscr{B}}\big]_\theta = \big[H^{s_0,p}, [H^{s_0,p}, H^{t,p}_{\mathscr{B}}]_{\theta_1}\big]_\theta = \big[H^{s_0,p}, H^{t,p}_{\mathscr{B}}\big]_{\theta\theta_1} = H^{s_\theta,p}_{\mathscr{B}}.$$

Step 2c. Assume that $s_1 < m_n + \frac{\gamma+1}{p}$. Fix $t \in (m_n + \frac{\gamma+1}{p}, s]$ and let $\theta_1 := (s_1 - s_0)/(t - s_0)$. Then again by reiteration and Step 2a, we obtain $[H^{s_0,p}, H^{s_1,p}_{\mathscr{B}}]_\theta = H^{s_\theta,p}_{\mathscr{B}}$. This completes the proof of (3.6.14).

Step 3. It remains to prove

$$H^{s_\theta,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X) = \big[H^{s_0,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X), H^{s_1,p}_{\mathscr{B}}(\mathbb{R}^d_+, w_\gamma; X)\big]_\theta.$$

Fix $\widetilde{s} < s_0$, with $\widetilde{s} > 0$ if $s_0 > 0$ and $\widetilde{s} > -1 + \frac{\gamma+1}{p}$ otherwise. Let $\theta_1 := (s_0 - \widetilde{s})/(s_1 - \widetilde{s})$. Then by reiteration and (3.6.14) twice, we obtain

$$\big[H^{s_0,p}_{\mathscr{B}}, H^{s_1,p}_{\mathscr{B}}\big]_\theta = \big[[H^{\widetilde{s},p}, H^{s_1,p}_{\mathscr{B}}]_{\theta_1}, H^{s_1,p}_{\mathscr{B}}\big]_\theta = \big[H^{\widetilde{s},p}, H^{s_1,p}_{\mathscr{B}}\big]_{(1-\theta)\theta_1+\theta} = H^{s_\theta,p}_{\mathscr{B}}.$$

This completes the proof of the theorem. □

4

Trace theory for intersections of weighted Sobolev spaces

This chapter is based on:

R. Denk and F. Roodenburg. Trace theory for parabolic boundary value problems with rough boundary conditions. *J. Evol. Equ.*, 26(3):80, 2026.

In addition, this chapter is complemented with some unpublished results.

Abstract. *We characterise the spatial trace spaces arising from intersections of weighted vector-valued Sobolev spaces, where the weights are powers of the distance to the boundary. These weighted function spaces are particularly suitable for treating boundary value problems where derivatives of the solution blow up at the boundary. First, we establish trace results for anisotropic Triebel–Lizorkin spaces. This allows us to derive similar results for intersections of weighted Sobolev spaces on the half-space. Finally, using a localisation argument based on the Dahlberg–Kenig–Stein pullback, we can transfer the trace results to bounded domains with minimal regularity assumptions.*

4.1. Introduction

The analysis of boundary value problems with inhomogeneous boundary data requires a precise understanding of the underlying function spaces and their spatial trace spaces. For instance, consider the heat equation on a bounded domain $\mathscr{O} \subseteq \mathbb{R}^d$ with Dirichlet boundary conditions, i.e.,

$$\begin{cases} \partial_t u - \Delta u = f & t > 0,\ x \in \mathscr{O}, \\ u|_{\partial\mathscr{O}} = g & t > 0, \\ u|_{t=0} = 0 & x \in \mathscr{O}. \end{cases} \tag{4.1.1}$$

A classical problem is to find necessary and sufficient conditions for the data f and g such that (4.1.1) is well-posed and has a solution

$$u \in W^{1,q}(\mathbb{R}_+; L^p(\mathscr{O})) \cap L^q(\mathbb{R}_+; W^{2,p}(\mathscr{O})), \qquad p, q \in (1,\infty), \tag{4.1.2}$$

that depends continuously on the data. It is well known that $f \in L^q(\mathbb{R}_+; L^p(\mathscr{O}))$ is a natural condition, while identifying the optimal function space for the boundary data g is more delicate. This problem has been extensively studied in the literature, first for $p = q$ in [176] and later in [68, 132, 269, 270, 271] for the more general case $p \neq q$. The case $p \neq q$ is particularly relevant for nonlinear problems, where one wants to exploit the scaling invariance of the problem, see, e.g., [99]. Moreover, we note that the end-point case $q = 1$ is studied in [226, 227].

For (4.1.1) on a bounded C^2-domain $\mathscr{O}$, it turns out that the boundary data g has to be in the intersection space

$$F^{1-\frac{1}{2p}}_{q,p}(\mathbb{R}_+; L^p(\partial\mathscr{O})) \cap L^q(\mathbb{R}_+; B^{2-\frac{1}{p}}_{p,p}(\partial\mathscr{O})), \tag{4.1.3}$$

where $F^s_{q,p}$ and $B^s_{p,p}$ denote the Triebel–Lizorkin and Besov spaces, respectively. The space (4.1.3) is the optimal space for the boundary data, meaning that the Dirichlet trace operator from the solution space (4.1.2) to the spatial trace space (4.1.3) is continuous and surjective.

In this chapter, we study mapping properties of trace operators in a weighted framework. Specifically, we consider spatial weights of the form $w^{\partial\mathscr{O}}_\gamma(x) := \operatorname{dist}(x, \partial\mathscr{O})^\gamma$ for some suitable $\gamma > -1$. These weights are well suited to control boundary singularities of solutions and their derivatives. As a special case of our main result in Theorem 4.5.2, we obtain the following weighted trace result, which is suitable for treating (4.1.1) in the weighted setting. Applications to partial differential equations such as (4.1.1) are postponed to Chapter 7.

Theorem 4.1.1. *Let $p, q \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1)$, $\gamma \in ((1-\kappa)p - 1, 2p-1) \setminus \{p-1\}$. Furthermore, let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Then the trace operator $\operatorname{Tr}^{\partial\mathscr{O}}$ from*

$$W^{1,q}(\mathbb{R}_+; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})) \cap L^q(\mathbb{R}_+; W^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$$

to

$$F^{1-\frac{\gamma+1}{2p}}_{q,p}(\mathbb{R}_+; L^p(\partial\mathscr{O})) \cap L^q(\mathbb{R}_+; B^{2-\frac{\gamma+1}{p}}_{p,p}(\partial\mathscr{O}))$$

is continuous and surjective. Moreover, there exists a continuous right inverse $\operatorname{ext}^{\partial\mathscr{O}}$ of $\operatorname{Tr}^{\partial\mathscr{O}}$.

In addition, we prove a variant of Theorem 4.1.1 with higher-order traces and more time and space regularity, which is required for treating higher-order parabolic equations with inhomogeneous boundary conditions. Furthermore, our approach also covers the vector-valued setting with temporal weights on bounded time intervals. The proof of Theorem 4.1.1 is based on an extension of the trace theory for weighted anisotropic Triebel–Lizorkin spaces as developed in [132, 184] and intersection representations of these spaces. We note that Theorem 4.1.1 (see also Theorem 4.5.2) advances the existing literature in several respects.

(i) We consider spaces with higher-order spatial Sobolev regularity. This allows us to deal with higher-order regularity of (4.1.1) in Chapter 7. It is important to note that the trace space is *independent* of the smoothness k, as is also indicated by trace theory for weighted Sobolev spaces, see, e.g., [140] or Chapter 3. The weighted setting is particularly convenient for higher-order regularity since it avoids any compatibility conditions in the underlying Banach spaces, which are necessary to obtain, for instance, sectoriality of the Laplacian, see [66] and the introduction of Chapter 5 for a more elaborate discussion.

(ii) The regularity of the domain $\mathscr{O}$ depends on the weight exponent γ, but *not* on the smoothness k.

(iii) In Theorem 4.5.2 we deal with higher-order traces $\overline{\mathrm{Tr}}_m^{\partial\mathscr{O}}$, which is in contrast to [190, Theorem 7.1], where the trace theory in the weighted setting was developed for second-order parabolic equations with Dirichlet boundary conditions. The higher-order traces model Dirichlet boundary conditions for higher-order parabolic boundary value problems, such as in [74]. In Chapter 8, we will use this general trace theory for studying a structurally damped plate equation with rough boundary conditions. In addition, we mention that in [123, 184] higher-order trace spaces are characterised for function spaces with mainly Muckenhoupt weights.

Outline

The outline of this chapter is as follows. In Section 4.2 we introduce the anisotropic function spaces and some of their properties. In Section 4.3 we first prove characterisations of higher-order trace spaces of anisotropic Triebel–Lizorkin spaces, which are used in Section 4.4 to obtain the trace spaces of intersections of weighted Sobolev spaces on the half-space. In Section 4.5 we apply a localisation procedure to transfer the trace results from the half-space to domains with minimal smoothness assumptions. Finally, in Section 4.6 we prove a trace theorem for more general boundary operators with space and time-dependent coefficients.

4.2. Anisotropic function spaces

Recall that (isotropic) Besov, Triebel–Lizorkin, Bessel potential and Sobolev spaces are defined in Section 3.2.1. We will now introduce anisotropic function spaces, where we mainly focus on Triebel–Lizorkin, Bessel potential and Sobolev spaces. For more details

on anisotropic spaces, we refer to [15, 132, 183].

Let $\ell \in \mathbb{N}_1$ and $\mathscr{d} = (d_1, \dots, d_\ell) \in \mathbb{N}_1^\ell$ such that $d := |\mathscr{d}| = \sum_{j=1}^\ell d_j$. The *$\mathscr{d}$-decomposition of* $\mathbb{R}^d$ is given by

$$\mathbb{R}^d = \mathbb{R}^{d_1} \times \cdots \times \mathbb{R}^{d_\ell},$$

and for $x \in \mathbb{R}^d$ we write $x = (x_1, \dots, x_\ell)$ and $x_j = (x_{j,1}, \dots, x_{j,d_j})$ where $x_j \in \mathbb{R}^{d_j}$ and $x_{j,i} \in \mathbb{R}$ for $j \in \{1, \dots, \ell\}$ and $i \in \{1, \dots, d_j\}$.

Let $\boldsymbol{p} = (p_1, \dots, p_\ell) \in [1,\infty)^\ell$ and $\boldsymbol{w} = (w_1, \dots, w_\ell)$, where w_j is a weight on $\mathbb{R}^{d_j}$ for $j \in \{1, \dots, \ell\}$. Furthermore, let X be a Banach space. The *weighted mixed-norm Lebesgue space* $L^{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d, \boldsymbol{w}; X)$ is the Bochner space consisting of all strongly measurable $f : \mathbb{R}^d \to X$ such that

$$\|f\|_{L^{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d,\boldsymbol{w};X)} := \Big(\int_{\mathbb{R}^{d_\ell}} \cdots \Big(\int_{\mathbb{R}^{d_1}} \|f(x)\|_X^{p_1} w_1(x_1) \, \mathrm{d}x_1 \Big)^{\frac{p_2}{p_1}} \cdots w_\ell(x_\ell) \, \mathrm{d}x_\ell \Big)^{\frac{1}{p_\ell}} < \infty.$$

To introduce anisotropic function spaces that measure smoothness as well, we first define the following concepts. Let $\mathbb{R}^d$ be $\mathscr{d}$-decomposed and $\boldsymbol{a} \in (0,\infty)^\ell$. For $\lambda > 0$ we define the *$(\mathscr{d}, \boldsymbol{a})$-anisotropic dilation* $\delta_\lambda^{(\mathscr{d},\boldsymbol{a})} := (\lambda^{a_1} x_1, \dots, \lambda^{a_\ell} x_\ell)$ for $x \in \mathbb{R}^d$ and the *$(\mathscr{d}, \boldsymbol{a})$-anisotropic distance function*

$$|x|_{\mathscr{d},\boldsymbol{a}} := \Big(\sum_{j=1}^\ell |x_j|^{\frac{2}{a_j}} \Big)^{\frac{1}{2}}, \qquad x \in \mathbb{R}^d.$$

In addition, let $0 < A < B < \infty$ and define the set of *anisotropic Littlewood–Paley sequences* $\Phi_{A,B}^{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^d)$ as the set of all $(\varphi_n)_{n\geq 0} \subseteq \mathscr{S}(\mathbb{R}^d)$ such that $\widehat{\varphi}_0 = \widehat{\varphi}$ and

$$\widehat{\varphi}_n(\xi) = \widehat{\varphi}\big(\delta_{2^{-n}}^{(\mathscr{d},\boldsymbol{a})}\xi\big) - \widehat{\varphi}\big(\delta_{2^{-n+1}}^{(\mathscr{d},\boldsymbol{a})}\xi\big), \qquad \xi \in \mathbb{R}^d, \, n \in \mathbb{N}_1,$$

where the generating function $\varphi \in \mathscr{S}(\mathbb{R}^d)$ satisfies $0 \leq \widehat{\varphi}(\xi) \leq 1$ for $\xi \in \mathbb{R}^d$, $\widehat{\varphi}(\xi) = 1$ if $|\xi|_{\mathscr{d},\boldsymbol{a}} \leq A$ and $\widehat{\varphi}(\xi) = 0$ if $|\xi|_{\mathscr{d},\boldsymbol{a}} \geq B$. Furthermore, we define $\Phi^{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^d) := \bigcup_{0<A<B<\infty} \Phi_{A,B}^{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^d)$.

Let $\boldsymbol{p} \in [1,\infty)^\ell$, $\boldsymbol{a} \in (0,\infty)^\ell$, $q \in [1,\infty]$, $s \in \mathbb{R}$, $\boldsymbol{w} \in \prod_{j=1}^\ell A_\infty(\mathbb{R}^{d_j})$ and let X be a Banach space. Let $\varphi \in \Phi^{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^d)$ and let $(S_n)_{n\geq 0} \subseteq \mathscr{L}(\mathscr{S}'(\mathbb{R}^d; X))$ be the associated family of convolution operators given by $S_n f := \varphi_n * f$. The *weighted anisotropic Triebel–Lizorkin space* $F_{\boldsymbol{p},q}^{s,\boldsymbol{a}}(\mathbb{R}^d, \boldsymbol{w}; X)$ is defined as the space of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ such that

$$\|f\|_{F_{\boldsymbol{p},q}^{s,\boldsymbol{a}}(\mathbb{R}^d,\boldsymbol{w};X)} := \|(2^{ns} S_n f)_{n\geq 0}\|_{L^{\boldsymbol{p},\mathscr{d}}(\mathbb{R}^d,\boldsymbol{w};\ell^q(X))} < \infty.$$

We note that this definition of the Triebel–Lizorkin space is independent of the choice of $\varphi \in \Phi^{\mathscr{d},\boldsymbol{a}}(\mathbb{R}^d)$.

Next, we define anisotropic Bessel potential spaces and Sobolev spaces. Let $\mathbb{R}^d$ be $\mathscr{d}$-decomposed as above, then for $j \in \{1, \dots, \ell\}$ we define the inclusion and projection

maps

$$\iota_{[d;j]} : \mathbb{R}^{d_j} \to \mathbb{R}^d, \qquad x_j \mapsto (0,\dots,0,x_j,0,\dots,0),$$
$$\pi_{[d;j]} : \mathbb{R}^d \to \mathbb{R}^{d_j}, \qquad (x_1,\dots,x_\ell) \mapsto x_j.$$

For $\sigma \in \mathbb{R}$ we define the operator $\mathscr{J}_\sigma^{[d;j]} \in \mathscr{L}(\mathscr{S}'(\mathbb{R}^d;X))$ by

$$\mathscr{J}_\sigma^{[d;j]} f := \mathscr{F}^{-1}\big((1+|\pi_{[d;j]}\cdot|^2)^{\frac{\sigma}{2}} \mathscr{F} f\big).$$

Furthermore, for $\boldsymbol{n} = (n_1,\dots,n_\ell) \in \mathbb{N}_0^\ell$ let $J_{\boldsymbol{n},d}$ be the set of multi-indices given by

$$J_{\boldsymbol{n},d} := \Big\{\alpha \in \bigcup_{j=1}^{\ell} \iota_{[d;j]}\mathbb{N}_0^{d_j} : |\alpha_j| \le n_j\Big\}.$$

Let $\boldsymbol{p} \in (1,\infty)^\ell$, $\boldsymbol{s} \in (0,\infty)^\ell$, $\boldsymbol{n} \in \mathbb{N}_0^\ell$, $\boldsymbol{w} \in \prod_{j=1}^{\ell} A_{p_j}(\mathbb{R}^{d_j})$ and let X be a Banach space. Then we define the *weighted anisotropic Bessel potential and Sobolev spaces* as

$$W_d^{\boldsymbol{n},\boldsymbol{p}}(\mathbb{R}^d,\boldsymbol{w};X) := \big\{f \in \mathscr{S}'(\mathbb{R}^d;X) : \partial^\alpha f \in L^{\boldsymbol{p},d}(\mathbb{R}^d,\boldsymbol{w};X),\ \alpha \in J_{\boldsymbol{n},d}\big\},$$
$$H_d^{\boldsymbol{s},\boldsymbol{p}}(\mathbb{R}^d,\boldsymbol{w};X) := \big\{f \in \mathscr{S}'(\mathbb{R}^d;X) : \mathscr{J}_{s_j}^{[d;j]} f \in L^{\boldsymbol{p},d}(\mathbb{R}^d,\boldsymbol{w};X),\ j \in \{1,\dots,\ell\}\big\},$$

endowed with the norms

$$\|f\|_{W_d^{\boldsymbol{n},\boldsymbol{p}}(\mathbb{R}^d,\boldsymbol{w};X)} := \sum_{\alpha\in J_{\boldsymbol{n},d}} \|\partial^\alpha f\|_{L^{\boldsymbol{p},d}(\mathbb{R}^d,\boldsymbol{w};X)},$$
$$\|f\|_{H_d^{\boldsymbol{s},\boldsymbol{p}}(\mathbb{R}^d,\boldsymbol{w};X)} := \sum_{j=1}^{\ell} \|\mathscr{J}_{s_j}^{[d;j]} f\|_{L^{\boldsymbol{p},d}(\mathbb{R}^d,\boldsymbol{w};X)}.$$

4.3. Traces of anisotropic Triebel–Lizorkin spaces

Similar to [132, 183, 184], we introduce two notions for the trace at the hyperplane $\{0\}\times\mathbb{R}^{d-1}$: the distributional trace and the so-called working definition of the trace.

The working definition of the trace. Let $\varphi \in \Phi^{d,\boldsymbol{a}}(\mathbb{R}^d)$ and let $(S_n)_{n\ge0} \subseteq \mathscr{L}(\mathscr{S}'(\mathbb{R}^d;X))$ be the associated family of convolution operators given by $S_n f := \varphi_n * f$. Let $\mathbb{S}_d \in \{\mathscr{S}'(\mathbb{R}^d;X), \mathscr{S}(\mathbb{R}^d;X)\}$, then for $f \in \mathbb{S}_d$ it holds that $f = \sum_{n=0}^\infty S_n f$ in $\mathbb{S}_d$ and thus

$$f|_{\{0\}\times\mathbb{R}^{d-1}} = \sum_{n=0}^{\infty} (S_n f)|_{\{0\}\times\mathbb{R}^{d-1}} \text{ in } \mathbb{S}_{d-1}.$$

By the Paley–Wiener–Schwartz theorem, $S_n f$ is smooth for every $f \in \mathscr{S}'(\mathbb{R}^d;X)$ and $n \in \mathbb{N}_0$ and therefore has a well-defined classical trace. We define the trace operator $\operatorname{Tr} : \mathscr{S}'(\mathbb{R}^d;X) \to \mathscr{S}'(\mathbb{R}^{d-1};X)$ by

$$\operatorname{Tr} f := \sum_{n=0}^{\infty} (S_n f)|_{\{0\}\times\mathbb{R}^{d-1}},$$

whenever this series converges in $\mathscr{S}'(\mathbb{R}^{d-1};X)$.

The distributional trace. Recall that by the Schwartz kernel theorem we have the canonical identification $\mathscr{D}'(\mathbb{R}\times\mathbb{R}^{d-1};X)=\mathscr{D}'(\mathbb{R};\mathscr{D}'(\mathbb{R}^{d-1};X))$. Hence, we have

$$C(\mathbb{R};\mathscr{D}'(\mathbb{R}^{d-1};X))\hookrightarrow\mathscr{D}'(\mathbb{R}^d;X)=\mathscr{D}'(\mathbb{R}\times\mathbb{R}^{d-1};X)=\mathscr{D}'(\mathbb{R};\mathscr{D}'(\mathbb{R}^{d-1};X)),$$

and we define the bounded evaluation map

$$r:C(\mathbb{R};\mathscr{D}'(\mathbb{R}^{d-1};X))\to\mathscr{D}'(\mathbb{R}^{d-1};X),\qquad f\mapsto \mathrm{ev}_0 f,$$

where $\mathrm{ev}_0 f:=f(0)$ denotes the evaluation at 0. Since

$$C(\mathbb{R}^d;X)=C(\mathbb{R}\times\mathbb{R}^{d-1};X)=C(\mathbb{R};C(\mathbb{R}^{d-1};X))\hookrightarrow C(\mathbb{R};\mathscr{D}'(\mathbb{R}^{d-1};X)),$$

the classical trace and the distributional trace r coincide on $C(\mathbb{R}^d;X)$, i.e.,

$$r:C(\mathbb{R}^d;X)\to C(\mathbb{R}^{d-1};X),\qquad f\mapsto f|_{\{0\}\times\mathbb{R}^{d-1}}.$$

Lemma 4.3.1. *Let $\ell\in\mathbb{N}_1$, $d=(1,d'')\in\mathbb{N}_1\times\mathbb{N}_1^{\ell-1}$, $\boldsymbol{a}=(a_1,\boldsymbol{a}'')\in(0,\infty)\times(0,\infty)^{\ell-1}$. Let $\rho\in\mathscr{S}(\mathbb{R})$ with $\rho(0)=1$ and $\operatorname{supp}\widehat{\rho}\subseteq[1,2]$ and let $(\phi_n)_{n\geq0}\in\Phi^{d'',\boldsymbol{a}''}(\mathbb{R}^{d-1})$. Then*

$$\operatorname{ext} g(x_1,\widetilde{x}):=\sum_{n=0}^{\infty}\rho(2^{na_1}x_1)(\phi_n*g)(\widetilde{x}),\qquad g\in\mathscr{S}'(\mathbb{R}^{d-1};X),$$

is a convergent series in $\mathscr{S}'(\mathbb{R}^d;X)$. Moreover,

$$\operatorname{ext}:\mathscr{S}'(\mathbb{R}^{d-1};X)\to\mathscr{S}'(\mathbb{R}^d;X)$$

is linear and continuous and

$$\operatorname{ext}:\mathscr{S}'(\mathbb{R}^{d-1};X)\to C_{\mathrm{b}}(\mathbb{R};\mathscr{S}'(\mathbb{R}^{d-1};X))$$

acts as a right inverse of $r:C(\mathbb{R};\mathscr{S}'(\mathbb{R}^{d-1};X))\to\mathscr{S}'(\mathbb{R}^{d-1};X)$.

Proof. All statements except the continuity of $\operatorname{ext}:\mathscr{S}'(\mathbb{R}^{d-1};X)\to\mathscr{S}'(\mathbb{R}^d;X)$ are shown in [183, Proposition 5.2.49], which generalises the scalar case in [132, Section 4.2]. To show the continuity, we use the linear operator $K:\mathscr{S}(\mathbb{R}^d)\to\mathscr{S}(\mathbb{R}^{d-1})$ from [132, Equation (4.18)], defined by

$$(K\varphi)(\widetilde{x}):=\sum_{n=0}^{\infty}\int_{\mathbb{R}}\rho(2^{na_1}x_1)\Big(\int_{\mathbb{R}^{d-1}}(\widetilde{\mathscr{F}}^{-1}\phi_n)(\widetilde{y})\varphi(x_1,\widetilde{x}-\widetilde{y})\,\mathrm{d}\widetilde{y}\Big)\,\mathrm{d}x_1.$$

Here, $\widetilde{\mathscr{F}}$ stands for the Fourier transform in $\mathbb{R}^{d-1}$. It is shown in [132, Equation (4.19)] that in fact $K\varphi\in\mathscr{S}(\mathbb{R}^{d-1})$ for each $\varphi\in\mathscr{S}(\mathbb{R}^d)$ and that $(\operatorname{ext} g)(\overline{\varphi})=g(\overline{K\varphi})$ for all $g\in\mathscr{S}'(\mathbb{R}^{d-1};X)$ and $\varphi\in\mathscr{S}(\mathbb{R}^d)$. By the definition of the locally convex topology on $\mathscr{S}'(\mathbb{R}^{d-1};X)$, this shows that $\operatorname{ext}:\mathscr{S}'(\mathbb{R}^{d-1};X)\to\mathscr{S}'(\mathbb{R}^d;X)$ is continuous. □

For $m\in\mathbb{N}_0$ we define the *m-th order trace operator* as $\mathrm{Tr}_m:=\mathrm{Tr}\circ\partial_1^m$. Furthermore, we define the vector of traces

$$\overline{\mathrm{Tr}}_m:=(\mathrm{Tr}_0,\dots,\mathrm{Tr}_m).$$

Moreover, to be consistent with the notation in [132, 183, 184] we write $\boldsymbol{\nu}=(\nu_1,\boldsymbol{\nu}'')\in\mathbb{R}\times\mathbb{R}^{\ell-1}$ for vectors of length $\ell\in\mathbb{N}_1$.

4.3.1. Trace theory for anisotropic spaces

We now characterise higher-order trace spaces of weighted anisotropic Triebel–Lizorkin spaces, extending the results in [132] (unweighted scalar-valued case) and [183, 184] (weighted vector-valued case for the zeroth-order trace). Trace theory for (unweighted) mixed norm spaces was developed in [28, 29, 30, 31, 36]. In particular, trace theory for anisotropic Triebel–Lizorkin spaces is contained in [92, 132] and [123, 183, 184] for the unweighted and weighted case, respectively.

By combining the arguments in [132, Corollary 2.7] and [184, Theorem 4.6], we obtain the characterisation of the higher-order trace space of anisotropic Triebel–Lizorkin spaces. We note that a version of the theorem below for $m = 0$ and more general weights is contained in [123, Proposition 3.7].

Theorem 4.3.2. *Let $\ell \in \mathbb{N}_1$, $d_1 = 1$, $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in [1,\infty]$, $\boldsymbol{a} \in (0,\infty)^\ell$, $\gamma \in (-1,\infty)$, $m \in \mathbb{N}_0$ and let X be a Banach space. Let $w_\gamma(x_1) = |x_1|^\gamma$, $\widetilde{w}_j \in A_{p_j}(\mathbb{R}^{d_j})$ for $j \in \{2,\dots,\ell\}$ and let $\boldsymbol{w} = (w_\gamma, \widetilde{w}_2, \dots, \widetilde{w}_\ell)$. Then, for $s > a_1(m + \frac{\gamma+1}{p_1})$, the trace operator*

$$\overline{\mathrm{Tr}}_m : F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^d, (w_\gamma, \widetilde{\boldsymbol{w}}''); X) \to \prod_{j=0}^{m} F^{s-a_1(j+\frac{\gamma+1}{p_1}),\boldsymbol{a}''}_{\boldsymbol{p}'',p_1,d''}(\mathbb{R}^{d-1}, \widetilde{\boldsymbol{w}}''; X)$$

is continuous and surjective. Moreover, there exists a continuous right inverse $\overline{\mathrm{ext}}_m$ of $\overline{\mathrm{Tr}}_m$.

Proof. The case $m = 0$ is contained in [184, Theorem 4.6] and the formula for the zeroth-order extension operator as given in Lemma 4.3.1 restricts to a coretraction corresponding to Tr on $F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^d, (w_\gamma, \widetilde{\boldsymbol{w}}''); X)$. Now, let $m \in \mathbb{N}_1$ and $j \in \{0,\dots,m\}$. Note that ∂_1^j has order ja_1 in the scale of anisotropic Triebel–Lizorkin spaces, see for instance [183, Proposition 5.2.29]. Hence, the continuity for Tr_j and $\overline{\mathrm{Tr}}_m$ follows from the case $m = 0$.

It remains to consider the extension operator, and we argue similarly as in [132, Corollary 2.7 & Section 4.3]. Let $(\phi_n)_{n\geq 0} \in \Phi^{d'',\boldsymbol{a}''}(\mathbb{R}^{d-1})$ and $\rho \in \mathscr{S}(\mathbb{R})$ with $\operatorname{supp}\widehat{\rho} \subseteq [1,2]$, $\rho(0) = 1$ and $\rho'(0) = \cdots = \rho^{(m)}(0) = 0$. Furthermore, let $\rho_j(x_1) := x_1^j \rho(x_1)/j!$ and define the higher-order extension operator

$$\mathrm{ext}_j\, g(x_1, \widetilde{x}) := \sum_{n=0}^{\infty} 2^{-jna_1} \rho_j(2^{na_1} x_1)(\phi_n * g)(\widetilde{x}), \qquad g \in \mathscr{S}'(\mathbb{R}^{d-1}; X).$$

Due to the particular choice of the ρ_j, Lemma 4.3.1 implies that the series converges in $\mathscr{S}'(\mathbb{R}^{d-1}; X)$ and that $\mathrm{ext}_j\, g$ is in $C_{\mathrm{b}}(\mathbb{R}; \mathscr{S}'(\mathbb{R}^{d-1}; X))$ and is continuous as a mapping from $\mathscr{S}'(\mathbb{R}^{d-1}; X)$ to $\mathscr{S}'(\mathbb{R}^d; X)$. Note that by definition we have for $j_1 \leq m$ that $\mathrm{Tr}_{j_1} \rho_j = \delta_{j_1 j}$ where $\delta_{j_1 j}$ is the Kronecker delta. Since $\partial_1^{j_1}$ is continuous on $\mathscr{S}'(\mathbb{R}^d; X)$ it follows that

$$\mathrm{Tr}_{j_1} \mathrm{ext}_j\, g = \delta_{j_1 j} g, \qquad g \in \mathscr{S}'(\mathbb{R}^{d-1}; X),\ j_1, j \in \{0,\dots,m\}. \tag{4.3.1}$$

Taking into account the additional power of 2 in the definition of ext_j (compared to the zeroth-order extension operator), we can redo part (II) of the proof of [184, Theorem 4.6] to obtain that

$$\mathrm{ext}_j : F^{s-a_1(j+\frac{\gamma+1}{p_1}),\boldsymbol{a}''}_{\boldsymbol{p}'',p_1,d''}(\mathbb{R}^{d-1}, \widetilde{\boldsymbol{w}}''; X) \to F^{s,\boldsymbol{a}}_{\boldsymbol{p},q,d}(\mathbb{R}^d, (w_\gamma, \widetilde{\boldsymbol{w}}''); X)$$

is continuous. Finally, defining

$$\overline{\operatorname{ext}}_m g := \sum_{j=0}^{m} \operatorname{ext}_j g_j, \qquad g = (g_0, \dots, g_m) \in \mathscr{S}'(\mathbb{R}^d; X)^{m+1},$$

gives the desired extension operator to $\overline{\operatorname{Tr}}_m$. Indeed, the continuity follows from the continuity properties of ext_j and $\overline{\operatorname{Tr}}_m \circ \overline{\operatorname{ext}}_m = \mathrm{id}$ follows from (4.3.1). □

The trace spaces of the weighted anisotropic Triebel–Lizorkin spaces are *independent* of the microscopic parameter $q \in [1,\infty]$. As a corollary, we also obtain the trace space for any normed space $\mathbb{F}$ that satisfies

$$F^{s,\boldsymbol{a}}_{\boldsymbol{p},1,d}(\mathbb{R}^d, (w_\gamma, \widetilde{\boldsymbol{w}}''); X) \hookrightarrow \mathbb{F} \hookrightarrow F^{s,\boldsymbol{a}}_{\boldsymbol{p},\infty,d}(\mathbb{R}^d, (w_\gamma, \widetilde{\boldsymbol{w}}''); X).$$

In view of these embeddings (see also [184, Equation (16)]), we obtain the following corollary for the spaces

$$\begin{aligned} \mathbb{F} &:= W^{\boldsymbol{n},\boldsymbol{p}}_{d}(\mathbb{R}^d, (w_\gamma, \widetilde{\boldsymbol{w}}''); X), && \boldsymbol{n} \in \mathbb{N}_1^\ell,\ \boldsymbol{n} = s\boldsymbol{a}^{-1}, \\ \mathbb{F} &:= H^{\boldsymbol{s},\boldsymbol{p}}_{d}(\mathbb{R}^d, (w_\gamma, \widetilde{\boldsymbol{w}}''); X), && \boldsymbol{s} \in (0,\infty)^\ell,\ \boldsymbol{s} = s\boldsymbol{a}^{-1}. \end{aligned} \tag{4.3.2}$$

Corollary 4.3.3. *Let $\ell \in \mathbb{N}_1$, $d_1 = 1$, $\boldsymbol{p} \in (1,\infty)^\ell$, $q \in [1,\infty]$, $\boldsymbol{a} \in (0,\infty)^\ell$, $\gamma \in (-1, p_1 - 1)$, $m \in \mathbb{N}_0$ and let X be a Banach space. Let $w_\gamma(x_1) = |x_1|^\gamma$, $\widetilde{w}_j \in A_{p_j}(\mathbb{R}^{d_j})$ for $j \in \{2,\dots,\ell\}$ and let $\boldsymbol{w} = (w_\gamma, \widetilde{w}_2, \dots, \widetilde{w}_\ell)$. Furthermore, let $\mathbb{F}$ be as in* (4.3.2)*. Then, for $s > a_1(m + \frac{\gamma+1}{p_1})$, the trace operator*

$$\overline{\operatorname{Tr}}_m : \mathbb{F} \to \prod_{j=0}^{m} F^{s - a_1(j+\frac{\gamma+1}{p_1}),\boldsymbol{a}''}_{\boldsymbol{p}'',p_1,d''}(\mathbb{R}^{d-1}, \widetilde{\boldsymbol{w}}''; X)$$

is a continuous and surjective operator. Moreover, there exists a continuous right inverse $\overline{\operatorname{ext}}_m$ of $\overline{\operatorname{Tr}}_m$.

4.4. Traces of intersections of weighted Sobolev spaces

In this section, we characterise the higher-order trace space of an intersection of weighted Sobolev spaces with spatial domain $\mathscr{O} = \mathbb{R}^d_+$ and with both bounded and unbounded time intervals. We consider spatial and temporal power weights of the form $w_\gamma(x) = |x_1|^\gamma$ and $v_\mu(t) = |t|^\mu$, respectively. The main result of this section reads as follows.

Theorem 4.4.1. *Let $p, q \in (1,\infty)$, $k, m \in \mathbb{N}_0$, $k_1, \ell \in \mathbb{N}_1$, $\gamma \in (-1, (k_1 - m)p - 1) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$, $\mu \in (-1, q-1)$, and let X be a* UMD *Banach space. Furthermore, let $I = \mathbb{R}$ or $I = (0,T)$ for some $T \in (0,\infty]$. Then the trace operator $\overline{\operatorname{Tr}}_m$ from*

$$W^{\ell,q}(I, v_\mu; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)) \cap L^q(I, v_\mu; W^{k+k_1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))$$

to

$$\prod_{j=0}^{m} F^{\ell - \frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}_{q,p}(I, v_\mu; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(I, v_\mu; B^{k_1 - j - \frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X))$$

is continuous and surjective. Moreover, there exists a continuous right inverse $\overline{\operatorname{ext}}_m$ of $\overline{\operatorname{Tr}}_m$.

For the notion of UMD Banach spaces, we refer to Section 2.4.

Remark 4.4.2. We make the following remarks concerning Theorem 4.4.1.

(i) The trace space in Theorem 4.4.1 does not depend on the smoothness parameter k. This agrees with known trace results for weighted Sobolev spaces as presented in Chapter 3.

(ii) If $I = \mathbb{R}$, then the result of Theorem 4.4.1 actually holds for any temporal weight $v \in A_q(\mathbb{R})$. The case $I = (0,T)$ follows from using appropriate extension operators, which are easier to construct if $v = v_\mu$ with $\mu \in (-1, q-1)$. We elaborate on the construction of extension operators in Remark 4.4.5.

Before we turn to the proof of Theorem 4.4.1, we need some additional results. First, we recall the following intersection representation for Triebel–Lizorkin spaces. A more general version can be found in [185], while the unweighted scalar-valued case is contained in [69, Proposition 3.23].

Theorem 4.4.3 ([183, Theorem 5.2.35])**.** *Let $p,q \in (1,\infty)$, $\boldsymbol{a} \in (0,\infty)^2$, $s > 0$, $\boldsymbol{w} \in A_p(\mathbb{R}^{d_1}) \times A_q(\mathbb{R}^{d_2})$ and let X be a Banach space. Then*

$$F^{s,\boldsymbol{a}}_{(p,q),p,d}(\mathbb{R}^d, \boldsymbol{w}; X) = F^{\frac{s}{a_2}}_{q,p}\big(\mathbb{R}^{d_2}, w_2; L^p(\mathbb{R}^{d_1}, w_1; X)\big) \cap L^q\big(\mathbb{R}^{d_2}, w_2; F^{\frac{s}{a_1}}_{p,p}(\mathbb{R}^{d_1}, w_1; X)\big).$$

We can now identify the trace space of an intersection of Sobolev and Bessel potential spaces with Muckenhoupt weights. Additionally, recall that $B^s_{p,p}(\mathbb{R}^{d-1}; X) = F^s_{p,p}(\mathbb{R}^{d-1}; X)$ for all $s \in \mathbb{R}$ and $p \in (1,\infty)$, see, for instance [126, Proposition 14.6.8].

Theorem 4.4.4. *Let $p,q \in (1,\infty)$, $m \in \mathbb{N}_0$, $n_1, n_2 \in \mathbb{N}_1$, $s_1, s_2 \in (0,\infty)$, $v \in A_q(\mathbb{R})$, $\gamma \in (-1, p-1)$ and let X be a Banach space. If $n_2, s_2 > m + \frac{\gamma+1}{p}$, then*

$$\begin{aligned}\overline{\mathrm{Tr}}_m : W^{n_1,q}(\mathbb{R}, v; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}, v; W^{n_2,p}(\mathbb{R}^d_+, w_\gamma; X)) \\ \to \prod_{j=0}^m F^{n_1 - \frac{n_1}{n_2}(j+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B^{n_2 - j - \frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X))\end{aligned}$$

and

$$\begin{aligned}\overline{\mathrm{Tr}}_m : H^{s_1,q}(\mathbb{R}, v; L^p(\mathbb{R}^d_+, w_\gamma; X)) \cap L^q(\mathbb{R}, v; H^{s_2,p}(\mathbb{R}^d_+, w_\gamma; X)) \\ \to \prod_{j=0}^m F^{s_1 - \frac{s_1}{s_2}(j+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B^{s_2 - j - \frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1}; X))\end{aligned}$$

are continuous and surjective. Moreover, there exists a continuous right inverse $\overline{\mathrm{ext}}_m$ of $\overline{\mathrm{Tr}}_m$.

Proof. We only consider the case of Sobolev spaces, since the proof for the Bessel potential spaces is completely similar. By a standard restriction argument, it suffices to consider the spatial domain $\mathbb{R}^d$ instead of $\mathbb{R}^d_+$. This is possible as we are in the range of Muckenhoupt

weights, and therefore extension operators are available (see [189, Section 5]). Note that we have the identification

$$\begin{aligned} &W^{(n_2,n_1),(p,q)}_{(d,1)}(\mathbb{R}^{d+1},(w_\gamma,v);X) \\ &\qquad = W^{n_1,q}(\mathbb{R},v;L^p(\mathbb{R}^d,w_\gamma;X))\cap L^q(\mathbb{R},v;W^{n_2,p}(\mathbb{R}^d,w_\gamma;X)), \end{aligned}$$

so that Corollary 4.3.3 and Theorem 4.4.3 imply that

$$\begin{aligned} \overline{\mathrm{Tr}}_m &: W^{n_1,q}(\mathbb{R},v;L^p(\mathbb{R}^d,w_\gamma;X))\cap L^q(\mathbb{R},v;W^{n_2,p}(\mathbb{R}^d,w_\gamma;X)) \\ &\to \prod_{j=0}^m F^{1-\frac{1}{n_2}(j+\frac{\gamma+1}{p}),(\frac{1}{n_2},\frac{1}{n_1})}_{(p,q),p,(d-1,1)}(\mathbb{R}^d,(1,v);X) \\ &= \prod_{j=0}^m F^{n_1-\frac{n_1}{n_2}(j+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X))\cap L^q(\mathbb{R},v;B^{n_2-j-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)) \end{aligned}$$

is continuous and surjective. Moreover, Corollary 4.3.3 gives the existence of an extension operator as well. □

We can now characterise the spatial trace space of an intersection of weighted Sobolev spaces as stated in Theorem 4.4.1.

Proof of Theorem 4.4.1. Step 1: the case $I=\mathbb{R}$. Let $v\in A_q(\mathbb{R})$. Let $r\in\{0,\dots,k_1-m-1\}$ be such that $\gamma\in(rp-1,(r+1)p-1)$. By [189, Proposition 5.5], standard properties of the complex interpolation method (see, e.g., [262, Equation (4) in Section 1.9.3]), [190, Theorem 3.18], Proposition 3.6.6 (using that X is UMD) and Hardy's inequality (see Lemma 2.2.2), we obtain

$$\begin{aligned} &W^{\ell,q}\big(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\big)\cap L^q\big(\mathbb{R},v;W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\big) \\ &\quad\hookrightarrow \big[H^{\ell,q}\big(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\big),L^q\big(\mathbb{R},v;W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\big)\big]_{\frac{r}{k_1}} \\ &\quad= H^{\ell(1-\frac{r}{k_1}),q}\big(\mathbb{R},v;[W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{\frac{r}{k_1}}\big) \\ &\quad= H^{\ell(1-\frac{r}{k_1}),q}\big(\mathbb{R},v;W^{k+r,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\big) \\ &\quad\hookrightarrow H^{\ell(1-\frac{r}{k_1}),q}\big(\mathbb{R},v;L^p(\mathbb{R}^d,w_{\gamma-rp};X)\big). \end{aligned}$$

By applying Hardy's inequality once more, we have

$$\begin{aligned} &W^{\ell,q}(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))\cap L^q(\mathbb{R},v;W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)) \\ &\quad\hookrightarrow H^{\ell(1-\frac{r}{k_1}),q}\big(\mathbb{R},v;L^p(\mathbb{R}^d,w_{\gamma-rp};X)\big)\cap L^q\big(\mathbb{R},v;W^{k_1-r,p}(\mathbb{R}^d_+,w_{\gamma-rp};X)\big). \end{aligned}$$

Since $\gamma-rp\in(-1,p-1)$ and $k_1-r>m+\frac{\gamma-rp+1}{p}$, Theorem 4.4.4 implies that $\overline{\mathrm{Tr}}_m$ is continuous and surjective from the latter space to

$$\prod_{j=0}^m F^{\ell(1-\frac{r}{k_1})-\frac{\ell(1-\frac{r}{k_1})}{k_1-r}(j+\frac{\gamma-rp+1}{p})}_{q,p}\big(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X)\big)\cap L^q\big(\mathbb{R},v;B^{k_1-r-j-\frac{\gamma-rp+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)\big),$$

which gives the desired trace space after rewriting. This shows that $\overline{\mathrm{Tr}}_m$ is continuous between the desired spaces.

To prove the existence and continuity of the extension operator, it suffices to establish the embedding

$$\begin{aligned} &F^{1,(a_1,a_2)}_{(p,q),1,(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}, (w_{\gamma+kp}, v); X) \\ &\hookrightarrow W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)), \end{aligned} \tag{4.4.1}$$

with

$$a_1 := \frac{1}{k+k_1} \quad \text{and} \quad a_2 := \frac{k_1}{\ell(k+k_1)}. \tag{4.4.2}$$

Indeed, by Theorems 4.3.2 and 4.4.3 we have that the extension operator $\overline{\mathrm{ext}}_m$ is continuous from

$$\prod_{j=0}^{m} F^{\frac{1}{a_2}(1-a_1(j+\frac{\gamma+kp+1}{p}))}_{q,p}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B^{\frac{1}{a_1}(1-a_1(j+\frac{\gamma+kp+1}{p}))}_{p,p}(\mathbb{R}^{d-1}; X))$$

to $F^{1,(a_1,a_2)}_{(p,q),1,(d,1)}(\mathbb{R}^d_+ \times \mathbb{R}, (w_{\gamma+kp}, v); X)$. Rewriting using (4.4.2) gives continuity of $\overline{\mathrm{ext}}_m$ between the desired spaces.

It remains to prove the embedding (4.4.1). Similar as in the proof of [190, Lemma 7.2] it holds that the restriction operator

$$R : \mathscr{D}'(\mathbb{R}^d \times \mathbb{R}, X) \to \mathscr{D}'(\mathbb{R}^d_+ \times \mathbb{R}; X), \qquad f \mapsto f|_{\mathbb{R}^d_+ \times \mathbb{R}},$$

restricts to a bounded linear operator on

$$R : F^{0,(a_1,a_2)}_{(p,q),1,(d,1)}(\mathbb{R}^d \times \mathbb{R}, (w_{\gamma+kp}, v); X) \to L^q(\mathbb{R}, v; L^p(\mathbb{R}^d_+, w_{\gamma+kp}; X)).$$

By [183, Proposition 5.2.29], using that $a_1(k+k_1) = 1$ and $a_1 k + a_2 \ell = 1$, it follows that

$$\begin{aligned} &R : F^{1,(a_1,a_2)}_{(p,q),1,(d,1)}(\mathbb{R}^d \times \mathbb{R}, (w_{\gamma+kp}, v); X) \\ &\to W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)), \end{aligned}$$

is bounded as well. This proves (4.4.1) and completes the proof if $I = \mathbb{R}$.

Step 2: the case $I = (0, T)$. This case follows from $I = \mathbb{R}$ and a suitable extension operator. For the convenience of the reader, we include the details of the proof. First, let $I = (0, \infty)$. By $\overline{\mathrm{Tr}}_m$ and $\overline{\mathrm{ext}}_m$ we denote the (spatial) trace and extension operators from Step 1. For $j \in \{0, \dots, m\}$ we denote by $\mathrm{Tr}^{\mathbb{R}_+}_j$ the (spatial) trace operator for functions defined on the time interval $I = \mathbb{R}_+$, i.e., $\mathrm{Tr}^{\mathbb{R}_+}_j f = f|_{\{0\}\times\mathbb{R}^{d-1}}$ for $f \in C^\infty_c(\overline{\mathbb{R}_+}; C^\infty_c(\overline{\mathbb{R}^d_+}; X))$. Furthermore, if R is the restriction operator from $\mathbb{R}$ to $\mathbb{R}_+$, then it is clear that

$$R\,\mathrm{Tr}_j f = \mathrm{Tr}^{\mathbb{R}_+}_j Rf, \qquad j \in \{0, \dots, m\},\ f \in C^\infty_c(\mathbb{R}; C^\infty_c(\overline{\mathbb{R}^d_+}; X)). \tag{4.4.3}$$

Let E be a universal extension operator from $\mathbb{R}_+$ to $\mathbb{R}$ such that

$$\begin{aligned} &E : W^{\ell,q}(\mathbb{R}_+, v_\mu; Y) \to W^{\ell,q}(\mathbb{R}, v_\mu; Y) \quad \text{and} \\ &E : F^{\ell-\frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R}_+, v_\mu; Y) \to F^{\ell-\frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R}, v_\mu; Y) \end{aligned} \tag{4.4.4}$$

are bounded for all parameters $p, q, k_1, \ell, j, \gamma$ as specified in the theorem and Banach spaces Y. Such an extension operator exists by redoing the proof of [263, Theorem 2.9.2] using the multiplier result [214, Theorem 1.3]. Note that one has to redo the proof of [214, Theorem 1.3] for the Triebel–Lizorkin and Sobolev spaces separately, since in the vector-valued case with UMD spaces the Sobolev scale is not contained in the Triebel–Lizorkin scale, see, e.g., [126, Theorem 14.7.9].

Moreover, it holds that $REf = f$ for $f \in C_c^\infty(\overline{\mathbb{R}_+}; C_c^\infty(\overline{\mathbb{R}_+^d}; X))$. Hence, with (4.4.3) it follows that

$$\operatorname{Tr}_j^{\mathbb{R}_+} f = R \operatorname{Tr}_j Ef, \qquad j \in \{0, \dots, m\},\ f \in C_c^\infty(\overline{\mathbb{R}_+}; C_c^\infty(\overline{\mathbb{R}_+^d}; X)).$$

By the properties of R and E, Step 1, and density of $C_c^\infty(\overline{\mathbb{R}_+}; C_c^\infty(\overline{\mathbb{R}_+^d}; X))$ in

$$W^{\ell,q}(\mathbb{R}_+, \nu_\mu; W^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}_+, \nu_\mu; W^{k+k_1,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X)),$$

we find that $\overline{\operatorname{Tr}}_m^{\mathbb{R}_+}$ extends to an operator as stated in the theorem and we have

$$\begin{aligned}
&\sum_{j=0}^m \| \operatorname{Tr}_j^{\mathbb{R}_+} f \|_{F_{q,p}^{\ell - \frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R}_+, \nu_\mu; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}_+, \nu_\mu; B_{p,p}^{k_1 - j - \frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X))} \\
&\quad \lesssim \sum_{j=0}^m \| \operatorname{Tr}_j Ef \|_{F_{q,p}^{\ell - \frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R}, \nu_\mu; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, \nu_\mu; B_{p,p}^{k_1 - j - \frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X))} \\
&\quad \lesssim \| Ef \|_{W^{\ell,q}(\mathbb{R}, \nu_\mu; W^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}, \nu_\mu; W^{k+k_1,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X))} \\
&\quad \lesssim \| f \|_{W^{\ell,q}(\mathbb{R}_+, \nu_\mu; W^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}_+, \nu_\mu; W^{k+k_1,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X))}.
\end{aligned}$$

Define the extension operator by

$$\overline{\operatorname{ext}}_m^{\mathbb{R}_+} := R \circ \overline{\operatorname{ext}}_m \circ E.$$

Then again by properties of R and E, and the continuity of the extension operator $\overline{\operatorname{ext}}_m$, we find the continuity of $\overline{\operatorname{ext}}_m^{\mathbb{R}_+}$ as stated in the theorem. It remains to prove that this extension is the right inverse to the above constructed trace operator $\overline{\operatorname{Tr}}_m^{\mathbb{R}_+}$. Indeed, for $i, j \in \{0, \dots, m\}$ we have using (4.4.3) twice and $RE = \mathrm{id}$ that

$$\begin{aligned}
\operatorname{Tr}_j^{\mathbb{R}_+} \operatorname{ext}_i^{\mathbb{R}_+} h &= R \operatorname{Tr}_j ER \operatorname{ext}_i Eh = R \operatorname{Tr}_j \operatorname{ext}_i Eh \\
&= \delta_{ij} REh = \delta_{ij} h, \quad h \in C_c^\infty(\overline{\mathbb{R}_+}; C_c^\infty(\mathbb{R}^{d-1}; X)),
\end{aligned}$$

and by density this gives the desired result.

Finally, for $I = (0, T)$ with $T \in (0, \infty)$ the result follows similarly by using a universal extension operator from $(0, T)$ to $\mathbb{R}$. To obtain such an extension operator one can proceed as in [11, Proposition 2.5]. □

Remark 4.4.5. There are alternative methods to obtain an extension operator satisfying (4.4.4).

(i) The Rychkov extension operator from [243] can also be used in the weighted and vector-valued setting.

(ii) One can also use the more classical extension operator for Sobolev spaces from [4, Theorem 5.19] (see [189, Lemma 5.1] for the weighted setting). The boundedness of this extension operator on Triebel–Lizorkin spaces is then obtained by the ℓ^q-interpolation method, see [186]. With this extension operator we expect that one can treat more general (Muckenhoupt) weights, instead of only power weights v_μ with $\mu \in (-1, q-1)$.

4.5. Localisation of traces to rough domains

We continue with localising the trace result from Theorem 4.4.1 to bounded domains. Localisation procedures for smooth domains are quite standard in the literature, see, e.g., [262, Section 3.6.1]. Since we deal with domains with low regularity, we provide the details of the localisation argument, which is based on the Dahlberg–Kenig–Stein pullback in Section 2.6. We first collect the required properties of this diffeomorphism in Section 4.5.1. In Section 4.5.2, we apply the localisation to prove our main trace theorem on domains.

4.5.1. Localisation and traces on domains

We introduce a localisation procedure for rough domains. Let $\mathscr{O}$ be a special $C_c^{r,\kappa}$-domain with $r \in \mathbb{N}_1$ and $\kappa \in [0,1]$. Then by Lemma 2.6.5, there exists a $C^{r,\kappa}$-diffeomorphism $\Phi : \mathscr{O} \to \mathbb{R}^d_+$ which straightens the boundary and preserves the distance to the boundary. Moreover, this diffeomorphism is smooth in the interior, but higher-order derivatives may blow up near the boundary. This diffeomorphism is known as the *Dahlberg–Kenig–Stein pullback*. In particular, Φ satisfies the following mapping properties. The proof of this proposition is postponed to Chapter 6, see Proposition 6.3.4.

Proposition 4.5.1. *Let $p \in (1,\infty)$, $r \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_1$ such that $k \geq r+\kappa$ and $\gamma \in ((k-(r+\kappa))p-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$. Furthermore, let $\mathscr{O}$ be a special $C_c^{r,\kappa}$-domain and let X be a Banach space. Let $\Phi : \mathscr{O} \to \mathbb{R}^d_+$ be the Dahlberg–Kenig–Stein pullback from Lemma 2.6.5 and consider the change of coordinate mappings*

$$\Phi_* : W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to W^{k,p}(\mathbb{R}^d_+, w_\gamma; X),$$

defined by $\Phi_* f := f \circ \Phi^{-1}$. Then Φ_* is an isomorphism of Banach spaces for which $(\Phi^{-1})_*$ acts as inverse.

Next, let $\mathscr{O}$ be a bounded $C^{r,\kappa}$-domain with $r \in \mathbb{N}_1$ and $\kappa \in [0,1]$. The following statements hold.

(i) There exists a finite open cover $(V_n)_{n=1}^N$ of $\partial\mathscr{O}$, together with special $C_c^{r,\kappa}$-domains $(\mathscr{O}_n)_{n=1}^N$, such that

$$\mathscr{O} \cap V_n = \mathscr{O}_n \cap V_n \qquad \text{and} \qquad \partial\mathscr{O} \cap V_n = \partial\mathscr{O}_n \cap V_n, \quad n \in \{1,\dots,N\}.$$

Without loss of generality, we may assume that each V_n is a ball.

(ii) There exist $\eta_0 \in C_c^\infty(\mathscr{O})$ and $\eta_n \in C_c^\infty(V_n)$ for all $n \in \{1,\dots,N\}$ such that $\sum_{n=0}^N \eta_n^2 = 1$ on $\mathscr{O}$ and $\sum_{n=1}^N \eta_n^2 = 1$ on $\partial\mathscr{O}$ (partition of unity).

For $n \in \{1,\dots,N\}$, let $\Phi_n : \mathscr{O}_n \to \mathbb{R}^d_+$ be the Dahlberg–Kenig–Stein pullback as introduced above. For any f defined on $\overline{\mathscr{O}_n}$, the change of coordinate mapping is $(\Phi_n)_* f := f \circ \Phi_n^{-1}$. Furthermore, for any g defined on $\partial\mathscr{O}_n$, we write $(\Phi_n(0,\cdot))_* g = g(\Phi_n^{-1}(0,\cdot))$ which defines a function on $\mathbb{R}^{d-1}$. Throughout this section, we fix $(V_n)_{n=1}^N$, $(\mathscr{O}_n)_{n=1}^N$, $(\eta_n)_{n=0}^N$ and $(\Phi_n)_{n=1}^N$ as introduced above.

Similar to [262, Definition 3.6.1], we define Besov spaces on the boundary. Let $p \in (1,\infty)$, $s > 0$, X a Banach space. Let $L^p(\partial\mathscr{O}; X)$ be the Lebesgue space with respect to the surface measure. Note that the norms $\|g\|_{L^p(\partial\mathscr{O};X)}$ and

$$\sum_{n=1}^N \|(\Phi_n(0,\cdot))_* \eta_n^2 g\|_{L^p(\mathbb{R}^{d-1};X)}, \qquad g \in L^p(\partial\mathscr{O}; X),$$

are equivalent. We define

$$B^s_{p,q}(\partial\mathscr{O}; X) := \big\{ g \in L^p(\partial\mathscr{O}; X) : (\Phi_n(0,\cdot))_* \eta_n^2 g \in B^s_{p,q}(\mathbb{R}^{d-1}; X)\ \forall n \in \{1,\dots,N\} \big\}, \tag{4.5.1}$$

equipped with the norm

$$\|g\|_{B^s_{p,q}(\partial\mathscr{O};X)} := \sum_{n=1}^N \|(\Phi_n(0,\cdot))_* \eta_n^2 g\|_{B^s_{p,q}(\mathbb{R}^{d-1};X)}, \qquad g \in B^s_{p,q}(\partial\mathscr{O}; X).$$

Note that this definition is independent of the partition of unity and the diffeomorphisms up to an equivalent norm. Moreover, we need the following spaces of smooth functions on the boundary. Let $L^1_{\mathrm{loc}}(\partial\mathscr{O}; X)$ be the Lebesgue space of locally integrable functions with respect to the surface measure. For $r \in \mathbb{N}_1$, $\kappa \in [0,1)$ and $\mathscr{O}$ a bounded $C^{r,\kappa}$-domain, we define

$$C^{r,\kappa}(\partial\mathscr{O}; X) := \big\{ g \in L^1_{\mathrm{loc}}(\partial\mathscr{O}; X) : (\Phi_n(0,\cdot))_* \eta_n^2 g \in C^{r,\kappa}(\mathbb{R}^{d-1}; X)\ \forall n \in \{1,\dots,N\} \big\}.$$

4.5.2. Traces of intersections of weighted Sobolev spaces on domains

We first define the spatial trace operator for smooth functions on domains. Let $r \in \mathbb{N}_1$ and let $\mathscr{O}$ be a special C^r_{c}-domain or a bounded C^r-domain. Furthermore, the unit inner normal vector on $\partial\mathscr{O}$ is denoted by $\boldsymbol{n}$. For $u \in C^r(\overline{\mathscr{O}}; X)$ we define the trace

$$\begin{aligned} \mathrm{Tr}_j^{\partial\mathscr{O}}\, u &= \sum_{|\alpha|=j} (\partial^\alpha u)|_{\partial\mathscr{O}}\, \boldsymbol{n}^\alpha, \qquad j \in \{0,\dots,r\}, \\ \overline{\mathrm{Tr}}_r^{\partial\mathscr{O}}\, u &= (\mathrm{Tr}_0^{\partial\mathscr{O}}\, u, \dots, \mathrm{Tr}_r^{\partial\mathscr{O}}\, u). \end{aligned} \tag{4.5.2}$$

The main trace result for intersections of weighted Sobolev spaces on domains reads as follows. The rest of this section is devoted to its proof.

Theorem 4.5.2. *Let $p, q \in (1,\infty)$, $k, m \in \mathbb{N}_0$, $k_1, \ell, r \in \mathbb{N}_1$ and $\kappa \in [0,1)$ such that $k_1 \geq r + \kappa > m$ and $\gamma \in ((k_1 - (r+\kappa))p - 1, (k_1 - m)p - 1) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$, $\mu \in (-1, q-1)$ and let X be a* UMD *Banach space. Furthermore, let $I = \mathbb{R}$ or $I = (0,T)$ for some $T \in (0,\infty]$ and let $\mathscr{O}$ be a bounded $C^{r,\kappa}$-domain. Then the trace operator $\overline{\mathrm{Tr}}_m^{\partial\mathscr{O}}$ from*

$$W^{\ell,q}(I, v_\mu; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)) \cap L^q(I, v_\mu; W^{k+k_1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))$$

to

$$\prod_{j=0}^{m} F_{q,p}^{\ell-\frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(I,v_\mu;L^p(\partial\mathscr{O};X)) \cap L^q(I,v_\mu;B_{p,p}^{k_1-j-\frac{\gamma+1}{p}}(\partial\mathscr{O};X)) \tag{4.5.3}$$

is continuous and surjective. Moreover, there exists a continuous right inverse $\overline{\mathrm{ext}}_m^{\partial\mathscr{O}}$ *of* $\overline{\mathrm{Tr}}_m^{\partial\mathscr{O}}$.

First of all, it should be noted that $\mathrm{Tr}_1^{\partial\mathscr{O}}$ coincides with the normal derivative $\partial_{\boldsymbol{n}} = \frac{\partial}{\partial \boldsymbol{n}}$, while for $j \geq 2$ the trace $\mathrm{Tr}_j^{\partial\mathscr{O}}$ is in general not equal to $\partial_{\boldsymbol{n}}^j$, see [140, Remark 2.9]. Nevertheless, one can construct a vector field $\widetilde{\boldsymbol{n}}$ which does satisfy this property. Indeed, if $\mathscr{O}$ is a special $C_{\mathrm{c}}^{r,\kappa}$-domain with $r \geq 2$ and $\kappa \in [0,1]$, then by [140, Lemma 2.17] there exists an extension $\widetilde{\boldsymbol{n}} \in C^{r-1,\kappa}(\mathscr{O})$ of $\boldsymbol{n}$ such that $\widetilde{\boldsymbol{n}} = \boldsymbol{n}$ on $\partial\mathscr{O}$, which satisfies for $u \in C^{r,\kappa}(\mathscr{O};X)$

$$\frac{\partial^j u}{\partial \widetilde{\boldsymbol{n}}^j} = \sum_{|\alpha|=j} (\partial^\alpha u)\,\widetilde{\boldsymbol{n}}^\alpha \qquad \text{on } \partial\mathscr{O},\ j \in \{1,\dots,r\}.$$

To prove Theorem 4.5.2, we use a covering of $\partial\mathscr{O}$ and a partition of unity to reduce to special domains $(\mathscr{O}_n)_{n=1}^N$. Then, on each special domain, the Dahlberg–Kenig–Stein pullback Φ_n can be applied to reduce again to the half-space, where we have a trace theorem available. However, the direction of the normal vector $\boldsymbol{n}$ is not preserved under the Dahlberg–Kenig–Stein pullback. Therefore, we need a trace result on $\mathbb{R}_+^d$ with a different non-tangential vector field $\mathscr{N}$ on the boundary. We first construct this vector field $\mathscr{N}$ that corresponds to $\boldsymbol{n}$ under the Dahlberg–Kenig–Stein pullback.

Let $\mathscr{O}$ be a special $C_{\mathrm{c}}^{r,\kappa}$-domain with $r \in \mathbb{N}_1$ and $\kappa \in [0,1]$. Furthermore, let $m < r+\kappa$ and let $\boldsymbol{n}$ be the unit inner normal vector on $\partial\mathscr{O}$. If $r = 1$, set $\widetilde{\boldsymbol{n}}(x) = \boldsymbol{n}(h(\widetilde{x}), x_2, \dots, x_d)$ for $x \in \overline{\mathscr{O}}$. If $r \geq 2$, then take $\widetilde{\boldsymbol{n}}$ from [140, Lemma 2.17] as explained above. Using the Dahlberg–Kenig–Stein pullback $\Phi : \mathscr{O} \to \mathbb{R}_+^d$ introduced in Section 4.5.1, we define the vector field $\mathscr{N} \in C^{r-1,\kappa}(\mathbb{R}_+^d)$ on $\overline{\mathbb{R}_+^d}$ as

$$\mathscr{N}(y) = (D\Phi)(\Phi^{-1}(y))\widetilde{\boldsymbol{n}}(\Phi^{-1}(y)), \qquad y \in \overline{\mathbb{R}_+^d}, \tag{4.5.4}$$

where $D\Phi$ denotes the Jacobian matrix of Φ. Furthermore, $\mathscr{N}(0,\cdot)$ is a non-tangential vector field on $\partial\mathbb{R}_+^d = \mathbb{R}^{d-1}$, which satisfies (see [140, Equation (19)])

$$\sum_{|\alpha|=j} (\partial^\alpha u)(x)\boldsymbol{n}^\alpha(x) = \sum_{|\alpha|=j} (\partial^\alpha u)(x)\widetilde{\boldsymbol{n}}^\alpha(x) = \frac{\partial^j v}{\partial \mathscr{N}^j}(\Phi(x)) \quad \text{on } \partial\mathscr{O}, \tag{4.5.5}$$

for $j \in \{0,\dots,m\}$, $u \in C^{r,\kappa}(\overline{\mathscr{O}};X)$, and $v(y) = u(\Phi^{-1}(y))$.

We will prove a variant of Theorem 4.4.1 on $\mathbb{R}_+^d$ for the vector field $\mathscr{N}$ on the boundary as defined in (4.5.4).

Proposition 4.5.3. *Let $p,q \in (1,\infty)$, $k,m \in \mathbb{N}_0$, $k_1,\ell,r \in \mathbb{N}_1$ and $\kappa \in [0,1)$ such that $k_1 \geq r+\kappa > m$ and $\gamma \in ((k_1-(r+\kappa))p-1, (k_1-m)p-1) \setminus \{jp-1 : j \in \mathbb{N}_1\}$. Furthermore, let $v \in A_q(\mathbb{R})$ and let X be a* UMD *Banach space. Let $\mathscr{N} \in C^{r-1,\kappa}(\mathbb{R}_+^d)$ be the vector field as in* (4.5.4)*. Then the trace operator*

$$\overline{\mathrm{Tr}}_m^{\mathscr{N}} := (\mathrm{id}, \partial_{\mathscr{N}}, \dots, \partial_{\mathscr{N}}^m)|_{\partial\mathbb{R}_+^d}$$

from

$$W^{\ell,q}(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))\cap L^q(\mathbb{R},v;W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))$$

to

$$\prod_{j=0}^{m} F_{q,p}^{\ell-\frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X))\cap L^q(\mathbb{R},v;B_{p,p}^{k_1-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X)) \tag{4.5.6}$$

is continuous and surjective. Moreover, there exists a continuous right inverse $\overline{\mathrm{ext}}_m^{\mathscr{N}}$ *of* $\overline{\mathrm{Tr}}_m^{\mathscr{N}}$.

Remark 4.5.4.

(i) In addition to the conditions in Theorem 4.4.1, there is a more restrictive lower bound on γ to ensure that the diffeomorphism Φ has the required mapping properties.

(ii) More generally, the proposition actually holds for any non-tangential vector field $\mathscr{N}$ on $\partial\mathbb{R}^d_+=\mathbb{R}^{d-1}$. We will only need the result for the vector field $\mathscr{N}$ obtained after applying the pullback Φ to the normal vector $\tilde{\boldsymbol{n}}$ at $\partial\mathscr{O}$, see (4.5.4).

(iii) Due to the low regularity of the domain $\mathscr{O}$ and the vector field $\mathscr{N}$, we cannot apply the argument as given in [262, Remark 3.6.1/3] to obtain the result of Proposition 4.5.3.

Proof. Step 1: a preliminary estimate. Let $\widetilde{\alpha}\in\mathbb{N}_0^{d-1}$ and $j\in\{0,\dots,m\}$. Then by Theorem 4.4.3 twice and the fact that the (spatial) $\partial^{\widetilde{\alpha}}$-derivatives are of order $\frac{|\widetilde{\alpha}|}{k_1}$ in the scale of anisotropic Triebel–Lizorkin spaces (e.g., [183, Proposition 5.2.29]), we obtain the estimate

$$\begin{aligned}
\|\partial^{\widetilde{\alpha}}g\|_{F_{q,p}^{\ell-\frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X))\cap L^q(\mathbb{R},v;B_{p,p}^{k_1-j-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X))}&\\
\lesssim \|\partial^{\widetilde{\alpha}}g\|_{F_{(p,q),p,(d-1,1)}^{1-\frac{1}{k_1}(j+\frac{\gamma+1}{p}),(\frac{1}{k_1},\frac{1}{\ell})}(\mathbb{R}^d,(1,v);X)}&\\
\lesssim \|g\|_{F_{(p,q),p,(d-1,1)}^{1-\frac{1}{k_1}(j-|\widetilde{\alpha}|+\frac{\gamma+1}{p}),(\frac{1}{k_1},\frac{1}{\ell})}(\mathbb{R}^d,(1,v);X)}&\\
\lesssim \|g\|_{F_{q,p}^{\ell-\frac{\ell}{k_1}(j-|\widetilde{\alpha}|+\frac{\gamma+1}{p})}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X))\cap L^q(\mathbb{R},v;B_{p,p}^{k_1-j+|\widetilde{\alpha}|-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X))}&,
\end{aligned} \tag{4.5.7}$$

for all g such that the right-hand side is finite.

Step 2: the trace operator. We prove the continuity of the trace. By density it suffices to take $u\in C_c^\infty(\mathbb{R};C_c^\infty(\overline{\mathbb{R}^d_+};X))$. Let $j\in\{0,\dots,m\}$ and let $\mathscr{N}\in C^{r-1,\kappa}(\mathbb{R}^d_+)$ be the non-tangential vector field from (4.5.4), then by the Leibniz rule we obtain that $\partial_{\mathscr{N}}^j u=(\mathscr{N}\cdot\nabla)^j u$ is a linear combination of terms of the form

$$(\partial^\alpha u)\prod_{i=1}^{j}(\partial^{\beta_i}\mathscr{N}_{r_i}) \quad \text{with } 0\le\sum_{i=1}^{j}|\beta_i|\le j-1 \text{ and } |\alpha|+\sum_{i=1}^{j}|\beta_i|=j,$$

where $\alpha=(\alpha_1,\widetilde{\alpha})\in\mathbb{N}_0\times\mathbb{N}_0^{d-1}$ satisfies $1\le|\alpha|\le j$ and $r_i\in\{1,\dots,d\}$ for all $i\in\{1,\dots,j\}$. Moreover, it holds that $(\partial^{\beta_i}\mathscr{N}_{r_i})(0,\cdot)\in C^{r-j,\kappa}(\mathbb{R}^{d-1})$ for all $i\in\{1,\dots,j\}$. In particular, it

follows that $(\partial^{\beta_i}\mathscr{N}_{r_i})(0,\cdot)$ are pointwise multipliers on $L^p(\mathbb{R}^{d-1};X)$ and $B^{k_1-j-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X)$ by [214, Proposition 5.4] (using that $\kappa\in[0,1)$) since $r-j+\kappa>k_1-j-\frac{\gamma+1}{p}$. This, together with (4.5.7) and Theorem 4.4.1, gives

$$\begin{aligned}
\|\partial^j_{\mathscr{N}}u|_{\partial\mathbb{R}^d_+}\|&_{F^{\ell-\frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X))\cap L^q(\mathbb{R},v;B^{k_1-j-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X))}\\
&\lesssim \sum_{|\alpha|\le j}\|\partial^{\widetilde{\alpha}}(\partial_1^{\alpha_1}u)(0,\cdot)\|_{F^{\ell-\frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X))\cap L^q(\mathbb{R},v;B^{k_1-j-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X))}\\
&\lesssim \sum_{|\alpha|\le j}\|(\partial_1^{\alpha_1}u)(0,\cdot)\|_{F^{\ell-\frac{\ell}{k_1}(\alpha_1+\frac{\gamma+1}{p})}_{q,p}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};X))\cap L^q(\mathbb{R},v;B^{k_1-\alpha_1-\frac{\gamma+1}{p}}_{p,p}(\mathbb{R}^{d-1};X))}\\
&\lesssim \|u\|_{W^{\ell,q}(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))\cap L^q(\mathbb{R},v;W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))},
\end{aligned}$$

where we have used that $|\widetilde{\alpha}|-j=|\alpha|-\alpha_1-j\le-\alpha_1$.

Step 3: the extension operator. We first prove that there exists an isomorphism I which is bounded on the trace space (4.5.6) and relates $\partial_{\mathscr{N}}$-derivatives to ∂_1-derivatives. Let $j\in\{0,\dots,m\}$, $u\in C^\infty_{\rm c}(\mathbb{R};C^\infty_{\rm c}(\overline{\mathbb{R}^d_+};X))$ and define

$$h_j:=(\partial_1^j u)(0,\cdot)\quad\text{ and }\quad g_j:=(\partial^j_{\mathscr{N}}u)(0,\cdot).$$

Again, by the Leibniz rule g_j can be written as a sum of $N_1^j h_j$ and terms of the form $(\partial^{\widetilde{\alpha}}h_{\alpha_1})\prod_{i=1}^j(\partial^{\beta_i}\mathscr{N}_{r_i})(0,\cdot)$ where $|\alpha|\le j$ with $\alpha_1\le j-1$ and $|\alpha|+\sum_{i=1}^j|\beta_i|=j$. Hence, there exists a lower triangular $(m+1)\times(m+1)$ matrix I with diagonal entries $I_{j,j}=\mathscr{N}_1^j$ such that $(g_0,\dots,g_m)^\top=I(h_0,\dots,h_m)^\top$. From the estimate in Step 2, it follows that I is bounded on the trace space (4.5.6). The invertibility of I follows from the fact that $\mathscr{N}$ is non-tangential, i.e., $\mathscr{N}_1\ge c>0$ on $\mathbb{R}^{d-1}$. The boundedness of I^{-1}, which is again a lower triangular matrix, can be proved similarly to the boundedness of I after recursively expressing the h_j in terms of the g_j.

Let $\overline{\rm ext}_m$ be the extension operator from Theorem 4.4.1 which satisfies $\overline{\rm Tr}_m\circ\overline{\rm ext}_m=\mathrm{id}$. We claim that $\overline{\rm ext}^{\mathscr{N}}_m:=\overline{\rm ext}_m\circ I^{-1}$ is the desired extension operator. Indeed, the continuity follows from boundedness of I^{-1} and Theorem 4.4.1. Furthermore, it holds that

$$\overline{\rm Tr}^{\mathscr{N}}_m\circ\overline{\rm ext}^{\mathscr{N}}_m=I\circ\overline{\rm Tr}_m\circ\overline{\rm ext}_m\circ I^{-1}=\mathrm{id}.$$

This completes the proof. □

We can now prove our main trace theorem.

Proof of Theorem 4.5.2. We first prove the theorem for $I=\mathbb{R}$ and in this case we can consider general Muckenhoupt weights $v\in A_q(\mathbb{R})$.

Let $(V_n)_{n=1}^N$, $(\mathscr{O}_n)_{n=1}^N$, $(\eta_n)_{n=0}^N$ and $(\Phi_n)_{n=1}^N$ be a finite covering of $\partial\mathscr{O}$ consisting of balls, the corresponding special $C^{r,\kappa}_{\rm c}$-domains, the partition of unity and the diffeomorphisms as introduced in Section 4.5.1, respectively. Furthermore, let $\mathscr{N}_n$ be the non-tangential vector fields corresponding to Φ_n as constructed in (4.5.4).

Step 1: the trace operator. We prove continuity of the trace. By density it suffices to take $u \in C_c^\infty(\mathbb{R}; C_c^{r,\kappa}(\overline{\mathscr{O}}; X))$. Set $v_n := (\Phi_n)_* \eta_n^2 u$ for $n \in \{1, \dots, N\}$. By Propositions 4.5.3 and 4.5.1 (using that $\gamma > (k_1 - (r+\kappa))p - 1$), we obtain for all $n \in \{1, \dots, N\}$ and $j \in \{0, \dots, m\}$ that

$$\begin{aligned}
\big\| (\partial_{\mathscr{N}_n}^j v_n)(0,\cdot) \big\|_{F_{q,p}^{\ell - \frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B_{p,p}^{k_1 - j - \frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X))} \\
&\lesssim \| v_n \|_{W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))} \\
&= \| (\Phi_n)_* \eta_n^2 u \|_{W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))} \\
&\lesssim \| \eta_n^2 u \|_{W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathscr{O}_n, w_{\gamma+kp}^{\partial\mathscr{O}_n}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathscr{O}_n, w_{\gamma+kp}^{\partial\mathscr{O}_n}; X))} \\
&\lesssim \| u \|_{W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}; X))},
\end{aligned}$$

where in the last step we have used that $\eta_n \in C_c^\infty(V_n)$, so that the weight $w_{\gamma+kp}^{\partial\mathscr{O}_n}$ can be replaced by $w_{\gamma+kp}^{\partial\mathscr{O}}$. Furthermore, by (4.5.5) we have

$$(\partial_{\mathscr{N}_n}^j v_n)(0, \widetilde{y}) = \sum_{|\alpha|=j} (\partial^\alpha \eta_n^2 u)(\Phi_n^{-1}(0, \widetilde{y})) \boldsymbol{n}^\alpha (\Phi_n^{-1}(0, \widetilde{y})), \qquad \widetilde{y} \in \mathbb{R}^{d-1}.$$

Hence, by definition of the trace operator (4.5.2) and the spaces on the boundary (4.5.1), this proves the desired estimate.

Step 2: the extension operator. Let $g = (g_0, \dots, g_m)$ be a function in the trace space (4.5.3). For all $n \in \{1, \dots, N\}$ and $j \in \{0, \dots, m\}$, set $h_{n,j} := (\eta_n^2 g_j)(\Phi_n^{-1}(0, \cdot))$ and $h_n := (h_{n,0}, \dots, h_{n,m})$. It holds that

$$h_{n,j} \in F_{q,p}^{\ell - \frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B_{p,p}^{k_1 - j - \frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X)).$$

Furthermore, let $\overline{\operatorname{ext}}_m^{\mathscr{N}_n}$ be the extension operator from Proposition 4.5.3 and set $v_n := \overline{\operatorname{ext}}_m^{\mathscr{N}_n} h_n$. Proposition 4.5.3 implies that

$$\begin{aligned}
&\| v_n \|_{W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))} \\
&\quad\lesssim \sum_{j=0}^m \| h_{n,j} \|_{F_{q,p}^{\ell - \frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X)) \cap L^q(\mathbb{R}, v; B_{p,p}^{k_1 - j - \frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; X))} \qquad (4.5.8) \\
&\quad\lesssim \sum_{j=0}^m \| g_j \|_{F_{q,p}^{\ell - \frac{\ell}{k_1}(j+\frac{\gamma+1}{p})}(\mathbb{R}, v; L^p(\partial\mathscr{O}; X)) \cap L^q(\mathbb{R}, v; B_{p,p}^{k_1 - j - \frac{\gamma+1}{p}}(\partial\mathscr{O}; X))},
\end{aligned}$$

and $(\partial_{\mathscr{N}_n}^j v_n)|_{\partial\mathbb{R}^d_+} = h_{n,j}$. Additionally, for $n \in \{1, \dots, N\}$ we define the cut-off function $\zeta_n \in C_c^\infty(\mathbb{R}^d)$ such that $\operatorname{supp} \zeta_n \subseteq V_n$ and $\zeta_n(x) = 1$ if $x \in B_n$, where B_n is a ball which is compactly contained in V_n and satisfies $\operatorname{supp} \eta_n \subseteq B_n$. We claim that

$$\overline{\operatorname{ext}}_m^{\partial\mathscr{O}} g(x) := u(x) := \sum_{n=1}^N \zeta_n(x) v_n(\Phi_n(x)), \qquad x \in \mathscr{O}, \qquad (4.5.9)$$

defines the desired extension operator. First, note that we may assume that $v_n(y) = 0$ if $y \in \overline{\mathbb{R}^d_+} \setminus \Phi_n(\mathscr{O} \cap B_n)$. In particular, $v_n(y) = 0$ if $\zeta_n(\Phi_n^{-1}(y)) \neq 1$. This, together with (4.5.5) and the properties of h_n, yields

$$\begin{aligned}\sum_{|\alpha|=j} (\partial^\alpha u)(x)\boldsymbol{n}^\alpha(x) &= \sum_{n=1}^N \zeta_n(x) \sum_{|\alpha|=j} \partial^\alpha (v_n(\Phi_n(x)))\boldsymbol{n}^\alpha(x) \\ &= \sum_{n=1}^N \zeta_n(x) \frac{\partial^j v_n}{\partial \mathscr{N}_n}(\Phi(x)) = \sum_{n=1}^N \eta_n^2 g_j = g_j, \qquad x \in \partial\mathscr{O},\ j \in \{0,\dots,m\}.\end{aligned}$$

It remains to prove the continuity of the extension operator. With (4.5.9) and Proposition 4.5.1 (using that $\gamma > (k_1 - (r+\kappa))p - 1$), we obtain

$$\begin{aligned}&\|\overline{\operatorname{ext}}_m^{\partial\mathscr{O}} g\|_{W^{\ell,q}(\mathbb{R},v;W^{k,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X))\cap L^q(\mathbb{R},v;W^{k+k_1,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X))} \\ &\quad\lesssim \sum_{n=1}^N \|(\Phi_n^{-1})_* v_n\|_{W^{\ell,q}(\mathbb{R},v;W^{k,p}(\mathscr{O}_n,w^{\partial\mathscr{O}_n}_{\gamma+kp};X))\cap L^q(\mathbb{R},v;W^{k+k_1,p}(\mathscr{O}_n,w^{\partial\mathscr{O}_n}_{\gamma+kp};X))} \\ &\quad\lesssim \sum_{n=1}^N \|v_n\|_{W^{\ell,q}(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))\cap L^q(\mathbb{R},v;W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))}\end{aligned}$$

and the required estimate follows using (4.5.8). This completes the proof for $I = \mathbb{R}$.

Finally, for $I = (0,T)$ with $T \in (0,\infty]$ the results follow similarly to Step 2 in the proof of Theorem 4.4.1. In this case, we should work with $C_c^\infty(\overline{I}; C_c^{r,\kappa}(\overline{\mathscr{O}}; X))$ and $C_c^\infty(\overline{I}; C^{r,\kappa}(\partial\mathscr{O}; X))$ as dense subsets, since the spaces $B^s_{p,q}(\partial\mathscr{O}; X)$ and $C^{r,\kappa}(\partial\mathscr{O}; X)$ (see Section 4.5.1) are defined using $C^{r,\kappa}$-diffeomorphisms. □

4.6. Trace theorem for boundary operators

In this section, we study differential operators at the boundary and prove a trace theorem for systems of normal boundary operators on intersections of weighted Sobolev spaces. For simplicity we only consider the case of the half-space similar as in Section 3.5.

We start by introducing the definition of space- and time-dependent boundary operators. Note that compared to Definition 3.5.1 the coefficients of the boundary operator may now also depend on time.

Definition 4.6.1 (Boundary operators)**.** Let $p \in (1,\infty)$, $k_1, \ell \in \mathbb{N}_1$, $\gamma \in (-1,\infty) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$, $m \in \mathbb{N}_0$ and let X, Y be Banach spaces. Then the *boundary operator of order m*, given by

$$\mathscr{B} := \sum_{|\alpha|\le m} b_\alpha \operatorname{Tr}\partial^\alpha \qquad \text{with } b_\alpha = b_\alpha(t,\widetilde{x}) \neq 0 \text{ for some } |\alpha| = m,$$

is of *type* $(p, k_1, \ell, \gamma, m, Y)$ if

(i) $\ell - \frac{\ell}{k_1}\big(m + \frac{\gamma+1}{p}\big) > 0$ and $k_1 - m - \frac{\gamma+1}{p} > 0$,

(ii) for all $|\alpha| \le m$ we have

$$b_\alpha \in C_b^{\sigma_t}\big(\mathbb{R}; L^\infty(\mathbb{R}^{d-1}; \mathscr{L}(X,Y))\big) \cap L^\infty\big(\mathbb{R}; C_b^{\sigma_x}(\mathbb{R}^{d-1}; \mathscr{L}(X,Y))\big)$$

for some $\sigma_t, \sigma_x > 0$ such that

$$\sigma_t > \ell - \tfrac{\ell}{k_1}\big(m + \tfrac{\gamma+1}{p}\big) \quad \text{and} \quad \sigma_x > k_1 - m - \tfrac{\gamma+1}{p}.$$

Note that for $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$, we have $\operatorname{Tr} \partial^\alpha = \partial^{\widetilde{\alpha}} \operatorname{Tr}_{\alpha_1}$ and therefore the boundary operator $\mathscr{B}$ can be rewritten as

$$\mathscr{B} = \sum_{j=0}^{m} b_j \operatorname{Tr}_j, \quad \text{where} \quad b_j(t, \widetilde{x}, \nabla_{\widetilde{x}}) := \sum_{|\widetilde{\alpha}| \le m-j} b_{(j,\widetilde{\alpha})}(t, \widetilde{x}) \partial^{\widetilde{\alpha}}. \tag{4.6.1}$$

Note that the leading order coefficient b_m is independent of $\nabla_{\widetilde{x}}$. We prove a lemma about the boundedness of boundary operators on an intersection of weighted Sobolev spaces.

Lemma 4.6.2. *Let $p, q \in (1,\infty)$, $k, m \in \mathbb{N}_0$, $k_1, \ell \in \mathbb{N}_1$, $\gamma \in (-1, (k_1 - m)p - 1) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$, $v \in A_q(\mathbb{R})$ and let X and Y be* UMD *Banach spaces. Let $\mathscr{B}$ be a boundary operator of type $(p, k_1, \ell, \gamma, m, Y)$. Then $\mathscr{B}$ from*

$$\mathbb{W} := W^{\ell,q}(\mathbb{R}, v; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)) \cap L^q(\mathbb{R}, v; W^{k+k_1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))$$

to

$$\mathbb{F}^m(Y) := F_{q,p}^{\ell - \frac{\ell}{k_1}(m + \frac{\gamma+1}{p})}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; Y)) \cap L^q(\mathbb{R}, v; B_{p,p}^{k_1 - m - \frac{\gamma+1}{p}}(\mathbb{R}^{d-1}; Y))$$

is bounded.

The proof of Lemma 4.6.2 is based on a pointwise multiplier result to estimate the coefficients of $\mathscr{B}$, see also Lemma 3.5.2 for the case of time-independent boundary operators. We first prove a pointwise multiplier result for intersections of Triebel–Lizorkin spaces.

Proposition 4.6.3. *Suppose that the conditions of Lemma 4.6.2 hold. Then for all $|\alpha| \le m$, we have*

$$\|b_\alpha f\|_{\mathbb{F}^m(Y)} \le C \|f\|_{\mathbb{F}^m(X)}, \qquad f \in \mathbb{F}^m(X).$$

Proof. Let $f \in \mathbb{F}^m(X)$. It suffices to consider both spaces in the intersection separately.

First, define the pointwise multiplier

$$M_{b_\alpha} : \mathbb{R} \to \mathscr{L}\big(L^p(\mathbb{R}^{d-1}; X), L^p(\mathbb{R}^{d-1}; Y)\big)$$

by $M_{b_\alpha}(t) g(\widetilde{x}) := b_\alpha(t, \widetilde{x}) g(\widetilde{x})$ for $g \in L^p(\mathbb{R}^{d-1}; X)$. Using the regularity of b_α it is straightforward to verify that

$$M_{b_\alpha} \in C_{\mathrm{b}}^{\sigma_t}\big(\mathbb{R}; \mathscr{L}\big(L^p(\mathbb{R}^{d-1}; X), L^p(\mathbb{R}^{d-1}; Y)\big)\big).$$

Now, by [214, Proposition 5.4], using that $\sigma_t > \ell - \frac{\ell}{k_1}(m + \frac{\gamma+1}{p})$, we obtain

$$\begin{aligned} \|M_{b_\alpha} f\|_{F_{q,p}^{\ell - \frac{\ell}{k_1}(m + \frac{\gamma+1}{p})}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; Y))} &\le C \, \|M_{b_\alpha}\|_{C_{\mathrm{b}}^{\sigma_t}(\mathbb{R}; \mathscr{L}(L^p(\mathbb{R}^{d-1}; X), L^p(\mathbb{R}^{d-1}; Y)))} \\ &\quad \cdot \|f\|_{F_{q,p}^{\ell - \frac{\ell}{k_1}(m + \frac{\gamma+1}{p})}(\mathbb{R}, v; L^p(\mathbb{R}^{d-1}; X))}. \end{aligned}$$

Similarly, by [214, Proposition 5.4], using that $\sigma_x > k_1 - m - \frac{\gamma+1}{p}$, we obtain for almost every $t \in \mathbb{R}$ that

$$\|b_\alpha(t,\cdot)f(t,\cdot)\|_{B_{p,p}^{k_1-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};Y)} \leq C\|b_\alpha(t,\cdot)\|_{C_{\mathrm{b}}^{\sigma_x}(\mathbb{R}^{d-1};\mathscr{L}(X,Y))}\|f(t,\cdot)\|_{B_{p,p}^{k_1-m-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};X)}.$$

Taking the $L^q(\mathbb{R},v)$-norm and using the regularity of b_α gives the result. □

We can now use the above multiplier result to prove continuity of the boundary operator.

Proof of Lemma 4.6.2. Recall that the boundary operator $\mathscr{B}$ is given as in (4.6.1). Fix $j \in \{0,\dots,m\}$ and $|\widetilde{\alpha}| \leq m-j$, then by Proposition 4.6.3, the intersection representation Theorem 4.4.3, the fact that the (spatial) $\partial^{\widetilde{\alpha}}$-derivatives are of order $\frac{|\widetilde{\alpha}|}{k_1}$ in the scale of anisotropic Triebel–Lizorkin spaces (e.g., [183, Proposition 5.2.29]), and Theorem 4.4.1, we obtain the estimate

$$\begin{aligned}
\|b_{(j,\widetilde{\alpha})}\partial^{\widetilde{\alpha}}\operatorname{Tr}_j f\|_{\mathbb{F}^m(Y)} &\lesssim \|\partial^{\widetilde{\alpha}}\operatorname{Tr}_j f\|_{\mathbb{F}^m(X)} \\
&\lesssim \|\partial^{\widetilde{\alpha}}\operatorname{Tr}_j f\|_{F_{(p,q),p,(d-1,1)}^{1-\frac{1}{k_1}(m+\frac{\gamma+1}{p}),(\frac{1}{k_1},\frac{1}{\ell})}(\mathbb{R}^d,(1,v);X)} \\
&\lesssim \|\operatorname{Tr}_j f\|_{F_{(p,q),p,(d-1,1)}^{1-\frac{1}{k_1}(m-|\widetilde{\alpha}|+\frac{\gamma+1}{p}),(\frac{1}{k_1},\frac{1}{\ell})}(\mathbb{R}^d,(1,v);X)} \\
&\lesssim \|\operatorname{Tr}_j f\|_{F_{(p,q),p,(d-1,1)}^{1-\frac{1}{k_1}(j+\frac{\gamma+1}{p}),(\frac{1}{k_1},\frac{1}{\ell})}(\mathbb{R}^d,(1,v);X)} \\
&\lesssim \|\operatorname{Tr}_j f\|_{\mathbb{F}^j(X)} \lesssim \|f\|_{\mathbb{W}}.
\end{aligned}$$

This completes the proof. □

Similar to Definitions 3.5.3 and 3.5.4, we can now define a *system of normal boundary operators* $\mathscr{B} := (\mathscr{B}^{m_0},\dots,\mathscr{B}^{m_n})$ *of type* $(p,k_1,\ell,\gamma,\overline{m},\overline{Y})$, where $0 \leq m_0 < m_1 < \cdots < m_n$ are integers for some $n \in \mathbb{N}_0$, $\overline{m} := (m_0,\dots,m_n)$, $\overline{Y} := (Y_0,\dots,Y_n)$ are Banach spaces and $\mathscr{B}^{m_i}$ are normal boundary operators of type $(p,k_1,\ell,\gamma,m_i,Y_i)$ for $i \in \{0,\dots,n\}$.

We obtain the following trace theorem for systems of normal boundary operators.

Theorem 4.6.4. *Let $p,q \in (1,\infty)$, $k \in \mathbb{N}_0$, $k_1,\ell \in \mathbb{N}_1$, $\gamma \in (-1,\infty)\setminus\{jp-1 : j \in \mathbb{N}_1\}$, $v \in A_q(\mathbb{R})$ and let X and $\overline{Y} = (Y_0,\dots,Y_n)$ be* UMD *Banach spaces. Let $\mathscr{B}$ be a system of normal boundary operators of type $(p,k_1,\ell,\gamma,\overline{m},\overline{Y})$. Then $\mathscr{B}$ from*

$$W^{\ell,q}(\mathbb{R},v;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)) \cap L^q(\mathbb{R},v;W^{k+k_1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))$$

to

$$\prod_{i=0}^n F_{q,p}^{\ell-\frac{\ell}{k_1}(m_i+\frac{\gamma+1}{p})}(\mathbb{R},v;L^p(\mathbb{R}^{d-1};Y_i)) \cap L^q(\mathbb{R},v;B_{p,p}^{k_1-m_i-\frac{\gamma+1}{p}}(\mathbb{R}^{d-1};Y_i))$$

is a continuous and surjective operator.

If $\mathscr{B}$ is a system of normal boundary operators of type $(p,k_1,\ell,\gamma,\overline{m},\overline{Y})$, then the condition $\gamma < (k_1-m_n)p-1$ holds by Definition 4.6.1.

Proof. The proof is analogous to the proof of Theorem 3.5.6. The continuity of $\mathscr{B}$ follows from Lemma 4.6.2. For the construction of the right inverse one can argue as in the proof of Theorem 3.5.6 using Theorem 4.4.1. □

Remark 4.6.5.

(i) The result of Theorem 4.6.4 can also be obtained on bounded domains via localisation arguments. Using an appropriate extension operator and an extension-restriction argument similar to the one in the proof of Theorem 4.4.1, the trace theorem can be obtained for $\mathbb{R}_+$ or bounded time intervals (using power weights v_μ with $\mu \in (-1, q-1)$).

(ii) It is not clear whether Theorem 4.6.4 has any application in the study of boundary value problems with inhomogeneous boundary conditions $\mathscr{B}u = g$. In fact, using the extension operator in Theorem 4.6.4, it suffices to consider the boundary value problem with homogeneous boundary conditions $\mathscr{B}u = 0$. However, in the case of general boundary operators, well-posedness is obtained via perturbation methods which require a nonzero right-hand side in the boundary conditions, see [67, 68].

II

Functional calculus

5

Functional calculus for the Laplacian on the half-space

This chapter is based on:

N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on the half-space. *J. Funct. Anal.*, 289(8):110985, 2025.

Abstract. *We consider the Laplace operator on the half-space with Dirichlet and Neumann boundary conditions. We prove that this operator admits a bounded H^∞-calculus on Sobolev spaces with power weights measuring the distance to the boundary. These weights do not necessarily belong to the class of Muckenhoupt A_p weights. We additionally study the corresponding Dirichlet and Neumann heat semigroup. It is shown that these semigroups, in contrast to the L^p-case, have polynomial growth.*

5.1. Introduction and main results

Since the development of the H^∞-calculus in [52, 204], motivated by the Kato square root problem (see [22, 205]), this holomorphic functional calculus turned out to provide an efficient way to tackle other problems in mathematics as well. For instance, in the theory of (stochastic) partial differential equations ((S)PDEs), the boundedness of the H^∞-calculus plays an important role. It can be used to obtain well-posedness and regularity of the, possibly nonlinear, (S)PDE. There is a vast literature on the H^∞-calculus for sectorial operators, see, e.g., the monographs and lecture notes [20, 67, 118, 125, 126, 136, 174, 216, 234]. For applications to PDEs, see, e.g., [26, 65, 69, 134, 175, 206, 261, 274] and to SPDEs, see, e.g., [11, 76, 122, 219, 221, 222].

A particular problem that can occur in the analysis of a (stochastic) PDE on a spatial domain $\mathcal{O} \subseteq \mathbb{R}^d$ is that the solution or its derivatives may exhibit blow-up behaviour near the boundary $\partial\mathcal{O}$. To solve the PDE on a Sobolev space $W^{k,p}(\mathcal{O})$, it is required to impose additional conditions such as smoothness of the domain and/or unnatural boundary conditions for the data.

(i) *Smoothness of the domain:* for regularity of second-order elliptic operators on $W^{k,p}(\mathcal{O})$ one needs that $\mathcal{O}$ is a C^{k+2}-domain, see [166, Chapter 9]. This condition is also present in regularity theory for parabolic SPDEs, see [160].

(ii) *Unnatural boundary conditions for the data:* as illustrated in [166, Chapter 9], one obtains regularity for second-order elliptic operators on $W^{k,p}(\mathcal{O})$, however the operator will not be sectorial. To obtain a sectorial operator, additional boundary conditions on the data need to be specified, see [66]. Furthermore, for higher-order regularity for the heat equation $\partial_t u - \Delta u = f$ on $\mathcal{O}$ with homogeneous Dirichlet boundary conditions, additional conditions on f need to be imposed as well, see, e.g., [89, Section 7.1.3].

To circumvent these additional conditions, weighted spaces for the solution are used with a spatial weight of the form $w_\gamma(x) := \operatorname{dist}(x, \partial\mathcal{O})^\gamma$ for some suitable $\gamma \in \mathbb{R}$. In this way, the solution is allowed to have a certain blow-up near the boundary. Weights are also commonly used for SPDEs as is motivated in [160, Examples 1.1 & 1.2].

Motivated by the applications to (S)PDEs mentioned above, it is natural to study the H^∞-calculus for differential operators on *inhomogeneous weighted Sobolev spaces*. In this chapter, we take up the study of two commonly used elliptic differential operators: the Laplace operator with Dirichlet and Neumann boundary conditions on the half-space $\mathbb{R}^d_+$. We consider Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ and Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ for $k \geq -1$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Our main results include the bounded H^∞-calculus for $-\Delta_{\mathrm{Dir}}$ and $-\Delta_{\mathrm{Neu}}$, and growth bounds on the corresponding semigroups. The mathematical statements of the main theorems are presented in Section 5.1.1.

The study of the H^∞-calculus for differential operators with Dirichlet and Neumann boundary conditions on Sobolev spaces has been taken up before, see [65, 134]. The H^∞-calculus on weighted Lebesgue spaces is considered for instance in [23, 24, 181, 200]. In these papers, the weights belong to the class of Muckenhoupt A_p weights. In [190], the first and fourth authors studied the Dirichlet Laplacian on weighted Lebesgue spaces

with weights outside the A_p range. The current chapter extends the H^∞-calculus results of [190] and we present the first theorems on inhomogeneous weighted Sobolev spaces with weights outside the A_p class.

Parabolic and elliptic differential equations on weighted spaces have already been studied extensively in the literature, see, e.g., [79, 80, 81, 144, 149, 157, 158, 164, 165] for the deterministic setting and [141, 142, 143, 167] for the stochastic setting. In these papers, weighted spaces are used to obtain (stochastic) maximal regularity for elliptic operators on *homogeneous weighted Sobolev spaces*. The new aspects of our approach, compared to the aforementioned works, are the following.

(i) We prove the boundedness of the H^∞-calculus, which gives the boundedness of many singular integral operators. In particular, it yields (stochastic) maximal regularity and bounded imaginary powers. Applications of the functional calculus to boundary value problems are presented in Chapter 7.

(ii) Using a scaling argument, we show in Chapter 7 that our results on inhomogeneous weighted Sobolev spaces recover maximal regularity for the heat equation on homogeneous spaces as well. In particular, we recover parts of the results for the heat equation in [80, 81, 164, 165]. Moreover, we can allow for weights in time.

(iii) We can treat power weights with exponents $\gamma \in (-1, 2p-1) \setminus \{p-1\}$, extending the typical range $\gamma \in (p-1, 2p-1)$ considered for higher-order regularity in the homogeneous setting with Dirichlet boundary conditions.

(iv) The main results are presented for both the Dirichlet and the Neumann Laplacian. The weighted higher-order regularity for the Neumann Laplacian is new. Weighted second-order regularity for elliptic and parabolic equations with Neumann boundary conditions is considered in [80, 81, 158].

In contrast to our study of the Laplacian on $\mathbb{R}^d_+$ on inhomogeneous weighted Sobolev spaces, more general elliptic differential operators on domains are considered in the aforementioned works on homogeneous weighted Sobolev spaces. Moreover, negative and fractional smoothness parameters were studied. In our case, we can also consider the following generalisations.

(i) The bounded H^∞-calculus obtained for the Laplacian on $\mathbb{R}^d_+$ can be transferred to bounded domains using perturbation theorems for the H^∞-calculus. In particular, for the Dirichlet Laplacian, we can use the weights to weaken the required smoothness of the boundary from C^{k+2} to $C^{1,\lambda}$ for $\lambda \in [0,1]$ depending on γ. This will be done in Chapter 6.

(ii) Our results for, e.g., the Dirichlet Laplacian remain true on the complex interpolation space

$$W^{k+\theta,p}(\mathbb{R}^d_+, w_{\gamma+(k+\theta)p}) := [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}), W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})]_\theta,$$

with $\theta \in (0,1)$. To characterise this complex interpolation space one can use the characterisations used in [164, 165, 197] for homogeneous weighted Sobolev spaces. Similarly, it is expected that negative smoothness can be obtained via duality.

For negative smoothness we do consider $-\Delta_{\mathrm{Dir}}$ on $W^{-1,p}(\mathbb{R}^d_+, w_\gamma)$ with $\gamma \in (-1, p-1)$. In this case, the H^∞-calculus can be derived from the calculus for $-\Delta_{\mathrm{Dir}}$ and $-\Delta_{\mathrm{Neu}}$ on $L^p(\mathbb{R}^d_+, w_\gamma)$.

(iii) For applications to stochastic PDEs, it is well known that a bounded H^∞-calculus yields stochastic maximal regularity in the setting without gradient noise, see [219]. We will investigate an example of a stochastic equation via the H^∞-calculus in Chapter 7.

5.1.1. Main results

We start with the definition of the inhomogeneous weighted Sobolev spaces on $\mathbb{R}^d_+$. Throughout this section we assume that $p \in (1,\infty)$ and $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$. Moreover, for $x \in \mathbb{R}^d_+$, we write $x = (x_1, \widetilde{x}) \in \mathbb{R}_+ \times \mathbb{R}^{d-1}$. Let $w_\gamma(x) := \mathrm{dist}(x, \partial\mathbb{R}^d_+)^\gamma = |x_1|^\gamma$, then we define the weighted Lebesgue space $L^p(\mathbb{R}^d_+, w_\gamma)$ as the space consisting of all strongly measurable $f : \mathbb{R}^d_+ \to \mathbb{C}$ such that

$$\|f\|_{L^p(\mathbb{R}^d_+, w_\gamma)} := \Big(\int_{\mathbb{R}^d_+} |f(x)|^p \, w_\gamma(x) \, \mathrm{d}x \Big)^{1/p} < \infty.$$

The associated k-th order weighted Sobolev space for $k \in \mathbb{N}_0$ is defined as

$$W^{k,p}(\mathbb{R}^d_+, w_\gamma) := \Big\{ f \in \mathscr{D}'(\mathbb{R}^d_+) : \forall |\alpha| \le k, \partial^\alpha f \in L^p(\mathbb{R}^d_+, w_\gamma) \Big\}.$$

For the definition of the weighted Sobolev space with $k = -1$, we refer to Section 5.4.3.

For $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$, we define the following spaces with Dirichlet and Neumann boundary conditions

$$W^{k,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma) := \Big\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma) : \mathrm{Tr}(f) = 0 \text{ if } k > \tfrac{\gamma+1}{p} \Big\},$$
$$W^{k,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_\gamma) := \Big\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma) : \mathrm{Tr}(\partial_1 f) = 0 \text{ if } k-1 > \tfrac{\gamma+1}{p} \Big\},$$

where Tr denotes the trace operator. We will elaborate in Section 5.3 on the existence of these traces. Equivalently, in many cases, these spaces with boundary conditions can be defined as the closure of test functions.

Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ be such that $\gamma + kp > -1$. The Dirichlet and Neumann Laplacian on $\mathbb{R}^d_+$ are defined as follows.

(i) The *Dirichlet Laplacian* Δ_{Dir} *on* $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ is defined by

$$\Delta_{\mathrm{Dir}} u := \sum_{j=1}^d \partial_j^2 u \quad \text{with} \quad D(\Delta_{\mathrm{Dir}}) := W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}).$$

(ii) The *Neumann Laplacian* Δ_{Neu} *on* $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ is defined by

$$\Delta_{\mathrm{Neu}} u := \sum_{j=1}^d \partial_j^2 u \quad \text{with} \quad D(\Delta_{\mathrm{Neu}}) := W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}).$$

In addition, we study the corresponding Dirichlet and Neumann heat semigroup. Let $G_z^d : \mathbb{R}^d \to \mathbb{C}$ be the standard heat kernel on $\mathbb{R}^d$, defined by

$$G_z^d(x) := \frac{1}{(4\pi z)^{d/2}} e^{\frac{-|x|^2}{4z}}, \qquad z \in \mathbb{C}_+. \tag{5.1.1}$$

For $z \in \mathbb{C}_+$ and $x, y \in \mathbb{R}^d_+$, we define the kernels

$$H_z^{d,\pm}(x,y) := G_z^d(x_1 - y_1, \widetilde{x} - \widetilde{y}) \pm G_z^d(x_1 + y_1, \widetilde{x} - \widetilde{y}).$$

The Dirichlet and Neumann heat semigroups T_{Dir} and T_{Neu} are defined by

$$\begin{aligned} T_{\mathrm{Dir}}(z)f(x) &:= H_z^{d,-} * f(x) := \int_{\mathbb{R}^d_+} H_z^{d,-}(x,y) f(y)\, \mathrm{d}y, \\ T_{\mathrm{Neu}}(z)f(x) &:= H_z^{d,+} * f(x) := \int_{\mathbb{R}^d_+} H_z^{d,+}(x,y) f(y)\, \mathrm{d}y, \end{aligned} \tag{5.1.2}$$

for any $f \in L^p(\mathbb{R}^d_+, w_\gamma)$ with γ such that the formulas are well defined.

The main results of this chapter are summarised in the following theorems. We start with the main result for the Dirichlet Laplacian (see Theorems 5.4.10, 5.5.1, 5.6.1 and 5.7.1).

Theorem 5.1.1 (Main results for the Dirichlet Laplacian)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ be such that $\gamma + kp > -1$. Then for Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ the following assertions hold for all $\lambda > 0$:*

(i) *$\lambda - \Delta_{\mathrm{Dir}}$ is sectorial of angle $\omega(\lambda - \Delta_{\mathrm{Dir}}) = 0$,*

(ii) *$(T_{\mathrm{Dir}}(z))_{z \in \Sigma_\sigma}$ with $\sigma \in (0, \frac{\pi}{2})$ is an analytic C_0-semigroup on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ which is generated by Δ_{Dir},*

(iii) *$\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) = 0$.*

In addition, the semigroup $T_{\mathrm{Dir}}(t)$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ satisfies the following growth properties:

(iv) *if $\gamma + kp \in (-1, 2p-1)$, then $T_{\mathrm{Dir}}(t)$ is bounded and assertions (i) and (iii) hold for $\lambda = 0$ as well,*

(v) *if $\gamma + kp > 2p - 1$, then $T_{\mathrm{Dir}}(t)$ has polynomial growth and for any $\varepsilon > 0$ there are constants $c, C > 0$ only depending on $p, k, \gamma, \varepsilon$ and d, such that*

$$c\big(1 + t^{\frac{\gamma+kp-2p+1}{2p}}\big) \le \|T_{\mathrm{Dir}}(t)\| \le C\big(1 + t^{\frac{\gamma+kp-2p+1+\varepsilon}{2p}}\big), \qquad t \ge 0.$$

Theorem 5.1.1 in the special case $k = 0$ has already been established by the first and fourth authors in [190, Theorem 4.1 & 5.7]. The case $k = 0$ is used as the basis for an induction argument to obtain Theorem 5.1.1 for general $k \in \mathbb{N}_0$. For this induction argument, we directly use the estimate from the definition of a bounded H^∞-calculus together with

perturbation arguments and commutator estimates. The fact that ∂_1 and Δ_{Dir} do not commute complicates the analysis.

Concerning the Neumann Laplacian we obtain the following results (see Theorems 5.5.2, 5.6.2 and 5.7.2).

Theorem 5.1.2 (Main results for the Neumann Laplacian)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ be such that $\gamma + kp > -1$. Then for Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ the following assertions hold for all $\lambda > 0$:*

(i) $\lambda - \Delta_{\mathrm{Neu}}$ *is sectorial of angle* $\omega(\lambda - \Delta_{\mathrm{Neu}}) = 0$,

(ii) $(T_{\mathrm{Neu}}(z))_{z\in\Sigma_\sigma}$ *with* $\sigma \in (0, \frac{\pi}{2})$ *is an analytic C_0-semigroup on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ which is generated by* Δ_{Neu},

(iii) $\lambda - \Delta_{\mathrm{Neu}}$ *has a bounded H^∞-calculus of angle* $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) = 0$.

In addition, the semigroup $T_{\mathrm{Neu}}(t)$ on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ satisfies the following growth properties:

(iv) if $\gamma + kp \in (-1, p-1)$, then $T_{\mathrm{Neu}}(t)$ is bounded and assertions (i) and (iii) hold for $\lambda = 0$ as well,

(v) if $\gamma + kp > p-1$, then $T_{\mathrm{Neu}}(t)$ has polynomial growth and for any $\varepsilon > 0$ there are constants $c, C > 0$ only depending on $p, k, \gamma, \varepsilon$ and d, such that

$$c\big(1 + t^{\frac{\gamma+kp-p+1}{2p}}\big) \le \|T_{\mathrm{Neu}}(t)\| \le C\big(1 + t^{\frac{\gamma+kp-p+1+\varepsilon}{2p}}\big), \qquad t \ge 0.$$

Theorem 5.1.2 will be derived in a similar way to Theorem 5.1.1, but we can also use the results from Theorem 5.1.1. This is illustrated by the fact that $\partial_1 R(\lambda, \Delta_{\mathrm{Neu}}) f = R(\lambda, \Delta_{\mathrm{Dir}})\partial_1 f$ for appropriate f.

Remark 5.1.3.

(i) Theorems 5.1.1 and 5.1.2 also hold for vector-valued weighted Sobolev spaces, which is the setting used throughout this chapter.

(ii) Theorems 5.1.1 and 5.1.2 are false for $\gamma \notin (-1, 2p-1)$. If $\gamma \ge 2p-1$, then the Laplacian is not sectorial (see Example 5.6.9). If $\gamma \le -1$, then the Laplacian will only be sectorial if additional boundary conditions are included in the domain of the operator, as is illustrated in [190, Section 5.5].

(iii) By complex interpolation in the parameter γ, we can obtain the statements in Theorem 5.1.1 for $\gamma = p-1$, although the domain will need to be changed. The existing interpolation theory for weighted spaces with boundary conditions is inadequate here.

(iv) The growth of the Dirichlet and Neumann heat semigroup is a result of the behaviour of the weight away from the boundary $x_1 = 0$. Replacing the weight $w_{\gamma+kp}$ by a weight which behaves like $w_{\gamma+kp}$ near zero and becomes constant at infinity, or,

considering a bounded domain, would result in a bounded semigroup. We refer to [163] and [208] for related results on the Dirichlet heat semigroup on weighted spaces.

(v) It is proved by the author in [238] that the upper bounds for the semigroups hold with $\varepsilon = 0$. However, with the methods available in this chapter, we can only achieve the result with an additional $\varepsilon > 0$.

As a consequence of Theorems 5.1.1 and 5.1.2, we obtain maximal regularity for the Poisson and heat equation. These results will be presented in Chapter 7.

Outline

The outline of this chapter is as follows. After introducing the functional calculus in Section 5.2, we study Sobolev spaces with power weights and their properties in Section 5.3. In Section 5.4 we collect some known and straightforward results concerning the H^∞-calculus for the Laplacian on lower-order weighted Sobolev spaces. Then, in Section 5.5 we study the sectoriality of the Laplacian on higher-order weighted Sobolev spaces and in Section 5.6 we prove growth estimates for the corresponding heat semigroups. Section 5.7 deals with the H^∞-calculus of the Dirichlet and Neumann Laplacian.

5.2. Functional calculus

We briefly recall the notions of sectoriality and functional calculus from Section 2.3 and prove some related preliminary results.

We state the following lemma on the H^∞-calculus for the Laplacian on Sobolev and Bessel potential spaces with weights $w \in A_p(\mathbb{R}^d)$. This result is folklore and follows easily from the H^∞-calculus on $L^p(\mathbb{R}^d, w; X)$ and lifting.

Let $s \in \mathbb{R}$, then the Bessel potential operator is

$$J_s f = (1-\Delta)^{\frac{s}{2}} f := \mathscr{F}^{-1}\big((1+|\cdot|^2)^{\frac{s}{2}} \mathscr{F} f\big), \quad f \in \mathscr{S}'(\mathbb{R}^d; X).$$

For $p \in (1,\infty)$, $s \in \mathbb{R}$ and $w \in A_p(\mathbb{R}^d)$, the weighted Bessel potential space $H^{s,p}(\mathbb{R}^d, w; X) \subseteq \mathscr{S}'(\mathbb{R}^d; X)$ is defined as the space consisting of all $f \in \mathscr{S}'(\mathbb{R}^d; X)$ such that $J_s f \in L^p(\mathbb{R}^d, w; X)$ and

$$\|f\|_{H^{s,p}(\mathbb{R}^d,w;X)} := \|J_s f\|_{L^p(\mathbb{R}^d,w;X)} < \infty.$$

Lemma 5.2.1. *Let $p \in (1,\infty)$, $s \in \mathbb{R}$, $w \in A_p(\mathbb{R}^d)$ and let X be a* UMD *Banach space. Then $-\Delta$ on $H^{s,p}(\mathbb{R}^d, w; X)$ with domain $H^{s+2,p}(\mathbb{R}^d, w; X)$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(-\Delta) = 0$. In particular, if $k \in \mathbb{N}_0$, then the same statement holds for $-\Delta$ on $W^{k,p}(\mathbb{R}^d, w; X)$ with domain $W^{k+2,p}(\mathbb{R}^d, w; X)$.*

Proof. Let $\omega \in (0,\pi)$. The case $s = 0$ follows from [214, Proposition 3.6(b)], i.e., there is a constant $C > 0$ such that for $f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$

$$\|f(-\Delta)u\|_{L^p(\mathbb{R}^d,w;X)} \le C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{L^p(\mathbb{R}^d,w;X)}, \qquad u \in L^p(\mathbb{R}^d, w; X). \tag{5.2.1}$$

Moreover, the H^∞-calculus of the Laplacian is given by (see [125, Theorem 10.2.25])

$$f(-\Delta)u = \mathscr{F}^{-1}\big(f(|\cdot|^2)\mathscr{F}u\big), \qquad u \in L^p(\mathbb{R}^d, w; X).$$

Therefore, by (5.2.1) and lifting we obtain for $u \in H^{s,p}(\mathbb{R}^d, w; X)$

$$\begin{aligned}\|f(-\Delta)u\|_{H^{s,p}(\mathbb{R}^d,w;X)} &= \|J_s f(-\Delta)u\|_{L^p(\mathbb{R}^d,w;X)} = \|f(-\Delta)J_s u\|_{L^p(\mathbb{R}^d,w;X)}\\ &\le C\|f\|_{H^\infty(\Sigma_\omega)}\|J_s u\|_{L^p(\mathbb{R}^d,w;X)} = C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{H^{s,p}(\mathbb{R}^d,w;X)},\end{aligned}$$

which proves boundedness of the H^∞-calculus on $H^{s,p}(\mathbb{R}^d, w; X)$. The last statement about the H^∞-calculus on $W^{k,p}(\mathbb{R}^d, w; X)$ follows from [214, Proposition 3.2]. □

We note that the UMD condition on X (see Section 2.4) is necessary for the H^∞-calculus on $L^p(\mathbb{R}^d; X)$, see [125, Section 10.5].

5.3. Weighted Sobolev spaces

In this section, we introduce the inhomogeneous weighted Sobolev spaces with boundary conditions and derive certain properties.

5.3.1. Sobolev spaces with boundary conditions

Recall the definitions of the weighted Lebesgue and Sobolev spaces from Section 2.2. We will now introduce weighted Sobolev spaces with vanishing traces.

For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and X a Banach space, we define the following spaces with vanishing traces

$$\begin{aligned}W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(\partial^\alpha f) = 0 \text{ if } k - |\alpha| > \tfrac{\gamma+1}{p}\Big\},\\ W_{\mathrm{Dir}}^{k,p}(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(f) = 0 \text{ if } k > \tfrac{\gamma+1}{p}\Big\},\\ W_{\mathrm{Neu}}^{k,p}(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(\partial_1 f) = 0 \text{ if } k-1 > \tfrac{\gamma+1}{p}\Big\}.\end{aligned} \tag{5.3.1}$$

For $w \in A_p(\mathbb{R}^d_+)$ the Sobolev spaces with boundary conditions are

$$\begin{aligned}W_0^{k,p}(\mathbb{R}^d_+, w; X) &:= \Big\{f \in W^{k,p}(\mathbb{R}^d_+, w; X) : \mathrm{Tr}(\partial^\alpha f) = 0 \text{ if } |\alpha| \le k-1\Big\},\\ W_{\mathrm{Dir}}^{k,p}(\mathbb{R}^d_+, w; X) &:= \Big\{f \in W^{k,p}(\mathbb{R}^d_+, w; X) : \mathrm{Tr}(f) = 0 \text{ if } k \ge 1\Big\},\\ W_{\mathrm{Neu}}^{k,p}(\mathbb{R}^d_+, w; X) &:= \Big\{f \in W^{k,p}(\mathbb{R}^d_+, w; X) : \mathrm{Tr}(\partial_1 f) = 0 \text{ if } k \ge 2\Big\}.\end{aligned} \tag{5.3.2}$$

We check that all the traces in the above definitions exist. For the traces in (5.3.2) this follows from (2.2.1). For (5.3.1), let $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ and $\alpha \in \mathbb{N}_0^d$ be such that $k - |\alpha| > \frac{\gamma+1}{p}$. Then for $\gamma \in (jp-1, (j+1)p-1)$ with $j \in \mathbb{N}_0$, note that $k - |\alpha| \ge j+1$ and thus by Hardy's inequality (Lemma 2.2.2) and (2.2.1)

$$\begin{aligned}W^{k-|\alpha|,p}(\mathbb{R}^d_+, w_\gamma; X) &\hookrightarrow W^{k-|\alpha|,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))\\ &\hookrightarrow W^{j+1,p}(\mathbb{R}_+, w_\gamma; L^p(\mathbb{R}^{d-1}; X))\\ &\hookrightarrow W^{1,p}(\mathbb{R}_+, w_{\gamma-jp}; L^p(\mathbb{R}^{d-1}; X)) \hookrightarrow C([0,\infty); L^p(\mathbb{R}^{d-1}; X)).\end{aligned}$$

We conclude that $\partial^\alpha f \in C([0,\infty); L^p(\mathbb{R}^{d-1}; X))$ and thus the spaces in (5.3.1) are well defined.

Remark 5.3.1. An important observation is that for $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ we have by definition

$$W^{k,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \qquad \text{if } \gamma > kp-1,$$
$$W^{k,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) \qquad \text{if } \gamma > (k-1)p-1.$$

Although we will not consider weights w_γ with $\gamma \le -1$, we can nonetheless define

$$W^{k,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}(\mathbb{R}^d_+, w_\gamma; X),$$

see [190, Lemma 3.1(2)].

5.3.2. Properties of weighted Sobolev spaces

We collect certain properties of weighted Sobolev spaces for later reference. We start with the following weighted Sobolev embeddings, which are a direct consequence of Lemma 2.2.2, see also [169, Section 8.8].

Corollary 5.3.2 (Hardy's inequality on $\mathbb{R}^d_+$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_1$, $\gamma \in \mathbb{R}$ and let X be a Banach space. Then*

$$W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X) \qquad \text{if } \gamma < p-1,$$
$$W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X) \qquad \text{if } \gamma > p-1,$$
$$W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow W^{k-1,p}_0(\mathbb{R}^d_+, w_{\gamma-p}; X) \qquad \text{if } \gamma \notin \{jp-1 : j \in \mathbb{N}_1\}.$$

We continue with a density result. To deal with the Neumann boundary condition we do not only need approximation by compactly supported functions, but density of functions of which only certain derivatives have compact support, is required as well. For $j \in \mathbb{N}_0$ and X a Banach space, define

$$C^\infty_{\mathrm{c},j}(\overline{\mathbb{R}^d_+}; X) := \{f \in C^\infty_{\mathrm{c}}(\overline{\mathbb{R}^d_+}; X) : \partial_1^j f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)\}. \tag{5.3.3}$$

Note that the condition $\partial_1^j f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ implies that $\partial^\alpha f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ for all $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ with $\alpha_1 \ge j$.

Lemma 5.3.3. *Let $p \in (1,\infty)$, $j,k \in \mathbb{N}_0$ such that $k \ge j$ and $\gamma > (k-j)p-1$, and let X be a Banach space. Then $C^\infty_{\mathrm{c},j}(\overline{\mathbb{R}^d_+}; X)$ is dense in $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$. If, in addition, X is reflexive and $k \ge j+1$, then the statement holds for $\gamma = (k-j)p-1$ as well.*

Remark 5.3.4. In particular, $C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X) = C^\infty_{\mathrm{c},0}(\overline{\mathbb{R}^d_+}; X)$ is dense in $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ if $\gamma > kp-1$, or $\gamma \ge kp-1$ when X is reflexive. The density of $C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ in $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ for $\gamma \in \mathbb{R} \setminus \{jp-1 : j \in \mathbb{N}_1\}$ with $\gamma > kp-1$ also follows from [190, Proposition 3.8] and Remark 5.3.1.

In general, it holds that $C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ is dense in $W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X)$ for $\gamma \in \mathbb{R} \setminus \{jp-1 : j \in \mathbb{N}_1\}$, see [190, Proposition 3.8]. However, density of $C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ in $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ is not true for all $\gamma \in \mathbb{R}$.

Proof. Take $\varepsilon > 0$ and fix $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$. By [169, Theorem 7.2 & Remark 11.12(iii)], which also holds in the vector-valued case, and a standard cut-off argument, we find a $g \in C_c^\infty(\overline{\mathbb{R}^d_+}; X)$ with its support in $[0,R] \times [-R,R]^{d-1}$ for some $R > 0$ such that

$$\|f - g\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} < \varepsilon. \tag{5.3.4}$$

Let $\phi \in C^\infty(\mathbb{R}_+)$ be such that $\phi = 0$ on $[0, \frac{1}{2}]$ and $\phi = 1$ on $[1, \infty)$ and set $\phi_n(x_1) := \phi(nx_1)$. We construct a sequence $(g_n)_{n\geq 1}$ as follows:

- If $j = 0$, define $g_n(x) := \phi_n(x_1) g(x)$.
- If $j \geq 1$, define

$$g_n(x) := \sum_{m=0}^{j-1} (\partial_1^m g)(1, \widetilde{x}) \frac{(x_1 - 1)^m}{m!} + \frac{1}{(j-1)!} \int_1^{x_1} (x_1 - t)^{j-1} \phi_n(t) \partial_1^j g(t, \widetilde{x}) \, \mathrm{d}t.$$

Note that by integration by parts $g_n(x) = g(x)$ for all $x \in \mathbb{R}^d_+$ with $x_1 \geq \frac{1}{n}$. For $|\alpha| \leq k$ with $\alpha_1 \leq j$ and $x \in \mathbb{R}^d_+$ with $x_1 < 1$ we have

$$|\partial^\alpha g_n(x)| \leq C \|g\|_{C^{k+j}(\mathbb{R}^d_+; X)} \|\phi\|_{L^\infty(\mathbb{R}_+)}.$$

Moreover, we have

$$\partial_1^j g_n(x) = \phi_n(x_1) \partial_1^j g(x), \qquad x \in \mathbb{R}^d_+, \tag{5.3.5}$$

so that in particular $\partial_1^j g_n \in C_c^\infty(\mathbb{R}^d_+; X)$. Let $K_R := [-R,R]^{d-1}$. Using the properties of g_n we obtain

$$\begin{aligned}
&\|g_n - g\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} \\
&\quad \leq \sum_{|\alpha| \leq k} \Big(\int_{K_R} \int_0^{\frac{1}{n}} \|\partial^\alpha g(x)\|_X^p x_1^\gamma \, \mathrm{d}x_1 \, \mathrm{d}\widetilde{x} \Big)^{\frac{1}{p}} \\
&\qquad + \sum_{\substack{|\alpha| \leq k \\ \alpha_1 \leq j}} \Big(\int_{K_R} \int_0^{\frac{1}{n}} \|\partial^\alpha g_n(x)\|_X^p x_1^\gamma \, \mathrm{d}x_1 \, \mathrm{d}\widetilde{x} \Big)^{\frac{1}{p}} \\
&\qquad + C \sum_{\substack{|\alpha| \leq k \\ \alpha_1 > j}} \sum_{m=0}^{\alpha_1 - j} \Big(\int_{K_R} \int_0^{\frac{1}{n}} n^{mp} |\phi^{(m)}(nx_1)|^p \|\partial^{(\alpha_1 - m, \widetilde{\alpha})} g(x)\|_X^p x_1^\gamma \, \mathrm{d}x_1 \, \mathrm{d}\widetilde{x} \Big)^{\frac{1}{p}} \\
&\quad \leq C n^{\frac{1}{p}((k-j)p - 1 - \gamma)} \cdot \Big(\tfrac{(2R)^{d-1}}{\gamma + 1} \Big)^{\frac{1}{p}} \|g\|_{C^{k+j}(\mathbb{R}^d_+; X)} \|\phi\|_{C^{k-j}(\mathbb{R}_+)},
\end{aligned}$$

where the sum for $\alpha_1 > j$ comes from taking $\alpha_1 - j$ derivatives of (5.3.5) and the product rule. Hence, as $\gamma > (k-j)p - 1$, taking n large enough we obtain using (5.3.4) that

$$\|f - g_n\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} \leq \|f - g\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} + \|g - g_n\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} < 2\varepsilon.$$

Now let $\gamma = (k-j)p - 1$ and X be a reflexive Banach space. Note that the sequence $(g_n)_{n\geq 1}$ is bounded in the reflexive space $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$. As a corollary of the Banach–Alaoglu theorem, $(g_n)_{n\geq 1}$ has a weakly convergent subsequence, say $g_{n_\ell} \to \widetilde{g}$ weakly in

$W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ as $\ell \to \infty$. Since $g_n \to f$ in $\mathscr{D}'(\mathbb{R}^d_+; X)$ as $n \to \infty$ as well, we find $\widetilde{g} = f$ by uniqueness of the limit. With the Hahn–Banach separation theorem, this implies that

$$f \in \overline{C^\infty_{c,j}(\overline{\mathbb{R}^d_+}; X)}^{\text{weak}} = \overline{C^\infty_{c,j}(\overline{\mathbb{R}^d_+}; X)}^{\|\cdot\|},$$

where the closures are taken in the weak and norm topology of $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$, respectively. □

Let $\theta \in \mathbb{R}$ and define the pointwise multiplication operator

$$M^\theta : C^\infty_c(\mathbb{R}^d_+; X) \to C^\infty_c(\mathbb{R}^d_+; X) \quad \text{by} \quad M^\theta u(x) = x_1^\theta \cdot u(x), \qquad x \in \mathbb{R}^d_+. \tag{5.3.6}$$

By duality the operator extends to $M^\theta : \mathscr{D}'(\mathbb{R}^d_+; X) \to \mathscr{D}'(\mathbb{R}^d_+; X)$ and $M^{-\theta}$ acts as inverse for M^θ on $\mathscr{D}'(\mathbb{R}^d_+; X)$. Moreover, we write $M := M^1$. We first study the boundedness properties of the multiplication operator M^j with $j \in \mathbb{N}_1$.

Lemma 5.3.5. *Let $p \in (1,\infty)$, $j \in \mathbb{N}_1$, $k \in \mathbb{N}_0$ and let X be a Banach space.*

(i) If $\gamma \in (jp-1, \infty)$, then

$$M^j : W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to W^{k,p}(\mathbb{R}^d_+, w_{\gamma - jp}; X) \quad \textit{is bounded.}$$

(ii) If $\gamma \in (-1,\infty) \setminus \{\ell p - 1 : \ell \in \mathbb{N}_1\}$, then

$$M^j : W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) \to W^{k,p}_0(\mathbb{R}^d_+, w_{\gamma - jp}; X) \quad \textit{is an isomorphism.}$$

Proof. It suffices to consider $\mathbb{R}_+$ instead of $\mathbb{R}^d_+$, since derivatives with respect to x_i with $i \in \{2, \dots, d\}$ commute with M^j. For (i) note that by the product rule

$$(M^j u)^{(i)} = \sum_{n=0}^{\min\{i,j\}} c_{n,i,j} M^{j-n} u^{(i-n)}, \qquad i \in \{0, \dots k\}.$$

Hence, by Hardy's inequality (Lemma 2.2.2 using $\gamma > jp - 1$)

$$\begin{aligned} \|M^j u\|_{W^{k,p}(\mathbb{R}_+, w_{\gamma-jp}; X)} &\lesssim \sum_{i=0}^{k} \sum_{n=0}^{\min\{i,j\}} \|M^{j-n} u^{(i-n)}\|_{L^p(\mathbb{R}_+, w_{\gamma-jp}; X)} \\ &= \sum_{i=0}^{k} \sum_{n=0}^{\min\{i,j\}} \|u^{(i-n)}\|_{L^p(\mathbb{R}_+, w_{\gamma-np}; X)} \lesssim \|u\|_{W^{k,p}(\mathbb{R}_+, w_\gamma; X)}. \end{aligned}$$

For (ii), by density of $C^\infty_c(\mathbb{R}_+; X)$ in $W^{k,p}_0(\mathbb{R}_+, w_\gamma; X)$ (see [190, Proposition 3.8]), we have that

$$M^j : W^{k,p}_0(\mathbb{R}_+, w_\gamma; X) \to W^{k,p}_0(\mathbb{R}_+, w_{\gamma - jp}; X)$$

is bounded. It remains to show that the inverse is also bounded, where by density it suffices to consider $u \in C^\infty_c(\mathbb{R}_+; X)$. Again by the product rule

$$(M^{-j} u)^{(i)} = \sum_{n=0}^{i} c_{n,i,j} M^{-j-n} u^{(i-n)},$$

so that by Hardy's inequality

$$\|M^{-j}u\|_{W^{k,p}(\mathbb{R}_+,w_\gamma;X)} \lesssim \sum_{i=0}^{k}\sum_{n=0}^{i} \|M^{-j-n}u^{(i-n)}\|_{L^p(\mathbb{R}_+,w_\gamma;X)} \lesssim \|u\|_{W^{k,p}(\mathbb{R}_+,w_{\gamma-jp};X)}. \qquad \square$$

Analogous to Lemma 5.3.5 we obtain the following result for M^θ with $\theta > 0$.

Lemma 5.3.6. *Let $p \in (1,\infty)$, $\theta > 0$, $k \in \mathbb{N}_0$ and let X be a Banach space. Moreover, let $\gamma, \gamma - \theta p \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$. Then*

$$M^\theta : W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \to W_0^{k,p}(\mathbb{R}^d_+, w_{\gamma-\theta p}; X) \quad \text{is an isomorphism.}$$

Proof. As in the proof of Lemma 5.3.5 it suffices to consider $\mathbb{R}_+$ and $u \in C_c^\infty(\mathbb{R}_+;X)$. The required estimates follow from the formula

$$(M^{\pm\theta}u)^{(i)} = \sum_{n=0}^{i} c_{n,i,\theta} M^{\pm\theta-n} u^{(i-n)}, \qquad i \in \{0,\dots,k\},$$

and Hardy's inequality (Lemma 2.2.2). $\square$

Using Lemma 5.3.5 we obtain a useful characterisation for induction arguments.

Lemma 5.3.7. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_1$, $\gamma \in ((k-1)p-1,\infty) \setminus \{jp-1 : j \geq k\}$ and let X be a Banach space. Then*

$$\|f\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)} \eqsim \sum_{|\alpha|\leq 1} \|M\partial^\alpha f\|_{W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma-p};X)}, \qquad f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X),$$

where the constant only depends on p, k, γ, d and X.

Proof. From Lemma 5.3.5 we have that

$$M : W_0^{k-1,p}(\mathbb{R}^d_+, w_\gamma; X) \to W_0^{k-1,p}(\mathbb{R}^d_+, w_{\gamma-p}; X)$$

is an isomorphism and hence for $g \in W_0^{k-1,p}(\mathbb{R}^d_+, w_\gamma; X)$ it holds

$$\begin{aligned} \|Mg\|_{W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma-p};X)} &\lesssim \|g\|_{W^{k-1,p}(\mathbb{R}^d_+,w_\gamma;X)} \quad \text{and} \\ \|g\|_{W^{k-1,p}(\mathbb{R}^d_+,w_\gamma;X)} &= \|M^{-1}Mg\|_{W^{k-1,p}(\mathbb{R}^d_+,w_\gamma;X)} \lesssim \|Mg\|_{W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma-p};X)}. \end{aligned}$$

Therefore, Remark 5.3.1 gives that for $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$, we have

$$\|f\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)} \eqsim \sum_{|\alpha|\leq 1} \|\partial^\alpha f\|_{W^{k-1,p}(\mathbb{R}^d_+,w_\gamma;X)} \eqsim \sum_{|\alpha|\leq 1} \|M\partial^\alpha f\|_{W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma-p};X)}. \qquad \square$$

We close this section with a complex interpolation result in the parameter γ, which will only play a role in Section 5.6.2 to improve the estimate on the growth of the heat semigroup. Complex interpolation of L^p-spaces with a change of measure dates back to Stein and Weiss [258, 259]. Related results on weighted Sobolev spaces can be found in [53, 133, 194].

Proposition 5.3.8. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $-1 < \gamma_0 < \gamma < \gamma_1 < \infty$ be such that $\gamma = (1-\theta)\gamma_0 + \theta\gamma_1 > kp - 1$ for some $\theta \in (0,1)$. Then*

$$\big[W^{k,p}(\mathbb{R}^d_+, w_{\gamma_0}; X), W^{k,p}(\mathbb{R}^d_+, w_{\gamma_1}; X)\big]_\theta = W^{k,p}(\mathbb{R}^d_+, w_\gamma; X).$$

Proof. For $k = 0$ the result is a special case of the Stein–Weiss theorem, see, e.g., [126, Theorem 14.3.1]. From now on assume that $k \in \mathbb{N}_1$ and we first prove the inclusion "$\hookrightarrow$". Let $N := \sum_{|\alpha|\le k} 1$ and define the operator

$$T : W^{k,p}(\mathbb{R}^d_+, w_{\gamma_0}; X) + W^{k,p}(\mathbb{R}^d_+, w_{\gamma_1}; X) \to L^p(\mathbb{R}^d_+, w_{\gamma_0}; X^N) + L^p(\mathbb{R}^d_+, w_{\gamma_1}; X^N),$$

by $Tf := (\partial^\alpha f)_{|\alpha|\le k}$. It is straightforward to verify that for $j \in \{0,1\}$ the mapping

$$T : W^{k,p}(\mathbb{R}^d_+, w_{\gamma_j}; X) \to L^p(\mathbb{R}^d_+, w_{\gamma_j}; X^N)$$

is bounded. Therefore, by properties of the complex interpolation method and the Stein–Weiss theorem, we have that

$$T : [W^{k,p}(\mathbb{R}^d_+, w_{\gamma_0}; X), W^{k,p}(\mathbb{R}^d_+, w_{\gamma_1}; X)]_\theta \to L^p(\mathbb{R}^d_+, w_\gamma; X^N)$$

is bounded and

$$\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} \lesssim \|Tf\|_{L^p(\mathbb{R}^d_+, w_\gamma; X^N)} \lesssim \|f\|_{[W^{k,p}(\mathbb{R}^d_+, w_{\gamma_0}; X), W^{k,p}(\mathbb{R}^d_+, w_{\gamma_1}; X)]_\theta},$$

where the constant only depends on $p, k, \gamma, \gamma_0, \gamma_1$ and d.

To prove the other inclusion "$\hookleftarrow$", note that $C_c^\infty(\mathbb{R}^d_+; X)$ is dense in $W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ by Lemma 5.3.3. Thus it suffices to show that for all $f \in C_c^\infty(\mathbb{R}^d_+; X)$

$$\|f\|_{[W^{k,p}(\mathbb{R}^d_+, w_{\gamma_0}; X), W^{k,p}(\mathbb{R}^d_+, w_{\gamma_1}; X)]_\theta} \lesssim \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)},$$

where the constant may depend on $p, k, \gamma, \gamma_0, \gamma_1$ and d. For $j \in \{0,1\}$ we define $\beta_j := \gamma + j(\gamma_1 - \gamma_0)$. Let $\mathbb{S} := \{s \in \mathbb{C} : 0 < \operatorname{Re} s < 1\}$ and for $z \in \overline{\mathbb{S}}$ define $T(z)\colon C_c^\infty(\mathbb{R}^d_+; X) \to W^{k,p}(\mathbb{R}^d_+, w_{\beta_0}; X) + W^{k,p}(\mathbb{R}^d_+, w_{\beta_1}; X)$ by

$$T(z)f(x) := e^{z^2-\theta^2} \cdot x_1^{-\frac{z}{p}(\gamma_1-\gamma_0)} \cdot f(x), \qquad x \in \mathbb{R}^d_+.$$

Then $T(\cdot)f$ is bounded and continuous on $\overline{\mathbb{S}}$ and analytic on $\mathbb{S}$ for all $f \in C_c^\infty(\mathbb{R}^d_+; X)$. Let $t \in \mathbb{R}$ and $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ with $|\alpha| \le k$. By the product rule and Hardy's inequality (Corollary 5.3.2 using that $\gamma > kp - 1$), we have

$$\begin{aligned}
\big\|\partial^\alpha T(j+\mathrm{i}t)f\big\|_{L^p(\mathbb{R}^d_+, w_{\beta_j}; X)} &\le e\|\partial^\alpha f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \\
&\quad + \sum_{n=0}^{\alpha_1-1} c_{n,\alpha_1} e^{1-t^2} \prod_{m=1}^{\alpha_1-n} \Big|\tfrac{1+|t|}{p}|\gamma_1-\gamma_0| - m + 1\Big| \|M^{n-\alpha_1}\partial_1^n \partial^{\widetilde{\alpha}} f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \\
&\lesssim \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)}
\end{aligned}$$

and therefore for $j \in \{0,1\}$

$$\sup_{t\in\mathbb{R}} \big\|T(j+\mathrm{i}t)f\big\|_{W^{k,p}(\mathbb{R}^d_+, w_{\beta_j}; X)} \lesssim \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)}.$$

Using that $C_c^\infty(\mathbb{R}^d_+;X)$ is dense in $W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)$, it follows by Stein interpolation [268, Theorem 2.1] that

$$\big\| T(\theta) f \big\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\beta_0};X),W^{k,p}(\mathbb{R}^d_+,w_{\beta_1};X)]_\theta} \lesssim \|f\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)}.$$

It remains to show that for $f \in C_c^\infty(\mathbb{R}^d_+;X)$ we have

$$\|f\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\gamma_0};X),W^{k,p}(\mathbb{R}^d_+,w_{\gamma_1};X)]_\theta} \lesssim \|T(\theta)f\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\beta_0};X),W^{k,p}(\mathbb{R}^d_+,w_{\beta_1};X)]_\theta}. \tag{5.3.7}$$

To this end, let $T^{-1}(\theta): C_c^\infty(\mathbb{R}^d_+;X) \to C_c^\infty(\mathbb{R}^d_+;X)$ be defined by

$$T^{-1}(\theta)g(x) := x_1^{\frac{\theta}{p}(\gamma_1-\gamma_0)} \cdot g(x), \qquad x \in \mathbb{R}^d_+.$$

Then we claim that for $g \in C_c^\infty(\mathbb{R}^d_+;X)$ we have

$$\|T^{-1}(\theta)g\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\gamma_0};X),W^{k,p}(\mathbb{R}^d_+,w_{\gamma_1};X)]_\theta} \lesssim \|g\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\beta_0};X),W^{k,p}(\mathbb{R}^d_+,w_{\beta_1};X)]_\theta}.$$

Applying this to $g := T(\theta)f \in C_c^\infty(\mathbb{R}^d_+;X)$ proves (5.3.7), which in turn proves the proposition.

To prove the claim, note that by Lemma 5.3.3 (using that $\beta_j > \gamma > kp-1$) and properties of the complex interpolation method, it suffices to prove for $g \in C_c^\infty(\mathbb{R}^d_+;X)$ and $j \in \{0,1\}$

$$\|T^{-1}(\theta)g\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma_j};X)} \lesssim \|g\|_{W^{k,p}(\mathbb{R}^d_+,w_{\beta_j};X)}.$$

Note that $\gamma_j + \theta(\gamma_1-\gamma_0) = \beta_j$, so for any $|\alpha| \le k$ we obtain by Hardy's inequality

$$\begin{aligned}\|\partial^\alpha T^{-1}(\theta)g\|_{L^p(\mathbb{R}^d_+,w_{\gamma_j},X)} &\lesssim \Big(\|\partial^\alpha g\|_{L^p(\mathbb{R}^d_+,w_{\beta_j},X)} + \sum_{n=0}^{\alpha_1-1} \|M^{n-\alpha_1}\partial_1^n \partial^{\tilde\alpha} g\|_{L^p(\mathbb{R}^d_+,w_{\beta_j},X)}\Big)\\ &\lesssim \|g\|_{W^{k,p}(\mathbb{R}^d_+,w_{\beta_j};X)}.\end{aligned}$$

This proves the claim and finishes the proof. □

5.4. The Laplacian on lower-order weighted Sobolev spaces

We first note some properties of the Dirichlet and Neumann Laplacian on the half-space in some known and easy cases. That is, we collect the results for the Dirichlet Laplacian on L^p from [190] in Section 5.4.1. Moreover, with a similar reflection technique as in [190], we derive boundedness of the H^∞-calculus for the Neumann Laplacian in the case of A_p weights in Section 5.4.2. Finally, in Section 5.4.3 we consider the weak setting for the Dirichlet Laplacian corresponding to $k=-1$ in Theorem 5.1.1.

Throughout this chapter the Dirichlet and Neumann Laplacian will be defined as follows.

Definition 5.4.1. Let $p \in (1,\infty)$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$, $k \in \mathbb{N}_0 \cup \{-1\}$ be such that $\gamma + kp > -1$ and let X be a Banach space. The Dirichlet and Neumann Laplacian on $\mathbb{R}^d_+$ are defined as follows.

(i) The *Dirichlet Laplacian* $\Delta_{\rm Dir}$ *on* $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is defined by

$$\Delta_{\rm Dir} u := \sum_{j=1}^{d} \partial_j^2 u \quad \text{with} \quad D(\Delta_{\rm Dir}) := W^{k+2,p}_{\rm Dir}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

For $k = -1$ we elaborate on the definition of $W^{-1,p}(\mathbb{R}^d_+, w_\gamma; X)$ with $\gamma \in (-1, p-1)$ in Section 5.4.3.

(ii) The *Neumann Laplacian* $\Delta_{\rm Neu}$ *on* $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is defined by

$$\Delta_{\rm Neu} u := \sum_{j=1}^{d} \partial_j^2 u \quad \text{with} \quad D(\Delta_{\rm Neu}) := W^{k+3,p}_{\rm Neu}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

Moreover, recall that the semigroups $T_{\rm Dir}$ and $T_{\rm Neu}$ are as defined in (5.1.2).

5.4.1. The strong setting for the Dirichlet Laplacian

The H^∞-calculus for the Dirichlet Laplacian on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ with $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ is already obtained in [190]. We will use this theorem (see Theorem 5.4.2 below) in Sections 5.5 and 5.7 as the basis for an induction argument to show that the Dirichlet Laplacian is sectorial and has a bounded H^∞-calculus on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ with $k \geq 1$ as well.

Theorem 5.4.2 ([190, Theorems 4.1 & 5.7])**.** *Let* $p \in (1, \infty)$, $w \in A_p(\mathbb{R}^d_+)$ *or* $w = w_\gamma$ *with* $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ *and let* X *be a* UMD *Banach space. Let* $\Delta_{\rm Dir}$ *on* $L^p(\mathbb{R}^d_+, w; X)$ *be as in Definition 5.4.1 with* $k = 0$ *and let* $T_{\rm Dir}$ *on* $L^p(\mathbb{R}^d_+, w; X)$ *be as in* (5.1.2)*. Then, the following assertions hold for all* $\lambda \geq 0$*:*

(i) $\lambda - \Delta_{\rm Dir}$ *is a sectorial operator of angle* $\omega(\lambda - \Delta_{\rm Dir}) = 0$*,*

(ii) $(T_{\rm Dir}(z))_{z \in \Sigma_\sigma}$ *with* $\sigma \in (0, \frac{\pi}{2})$ *is a bounded analytic* C_0*-semigroup on* $L^p(\mathbb{R}^d_+, w; X)$ *which is generated by* $\Delta_{\rm Dir}$*,*

(iii) $\lambda - \Delta_{\rm Dir}$ *has a bounded* H^∞*-calculus of angle* $\omega_{H^\infty}(\lambda - \Delta_{\rm Dir}) = 0$*.*

Remark 5.4.3 (The case $\gamma = p-1$). We will not consider the special case $\gamma = p-1$ since Hardy's inequality fails in this case. Nonetheless, Theorem 5.4.2 remains true if one considers $\Delta_{\rm Dir}$ on $L^p(\mathbb{R}^d_+, w_{p-1}; X)$ with the different domain

$$D(\Delta_{\rm Dir}) := \overline{\big\{ f \in C_c^\infty(\overline{\mathbb{R}^d_+}; X) : f|_{\partial \mathbb{R}^d_+} = 0 \big\}}^{W^{2,p}(\mathbb{R}^d_+, w_{p-1}; X)}.$$

This can be proved using [190, Propositions 5.1 & 5.3] which do hold for $\gamma = p-1$, and interpolation (Proposition 5.3.8).

The proof of Theorem 5.4.2 for $w \in A_p(\mathbb{R}^d_+)$, which includes w_γ with $\gamma \in (-1, p-1)$, goes via an odd reflection to $\mathbb{R}^d$ and uses the H^∞-calculus of Δ on $L^p(\mathbb{R}^d, w; X)$, see [190, Theorem 4.1]. Since the kernel of the Dirichlet heat semigroup has a zero of order one at the boundary, the range for γ can be extended.

5.4.2. The Neumann Laplacian in the A_p setting

Let $p \in (1,\infty)$ and $w \in A_p(\mathbb{R}^d_+)$ or, equivalently, $w \in A_p(\mathbb{R}^d)$ and w is even. Using a similar reflection technique as in the proof of Theorem 5.4.2, we prove that the Neumann Laplacian has a bounded H^∞-calculus on $L^p(\mathbb{R}^d_+, w; X)$ and $W^{1,p}(\mathbb{R}^d_+, w; X)$. To reflect the Neumann boundary condition we need an even extension in the first variable.

For $f \in L^p(\mathbb{R}^d_+, w; X)$ and $y = (y_1, \widetilde{y}) \in \mathbb{R} \times \mathbb{R}^{d-1}$ we define the odd and even extensions by

$$\begin{aligned} E_{\mathrm{odd}} f(y) &:= f_{\mathrm{odd}}(y) := \mathrm{sign}(y_1) f(|y_1|, \widetilde{y}) \\ E_{\mathrm{even}} f(y) &:= f_{\mathrm{even}}(y) := f(|y_1|, \widetilde{y}). \end{aligned} \tag{5.4.1}$$

Moreover, for $k \in \mathbb{N}_0$ let $W^{k,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X)$ and $W^{k,p}_{\mathrm{even}}(\mathbb{R}^d, w; X)$ be the closed subspaces of all odd and even functions in $W^{k,p}(\mathbb{R}^d, w; X)$, respectively.

We have the following lemma on odd and even extensions.

Lemma 5.4.4. *Let $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d)$ be even and let X be a Banach space. Then*

$$\begin{aligned} E_{\mathrm{odd}} &: W^{k,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X) \to W^{k,p}_{\mathrm{odd}}(\mathbb{R}^d, w; X), && k \in \{0,1,2\}, \\ E_{\mathrm{even}} &: W^{k,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w; X) \to W^{k,p}_{\mathrm{even}}(\mathbb{R}^d, w; X), && k \in \{0,1,2,3\}, \end{aligned}$$

are isomorphisms and

$$\begin{aligned} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w; X)} &\le \|E_{\mathrm{odd}} u\|_{W^{k,p}(\mathbb{R}^d, w; X)} \le 2^{\frac{1}{p}} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w; X)}, && k \in \{0,1,2\}, \\ \|u\|_{W^{k,p}(\mathbb{R}^d_+, w; X)} &\le \|E_{\mathrm{even}} u\|_{W^{k,p}(\mathbb{R}^d, w; X)} \le 2^{\frac{1}{p}} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w; X)} && k \in \{0,1,2,3\}. \end{aligned}$$

Remark 5.4.5. It should be noted that Lemma 5.4.4 does not hold for odd extensions and $k \ge 3$, or even extensions and $k \ge 4$ unless additional boundary conditions are specified (cf. Remark 5.1.3(ii)).

Proof. The statement for the odd extensions is proved in [190, Lemma 4.2], thus it remains to prove the statement for the even extensions. It is straightforward to verify the case $k = 0$. For $k = 1$ the Neumann trace does not exist and therefore, arguing similar as in [190, Lemma 4.2], we obtain for $u \in W^{1,p}(\mathbb{R}^d_+, w; X)$ the formula

$$\partial^\alpha u_{\mathrm{even}}(x) = \mathrm{sign}(x_1)^{\alpha_1} (\partial^\alpha u)(|x_1|, x), \tag{5.4.2}$$

where $\alpha = (\alpha_1, \widetilde{\alpha})$ and $|\alpha| \le 1$. This proves that $u_{\mathrm{even}} \in W^{1,p}(\mathbb{R}^d, w; X)$ and that the required estimates hold. This finishes the case $k = 1$.

Note that $\mathrm{sign}(x_1)$ in (5.4.2) is not differentiable so we cannot differentiate once more. Instead, if $u \in W^{2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w; X)$, then $\partial_1 u \in W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X)$ and we can use that $\partial_1 u_{\mathrm{even}} =$

$(\partial_1 u)_{\mathrm{odd}}$ to obtain

$$\begin{aligned}
&\|u_{\mathrm{even}}\|_{W^{2,p}(\mathbb{R}^d,w;X)} \\
&\quad = \|u_{\mathrm{even}}\|_{W^{1,p}(\mathbb{R}^d,w;X)} + \|(\partial_1 u)_{\mathrm{odd}}\|_{W^{1,p}(\mathbb{R}^d,w;X)} + \sum_{j=2}^{d} \|\partial_j u_{\mathrm{even}}\|_{W^{1,p}(\mathbb{R}^d,w;X)} \\
&\quad \leq 2^{\frac{1}{p}} \big(\|u\|_{W^{1,p}(\mathbb{R}^d_+,w;X)} + \|\partial_1 u\|_{W^{1,p}(\mathbb{R}^d_+,w;X)} + \sum_{j=2}^{d} \|\partial_j u\|_{W^{1,p}(\mathbb{R}^d_+,w;X)} \big) \\
&\quad = 2^{\frac{1}{p}} \|u\|_{W^{2,p}(\mathbb{R}^d_+,w;X)},
\end{aligned}$$

using the estimates for the odd and even extension with $k = 1$. Similarly, for $k = 3$, the estimate follows using the estimates for the odd and even extensions with $k = 2$.

Conversely, let $k \in \{2,3\}$ be fixed and let $u_{\mathrm{even}} \in W^{k,p}(\mathbb{R}^d, w; X)$. Then by [190, Lemma 3.6] there exists a sequence $(u_n)_{n\geq 1} \subseteq C_c^\infty(\mathbb{R}^d; X)$ such that $u_n \to u_{\mathrm{even}}$ in $W^{k,p}(\mathbb{R}^d, w; X)$ as $n \to \infty$. In addition, $u_n(-\cdot,\cdot) \to u_{\mathrm{even}}(-\cdot,\cdot) = u_{\mathrm{even}}(\cdot,\cdot)$ in $W^{k,p}(\mathbb{R}^d, w; X)$. This implies that the sequence $v_n := \frac{1}{2}(u_n + u_n(-\cdot,\cdot))$ satisfies

$$v_n \in C_c^\infty(\mathbb{R}^d; X), \quad (\partial_1 v_n)(0,\cdot) = 0 \quad \text{and} \quad v_n \to u_{\mathrm{even}} \text{ in } W^{k,p}(\mathbb{R}^d, w; X).$$

Continuity of the trace implies that $u = u_{\mathrm{even}}|_{\mathbb{R}^d_+}$ satisfies $\mathrm{Tr}(\partial_1 u) = 0$. This proves that $u \in W^{k,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w; X)$ since the norm estimates are again clear. □

Using the even extensions we can obtain the H^∞-calculus for the Laplacian on $\mathbb{R}^d_+$ with Neumann boundary conditions.

Theorem 5.4.6. *Let $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d_+)$ and let X be a* UMD *Banach space. For $\ell \in \{0,1\}$, define the Neumann Laplacian Δ_{Neu} on $W^{\ell,p}(\mathbb{R}^d_+, w; X)$ with domain $D(\Delta_{\mathrm{Neu}}) = W^{\ell+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w; X)$ and let T_{Neu} on $W^{\ell,p}(\mathbb{R}^d_+, w; X)$ be as in* (5.1.2)*. The following assertions hold for all $\lambda \geq 0$:*

(i) $\lambda - \Delta_{\mathrm{Neu}}$ *is a sectorial operator of angle* $\omega(\lambda - \Delta_{\mathrm{Neu}}) = 0$,

(ii) $(T_{\mathrm{Neu}}(z))_{z\in\Sigma_\sigma}$ *with* $\sigma \in (0, \frac{\pi}{2})$ *is a bounded analytic C_0-semigroup on* $W^{\ell,p}(\mathbb{R}^d_+, w; X)$ *which is generated by* Δ_{Neu},

(iii) $\lambda - \Delta_{\mathrm{Neu}}$ *has a bounded H^∞-calculus of angle* $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) = 0$.

Proof. This proof goes via a reflection argument from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ and using the H^∞-calculus for Δ on $L^p(\mathbb{R}^d, w; X)$ and $W^{1,p}(\mathbb{R}^d, w; X)$, see Lemma 5.2.1. The argument is completely similar to [190, Section 4] for the Dirichlet case if one considers the even extension from Lemma 5.4.4 instead of the odd extension. □

We state two corollaries of Theorems 5.4.2 and 5.4.6 concerning the Dirichlet and Neumann resolvent equation. Similar results on $L^p(\mathbb{R}^d_+, w; \mathbb{C})$ with $w \in A_p(\mathbb{R}^d_+)$ are already contained in [80, 81].

Corollary 5.4.7 ([190], Corollary 4.3 & 5.8)**.** *Let $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d_+)$ or $w = w_\gamma$ with $\gamma \in (-1,2p-1)\setminus\{p-1\}$ and let X be a* UMD *Banach space. Then for all $f \in L^p(\mathbb{R}^d_+,w;X)$ and $\lambda \in \Sigma_{\pi-\omega}$ with $\omega \in (0,\pi)$ there exists a unique $u \in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w;X)$ such that $\lambda u - \Delta_{\mathrm{Dir}} u = f$. Moreover, this solution satisfies*

$$\sum_{|\beta|\le 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{L^p(\mathbb{R}^d_+,w;X)} \le C\|f\|_{L^p(\mathbb{R}^d_+,w;X)},$$

where the constant C only depends on p, w, ω, d and X.

For the Neumann Laplacian we have the following result.

Corollary 5.4.8. *Let $p \in (1,\infty)$, $\ell \in \{0,1\}$, $w \in A_p(\mathbb{R}^d_+)$ and let X be a* UMD *Banach space. Then for all $f \in W^{\ell,p}(\mathbb{R}^d_+,w;X)$ and $\lambda \in \Sigma_{\pi-\omega}$ with $\omega \in (0,\pi)$ there exists a unique $u \in W^{\ell+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+,w;X)$ such that $\lambda u - \Delta_{\mathrm{Neu}} u = f$. Moreover, this solution satisfies*

$$\sum_{|\beta|\le 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{\ell,p}(\mathbb{R}^d_+,w;X)} \le C\|f\|_{W^{\ell,p}(\mathbb{R}^d_+,w;X)},$$

where the constant C only depends on p, ℓ, w, ω, d and X.

Proof. This follows from Theorem 5.4.6 using the same argument as in [190, Corollary 4.3]. □

5.4.3. The weak setting for the Dirichlet Laplacian

In this section, we prove Theorem 5.1.1 with $k = -1$, which corresponds to the weak setting for the Dirichlet Laplacian. This follows from combining the results for the Dirichlet and Neumann Laplacian in the strong setting.

The space $W^{-1,p}(\mathbb{R}^d_+,w;X)$ with $w \in A_p(\mathbb{R}^d_+)$ consists of all $f \in \mathscr{S}'(\mathbb{R}^d_+;X)$ such that

$$f = f_0 + \sum_{j=1}^d \partial_j f_j \quad \text{with} \quad f_0, f_j \in L^p(\mathbb{R}^d_+,w;X) \text{ for all } j \in \{1,\dots,d\} \tag{5.4.3}$$

equipped with the norm

$$\|f\|_{W^{-1,p}(\mathbb{R}^d_+,w;X)} := \inf\Big\{ \sum_{j=0}^d \|f_j\|_{L^p(\mathbb{R}^d_+,w;X)} : (5.4.3) \text{ holds}\Big\}.$$

As usual, the derivatives are understood in the sense of distributions, i.e., (5.4.3) reads

$$f(\varphi) = f_0(\varphi) - \sum_{j=1}^d f_j(\partial_j\varphi), \qquad \varphi \in \mathscr{S}(\mathbb{R}^d_+;X).$$

For $f \in W^{-1,p}(\mathbb{R}^d_+,w;X)$ with $w \in A_p(\mathbb{R}^d_+)$ we define the Dirichlet heat semigroup as

$$(T_{\mathrm{Dir}}(z)f)(\varphi) := f(T_{\mathrm{Dir}}(z)\varphi), \qquad \varphi \in \mathscr{S}(\mathbb{R}^d_+;X). \tag{5.4.4}$$

This definition coincides with the definition for functions $f \in L^p(\mathbb{R}^d_+, w; X)$. Moreover, for $f \in W^{-1,p}(\mathbb{R}^d_+, w; X)$ and $\varphi \in \mathscr{S}(\mathbb{R}^d_+; X)$ it holds that

$$\begin{aligned}(T_{\mathrm{Dir}}(z)f)(\varphi) &= (T_{\mathrm{Dir}}(z)f_0)(\varphi) + (\partial_1 T_{\mathrm{Neu}}(z)f_1)(\varphi) + \sum_{j=2}^d (\partial_j T_{\mathrm{Dir}}(z)f_j)(\varphi) \\ &= (T_{\mathrm{Dir}}(z)f_0)(\varphi) - (T_{\mathrm{Neu}}(z)f_1)(\partial_1\varphi) - \sum_{j=2}^d (T_{\mathrm{Dir}}(z)f_j)(\partial_j\varphi).\end{aligned} \tag{5.4.5}$$

Using Corollaries 5.4.7 and 5.4.8 we can derive elliptic regularity for the Dirichlet Laplacian on $W^{-1,p}(\mathbb{R}^d_+, w; X)$. For a similar result on $\mathbb{R}^d$ without weights, we refer to [166, Section 4.4].

Proposition 5.4.9. *Let $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d_+)$ and let X be a* UMD *Banach space. Then for all $f_j \in L^p(\mathbb{R}^d_+, w; X)$ with $j \in \{0,\dots,d\}$ and $\lambda \in \Sigma_{\pi-\omega}$ with $\omega \in (0,\pi)$, there exists a unique $u \in W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X)$ such that $\lambda u - \Delta_{\mathrm{Dir}} u = f_0 + \sum_{j=1}^d \partial_j f_j$. Moreover, this solution satisfies*

$$\sum_{|\beta|\le 1} |\lambda|^{\frac{1}{2}-\frac{|\beta|}{2}} \|\partial^\beta u\|_{L^p(\mathbb{R}^d_+,w;X)} \le C|\lambda|^{-\frac{1}{2}} \|f_0\|_{L^p(\mathbb{R}^d_+,w;X)} + C\sum_{j=1}^d \|f_j\|_{L^p(\mathbb{R}^d_+,w;X)},$$

where the constant C only depends on p, w, ω, d and X.

Proof. For the existence and the estimate it suffices to consider the equations

$$\begin{aligned} \lambda v_0 - \Delta_{\mathrm{Dir}} v_0 &= f_0, && v_0(0,\cdot) = 0, \\ \lambda v_1 - \Delta_{\mathrm{Neu}} v_1 &= f_1, && (\partial_1 v_1)(0,\cdot) = 0, \\ \lambda v_j - \Delta_{\mathrm{Dir}} v_j &= f_j, && v_j(0,\cdot) = 0 \text{ for } j \in \{2,\dots,d\}. \end{aligned} \tag{5.4.6}$$

Corollaries 5.4.7 and 5.4.8 yield solvability of the above equations and the estimates

$$\sum_{|\beta|\le 1} |\lambda|^{\frac{1}{2}-\frac{|\beta|}{2}} \|\partial^\beta v_0\|_{L^p(\mathbb{R}^d_+,w;X)} \le C|\lambda|^{-\frac{1}{2}} \|f_0\|_{L^p(\mathbb{R}^d_+,w;X)} \tag{5.4.7}$$

and

$$\sum_{1\le|\beta|\le 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta v_j\|_{L^p(\mathbb{R}^d_+,w;X)} \le C\|f_j\|_{L^p(\mathbb{R}^d_+,w;X)}, \quad j \in \{1,\dots,d\}. \tag{5.4.8}$$

Then $u := v_0 + \sum_{j=1}^d \partial_j v_j \in W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X)$ satisfies $\lambda u - \Delta_{\mathrm{Dir}} u = f_0 + \sum_{j=1}^d \partial_j f_j$ and

$$\begin{aligned}\sum_{|\beta|\le 1} |\lambda|^{\frac{1}{2}-\frac{|\beta|}{2}} \|\partial^\beta u\|_{L^p(\mathbb{R}^d_+,w;X)} &\le \sum_{|\beta|\le 1} |\lambda|^{\frac{1}{2}-\frac{|\beta|}{2}} \Big(\|\partial^\beta v_0\|_{L^p(\mathbb{R}^d_+,w;X)} + \sum_{j=1}^d \|\partial^\beta \partial_j v_j\|_{L^p(\mathbb{R}^d_+,w;X)} \Big) \\ &\le C|\lambda|^{-\frac{1}{2}} \|f_0\|_{L^p(\mathbb{R}^d_+,w;X)} + C\sum_{j=1}^d \|f_j\|_{L^p(\mathbb{R}^d_+,w;X)},\end{aligned}$$

using (5.4.7) and (5.4.8). Finally, since $\lambda u - \Delta_{\mathrm{Dir}} u = 0$ has a solution $u \in W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X)$, we obtain by Lemma 5.4.4 that $u_{\mathrm{odd}} \in W^{1,p}(\mathbb{R}^d, w; X) \hookrightarrow \mathscr{S}'(\mathbb{R}^d; X)$ and satisfies $\lambda u_{\mathrm{odd}} - \Delta u_{\mathrm{odd}} = 0$ on $\mathbb{R}^d$. By employing the Fourier transform it follows that $u_{\mathrm{odd}} = 0$ and thus $u = 0$ as well. This proves the uniqueness and finishes the proof. □

Sectoriality and boundedness of the H^∞-calculus for $-\Delta_{\mathrm{Dir}}$ on $W^{-1,p}(\mathbb{R}^d_+, w; X)$ can be derived from the same properties of $-\Delta_{\mathrm{Dir}}$ and $-\Delta_{\mathrm{Neu}}$ on $L^p(\mathbb{R}^d_+, w; X)$. We close this section with the main result for the Dirichlet Laplacian in the weak setting.

Theorem 5.4.10. *Let $p \in (1,\infty)$, $w \in A_p(\mathbb{R}^d_+)$ and let X be a* UMD *Banach space. Define the Dirichlet Laplacian Δ_{Dir} on $W^{-1,p}(\mathbb{R}^d_+, w; X)$ with domain $D(\Delta_{\mathrm{Dir}}) = W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w; X)$ and let T_{Dir} on $W^{-1,p}(\mathbb{R}^d_+, w; X)$ be as in* (5.4.4)*. The following assertions hold for all $\lambda \geq 0$:*

(i) $\lambda - \Delta_{\mathrm{Dir}}$ *is a sectorial operator of angle $\omega(\lambda - \Delta_{\mathrm{Dir}}) = 0$,*

(ii) $(T_{\mathrm{Dir}}(z))_{z\in\Sigma_\sigma}$ *with $\sigma \in (0, \frac{\pi}{2})$ is a bounded analytic C_0-semigroup on $W^{-1,p}(\mathbb{R}^d_+, w; X)$ which is generated by Δ_{Dir},*

(iii) $\lambda - \Delta_{\mathrm{Dir}}$ *has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) = 0$.*

Proof. Let $f \in W^{-1,p}(\mathbb{R}^d_+, w; X)$ such that $f = f_0 + \sum_{j=1}^d \partial_j f_j$ with $f_j \in L^p(\mathbb{R}^d_+, w; X)$ for $j \in \{0, \dots, d\}$. Moreover, let v_j be the solutions to the equations in (5.4.6).

To prove (i), note that by Corollaries 5.4.7 and 5.4.8, we obtain for $z \in \Sigma_{\pi-\omega}$ with $\omega \in (0,\pi)$

$$|z|\|R(z,\Delta_{\mathrm{Dir}})f\|_{W^{-1,p}(\mathbb{R}^d_+,w;X)} \leq |z| \sum_{j=0}^d \|v_j\|_{L^p(\mathbb{R}^d_+,w;X)} \leq C \sum_{j=0}^d \|f_j\|_{L^p(\mathbb{R}^d_+,w;X)}.$$

Taking the infimum over all possible f_j yields that $-\Delta_{\mathrm{Dir}}$ is sectorial of angle 0. Together with [126, Proposition 16.2.1] this proves (i).

We continue with the proof of (ii). By (i) and [125, Theorem G.5.2] we have that Δ_{Dir} generates the bounded analytic C_0-semigroup $(S(z))_{z\in\Sigma_\sigma}$ and

$$S(z)f = \frac{1}{2\pi \mathrm{i}} \int_\Gamma e^{zs} R(s,\Delta_{\mathrm{Dir}}) f \, \mathrm{d}s, \qquad z \in \Sigma_\sigma, \ f \in W^{-1,p}(\mathbb{R}^d_+, w; X), \tag{5.4.9}$$

where Γ is the upwards orientated boundary of $\Sigma_{\sigma'} \setminus \overline{B(0,r)}$ for some $r > 0$ and $\sigma' \in (\frac{\pi}{2} + |\arg z|, \frac{\pi}{2} + \sigma)$.

We show that $S(z) = T_{\mathrm{Dir}}(z)$ on $W^{-1,p}(\mathbb{R}^d_+, w; X)$ for $z \in \Sigma_\sigma$. It follows from (5.4.9), Theorems 5.4.2(ii) and 5.4.6(ii), commuting ∂_1 and the resolvent, and (5.4.5) that

$$\begin{aligned} S(z)f &= \frac{1}{2\pi \mathrm{i}} \int_\Gamma e^{zs} \Big(R(s,\Delta_{\mathrm{Dir}}) f_0 + \partial_1 R(s,\Delta_{\mathrm{Neu}}) f_1 + \sum_{j=2}^d \partial_j R(s,\Delta_{\mathrm{Dir}}) f_j \Big) \, \mathrm{d}s \\ &= T_{\mathrm{Dir}}(z) f_0 + \partial_1 T_{\mathrm{Neu}}(z) f_1 + \sum_{j=2}^d \partial_j T_{\mathrm{Dir}}(z) f_j = T_{\mathrm{Dir}}(z) f, \end{aligned}$$

which completes the proof of (ii).

Finally, to prove (iii) let $\omega \in (0,\pi)$ and $\phi \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$. By Theorems 5.4.2 and 5.4.6, we obtain

$$\begin{aligned} \|\phi(-\Delta_{\mathrm{Dir}})f\|_{W^{-1,p}(\mathbb{R}^d_+,w;X)} &\leq \|\phi(-\Delta_{\mathrm{Dir}})f_0\|_{L^p(\mathbb{R}^d_+,w;X)} + \|\phi(-\Delta_{\mathrm{Neu}})f_1\|_{L^p(\mathbb{R}^d_+,w;X)} \\ &\quad + \sum_{j=2}^d \|\phi(-\Delta_{\mathrm{Dir}})f_j\|_{L^p(\mathbb{R}^d_+,w;X)} \\ &\leq C\|\phi\|_{H^\infty(\Sigma_\omega)} \sum_{j=0}^d \|f_j\|_{L^p(\mathbb{R}^d_+,w;X)}. \end{aligned}$$

Taking the infimum over all possible f_j yields that $-\Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus of angle 0. Together with [126, Proposition 16.2.6] this proves (iii). □

5.5. Sectoriality of the Dirichlet and Neumann Laplacian

In this section we study the sectoriality of $-\Delta_{\mathrm{Dir}}$ and $-\Delta_{\mathrm{Neu}}$ on weighted Sobolev spaces of arbitrary order $k \in \mathbb{N}_0$. The results of this section are summarised in the following theorems.

Theorem 5.5.1 (Sectoriality of $\lambda - \Delta_{\mathrm{Dir}}$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Assume that either*

(i) $\gamma + kp \in (-1, 2p-1)$ *and* $\lambda \geq 0$*, or,*

(ii) $\gamma + kp > 2p-1$ *and* $\lambda > 0$.

Then $\lambda - \Delta_{\mathrm{Dir}}$ is sectorial of angle $\omega(\lambda - \Delta_{\mathrm{Dir}}) = 0$.

Theorem 5.5.2 (Sectoriality of $\lambda - \Delta_{\mathrm{Neu}}$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Assume that either*

(i) $\gamma + kp \in (-1, p-1)$ *and* $\lambda \geq 0$*, or,*

(ii) $\gamma + kp > p-1$ *and* $\lambda > 0$.

Then $\lambda - \Delta_{\mathrm{Neu}}$ is sectorial of angle $\omega(\lambda - \Delta_{\mathrm{Neu}}) = 0$.

Theorems 5.5.1 and 5.5.2 will be proved using elliptic regularity for the corresponding resolvent equations. The elliptic regularity for the Dirichlet and Neumann resolvent equation is shown using an induction argument on k in Sections 5.5.1 and 5.5.2, respectively. Finally, in Section 5.5.3 the elliptic regularity is used to finish the proofs of Theorems 5.5.1 and 5.5.2.

5.5.1. Elliptic regularity for the Dirichlet Laplacian

We start with a preliminary result on the solvability of the resolvent equation if the right hand side is compactly supported. Recall that we defined the Schwartz space on $\mathbb{R}^d_+$ as $\mathscr{S}(\mathbb{R}^d_+; X) := \{u|_{\mathbb{R}^d_+} : u \in \mathscr{S}(\mathbb{R}^d; X)\}$.

Lemma 5.5.3. *Let X be a* UMD *Banach space. Then for all $f \in C_c^\infty(\mathbb{R}^d_+; X)$ and $\lambda \in \Sigma_{\pi-\omega}$ with $\omega \in (0,\pi)$ there exists a unique $u \in \mathscr{S}(\mathbb{R}^d_+; X)$ such that*

$$\lambda u - \Delta u = f, \quad u(0,\cdot) = 0.$$

Proof. Let f_{odd} be the odd extension of f as in (5.4.1). Then $f_{\mathrm{odd}} \in C_c^\infty(\mathbb{R}^d; X) \subseteq \mathscr{S}(\mathbb{R}^d; X)$ and taking the Fourier transform of the resolvent equation $\lambda u - \Delta u = f_{\mathrm{odd}}$ on $\mathbb{R}^d$ yields a solution $u_{\mathrm{odd}} \in \mathscr{S}(\mathbb{R}^d; X)$ given by

$$u_{\mathrm{odd}}(x_1, \widetilde{x}) = \Big(\mathscr{F}^{-1}\big[\xi \mapsto \tfrac{(\mathscr{F} f_{\mathrm{odd}})(\xi)}{\lambda + |\xi|^2}\big]\Big)(x_1, \widetilde{x}), \quad x, \xi \in \mathbb{R}^d,\ \lambda \in \Sigma_{\pi-\omega}.$$

Note that $-u_{\mathrm{odd}}(-x_1,\widetilde{x})$ also solves the resolvent equation on $\mathbb{R}^d$. By uniqueness of the solution we obtain $u_{\mathrm{odd}}(x_1,\widetilde{x}) = -u_{\mathrm{odd}}(-x_1,\widetilde{x})$ and therefore $u_{\mathrm{odd}}(0,\widetilde{x}) = 0$. Thus the restriction $u := u_{\mathrm{odd}}|_{\mathbb{R}^d_+}$ solves $\lambda u - \Delta u = f$ on $\mathbb{R}^d_+$ with $u(0,\cdot) = 0$. The uniqueness follows from Corollary 5.4.7. □

Recall that the pointwise multiplication operator M is defined in (5.3.6). By $[A_1, A_2] = A_1A_2 - A_2A_1$ we denote the commutator of two operators A_1 and A_2. For $a \in \mathbb{R}$ we use the notation $(a)_+ = a$ if $a \geq 0$ and $(a)_+ = 0$ otherwise. Moreover, note that for all $u \in \mathscr{S}(\mathbb{R}^d_+; X)$ we have the commutation relation

$$[M^j\partial^\alpha, \Delta]u = -j(j-1)M^{j-2}\partial^\alpha u - 2jM^{j-1}\partial_1\partial^\alpha u, \qquad j \in \mathbb{N}_0,\ \alpha \in \mathbb{N}_0^d. \tag{5.5.1}$$

The next proposition provides the key argument for proving the sectoriality of the Dirichlet Laplacian on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ for $k \geq 1$. A version of the proposition below with improved growth $g_{k,\gamma}$ on the right hand side of the estimate will be obtained in Section 5.6.2.

Proposition 5.5.4. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$, $\omega \in (0,\pi)$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then for all $f \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and $\lambda \in \Sigma_{\pi-\omega}$, there exists a unique $u \in W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $\lambda u - \Delta_{\mathrm{Dir}} u = f$. Moreover, this solution satisfies*

$$\sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C g_{k,\gamma}(\lambda) \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}, \tag{5.5.2}$$

where

$$g_{k,\gamma}(\lambda) := \begin{cases} 1 + |\lambda|^{-\frac{(k-1)_+}{2}} & \text{if } \gamma \in (-1, p-1) \\ 1 + |\lambda|^{-\frac{k}{2}} & \text{if } \gamma \in (p-1, 2p-1) \end{cases}, \tag{5.5.3}$$

and the constant C only depends on p, k, γ, ω, d and X.

Proof. For any $k \in \mathbb{N}_1$ the uniqueness of $u \in W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is clear from Corollary 5.4.7 and Hardy's inequality (Corollary 5.3.2). The proof for the existence and the estimate goes by induction on $k \geq 0$. The case $k = 0$ is stated in Corollary 5.4.7. Assume that the statement of the proposition holds for a fixed $k \in \mathbb{N}_0$ and for all $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. It remains to prove the statement for $k+1$. Throughout the proof, C denotes a constant only depending on p, k, γ, ω, d and X, which may change from line to line.

First, take $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$. Lemma 5.5.3 implies that

$$\lambda u - \Delta_{\mathrm{Dir}} u = f, \qquad u(0,\cdot) = 0, \tag{5.5.4}$$

has a unique solution $u \in \mathscr{S}(\mathbb{R}^d_+; X)$. For $|\alpha| \leq k+1$, let $v_\alpha = M^{k+1}\partial^\alpha u$. Then we have $v_\alpha(0,\cdot) = 0$ and by (5.5.1) and (5.5.4)

$$\lambda v_\alpha - \Delta_{\mathrm{Dir}} v_\alpha = M^{k+1}\partial^\alpha f - k(k+1)M^{k-1}\partial^\alpha u - 2(k+1)M^k\partial_1\partial^\alpha u. \tag{5.5.5}$$

Since $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ and $u \in \mathscr{S}(\mathbb{R}^d_+; X)$, we find that the right-hand side of (5.5.5) is an element of $L^p(\mathbb{R}^d_+, w_\gamma; X)$. Therefore, the case $k = 0$ (see Corollary 5.4.7) implies that

$v_\alpha \in W^{2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X)$ and by Hardy's inequality (if $k \geq 1$) it holds that

$$\begin{aligned}
&\sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta v_\alpha\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \\
&\qquad \leq C\|M^{k+1}\partial^\alpha f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \\
&\qquad\quad + C\big(k\|M^{k-1}\partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} + \|M^k \partial_1 \partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)}\big) \\
&\qquad \leq C\|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)} + C\sum_{1\leq|\delta|\leq 2} \|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}.
\end{aligned} \tag{5.5.6}$$

If $\gamma \in (-1, p-1)$, then note that $\gamma + (k+1)p \in ((k+1)p-1, (k+2)p-1)$. With Hardy's inequality (Lemma 2.2.2) and the induction hypothesis twice (once with γ replaced by $\gamma + p$), we find

$$\begin{aligned}
&\sum_{1\leq|\delta|\leq 2} \|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\quad \leq C\sum_{|\delta|=1}\sum_{|\beta|\leq k} \|\partial_1\partial^\beta\partial^\delta u\|_{L^p(\mathbb{R}^d_+, w_{\gamma+p+kp}; X)} + \sum_{|\delta|=2} \|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\quad \leq C\sum_{|\delta|=2} \big(\|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+p+kp}; X)} + \|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}\big) \\
&\quad \leq C\big(1+|\lambda|^{-\frac{k}{2}}\big)\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+p+kp}; X)} \\
&\qquad\quad + C\big(1+|\lambda|^{-\frac{(k-1)_+}{2}}\big)\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\quad \leq C\big(1+|\lambda|^{-\frac{k}{2}}\big)\|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)},
\end{aligned} \tag{5.5.7}$$

using Hardy's inequality (Corollary 5.3.2) once more in the last estimate. If $\gamma \in (p-1, 2p-1)$, then the induction hypothesis and Hardy's inequality yield

$$\begin{aligned}
\sum_{1\leq|\delta|\leq 2} \|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &\leq C\big(1+|\lambda|^{-\frac{1}{2}}\big) g_{k,\gamma}(\lambda)\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\leq C\big(1+|\lambda|^{-\frac{k+1}{2}}\big)\|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)}.
\end{aligned} \tag{5.5.8}$$

From the above estimates, the definition of v_α and the fact that $|\alpha| \leq k+1$ was arbitrary, we further estimate (5.5.6) as

$$\sum_{|\beta|\leq 2}\sum_{|\alpha|\leq k+1} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta M^{k+1}\partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \leq C g_{k+1,\gamma}(\lambda)\|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)}. \tag{5.5.9}$$

From (5.5.9) we obtain

$$\begin{aligned}
&\sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)} \\
&\quad \leq \sum_{|\beta|\leq 2}\sum_{|\alpha|\leq k+1} |\lambda|^{1-\frac{|\beta|}{2}} \Big(\|\partial^\beta M^{k+1}\partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} + \|[M^{k+1}, \partial_1^{|\beta|}]\partial^\alpha u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)}\Big) \\
&\quad \leq C g_{k+1,\gamma}(\lambda)\|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)} + C\sum_{j\in\{1,2\}}\sum_{j-1\leq|\delta|\leq j} |\lambda|^{1-\frac{j}{2}} \|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},
\end{aligned}$$

where in the last step we also used Hardy's inequality and

$$[M^{k+1},\partial_1]\partial^\alpha u = -(k+1)M^k\partial^\alpha u,$$
$$[M^{k+1},\partial_1^2]\partial^\alpha u = -k(k+1)M^{k-1}\partial^\alpha u - 2(k+1)M^k\partial_1\partial^\alpha u.$$

To estimate the last sums, consider the cases $\gamma\in(-1,p-1)$ and $\gamma\in(p-1,2p-1)$ separately. Then arguing similar as in (5.5.7) and (5.5.8), respectively, gives

$$\sum_{j\in\{1,2\}}\sum_{j-1\le|\delta|\le j}|\lambda|^{1-\frac{j}{2}}\|\partial^\delta u\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} \le Cg_{k+1,\gamma}(\lambda)\|f\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)}. \quad (5.5.10)$$

This proves the required estimate for $f\in C_c^\infty(\mathbb{R}^d_+;X)$.

To conclude, let $f\in W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)$. Then by Lemma 5.3.3 there exists a sequence $(f_n)_{n\ge1}\subseteq C_c^\infty(\mathbb{R}^d_+;X)$ such that

$$f_n\to f \quad \text{in } W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X) \quad \text{as } n\to\infty.$$

Every f_n defines a $u_n\in W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)$ such that $\lambda u_n-\Delta_{\mathrm{Dir}}u_n=f_n$ and

$$\|u_n-u_m\|_{W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)}\lesssim\|f_n-f_m\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)}\to0 \quad \text{as } m,n\to\infty.$$

Completeness of $W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)$ implies that u_n converges to some

$$u\in W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X).$$

Moreover,

$$\|(\lambda-\Delta_{\mathrm{Dir}})(u_n-u)\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)}\lesssim\|u_n-u\|_{W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)}\to0,$$

as $n\to\infty$. Therefore, $(\lambda-\Delta_{\mathrm{Dir}})u=\lim_{n\to\infty}(\lambda-\Delta_{\mathrm{Dir}})u_n=\lim_{n\to\infty}f_n=f$. Thus, for any $f\in W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)$ there exists a $u\in W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+,w_{\gamma+(k+1)p};X)$ solving $\lambda u-\Delta_{\mathrm{Dir}}u=f$ and the required estimate holds. This finishes the proof. □

5.5.2. Elliptic regularity for the Neumann Laplacian

We continue with elliptic regularity for the Laplace operator with Neumann boundary conditions. Analogous to Lemma 5.5.3 the Neumann resolvent equation can easily be solved if the right hand side f is in $C_{c,1}^\infty(\overline{\mathbb{R}^d_+};X)$, i.e., $\partial^\alpha f\in C_c^\infty(\mathbb{R}^d_+;X)$ for $\alpha=(\alpha_1,\widetilde{\alpha})\in\mathbb{N}_1\times\mathbb{N}_0^{d-1}$, see (5.3.3).

Lemma 5.5.5. *Let X be a* UMD *Banach space. Then for all $f\in C_{c,1}^\infty(\overline{\mathbb{R}^d_+};X)$ and $\lambda\in\Sigma_{\pi-\omega}$ with $\omega\in(0,\pi)$ there exists a unique $u\in\mathscr{S}(\mathbb{R}^d_+;X)$ such that*

$$\lambda u-\Delta u=f,\quad(\partial_1u)(0,\cdot)=0.$$

Proof. This follows similarly as in Lemma 5.5.3 using an even extension from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ instead of an odd extension. Note that this extension satisfies $f_{\mathrm{even}}\in C_c^\infty(\mathbb{R}^d;X)\subseteq\mathscr{S}(\mathbb{R}^d;X)$. □

The following proposition contains the key result for proving the sectoriality of the Neumann Laplacian. A version of the proposition below with improved growth $g_{k+1,\gamma}$ on the right hand side of the estimate will be stated in Section 5.6.2.

Proposition 5.5.6. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$ such that $\gamma + kp > -1$, $\omega \in (0,\pi)$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then for all $f \in W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and $\lambda \in \Sigma_{\pi-\omega}$, there exists a unique $u \in W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $\lambda u - \Delta_{\mathrm{Neu}} u = f$. Moreover, this solution satisfies*

$$\sum_{|\beta|\le 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \le C g_{k+1,\gamma}(\lambda) \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where (cf. (5.5.3)*)*

$$g_{k+1,\gamma}(\lambda) = \begin{cases} 1 + |\lambda|^{-\frac{k}{2}} & \text{if } \gamma \in (-1, p-1) \\ 1 + |\lambda|^{-\frac{k+1}{2}} & \text{if } \gamma \in (p-1, 2p-1) \end{cases},$$

and the constant C only depends on p, k, γ, ω, d and X.

Proof. Throughout the proof, C denotes a constant only depending on p, k, γ, ω, d and X, which may change from line to line. The proof proceeds in two steps. We first prove the case $\gamma \in (p-1, 2p-1)$ and $k \ge -1$ with induction similar to the proof of Proposition 5.5.4. Secondly, we prove the case $\gamma \in (-1, p-1)$ and $k \ge 0$ using the previous step.

Step 1: the case $\gamma \in (p-1, 2p-1)$ and $k \ge -1$. Let $\gamma \in (p-1, 2p-1)$. The case $k = -1$ is stated in Corollary 5.4.8. Assume that the statement of the proposition holds for all $\gamma \in (p-1, 2p-1)$ and a fixed $k \in \mathbb{N}_0 \cup \{-1\}$. We prove the statement of the proposition for $k+1$.

First of all, note that uniqueness of $u \in W^{k+4,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)$ follows from Corollary 5.4.8 and Hardy's inequality (Corollary 5.3.2). Take $f \in C^\infty_c(\mathbb{R}^d_+; X)$ and let $u \in \mathscr{S}(\mathbb{R}^d_+; X)$ be the unique solution to

$$\lambda u - \Delta_{\mathrm{Neu}} u = f, \qquad (\partial_1 u)(0,\cdot) = 0, \tag{5.5.11}$$

see Lemma 5.5.5. For $|\alpha| \le k+2$ define $v_\alpha = M^{k+2}\partial^\alpha u$. Then v_α satisfies the Dirichlet boundary condition $v_\alpha(0,\cdot) = 0$ and the equation

$$\lambda v_\alpha - \Delta_{\mathrm{Dir}} v_\alpha = M^{k+2}\partial^\alpha f - (k+1)(k+2)M^k \partial^\alpha u - 2(k+2)M^{k+1}\partial_1 \partial^\alpha u.$$

Applying Corollary 5.4.7 with $\gamma - p \in (-1, p-1)$ yields the estimate (5.5.6) with k and γ replaced by $k+1$ and $\gamma - p$, respectively. The rest of the argument is analogous to the Dirichlet case, see (5.5.8)-(5.5.10). For $f \in C^\infty_c(\mathbb{R}^d_+; X)$ we obtain the estimate

$$\sum_{|\beta|\le 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)} \le C\big(1 + |\lambda|^{-\frac{k+2}{2}}\big) \|f\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)}.$$

By a density argument similar to the one in the proof of Proposition 5.5.4 the induction is finished. This proves the statement of the proposition for $k \in \mathbb{N}_0 \cup \{-1\}$ and

$\gamma \in (p-1, 2p-1)$.

Step 2: the case $\gamma \in (-1, p-1)$ and $k \geq 0$. Let $\gamma \in (-1, p-1)$ and note that for $k = 0$ the result is already contained in Theorem 5.4.6. Therefore, we let $k \in \mathbb{N}_1$ from now on. Then uniqueness of $u \in W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ follows from the uniqueness in $W^{3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_\gamma; X)$ (case $k = 0$) and Hardy's inequality.

To continue, take $f \in C^\infty_{\mathrm{c},1}(\overline{\mathbb{R}^d_+}; X)$. Lemma 5.5.5 implies that the resolvent equation (5.5.11) has a unique solution $u \in \mathscr{S}(\mathbb{R}^d_+; X)$. Let $j \in \{1, \dots, d\}$ and define $v_j = \partial_j u$. If $j = 1$, then v_1 satisfies

$$\lambda v_1 - \Delta_{\mathrm{Dir}} v_1 = \partial_1 f, \qquad v_1(0, \cdot) = 0.$$

If $j \in \{2, \dots, d\}$, then v_j satisfies

$$\lambda v_j - \Delta_{\mathrm{Neu}} v_j = \partial_j f, \qquad (\partial_1 v_j)(0, \cdot) = 0.$$

Applying Step 1 for $k-1$ and $\gamma + p \in (p-1, 2p-1)$ to estimate v_j for $j \in \{2, \dots, d\}$ and Proposition 5.5.4 to estimate v_1, yields

$$\begin{aligned}
&\sum_{|\beta| \leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\quad = \sum_{|\beta| \leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \big(\|\partial^\beta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} + \sum_{j=1}^d \|\partial^\beta v_j\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \big) \\
&\quad \leq C\big(1 + |\lambda|^{-\frac{k}{2}}\big) \big(\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} + \sum_{j=1}^d \|\partial_j f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \big) \\
&\quad \leq C\big(1 + |\lambda|^{-\frac{k}{2}}\big) \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}.
\end{aligned}$$

From Lemma 5.3.3 we have that $C^\infty_{\mathrm{c},1}(\overline{\mathbb{R}^d_+}; X)$ is dense in $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$. Therefore, a similar density argument as in the proof of Proposition 5.5.4 yields that for any $f \in W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ there exists a $u \in W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ solving (5.5.11) and the required estimate holds. This finishes the proof. □

5.5.3. Sectoriality and the proof of Theorems 5.5.1 and 5.5.2

The sectoriality of the Dirichlet and Neumann Laplacian in Theorems 5.5.1 and 5.5.2 is now an easy consequence of Propositions 5.5.4 and 5.5.6.

Proof of Theorems 5.5.1 and 5.5.2. Let $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and $k \in \mathbb{N}_0 \cup \{-1\}$ be such that $\gamma + kp > -1$. We first consider the Dirichlet Laplacian. Assume that $\gamma + kp \in (-1, 2p-1)$ and $\lambda \geq 0$, or, $\gamma + kp > 2p-1$ and $\lambda > 0$. We prove that $\lambda - \Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ with domain $W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is sectorial of angle zero. Note that this operator is closed by the estimate in Proposition 5.5.4 (see Remark 2.3.2). The resolvent estimate for $\lambda - \Delta_{\mathrm{Dir}}$ now follows from Proposition 5.5.4 (corresponding to $\beta = 0$ below, the estimates

for $|\beta| \in \{1,2\}$ will be needed in Section 5.7). Indeed, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{C}\setminus\overline{\Sigma_\omega}} \sum_{|\beta|\le 2} |z|^{1-\frac{|\beta|}{2}} \|\partial^\beta R(z,\lambda-\Delta_{\mathrm{Dir}})f\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} \\
&\quad = \sup_{z\in\lambda+\Sigma_{\pi-\omega}} \sum_{|\beta|\le 2} |z-\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta R(z,\Delta_{\mathrm{Dir}})f\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} \\
&\quad \le C \sup_{z\in\lambda+\Sigma_{\pi-\omega}} \sum_{|\beta|\le 2} |z|^{1-\frac{|\beta|}{2}} \|\partial^\beta R(z,\Delta_{\mathrm{Dir}})f\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} \\
&\quad \le C \sup_{z\in\lambda+\Sigma_{\pi-\omega}} g_{k,\gamma}(z)\|f\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} = \widetilde{C} g_{k,\gamma}(\lambda)\|f\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)},
\end{aligned} \tag{5.5.12}$$

using that for $z-\lambda \in \Sigma_{\pi-\omega}$

$$|z-\lambda| \le \begin{cases} |z| & \text{if } \omega \in [\frac{\pi}{2},\pi) \\ \frac{|z|}{\sin\omega} & \text{if } \omega \in (0,\frac{\pi}{2}] \end{cases}.$$

Note that $g_{k,\gamma}(\lambda) = 1$ if $\gamma + kp < 2p-1$, so that the constant in the sectoriality estimate is independent of λ. Thus $-\Delta_{\mathrm{Dir}}$ is also sectorial in this case.

Furthermore, by Proposition 5.5.4 the mapping

$$\lambda - \Delta_{\mathrm{Dir}} : W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \to W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$$

is surjective. The sectoriality now follows from Remark 2.3.2. This finishes the proof of Theorem 5.5.1

The proof of Theorem 5.5.2 for the Neumann Laplacian on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is similar to the Dirichlet case using Proposition 5.5.6. □

5.6. The Dirichlet and Neumann heat semigroup

In this section, we study the growth of the Dirichlet and Neumann heat semigroup. Recall that the odd and even extensions from $\mathbb{R}^d_+$ to $\mathbb{R}^d$ are defined as in (5.4.1). Let $z \in \mathbb{C}_+$ and recall from Section 5.1.1 that we defined the kernels

$$H^{d,\pm}_z(x,y) := G^d_z(x_1-y_1,\widetilde{x}-\widetilde{y}) \pm G^d_z(x_1+y_1,\widetilde{x}-\widetilde{y}), \qquad x,y \in \mathbb{R}^d_+, \tag{5.6.1}$$

where G^d_z is the standard heat kernel on $\mathbb{R}^d$, see (5.1.1). The Dirichlet heat semigroup T_{Dir} is defined for $f \in L^p(\mathbb{R}^d_+, w_\gamma; X)$ with $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ by

$$T_{\mathrm{Dir}}(z)f(x) := H^{d,-}_z * f(x) := \int_{\mathbb{R}^d_+} H^{d,-}_z(x,y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} G^d_z(x-y) f_{\mathrm{odd}}(y)\,\mathrm{d}y, \tag{5.6.2}$$

and the Neumann heat semigroup T_{Neu} is defined for $f \in W^{\ell,p}(\mathbb{R}^d_+, w_\gamma; X)$ with $\gamma \in (-1, p-1)$ and $\ell \in \{0,1\}$ by

$$T_{\mathrm{Neu}}(z)f(x) := H^{d,+}_z * f(x) := \int_{\mathbb{R}^d_+} H^{d,+}_z(x,y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^d} G^d_z(x-y) f_{\mathrm{even}}(y)\,\mathrm{d}y. \tag{5.6.3}$$

Note that the above definitions of the semigroups are well defined by Theorems 5.4.2 and 5.4.6.

We prove the following theorems concerning the heat semigroups.

Theorem 5.6.1 (Dirichlet heat semigroup)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then $(T_{\mathrm{Dir}}(z))_{z\in\Sigma_\sigma}$ with $\sigma \in (0, \frac{\pi}{2})$ is an analytic C_0-semigroup on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ which is generated by Δ_{Dir}. Moreover, the heat semigroup $T_{\mathrm{Dir}}(t)$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ satisfies the following growth properties:*

(i) if $\gamma + kp \in (-1, 2p-1)$, then $T_{\mathrm{Dir}}(t)$ is bounded,

(ii) if $\gamma + kp > 2p-1$, then $T_{\mathrm{Dir}}(t)$ has polynomial growth and for any $\varepsilon > 0$ there are constants $c, C > 0$ only depending on $p, k, \gamma, \varepsilon, d$ and X, such that

$$c\big(1 + t^{\frac{\gamma+kp-2p+1}{2p}}\big) \le \|T_{\mathrm{Dir}}(t)\| \le C\big(1 + t^{\frac{\gamma+kp-2p+1+\varepsilon}{2p}}\big), \qquad t \ge 0.$$

Theorem 5.6.2 (Neumann heat semigroup)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ such that $\gamma + kp > -1$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then $(T_{\mathrm{Neu}}(z))_{z\in\Sigma_\sigma}$ with $\sigma \in (0, \frac{\pi}{2})$ is an analytic C_0-semigroup on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ which is generated by Δ_{Neu}. Moreover, the heat semigroup $T_{\mathrm{Neu}}(t)$ on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ satisfies the following growth properties:*

(i) if $\gamma + kp \in (-1, p-1)$, then $T_{\mathrm{Neu}}(t)$ is bounded,

(ii) if $\gamma + kp > p-1$, then $T_{\mathrm{Neu}}(t)$ has polynomial growth and for any $\varepsilon > 0$ there are constants $c, C > 0$ only depending on $p, k, \gamma, \varepsilon, d$ and X, such that

$$c\big(1 + t^{\frac{\gamma+kp-p+1}{2p}}\big) \le \|T_{\mathrm{Neu}}(t)\| \le C\big(1 + t^{\frac{\gamma+kp-p+1+\varepsilon}{2p}}\big), \qquad t \ge 0.$$

From Theorems 5.5.1 and 5.5.2 together with [125, Theorem G.5.2], the statements in Theorems 5.6.1(i) and 5.6.2(i) immediately follow. In Section 5.6.1 we prove that the generators of the Dirichlet and Neumann heat semigroup are Δ_{Dir} and Δ_{Neu}, respectively. Finally, in Section 5.6.2 we study the growth of the semigroups and prove Theorems 5.6.1(i) and 5.6.2(i).

5.6.1. Generators of the Dirichlet and Neumann heat semigroup

We start with some preliminaries about the consistency of the resolvents and the semigroups.

Let X_0 and X_1 be two compatible Banach spaces and suppose that $B_0 \in \mathscr{L}(X_0)$ and $B_1 \in \mathscr{L}(X_1)$. Then we call the operators B_0 and B_1 *consistent* if

$$B_0 u = B_1 u \qquad \text{for all } u \in X_0 \cap X_1.$$

For $z \in \Sigma \subseteq \mathbb{C}$ the two families of operators $B_0(z) \in \mathscr{L}(X_0)$ and $B_1(z) \in \mathscr{L}(X_1)$ are called consistent if $B_0(z)$ and $B_1(z)$ are consistent for all $z \in \Sigma$.

From Theorems 5.4.2(ii) and 5.4.6(ii) we obtain an easy corollary on the consistency of the semigroups.

Corollary 5.6.3. *Let $j \in \{0,1\}$. Let $p_j \in (1,\infty)$, $\sigma \in (0, \frac{\pi}{2})$ and let X be a* UMD *Banach space. Let $z \in \Sigma_\sigma$, then the following assertions hold.*

(i) If $\gamma_j \in (-1, 2p_j - 1) \setminus \{p_j - 1\}$, then $e^{z\Delta_{\mathrm{Dir}}}$ on $L^{p_j}(\mathbb{R}^d_+, w_{\gamma_j}; X)$ for $j \in \{0,1\}$ are consistent.

(ii) If $\gamma_j \in (-1, p_j - 1)$ and $\ell_j \in \{0,1\}$, then $e^{z\Delta_{\mathrm{Neu}}}$ on $W^{\ell_j, p_j}(\mathbb{R}^d_+, w_{\gamma_j}; X)$ for $j \in \{0,1\}$ are consistent.

To prove Theorems 5.6.1 and 5.6.2 we need the semigroups to be consistent on the weighted Sobolev spaces that we consider. To this end, we prove the consistency of the resolvents. We start with the consistency of the resolvents for the Dirichlet Laplacian.

Lemma 5.6.4 (Consistency of Dirichlet resolvents)**.** *Let $j \in \{0,1\}$. Let $p_j \in (1,\infty)$, $k_j \in \mathbb{N}_0$, $\gamma_j \in (-1, 2p_j - 1) \setminus \{p_j - 1\}$ and let X be a* UMD *Banach space. Let $A_j := \Delta_{\mathrm{Dir}}$ on $W^{k_j, p_j}(\mathbb{R}^d_+, w_{\gamma_j + k_j p_j}; X)$ be as in Definition 5.4.1. Then the resolvents $R(z, A_j - \lambda)$ for $j \in \{0,1\}$ are consistent for all $\lambda \geq 0$ and $z \in \Sigma_{\pi-\omega}$ with $\omega \in (0,\pi)$.*

Proof. First, let $k_0 = k_1 = 0$ and $\lambda = 0$, then by [19, Proposition 2.4] and Corollaries 5.4.7 and 5.6.3, we have that the resolvents are consistent. If $\lambda > 0$, then consistency of the resolvents for $\lambda = 0$ implies

$$R(z, A_0 - \lambda) = R(z + \lambda, \Delta_{\mathrm{Dir}}) = R(z, A_1 - \lambda), \tag{5.6.4}$$

since $z + \lambda \in \lambda + \Sigma_{\pi-\omega} \subseteq \Sigma_{\pi-\omega}$. This proves the case $k_0 = k_1 = 0$.

To prove the general case it now suffices to prove that for fixed $j \in \{0,1\}$ the resolvents on $W^{k_j, p_j}(\mathbb{R}^d_+, w_{\gamma_j + k_j p_j}; X)$ and $L^{p_j}(\mathbb{R}^d_+, w_{\gamma_j}; X)$ are consistent. By Proposition 5.5.4 the resolvent equation

$$(z + \lambda)u - \Delta_{\mathrm{Dir}} u = f, \qquad z \in \Sigma_{\pi-\omega},\ f \in L^{p_j}(\mathbb{R}^d_+, w_{\gamma_j}; X) \cap W^{k_j, p_j}(\mathbb{R}^d_+, w_{\gamma_j + k_j p_j}; X)$$

has unique solutions $u_0 \in W^{2, p_j}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma_j}; X)$ and $u_1 \in W^{k_j+2, p_j}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma_j + k_j p_j}; X)$. Now, by Hardy's inequality (Corollary 5.3.2) it follows that $u_0 = u_1$. This proves the lemma. □

Moreover, we have a similar lemma on the consistency of the Neumann resolvents.

Lemma 5.6.5 (Consistency of Neumann resolvents)**.** *Let $j \in \{0,1\}$. Let $p_j \in (1,\infty)$, $k_j \in \mathbb{N}_0$, $\ell_j \in \{0,1\}$, $\gamma_j \in (-1, p_j - 1)$ and let X be a* UMD *Banach space. Let $A_j := \Delta_{\mathrm{Neu}}$ on $W^{k_j + \ell_j, p_j}(\mathbb{R}^d_+, w_{\gamma_j + k_j p_j}; X)$ be as in Definition 5.4.1. Then the resolvents $R(z, A_j - \lambda)$ for $j \in \{0,1\}$ are consistent for all $\lambda \geq 0$ and $z \in \Sigma_{\pi-\omega}$ with $\omega \in (0,\pi)$.*

Proof. By [19, Proposition 2.4] and Corollaries 5.4.8 and 5.6.3, the case $k_0 = k_1 = 0$ with $\lambda = 0$ follows. Arguing as in (5.6.4) gives the result for $\lambda > 0$ as well.

To prove the general case, it now suffices to prove that for fixed $j \in \{0,1\}$ the resolvents on $W^{k_j + \ell_j, p_j}(\mathbb{R}^d_+, w_{\gamma_j + k_j p_j}; X)$ and $W^{\ell_j, p_j}(\mathbb{R}^d_+, w_{\gamma_j}; X)$ are consistent. This follows similarly as in the proof of Lemma 5.6.4 using Proposition 5.5.6 and Hardy's inequality. □

We can now prove that the Dirichlet and Neumann semigroup are generated by Δ_{Dir} and Δ_{Neu}, as is stated in Theorems 5.6.1 and 5.6.2, respectively.

Proof of Theorems 5.6.1 and 5.6.2: generator identification. Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and let $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. We first prove that $(T_{\mathrm{Dir}}(z))_{z\in\Sigma_\sigma}$ with $\sigma \in (0, \frac{\pi}{2})$ is an analytic C_0-semigroup on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ generated by Δ_{Dir}. Let

$$A_0 := \Delta_{\mathrm{Dir}} \quad \text{on } L^p(\mathbb{R}^d_+, w_\gamma; X) \quad \text{and} \quad A_1 := \Delta_{\mathrm{Dir}} \quad \text{on } W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$$

be as in Definition 5.4.1. By Theorem 5.5.1 and [125, Theorem G.5.2] we have that $A_1 - \lambda$ with $\lambda > 0$ generates the bounded analytic C_0-semigroup $(e^{-\lambda z}S(z))_{z\in\Sigma_\sigma}$ and

$$e^{-\lambda z}S(z)f = \frac{1}{2\pi \mathrm{i}}\int_\Gamma e^{zs}R(s+\lambda, A_1)f \,\mathrm{d}s, \qquad z \in \Sigma_\sigma,\ f \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), \tag{5.6.5}$$

where Γ is the upwards orientated boundary of $\Sigma_{\sigma'} \setminus \overline{B(0,r)}$ for some $r > 0$ and $\sigma' \in (\frac{\pi}{2} + |\arg z|, \frac{\pi}{2} + \sigma)$.

We show that $S(z) = T_{\mathrm{Dir}}(z)$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ for $z \in \Sigma_\sigma$. By (5.6.5), Lemma 5.6.4 and Theorem 5.4.2(ii) we obtain for $f \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$

$$e^{-\lambda z}S(z)f = \frac{1}{2\pi \mathrm{i}}\int_\Gamma e^{zs}R(s+\lambda, A_0)f \,\mathrm{d}s = \frac{1}{2\pi \mathrm{i}}\int_\Gamma e^{(s-\lambda)z}R(s, A_0)f \,\mathrm{d}s = e^{-\lambda z}T_{\mathrm{Dir}}(z)f.$$

Therefore, $S(z) = T_{\mathrm{Dir}}(z)$ and thus $T_{\mathrm{Dir}}(z)$ is the analytic C_0-semigroup generated by A_1. This completes the Dirichlet case from Theorem 5.6.1.

We continue with the Neumann Laplacian. If $\gamma \in (-1, p-1)$ and $k \in \mathbb{N}_0$, then define

$$A_0 := \Delta_{\mathrm{Neu}} \quad \text{on } W^{1,p}(\mathbb{R}^d_+, w_\gamma; X) \quad \text{and} \quad A_1 := \Delta_{\mathrm{Neu}} \quad \text{on } W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

If $\gamma \in (p-1, 2p-1)$ and $k \in \mathbb{N}_0 \cup \{-1\}$, then define

$$A_0 := \Delta_{\mathrm{Neu}} \quad \text{on } L^p(\mathbb{R}^d_+, w_{\gamma-p}; X) \quad \text{and} \quad A_1 := \Delta_{\mathrm{Neu}} \quad \text{on } W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

For both cases, we can proceed similarly as for the Dirichlet Laplacian using Theorem 5.5.2, Lemma 5.6.5 and Theorem 5.4.6(ii). This proves that $(T_{\mathrm{Neu}}(z))_{z\in\Sigma_\sigma}$ with $\sigma \in (0, \frac{\pi}{2})$ is an analytic C_0-semigroup on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ which is generated by Δ_{Neu}, as stated in Theorem 5.6.2. □

5.6.2. Growth of the Dirichlet and Neumann semigroups

We proceed with investigating the growth of the Dirichlet and Neumann heat semigroup on weighted Sobolev spaces and we prove Theorems 5.6.1(ii) and 5.6.2(ii).

Before turning to the growth of the semigroups, we first reconsider the elliptic regularity estimates from Propositions 5.5.4 and 5.5.6. In particular, using complex interpolation we can improve these estimates with a sharper function $g_{k,\gamma}(\lambda)$ on the right-hand side of the estimates. This will allow us to find sharper growth rates for the semigroups later on.

Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1)$ and let $\varepsilon > 0$. Then we define

$$h_{k,\gamma,\varepsilon}(\lambda) := \begin{cases} 1 & \text{if } \gamma + kp \in (-1, 2p-1) \\ 1 + |\lambda|^{-\frac{\gamma+kp-2p+1+\varepsilon}{2p}} & \text{if } \gamma + kp > 2p-1 \end{cases}. \tag{5.6.6}$$

For the Dirichlet Laplacian, we obtain the following improved elliptic regularity result.

Proposition 5.6.6 (Elliptic regularity for the Dirichlet Laplacian)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$, $\varepsilon > 0$, $\omega \in (0,\pi)$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then for all $f \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and $\lambda \in \Sigma_{\pi-\omega}$, there exists a unique $u \in W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $\lambda u - \Delta_{\mathrm{Dir}} u = f$. Moreover, this solution satisfies*

$$\sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C h_{k,\gamma,\varepsilon}(\lambda) \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where $h_{k,\gamma,\varepsilon}$ is defined in (5.6.6) *and the constant C only depends on $p, k, \gamma, \varepsilon, \omega, d$ and X.*

Proof. By Proposition 5.5.4 we only have to prove the estimate for $k \in \mathbb{N}_1$. First, let $\gamma \in (p-1, 2p-1)$ and without loss of generality we may assume $\varepsilon \in (0, 2p-1-\gamma)$. Define $\gamma_0 = p-1-\varepsilon \in (-1, p-1)$ and $\gamma_1 = 2p-1-\varepsilon \in (p-1, 2p-1)$. Then $\gamma = (1-\theta)\gamma_0 + \theta\gamma_1$ with $\theta = (\gamma - p + 1 + \varepsilon)/p$. Then Proposition 5.3.8 twice, properties of the complex interpolation method, Lemma 5.6.4 and the estimate in Proposition 5.5.4, yield

$$\begin{aligned}
&\sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\quad \leq C \sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{[W^{k,p}(\mathbb{R}^d_+, w_{\gamma_0+kp}; X), W^{k,p}(\mathbb{R}^d_+, w_{\gamma_1+kp}; X)]_\theta} \\
&\quad \leq C (g_{k,\gamma_0}(\lambda))^{1-\theta} (g_{k,\gamma_1}(\lambda))^{\theta} \|f\|_{[W^{k,p}(\mathbb{R}^d_+, w_{\gamma_0+kp}; X), W^{k,p}(\mathbb{R}^d_+, w_{\gamma_1+kp}; X)]_\theta} \\
&\quad \leq C \big(1 + |\lambda|^{-\frac{k-1}{2}}\big)^{1-\theta} \big(1 + |\lambda|^{-\frac{k}{2}}\big)^{\theta} \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\quad \leq C \big(1 + |\lambda|^{-\frac{\gamma+kp-2p+1+\varepsilon}{2p}}\big) \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},
\end{aligned}$$

where we used that

$$-\frac{k-1}{2}(1-\theta) - \frac{k}{2}\theta = -\frac{1}{2}(k-1+\theta) = -\frac{\gamma+kp-2p+1+\varepsilon}{2p}.$$

If $\gamma \in (-1, p-1)$, then by inspection of the inductive proof of Proposition 5.5.4, we see that on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ the function $g_{k,\gamma}$ on the right-hand side of the elliptic regularity estimate (5.5.2) is determined by $g_{k,\gamma}(\lambda) = g_{k-1,\gamma+p}(\lambda)$. Thus in a similar way, we obtain for $\gamma \in (-1, p-1)$ and $k \geq 2$ the estimate

$$\sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C h_{k-1,\gamma+p,\varepsilon}(\lambda) \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

and the result follows upon noting that $h_{k-1,\gamma+p,\varepsilon}(\lambda) = h_{k,\gamma,\varepsilon}(\lambda)$. □

In a similar way, there is the following improved elliptic regularity result for Δ_{Neu}.

Proposition 5.6.7 (Elliptic regularity for the Neumann Laplacian)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$ and $\gamma \in (-1,2p-1)\setminus\{p-1\}$ be such that $\gamma + kp > -1$. Moreover, let $\varepsilon > 0$, $\omega \in (0,\pi)$*

and let X be a UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then for all $f \in W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and $\lambda \in \Sigma_{\pi-\omega}$, there exists a unique $u \in W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $\lambda u - \Delta_{\mathrm{Neu}} u = f$. Moreover, this solution satisfies*

$$\sum_{|\beta|\le 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \le C h_{k+1,\gamma,\varepsilon}(\lambda) \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where $h_{k+1,\gamma,\varepsilon}$ is defined in (5.6.6) *and the constant C only depends on $p, k, \gamma, \varepsilon, \omega, d$ and X.*

Proof. The proof uses Proposition 5.5.6 and is analogous to the proof of Proposition 5.6.6. □

With the elliptic regularity estimates from Propositions 5.6.6 and 5.6.7 at hand, we can establish (almost) optimal growth bounds for the Dirichlet and Neumann heat semigroup.

Proof of Theorems 5.6.1(ii) and 5.6.2(ii). We start with the Dirichlet case, i.e., we prove that for $\gamma + kp > 2p - 1$, the semigroup $T_{\mathrm{Dir}}(t)$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ grows polynomially and there are constants $c, C > 0$ only depending on $p, k, \gamma, \varepsilon, d$ and X, such that

$$c\big(1 + t^{\frac{\gamma+kp-2p+1}{2p}}\big) \le \|T_{\mathrm{Dir}}(t)\| \le C\big(1 + t^{\frac{\gamma+kp-2p+1+\varepsilon}{2p}}\big), \qquad t \ge 0, \varepsilon > 0. \tag{5.6.7}$$

By Proposition 5.6.6, Lemma 2.3.3 and the fact that Δ_{Dir} generates T_{Dir}, it follows that $T_{\mathrm{Dir}}(t)$ grows at most polynomially and the upper bound in (5.6.7) holds.

It remains to show that $T_{\mathrm{Dir}}(t)$ grows at least polynomially. We provide an example of a function $h \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $\|T_{\mathrm{Dir}}(t)h\|_{L^p(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \ge c_{p,k,\gamma}\, t^{\frac{\gamma+kp-2p+1}{2p}}$.

We first construct a $\zeta \in W^{k,p}(\mathbb{R}_+, w_{\gamma+kp})$ such that $\|T_{\mathrm{Dir}}(t)\zeta\|_{L^p(\mathbb{R}_+, w_{\gamma+kp})} \ge c_{p,k,\gamma}\, t^{\frac{\gamma+kp-2p+1}{2p}}$. Define a cut-off function by

$$\zeta \in C^\infty(\mathbb{R}_+) \quad \text{such that} \quad 0 \le \zeta \le 1 \quad \text{and} \quad \zeta(x) = \begin{cases} 1 & \text{if } x \le \frac{1}{2} \\ 0 & \text{if } x \ge \frac{3}{4} \end{cases}. \tag{5.6.8}$$

Note that $\zeta \in W^{k,p}(\mathbb{R}_+, w_{\gamma+kp})$. Let $t \ge 1$ and using that $1 - e^{-a} \ge \frac{1}{2}\min\{a, 1\}$ for $a \ge 0$, we obtain

$$\begin{aligned}
\|T_{\mathrm{Dir}}(t)\zeta\|^p_{L^p(\mathbb{R}_+, w_{\gamma+kp})} &\ge \int_0^\infty x^{\gamma+kp} \Big| \int_0^{\frac{1}{2}} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} \Big(1 - e^{-\frac{xy}{t}}\Big)\, \mathrm{d}y \Big|^p \,\mathrm{d}x \\
&\ge \int_0^{\sqrt{t}} x^{\gamma+kp} \Big| \int_0^{\frac{1}{2}} \frac{1}{2} \frac{1}{\sqrt{4\pi t}} e^{-\frac{|x-y|^2}{4t}} \min\Big\{1, \frac{xy}{t}\Big\}\, \mathrm{d}y \Big|^p \,\mathrm{d}x \\
&\overset{t\ge 1}{\ge} (4\sqrt{\pi})^{-p} t^{-\frac{3p}{2}} \int_0^{\sqrt{t}} x^{\gamma+(k+1)p} \Big| \int_0^{\frac{1}{2}} e^{-\frac{|x-y|^2}{4t}} y\, \mathrm{d}y \Big|^p \,\mathrm{d}x \\
&\overset{x \mapsto \sqrt{t}x}{=} (4\sqrt{\pi})^{-p} t^{\frac{\gamma+kp-2p+1}{2}} \int_0^1 x^{\gamma+(k+1)p} \Big| \int_0^{\frac{1}{2}} e^{-\frac{|x-\frac{y}{\sqrt{t}}|^2}{4}} y\, \mathrm{d}y \Big|^p \,\mathrm{d}x \\
&\ge (4\sqrt{\pi})^{-p} t^{\frac{\gamma+kp-2p+1}{2}} \int_0^1 x^{\gamma+(k+1)p} \Big| \int_0^{\frac{1}{2}} e^{-\frac{1}{4}} y\, \mathrm{d}y \Big|^p \,\mathrm{d}x \\
&= c_{p,k,\gamma}\, t^{\frac{\gamma+kp-2p+1}{2}}.
\end{aligned}$$

Now, take $0 \neq g \in \mathscr{S}(\mathbb{R}^{d-1})$ and set $g_t(\widetilde{x}) := t^{-\frac{d-1}{2p}} g(t^{-\frac{1}{2}}\widetilde{x})$ for $t > 0$ and $\widetilde{x} \in \mathbb{R}^{d-1}$. Fix $x_0 \in X$ with $\|x_0\| = 1$. Then $h_t := \zeta \otimes g_t x_0$ satisfies $\sup_{t\geq 1} \|h_t\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} < \infty$. Using the properties of the heat semigroup and the one-dimensional lower bound, we obtain for $t \geq 1$ that

$$\begin{aligned}\|T^d_{\mathrm{Dir}}(t)h_t\|_{L^p(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &= \|T^1_{\mathrm{Dir}}(t)\zeta\|_{L^p(\mathbb{R}_+, w_{\gamma+kp})}\|T^{d-1}(t)g_t x_0\|_{L^p(\mathbb{R}^{d-1}; X)} \\ &= \|T^1_{\mathrm{Dir}}(t)\zeta\|_{L^p(\mathbb{R}_+, w_{\gamma+kp})}\|T^{d-1}(1)g x_0\|_{L^p(\mathbb{R}^{d-1}; X)} \\ &\gtrsim c_{p,k,\gamma}\, t^{\frac{\gamma+kp-2p+1}{2p}},\end{aligned}$$

where T^d_{Dir} is the Dirichlet heat semigroup on $\mathbb{R}^d_+$ and T^{d-1} is the heat semigroup on $\mathbb{R}^{d-1}$.

Therefore, if $\gamma + kp > 2p - 1$, then the semigroup $T_{\mathrm{Dir}}(t)$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ grows at least polynomially and $\|T_{\mathrm{Dir}}(t)\| \geq 1$ for all $t \geq 0$. This implies the lower bound in (5.6.7) and finishes the proof of Theorem 5.6.1(ii).

It remains to prove Theorem 5.6.2(ii), i.e., for $\gamma + kp > p - 1$ the semigroup $T_{\mathrm{Neu}}(t)$ on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ grows polynomially and there are constants $c, C > 0$ only depending on $p, k, \gamma, \varepsilon, d$ and X, such that

$$c\big(1 + t^{\frac{\gamma+kp-p+1}{2p}}\big) \leq \|T_{\mathrm{Neu}}(t)\| \leq C\big(1 + t^{\frac{\gamma+kp-p+1+\varepsilon}{2p}}\big), \qquad t \geq 0,\, \varepsilon > 0. \tag{5.6.9}$$

The upper bound in (5.6.9) follows from Proposition 5.6.7, Lemma 2.3.3 and the fact that Δ_{Neu} generates T_{Neu}. The lower estimate follows from the same example as in the Dirichlet case above. Indeed, we obtain for $t \geq 1$

$$\begin{aligned}\|T_{\mathrm{Neu}}(t)\zeta\|^p_{L^p(\mathbb{R}_+, w_{\gamma+kp})} &\geq (4\sqrt{\pi})^{-p} t^{-\frac{p}{2}} \int_0^{\sqrt{t}} x^{\gamma+kp} \Big| \int_0^{\frac{1}{2}} e^{-\frac{|x-y|^2}{4t}}\, \mathrm{d}y\Big|^p\, \mathrm{d}x \\ &\overset{x\mapsto\sqrt{t}x}{=} c_{p,k,\gamma}\, t^{\frac{\gamma+kp-p+1}{2}} \int_0^1 x^{\gamma+kp} \Big| \int_0^{\frac{1}{2}} e^{-\frac{|x-\frac{y}{\sqrt{t}}|^2}{4}}\, \mathrm{d}y\Big|^p\, \mathrm{d}x \\ &\geq \widetilde{c}_{p,k,\gamma}\, t^{\frac{\gamma+kp-p+1}{2}}.\end{aligned}$$

This finishes the proof of Theorem 5.6.2(ii). □

Remark 5.6.8. By inspection of the above proof, it follows that the example for the polynomial growth fails if the weight $w_{\gamma+kp}$ is replaced by a weight that behaves as $w_{\gamma+kp}$ near zero but becomes constant as $x_1 \to \infty$. Using such a weight we expect that the corresponding heat semigroup is actually bounded. Similarly, on bounded domains the semigroup will be bounded as well.

We end this section with an example that shows that the range for γ in Theorems 5.1.1 and 5.1.2 is optimal in the sense that the theorems fail for $\gamma \geq 2p - 1$. To this end, recall that

$$\int_0^{\frac{1}{2}} y^\alpha |\log(y)|^\beta\, \mathrm{d}y < \infty \tag{5.6.10}$$

if and only if $\alpha > -1$ and $\beta \in \mathbb{R}$, or, $\alpha = -1$ and $\beta < -1$.

Example 5.6.9. Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and $\gamma \geq 2p-1$. We provide an example of a function $h \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $T_{\mathrm{Dir}}(t)h \notin W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ for all $t > 0$. By setting $h = f \otimes g$ with $g \in \mathscr{S}(\mathbb{R}^{d-1}; X)$ if $d \geq 2$, it suffices to find an $f \in W^{k,p}(\mathbb{R}_+, w_{\gamma+kp})$ such that $T_{\mathrm{Dir}}(t)f \notin W^{k,p}(\mathbb{R}_+, w_{\gamma+kp})$.

Let ζ be as in (5.6.8) and define

$$f(x) := x^{-2}|\log(x)|^{-\frac{p+1}{2p}}\zeta(x), \qquad x \in \mathbb{R}_+.$$

A straightforward computation shows that for $j \in \{0,\dots,k\}$ we have $f^{(j)} \in L^p(\mathbb{R}_+, w_{\gamma+kp})$ and thus $f \in W^{k,p}(\mathbb{R}_+, w_{\gamma+kp})$. Note that the heat kernel satisfies $H^{1,-}_t \geq 0$, so that for $t > 0$ and $x \in (0, \frac{1}{2})$ we obtain

$$\begin{aligned} T_{\mathrm{Dir}}(t)f(x) &\geq \frac{1}{\sqrt{4\pi t}}\int_0^{\frac{1}{2}} e^{-\frac{|x-y|^2}{4t}}\big(1-e^{-\frac{xy}{t}}\big)y^{-2}|\log(y)|^{-\frac{p+1}{2p}}\,\mathrm{d}y \\ &\geq \frac{1}{\sqrt{4\pi t}}c_{t,x}\int_0^{\frac{1}{2}} y^{-1}|\log(y)|^{-\frac{p+1}{2p}}\,\mathrm{d}y \overset{(5.6.10)}{=} \infty, \end{aligned}$$

where

$$c_{t,x} := \inf_{y\in(0,\frac{1}{2})} y^{-1}e^{-\frac{|x-y|^2}{4t}}\left(1-e^{-\frac{xy}{t}}\right) > 0, \qquad \text{for } x \in \left(0,\tfrac{1}{2}\right) \text{ and } t > 0.$$

Therefore, $T_{\mathrm{Dir}}(t)f(x) = \infty$ on $(0,\frac{1}{2})$ and in particular $T_{\mathrm{Dir}}(t)f \notin W^{k,p}(\mathbb{R}_+, w_{\gamma+kp})$.

Analogously, for the Neumann Laplacian the example $f(x) := x^{-1}|\log(x)|^{-\frac{p+1}{2p}}\zeta(x)$ shows that the range for γ in Theorem 5.1.2 is optimal.

5.7. The H^∞-calculus for the Dirichlet and Neumann Laplacian

In this section, we prove the following theorems concerning the H^∞-calculus.

Theorem 5.7.1 (H^∞-calculus for $\lambda - \Delta_{\mathrm{Dir}}$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Assume that either*

(i) $\gamma + kp \in (-1, 2p-1)$ and $\lambda \geq 0$, or,

(ii) $\gamma + kp > 2p-1$ and $\lambda > 0$.

Then $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) = 0$.

Theorem 5.7.2 (H^∞-calculus for $\lambda - \Delta_{\mathrm{Neu}}$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Assume that either*

(i) $\gamma + kp \in (-1, p-1)$ and $\lambda \geq 0$, or,

(ii) $\gamma + kp > p-1$ and $\lambda > 0$.

Then $\lambda - \Delta_{\mathrm{Neu}}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) = 0$.

In Sections 5.7.1 and 5.7.2 below we prove some preliminary estimates on the Dirichlet and Neumann resolvent, respectively. In Section 5.7.3 we prove Theorems 5.7.1 and 5.7.2.

5.7.1. Preliminary estimates for the Dirichlet resolvent

We start with a preliminary lemma on commutators of the Dirichlet resolvent and derivatives.

Lemma 5.7.3. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and let X be a Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then for all $u \in C_c^\infty(\mathbb{R}^d_+; X)$ and $z \in \rho(\Delta_{\mathrm{Dir}})$ we have*

(i) $[\partial_j^n, R(z,\Delta_{\mathrm{Dir}})]u = 0$, $\quad n \in \mathbb{N}_1$, $j \in \{2,\dots,d\}$,

(ii) $[\partial_1^2, R(z,\Delta_{\mathrm{Dir}})]u = 0$,

(iii) $[M\partial_1^\ell, R(z,\Delta_{\mathrm{Dir}})]u = -2R(z,\Delta_{\mathrm{Dir}})\partial_1^{\ell+1}R(z,\Delta_{\mathrm{Dir}})u$, $\quad \ell \in \{0,1\}$.

It should be noted that $R(z,\Delta_{\mathrm{Dir}})$ and ∂_1 do not commute since $\partial_1 R(z,\Delta_{\mathrm{Dir}})u$ does not satisfy the Dirichlet boundary condition. This causes various complications in the proof of Theorem 5.7.1.

Proof. Let $j \in \{2,\dots,d\}$, then using that Δ_{Dir} and ∂_j commute gives

$$\begin{aligned}(z-\Delta_{\mathrm{Dir}})[\partial_j, R(z,\Delta_{\mathrm{Dir}})]u &= (z-\Delta_{\mathrm{Dir}})\partial_j R(z,\Delta_{\mathrm{Dir}})u - \partial_j u\\ &= \partial_j(z-\Delta_{\mathrm{Dir}})R(z,\Delta_{\mathrm{Dir}})u - \partial_j u = 0,\end{aligned}$$

since $\partial_j R(z,\Delta_{\mathrm{Dir}})u \in D(\Delta_{\mathrm{Dir}})$ by Lemma 5.5.3. This proves (i) for $n=1$ and the general case follows by iteration. Note that $\partial_1^2 = \Delta_{\mathrm{Dir}} - \sum_{j=2}^d \partial_j^2$, so (ii) follows from (i) and the fact that $R(z,\Delta_{\mathrm{Dir}})$ and Δ_{Dir} commute.

Finally, using (5.5.1) and $M\partial_1^\ell R(z,\Delta_{\mathrm{Dir}})u \in D(\Delta_{\mathrm{Dir}})$, we compute

$$\begin{aligned}&(z-\Delta_{\mathrm{Dir}})[M\partial_1^\ell, R(z,\Delta_{\mathrm{Dir}})]u\\ &\quad= zM\partial_1^\ell R(z,\Delta_{\mathrm{Dir}})u - \Delta_{\mathrm{Dir}}M\partial_1^\ell R(z,\Delta_{\mathrm{Dir}})u - M\partial_1^\ell u\\ &\quad= zM\partial_1^\ell R(z,\Delta_{\mathrm{Dir}})u - M\Delta\partial_1^\ell R(z,\Delta_{\mathrm{Dir}})u - 2\partial_1^{\ell+1}R(z,\Delta_{\mathrm{Dir}})u - M\partial_1^\ell u\\ &\quad= M\partial_1^\ell(z-\Delta_{\mathrm{Dir}})R(z,\Delta_{\mathrm{Dir}})u - 2\partial_1^{\ell+1}R(z,\Delta_{\mathrm{Dir}})u - M\partial_1^\ell u\\ &\quad= -2\partial_1^{\ell+1}R(z,\Delta_{\mathrm{Dir}})u,\end{aligned}$$

which proves (iii). □

In the following two lemmas, we prove estimates for the commutators in Lemma 5.7.3(iii), which are required in the proof of Theorem 5.7.1. We start with an estimate for the commutator in Lemma 5.7.3(iii) with $\ell = 0$.

Lemma 5.7.4. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$, $\lambda > 0$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Let $0 < \nu < \omega < \pi$ and let Γ_ν be the downwards orientated boundary of $\Sigma_\nu \setminus B(0, \frac{\lambda}{2})$. Then there exists a $C > 0$ such that*

$$\begin{aligned}&\Big\| \int_{\Gamma_\nu} f(z)R(z,\lambda-\Delta_{\mathrm{Dir}})\partial_1 R(z,\lambda-\Delta_{\mathrm{Dir}})u \,\mathrm{d}z \Big\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}\\ &\qquad\le C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},\end{aligned}$$

for all $f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$ and $u \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$. Moreover, the constant C only depends on $p, k, \gamma, \lambda, \omega, \nu, d$ and X.

Proof. From (5.5.12) in the proof of Theorem 5.5.1 we obtain for any $\sigma \in (0,\pi)$

$$\sup_{z \in \mathbb{C}\setminus\overline{\Sigma_\sigma}} |z| \|R(z, \lambda - \Delta_{\mathrm{Dir}})\| < \infty \quad \text{and} \quad \sup_{z \in \mathbb{C}\setminus\overline{\Sigma_\sigma}} |z|^{\frac{1}{2}} \|\partial_1 R(z, \lambda - \Delta_{\mathrm{Dir}})\| < \infty.$$

Therefore,

$$\begin{aligned}
&\Big\| \int_{\Gamma_\nu} f(z) R(z, \lambda - \Delta_{\mathrm{Dir}}) \partial_1 R(z, \lambda - \Delta_{\mathrm{Dir}}) u \, \mathrm{d}z \Big\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\quad \le \|f\|_{H^\infty(\Sigma_\omega)} \int_{\Gamma_\nu} \|zR(z, \lambda - \Delta_{\mathrm{Dir}})\| \|z^{\frac{1}{2}} \partial_1 R(z, \lambda - \Delta_{\mathrm{Dir}})\| \|u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \frac{|\mathrm{d}z|}{|z|^{\frac{3}{2}}} \\
&\quad \le C \|f\|_{H^\infty(\Sigma_\omega)} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}.
\end{aligned}$$

□

To continue, we prove an estimate for the commutator in Lemma 5.7.3(iii) with $\ell = 1$ under the additional assumption that $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus.

Lemma 5.7.5. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$, $\lambda > 0$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Let $0 < \nu < \omega < \sigma < \pi$ and let Γ_ν be the downwards orientated boundary of $\Sigma_\nu \setminus B(0, \frac{\lambda}{2})$. Assume that $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded $H^\infty(\Sigma_\omega)$-calculus on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$. Then there exists a $C > 0$ such that*

$$\Big\| \int_{\Gamma_\nu} f(z) R^2(z, \lambda - \Delta_{\mathrm{Dir}}) \partial_1^2 u \, \mathrm{d}z \Big\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \le C \|f\|_{H^\infty(\Sigma_\sigma)} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

for all $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $u \in W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$. Moreover, the constant C only depends on $p, k, \gamma, \lambda, \omega, \nu, \sigma, d$ and X. In addition, if $k = 0$, then the constant C can be taken independent of λ.

Proof. Let $g(z) = zf'(z)$. We first claim that $g \in H^\infty(\Sigma_\omega)$. Indeed, fix a $\sigma' \in (\omega, \sigma)$, then by Cauchy's differentiation formula and the substitution $s \mapsto s|z|$

$$\begin{aligned}
\sup_{z \in \Sigma_\omega} |zf'(z)| &\le \sup_{z \in \Sigma_\omega} \Big| \int_{\partial\Sigma_{\sigma'}} \frac{zf(s)}{(s-z)^2} \, \mathrm{d}s \Big| \\
&\le \|f\|_{H^\infty(\Sigma_\sigma)} \sup_{z \in \Sigma_\omega} \int_{\partial\Sigma_{\sigma'}} \Big| \frac{s}{|z|} - \frac{z}{|z|} \Big|^{-2} |z|^{-1} |\mathrm{d}s| \\
&\le \|f\|_{H^\infty(\Sigma_\sigma)} \sup_{z \in \Sigma_\omega} \int_{\partial\Sigma_{\sigma'}} \Big| s - \frac{z}{|z|} \Big|^{-2} |\mathrm{d}s| \\
&\le C_{\omega,\sigma} \|f\|_{H^\infty(\Sigma_\sigma)}.
\end{aligned}$$

Secondly, we claim that for $A_{\mathrm{Dir}} := \lambda - \Delta_{\mathrm{Dir}}$ and $v \in \mathscr{S}(\mathbb{R}^d_+; X)$ with $v(0,\cdot) = 0$, we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\nu} f(z) A_{\mathrm{Dir}} R^2(z, A_{\mathrm{Dir}}) v \, \mathrm{d}z = g(A_{\mathrm{Dir}}) v,$$

where $g(A_{\mathrm{Dir}})$ is defined through the extended H^∞-calculus, see [126, Section 15.1]. Define the regulariser

$$\zeta_n(z) := \frac{n}{n+z} - \frac{1}{1+nz} \qquad \text{for } z \in \Sigma_\omega \text{ and } n \in \mathbb{N}_1,$$

and for $\nu' \in (0,\nu)$ define the downwards orientated contour $\Gamma_{\nu'} = \partial(\Sigma_{\nu'} \setminus B(0, \frac{3\lambda}{4}))$. Then using the dominated convergence theorem, the functional calculus for $A_{\mathrm{Dir}} R^2(z, A_{\mathrm{Dir}})$, Fubini's theorem and Cauchy's differentiation formula, we obtain

$$\begin{aligned}
\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\nu} f(z) A_{\mathrm{Dir}} R^2(z, A_{\mathrm{Dir}}) v \, \mathrm{d}z &= \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_\nu} \zeta_n(z) f(z) A_{\mathrm{Dir}} R^2(z, A_{\mathrm{Dir}}) v \, \mathrm{d}z \\
&= \lim_{n\to\infty} \frac{1}{(2\pi \mathrm{i})^2} \int_{\Gamma_\nu} \int_{\Gamma_{\nu'}} \frac{\zeta_n(z) f(z)}{(z-s)^2} s R(s, A_{\mathrm{Dir}}) v \, \mathrm{d}s \, \mathrm{d}z \\
&= \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\nu'}} (\zeta_n f)'(s) s R(s, A_{\mathrm{Dir}}) v \, \mathrm{d}s \\
&= \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\nu'}} \zeta_n(s) f'(s) s R(s, A_{\mathrm{Dir}}) v \, \mathrm{d}s \\
&\quad + \lim_{n\to\infty} \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\nu'}} \zeta_n'(s) f(s) s R(s, A_{\mathrm{Dir}}) v \, \mathrm{d}s \\
&= g(A_{\mathrm{Dir}}) v,
\end{aligned}$$

where in the last step we used for the first term [125, Theorem 10.2.13] and for the second term we used that $|s\zeta_n'(s)|$ is uniformly bounded on $\Gamma_{\nu'}$, the dominated convergence theorem and $\zeta_n' \to 0$ pointwise as $n \to \infty$. This proves the second claim.

Let $u \in C_c^\infty(\mathbb{R}^d_+; X)$. Then by invertibility of A_{Dir}, introducing an additional $A_{\mathrm{Dir}} A_{\mathrm{Dir}}^{-1}$ and using the claims, we obtain

$$\begin{aligned}
&\Big\| \int_{\Gamma_\nu} f(z) R^2(z, A_{\mathrm{Dir}}) A_{\mathrm{Dir}} A_{\mathrm{Dir}}^{-1} \partial_1^2 u \, \mathrm{d}z \Big\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\qquad = \Big\| \int_{\Gamma_\nu} f(z) A_{\mathrm{Dir}} R^2(z, A_{\mathrm{Dir}}) A_{\mathrm{Dir}}^{-1} \partial_1^2 u \, \mathrm{d}z \Big\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\qquad = 2\pi \| g(A_{\mathrm{Dir}}) A_{\mathrm{Dir}}^{-1} \partial_1^2 u \|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\qquad \le C \| g \|_{H^\infty(\Sigma_\omega)} \| A_{\mathrm{Dir}}^{-1} \partial_1^2 u \|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\qquad \le C_{\omega,\sigma} \| f \|_{H^\infty(\Sigma_\sigma)} \| A_{\mathrm{Dir}}^{-1} \partial_1^2 u \|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},
\end{aligned} \tag{5.7.1}$$

where in the penultimate step we used the $H^\infty(\Sigma_\omega)$-calculus of A_{Dir}. From Lemma 5.7.3(ii) and Proposition 5.5.4 we find

$$\begin{aligned}
\| A_{\mathrm{Dir}}^{-1} \partial_1^2 u \|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &= \| \partial_1^2 R(\lambda, \Delta_{\mathrm{Dir}}) u \|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\
&\le C g_{k,\gamma}(\lambda) \| u \|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},
\end{aligned} \tag{5.7.2}$$

where C is independent of λ and $g_{k,\gamma}$ is as in (5.5.3). By density (Lemma 5.3.3) the desired estimate follows.

Finally, assume that $k = 0$. Then we obtain (5.7.1) with a constant independent of λ. Indeed, this follows from using the H^∞-calculus on $L^p(\mathbb{R}^d_+, w_\gamma; X)$, see Theorem 5.4.2 and [126, Proposition 16.2.6]. Moreover, (5.7.2) also holds with a constant independent of λ by (5.5.3). This finishes the proof. □

5.7.2. Preliminary estimates for the Neumann resolvent

In this section, we provide some preliminary estimates required for the proof of the H^∞-calculus for the Neumann Laplacian. This proof will be given in the next section, relying on the H^∞-calculus for the Dirichlet Laplacian on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ with $k \in \mathbb{N}_1$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$.

We start with a lemma on commutators for the Neumann Laplacian.

Lemma 5.7.6. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1,2p-1) \setminus \{p-1\}$ and let X be a Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then for all $u \in C^\infty_{\mathrm{c},1}(\overline{\mathbb{R}^d_+}; X)$ and $z \in \rho(\Delta_{\mathrm{Neu}})$, we have*

(i) $[\partial_j^n, R(z,\Delta_{\mathrm{Neu}})]u = 0, \qquad n \in \mathbb{N}_1,\ j \in \{2,\dots,d\},$

(ii) $[\partial_1^2, R(z,\Delta_{\mathrm{Neu}})]u = 0,$

In addition, let $k \in \mathbb{N}_0$, Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$. Then, for all $u \in C^\infty_{\mathrm{c},1}(\overline{\mathbb{R}^d_+}; X)$ and $z \in \rho(\Delta_{\mathrm{Neu}}) \cap \rho(\Delta_{\mathrm{Dir}})$, we have

(iii) $\partial_1 R(z,\Delta_{\mathrm{Neu}})u = R(z,\Delta_{\mathrm{Dir}})\partial_1 u.$

Proof. The statements (i) and (ii) follow similarly as in Lemma 5.7.3 using Lemma 5.5.5.

For (iii), set $\widetilde{u} := R(z,\Delta_{\mathrm{Neu}})u$, then $\widetilde{u} \in \mathscr{S}(\mathbb{R}^d_+; X)$ with $(\partial_1\widetilde{u})(0,\cdot) = 0$ and $z\widetilde{u} - \Delta_{\mathrm{Neu}}\widetilde{u} = u$ by Lemma 5.5.5. Moreover, it holds that $(z-\Delta_{\mathrm{Dir}})\partial_1\widetilde{u} = \partial_1(z-\Delta_{\mathrm{Neu}})\widetilde{u} = \partial_1 u$ and applying the Dirichlet resolvent gives the result. □

We continue with an estimate for the commutator of M and the Neumann resolvent.

Lemma 5.7.7. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (p-1, 2p-1)$, $\lambda > 0$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and Δ_{Dir} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Then, for all $u \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ and $z \in \rho(\Delta_{\mathrm{Neu}}) \cap \rho(\Delta_{\mathrm{Dir}})$, we have*

$$MR(z,\Delta_{\mathrm{Neu}})u = R(z,\Delta_{\mathrm{Dir}})Mu - 2R(z,\Delta_{\mathrm{Dir}})\partial_1 R(z,\Delta_{\mathrm{Neu}})u.$$

Moreover, there exists a $C > 0$ such that

$$\begin{aligned}\|Mf(\lambda-\Delta_{\mathrm{Neu}})u\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &\le \|f(\lambda-\Delta_{\mathrm{Dir}})Mu\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\ &\quad + C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},\end{aligned}$$

for all $f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$ with $\omega \in (0,\pi)$. Moreover, the constant C only depends on $p, k, \gamma, \lambda, \omega, \nu, d$ and X.

Proof. Note that $R(z,\Delta_{\mathrm{Dir}})Mu \in D(\Delta_{\mathrm{Dir}})$ by Lemma 5.5.3 and $MR(z,\Delta_{\mathrm{Neu}})u \in D(\Delta_{\mathrm{Dir}})$ by Lemma 5.5.5. Therefore, arguing as in Lemma 5.7.3(iii), we obtain

$$(z-\Delta_{\mathrm{Dir}})\big[MR(z,\Delta_{\mathrm{Neu}})u - R(z,\Delta_{\mathrm{Dir}})Mu\big] = -2\partial_1 R(z,\Delta_{\mathrm{Neu}})u.$$

To prove the estimate, let $0<\nu<\omega<\pi$ and Γ_ν be the downwards orientated boundary of $\Sigma_\nu \setminus B(0,\frac{\lambda}{2})$. Then

$$\begin{aligned}
&\|Mf(\lambda-\Delta_{\mathrm{Neu}})u\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}\\
&\quad\le \|f(\lambda-\Delta_{\mathrm{Dir}})Mu\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}\\
&\qquad + 2\Big\|\int_{\Gamma_\nu} f(z)R(z,\lambda-\Delta_{\mathrm{Dir}})\partial_1 R(z,\lambda-\Delta_{\mathrm{Neu}})u\,\mathrm{d}z\Big\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma-p+(k+1)p};X)}
\end{aligned}$$

and the latter term can be estimated similarly as in the proof of Lemma 5.7.4 using Theorem 5.5.1 with $k+1$ and $\gamma-p\in(-1,p-1)$, and Theorem 5.5.2. □

5.7.3. Proofs of Theorems 5.7.1 and 5.7.2

Before turning to the proof of Theorem 5.7.1, we prove an estimate for the derivatives in directions $2,\dots,d$ of the functional calculus on the spaces $W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)$. For this we will use a perturbation argument in the parameter γ inspired by [190, Proposition 5.3].

Lemma 5.7.8. *Let $p\in(1,\infty)$, $k\in\mathbb{N}_0$ and let X be a* UMD *Banach space. Let $\lambda>0$ and consider either of the following cases:*

(i) $\gamma\in(-1,p-1)$ *and* $A:=\lambda-\Delta_{\mathrm{Dir}}$, *or,*

(ii) $\gamma\in(p-1,2p-1)$ *and* $A:=\lambda-\Delta_{\mathrm{Dir}}$, *or,*

(iii) $\gamma\in(-1,p-1)$ *and* $A:=\lambda-\Delta_{\mathrm{Neu}}$,

where Δ_{Dir} and Δ_{Neu} on $W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)$ are as in Definition 5.4.1. Let $\omega\in(0,\frac{\pi}{2})$ and $\alpha=(0,\widetilde{\alpha})\in\mathbb{N}_0\times\mathbb{N}_0^{d-1}$ with $|\alpha|=k$. Then for all $f\in H^1(\Sigma_\omega)\cap H^\infty(\Sigma_\omega)$ and $u\in C_{\mathrm{c}}^\infty(\mathbb{R}^d_+;X)$ we have

$$\|\partial^{\widetilde{\alpha}} f(A)u\|_{L^p(\mathbb{R}^d_+,w_{\gamma+kp};X)} \le C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)},$$

where the constant C only depends on $p,k,\gamma,\omega,\alpha,d$ and X.

Proof. Throughout the proof, C denotes a constant only depending on $p,k,\gamma,\omega,\alpha,d$ and X, which may change from line to line. We proceed by induction on $k\ge 0$. The case $k=0$ follows from Theorems 5.4.2 and 5.4.6. Let $k\in\mathbb{N}_1$ be fixed. Assume that the statement of the lemma holds with k replaced by ℓ for all $\ell\in\{0,\dots,k-1\}$. We prove the estimate as stated in the lemma.

Let $\alpha = (0,\widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ be such that $|\alpha| = k$. For any $\theta \in (0,1)$ set $g := M^{(k+1)\theta} u$ and $\psi(x_1,y_1) := \big(\frac{x_1^\theta}{y_1^\theta} - 1\big)^{k+1}$. Then for $x \in \mathbb{R}^d_+$ we have

$$\begin{aligned} f(A)u(x) &= x_1^{-(k+1)\theta} f(A)\big((x_1^\theta - M^\theta)^{k+1} u\big)(x) \\ &\quad + x_1^{-(k+1)\theta} \sum_{j=0}^{k} c_{j,k} f(A)\big(x_1^{j\theta} M^{(k+1-j)\theta} u\big)(x) \\ &= x_1^{-(k+1)\theta} f(A)(\psi(x_1,\cdot)g)(x) \\ &\quad - \sum_{j=0}^{k} c_{j,k} x_1^{-(k+1-j)\theta} f(A)(M^{(k+1-j)\theta} u)(x), \end{aligned} \tag{5.7.3}$$

where $c_{j,k} = (-1)^{k+1-j}\binom{k+1}{j}$. For the latter sum, we find

$$\begin{aligned} &\Big\| x \mapsto \partial^{\widetilde{\alpha}} \sum_{j=0}^{k} c_{j,k} x_1^{-(k+1-j)\theta} f(A)(M^{(k+1-j)\theta} u)(x) \Big\|_{L^p(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\ &\qquad \le C \sum_{j=0}^{k} \|\partial^{\widetilde{\alpha}} f(A) M^{(k+1-j)\theta} u\|_{L^p(\mathbb{R}^d_+, w_{\gamma_j}; X)}, \end{aligned} \tag{5.7.4}$$

where $\gamma_j := \gamma + kp - (k+1-j)\theta p$. Now choose $\theta \in (0,1)$ such that

$$\begin{aligned} &\gamma_0 \in (-1, p-1) \quad \text{and} \quad \gamma_j \in ((j-1)p-1, jp-1) \qquad \text{if } \gamma \in (-1,p-1), \\ &\gamma_0 \in (p-1, 2p-1) \quad \text{and} \quad \gamma_j \in (jp-1, (j+1)p-1) \qquad \text{if } \gamma \in (p-1,2p-1). \end{aligned} \tag{5.7.5}$$

The conditions $\gamma_0, \gamma_1 \in (-1, p-1)$ and $\gamma_0, \gamma_1 \in (p-1, 2p-1)$, respectively, lead to

$$\begin{aligned} &\theta \in \big(\tfrac{\gamma+(k-1)p+1}{kp}, \tfrac{\gamma+kp+1}{kp}\big) \cap \big(\tfrac{\gamma+(k-1)p+1}{(k+1)p}, \tfrac{\gamma+kp+1}{(k+1)p}\big) \qquad \text{if } \gamma \in (-1,p-1), \\ &\theta \in \big(\tfrac{\gamma+(k-2)p+1}{kp}, \tfrac{\gamma+(k-1)p+1}{kp}\big) \cap \big(\tfrac{\gamma+(k-2)p+1}{(k+1)p}, \tfrac{\gamma+(k-1)p+1}{(k+1)p}\big) \qquad \text{if } \gamma \in (p-1,2p-1). \end{aligned}$$

Therefore, we can take

$$\begin{aligned} &\theta \in \big(\tfrac{\gamma+(k-1)p+1}{kp}, \tfrac{\gamma+kp+1}{(k+1)p}\big) \qquad \text{if } \gamma \in (-1,p-1), \\ &\theta \in \big(\tfrac{\gamma+(k-2)p+1}{kp}, \tfrac{\gamma+(k-1)p+1}{(k+1)p}\big) \qquad \text{if } \gamma \in (p-1,2p-1), \end{aligned} \tag{5.7.6}$$

and it is straightforward to verify that these intervals are non-empty for the given ranges of γ. Moreover, if (5.7.6) holds, then the other conditions on γ_j for $j \in \{2,\dots,k\}$ in (5.7.5) are automatically satisfied.

Write $\widetilde{\alpha} = \widetilde{\beta} + \widetilde{\delta}$ for some $\widetilde{\beta}, \widetilde{\delta} \in \mathbb{N}_0^{d-1}$ such that $|\widetilde{\beta}| = k-(j-1)$ and $|\widetilde{\delta}| = j-1$. Using that derivatives on $\mathbb{R}^{d-1}$ commute with the resolvent (Lemmas 5.7.3(i) and 5.7.6(i)) and that the resolvents are consistent (Lemmas 5.6.4 and 5.6.5), we can estimate the right-hand

side of (5.7.4) as

$$\begin{aligned}
&\sum_{j=0}^{k} \|\partial^{\widetilde{\alpha}} f(A) M^{(k+1-j)\theta} u\|_{L^p(\mathbb{R}^d_+, w_{\gamma_j}; X)} \\
&\quad \le \|f(A)\partial^{\widetilde{\alpha}} M^{(k+1)\theta} u\|_{L^p(\mathbb{R}^d_+, w_{\gamma_0}; X)} + \sum_{j=1}^{k} \|\partial^{\widetilde{\delta}} f(A)\partial^{\widetilde{\beta}} M^{(k+1-j)\theta} u\|_{L^p(\mathbb{R}^d_+, w_{\gamma_j}; X)} \\
&\quad \le C\|f\|_{H^\infty(\Sigma_\omega)} \Big(\|\partial^{\widetilde{\alpha}} M^{(k+1)\theta} u\|_{L^p(\mathbb{R}^d_+, w_{\gamma_0}; X)} + \sum_{j=1}^{k} \|\partial^{\beta} M^{(k+1-j)\theta} u\|_{W^{j-1,p}(\mathbb{R}^d_+, w_{\gamma_j}; X)} \Big) \\
&\quad \le C\|f\|_{H^\infty(\Sigma_\omega)} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},
\end{aligned}$$

using the induction hypothesis in the penultimate step, which is allowed by (5.7.5) and (5.7.6). In addition, we have applied Lemma 5.3.6 in the last step.

In view of (5.7.3), it remains to show the estimate

$$\begin{aligned}
\|x \mapsto x_1^{-(k+1)\theta} \partial^{\widetilde{\alpha}} f(A)(\psi(x_1,\cdot) g)(x)\|_{L^p(\mathbb{R}^d_+, w_{\gamma+kp}; X)} & \\
\le C\|f\|_{H^\infty(\Sigma_\omega)} \|g\|_{L^p(\mathbb{R}^d_+, w_{\gamma-(k+1)\theta p}; X)} & \\
\le C\|f\|_{H^\infty(\Sigma_\omega)} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}, &
\end{aligned} \tag{5.7.7}$$

where the last estimate follows from Hardy's inequality (Corollary 5.3.2).

Let $\nu \in (0, \omega)$ so that

$$f(A) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_- \cup \Gamma_+} f(z) R(z, A)\, \mathrm{d}z,$$

where $\Gamma_\pm = \{re^{\pm \mathrm{i}\nu} : r > 0\}$ are both orientated downwards. To prove (5.7.7) we estimate the integrals over $\Gamma_\pm$ separately and by symmetry it suffices to consider only Γ_+. We define $\delta = \frac{\pi - \nu}{2}$ and write $z = re^{\mathrm{i}\nu}$ with $r > 0$. Then we have the Laplace transform representation

$$\begin{aligned}
R(z, A) &= -(re^{\mathrm{i}(\nu-\pi)} + A)^{-1} = -e^{\mathrm{i}\delta}(re^{-\mathrm{i}\delta} + e^{\mathrm{i}\delta} A)^{-1} \\
&= -e^{\mathrm{i}\delta} \int_0^\infty e^{-tre^{-\mathrm{i}\delta}} e^{-t\lambda e^{\mathrm{i}\delta}} T(te^{\mathrm{i}\delta})\, \mathrm{d}t,
\end{aligned} \tag{5.7.8}$$

where T is the Dirichlet or Neumann heat semigroup from (5.6.2) and (5.6.3), respectively. For $x, y \in \mathbb{R}^d_+$ and $z \in \mathbb{C}_+$, recall that G^d_z is the standard heat kernel on $\mathbb{R}^d$ (see (5.1.1)) and the Dirichlet and Neumann heat kernels are given by $H^{d,\pm}_z(x, y) := G^d_z(x_1 - y_1, \widetilde{x} - \widetilde{y}) \pm G^d_z(x_1 + y_1, \widetilde{x} - \widetilde{y})$, see (5.6.1). It holds that $H^{d,\pm}_z(x, y) = H^{1,\pm}_z(x_1, y_1) G^{d-1}_z(\widetilde{x} - \widetilde{y})$. Moreover, for $|\widetilde{\alpha}| = k$ and $z \in \mathbb{C}_+$ we have the following estimate on derivatives of G^{d-1}_z

$$|\partial^{\widetilde{\alpha}}_{\widetilde{x}} G^{d-1}_z(\widetilde{x} - \widetilde{y})| \le C_{d,k} \Big(|z|^{-\frac{k}{2}} + \frac{|\widetilde{x} - \widetilde{y}|^k}{|z|^k} \Big) |G^{d-1}_z(\widetilde{x} - \widetilde{y})|, \qquad \widetilde{x}, \widetilde{y} \in \mathbb{R}^{d-1}. \tag{5.7.9}$$

Indeed, let $q(s) = e^{-s^2}$ with $s \in \mathbb{C}$ which satisfies $q^{(n)}(s) = p(s) q(s)$ where $p(s)$ is a polynomial of degree $n \in \mathbb{N}_0$. Then, $|q^{(n)}(s)| \le C_n (1 + |s|^2)^{\frac{n}{2}} |q(s)|$ and upon noting that $G^1_z(x - y) = (4\pi z)^{-\frac{1}{2}} q(\frac{x-y}{2\sqrt{z}})$ for $x, y \in \mathbb{R}$, we find with the chain rule

$$\Big| \frac{\mathrm{d}^n}{\mathrm{d}x^n} G^1_z(x - y) \Big| \le C_n |z|^{-\frac{n}{2}} \Big(1 + \frac{|x - y|^2}{|z|} \Big)^{\frac{n}{2}} |G^1_z(x - y)|, \qquad x, y \in \mathbb{R},\ n \in \mathbb{N}_0.$$

Write $\widetilde{\alpha} = (\alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^{d-1}$, then from $G_z^{d-1}(\widetilde{x}-\widetilde{y}) = \prod_{j=2}^d G_z^1(x_j - y_j)$ and the one dimensional estimate above, we find

$$\begin{aligned}
|\partial_{\widetilde{x}}^{\widetilde{\alpha}} G_z^{d-1}(\widetilde{x}-\widetilde{y})| &\le C_k \prod_{j=2}^d |z|^{-\widetilde{\alpha}_j/2} \Big(1 + \frac{|x_j - y_j|^2}{|z|}\Big)^{\widetilde{\alpha}_j/2} |G_z^1(x_j - y_j)| \\
&\le C_{d,k} |z|^{-\frac{k}{2}} \Big(1 + \frac{|\widetilde{x}-\widetilde{y}|^2}{|z|}\Big)^{\frac{k}{2}} |G_z^{d-1}(\widetilde{x}-\widetilde{y})| \\
&\le C_{d,k} \Big(|z|^{-\frac{k}{2}} + \frac{|\widetilde{x}-\widetilde{y}|^k}{|z|^k}\Big) |G_z^{d-1}(\widetilde{x}-\widetilde{y})|,
\end{aligned}$$

which proves (5.7.9).

Let $h \in C_c^\infty(\mathbb{R}_+^d; X)$. It follows from (5.7.9) that

$$\begin{aligned}
\|\partial^{\widetilde{\alpha}} T(te^{\mathrm{i}\delta}) h(x)\|_X &\le \int_{\mathbb{R}_+^d} |H_{te^{\mathrm{i}\delta}}^{1,\pm}(x_1, y_1)| |\partial_{\widetilde{x}}^{\widetilde{\alpha}} G_{te^{\mathrm{i}\delta}}^{d-1}(\widetilde{x}-\widetilde{y})| \|h(y)\|_X \,\mathrm{d}y \\
&\le C_{\delta,d,k} \int_{\mathbb{R}_+^d} \Big(t^{-\frac{k}{2}} + \frac{|\widetilde{x}-\widetilde{y}|^k}{t^k}\Big) |H_{te^{\mathrm{i}\delta}}^{1,\pm}(x_1, y_1)| |G_{te^{\mathrm{i}\delta}}^{d-1}(\widetilde{x}-\widetilde{y})| \|h(y)\|_X \,\mathrm{d}y \\
&\le C_{\delta,d,k} \sum_{m\in\{0,1\}} \int_{\mathbb{R}_+^d} \frac{|\widetilde{x}-\widetilde{y}|^{km}}{t^{\frac{(m+1)k}{2}}} H_{t\cos^{-1}(\delta)}^{d,\pm}(x,y) \|h(y)\|_X \,\mathrm{d}y,
\end{aligned} \tag{5.7.10}$$

where it was used that

$$\begin{aligned}
|H_{te^{\mathrm{i}\delta}}^{1,-}(x_1, y_1)| &= (4\pi t)^{-\frac{1}{2}} \Big| e^{-\frac{e^{-\mathrm{i}\delta}|x_1 - y_1|^2}{4t}} \Big| 1 - e^{\frac{-e^{-\mathrm{i}\delta} x_1 y_1}{t}} \Big| \\
&= (4\pi t)^{-\frac{1}{2}} e^{-\frac{\cos(\delta)|x_1 - y_1|^2}{4t}} \Big| \int_0^{\frac{x_1 y_1}{t}} e^{-e^{-\mathrm{i}\delta} s} \,\mathrm{d}s \Big| \\
&\le \frac{1}{\cos(\delta)} \frac{1}{\sqrt{4\pi t}} e^{-\frac{\cos(\delta)|x_1 - y_1|^2}{4t}} \big(1 - e^{-\frac{\cos(\delta) x_1 y_1}{t}}\big) = C_\delta H_{t\cos^{-1}(\delta)}^{1,-}(x_1, y_1), \\
|H_{te^{\mathrm{i}\delta}}^{1,+}(x_1, y_1)| &= (4\pi t)^{-\frac{1}{2}} e^{-\frac{\cos(\delta)|x_1 - y_1|^2}{4t}} \big(1 + e^{-\frac{\cos(\delta) x_1 y_1}{t}}\big) = H_{t\cos^{-1}(\delta)}^{1,+}(x_1, y_1), \\
|G_{te^{\mathrm{i}\delta}}^{d-1}(\widetilde{x}-\widetilde{y})| &= (4\pi t)^{-\frac{(d-1)}{2}} e^{-\frac{\cos(\delta)|\widetilde{x}-\widetilde{y}|^2}{4t}} = G_{t\cos^{-1}(\delta)}^{d-1}(\widetilde{x}-\widetilde{y}).
\end{aligned}$$

So by (5.7.8), (5.7.10) and the substitution $t \mapsto \cos(\delta) t$, we find

$$\begin{aligned}
&\Big\| x \mapsto \int_{\Gamma_+} f(z) \partial^{\widetilde{\alpha}} R(z, A)(\psi(x_1, \cdot) g)(x) \,\mathrm{d}z \Big\|_X \\
&\le \|f\|_{H^\infty(\Sigma_\omega)} \int_0^\infty \int_0^\infty e^{-tr\cos(\delta)} \|\partial^{\widetilde{\alpha}} T(te^{\mathrm{i}\delta})(\psi(x_1, \cdot) g)(x)\|_X \,\mathrm{d}t \,\mathrm{d}r \\
&= \frac{1}{\cos(\delta)} \|f\|_{H^\infty(\Sigma_\omega)} \int_0^\infty \|\partial^{\widetilde{\alpha}} T(te^{\mathrm{i}\delta})(\psi(x_1, \cdot) g)(x)\|_X \frac{\mathrm{d}t}{t} \\
&\le C_{\delta,d,k} \|f\|_{H^\infty(\Sigma_\omega)} \sum_{m\in\{0,1\}} \int_0^\infty \int_{\mathbb{R}_+^d} \frac{|\widetilde{x}-\widetilde{y}|^{km}}{t^{\frac{(m+1)k}{2}}} H_t^{d,\pm}(x,y) |\psi(x_1, y_1)| \|g(y)\|_X \,\mathrm{d}y \frac{\mathrm{d}t}{t}.
\end{aligned} \tag{5.7.11}$$

For any $x \in \mathbb{R}^d$ we have

$$\begin{aligned}\int_0^\infty t^{-\frac{(m+1)k}{2}} G_t^d(x)\,\frac{\mathrm{d}t}{t} &= C\int_0^\infty t^{-\frac{(m+1)k}{2}-\frac{d}{2}} e^{-\frac{|x|^2}{4t}}\,\frac{\mathrm{d}t}{t}\\ &= C|x|^{-d-(m+1)k}\int_0^\infty s^{\frac{(m+1)k}{2}+\frac{d}{2}} e^{-\frac{s}{4}}\,\frac{\mathrm{d}s}{s}\\ &= C_{d,k}|x|^{-d-(m+1)k}.\end{aligned}$$

Therefore, after the substitution $\widetilde{y} \mapsto \widetilde{x}-\widetilde{y}$ we obtain for $m \in \{0,1\}$

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^d_+} \frac{|\widetilde{x}-\widetilde{y}|^{km}}{t^{\frac{(m+1)k}{2}}} H_t^{d,\pm}(x,y)|\psi(x_1,y_1)|\|g(y)\|_X\,\mathrm{d}y\,\frac{\mathrm{d}t}{t}\\ &= \int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\int_0^\infty t^{-\frac{(m+1)k}{2}}\big(G_t^d(x_1-y_1,\widetilde{y}) \pm G_t^d(x_1+y_1,\widetilde{y})\big)\,\frac{\mathrm{d}t}{t}|\psi(x_1,y_1)|\|g(y_1,\widetilde{x}-\widetilde{y})\|_X\,\mathrm{d}y\\ &= C_{d,k}\int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\Big[\frac{1}{|(\frac{x_1}{y_1}-1,\frac{\widetilde{y}}{y_1})|^{d+(m+1)k}} \pm \frac{1}{|(\frac{x_1}{y_1}+1,\frac{\widetilde{y}}{y_1})|^{d+(m+1)k}}\Big]\\ &\qquad \cdot\Big|\frac{x_1^\theta}{y_1^\theta}-1\Big|^{k+1}\|g(y_1,\widetilde{x}-\widetilde{y})\|_X\,\frac{\mathrm{d}y}{y_1^{d+(m+1)k}}\\ &= \int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\ell_m^\pm\big(\frac{x_1}{y_1},\widetilde{y}\big)\|g(y_1,\widetilde{x}-y_1\widetilde{y})\|_X\,\frac{\mathrm{d}y}{y_1^{k+1}} =: L_m(x)\end{aligned}$$

performing the substitution $\widetilde{y} \mapsto y_1\widetilde{y}$ in the last step, where

$$\ell_m^\pm(v_1,\widetilde{v}) := \Big[\frac{1}{|(v_1-1,\widetilde{v})|^{d+(m+1)k}} \pm \frac{1}{|(v_1+1,\widetilde{v})|^{d+(m+1)k}}\Big]|v_1^\theta-1|^{k+1}, \quad (v_1,\widetilde{v}) \in \mathbb{R}^d_+.$$

Recall that $\gamma_0 = \gamma + kp - (k+1)\theta p$ and to show (5.7.7) we estimate the $L^p(\mathbb{R}^d_+, w_{\gamma_0})$-norm of the left-hand side of (5.7.11). To this end, it remains to bound $\|L_m\|_{L^p(\mathbb{R}^d_+,w_{\gamma_0})}$ for $m \in \{0,1\}$. With Minkowski's inequality and the substitutions $\widetilde{x} \mapsto \widetilde{x}+y_1\widetilde{y}$, $y_1 \mapsto x_1 y_1^{-1}$ and $x_1 \mapsto x_1 y_1$, we obtain

$$\begin{aligned}\|L_m\|_{L^p(\mathbb{R}^d_+,w_{\gamma_0})} &\lesssim \Big\|x_1 \mapsto \int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\ell_m^\pm\big(\frac{x_1}{y_1},\widetilde{y}\big)\|g(y_1,\cdot-y_1\widetilde{y})\|_{L^p(\mathbb{R}^{d-1};X)}\,\frac{\mathrm{d}y}{y_1^{k+1}}\Big\|_{L^p(\mathbb{R}_+,w_{\gamma_0})}\\ &= \Big\|x_1 \mapsto \int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\ell_m^\pm\big(\frac{x_1}{y_1},\widetilde{y}\big)\|g(y_1,\cdot)\|_{L^p(\mathbb{R}^{d-1};X)}\,\frac{\mathrm{d}y}{y_1^{k+1}}\Big\|_{L^p(\mathbb{R}_+,w_{\gamma_0})}\\ &= \Big\|x_1 \mapsto \int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\ell_m^\pm\big(y_1,\widetilde{y}\big)\Big\|g\big(\frac{x_1}{y_1},\cdot\big)\Big\|_{L^p(\mathbb{R}^{d-1};X)}x_1^{-k}y_1^{k-1}\,\mathrm{d}y\Big\|_{L^p(\mathbb{R}_+,w_{\gamma_0})}\\ &\le \int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\ell_m^\pm\big(y_1,\widetilde{y}\big)\Big\|x_1 \mapsto x_1^{-k}\Big\|g\big(\frac{x_1}{y_1},\cdot\big)\Big\|_{L^p(\mathbb{R}^{d-1};X)}\Big\|_{L^p(\mathbb{R}_+,w_{\gamma_0})}y_1^{k-1}\,\mathrm{d}y\\ &= \|g\|_{L^p(\mathbb{R}^d_+,w_{\gamma_0-kp};X)}\int_{\mathbb{R}^d_+} |\widetilde{y}|^{km}\ell_m^\pm\big(y_1,\widetilde{y}\big)y_1^{\frac{\gamma+1}{p}-(k+1)\theta+k-1}\,\mathrm{d}y,\end{aligned}$$

which yields (5.7.7) if the latter integral is finite for the given ranges of γ. Indeed, note that for any $a \in \mathbb{R}$ we have using the substitution $\widetilde{y} \mapsto a\widetilde{y}$

$$\int_{\mathbb{R}^{d-1}} \frac{|\widetilde{y}|^{km}}{|(a,\widetilde{y})|^{d+(m+1)k}}\,\mathrm{d}\widetilde{y} = \frac{1}{|a|^{k+1}}\int_{\mathbb{R}^{d-1}} \frac{|\widetilde{y}|^{km}}{|(1,\widetilde{y})|^{d+(m+1)k}}\,\mathrm{d}\widetilde{y} = \frac{C_{d,k,m}}{|a|^{k+1}},$$

therefore

$$\begin{aligned}\int_{\mathbb{R}^d_+} |\widetilde{y}|^{km} \ell^{\pm}_m(y_1,\widetilde{y}) y_1^{\frac{\gamma+1}{p}-(k+1)\theta+k-1}\,\mathrm{d}y \\ &= C_{d,k,m}\int_{\mathbb{R}_+}\Big[\frac{1}{|y_1-1|^{k+1}} \pm \frac{1}{|y_1+1|^{k+1}}\Big]|y_1^\theta-1|^{k+1} y_1^{\frac{\gamma+1}{p}-(k+1)\theta+k-1}\,\mathrm{d}y_1 \\ &=: C_{d,k,m}\int_{\mathbb{R}_+} \ell^{\pm}(y_1)|y_1^\theta-1|^{k+1} y_1^{\frac{\gamma+1}{p}-(k+1)\theta+k-1}\,\mathrm{d}y_1. \end{aligned} \tag{5.7.12}$$

For $y_1 \in (0,\frac{1}{2})$ we have $\ell^{\pm}(y_1) \le C$ and $|y_1^\theta-1|^{k+1} \le C$ since $\theta>0$. Moreover, by (5.7.6) it follows

$$\frac{\gamma+1}{p}-(k+1)\theta+k-1 > -1,$$

so that the integral in (5.7.12) over $y_1 \in (0,\frac{1}{2})$ is finite. For $y_1 \in (\frac{1}{2},\frac{3}{2})$ it follows by the mean value theorem that there is no singularity at $y_1=1$ and that the integral in (5.7.12) over the interval $(\frac{1}{2},\frac{3}{2})$ is finite. Finally, if $y_1>\frac{3}{2}$, then $(y_1^\theta-1)^{k+1} \le C y_1^{(k+1)\theta}$ and

$$\ell^{\pm}(y_1) = \frac{(y_1+1)^{k+1} \pm (y_1-1)^{k+1}}{(y_1-1)^{k+1}(y_1+1)^{k+1}} \le C \begin{cases} \frac{1}{y_1^{k+1}} & \text{if } A=\lambda-\Delta_{\mathrm{Neu}} \\ \frac{1}{y_1^{k+2}} & \text{if } A=\lambda-\Delta_{\mathrm{Dir}} \end{cases},$$

where we recall that the Dirichlet and Neumann Laplacian correspond to ℓ^- and ℓ^+, respectively. Therefore, using that $\gamma \in (-1,2p-1)\setminus\{p-1\}$ for the Dirichlet Laplacian, we obtain

$$\int_{\frac{3}{2}}^\infty \ell^-(y_1)|y_1^\theta-1|^{k+1} y_1^{\frac{\gamma+1}{p}-(k+1)\theta+k-1}\,\mathrm{d}y_1 \le C\int_{\frac{3}{2}}^\infty y_1^{\frac{\gamma+1}{p}-3}\,\mathrm{d}y_1 < \infty.$$

Similarly, using that $\gamma \in (-1,p-1)$ for the Neumann Laplacian, we obtain

$$\int_{\frac{3}{2}}^\infty \ell^+(y_1)|y_1^\theta-1|^{k+1} y_1^{\frac{\gamma+1}{p}-(k+1)\theta+k-1}\,\mathrm{d}y_1 \le C\int_{\frac{3}{2}}^\infty y_1^{\frac{\gamma+1}{p}-2}\,\mathrm{d}y_1 < \infty.$$

This proves (5.7.7) and therefore this finishes the proof. □

We can now give the proof of the bounded H^∞-calculus for the Dirichlet Laplacian using Lemmas 5.7.3-5.7.5 and 5.7.8.

Proof of Theorem 5.7.1. For $\lambda>0$, we show by induction on $k\ge 0$ that $A_{\mathrm{Dir}} := \lambda-\Delta_{\mathrm{Dir}}$ on $W^{k,p}(w_{\gamma+kp}) := W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(A_{\mathrm{Dir}})=0$. Throughout the proof, C denotes a constant only depending on $p,k,\gamma,\lambda,\omega,d$ and X, which may change from line to line.

The case $k=0$ is stated in Theorem 5.4.2. For some fixed $k\in\mathbb{N}_0$ we assume that A_{Dir} on $W^{k,p}(w_{\gamma+kp})$ has a bounded H^∞-calculus of angle 0 for all $\gamma\in(-1,2p-1)\setminus\{p-1\}$. That is, for $\omega\in(0,\frac{\pi}{2})$, we have the estimate

$$\|f(A_{\mathrm{Dir}})u\|_{W^{k,p}(w_{\gamma+kp})} \le C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k,p}(w_{\gamma+kp})}, \tag{5.7.13}$$

for all $\gamma \in (-1, 2p-1) \setminus \{p-1\}$, $f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$ and $u \in W^{k,p}(w_{\gamma+kp})$. We prove that A_{Dir} has a bounded $H^\infty(\Sigma_\sigma)$-calculus for $\sigma \in (\omega, \frac{\pi}{2})$ on $W^{k+1,p}(w_{\gamma+(k+1)p})$.

Let $0 < \nu < \omega < \sigma < \frac{\pi}{2}$, $f \in H^1(\Sigma_\sigma) \cap H^\infty(\Sigma_\sigma)$ and $u \in C_c^\infty(\mathbb{R}^d_+; X)$. By writing $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ we obtain

$$\begin{aligned}\|f(A_{\mathrm{Dir}})u\|_{W^{k+1,p}(w_{\gamma+(k+1)p})} &= \|f(A_{\mathrm{Dir}})u\|_{W^{k,p}(w_{\gamma+(k+1)p})} + \sum_{\substack{|\alpha|=k+1\\ \alpha_1\geq 1}} \|\partial^\alpha f(A_{\mathrm{Dir}})u\|_{L^p(w_{\gamma+(k+1)p})} \\ &\quad + \sum_{|(0,\widetilde{\alpha})|=k+1} \|\partial^{\widetilde{\alpha}} f(A_{\mathrm{Dir}})u\|_{L^p(w_{\gamma+(k+1)p})} =: T_1 + T_2 + T_3.\end{aligned}$$

To apply the induction hypothesis we use that all the resolvents on the spaces we consider are consistent by Lemma 5.6.4. Furthermore, we define the downwards orientated contour $\Gamma_\nu = \partial(\Sigma_\nu \setminus B(0, \frac{\lambda}{2}))$, see Remark 2.3.5.

By Lemma 5.3.5 (using Remark 5.3.1), Lemma 5.7.3(iii) and 5.7.4, the induction hypothesis (5.7.13) and Hardy's inequality (Corollary 5.3.2), it follows that

$$\begin{aligned}T_1 &= \|M^{-1}Mf(A_{\mathrm{Dir}})u\|_{W^{k,p}(w_{\gamma+(k+1)p})} \lesssim \|Mf(A_{\mathrm{Dir}})u\|_{W^{k,p}(w_{\gamma+kp})} \\ &\leq \|f(A_{\mathrm{Dir}})Mu\|_{W^{k,p}(w_{\gamma+kp})} + 2\Big\|\int_{\Gamma_\nu} f(z)R(z,A_{\mathrm{Dir}})\partial_1 R(z,A_{\mathrm{Dir}})u\,\mathrm{d}z\Big\|_{W^{k,p}(w_{\gamma+kp})} \\ &\leq C\|f\|_{H^\infty(\Sigma_\omega)}\big(\|Mu\|_{W^{k,p}(w_{\gamma+kp})} + \|u\|_{W^{k,p}(w_{\gamma+kp})}\big) \\ &\leq C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(w_{\gamma+(k+1)p})}.\end{aligned}$$

We continue with T_2, so let $\alpha = (\alpha_1, \widetilde{\alpha})$ with $|\alpha| = k+1$ and $\alpha_1 \geq 1$. Using that $M\partial_1\partial_1^\ell v = \partial_1^\ell M\partial_1 v - \ell\partial_1^\ell v$ for $\ell = \alpha_1 - 1$, we compute

$$\begin{aligned}&\|\partial_1^{\alpha_1}\partial^{\widetilde{\alpha}} f(A_{\mathrm{Dir}})u\|_{L^p(w_{\gamma+(k+1)p})} \\ &\qquad = \|M\partial_1\partial_1^{\alpha_1-1}\partial^{\widetilde{\alpha}} f(A_{\mathrm{Dir}})u\|_{L^p(w_{\gamma+kp})} \\ &\qquad \leq \|M\partial_1 f(A_{\mathrm{Dir}})u\|_{W^{k,p}(w_{\gamma+kp})} + (\alpha_1-1)\|f(A_{\mathrm{Dir}})u\|_{W^{k,p}(w_{\gamma+kp})}.\end{aligned}$$

The second term can be estimated with the induction hypothesis (5.7.13) and Hardy's inequality. For the first term, by Lemma 5.7.3(ii)+(iii), the induction hypothesis (5.7.13), Lemma 5.7.5 and Hardy's inequality, we obtain

$$\begin{aligned}&\|M\partial_1 f(A_{\mathrm{Dir}})u\|_{W^{k,p}(w_{\gamma+kp})} \\ &\quad \leq \|f(A_{\mathrm{Dir}})M\partial_1 u\|_{W^{k,p}(w_{\gamma+kp})} + 2\Big\|\int_{\Gamma_\nu} f(z)R^2(z,A_{\mathrm{Dir}})\partial_1^2 u\,\mathrm{d}z\Big\|_{W^{k,p}(w_{\gamma+kp})} \\ &\quad \leq C\|f\|_{H^\infty(\Sigma_\omega)}\|M\partial_1 u\|_{W^{k,p}(w_{\gamma+kp})} + C\|f\|_{H^\infty(\Sigma_\sigma)}\|u\|_{W^{k,p}(w_{\gamma+kp})} \\ &\quad \leq C\|f\|_{H^\infty(\Sigma_\sigma)}\|u\|_{W^{k+1,p}(w_{\gamma+(k+1)p})}.\end{aligned} \tag{5.7.14}$$

This shows the estimate

$$T_2 \leq C\|f\|_{H^\infty(\Sigma_\sigma)}\|u\|_{W^{k+1,p}(w_{\gamma+(k+1)p})}.$$

From Lemma 5.7.8(i)+(ii) it immediately follows that

$$T_3 = \sum_{|(0,\widetilde{\alpha})|=k+1} \|\partial^{\widetilde{\alpha}} f(A_{\mathrm{Dir}})u\|_{L^p(w_{\gamma+(k+1)p})} \leq C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(w_{\gamma+(k+1)p})}.$$

Combining the above estimates for T_1, T_2 and T_3 yields that for any $0 < \omega < \sigma < \frac{\pi}{2}$

$$\|f(A_{\mathrm{Dir}})u\|_{W^{k+1,p}(w_{\gamma+(k+1)p})} \leq C\|f\|_{H^\infty(\Sigma_\sigma)}\|u\|_{W^{k+1,p}(w_{\gamma+(k+1)p})}, \quad u \in C_c^\infty(\mathbb{R}^d_+; X)$$

and by density (Lemma 5.3.3) this estimate extends to $u \in W^{k+1,p}(w_{\gamma+(k+1)p})$. Since ω and σ were arbitrary, this proves that A_{Dir} has a bounded H^∞-calculus on $W^{k+1,p}(w_{\gamma+(k+1)p})$ of angle 0. This finishes the induction argument.

Finally, we show that if $\gamma + kp \in (-1, 2p-1)$, then we can allow for $\lambda = 0$. For $k = 0$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ the result is already contained in Theorem 5.4.2. Therefore, it remains to consider $k = 1$ and $\gamma \in (-1, p-1)$. From here on, let $C > 0$ be a constant that is independent of λ. Note that $\gamma + p \in (p-1, 2p-1)$, so that by Theorem 5.4.2 and [126, Proposition 16.2.6]

$$T_1 = \|f(A_{\mathrm{Dir}})u\|_{L^p(w_{\gamma+p})} \leq C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{L^p(w_{\gamma+p})}.$$

Using Theorem 5.4.2 instead of (5.7.13) in (5.7.14) with $k = 0$ yields

$$T_2 = \|M\partial_1 f(A_{\mathrm{Dir}})u\|_{L^p(w_\gamma)} \leq C\|f\|_{H^\infty(\Sigma_\sigma)}\|u\|_{W^{1,p}(w_{\gamma+p})},$$

using that the constant in Lemma 5.7.5 is independent of λ. Finally, by Lemma 5.7.3(i) and Theorem 5.4.2 we obtain

$$\begin{aligned} T_3 = \sum_{j=2}^d \|f(A_{\mathrm{Dir}})\partial_j u\|_{L^p(w_{\gamma+p})} &\leq C\|f\|_{H^\infty(\Sigma_\omega)} \sum_{j=2}^d \|\partial_j u\|_{L^p(w_{\gamma+p})} \\ &\leq C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{1,p}(w_{\gamma+p})}. \end{aligned}$$

By combining the above estimates, we have for $\gamma \in (-1, p-1)$ and for any $\lambda > 0$

$$\|f(\lambda - \Delta_{\mathrm{Dir}})u\|_{W^{1,p}(w_{\gamma+p})} \leq C\|f\|_{H^\infty(\Sigma_\sigma)}\|u\|_{W^{1,p}(w_{\gamma+p})},$$

where the constant C is independent of λ. Letting $\lambda \downarrow 0$ and using the dominated convergence theorem yields that $-\Delta_{\mathrm{Dir}}$ on $W^{1,p}(w_{\gamma+p})$ has a bounded H^∞-calculus of angle 0. □

We conclude this section with the proof of the H^∞-calculus for the Neumann Laplacian, which can be derived from Theorem 5.7.1 for the Dirichlet Laplacian and Lemmas 5.7.6-5.7.8.

Proof of Theorem 5.7.2. We prove that $A_{\mathrm{Neu}} := \lambda - \Delta_{\mathrm{Neu}}$ on

$$W^{k+1,p}(w_{\gamma+kp}) := W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$$

has a bounded H^∞-calculus of angle $\omega_{H^\infty}(A_{\mathrm{Neu}}) = 0$ for $\lambda > 0$. Note that if $\gamma + kp \in (-1, p-1)$, then the result is already contained in Theorem 5.4.6 and holds for $\lambda = 0$.

We first consider the case $k \in \mathbb{N}_0$ and $\gamma \in (p-1, 2p-1)$. Let $0 < \nu < \omega < \frac{\pi}{2}$, $f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$ and $u \in C_c^\infty(\mathbb{R}^d_+; X)$. By writing $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ we obtain

$$\begin{aligned}\|f(A_{\mathrm{Neu}})u\|_{W^{k+1,p}(w_{\gamma+kp})} &= \|f(A_{\mathrm{Neu}})u\|_{W^{k,p}(w_{\gamma+kp})} \\ &\quad + \sum_{\substack{|\alpha|=k+1 \\ \alpha_1 \geq 1}} \|\partial^\alpha f(A_{\mathrm{Neu}})u\|_{L^p(w_{\gamma+kp})} \\ &\quad + \sum_{|(0,\widetilde{\alpha})|=k+1} \|\partial^{\widetilde{\alpha}} f(A_{\mathrm{Neu}})u\|_{L^p(w_{\gamma+kp})} =: T_1 + T_2 + T_3.\end{aligned}$$

We use that all the resolvents on the spaces we consider are consistent by Lemma 5.6.5. Throughout the proof, C denotes a constant only depending on $p, k, \gamma, \lambda, \omega, d$ and X, which may change from line to line. Moreover, we write $A_{\mathrm{Dir}} := \lambda - \Delta_{\mathrm{Dir}}$.

By Lemma 5.3.5 (using Remark 5.3.1 and $\gamma > p-1$), Lemma 5.7.7 and Theorem 5.7.1 with $\gamma - p \in (-1, p-1)$, it follows that

$$\begin{aligned}T_1 &= \|M^{-1}Mf(A_{\mathrm{Neu}})u\|_{W^{k,p}(w_{\gamma+kp})} \lesssim \|Mf(A_{\mathrm{Neu}})u\|_{W^{k,p}(w_{\gamma-p+kp})} \\ &\leq \|f(A_{\mathrm{Dir}})Mu\|_{W^{k,p}(w_{\gamma-p+kp})} + C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k,p}(w_{\gamma-p+kp})} \\ &\leq C\|f\|_{H^\infty(\Sigma_\omega)}\big(\|Mu\|_{W^{k,p}(w_{\gamma-p+kp})} + \|u\|_{W^{k,p}(w_{\gamma-p+kp})}\big) \\ &\leq C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(w_{\gamma+kp})},\end{aligned}$$

where we used Hardy's inequality (Corollary 5.3.2) in the last step. By Lemma 5.7.6(iii) and Theorem 5.7.1, we obtain

$$\begin{aligned}T_2 &= \sum_{|\alpha|=k} \|\partial^\alpha \partial_1 f(A_{\mathrm{Neu}})u\|_{L^p(w_{\gamma+kp})} \leq \|f(A_{\mathrm{Dir}})\partial_1 u\|_{W^{k,p}(w_{\gamma+kp})} \\ &\leq C\|f\|_{H^\infty(\Sigma_\omega)}\|\partial_1 u\|_{W^{k,p}(w_{\gamma+kp})} \leq C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(w_{\gamma+kp})}.\end{aligned}$$

Finally, using Lemma 5.7.8(iii) with $k+1$ and $\gamma - p \in (-1, p-1)$, gives

$$T_3 = \sum_{|(0,\widetilde{\alpha})|=k+1} \|\partial^{\widetilde{\alpha}} f(A_{\mathrm{Neu}})u\|_{L^p(w_{\gamma-p+(k+1)p})} \leq C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(w_{\gamma+kp})}.$$

The above estimates and density (Lemma 5.3.3) prove boundedness of the H^∞-calculus for A_{Neu} on $W^{k+1,p}(w_{\gamma+kp})$ with $k \in \mathbb{N}_1$ and $\gamma \in (p-1, 2p-1)$.

It remains to consider the case $k \in \mathbb{N}_1$ and $\gamma \in (-1, p-1)$. Let $\omega \in (0, \frac{\pi}{2})$, $f \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$ and $u \in C_{c,1}^\infty(\overline{\mathbb{R}^d_+}; X)$. Then writing $\alpha = (0, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ gives

$$\|f(A_{\mathrm{Neu}})u\|_{W^{k+1,p}(w_{\gamma+kp})} = \sum_{|(0,\widetilde{\alpha})|\leq 1} \|\partial^\alpha f(A_{\mathrm{Neu}})u\|_{W^{k,p}(w_{\gamma+kp})} + \|\partial_1 f(A_{\mathrm{Neu}})u\|_{W^{k,p}(w_{\gamma+kp})}.$$

We estimate these terms separately using that the resolvents on the spaces we consider are consistent by Lemma 5.6.5. Moreover, note that $\gamma + p \in (p-1, 2p-1)$, so that $W^{k,p}(w_{\gamma+kp}) = W^{(k-1)+1,p}(w_{\gamma+p+(k-1)p})$ and we can apply the result proved above with

$k-1$. Together with Lemma 5.7.6(i), we obtain

$$\begin{aligned}\sum_{|(0,\widetilde{\alpha})|\le 1}\|\partial^\alpha f(A_{\mathrm{Neu}})u\|_{W^{k,p}(w_{\gamma+kp})} &= \sum_{|(0,\widetilde{\alpha})|\le 1}\|f(A_{\mathrm{Neu}})\partial^\alpha u\|_{W^{(k-1)+1,p}(w_{\gamma+p+(k-1)p})}\\ &\le C\|f\|_{H^\infty(\Sigma_\omega)}\sum_{|(0,\widetilde{\alpha})|\le 1}\|\partial^\alpha u\|_{W^{(k-1)+1,p}(w_{\gamma+p+(k-1)p})}\\ &\le C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(w_{\gamma+kp})}.\end{aligned}$$

Moreover, using Lemma 5.7.6(iii) and Theorem 5.7.1 for $\gamma\in(-1,p-1)$, yield

$$\begin{aligned}\|\partial_1 f(A_{\mathrm{Neu}})u\|_{W^{k,p}(w_{\gamma+kp})} &= \|f(A_{\mathrm{Dir}})\partial_1 u\|_{W^{k,p}(w_{\gamma+kp})}\\ &\le C\|f\|_{H^\infty(\Sigma_\omega)}\|\partial_1 u\|_{W^{k,p}(w_{\gamma+kp})} \le C\|f\|_{H^\infty(\Sigma_\omega)}\|u\|_{W^{k+1,p}(w_{\gamma+kp})}.\end{aligned}$$

The above estimates and density (see Lemma 5.3.3) prove boundedness of the H^∞-calculus for A_{Neu} in the case $\gamma\in(-1,p-1)$. This completes the proof. □

6

Functional calculus for the Laplacian on rough domains

This chapter is based on:

N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on rough domains. *J. Differ. Equ.*, 454:113884, 2026.

Abstract. *We study the Laplace operator on domains subject to Dirichlet or Neumann boundary conditions. We show that these operators admit a bounded H^∞-functional calculus on weighted Sobolev spaces, where the weights are powers of the distance to the boundary. Our analysis applies to bounded $C^{1,\kappa}$-domains with $\kappa \in [0,1]$, revealing a crucial trade-off: lower domain regularity can be compensated by enlarging the weight exponent.*

6.1. Introduction

This chapter contributes to the extensive study of the Laplace operator on domains with minimal boundary regularity (often referred to as rough domains), see, e.g., [128, 130, 131, 267, 277] and the monographs [111, 203] and references therein. In particular, we are interested in the H^∞-functional calculus for the Laplacian on inhomogeneous weighted Sobolev spaces. The H^∞-functional calculus provides a powerful framework for establishing well-posedness and regularity results for (possibly nonlinear) partial and stochastic partial differential equations ((S)PDEs). Therefore, the H^∞-calculus for sectorial operators is widely studied, see for instance [67, 125, 126, 174] and the references therein. Applications to PDEs and SPDEs can, e.g., be found in [65, 69, 134, 175, 234, 274] and [11, 13, 219, 221], respectively.

Given a bounded C^2-domain $\mathcal{O} \subseteq \mathbb{R}^d$, it is well known that the Laplacian with Dirichlet boundary conditions on $L^p(\mathcal{O})$ with $p \in (1,\infty)$ and domain $W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O})$ generates an analytic C_0-semigroup, has the maximal regularity property and admits a bounded H^∞-functional calculus. However, if the regularity of $\mathcal{O}$ is too low (say Lipschitz or C^1), these properties fail and explicit counterexamples can be constructed, see [60, 203]. In such counterexamples, the derivatives of the solutions to the resolvent equation

$$\begin{aligned} \lambda u - \Delta u &= f, \\ u|_{\partial\mathcal{O}} &= 0, \end{aligned}$$

can drastically blow up near the boundary $\partial\mathcal{O}$. As a consequence, the canonical domain of the Dirichlet Laplacian on $L^p(\mathcal{O})$ is no longer a closed subspace of $W^{2,p}(\mathcal{O})$. Moreover, if one is interested in higher-order Sobolev regularity of the solution u, then more smoothness of $\mathcal{O}$ is needed (see [89, 166]), and additional boundary conditions for the data f (compatibility conditions) need to be imposed (see [66]). These additional boundary conditions for the data occur, in particular, in the study of mixed-order systems (see [74]).

To set up a satisfying well-posedness and regularity theory for PDEs *without* such additional regularity or compatibility conditions, one can use a weighted function space for the solution u. In particular, one can consider spatial weights of the form $w_\gamma^{\partial\mathcal{O}}(x) := \operatorname{dist}(x,\partial\mathcal{O})^\gamma$ for some suitable $\gamma \in \mathbb{R}$, which compensate the blow-up of the derivatives of the solution near $\partial\mathcal{O}$ and relax compatibility conditions. Partial differential equations on weighted spaces have already been studied extensively, see for instance [79, 80, 81, 144, 150, 164, 165, 218] for deterministic equations and [141, 142, 143, 160, 167] for stochastic equations.

As stated, we are interested in the H^∞-functional calculus for the Laplacian on inhomogeneous weighted Sobolev spaces of order $k \in \mathbb{N}_0$. This was studied in [190] and Chapter 5 for the Dirichlet and Neumann Laplacian on the half-space $\mathbb{R}^d_+$. In the present chapter, we extend the results to bounded domains $\mathcal{O}$ with minimal smoothness, while ensuring that the canonical domain of the Laplacian is a closed subspace of a weighted Sobolev space of order $k+2$.

Our main result for the Dirichlet Laplacian is as follows, see Theorems 6.6.2 and 6.6.4. For the definition of the involved spaces, the reader is referred to Section 6.3.

Theorem 6.1.1 (H^∞-calculus for the Dirichlet Laplacian)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$ and $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. Furthermore, suppose that*

$$\kappa > 1 - \tfrac{\gamma+1}{p} \qquad \text{or, equivalently} \qquad \gamma > (1-\kappa)p - 1$$

and $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain. Then for all $\lambda \geq 0$ the operator

$$\lambda - \Delta_{\mathrm{Dir}} \quad \text{on} \quad W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \quad \text{with} \quad D(\Delta_{\mathrm{Dir}}) = W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$$

has a bounded H^∞-calculus of angle zero.

Theorem 6.1.1 generalises the result in [190, Theorem 6.1], which is restricted to the case $k = 0$ and to bounded C^2-domains. Theorem 6.1.1 allows for bounded C^1-domains if $\gamma \in (p-1, 2p-1)$, while for $\gamma \in (-1, p-1)$ we obtain that the smoothness of the domain may depend on the weight: if the power of the weight is larger, then a rougher domain is allowed. The smoothness parameter κ is almost optimal. Indeed, solving the Dirichlet problem in the scale of weighted Sobolev spaces with a gain of two derivatives for the solution requires the boundary of the domain to have $W^{2-(\gamma+1)/p,p}$-smoothness, see [203, Theorem 15.6.1 applied to $\ell = 2-(\gamma+1)/p$] and [203, Section 14.6.1] for an explicit counterexample with C^1-domains. Furthermore, for $\gamma = p-1$ the domain characterisation in Theorem 6.1.1 in terms of spaces with vanishing traces fails, see Remark 5.4.3, and for this reason we omit this case.

Concerning the Neumann Laplacian on bounded domains, we prove the following result, see Theorems 6.6.3 and 6.6.5.

Theorem 6.1.2 (H^∞-calculus for the Neumann Laplacian)**.** *Let $p \in (1,\infty)$ and $\kappa \in (0,1]$. Furthermore, suppose that either*

(i) $k \in \mathbb{N}_0$, $\gamma \in (p-1, 2p-1)$, $\kappa > 2 - \tfrac{\gamma+1}{p}$ *and $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain, or,*

(ii) $k \in \mathbb{N}_1$, $\gamma \in (-1, p-1)$, $\kappa > 1 - \tfrac{\gamma+1}{p}$ *and $\mathscr{O}$ is a bounded $C^{2,\kappa}$-domain.*

Then for all $\lambda > 0$ the operator

$$\lambda - \Delta_{\mathrm{Neu}} \quad \text{on} \quad W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+(k-1)p}) \quad \text{with} \quad D(\Delta_{\mathrm{Neu}}) = W^{k+2,p}_{\mathrm{Neu}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+(k-1)p})$$

has a bounded H^∞-calculus of angle zero. Moreover, using function spaces modulo constants gives the result for all $\lambda \geq 0$.

Note that, compared to Theorem 6.1.1, the Sobolev spaces in Theorem 6.1.2 have a smaller weight exponent, which is consistent with Theorem 5.1.2. Figure 6.1 visualises the parameters of the spaces in Theorem 6.1.1 and 6.1.2 where we obtain a bounded H^∞-calculus. Similar to the case of Dirichlet boundary conditions, we expect that the regularity of the domain in Theorem 6.1.2 is almost optimal as well, see [203, Section 15.6] for some related results in this direction.

The main novelties of our results are the following.

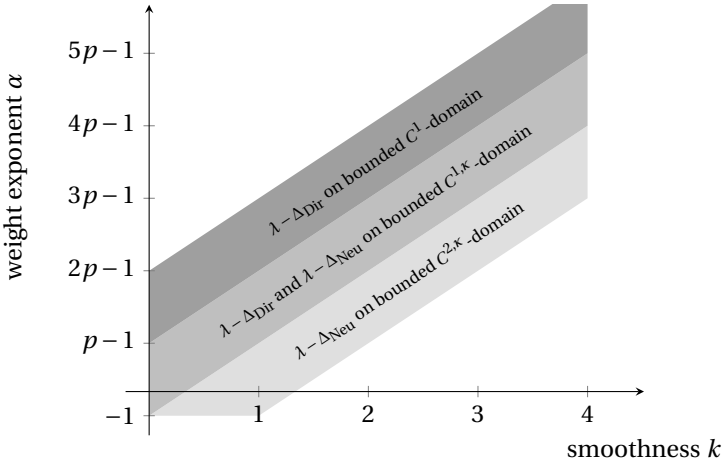


Figure 6.1.: The spaces $W^{k,p}(\mathscr{O}, w_\alpha^{\partial\mathscr{O}})$ where $\lambda - \Delta_{\mathrm{Dir}}$ and $\lambda - \Delta_{\mathrm{Neu}}$ as in Theorems 6.1.1 and 6.1.2 (with $\alpha = \gamma + kp$ and $\alpha = \gamma + (k-1)p$, respectively) admit a bounded H^∞-calculus.

(i) We prove the boundedness of the H^∞-calculus, which is, in general, much harder to prove than maximal regularity and yields the boundedness of many singular integral operators [137]. In particular, boundedness of the H^∞-calculus implies (stochastic) maximal regularity [126, 219]. Maximal regularity and higher-order regularity results for the heat equation with Dirichlet and Neumann boundary conditions are studied in Chapter 7. In particular, we can recover some maximal regularity results for the Dirichlet Laplacian from [149] (for bounded C^1-domains) and [157] (for bounded $C^{1,\kappa}$-domains and $k = 0$). For the latter case, our results with $k \geq 1$ are new. The Neumann Laplacian on the half-space is studied on weighted Sobolev spaces in [80, 81] (for $k = 0$) and in Chapter 5, but a systematic study on bounded domains seems to be unavailable until now.

(ii) The smoothness of the domain $\mathscr{O}$ in Theorems 6.1.1 and 6.1.2 is *independent* of the smoothness k of the Sobolev space. The reason for this is that we do not use the standard localisation procedure from the half-space to domains (see, e.g., [67, 89, 166]). This standard localisation procedure typically works for C^{k+2}-domains. Instead, we apply a more sophisticated C^1-diffeomorphism suitable for the weighted setting. We discuss this in more detail below.

The key ingredient in the proofs of Theorems 6.1.1 and 6.1.2 is the perturbation of the H^∞-calculus on the half-space (obtained in Chapter 5) to *special domains*, i.e. domains above the graph of a function with compact support. A common method is to relate the Laplacian on the half-space and on a special domain via a diffeomorphism. However, due to the low regularity of the domain, we cannot use the standard diffeomorphism as in, e.g., [67, 89, 166, 275]. Instead, we construct a variant of the *Dahlberg–Kenig–Stein pullback*, based on ideas in [140, 149, 275]. This diffeomorphism straightens the boundary,

preserves the distance to the boundary and leaves the direction of the normal vector at the boundary invariant. Moreover, higher-order derivatives exist, but blow up near the boundary of the domain. This blow-up is compensated by the weights in our spaces.

With estimates on this diffeomorphism at hand, we can employ perturbation theorems for the H^∞-calculus to extend the results to special domains. Another difficulty arising in this perturbation argument is that, if the regularity of the domain is too low, then the perturbations are of the same order as the Laplacian. It is known that the H^∞-calculus is not stable under small perturbations [207]. Additionally, we need the perturbations to be well behaved with respect to a fractional power of the original operator. This requires the identification of certain complex interpolation spaces and fractional domains to perform the perturbation argument. Finally, by another localisation argument, based on lower-order perturbations, the H^∞-calculus on special domains is transferred to bounded domains.

We comment on some related and open problems. Theorems 6.1.1 and 6.1.2 provide the bounded H^∞-calculus on Sobolev spaces with integer smoothness, and with complex interpolation, the bounded H^∞-calculus can also be obtained on spaces with fractional smoothness. However, an intrinsic characterisation of these complex interpolation spaces seems unavailable. Furthermore, we expect that our results can be extended to spaces with negative smoothness via duality. Some results for the weak (Dirichlet) Laplacian on weighted spaces are contained in [38, 228].

An interesting question regarding the smoothness of the domain is whether for $\gamma \in (p-1, 2p-1)$ the assumption of C^1-domains can be weakened to Lipschitz domains. In general, the analysis for Lipschitz domains becomes much more involved and different techniques are required than for C^1-domains, see for instance [128, 130, 131, 277] and the references therein. We believe that our method should work for domains with a small Lipschitz character. The H^∞-calculus on Lipschitz domains could be important for studying SPDEs in the weighted setting, see [145, 146, 147, 148], where the range of weights is significantly smaller than $\gamma \in (p-1, 2p-1)$.

Outline

The outline of this chapter is as follows. In Section 6.2 we recall some perturbation results for the functional calculus. In Section 6.3 we study weighted Sobolev spaces on domains and prove characterisations for these spaces. In Section 6.4 we prove results on the fractional domains of the Laplacian on the half-space, which are required for perturbation of the H^∞-calculus. In Section 6.5 we perturb the H^∞-calculus from the half-space to special domains, and in Section 6.6 we perform a localisation procedure to obtain the H^∞-calculus on bounded domains.

6.2. Perturbation theorems for functional calculus

In this section, we recall some perturbation theorems for the H^∞-calculus that are required later on in this chapter. To do so, we first list some properties about fractional domains.

6.2.1. Fractional domains

Let A be a sectorial operator and let $\alpha \in \mathbb{C}$. To define fractional powers A^α, we need a functional calculus allowing for holomorphic functions of polynomial growth. This is known as the *extended functional calculus* and the reader is referred to [126, Chapter 15] or [174, Appendix 15.C] for a detailed study of extended functional calculi and fractional powers. In particular, A^α is again sectorial.

A sectorial operator A on a Banach space X has *bounded imaginary powers* (BIP) if A^{is} extends to a bounded operator on X for every $s \in \mathbb{R}$. The angle is given by $\omega_{\mathrm{BIP}}(A) := \inf\{\omega \in \mathbb{R} : \sup_{s\in\mathbb{R}} e^{-\omega|s|}\|A^{is}\| < \infty\}$. Moreover, a bounded H^∞-calculus implies BIP and $\omega_{\mathrm{BIP}}(A) \le \omega_{H^\infty}(A)$, see [126, Section 15.3].

We recall a result on the interpolation of fractional domains. For details on interpolation theory, the reader is referred to [27] and [262].

Proposition 6.2.1 ([126, Corollary 15.3.10])**.** *Let A be a sectorial operator on a Banach space X and assume that A has* BIP. *Then for all $\theta \in (0,1)$ and $0 \le \alpha < \beta$ we have*

$$D(A^{(1-\theta)\alpha+\theta\beta}) = [D(A^\alpha), D(A^\beta)]_\theta.$$

Moreover, by [126, Proposition 15.2.12] we have for a sectorial operator A that $D((\lambda + A)^\alpha) = D(A^\alpha)$ for all $\lambda \ge 0$ and $\alpha > 0$.

6.2.2. Perturbation of the H^∞-calculus

We collect some known perturbation results for the H^∞-calculus. For further perturbation results for the H^∞-calculus, the reader is referred to [126, 134, 136, 174]. We start with a result for shifting the H^∞-calculus.

Proposition 6.2.2 ([126, Proposition 16.2.6])**.** *Let A be a sectorial operator on a Banach space X and let $\omega \in (\omega(A), \pi)$.*

(i) If A has a bounded $H^\infty(\Sigma_\omega)$-calculus, then $\lambda + A$ has a bounded $H^\infty(\Sigma_\omega)$-calculus for all $\lambda > 0$. Moreover, the constant in the estimate for the H^∞-calculus can be taken independent of λ.

(ii) If $\lambda_0 + A$ has a bounded $H^\infty(\Sigma_\omega)$-calculus for some $\lambda_0 > 0$, then $\lambda + A$ has a bounded $H^\infty(\Sigma_\omega)$-calculus for all $\lambda > 0$.

In the case of a lower-order perturbation, we have the following result.

Theorem 6.2.3 ([126, Theorem 16.2.7])**.** *Let A be a sectorial operator on a Banach space X. Let $\omega \in (\omega(A), \pi)$ and assume that A has a bounded $H^\infty(\Sigma_\omega)$-calculus. Let $\alpha \in (0,1)$ and assume that B is a linear operator on X such that $D(B) \supseteq D(A^\alpha)$ and*

$$\|Bu\|_X \le C\|A^\alpha u\|_X, \qquad u \in D(A), \tag{6.2.1}$$

for some $C > 0$. Then there exists a $\lambda \ge 0$ such that $\lambda + A + B$ with $D(\lambda + A + B) = D(A)$ has a bounded $H^\infty(\Sigma_\omega)$-calculus.

To extend the H^∞-calculus of the Laplacian on $\mathbb{R}^d_+$ to domains in Sections 6.5 and 6.6, we need to deal with perturbations that are not of lower order. Unfortunately, the H^∞-calculus is not stable under small perturbations, as shown in a counterexample by McIntosh and Yagi [207]. Instead, for the H^∞-calculus, one has statements of the following type, in which the perturbation is in addition required to be well behaved with respect to a fractional power of the original operator.

Theorem 6.2.4 ([126, Theorem 16.2.8])**.** *Let A be a sectorial operator on a Banach space X such that $0 \in \rho(A)$. Let $\omega \in (\omega(A), \pi)$ and assume that A has a bounded $H^\infty(\Sigma_\omega)$-calculus. Let B be a linear operator on X such that $D(B) \supseteq D(A)$. Suppose that there is an $\eta > 0$ such that*

(i) $\|Bu\|_X \leq \eta \|Au\|_X, \qquad u \in D(A).$

Moreover, suppose that at least one of the following relative bounds is satisfied:

(ii) there exists an $\alpha \in (0,1)$ such that $B(D(A^{1+\alpha})) \subseteq D(A^\alpha)$ and

$$\|A^\alpha Bu\|_X \leq C\|A^{1+\alpha}u\|_X, \qquad u \in D(A^{1+\alpha}),$$

(iii) there exists an $\alpha \in (0,1)$ such that

$$\|A^{-\alpha}Bu\|_X \leq C\|A^{1-\alpha}u\|_X, \qquad u \in D(A^{1-\alpha}),$$

for some $C > 0$. Then there exists an $\tilde{\eta} > 0$ such that, if (i) holds with $\eta < \tilde{\eta}$, then $A + B$ with $D(A+B) = D(A)$ has a bounded $H^\infty(\Sigma_\omega)$-calculus.

Remark 6.2.5. Theorem 6.2.4 is taken from [126, Theorem 16.2.8], where it should be noted that their condition of R-sectoriality on B is redundant, see also [173] and the errata to [126]. A version of Theorem 6.2.4 for positive fractional powers also appeared in [65, Theorem 3.2].

6.3. Weighted Sobolev spaces and trace characterisations

Recall the definitions of the weighted Lebesgue and Sobolev spaces from Section 2.2. We will now introduce weighted Sobolev spaces with vanishing traces by means of taking the closure of a certain space of test functions.

Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma > -1$ and let X be a Banach space. To impose zero boundary conditions, we define

$$\mathring{W}^{k,p}_0(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X) := \overline{C^\infty_{\mathrm{c}}(\mathscr{O}; X)}^{W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma; X)}. \tag{6.3.1}$$

Furthermore, to impose Dirichlet and Neumann boundary conditions, we set

$$C^\infty_{\mathrm{c,Dir}}(\overline{\mathscr{O}}; X) := C^\infty(\mathscr{O}; X) \cap \{f \in C_{\mathrm{c}}(\overline{\mathscr{O}}; X) : f|_{\partial\mathscr{O}} = 0\},$$
$$C^\infty_{\mathrm{c,Neu}}(\overline{\mathscr{O}}; X) := C^\infty(\mathscr{O}; X) \cap \{f \in C^1_{\mathrm{c}}(\overline{\mathscr{O}}; X) : (\partial_{\boldsymbol{n}} f)|_{\partial\mathscr{O}} = 0\},$$

which contain functions that are smooth in the interior of $\mathscr{O}$, satisfy the boundary condition and have compact support at infinity (in the case of unbounded domains). Here, $\boldsymbol{n}$ denotes the inward unit normal vector at $\partial\mathscr{O}$ and $\partial_{\boldsymbol{n}} = \boldsymbol{n}\cdot\nabla$. We define

$$\begin{aligned} \mathring{W}^{k,p}_{\mathrm{Dir}}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) &:= \overline{C^\infty_{\mathrm{c,Dir}}(\overline{\mathscr{O}}; X)}^{W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)}, \\ \mathring{W}^{k,p}_{\mathrm{Neu}}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) &:= \overline{C^\infty_{\mathrm{c,Neu}}(\overline{\mathscr{O}}; X)}^{W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)}. \end{aligned} \tag{6.3.2}$$

The notation $\mathring{W}^{k,p}_0$, $\mathring{W}^{k,p}_{\mathrm{Dir}}$ and $\mathring{W}^{k,p}_{\mathrm{Neu}}$ as in (6.3.1) and (6.3.2) will mean that the spaces are defined as the closure of some space of test functions. Alternative characterisations of these spaces with boundary conditions in terms of traces (which will be denoted by $W^{k,p}_0$, $W^{k,p}_{\mathrm{Dir}}$ and $W^{k,p}_{\mathrm{Neu}}$) are derived in Sections 6.3.1, 6.3.2 and 6.3.3. The characterisations involving traces are also used in [190] and Chapter 5 to define Sobolev spaces with boundary conditions.

6.3.1. Trace characterisations for weighted Sobolev spaces on the half-space

In the following three sections, we present characterisations of the spaces in (6.3.1) and (6.3.2) as closed subspaces of $W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ with vanishing traces. In this section, we start with the special case $\mathscr{O} = \mathbb{R}^d_+$.

For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and X a Banach space, we define the following spaces with vanishing traces

$$\begin{aligned} W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(\partial^\alpha f) = 0 \text{ if } k - |\alpha| > \tfrac{\gamma+1}{p} \Big\}, \\ W^{k,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(f) = 0 \text{ if } k > \tfrac{\gamma+1}{p} \Big\}, \\ W^{k,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(\partial_1 f) = 0 \text{ if } k - 1 > \tfrac{\gamma+1}{p} \Big\}. \end{aligned}$$

All the traces in the above definitions are well defined, see Section 5.3.1. Although we will not consider weights w_γ with $\gamma \le -1$, we can nonetheless define

$$W^{k,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) := W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X) := W^{k,p}(\mathbb{R}^d_+, w_\gamma; X),$$

see [190, Lemma 3.1(2)].

In [190] the above spaces are also used to define weighted Sobolev spaces on domains. However, since we consider domains with low regularity, we cannot do this, as will be explained in Remark 6.3.8. Therefore, we first defined the Sobolev spaces as the closure of test functions in (6.3.1) and (6.3.2). The following proposition relates the spaces $W^{k,p}_{\mathrm{BC}}$ and $\mathring{W}^{k,p}_{\mathrm{BC}}$, where $\mathrm{BC} \in \{0, \mathrm{Dir}, \mathrm{Neu}\}$ stands for boundary conditions. That is, we prove that certain classes of test functions are dense in Sobolev spaces with zero trace conditions.

Proposition 6.3.1 (Trace characterisation on $\mathbb{R}^d_+$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and let X be a Banach space. For* BC $\in \{0, \mathrm{Dir}, \mathrm{Neu}\}$ *we have the trace characterisations*

$$\mathring{W}^{k,p}_{\mathrm{BC}}(\mathbb{R}^d_+, w_\gamma; X) = W^{k,p}_{\mathrm{BC}}(\mathbb{R}^d_+, w_\gamma; X).$$

Proof. From [190, Proposition 3.8] we have that $C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$ is dense in $W^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X)$ and therefore the trace characterisation for $\mathring{W}^{k,p}_0(\mathbb{R}^d_+, w_\gamma; X)$ follows.

Let (BC, j) $\in \{(\mathrm{Dir}, 0), (\mathrm{Neu}, 1)\}$. Then Proposition 3.4.12 implies that

$$\overline{\{f \in C^\infty_{\mathrm{c}}(\overline{\mathbb{R}^d_+}; X) : (\partial_1^j f)|_{\partial\mathbb{R}^d_+} = 0\}}^{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} = W^{k,p}_{\mathrm{BC}}(\mathbb{R}^d_+, w_\gamma; X).$$

Since

$$\{f \in C^\infty_{\mathrm{c}}(\overline{\mathbb{R}^d_+}; X) : (\partial_1^j f)|_{\partial\mathbb{R}^d_+} = 0\} \subseteq C^\infty_{\mathrm{c,BC}}(\overline{\mathbb{R}^d_+}; X),$$

the trace characterisations for the Dirichlet and Neumann boundary conditions follow. □

Before we continue with trace characterisations on domains, we record some variants of Hardy's inequality, see Lemma 2.2.2 and Corollary 5.3.2 for the classical formulation of this inequality. As a consequence of Corollary 5.3.2, we obtain the following non-sharp Hardy's inequality.

Lemma 6.3.2. *Let $p \in (1,\infty)$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$, $s \in [0,\infty)$ such that $\gamma > sp-1$ and let X be a Banach space. Then for any integer $k \geq s$ it holds that*

$$W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \hookrightarrow L^p(\mathbb{R}^d_+, w_{\gamma-sp}; X).$$

Proof. Let $\varphi_1, \varphi_2 \in C^\infty(\mathbb{R}_+; [0,1])$ such that $\varphi_1(x_1) = 0$ for $x_1 \geq 2$ and $\varphi_2(x_1) = 0$ for $x_1 \leq 1$. In addition, take φ_1 and φ_2 such that $\varphi_1 + \varphi_2 = 1$. Let $f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$, with Hardy's inequality (Corollary 5.3.2 using that $\gamma > sp-1$) we obtain

$$\begin{aligned}\|f\|_{L^p(\mathbb{R}^d_+, w_{\gamma-sp}; X)} &\leq \|f\varphi_1\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+(k-s)p}; X)} + \|f\varphi_2\|_{L^p(\mathbb{R}^d_+, w_{\gamma-sp}; X)} \\ &\lesssim \|f\varphi_1\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)} + \|f\varphi_2\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \lesssim \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)},\end{aligned}$$

where we have used that $w_{\gamma+(k-s)p}(x) \lesssim w_\gamma(x)$ for $x_1 \leq 2$ (since $k \geq s$) and $w_{\gamma-sp}(x) \lesssim w_\gamma(x)$ for $x_1 \geq 1$. □

Occasionally, we also need a sharp Hardy's inequality with fractional smoothness. We use complex interpolation to deal with spaces with fractional smoothness and weights $w_{\gamma+kp}$ with $\gamma \in (-1, p-1)$ and $k \in \mathbb{N}_1$ outside the Muckenhoupt class.

Lemma 6.3.3. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, p-1)$, $s \in [0,1)$ such that $\gamma > sp-1$ and let X be a* UMD *Banach space. Then*

$$[W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{s+1}{2}} \hookrightarrow W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k-s)p}; X).$$

Proof. For $s = 0$ the result follows from Proposition 3.6.3, so from now on we assume $s \in (0,1)$. We start with the case $k = 0$. Let $|\alpha| \le 1$, then by [189, Lemma 3.7] (which also holds on $\mathbb{R}^d_+$) and [189, Propositions 5.5 & 5.6], we obtain

$$\begin{aligned}\|\partial^\alpha f\|_{L^p(\mathbb{R}^d_+, w_{\gamma-sp}; X)} &\lesssim \|\partial^\alpha f\|_{H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)} \lesssim \|f\|_{H^{s+1,p}(\mathbb{R}^d_+, w_\gamma; X)} \\ &\eqsim \|f\|_{[L^p(\mathbb{R}^d_+, w_\gamma; X), W^{2,p}(\mathbb{R}^d_+, w_\gamma; X)]_{\frac{s+1}{2}}},\end{aligned}$$

where $H^{s,p}(\mathbb{R}^d_+, w_\gamma; X)$ is a weighted Bessel potential space, see [189, Section 3]. For $k \ge 1$, we proceed by induction. Assume that the statement of the lemma holds for some $k \in \mathbb{N}_0$, then it remains to prove the statement for $k+1$. We recall from Section 5.3.2 that M is the pointwise multiplication operator given by $Mu(x) = x_1 u(x)$ for $x \in \mathbb{R}^d_+$. Then by Lemma 5.3.7 (using that $\gamma > sp - 1$) and the induction hypothesis, we obtain

$$\begin{aligned}\|f\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+(k+1-s)p}; X)} &\eqsim \sum_{|\beta|\le 1} \|M\partial^\beta f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k-s)p}; X)} \\ &\lesssim \sum_{|\beta|\le 1} \|M\partial^\beta f\|_{[W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{s+1}{2}}} \\ &\lesssim \|f\|_{[W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X), W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X)]_{\frac{s+1}{2}}},\end{aligned}$$

where the last estimate follows from the fact that for $|\beta| \le 1$ the operators

$$\begin{aligned}&M\partial^\beta : W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X) \to W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \quad \text{and} \\ &M\partial^\beta : W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}; X) \to W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\end{aligned}$$

are bounded, see Lemma 5.3.5. □

6.3.2. Trace characterisations for weighted Sobolev spaces on special domains

For $\mathscr{O} = \mathbb{R}^d_+$ we have shown in Proposition 6.3.1 that the definition of weighted Sobolev spaces in (6.3.1) and (6.3.2) is equivalent to setting certain traces to zero. To define Sobolev spaces with vanishing traces for a special $C^{\ell,\kappa}_{\mathrm{c}}$-domain $\mathscr{O}$, we will employ the diffeomorphisms $\Phi, \Psi : \mathscr{O} \to \mathbb{R}^d_+$ from Lemmas 2.6.5 and 2.6.6 to construct isomorphisms between Sobolev spaces on $\mathscr{O}$ and $\mathbb{R}^d_+$. Which diffeomorphism we use depends on the boundary conditions. Throughout the rest of this chapter, we will always use the diffeomorphism Φ from Lemma 2.6.5 for Dirichlet boundary conditions and the diffeomorphism Ψ from Lemma 2.6.6 for Neumann boundary conditions. The diffeomorphism Φ is not applicable for Neumann boundary conditions, since it does not preserve the direction of the normal vector.

Proposition 6.3.4. *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k - (\ell+\kappa))_+ p - 1$.*

(i) Let $\mathscr{O}$ be a special $C^{\ell,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \le 1$. Let $\Phi\colon \mathscr{O} \to \mathbb{R}^d_+$ be as in Lemma 2.6.5 and consider the change of coordinates mappings

$$\begin{aligned}&\Phi_*\colon W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to W^{k,p}(\mathbb{R}^d_+, w_\gamma; X), \\ &\Phi_*\colon \mathring{W}^{k,p}_{\mathrm{BC}}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to \mathring{W}^{k,p}_{\mathrm{BC}}(\mathbb{R}^d_+, w_\gamma; X) \quad \text{for } \mathrm{BC} \in \{0, \mathrm{Dir}\},\end{aligned}$$

defined by $\Phi_* f := f \circ \Phi^{-1}$.

(ii) Let $\mathscr{O}$ *be a special* $C^{\ell,\kappa}_{\mathrm{c}}$*-domain with* $[\mathscr{O}]_{C^{\ell,\kappa}} \le \Lambda$*, where* $\Lambda \in (0,1)$ *is as in Lemma 2.6.6. Let* $\Psi\colon \mathscr{O} \to \mathbb{R}^d_+$ *be as in Lemma 2.6.6 and consider the change of coordinates mappings*

$$\Psi_*\colon W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to W^{k,p}(\mathbb{R}^d_+, w_\gamma; X), \tag{6.3.3a}$$

$$\Psi_*\colon \mathring{W}^{k,p}_{\mathrm{BC}}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to \mathring{W}^{k,p}_{\mathrm{BC}}(\mathbb{R}^d_+, w_\gamma; X) \quad \text{for } \mathrm{BC} \in \{0, \mathrm{Dir}, \mathrm{Neu}\}, \tag{6.3.3b}$$

defined by $\Psi_* f := f \circ \Psi^{-1}$.

Then Φ_* *and* Ψ_* *are isomorphisms of Banach spaces for which* $(\Phi^{-1})_*$ *and* $(\Psi^{-1})_*$*, respectively, act as inverses.*

Proof. We give the proof of (ii) and the proof of (i) is similar using Lemma 2.6.5 instead of Lemma 2.6.6.

Step 1: proof of (6.3.3a). We start with some preparations. Let $k \in \mathbb{N}_1$ and $f \in C^{\ell,\kappa}_{\mathrm{c}}(\overline{\mathscr{O}}; X)$. Note that by Lemma 2.6.6 we have that $\Psi_* f \in C^{\ell,\kappa}_{\mathrm{c}}(\overline{\mathbb{R}^d_+}; X)$. Let $\alpha \in \mathbb{N}_0^d \setminus \{0\}$ with $|\alpha| \le k$, then by [50, Theorem 2.1] we have the multivariate Faà di Bruno's formula

$$\partial^\alpha \Psi_* f = \sum_{1 \le |\beta| \le |\alpha|} (\Psi_* \partial^\beta f) \sum_{s=1}^{|\alpha|} \sum_{p_s(\alpha,\beta)} \prod_{j=1}^{s} c_{\alpha,\boldsymbol{k}_j,\boldsymbol{\ell}_j} [\partial^{\boldsymbol{\ell}_j} \Psi^{-1}]^{\boldsymbol{k}_j},$$

for some constants $c_{\alpha,\boldsymbol{k}_j,\boldsymbol{\ell}_j}$ and sets $p_s(\alpha,\beta)$ contained in

$$\Big\{(\boldsymbol{k}_1,\dots,\boldsymbol{k}_s;\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_s) \in (\mathbb{N}_0^d \setminus \{0\})^s \times (\mathbb{N}_0^d \setminus \{0\})^s : \sum_{j=1}^{s} |\boldsymbol{k}_j| = |\beta|, \sum_{j=1}^{s} |\boldsymbol{k}_j||\boldsymbol{\ell}_j| = |\alpha|\Big\}. \tag{6.3.4}$$

Therefore, we have

$$\begin{aligned}
\|\partial^\alpha \Psi_* f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} &\lesssim \sum_{1 \le |\beta| \le |\alpha|} \sum_{s=1}^{|\alpha|} \sum_{p_s(\alpha,\beta)} \|(\Psi_* \partial^\beta f) \prod_{j=1}^{s} [\partial^{\boldsymbol{\ell}_j} \Psi^{-1}]^{\boldsymbol{k}_j}\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \\
&\lesssim \sum_{1 \le |\beta| \le |\alpha|} \sum_{s=1}^{|\alpha|} \sum_{p_s(\alpha,\beta)} \|\Psi_* \partial^\beta f\|_{L^p(\mathbb{R}^d_+, w_{\gamma - \sum_{j=1}^s (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j| p}; X)} \\
&\quad \cdot \prod_{j=1}^{s} \|y \mapsto y_1^{(|\boldsymbol{\ell}_j| - (\ell+\kappa))_+} \partial^{\boldsymbol{\ell}_j} \Psi^{-1}(y)\|^{|\boldsymbol{k}_j|}_{L^\infty(\mathbb{R}^d_+; \mathbb{R}^d)}.
\end{aligned} \tag{6.3.5}$$

From Lemma 2.6.6(v) we obtain

$$\prod_{j=1}^{s} \|y \mapsto y_1^{(|\boldsymbol{\ell}_j| - (\ell+\kappa))_+} \partial^{\boldsymbol{\ell}_j} \Psi^{-1}(y)\|^{|\boldsymbol{k}_j|}_{L^\infty(\mathbb{R}^d_+; \mathbb{R}^d)} \lesssim 1. \tag{6.3.6}$$

Step 1a: proof of (6.3.3a) *if* $\ell + \kappa \ge k$. If $k = 0$, then (6.3.3a) follows immediately from Lemma 2.6.6. Let $k \in \mathbb{N}_1$ and note that $|\boldsymbol{\ell}_j| \le |\alpha| \le k \le \ell + \kappa$. Therefore, $(|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ = 0$ in (6.3.5) and the case $k = 0$ implies

$$\|\Psi_* \partial^\beta f\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \lesssim \|\partial^\beta f\|_{L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)} \le \|f\|_{W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)}, \qquad 1 \le |\beta| \le |\alpha|, \tag{6.3.7}$$

and we find

$$\|\Psi_* f\|_{W^{k,p}(\mathbb{R}^d_+,w_\gamma;X)} \lesssim \|f\|_{W^{k,p}(\mathscr{O},w_\gamma^{\partial\mathscr{O}};X)}, \qquad f \in C_{\mathrm{c}}^{\ell,\kappa}(\overline{\mathscr{O}};X),$$

and by density the estimate extends to $f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}};X)$. Recall from Lemma 2.6.6 that Ψ is invertible and thus $(\Psi^{-1})_*$ is the inverse of Ψ_*. The estimate for the inverse $(\Psi^{-1})_*$ can be shown using similar estimates as in (6.3.5), (6.3.6) and (6.3.7). This shows that Ψ_* in (6.3.3a) is an isomorphism if $\ell+\kappa \geq k$.

Step 1b: proof of (6.3.3a) *if* $\ell+\kappa < k$. We claim that in (6.3.5) we have

$$\gamma - \sum_{j=1}^{s} (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j| p > -1. \tag{6.3.8}$$

Indeed, if $|\boldsymbol{\ell}_j| \leq \ell+\kappa$ for all $j \in \{1,\dots,s\}$, then

$$\gamma - \sum_{j=1}^{s} (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j| p = \gamma > (k-(\ell+\kappa))p - 1 > -1,$$

and if $|\boldsymbol{\ell}_{j_0}| > \ell+\kappa$ for some $j_0 \in \{1,\dots,s\}$, then with (6.3.4) we obtain

$$\begin{aligned}
\gamma - \sum_{j=1}^{s} (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j| p &= \gamma - \Big(\sum_{\substack{j=1\\ j\neq j_0}}^{s} (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j| + (|\boldsymbol{\ell}_{j_0}| - (\ell+\kappa))|\boldsymbol{k}_{j_0}| \Big) p \\
&\geq \gamma - \Big(\sum_{\substack{j=1\\ j\neq j_0}}^{s} |\boldsymbol{\ell}_j||\boldsymbol{k}_j| + |\boldsymbol{\ell}_{j_0}||\boldsymbol{k}_{j_0}| - (\ell+\kappa) \Big) p \\
&= \gamma - (|\alpha| - (\ell+\kappa))p \geq \gamma - (k-(\ell+\kappa))p > -1.
\end{aligned}$$

Moreover, again by (6.3.4) we have

$$\sum_{j=1}^{s} (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j| \leq \sum_{j=1}^{s} |\boldsymbol{\ell}_j||\boldsymbol{k}_j| - |\beta| = |\alpha| - |\beta| \leq k - |\beta|. \tag{6.3.9}$$

Therefore, by Lemma 6.3.2 (using (6.3.8) and (6.3.9)) and Step 1a, we have for $1 \leq |\beta| \leq |\alpha| \leq k = \ell+1$ that

$$\begin{aligned}
\|\Psi_* \partial^\beta f\|_{L^p(\mathbb{R}^d_+, w_{\gamma-\sum_{j=1}^s (|\boldsymbol{\ell}_j|-(\ell+\kappa))_+|\boldsymbol{k}_j| p};X)} &\lesssim \|\Psi_* \partial^\beta f\|_{W^{k-|\beta|,p}(\mathbb{R}^d_+, w_\gamma;X)} \\
&\lesssim \|\partial^\beta f\|_{W^{k-|\beta|,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}};X)} \\
&\lesssim \|f\|_{W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}};X)}, \qquad f \in C_{\mathrm{c}}^{\ell,\kappa}(\overline{\mathscr{O}};X).
\end{aligned} \tag{6.3.10}$$

Now, density and (6.3.5), (6.3.6) and (6.3.10) yield that

$$\Psi_* : W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}};X) \to W^{k,p}(\mathbb{R}^d_+, w_\gamma;X) \tag{6.3.11}$$

is bounded for $k = \ell+1$.

The general case $k \geq \ell + 1$ follows by induction on k. Assume that (6.3.11) holds for some $k \geq \ell + 1$ and let $1 \leq |\beta| \leq |\alpha| \leq k + 1$. Using the induction hypothesis instead of Step 1a in (6.3.10), we obtain the estimate

$$\|\Psi_* \partial^\beta f\|_{L^p(\mathbb{R}^d_+, w_{\gamma - \sum_{j=1}^s (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j|p}; X)} \lesssim \|f\|_{W^{k+1,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)},$$

which proves (6.3.11) for $k \geq \ell + 1$.

The estimate for the inverse can be shown directly using similar estimates as in (6.3.5) and (6.3.6), together with the estimate

$$\begin{aligned}
\|(\Psi^{-1})_* \partial^\beta f\|_{L^p(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma - \sum_{j=1}^s (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j|p}; X)} &\lesssim \|\partial^\beta f\|_{L^p(\mathbb{R}^d_+, w_{\gamma - \sum_{j=1}^s (|\boldsymbol{\ell}_j| - (\ell+\kappa))_+ |\boldsymbol{k}_j|p}; X)} \\
&\lesssim \|\partial^\beta f\|_{W^{k-|\beta|,p}(\mathbb{R}^d_+, w_\gamma; X)} \\
&\lesssim \|f\|_{W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)}, \qquad f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X),
\end{aligned}$$

which follows from Step 1a and Lemma 6.3.2. This completes the proof of (6.3.3a).

Step 2: proof of (6.3.3b). The proof (6.3.3b) is similar to the proof of (6.3.3a) if we work with a suitable dense subspace, i.e.,

- if BC = 0, take $f \in C_c^\infty(\mathscr{O}; X)$,
- if BC $\in$ {Dir, Neu}, take $f \in C^\infty_{c,\mathrm{BC}}(\overline{\mathscr{O}}; X)$,

see (6.3.1) and (6.3.2). Note that in both cases Lemma 2.6.6 ensures that $\Psi_* f$ is in the respective dense subspace on $\mathbb{R}^d_+$. In particular, for the Neumann boundary condition, we have

$$(\partial_1 \Psi_* f)|_{\partial\mathbb{R}^d_+} = (\Psi_*(\partial_\nu f))|_{\partial\mathbb{R}^d_+},$$

where $\nu(y) = \nu(\widetilde{y}) = (1, -\nabla_{\widetilde{y}} h(\widetilde{y}))^\top$ is the inward normal direction. Indeed, this follows from Lemma 2.6.6 since

$$\begin{aligned}
\partial_{y_1} f(\Psi^{-1}(y)) &= (\nabla f)(\Psi^{-1}(y)) \cdot \partial_{y_1} \Psi^{-1}(y) \\
&= (\nabla f)(\Psi^{-1}(y)) \cdot \nu(\Psi^{-1}(y)) = (\partial_\nu f)(\Psi^{-1}(y)), \qquad y = (0, \widetilde{y}) \in \partial\mathbb{R}^d_+,
\end{aligned}$$

where we recall from the construction of Ψ^{-1} in the proof of Lemma 2.6.6 that $\partial_{y_1} \Psi^{-1}(y) = (1, -\nabla_{\widetilde{y}} h(\widetilde{y}))^\top = \nu(\widetilde{y}) = \nu(\Psi^{-1}(y))$ if $y = (0, \widetilde{y}) \in \partial\mathbb{R}^d_+$. Furthermore, note that the conditions $(\partial_{\boldsymbol{n}} f)|_{\partial\mathscr{O}} = 0$ and $(\partial_\nu f)|_{\partial\mathscr{O}} = 0$ are equivalent. □

Remark 6.3.5. By inspection of the proof of Proposition 6.3.4, we see that for BC = 0 no additional conditions on γ are necessary since Hardy's inequality always applies in this case. That is, we can allow for any $\gamma \in (-1, \infty) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$. Furthermore, we expect that for Dirichlet boundary conditions, the range for γ can also be improved, although we will not need this.

We define the following spaces with vanishing traces at the boundary of a special $C_c^{\ell,\kappa}$-domain.

Definition 6.3.6. Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k-(\ell+\kappa))_+ p - 1$ and let $\mathscr{O}$ be a special $C_c^{\ell,\kappa}$-domain. If $[\mathscr{O}]_{C^{\ell,\kappa}} \leq 1$, let Φ_* be the isomorphism from Proposition 6.3.4(i) and define

$$W_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \Big\{ f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \mathrm{Tr}(\partial^\alpha(\Phi_* f)) = 0 \text{ if } k - |\alpha| > \tfrac{\gamma+1}{p} \Big\},$$
$$W_{\mathrm{Dir}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \Big\{ f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \mathrm{Tr}(\Phi_* f) = 0 \text{ if } k > \tfrac{\gamma+1}{p} \Big\}.$$

If $[\mathscr{O}]_{C^{\ell,\kappa}} \leq \Lambda$, where $\Lambda \in (0,1)$ is as in Lemma 2.6.6, let Ψ_* be the isomorphism from Proposition 6.3.4(ii) and define

$$W_{\mathrm{Neu}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \Big\{ f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \mathrm{Tr}(\partial_1(\Psi_* f)) = 0 \text{ if } k - 1 > \tfrac{\gamma+1}{p} \Big\}.$$

The above spaces are well defined by Proposition 6.3.4. Furthermore, by Lemmas 2.6.5 and 2.6.6, the definitions of the above spaces are consistent in the sense that viewing $\mathscr{O}$ as either a special $C_c^{\ell,\kappa}$-domain or a special C_c^1-domain yields the same space. Moreover, the condition $\mathrm{Tr}(\partial_1(\Psi_* f)) = 0$ correctly models the Neumann boundary condition, since Ψ leaves the direction of the normal vector invariant, see Lemma 2.6.6(iv). Finally, we note that the spaces $W_0^{k,p}$ and $W_{\mathrm{Dir}}^{k,p}$ could also be defined using Ψ_* instead of Φ_*, yielding an equivalent definition by Proposition 6.3.7 below.

Similar to Proposition 6.3.1 we can now characterise the spaces $\mathring{W}_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ in terms of vanishing traces with the aid of the isomorphisms from Proposition 6.3.4.

Proposition 6.3.7 (Trace characterisation on special domains)**.** *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k-(\ell+\kappa))_+ p - 1$.*

(i) Let $\mathscr{O}$ be a special $C_c^{\ell,\kappa}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \leq 1$ and let Φ_* be the isomorphism from Proposition 6.3.4. Let $\mathrm{BC} \in \{0, \mathrm{Dir}\}$ and let $W_{\mathrm{BC}}^{k,p}$ be defined using Φ_*. Then we have the trace characterisations

$$\mathring{W}_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) = W_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X).$$

(ii) Let $\mathscr{O}$ be a special $C_c^{\ell,\kappa}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \leq \Lambda$, where $\Lambda \in (0,1)$ is as in Lemma 2.6.6, and let Ψ_* be the isomorphism from Proposition 6.3.4. Let $\mathrm{BC} \in \{0, \mathrm{Dir}, \mathrm{Neu}\}$ and let $W_{\mathrm{BC}}^{k,p}$ be defined using Ψ_*. Then we have the trace characterisations

$$\mathring{W}_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) = W_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X).$$

Proof. We only prove (ii). The proof of (i) is similar. Let $\mathrm{BC} \in \{0, \mathrm{Dir}, \mathrm{Neu}\}$ and $f \in \mathring{W}_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$, then by Propositions 6.3.1 and 6.3.4 we have $\Psi_* f \in \mathring{W}_{\mathrm{BC}}^{k,p}(\mathbb{R}^d_+, w_\gamma; X) = W_{\mathrm{BC}}^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$. This implies that all the required traces of $\Psi_* f$ are zero. Moreover, since $\Psi_* f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ it follows by Proposition 6.3.4 that $f = (\Psi^{-1})_* \Psi_* f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ as well. This proves that $f \in W_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$. The other inclusion is similar. □

Remark 6.3.8. If $h \in C_c^{\ell,\kappa}(\mathbb{R}^{d-1})$ is associated with the special $C_c^{\ell,\kappa}$-domain, then the *classical* diffeomorphism $\Phi_{\mathrm{cl}} : \overline{\mathscr{O}} \to \overline{\mathbb{R}^d_+}$ given by

$$\Phi_{\mathrm{cl}}(x) = (x_1 - h(\widetilde{x}), \widetilde{x}), \qquad x = (x_1, \widetilde{x}) \in \overline{\mathscr{O}},$$

defines a $C^{\ell,\kappa}$-diffeomorphism. Moreover, the change of coordinates mapping $(\Phi_{\mathrm{cl}})_*$ becomes an isomorphism between $W^{k,p}_{\mathrm{BC}}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ and $W^{k,p}_{\mathrm{BC}}(\mathbb{R}^d_+, w_\gamma; X)$ for $\ell \geq k$ and $\mathrm{BC} \in \{0, \mathrm{Dir}\}$. In [190, Section 3.2], this isomorphism is used to define weighted Sobolev spaces on domains. However, for $\ell < k$ or Neumann boundary conditions, this isomorphism is not sufficient, which is why we have employed the diffeomorphisms Φ and Ψ from Lemma 2.6.5 and 2.6.6 to define weighted Sobolev spaces with vanishing traces.

6.3.3. Trace characterisations for weighted Sobolev spaces on bounded domains

In this section, we define Sobolev spaces with vanishing traces for bounded domains $\mathscr{O}$. To this end, we will employ a localisation procedure to relate spaces on bounded domains with spaces on special domains. We start with a lemma containing a decomposition of weighted Sobolev spaces, see also [190, Section 2.2].

Lemma 6.3.9. *Let $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$ and let $\mathscr{O} \subseteq \mathbb{R}^d$ be a bounded $C^{\ell,\kappa}$-domain. Then for any $\delta > 0$, the following statements hold.*

(i) For all $\varepsilon \in (0,\kappa)$ there exists a finite open cover $(V_n)_{n=1}^N$ of $\partial\mathscr{O}$, together with special $C_c^{\ell,\kappa}$-domains $(\mathscr{O}_n)_{n=1}^N$ which satisfy $[\mathscr{O}_n]_{C^{\ell,\kappa-\varepsilon}} < \delta$, such that

$$\mathscr{O} \cap V_n = \mathscr{O}_n \cap V_n \qquad \text{and} \qquad \partial\mathscr{O} \cap V_n = \partial\mathscr{O}_n \cap V_n, \quad n \in \{1,\dots,N\}.$$

If $\kappa = 0$, then the special C_c^ℓ-domains $(\mathscr{O}_n)_{n=1}^N$ can be chosen such that $[\mathscr{O}_n]_{C^\ell} < \delta$.

(ii) There exist $\eta_0 \in C_c^\infty(\mathscr{O})$ and $\eta_n \in C_c^\infty(V_n)$ for $n \in \{1,\dots,N\}$ such that $0 \leq \eta_n \leq 1$ for $n \in \{0,\dots,N\}$ and $\sum_{n=0}^N \eta_n^2 = 1$ on $\mathscr{O}$ (partition of unity).

(iii) For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ and X a Banach space, the space $W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ has the direct sum decomposition

$$\mathscr{W}^{k,p}_\gamma := W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^N W^{k,p}(\mathscr{O}_n, w_\gamma^{\partial\mathscr{O}_n}; X). \tag{6.3.12}$$

Moreover, the mappings

$$\mathscr{I} : W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to \mathscr{W}^{k,p}_\gamma \quad \text{and} \quad \mathscr{P} : \mathscr{W}^{k,p}_\gamma \to W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$$

given by

$$\mathscr{I} f := (\eta_n f)_{n=0}^N \quad \text{and} \quad \mathscr{P}(f_n)_{n=0}^N := \sum_{n=0}^N \eta_n f_n, \tag{6.3.13}$$

are continuous and satisfy $\mathscr{P}\mathscr{I} = \mathrm{id}$. Thus, $\mathscr{P}$ is a retraction with coretraction $\mathscr{I}$.

Proof. We note that the result in (i) follows from the discussion after Definition 2.6.1. The partition of unity in (ii) is standard, see for instance [166, Section 8.4] (noting that a C^2-domain is not required for constructing the partition of unity). Finally, using the partition of unity and the (co)retraction in (6.3.13), the direct sum decomposition in (iii) follows. Indeed, $\eta_0 \in C_c^\infty(\mathscr{O})$ and we can extend to the full space $\mathbb{R}^d$ without a weight since there is no boundary. Furthermore, for $n \in \{1,\dots,N\}$ we have $\eta_n \in C_c^\infty(V_n)$, so the weight $w_\gamma^{\partial\mathscr{O}}(x)$ can be replaced by $w_\gamma^{\partial\mathscr{O}_n}(x)$ for $x \in \mathscr{O}_n$. □

With Lemma 6.3.9 we can now define traces of functions in $W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ if $\mathscr{O}$ is a bounded $C^{\ell,\kappa}$-domain. Furthermore, we define the following spaces with vanishing traces at the boundary.

Definition 6.3.10. Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k-(\ell+\kappa))_+ p - 1$. Moreover, let $\mathscr{O}$ be a bounded $C^{\ell,\kappa}$-domain, let $(\mathscr{O}_n)_{n=1}^N$ be special $C_c^{\ell,\kappa}$-domains and let $\mathscr{I}$ be the coretraction from Lemma 6.3.9. We define

$$W_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \Big\{ f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \mathscr{I} f \in W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^N W_0^{k,p}(\mathscr{O}_n, w_\gamma^{\partial\mathscr{O}_n}; X) \Big\},$$

$$W_{\mathrm{Dir}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \Big\{ f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \mathscr{I} f \in W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^N W_{\mathrm{Dir}}^{k,p}(\mathscr{O}_n, w_\gamma^{\partial\mathscr{O}_n}; X) \Big\},$$

$$W_{\mathrm{Neu}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \Big\{ f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \mathscr{I} f \in W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^N W_{\mathrm{Neu}}^{k,p}(\mathscr{O}_n, w_\gamma^{\partial\mathscr{O}_n}; X) \Big\}.$$

Note that the above spaces are well defined by Lemma 6.3.9 and Definition 6.3.6. Moreover, the definitions are independent of the chosen covering of $\partial\mathscr{O}$ and the partition of unity in Lemma 6.3.9.

Similar to Propositions 6.3.1 and 6.3.7 we can now relate the spaces $\mathring{W}_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ and $W_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ for bounded domains.

Proposition 6.3.11 (Trace characterisation on bounded domains)**.** *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k-(\ell+\kappa))_+ p - 1$. Moreover, let $\mathscr{O}$ be a bounded $C^{\ell,\kappa}$-domain. For* $\mathrm{BC} \in \{0, \mathrm{Dir}, \mathrm{Neu}\}$ *we have the trace characterisations*

$$\mathring{W}_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) = W_{\mathrm{BC}}^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X).$$

Proof. We only prove the statement for $\mathrm{BC} = 0$ since the proof for the other cases is similar. Let $f \in W_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$. Proposition 6.3.7 and the fact that $C_c^\infty(\mathbb{R}^d; X)$ is dense in $W^{k,p}(\mathbb{R}^d; X)$, allows us to approximate $\mathscr{I} f$ by a sequence $g := (g_{0,m}, g_{1,m}, \dots, g_{N,m})_{m\geq 1}$ where $(g_{0,m})_{m\geq 1} \subseteq C_c^\infty(\mathbb{R}^d; X)$ and $(g_{n,m})_{m\geq 1} \subseteq C_c^\infty(\mathscr{O}_n; X)$ for all $n \in \{1,\dots,N\}$. Using Lemma 6.3.9 we see that $f = \mathscr{P}\mathscr{I} f$ can be approximated by the sequence $\mathscr{P} g \subseteq C_c^\infty(\mathscr{O}; X)$. □

6.3.4. Complex interpolation of weighted Sobolev spaces

To conclude this section, we recall the following two interpolation results for weighted Sobolev spaces on $\mathbb{R}^d_+$ with boundary conditions from Chapter 3, which also hold for special and bounded domains by the results from Sections 6.3.1, 6.3.2 and 6.3.3.

Proposition 6.3.12. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Moreover, let $\mathscr{O}$ be a special $C^{1,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{1,\kappa}} \leq 1$ or a bounded $C^{1,\kappa}$-domain. Then*

$$[W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X), W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)]_{\frac{1}{2}} = W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X).$$

Proposition 6.3.13. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p-1, p-1)$, $j \in \{0,1\}$ and let X be a* UMD *Banach space. Moreover, let $\mathscr{O}$ be a special $C^{j+1,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{j+1,\kappa}} \leq \Lambda$, where $\Lambda \in (0,1)$ is as in Lemma 2.6.6 or a bounded $C^{j+1,\kappa}$-domain. Then*

$$[W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X), W^{k+2+j,p}_{\mathrm{Neu}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)]_{\frac{1}{2}} = W^{k+1+j,p}_{\mathrm{Neu}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X).$$

Proof of Propositions 6.3.12 and 6.3.13. By Propositions 6.3.4, 6.3.7 and Lemma 6.3.9, it suffices to prove the statements for $\mathscr{O} = \mathbb{R}^d_+$, which follows from Theorem 3.6.5. □

We remark that in the above two propositions the conditions on $[\mathscr{O}]_{C^{j+1,\kappa}}$ can be omitted and in this case the implicit constants will depend on the domain.

6.4. Fractional domains of the Laplacian on the half-space

In this section, we establish properties of the Laplacian on the half-space that are required for Sections 6.5 and 6.6. There, we will transfer the H^∞-calculus for the Laplacian from $\mathbb{R}^d_+$ to domains using the perturbation results in Section 6.2. The aim of the present section is to recall the bounded H^∞-calculus for the Laplacian on $\mathbb{R}^d_+$ from Chapter 5 and to characterise the relevant fractional domains and interpolation spaces. These characterisations are one of the key ingredients in the perturbation theorems in Section 6.5.

Throughout this section, the Dirichlet and Neumann Laplacian on $\mathbb{R}^d_+$ will be defined as follows.

Definition 6.4.1. Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and let X be a UMD Banach space.

(i) Let $\gamma \in (-1, 2p-1) \setminus \{p-1\}$. The *Dirichlet Laplacian* Δ_{Dir} *on* $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is defined by

$$\Delta_{\mathrm{Dir}} u := \Delta u \quad \text{with} \quad D(\Delta_{\mathrm{Dir}}) := W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

(ii) Let $\gamma \in (-1, p-1)$ and $j \in \{0,1\}$. The *Neumann Laplacian* Δ_{Neu} *on* $W^{k+j,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is defined by

$$\Delta_{\mathrm{Neu}} u := \Delta u \quad \text{with} \quad D(\Delta_{\mathrm{Neu}}) := W^{k+j+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

Note that equivalently we can write Δ_{Neu} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+(k-1)p}; X)$ where $k \in \mathbb{N}_0$ and $\gamma \in (p-1, 2p-1)$, or, $k \in \mathbb{N}_1$ and $\gamma \in (-1, p-1)$. This matches the notation in Theorem 6.1.2.

We recall from Chapter 5 that these Laplace operators admit a bounded H^∞-calculus.

Theorem 6.4.2. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 6.4.1(i). Then for all $\lambda > 0$ we have that*

(i) $\lambda - \Delta_{\mathrm{Dir}}$ is sectorial of angle $\omega(\lambda - \Delta_{\mathrm{Dir}}) = 0$,

(ii) $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) = 0$.

Moreover, the statements hold for $\lambda = 0$ as well if $\gamma + kp \in (-1, 2p-1)$.

Theorem 6.4.3. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, p-1)$, $j \in \{0,1\}$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+j,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 6.4.1(ii). Then for all $\lambda > 0$ we have that*

(i) $\lambda - \Delta_{\mathrm{Neu}}$ is sectorial of angle $\omega(\lambda - \Delta_{\mathrm{Neu}}) = 0$,

(ii) $\lambda - \Delta_{\mathrm{Neu}}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) = 0$.

Moreover, the statements hold for $\lambda = 0$ as well if $k = 0$.

Remark 6.4.4. The domain $D(A)$ of an operator A on a Banach space Y is endowed with the graph norm $\|u\|_Y + \|Au\|_Y$ for $u \in D(A)$. It follows from Theorems 6.4.2 and 6.4.3 that the graph norm is equivalent to the norm of the domain in Definition 6.4.1. Under the conditions of Theorem 6.4.2, we have for the Dirichlet Laplacian that

$$\begin{aligned}\|u\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &\eqsim_{p,k,\gamma,\lambda,X} \|u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} + \|(\lambda - \Delta_{\mathrm{Dir}})u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\ &\eqsim_{p,k,\gamma,\lambda,X} \|(\lambda - \Delta_{\mathrm{Dir}})u\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}, \quad u \in W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X),\end{aligned}$$

where the latter identity only holds for $\lambda > 0$. A similar norm equivalence holds for the Neumann Laplacian.

To transfer the H^∞-calculus for the Laplacian from $\mathbb{R}^d_+$ to domains, we need to identify certain fractional domains and interpolation spaces. This will be done in Section 6.4.1 and 6.4.2 for the Dirichlet and Neumann Laplacian, respectively. We additionally define for $\gamma \in (-1,\infty) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$ and $k \in \mathbb{N}_0$ the following weighted Sobolev spaces with boundary conditions (cf. [190, Section 6.3])

$$\begin{aligned}W^{k,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{u \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(\Delta^j u) = 0, \forall\, j < \tfrac{1}{2}\big(k - \tfrac{\gamma+1}{p}\big)\Big\}, \\ W^{k,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_\gamma; X) &:= \Big\{u \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(\Delta^j \partial_1 u) = 0, \forall\, j < \tfrac{1}{2}\big(k - 1 - \tfrac{\gamma+1}{p}\big)\Big\}.\end{aligned}$$

6.4.1. Fractional domains for the Dirichlet Laplacian

We begin with an elliptic regularity result for the shifted Dirichlet Laplacian on spaces with additional boundary conditions.

Lemma 6.4.5. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$, $\lambda > 0$ and let X be a* UMD *Banach space. Then for all $f \in W^{k+1,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ there exists a unique $u \in W^{k+3,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $\lambda u - \Delta u = f$. Moreover, this solution satisfies*

$$\|u\|_{W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where the constant $C > 0$ only depends on p, k, γ, λ, d and X.

Proof. Step 1: the case $\gamma \in (-1, p-1)$. Let $\gamma \in (-1, p-1)$ and note that

$$W^{k+1,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{k+1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{k+1,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X),$$

which has $C^\infty_c(\mathbb{R}^d_+; X)$ as a dense subspace, see Proposition 6.3.1. We claim that for $f \in C^\infty_c(\mathbb{R}^d_+; X)$ there exists a unique solution $u \in \mathscr{S}(\mathbb{R}^d_+; X)$ to $\lambda u - \Delta u = f$ on $\mathbb{R}^d_+$ that satisfies $u(0,\cdot) = (\Delta u)(0,\cdot) = 0$. Indeed, by the proof of Lemma 5.5.3 we obtain an odd function $\overline{u} \in \mathscr{S}(\mathbb{R}^d; X)$ which solves $\lambda \overline{u} - \Delta \overline{u} = f_{\mathrm{odd}} \in \mathscr{S}(\mathbb{R}^d; X)$ on $\mathbb{R}^d$. We recall from Section 5.4.2 that $f_{\mathrm{odd}}(x) = \mathrm{sign}(x_1) f(|x_1|, \widetilde{x})$ for $x \in \mathbb{R}^d$ is the odd extension of f with respect to $x_1 = 0$. Since $\overline{u}$ is odd, it follows that $\Delta \overline{u}$ is odd as well. Then $u := \overline{u}|_{\mathbb{R}^d_+} \in \mathscr{S}(\mathbb{R}^d_+; X)$ is a solution to $\lambda u - \Delta u = f$ on $\mathbb{R}^d_+$ and satisfies $u(0,\cdot) = (\Delta u)(0,\cdot) = 0$. The uniqueness follows from [190, Corollary 4.3]. This proves the claim.

Let $f \in C^\infty_c(\mathbb{R}^d_+; X)$ and let $u \in \mathscr{S}(\mathbb{R}^d_+; X)$ be the solution to $\lambda u - \Delta u = f$ as follows from the claim. In particular, we have that $\mathrm{Tr}(\partial_1^2 u) = 0$. We define $v_0 := u$ and $v_j := \partial_j u$ for $j \in \{1, \dots, d\}$. These functions satisfy the equations

$$\begin{aligned}
&\lambda v_0 - \Delta v_0 = f && v_0(0,\cdot) = u(0,\cdot) = 0,\\
&\lambda v_1 - \Delta v_1 = \partial_1 f && (\partial_1 v_1)(0,\cdot) = (\partial_1^2 u)(0,\cdot) = 0,\\
&\lambda v_j - \Delta v_j = \partial_j f && v_j(0,\cdot) = 0, \quad j \in \{2, \dots, d\}.
\end{aligned}$$

Therefore, by Propositions 5.5.4 and 5.5.6 we have for $j \in \{0, \dots, d\}$ the estimates

$$\|v_j\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where the constant $C > 0$ only depends on p, k, γ, λ, d and X. This implies that

$$\|u\|_{W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \eqsim \sum_{j=0}^{d} \|v_j\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \lesssim \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where the constant $C > 0$ only depends on p, k, γ, λ, d and X. A density argument, similar to the proof of Proposition 5.5.4, yields the desired result for the case $\gamma \in (-1, p-1)$. Note that the uniqueness of $u \in W^{k+3,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \hookrightarrow W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ follows from Proposition 5.5.4.

Step 2: the case $\gamma \in (p-1, 2p-1)$. Note that for $\gamma \in (p-1, 2p-1)$ we have

$$W^{k+1,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma-p+(k+1)p}; X).$$

Since $\gamma - p \in (-1, p-1)$ and

$$W^{k+3,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{(k+1)+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma-p+(k+1)p}; X),$$

the result follows from Theorem 6.4.2 (see also Proposition 5.5.4). □

We can now proceed with characterising fractional domains of the Dirichlet Laplacian.

Proposition 6.4.6. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,2p-1) \setminus \{p-1\}$, $\lambda > 0$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ as in Definition 6.4.1. Then*

$$D\big((\lambda - \Delta_{\mathrm{Dir}})^{\frac{1}{2}}\big) = W^{k+1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X),$$
$$D\big((\lambda - \Delta_{\mathrm{Dir}})^{\frac{3}{2}}\big) = W^{k+3,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

Proof. We write $A_{\mathrm{Dir}} := \lambda - \Delta_{\mathrm{Dir}}$. For $\gamma \in (-1,2p-1) \setminus \{p-1\}$ it holds that A_{Dir} has BIP by Theorem 6.4.2, so Propositions 6.2.1 and 6.3.12 imply

$$D(A^{\frac{1}{2}}_{\mathrm{Dir}}) = [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{1}{2}} = W^{k+1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

By [126, Theorem 15.2.5] and the characterisation of $D(A^{\frac{1}{2}}_{\mathrm{Dir}})$ we find

$$\begin{aligned} D(A^{\frac{3}{2}}_{\mathrm{Dir}}) &= \{u \in D(A_{\mathrm{Dir}}) : A_{\mathrm{Dir}} u \in D(A^{\frac{1}{2}}_{\mathrm{Dir}})\} \\ &= \{u \in W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) : A_{\mathrm{Dir}} u \in W^{k+1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\}. \end{aligned} \tag{6.4.1}$$

It is straightforward to check that the embedding $W^{k+3,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \hookrightarrow D(A^{\frac{3}{2}}_{\mathrm{Dir}})$ holds. The converse embedding follows from (6.4.1) and Lemma 6.4.5. □

As a consequence of Proposition 6.4.6, we can characterise the fractional domains as complex interpolation spaces as well.

Corollary 6.4.7. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $k_0, k_1 \in \{0,1,2,3\}$, $\theta \in (0,1)$ and let X be a* UMD *Banach space. For $\lambda > 0$ and Δ_{Dir} on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 6.4.1.*

(i) If $\gamma \in (-1, p-1)$, then

$$D\big((\lambda - \Delta_{\mathrm{Dir}})^{\frac{(1-\theta)k_0+\theta k_1}{2}}\big) = \big[W^{k+k_0,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+k_1,p}_{\Delta,\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\big]_\theta.$$

(ii) If $\gamma \in (p-1, 2p-1)$, then

$$D\big((\lambda - \Delta_{\mathrm{Dir}})^{\frac{(1-\theta)k_0+\theta k_1}{2}}\big) = \big[W^{k+k_0,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+k_1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\big]_\theta.$$

Proof. The fractional domains of the shifted Dirichlet Laplacian on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ form a complex interpolation scale by Proposition 6.2.1 and Theorem 6.4.2, so the statements are a direct consequence of Proposition 6.4.6. □

We close this section about the Dirichlet Laplacian with a complex interpolation identification, which follows from reiteration and the work of Šneĭberg [254, 255] on the openness of the set of $\theta \in (0,1)$ for which a bounded operator $T\colon [X_0, X_1]_\theta \to [Y_0, Y_1]_\theta$ is invertible.

Proposition 6.4.8. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $k_0 \in \{0,1,2\}$, $\gamma \in (p-1, 2p-1)$ and let X be a* UMD *Banach space. Then there exists an $\varepsilon > 0$ such that for all $\theta \in \big(0, \frac{2-k_0}{3-k_0} + \varepsilon\big)$ we have*

$$\begin{aligned}\big[W^{k+k_0,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\big]_\theta \\ = \big[W^{k+k_0,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\big]_\theta.\end{aligned}$$

Proof. Let $\lambda > 0$ and define $A_{\mathrm{Dir}} := \lambda - \Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ as in Definition 6.4.1. First consider the case $k_0 = 0$ and $\theta = \frac{2}{3}$, in which case we have by Corollary 6.4.7 and Proposition 3.6.2

$$\begin{aligned}[W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{2}{3}} &= D(A_{\mathrm{Dir}}) \\ &= W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{k+2,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \\ &= [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{2}{3}}.\end{aligned} \tag{6.4.2}$$

Next, for $\theta \in (0, \frac{2}{3})$, we set $\widetilde{\theta} = \theta \cdot \frac{3}{2} \in (0,1)$. Then, by reiteration for the complex interpolation method (see [27, Theorem 4.6.1]) and (6.4.2) we have

$$\begin{aligned}&[W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_\theta \\ &\quad = \big[W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{2}{3}}\big]_{\widetilde{\theta}} \\ &\quad = \big[W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{2}{3}}\big]_{\widetilde{\theta}} \\ &\quad = [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_\theta.\end{aligned}$$

Note that the identity mapping is bounded on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and

$$\mathrm{id} : W^{k+3,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \to W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \quad \text{is bounded.}$$

Moreover, we have proved that it is invertible as a mapping

$$\begin{aligned}\mathrm{id} : [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{0}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_\theta \\ \to [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_\theta\end{aligned}$$

for $\theta \in (0, \frac{2}{3}]$. Since the collection of $\theta \in (0,1)$ for which this mapping is invertible is open (see [87, Theorem 1.3.24]), the proposition in the case $k_0 = 0$ follows.

Finally, for $k_0 \in \{1,2\}$, let $\varepsilon > 0$ be such that the proposition holds for $k_0 = 0$ and fix $\theta \in \big(0, \frac{2-k_0}{3-k_0} + \varepsilon\big)$. Then we have

$$(1-\theta)\tfrac{k_0}{3} + \theta = \tfrac{k_0}{3} + (\tfrac{3-k_0}{3})\theta < \tfrac{k_0}{3} + \tfrac{2-k_0}{3} + \varepsilon = \tfrac{2}{3} + \varepsilon.$$

Therefore, using Proposition 3.6.2, reiteration for the complex interpolation method and

the case $k_0 = 0$, we obtain

$$\begin{aligned}
&[W_0^{k+k_0,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W_0^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_\theta \\
&\qquad = \big[[W_0^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W_0^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{k_0}{3}}, W_0^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\big]_\theta \\
&\qquad = [W_0^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W_0^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{(1-\theta)\frac{k_0}{3}+\theta} \\
&\qquad = [W_{\mathrm{Dir}}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W_{\mathrm{Dir}}^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{(1-\theta)\frac{k_0}{3}+\theta}.
\end{aligned}$$

Using Corollary 6.4.7 two more times, we have

$$\begin{aligned}
&[W_{\mathrm{Dir}}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W_{\mathrm{Dir}}^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{(1-\theta)\frac{k_0}{3}+\theta} \\
&\qquad = D(A_{\mathrm{Dir}}^{(1-\theta)\frac{k_0}{2}+\frac{3}{2}\theta}) \\
&\qquad = [W_{\mathrm{Dir}}^{k+k_0,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W_{\mathrm{Dir}}^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_\theta,
\end{aligned}$$

proving the proposition. □

Remark 6.4.9. We conjecture that, e.g., in the case $k = k_0 = 0$, there is actually the equality of complex interpolation spaces

$$[L^p(\mathbb{R}^d_+, w_\gamma; X), W_{\mathrm{Dir}}^{3,p}(\mathbb{R}^d_+, w_\gamma; X)]_\theta = [L^p(\mathbb{R}^d_+, w_\gamma; X), W_0^{3,p}(\mathbb{R}^d_+, w_\gamma; X)]_\theta \tag{6.4.3}$$

for all $\theta \in \big(0, \frac{1}{3}(1+\frac{\gamma+1}{p})\big)$, which is suggested by results on interpolation with boundary conditions as studied in Chapter 3 and [189]. However, at the moment, the case $\gamma \in (p-1, 2p-1)$ of (6.4.3) for the parameter range $\theta \in \big(\frac{2}{3}+\varepsilon, \frac{1}{3}\big(1+\frac{\gamma+1}{p}\big)\big)$ is an interesting open problem that seems to require a novel approach to interpolation with boundary conditions.

6.4.2. Fractional domains for the Neumann Laplacian

Similar to the Dirichlet Laplacian above, we now characterise fractional domains for the Neumann Laplacian. The proofs are similar to those in Section 6.4.1, but for the convenience of the reader, we provide the details.

Lemma 6.4.10. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ such that $\gamma + kp > -1$, $\lambda > 0$ and let X be a* UMD *Banach space. Then for all $f \in W^{k+2,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ there exists a unique $u \in W^{k+4,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ such that $\lambda u - \Delta_{\mathrm{Neu}} u = f$. Moreover, this solution satisfies*

$$\|u\|_{W^{k+4,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \|f\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where the constant $C > 0$ only depends on p, k, γ, λ, d and X.

Proof. Step 1: the case $\gamma \in (p-1, 2p-1)$ and $k \geq -1$. Note that for $\gamma \in (p-1, 2p-1)$ we have

$$W^{k+2,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma-p+(k+1)p}; X).$$

Since $\gamma - p \in (-1, p-1)$ and

$$W^{k+4,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{(k+2)+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma-p+(k+1)p}; X),$$

the result follows from Theorem 6.4.3 (see also Proposition 5.5.6).

Step 2: the case $\gamma \in (-1, p-1)$ and $k \geq 0$. Note that for $\gamma \in (-1, p-1)$ we have

$$W^{k+2,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = W^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X),$$

which has

$$C^\infty_{\mathrm{c},1}(\overline{\mathbb{R}^d_+}; X) := \{f \in C^\infty_{\mathrm{c}}(\overline{\mathbb{R}^d_+}; X) : \partial_1 f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)\}$$

as a dense subspace, see Proposition 3.4.13. For $f \in C^\infty_{\mathrm{c},1}(\overline{\mathbb{R}^d_+}; X)$ there exists a unique solution $u \in \mathscr{S}(\mathbb{R}^d_+; X)$ to $\lambda u - \Delta_{\mathrm{Neu}} u = f$ on $\mathbb{R}^d_+$ that satisfies $(\partial_1 u)(0,\cdot) = (\Delta \partial_1 u)(0,\cdot) = 0$. This can be proved similarly to Lemma 6.4.5 now using an even extension (cf. Lemma 5.5.5).

Take $f \in C^\infty_{\mathrm{c},1}(\overline{\mathbb{R}^d_+}; X)$ and let $u \in \mathscr{S}(\mathbb{R}^d_+; X)$ be the solution to $\lambda u - \Delta_{\mathrm{Neu}} u = f$ as above. We define $v_0 := u$ and $v_j := \partial_j u$ for $j \in \{1, \dots, d\}$. These functions satisfy the estimates

$$\begin{aligned}
&\lambda v_0 - \Delta v_0 = f && (\partial_1 v_0)(0,\cdot) = 0,\\
&\lambda v_1 - \Delta v_1 = \partial_1 f && v_1(0,\cdot) = 0,\\
&\lambda v_j - \Delta v_j = \partial_j f && (\partial_1 v_j)(0,\cdot) = 0, \quad j \in \{2, \dots, d\}.
\end{aligned}$$

If $j = 1$, then by Lemma 6.4.5 (using that $(\partial_1 f)|_{\partial\mathbb{R}^d_+} = 0$) we have the estimate

$$\|v_1\|_{W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \|\partial_1 f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}. \tag{6.4.4}$$

If $j \in \{2, \dots, d\}$, then applying Step 1 with $k-1$ and $\gamma + p \in (p-1, 2p-1)$, yields

$$\begin{aligned}
\|v_j\|_{W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &= \|v_j\|_{W^{(k-1)+4,p}(\mathbb{R}^d_+, w_{\gamma+p+(k-1)p}; X)}\\
&\leq C \|\partial_j f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},
\end{aligned} \tag{6.4.5}$$

and similarly for $j = 0$ we obtain

$$\|v_j\|_{W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}. \tag{6.4.6}$$

The estimates (6.4.4), (6.4.5) and (6.4.6) imply that

$$\begin{aligned}
\|u\|_{W^{k+4,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &\eqsim \sum_{j=0}^d \|v_j\|_{W^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}\\
&\lesssim \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} + \sum_{j=1}^d \|\partial_j f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}\\
&\lesssim \|f\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},
\end{aligned}$$

where the constant only depends on p, k, γ, λ, d and X. A density argument, similar to the proof of Proposition 5.5.4, yields the result. Note that the uniqueness of $u \in W^{k+4,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \hookrightarrow W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ follows from Proposition 5.5.6. □

We continue with the characterisation of fractional domains of the Neumann Laplacian.

Proposition 6.4.11. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$ such that $\gamma + kp > -1$, $\lambda > 0$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ as in Definition 6.4.1. Then*

$$D\big((\lambda - \Delta_{\mathrm{Neu}})^{\frac{1}{2}}\big) = W^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X),$$
$$D\big((\lambda - \Delta_{\mathrm{Neu}})^{\frac{3}{2}}\big) = W^{k+4,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

Proof. We write $A_{\mathrm{Neu}} := \lambda - \Delta_{\mathrm{Neu}}$. For $\gamma \in (-1,2p-1)\setminus\{p-1\}$ it holds that A_{Neu} has BIP by Theorem 6.4.3, so Propositions 6.2.1 and 6.3.13 imply

$$D(A^{\frac{1}{2}}_{\mathrm{Neu}}) = [W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{\frac{1}{2}} = W^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

By [126, Theorem 15.2.5] and the characterisation of $D(A^{\frac{1}{2}}_{\mathrm{Neu}})$ we find

$$\begin{aligned} D(A^{\frac{3}{2}}_{\mathrm{Neu}}) &= \{u \in D(A_{\mathrm{Neu}}) : A_{\mathrm{Neu}} u \in D(A^{\frac{1}{2}}_{\mathrm{Neu}})\} \\ &= \{u \in W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) : A_{\mathrm{Neu}} u \in W^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\}. \end{aligned}$$

From this, the embedding $W^{k+4,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \hookrightarrow D(A^{\frac{3}{2}}_{\mathrm{Neu}})$ is straightforward and the converse embedding follows from Lemma 6.4.10. □

In contrast to the Dirichlet case, we do not need a version of Proposition 6.4.8 for the Neumann Laplacian. This is simply due to the fact that we cannot consider the Neumann Laplacian on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ with $\gamma > p-1$, see Theorem 6.4.3.

6.5. Functional calculus for the Laplacian on special domains

To derive the H^∞-calculus for the Dirichlet and Neumann Laplacian on bounded domains, we will proceed in two steps:

1. Use the H^∞-calculus for the Laplacian on the half-space (Theorems 6.4.2 and 6.4.3) and known perturbation theorems for the H^∞-calculus (Section 6.2) to obtain the H^∞-calculus for the Laplacian on special domains of the form $\mathcal{O} := \{x \in \mathbb{R}^d : x_1 > h(\widetilde{x})\}$ for a compactly supported function h on $\mathbb{R}^{d-1}$ (see Definition 2.6.1).

2. Perform a localisation procedure to transfer the H^∞-calculus for the Laplacian on special domains to bounded domains.

In this section, we will perform Step 1, while Step 2 is postponed to Section 6.6. While localisation procedures are standard in the literature (see, e.g., [67, 89, 166]), the low regularity of the domains considered here leads to perturbation terms that, in some cases, are of the same order as the Laplacian. Therefore, we employ a localisation procedure that is different from the standard procedure as in the aforementioned literature. This leads to a far-reaching generalisation of the results in [190, Theorem 6.1] where exclusively

bounded C^2-domains are considered for only the L^p-case (i.e., $k=0$).

We begin by defining the Laplacian on special domains. Recall that weighted Sobolev spaces on special domains with vanishing boundary conditions are defined in Definition 6.3.6.

Definition 6.5.1. Let $p\in(1,\infty)$, $k\in\mathbb{N}_0$, $\kappa\in[0,1]$ and let X be a UMD Banach space.

(i) Let $\gamma\in((1-\kappa)p-1,2p-1)\setminus\{p-1\}$ and $\mathscr{O}$ a special $C^{1,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{1,\kappa}}\leq 1$. The *Dirichlet Laplacian* Δ_{Dir} *on* $W^{k,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X)$ is defined by

$$\Delta_{\mathrm{Dir}}u:=\Delta u\quad\text{ with }\quad D(\Delta_{\mathrm{Dir}}):=W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X).$$

(ii) Let $\gamma\in((1-\kappa)p-1,p-1)$, $j\in\{0,1\}$ and $\mathscr{O}$ a special $C^{j+1,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{j+1,\kappa}}\leq\Lambda$, where $\Lambda\in(0,1)$ is as in Lemma 2.6.6. The *Neumann Laplacian* Δ_{Neu} *on* $W^{k+j,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X)$ is defined by

$$\Delta_{\mathrm{Neu}}u:=\Delta u\quad\text{ with }\quad D(\Delta_{\mathrm{Neu}}):=W^{k+j+2,p}_{\mathrm{Neu}}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X).$$

Moreover, the Dirichlet and Neumann Laplacian on $\mathbb{R}^d_+$ as in Definition 6.4.1 will be denoted by $\Delta^{\mathbb{R}^d_+}_{\mathrm{Dir}}$ and $\Delta^{\mathbb{R}^d_+}_{\mathrm{Neu}}$, respectively.

The main results from this section on the H^∞-calculus for the Laplacian on special domains are summarised in the following two theorems.

Theorem 6.5.2 (H^∞-calculus for $\lambda-\Delta_{\mathrm{Dir}}$ on special domains)**.** *Let $p\in(1,\infty)$, $k\in\mathbb{N}_0$, $\kappa\in[0,1]$, $\gamma\in((1-\kappa)p-1,2p-1)\setminus\{p-1\}$, $\lambda>0$ and let X be a* UMD *Banach space. Moreover, assume that $\mathscr{O}$ is a special $C^{1,\kappa}_{\mathrm{c}}$-domain. Then there exists a $\delta\in(0,1)$ such that if $[\mathscr{O}]_{C^{1,\kappa}}<\delta$, then $\lambda-\Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X)$ as in Definition 6.5.1 has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda-\Delta_{\mathrm{Dir}})=0$.*

Theorem 6.5.3 (H^∞-calculus for $\lambda-\Delta_{\mathrm{Neu}}$ on special domains)**.** *Let $p\in(1,\infty)$, $k\in\mathbb{N}_0$, $\kappa\in(0,1]$, $\gamma\in((1-\kappa)p-1,p-1)$, $j\in\{0,1\}$, $\lambda>0$ and let X be a* UMD *Banach space. Moreover, assume that $\mathscr{O}$ is a special $C^{j+1,\kappa}_{\mathrm{c}}$-domain. Then there exists a $\delta\in(0,1)$ such that if $[\mathscr{O}]_{C^{j+1,\kappa}}<\delta$, then $\lambda-\Delta_{\mathrm{Neu}}$ on $W^{k+j,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X)$ as in Definition 6.5.1 has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda-\Delta_{\mathrm{Neu}})=0$.*

Remark 6.5.4. Similar to Theorems 6.4.2 and 6.4.3, we expect that Theorems 6.5.2 and 6.5.3 also hold for $\lambda=0$ if $\gamma+kp$ is small. We will not consider this minor improvement of the theorems here, since in Section 6.6 we consider bounded domains and use properties of the spectrum to obtain the H^∞-calculus with $\lambda=0$.

The proofs of Theorems 6.5.2 and 6.5.3 are given in Section 6.5.2 after having established some preliminary estimates in Section 6.5.1.

6.5.1. Preliminary estimates

In the proofs of Theorems 6.5.2 and 6.5.3, we derive the H^∞-calculus on special domains by perturbing the corresponding calculus for the Laplacian on the half-space. To relate the Laplacian on special domains and the half-space, we use the diffeomorphisms Φ and Ψ from Lemmas 2.6.5 and 2.6.6 for the Dirichlet and Neumann Laplacian, respectively. The diffeomorphism Φ is easier to deal with, but it does not suffice for the Neumann Laplacian since it does not preserve the direction of the normal vector at the boundary, see Section 2.6.

First, consider the case of Dirichlet boundary conditions. Let $\mathscr{O}$ be a special C^1_{c}-domain and recall that $\Phi_* f = f \circ \Phi^{-1}$ for $f \in L^1_{\mathrm{loc}}(\mathscr{O}; X)$. Define $\Delta^\Phi : W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+; X) \to L^1_{\mathrm{loc}}(\mathbb{R}^d_+; X)$ by

$$\Delta^\Phi := \Phi_* \circ \Delta \circ (\Phi^{-1})_*.$$

Making use of the explicit form of the diffeomorphism $\Phi(x) = (x_1 - h_1(x), \widetilde{x})$ for $x \in \mathscr{O}$ (see Lemma 2.6.5), an elementary computation involving the chain rule shows that $\Delta^\Phi = \Delta + B$, where the perturbation B is given by

$$B = -2((\nabla h_1) \circ \Phi^{-1}) \cdot \nabla \partial_1 + |(\nabla h_1) \circ \Phi^{-1}|^2 \partial_1^2 - ((\Delta h_1) \circ \Phi^{-1}) \partial_1. \tag{6.5.1}$$

Note that the first two perturbation terms in (6.5.1) are second-order differential operators since $(\nabla h_1) \circ \Phi^{-1}$ is bounded on $\mathbb{R}^d_+$ if $\mathscr{O}$ is a special C^1_{c}-domain, see Lemma 2.6.5. The order of the latter perturbation term in (6.5.1) depends on the smoothness of the domain.

- If $\mathscr{O}$ is a special C^2_{c}-domain, then $(\Delta h_1) \circ \Phi^{-1}$ is bounded on $\mathbb{R}^d_+$ and thus the last term in (6.5.1) is a lower-order perturbation term.
- If $\mathscr{O}$ is a special C^1_{c}-domain, then $(\Delta h_1)(\Phi^{-1}(y))$ blows up like y_1^{-1} in the neighbourhood of $y_1 = 0$, see Lemma 2.6.5. Therefore, estimating, say, the $L^p(\mathbb{R}^d_+, w_\gamma)$-norm of $((\Delta h_1) \circ \Phi^{-1})\partial_1$ gives that the weight exponent effectively decreases. However, this loss can be compensated by applying Hardy's inequality, which allows us to recover the original weight w_γ. In this way, we also obtain an additional derivative from Hardy's inequality, meaning that all three perturbation terms in (6.5.1) have the same order.

This demonstrates that if the smoothness of the domain is low, then the last perturbation term in (6.5.1) is more difficult to deal with.

Similarly, for Neumann boundary conditions, let Ψ be the diffeomorphism from Lemma 2.6.6 and define $\Delta^\Psi : W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+; X) \to L^1_{\mathrm{loc}}(\mathbb{R}^d_+; X)$ by

$$\Delta^\Psi := \Psi_* \circ \Delta \circ (\Psi^{-1})_*.$$

Recall that $\Psi(x) = (x_1 - h_1(x), \widetilde{x} - \widetilde{h}_1(\widetilde{x}))$ for $x \in \mathscr{O}$ and we write $h = (h_1, \widetilde{h}_1) =: (h_1, h_2, \dots, h_d)$. Another tedious, but elementary calculation with the chain rule shows that $\Delta^\Psi = \Delta + B$ with

$$B = \sum_{i,j=1}^d \big[\big(-H_{i,j} - H_{j,i} + (HH^\top)_{i,j}\big) \circ \Psi^{-1}\big] \partial_{ij}^2 - \sum_{j=1}^d \big[\Delta h_j \circ \Psi^{-1}\big] \partial_j,$$

where $H := Dh$ is the Jacobi matrix of h. Compared to the case of Dirichlet boundary conditions in (6.5.1), we have more perturbation terms, since the diffeomorphism Ψ is more involved. To simplify the estimates later on, we simply note that B is a linear combination of terms of the following forms

$$\begin{aligned} &\big[(\partial^{\nu_1} h_{i_1}) \circ \Psi^{-1}\big]\partial^{\mu}, && |\nu_1| = 1, |\mu| = 2, \\ \big[(\partial^{\nu_1} h_{i_1}) \circ \Psi^{-1}\big]&\big[(\partial^{\nu_2} h_{i_2}) \circ \Psi^{-1}\big]\partial^{\mu}, && |\nu_1| = |\nu_2| = 1, |\mu| = 2, \\ &\big[(\partial^{\nu_1} h_{i_1}) \circ \Psi^{-1}\big]\partial^{\mu}, && |\nu_1| = 2, |\mu| = 1, \end{aligned} \tag{6.5.2}$$

where $i_1, i_2 \in \{1, \dots, d\}$. Note that the perturbation terms in (6.5.1) can also be written as in (6.5.2) with $i_1 = i_2 = 1$ and Φ instead of Ψ.

In the following lemmas we provide precise estimates for the perturbation terms. We note that these estimates work for both diffeomorphisms Φ and Ψ since they have the same regularity properties. Throughout the rest of this subsection, we let $\mathscr{O}$ be a special C^1_c-domain and take the following standing assumptions:

- In the case of Φ: $[\mathscr{O}]_{C^1} \le 1$ and take h_1 as in Lemma 2.6.5.
- In the case of Ψ: $[\mathscr{O}]_{C^1} \le \Lambda$, where $\Lambda \in (0,1)$ as in Lemma 2.6.6, take h_1 and $\widetilde{h}_1$ as Lemma 2.6.6 and set $(h_1, h_2, \dots, h_d) := (h_1, \widetilde{h}_1)$.

We start with the estimates for the perturbation terms with $|\mu| = 2$.

Lemma 6.5.5 (Estimates for $|\mu| = 2$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and let X be a Banach space. Take $|\mu| = 2$, $|\nu_1| = |\nu_2| = 1$ and $i_1, i_2 \in \{1, \dots, d\}$. Let $\Upsilon \in \{\Phi, \Psi\}$ and define*

$$\begin{aligned} P_1 &:= \big[(\partial^{\nu_1} h_{i_1}) \circ \Upsilon^{-1}\big]\partial^{\mu}, \\ P_2 &:= \big[(\partial^{\nu_1} h_{i_1}) \circ \Upsilon^{-1}\big]\big[(\partial^{\nu_2} h_{i_2}) \circ \Upsilon^{-1}\big]\partial^{\mu}. \end{aligned}$$

(i) If $\gamma \in (p-1, 2p-1)$, then for $n \in \{0,1\}$ and $u \in W^{k+2+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ it holds that

$$\|P_m u\|_{W^{k+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \le C \cdot [\mathscr{O}]_{C^1} \cdot \|u\|_{W^{k+2+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}, \qquad m \in \{1,2\}.$$

(ii) If $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p - 1, p-1)$, $j \in \{0,1\}$ and $\mathscr{O}$ is a special $C^{j+1,\kappa}_c$-domain, then for $n \in \{0,1\}$ and $u \in W^{k+2+j+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ it holds that

$$\|P_m u\|_{W^{k+j+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \le C \cdot [\mathscr{O}]_{C^{j+1,\kappa}} \cdot \|u\|_{W^{k+2+j+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}, \qquad m \in \{1,2\}.$$

In all cases, the constant $C > 0$ only depends on p, k, γ, κ, d and X.

Proof. For notational convenience we write $W^{k,p}(w_\gamma) := W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$.

Step 1: preparations. We prove the estimates for P_1, where, from now on, we omit the subscripts from ν and i. The estimates for P_2 are derived in a similar way.

For $\alpha \in \mathbb{N}_0^d$ and some regular enough u we obtain with the product rule that

$$\begin{aligned} &\|\partial^{\alpha}[((\partial^{\nu} h_i) \circ \Upsilon^{-1})\partial^{\mu} u]\|_{L^p(w_{\gamma+kp})} \\ &\qquad \lesssim \sum_{\beta \le \alpha} \big\| [\partial^{\beta}((\partial^{\nu} h_i) \circ \Upsilon^{-1})][\partial^{\alpha-\beta}\partial^{\mu} u] \big\|_{L^p(w_{\gamma+kp})}. \end{aligned} \tag{6.5.3}$$

In the case that $|\alpha|,|\beta|\geq 1$ and $y\in\mathbb{R}^d_+$, the multivariate Faà di Bruno's formula [50, Theorem 2.1] implies

$$|\partial_y^\beta(\partial^\nu h_i)(\Upsilon^{-1}(y))|\lesssim \sum_{1\leq|\delta|\leq|\beta|}|(\partial^\delta\partial^\nu h_i)(\Upsilon^{-1}(y))|\sum_{s=1}^{|\beta|}\sum_{p_s(\beta,\delta)}\prod_{m=1}^{s}|\partial^{\boldsymbol{\ell}_m}\Upsilon^{-1}(y)|^{\boldsymbol{k}_m}, \tag{6.5.4}$$

where the sets $p_s(\beta,\delta)$ are contained in

$$\Big\{(\boldsymbol{k}_1,\dots,\boldsymbol{k}_s;\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_s)\in(\mathbb{N}_0^d\setminus\{0\})^s\times(\mathbb{N}_0^d\setminus\{0\})^s:\sum_{m=1}^{s}|\boldsymbol{k}_m|=|\delta|,\sum_{m=1}^{s}|\boldsymbol{k}_m||\boldsymbol{\ell}_m|=|\beta|\Big\}. \tag{6.5.5}$$

By Lemma 2.6.5(ii)+(iv) for $\Upsilon=\Phi$ and Lemma 2.6.6(ii)+(v) for $\Upsilon=\Psi$, we have the estimate

$$|(\partial^\delta\partial^\nu h_i)(\Upsilon^{-1}(y))|\lesssim\frac{[\mathscr{O}]_{C^{j+1,\kappa}}}{\operatorname{dist}(\Upsilon^{-1}(y),\partial\mathscr{O})^{(|\delta|-j-\kappa)_+}}\lesssim\frac{[\mathscr{O}]_{C^{j+1,\kappa}}}{y_1^{(|\delta|-j-\kappa)_+}}, \tag{6.5.6}$$

for all $\kappa\in[0,1]$, $j\in\{0,1\}$, $\delta\in\mathbb{N}_0^d$, $|\nu|=1$ and $y\in\mathbb{R}^d_+$. Moreover, by the same lemmas we also have the (non-optimal) estimate

$$|\partial^{\boldsymbol{\ell}}\Upsilon^{-1}(y)|\lesssim\frac{[\mathscr{O}]_{C^{j+1}}}{y_1^{(|\boldsymbol{\ell}|-j-1)_+}}, \tag{6.5.7}$$

for all $j\in\{0,1\}$, $\boldsymbol{\ell}\in\mathbb{N}_0^d$ and $y\in\mathbb{R}^d_+$.

Step 2: proof of (i). Let $\gamma\in(p-1,2p-1)$, $n\in\{0,1\}$ and $\mathscr{O}$ a special C^1_{c}-domain. To prove (i) we need to consider (6.5.3) with $|\alpha|\leq k+n$. If $\beta=0$ in (6.5.3), then it follows from (6.5.6) that

$$\|((\partial^\nu h_i)\circ\Upsilon^{-1})(\partial^\alpha\partial^\mu u)\|_{L^p(w_{\gamma+kp})}\lesssim[\mathscr{O}]_{C^1}\|u\|_{W^{k+2+n,p}(w_{\gamma+kp})}.$$

By (6.5.4), (6.5.6) and (6.5.7), we have for $\beta\leq\alpha$ with $|\alpha|,|\beta|\geq1$ that (6.5.3) can be further estimated as

$$\begin{aligned}
&\big\|[\partial^\beta((\partial^\nu h_i)\circ\Upsilon^{-1})][\partial^{\alpha-\beta}\partial^\mu u]\big\|_{L^p(w_{\gamma+kp})}\\
&\lesssim[\mathscr{O}]_{C^1}\sum_{1\leq|\delta|\leq|\beta|}\sum_{s=1}^{|\beta|}\sum_{p_s(\beta,\delta)}\|\partial^{\alpha-\beta}\partial^\mu u\|_{L^p(w_{\gamma+kp-|\delta|p-\sum_{m=1}^{s}(|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p})}\\
&\lesssim[\mathscr{O}]_{C^1}\|\partial^{\alpha-\beta}\partial^\mu u\|_{W^{|\beta|,p}(w_{\gamma+kp})}\lesssim[\mathscr{O}]_{C^1}\|u\|_{W^{k+2+n,p}(w_{\gamma+kp})},
\end{aligned}$$

where we have applied Hardy's inequality (Corollary 5.3.2) $|\beta|$ times using that

$$\gamma+kp-|\delta|p-\sum_{m=1}^{s}(|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p\overset{(6.5.5)}{=}\gamma+kp-|\beta|p>(1-n)p-1\geq-1,$$

since $\gamma>p-1$, $|\beta|\leq k+n$ and $n\in\{0,1\}$. This completes the proof of (i).

Step 3: proof of (ii). Let $\kappa\in(0,1]$, $\gamma\in((1-\kappa)p-1,p-1)$, $n\in\{0,1\}$, $j\in\{0,1\}$ and $\mathscr{O}$ a special $C^{j+1,\kappa}_{\mathrm{c}}$-domain. Consider (6.5.3) with $|\alpha|\leq k+j+n$. In the case that $\beta=0$ it follows from (6.5.6) that

$$\|((\partial^\nu h_i)\circ\Upsilon^{-1})(\partial^\alpha\partial^\mu u)\|_{L^p(w_{\gamma+kp})}\lesssim[\mathscr{O}]_{C^{j+1,\kappa}}\|u\|_{W^{k+2+j+n,p}(w_{\gamma+kp})}.$$

By (6.5.4), (6.5.6) and (6.5.7), we have for $\beta \le \alpha$ with $|\alpha|, |\beta| \ge 1$ that (6.5.3) can be further estimated as

$$\begin{aligned}
&\big\| [\partial^\beta ((\partial^\nu h_i) \circ \Upsilon^{-1})] [\partial^{\alpha-\beta}\partial^\mu u] \big\|_{L^p(w_{\gamma+kp})} \\
&\lesssim [\mathscr{O}]_{C^{j+1,\kappa}} \sum_{1\le|\delta|\le j} \sum_{s=1}^{|\beta|} \sum_{p_s(\beta,\delta)} \|\partial^{\alpha-\beta}\partial^\mu u\|_{L^p(w_{\gamma+kp-\sum_{m=1}^s (|\ell_m|-(j+1))_+|k_m|p)}} \\
&\quad + [\mathscr{O}]_{C^{j+1,\kappa}} \sum_{j+1\le|\delta|\le|\beta|} \sum_{s=1}^{|\beta|} \sum_{p_s(\beta,\delta)} \|\partial^{\alpha-\beta}\partial^\mu u\|_{L^p(w_{\gamma+kp-(|\delta|-j-\kappa)p-\sum_{m=1}^s (|\ell_m|-1)|k_m|p)}},
\end{aligned} \tag{6.5.8}$$

where the sum over $1 \le |\delta| \le j$ is only present if $j = 1$ and in this case we have $(|\delta|-j-\kappa)_+ = 0$. We first consider the case $j+1 \le |\delta| \le |\beta|$ for $j \in \{0,1\}$. Note that by (6.5.5) we have

$$\begin{aligned}
\gamma + kp - (|\delta| - j - \kappa)p - \sum_{m=1}^s (|\ell_m| - 1)|k_m|p &= \gamma + kp - \big(|\beta| - j - \kappa\big)p \\
&> (1-n)p - 1 \ge -1.
\end{aligned}$$

Therefore, Lemma 6.3.2 applied with $s = |\beta| - j - \kappa \le |\beta|$ yields

$$\|\partial^{\alpha-\beta}\partial^\mu u\|_{L^p(w_{\gamma+kp-(|\beta|-j-\kappa)p})} \lesssim \|\partial^{\alpha-\beta}\partial^\mu u\|_{W^{|\beta|,p}(w_{\gamma+kp})} \le \|u\|_{W^{k+2+j+n,p}(w_{\gamma+kp})}.$$

In the case that $j = 1$, we additionally estimate the sum over $|\delta| = 1$ in (6.5.8). In the case that $|\ell_m| \le j+1 = 2$ for all $m \in \{1,\dots,s\}$, we have $(|\ell_m|-2)_+ = 0$ and

$$\|\partial^{\alpha-\beta}\partial^\mu u\|_{L^p(w_{\gamma+kp})} \lesssim \|u\|_{W^{k+3+n,p}(w_{\gamma+kp})}.$$

If there exists an $m_0 \in \{1,\dots,s\}$ such that $|\ell_{m_0}| > 2$, then it follows from (6.5.5) and $|\beta| \le k+1+n$ that

$$\begin{aligned}
\gamma + kp - \sum_{m=1}^s (|\ell_m|-2)_+|k_m|p &= \gamma + kp - \Big(\sum_{\substack{m=1\\ m\ne m_0}}^s (|\ell_m|-2)_+|k_m| + (|\ell_{m_0}|-2)|k_{m_0}| \Big)p \\
&\ge \gamma + kp - \Big(\sum_{\substack{m=1\\ m\ne m_0}}^s |\ell_m||k_m| + |\ell_{m_0}||k_{m_0}| - 2|k_{m_0}| \Big)p \\
&\ge \gamma + kp - |\beta|p + 2p > (2-n-\kappa)p - 1 \ge -1.
\end{aligned}$$

Therefore, Lemma 6.3.2 (applied with s replaced by $\sum_{m=1}^s (|\ell_m|-2)_+|k_m| \le |\beta|$), yields

$$\|\partial^{\alpha-\beta}\partial^\mu u\|_{L^p(w_{\gamma+kp-\sum_{m=1}^s (|\ell_m|-2)_+|k_m|p)}} \lesssim \|\partial^{\alpha-\beta}\partial^\mu u\|_{W^{|\beta|,p}(w_{\gamma+kp})} \le \|u\|_{W^{k+3+n,p}(w_{\gamma+kp})}.$$

This finishes the proof of (ii). □

We continue with some preliminary estimates for the perturbation term with $|\mu| = 1$.

Lemma 6.5.6 (Estimates for $|\mu| = 1$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and let X be a* UMD *Banach space. Take $|\mu| = 1$, $|\nu| = 2$ and $i \in \{1,\dots,d\}$. Let $\Upsilon \in \{\Phi, \Psi\}$ and define*

$$P := \big[(\partial^\nu h_i) \circ \Upsilon^{-1}\big]\partial^\mu,$$

(i) If $\gamma \in (p-1, 2p-1)$, then for $n \in \{0,1\}$ it holds that

$$\|Pu\|_{W^{k+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \cdot [\mathscr{O}]_{C^1} \cdot \|u\|_{W^{k+2+n,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

for

$$u \in \begin{cases} W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X) & \text{if } n = 0, \\ W^{k+3,p}_0(\mathbb{R}^d_+, w_{\gamma+kp}; X) & \text{if } n = 1. \end{cases}$$

(ii) If $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p - 1, p-1)$ and $\mathscr{O}$ is a special $C^{1,\kappa}_c$-domain, then for $u \in [W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{1-\frac{\kappa}{2}}$ it holds that

$$\|Pu\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \cdot [\mathscr{O}]_{C^{1,\kappa}} \cdot \|u\|_{[W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)]_{1-\frac{\kappa}{2}}}.$$

In particular, for $u \in W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ it holds that

$$\|Pu\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \cdot [\mathscr{O}]_{C^{1,\kappa}} \cdot \|u\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}.$$

(iii) If $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p - 1, p-1)$ and $\mathscr{O}$ is a special $C^{2,\kappa}_c$-domain, then for $u \in W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ it holds that

$$\|Pu\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \cdot [\mathscr{O}]_{C^{2,\kappa}} \cdot \|u\|_{W^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}.$$

In all cases, the constant $C > 0$ only depends on p, k, γ, κ, d and X.

Note that in Lemma 6.5.6(i) with $n = 1$, we need two traces of u to be zero. This will not be a problem later on, since the Neumann trace will disappear in the complex interpolation space, see Step 1 in the proof of Theorem 6.5.2.

Proof. For notational convenience we write $W^{k,p}(w_\gamma) := W^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$.

Step 1: preparations. For $\alpha \in \mathbb{N}^d_0$ and some regular enough u we obtain with the product rule that

$$\begin{aligned} &\|\partial^\alpha[((\partial^\nu h_i)\circ\Upsilon^{-1})\partial^\mu u]\|_{L^p(w_{\gamma+kp})} \\ &\qquad \lesssim \sum_{\beta\leq\alpha} \big\|[\partial^\beta((\partial^\nu h_i)\circ\Upsilon^{-1})][\partial^{\alpha-\beta}\partial^\mu u]\big\|_{L^p(w_{\gamma+kp})}. \end{aligned} \tag{6.5.9}$$

In the case that $|\alpha|, |\beta| \geq 1$ and $y \in \mathbb{R}^d_+$, the multivariate Faà di Bruno's formula [50, Theorem 2.1] implies

$$|\partial^\beta_y(\partial^\nu h_i)(\Upsilon^{-1}(y))| \lesssim \sum_{1\leq|\delta|\leq|\beta|} |(\partial^\delta\partial^\nu h_i)(\Upsilon^{-1}(y))| \sum_{s=1}^{|\beta|} \sum_{p_s(\beta,\delta)} \prod_{m=1}^{s} |\partial^{\ell_m}\Upsilon^{-1}(y)|^{k_m}, \tag{6.5.10}$$

where the sets $p_s(\beta,\delta)$ are given as in (6.5.5). By Lemma 2.6.5(ii)+(iv) for $\Upsilon = \Phi$ and Lemma 2.6.6(ii)+(v) for $\Upsilon = \Psi$, we have the estimate

$$|(\partial^\delta\partial^\nu h_i)(\Upsilon^{-1}(y))| \lesssim \frac{[\mathscr{O}]_{C^{j+1,\kappa}}}{\operatorname{dist}(\Upsilon^{-1}(y), \partial\mathscr{O})^{(|\delta|+1-j-\kappa)_+}} \lesssim \frac{[\mathscr{O}]_{C^{j+1,\kappa}}}{y_1^{(|\delta|+1-j-\kappa)_+}}, \tag{6.5.11}$$

for all $\kappa \in [0,1]$, $j \in \{0,1\}$, $\delta \in \mathbb{N}_0^d$, $|\nu| = 2$ and $y \in \mathbb{R}_+^d$. Moreover, by the same lemmas we also have the (non-optimal) estimate

$$|\partial^{\boldsymbol{\ell}} \Upsilon^{-1}(y)| \lesssim \frac{[\mathscr{O}]_{C^1}}{y_1^{(|\boldsymbol{\ell}|-1)_+}}, \tag{6.5.12}$$

for all $\boldsymbol{\ell} \in \mathbb{N}_0^d$ and $y \in \mathbb{R}_+^d$.

Step 2: proof of (i). Let $\gamma \in (p-1, 2p-1)$, $n \in \{0,1\}$ and $\mathscr{O}$ a special C_{c}^1-domain. To prove (i) we need to consider (6.5.9) with $|\alpha| \le k+n$. If $\beta = 0$ in (6.5.9), then it follows from (6.5.11) and Hardy's inequality (Corollary 5.3.2, using that $\gamma + (k-1)p > -1$) that

$$\|((\partial^\nu h_i) \circ \Upsilon^{-1})(\partial^\alpha \partial^\mu u)\|_{L^p(w_{\gamma+kp})} \lesssim [\mathscr{O}]_{C^1} \|\partial^\alpha \partial^\mu u\|_{L^p(w_{\gamma+(k-1)p})} \lesssim [\mathscr{O}]_{C^1} \|u\|_{W^{k+2+n,p}(w_{\gamma+kp})}.$$

By (6.5.10), (6.5.11) and (6.5.12), we have for $\beta \le \alpha$ with $|\alpha|, |\beta| \ge 1$ that (6.5.9) can be further estimated as

$$\begin{aligned}
&\big\| [\partial^\beta ((\partial^\nu h_i) \circ \Upsilon^{-1})][\partial^{\alpha-\beta} \partial^\mu u] \big\|_{L^p(w_{\gamma+kp})} \\
&\lesssim [\mathscr{O}]_{C^1} \sum_{1 \le |\delta| \le |\beta|} \sum_{s=1}^{|\beta|} \sum_{p_s(\beta,\delta)} \|\partial^{\alpha-\beta} \partial^\mu u\|_{L^p(w_{\gamma+kp-(|\delta|+1)p-\sum_{m=1}^s (|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p})} \\
&\lesssim [\mathscr{O}]_{C^1} \|\partial^{\alpha-\beta} \partial^\mu u\|_{W^{|\beta|+1,p}(w_{\gamma+kp})} \lesssim [\mathscr{O}]_{C^1} \|u\|_{W^{k+2+n,p}(w_{\gamma+kp})},
\end{aligned}$$

where we have applied Hardy's inequality $|\beta|+1$ times using that

$$\gamma + kp - (|\delta|+1)p - \sum_{m=1}^s (|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p \overset{(6.5.5)}{=} \gamma + kp - (|\beta|+1)p > -np - 1,$$

since $\gamma > p-1$, $|\beta| \le k+n$ and $n \in \{0,1\}$. This shows that for $n = 1$ we need to take $u \in W_0^{k+3,p}(w_{\gamma+kp})$ by Hardy's inequality. This completes the proof of (i).

Step 3: proof of (ii). Let $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p-1, p-1)$ and $\mathscr{O}$ a special $C_{\mathrm{c}}^{1,\kappa}$-domain. Consider (6.5.9) with $|\alpha| \le k$. If $\beta = 0$ in (6.5.9), then it follows from (6.5.11) and Lemma 6.3.3 (applied to $s = 1-\kappa$ and using that X is UMD) that

$$\begin{aligned}
\|((\partial^\nu h_i) \circ \Upsilon^{-1})(\partial^\alpha \partial^\mu u)\|_{L^p(w_{\gamma+kp})} &\lesssim [\mathscr{O}]_{C^{1,\kappa}} \|\partial^\alpha \partial^\mu u\|_{L^p(w_{\gamma+kp-(1-\kappa)p})} \\
&\lesssim [\mathscr{O}]_{C^{1,\kappa}} \|u\|_{W^{k+1,p}(w_{\gamma+kp-(1-\kappa)p})} \\
&\lesssim [\mathscr{O}]_{C^{1,\kappa}} \|u\|_{[W^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X), W^{k+2,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X)]_{1-\frac{\kappa}{2}}}.
\end{aligned}$$

By (6.5.10), (6.5.11) and (6.5.12), we have for $\beta \le \alpha$ with $|\alpha|, |\beta| \ge 1$ that (6.5.9) can be further estimated as

$$\begin{aligned}
&\big\| [\partial^\beta ((\partial^\nu h_i) \circ \Upsilon^{-1})][\partial^{\alpha-\beta} \partial^\mu u] \big\|_{L^p(w_{\gamma+kp})} \\
&\lesssim [\mathscr{O}]_{C^{1,\kappa}} \sum_{1 \le |\delta| \le |\beta|} \sum_{s=1}^{|\beta|} \sum_{p_s(\beta,\delta)} \|\partial^{\alpha-\beta} \partial^\mu u\|_{L^p(w_{\gamma+kp-(|\delta|+1-\kappa)p-\sum_{m=1}^s (|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p})}.
\end{aligned}$$

By (6.5.5) and the Hardy inequalities from Lemmas 2.2.2 (using $\gamma+kp-(|\beta|+1-\kappa)p>-1$) and 6.3.3, we obtain

$$\begin{aligned}\|\partial^{\alpha-\beta}\partial^{\mu}u\|_{L^p(w_{\gamma+kp-(|\delta|+1-\kappa)p-\sum_{m=1}^{s}(|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p})} &= \|\partial^{\alpha-\beta}\partial^{\mu}u\|_{L^p(w_{\gamma+kp-(|\beta|+1-\kappa)p})}\\ &\lesssim \|u\|_{W^{k+1,p}(w_{\gamma+kp-(1-\kappa)p})}\\ &\lesssim \|u\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{1-\frac{\kappa}{2}}}.\end{aligned}$$

This completes the proof of (ii).

Step 4: proof of (iii). Let $\kappa\in(0,1]$, $\gamma\in((1-\kappa)p-1,p-1)$ and $\mathscr{O}$ a special $C^{2,\kappa}_{\mathrm{c}}$-domain. Consider (6.5.9) with $|\alpha|\le k+1$. If $\beta=0$ in (6.5.9), then it follows from (6.5.11) that

$$\|((\partial^{\nu}h_i)\circ\Upsilon^{-1})(\partial^{\alpha}\partial^{\mu}u)\|_{L^p(w_{\gamma+kp})}\lesssim [\mathscr{O}]_{C^{2,\kappa}}\|\partial^{\alpha}\partial^{\mu}u\|_{L^p(w_{\gamma+kp})}\lesssim [\mathscr{O}]_{C^{2,\kappa}}\|u\|_{W^{k+2,p}(w_{\gamma+kp})}.$$

By (6.5.10), (6.5.11) and (6.5.12), we have for $\beta\le\alpha$ with $|\alpha|,|\beta|\ge 1$ that (6.5.9) can be further estimated as

$$\begin{aligned}&\big\|[\partial^{\beta}((\partial^{\nu}h_i)\circ\Upsilon^{-1})][\partial^{\alpha-\beta}\partial^{\mu}u]\big\|_{L^p(w_{\gamma+kp})}\\ &\lesssim [\mathscr{O}]_{C^{2,\kappa}}\sum_{1\le|\delta|\le|\beta|}\sum_{s=1}^{|\beta|}\sum_{p_s(\beta,\delta)}\|\partial^{\alpha-\beta}\partial^{\mu}u\|_{L^p(w_{\gamma+kp-(|\delta|-\kappa)p-\sum_{m=1}^{s}(|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p})}\\ &\lesssim [\mathscr{O}]_{C^{2,\kappa}}\|\partial^{\alpha-\beta}\partial^{\mu}u\|_{W^{|\beta|,p}(w_{\gamma+kp})}\lesssim [\mathscr{O}]_{C^{2,\kappa}}\|u\|_{W^{k+2,p}(w_{\gamma+kp})},\end{aligned}$$

where we have used Lemma 6.3.2 with s replaced by $|\beta|-\kappa$ and that

$$\gamma+kp-(|\delta|-\kappa)p-\sum_{m=1}^{s}(|\boldsymbol{\ell}_m|-1)|\boldsymbol{k}_m|p=\gamma+kp-|\beta|p+\kappa p>-1.$$

This finishes the proof of (iii). □

The fact that we need boundary conditions in the spaces in parts of Lemma 6.5.6 will complicate the proof of perturbing the H^{∞}-calculus in Section 6.5.2. In particular, for the Dirichlet Laplacian on special $C^{1,\kappa}_{\mathrm{c}}$-domains, we need an additional estimate, which we obtain via extrapolation spaces and the adjoint operator.

Let $p\in(1,\infty)$, $\gamma\in\mathbb{R}$, $\mathscr{O}\subseteq\mathbb{R}^d$ open and let X be a reflexive Banach space (which is implied by the UMD condition). Then $L^p(\mathscr{O},w_{\gamma}^{\partial\mathscr{O}};X)$ is reflexive and with the unweighted pairing

$$\langle f,g\rangle_{L^p(\mathscr{O},w_{\gamma}^{\partial\mathscr{O}};X)\times(L^p(\mathscr{O},w_{\gamma}^{\partial\mathscr{O}};X))'}=\int_{\mathscr{O}}\langle f(x),g(x)\rangle_{X\times X'}\,\mathrm{d}x,$$

its dual space is

$$(L^p(\mathscr{O},w_{\gamma}^{\partial\mathscr{O}};X))'=L^{p'}(\mathscr{O},w_{\gamma'}^{\partial\mathscr{O}};X'),$$

where $p'=p/(p-1)$ and $\gamma'=-\gamma/(p-1)$. Note that if $\gamma\in(-1,p-1)$, then $\gamma'\in(-1,p'-1)$.

We have the following characterisation of the adjoint operator of the Dirichlet Laplacian. We note that for $\gamma\in(p-1,2p-1)$ the characterisation of the domain of the adjoint is more sophisticated, see [190, Proposition 6.6].

Proposition 6.5.7 ([190, Proposition 6.5]). *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and let X be a* UMD *Banach space. Let $A_{p,\gamma,X} := \Delta_{\mathrm{Dir}}^{\mathbb{R}^d_+}$ on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ be the Dirichlet Laplacian as in Definition 6.4.1. Then the adjoint operator is $(A_{p,\gamma,X})' = A_{p',\gamma',X'}$.*

To continue, we briefly recall the extrapolation scales, see [174, Appendix E] or [14, Chapter 5] for more details. Let A be a sectorial operator on a Banach space Y such that $0 \in \rho(A)$. Then for any $\alpha \in \mathbb{R}$, we can define the scale of extrapolation spaces

$$(E_{\alpha,A}, \|\cdot\|_{E_{\alpha,A}}) = \begin{cases} (D(A^\alpha), \|A^\alpha \cdot\|_Y) & \text{if } \alpha \geq 0, \\ (Y, \|A^\alpha \cdot\|_Y)^\sim & \text{if } \alpha < 0. \end{cases}$$

where $\sim$ denotes the completion of the space. Let A' denote the adjoint of A. In the case that Y is reflexive and $\alpha \in \mathbb{R}$, the extrapolation scale satisfies

$$E_{-\alpha,A} = (E_{\alpha,A'})', \tag{6.5.13}$$

with respect to the duality $\langle Y, Y'\rangle$.

With the extrapolation scale and the characterisation of the adjoint, we can prove the following estimate for the perturbation terms coming from the Dirichlet Laplacian in (6.5.1) on weighted Lebesgue spaces.

Lemma 6.5.8. *Let $p \in (1,\infty)$, $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p-1, p-1)$ and let X be a* UMD *Banach space. Let $\Delta_{\mathrm{Dir}}^{\mathbb{R}^d_+}$ on $L^p(\mathbb{R}^d_+, w_\gamma; X)$ be the Dirichlet Laplacian as in Definition 6.4.1. Assume that $\mathscr{O}$ is a special $C^{1,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{1,\kappa}} \leq 1$. Then the perturbation B in* (6.5.1) *satisfies*

$$\|(\lambda - \Delta_{\mathrm{Dir}}^{\mathbb{R}^d_+})^{-\frac{1}{2}} Bu\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \leq C \cdot [\mathscr{O}]_{C^{1,\kappa}} \cdot \|(\lambda - \Delta_{\mathrm{Dir}}^{\mathbb{R}^d_+})^{\frac{1}{2}} u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)},$$

for all $\lambda > 0$ and $u \in W^{1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_\gamma; X)$.

Proof. We write $A := \lambda - \Delta_{\mathrm{Dir}}^{\mathbb{R}^d_+}$. Note that (6.5.13), Propositions 6.5.7 and 6.4.6 imply that

$$\|A^{-\frac{1}{2}} Bu\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)} \eqsim \sup \big| \langle Bu, v\rangle_{L^p(\mathbb{R}^d_+, w_\gamma; X) \times L^{p'}(\mathbb{R}^d_+, w_{\gamma'}; X')} \big|,$$

where the supremum is taken over all $v \in E_{\frac{1}{2},A'} = D((A')^{\frac{1}{2}}) = W^{1,p'}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma'}; X')$ with $\|v\|_{W^{1,p'}(\mathbb{R}^d_+, w_{\gamma'}; X')} \leq 1$. Fix such a $v \in W^{1,p'}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma'}; X')$. Recall from (6.5.1) that the perturbation terms in B are of the form $((\partial^\nu h_1) \circ \Phi^{-1})^r \partial^\mu \partial_1$ with $|\mu| = |\nu| = 1$ and $r \in \{1,2\}$, or, with $|\nu| = 2$, $|\mu| = 0$ and $r = 1$. Therefore, by Lemma 2.6.5(iv), integration by parts, Hölder's inequality, the proof of Lemma 6.3.2 (using the Dirichlet boundary condition to apply

Hardy's inequality) and Proposition 6.4.6, we obtain

$$\begin{aligned}
&\big|\langle Bu, v\rangle_{L^p(\mathbb{R}^d_+, w_\gamma; X)\times L^{p'}(\mathbb{R}^d_+, w_{\gamma'}; X')}\big| \\
&\lesssim [\mathscr{O}]_{C^{1,\kappa}}\Big(\sum_{|\mu|=1}\int_{\mathbb{R}^d_+}|\langle \partial^\mu \partial_1 u, v\rangle_{X\times X'}|\,\mathrm{d}x + \int_{\mathbb{R}^d_+} x_1^{-(1-\kappa)}|\langle \partial_1 u, v\rangle_{X\times X'}|\,\mathrm{d}x\Big) \\
&\leq [\mathscr{O}]_{C^{1,\kappa}}\Big(\int_{\mathbb{R}^d_+} x_1^{\gamma}\|\partial_1 u\|_X^p\,\mathrm{d}x\Big)^{\frac{1}{p}} \\
&\qquad\cdot\Big[\sum_{|\mu|=1}\Big(\int_{\mathbb{R}^d_+} x_1^{\gamma'}\|\partial^\mu v\|_{X'}^{p'}\,\mathrm{d}x\Big)^{\frac{1}{p'}} + \Big(\int_{\mathbb{R}^d_+} x_1^{\gamma'-(1-\kappa)p'}\|v\|_{X'}^{p'}\,\mathrm{d}x\Big)^{\frac{1}{p'}}\Big] \\
&\lesssim [\mathscr{O}]_{C^{1,\kappa}}\|u\|_{W^{1,p}(\mathbb{R}^d_+, w_\gamma; X)}\|v\|_{W^{1,p'}(\mathbb{R}^d_+, w_{\gamma'}; X')} \lesssim [\mathscr{O}]_{C^{1,\kappa}}\|A^{\frac{1}{2}}u\|_{L^p(\mathbb{R}^d_+, w_\gamma; X)}.
\end{aligned}$$

This proves the desired estimate. □

6.5.2. The proofs of Theorems 6.5.2 and 6.5.3

With the preliminary estimates on the perturbation terms, we can now continue with proving the boundedness of the H^∞-calculus for the Laplacian on special domains. We start with the proof of Theorem 6.5.2 for the Dirichlet Laplacian.

Proof of Theorem 6.5.2. Let $\mathscr{O}$ be a special domain as specified in the theorem, which is of the form

$$\mathscr{O} = \{x \in \mathbb{R}^d : x_1 > h(\widetilde{x})\},$$

and let h_1 and Φ be as in Lemma 2.6.5. Recall that we introduced $\Delta^\Phi : W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+; X) \to L^1_{\mathrm{loc}}(\mathbb{R}^d_+; X)$ given by

$$\begin{aligned}
\Delta^\Phi &:= \Phi_* \circ \Delta \circ (\Phi^{-1})_* \\
&= \Delta - 2((\nabla h_1)\circ\Phi^{-1})\cdot\nabla\partial_1 + |(\nabla h_1)\circ\Phi^{-1}|^2\partial_1^2 - ((\Delta h_1)\circ\Phi^{-1})\partial_1 \\
&=: \Delta + B_1 + B_2 + B_3.
\end{aligned}$$

Let $-\Delta^\Phi_{\mathrm{Dir}}$ denote the realisation of Δ^Φ in $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ with domain $D(-\Delta^\Phi_{\mathrm{Dir}}) = W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$. Due to the isomorphisms in Proposition 6.3.4, the trace characterisation in Proposition 6.3.7 and standard properties of the H^∞-calculus, the desired statements in Theorem 6.5.2 for $-\Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ are equivalent to the corresponding statements for $-\Delta^\Phi_{\mathrm{Dir}}$ on $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$. We will apply the perturbation theorems from Section 6.2.2 to show that the H^∞-calculus for the Laplacian on the half-space is preserved under the perturbation $B := B_1 + B_2 + B_3$. We note that the estimates for B_1 and B_2 are covered in Lemma 6.5.5 and for B_3 in Lemma 6.5.6.

Step 1: the case $\gamma \in (p-1, 2p-1)$. Let $\gamma \in (p-1, 2p-1)$ and let $\mathscr{O}$ be a special C^1_{c}-domain. Let $\lambda > 0$ and we write $A_{\mathrm{Dir}} := \lambda - \Delta^{\mathbb{R}^d_+}_{\mathrm{Dir}}$. We apply Theorem 6.2.4 to show that $\lambda - (\Delta^{\mathbb{R}^d_+}_{\mathrm{Dir}} + B)$ has a bounded H^∞-calculus. Let $u \in D(A_{\mathrm{Dir}}) = W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$, then by Lemma

6.5.5(i), Lemma 6.5.6(i) and Remark 6.4.4, we have

$$\begin{aligned}\|Bu\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} &\lesssim [\mathscr{O}]_{C^1}\|u\|_{W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}\\ &\eqsim [\mathscr{O}]_{C^1}\|A_{\mathrm{Dir}}u\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)},\end{aligned}$$

which shows condition (i) of Theorem 6.2.4. To show that condition (ii) of Theorem 6.2.4 holds, note that by Lemma 6.5.5(i) and Lemma 6.5.6(i) we have that

$$\begin{aligned}&B\colon W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\to W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X) \quad \text{and}\\ &B\colon W_0^{k+3,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\to W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)\end{aligned} \tag{6.5.14}$$

are bounded operators. Take $\theta\in(0,\frac{1}{2})$ such that Proposition 6.4.8 for $k_0=2$ holds and let $u\in D(A_{\mathrm{Dir}}^{1+\theta})$. Then, by Corollary 6.4.7 twice, properties of the complex interpolation method using (6.5.14), Proposition 6.4.8 and the invertibility of A_{Dir} we have

$$\begin{aligned}\|A_{\mathrm{Dir}}^{\theta}Bu\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}\le\|Bu\|_{D(A_{\mathrm{Dir}}^{\theta})} &\eqsim \|Bu\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{2\theta}}\\ &\lesssim \|u\|_{[W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W_0^{k+3,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{2\theta}}\\ &\lesssim \|u\|_{[W_0^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W_0^{k+3,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{2\theta}}\\ &\eqsim \|u\|_{[W_{\mathrm{Dir}}^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W_{\mathrm{Dir}}^{k+3,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{2\theta}}\\ &\eqsim \|u\|_{D(A_{\mathrm{Dir}}^{1+\theta})}\eqsim\|A_{\mathrm{Dir}}^{1+\theta}u\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}.\end{aligned}$$

This shows condition (ii) of Theorem 6.2.4. Therefore, Theorems 6.4.2 and 6.2.4 give that $\lambda-\Delta_{\mathrm{Dir}}^{\Phi}$ has a bounded H^∞-calculus of angle zero if $[\mathscr{O}]_{C^1}$ is small enough.

Step 2: the case $\gamma\in((1-\kappa)p-1,p-1)$. Let $\kappa\in(0,1]$, $\gamma\in((1-\kappa)p-1,p-1)$ and let $\mathscr{O}$ be a special $C_{\mathrm{c}}^{1,\kappa}$-domain. We apply Theorem 6.2.4 to show that $\lambda-(\Delta_{\mathrm{Dir}}^{\mathbb{R}^d_+}+B)$ has a bounded H^∞-calculus. Let $u\in D(A_{\mathrm{Dir}})=W_{\mathrm{Dir}}^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)$. Then by Lemma 6.5.5(ii), Lemma 6.5.6(ii) and Remark 6.4.4, we have

$$\begin{aligned}\|Bu\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} &\lesssim [\mathscr{O}]_{C^{1,\kappa}}\|u\|_{W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}\\ &\eqsim [\mathscr{O}]_{C^{1,\kappa}}\|A_{\mathrm{Dir}}u\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp})}.\end{aligned}$$

Thus, condition (i) of Theorem 6.2.4 is satisfied. To continue, we verify condition (iii) of Theorem 6.2.4 for $\alpha=\frac{1}{2}$. If $k=0$, then the required estimate follows from Lemma 6.5.8. If $k\in\mathbb{N}_1$, then by Proposition 6.3.12 and Corollary 6.4.7, we have

$$\begin{aligned}W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)&=[W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma+p+(k-1)p};X),W_{\mathrm{Dir}}^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+p+(k-1)p};X)]_{\frac{1}{2}}\\ &=D(\widetilde{A}_{\mathrm{Dir}}^{\frac{1}{2}}),\end{aligned}$$

where $\widetilde{A}_{\mathrm{Dir}}:=\lambda-\Delta_{\mathrm{Dir}}^{\mathbb{R}^d_+}$ on $W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma+p+(k-1)p};X)$. Moreover, note that by definition of the fractional powers and Lemma 5.6.4, it follows that the fractional powers of A_{Dir} and

$\widetilde{A}_{\mathrm{Dir}}$ are consistent. Therefore, together with Lemma 6.5.5(i), Lemma 6.5.6(i) and Remark 6.4.4, we obtain

$$\begin{aligned}\|A_{\mathrm{Dir}}^{-\frac{1}{2}}Bu\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} &\eqsim \|\widetilde{A}_{\mathrm{Dir}}^{\frac{1}{2}}A_{\mathrm{Dir}}^{-\frac{1}{2}}Bu\|_{W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma+p+(k-1)p};X)}\\ &= \|Bu\|_{W^{k-1,p}(\mathbb{R}^d_+,w_{\gamma+p+(k-1)p};X)}\\ &\lesssim \|u\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} \eqsim \|A_{\mathrm{Dir}}^{\frac{1}{2}}u\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)},\end{aligned}$$

for $u \in D(A_{\mathrm{Dir}}^{\frac{1}{2}}) = W^{k+1,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp};X)$. Therefore, Theorems 6.4.2 and 6.2.4 give that $\lambda - \Delta^{\Phi}_{\mathrm{Dir}}$ has a bounded H^∞-calculus of angle zero if $[\mathscr{O}]_{C^{1,\kappa}}$ is small enough. □

We conclude this section with the proof of Theorem 6.5.3 about the H^∞-calculus for the Neumann Laplacian.

Proof of Theorem 6.5.3. Let $\mathscr{O}$ be a special $C_{\mathrm{c}}^{j+1,\kappa}$-domain with $[\mathscr{O}]_{C^{j+1,\kappa}} \le \Lambda$, where $\Lambda \in (0,1)$ is as in Lemma 2.6.6. Let Ψ be the diffeomorphism from Lemma 2.6.6 and define $\Delta^\Psi : W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+;X) \to L^1_{\mathrm{loc}}(\mathbb{R}^d_+;X)$ by $\Delta^\Psi := \Psi_* \circ \Delta \circ (\Psi^{-1})_*$. Recall that $\Delta^\Psi = \Delta + B$, where the perturbation term B is a linear combination of terms of the form (6.5.2). In particular, we define B_1, B_2 and B_3 as, respectively, all the perturbation terms in B of the form

$$\begin{aligned}\big[(\partial^{\nu_1} h_{i_1}) \circ \Psi^{-1}\big]\partial^\mu, &\qquad |\nu_1| = 1, |\mu| = 2,\\ \big[(\partial^{\nu_1} h_{i_1}) \circ \Psi^{-1}\big]\big[(\partial^{\nu_2} h_{i_2}) \circ \Psi^{-1}\big]\partial^\mu, &\qquad |\nu_1| = |\nu_2| = 1, |\mu| = 2,\\ \big[(\partial^{\nu_1} h_{i_1}) \circ \Psi^{-1}\big]\partial^\mu, &\qquad |\nu_1| = 2, |\mu| = 1,\end{aligned}$$

where $i_1, i_2 \in \{1,\dots,d\}$.

For $j \in \{0,1\}$ let $-\Delta^\Psi_{\mathrm{Neu}}$ denote the realisation of $-\Delta^\Psi$ in $W^{k+j,p}(\mathbb{R}^d_+, w_{\gamma+kp};X)$ with domain $D(\Delta^\Psi_{\mathrm{Neu}}) = W^{k+2+j,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp};X)$. Due to the isomorphisms in Proposition 6.3.4, the trace characterisation in Proposition 6.3.7 and standard properties of the H^∞-calculus, the desired statements in Theorem 6.5.3 for $-\Delta_{\mathrm{Neu}}$ on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp};X)$ are equivalent to the corresponding statements for $-\Delta^\Psi_{\mathrm{Neu}}$ on $W^{k+j,p}(\mathbb{R}^d_+, w_{\gamma+kp};X)$. We will apply the perturbation theorems from Section 6.2.2 to show that the H^∞-calculus for the Laplacian on the half-space is preserved under the perturbation $B = B_1 + B_2 + B_3$. We note that the estimates for B_1 and B_2 are covered in Lemma 6.5.5 and for B_3 in Lemma 6.5.6.

Let $\lambda > 0$ and we write $A_{\mathrm{Neu}} := \lambda - \Delta^{\mathbb{R}^d_+}_{\mathrm{Neu}}$. We first apply Theorem 6.2.4 to show that $\lambda - (\Delta^{\mathbb{R}^d_+}_{\mathrm{Neu}} + B_1 + B_2)$ has a bounded H^∞-calculus on $W^{k+j,p}(\mathbb{R}^d_+, w_{\gamma+kp};X)$ for $k \in \mathbb{N}_0$. Let $u \in D(A_{\mathrm{Neu}}) = W^{k+j+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp};X)$. Then by Lemma 6.5.5(ii) and Remark 6.4.4, we have

$$\begin{aligned}\|B_1u + B_2u\|_{W^{k+j,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} &\lesssim [\mathscr{O}]_{C^{j+1,\kappa}}\|u\|_{W^{k+j+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}\\ &\eqsim [\mathscr{O}]_{C^{j+1,\kappa}}\|A_{\mathrm{Neu}}u\|_{W^{k+j,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)},\end{aligned}$$

which shows condition (i) of Theorem 6.2.4. Next, we verify condition (ii) of Theorem 6.2.4 for $\alpha = \frac{1}{2}$. Let $u \in D(A_{\mathrm{Neu}}^{\frac{3}{2}}) = W^{k+j+3,p}_{\Delta,\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp};X)$, then by Proposition 6.4.11, Lemma

6.5.5(ii) and the invertibility of A_{Neu}, we have

$$\begin{aligned}\|A_{\mathrm{Neu}}^{\frac{1}{2}}(B_1+B_2)u\|_{W^{k+j,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} &\leq \|B_1u+B_2u\|_{D(A_{\mathrm{Neu}}^{\frac{1}{2}})} \lesssim \|u\|_{W^{k+j+3,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}\\ &\eqsim \|u\|_{D(A_{\mathrm{Neu}}^{\frac{3}{2}})} \eqsim \|A_{\mathrm{Neu}}^{\frac{3}{2}}u\|_{W^{k+j,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}.\end{aligned}$$

Therefore, Theorems 6.4.3 and 6.2.4 give that $\lambda-(\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}+B_1+B_2)$ has a bounded H^∞-calculus of angle zero if $[\mathscr{O}]_{C^{j+1,\kappa}}$ is small enough.

To obtain that $\lambda-\Delta_{\mathrm{Neu}}^{\Psi}$ has a bounded H^∞-calculus, it remains to apply Theorem 6.2.3 to the lower-order perturbation B_3. First, for $j=0$ we apply Theorem 6.2.3 with $\alpha=1-\frac{\kappa}{2}\in[\frac{1}{2},1)$. For $u\in D(\lambda-(\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}+B_1+B_2))=D(A_{\mathrm{Neu}})$ we obtain with Lemma 6.5.6(ii) and Proposition 6.2.1 (using Theorem 6.4.3) that

$$\|B_3u\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} \lesssim \|u\|_{[W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{1-\frac{\kappa}{2}}} \lesssim \|u\|_{D(A_{\mathrm{Neu}}^{1-\frac{\kappa}{2}})}.$$

For $j=1$ we apply Theorem 6.2.3 with $\alpha=\frac{1}{2}$. For $u\in D(\lambda-(\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}+B_1+B_2))=D(A_{\mathrm{Neu}})$ we obtain with Lemma 6.5.6(iii) that

$$\|B_3u\|_{W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)} \lesssim \|u\|_{W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)}.$$

Observe that by Proposition 6.3.13, the bounded H^∞-calculus for $\lambda-(\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}+B_1+B_2)$ and Proposition 6.2.1, we obtain

$$\begin{aligned}W_{\mathrm{Neu}}^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp};X) &= [W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),W_{\mathrm{Neu}}^{k+3,p}(\mathbb{R}^d_+,w_{\gamma+kp};X)]_{\frac{1}{2}}\\ &= [W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X),D(\lambda-(\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}+B_1+B_2))]_{\frac{1}{2}}\\ &= D\big((\lambda-(\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}+B_1+B_2))^{\frac{1}{2}}\big).\end{aligned}$$

This shows the required estimate (6.2.1) for both $j=0$ and $j=1$. Therefore, the bounded H^∞-calculus for $\lambda-(\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}+B_1+B_2)$, Theorem 6.2.3 and Proposition 6.2.2(ii) show that $\lambda-\Delta_{\mathrm{Neu}}^{\Psi}$ has a bounded H^∞-calculus of angle zero if $[\mathscr{O}]_{C^{j+1,\kappa}}$ is small enough. Note that the application of Proposition 6.2.2(ii) requires sectoriality of $\lambda-\Delta_{\mathrm{Neu}}^{\Psi}$ for all $\lambda>0$, which can be obtained from [126, Theorem 16.2.3(2)] applied to the operator $A=\lambda-\Delta_{\mathrm{Neu}}^{\mathbb{R}^d_+}$, provided that $[\mathscr{O}]_{C^{j+1,\kappa}}$ is small enough. □

6.6. Functional calculus for the Laplacian on bounded domains

In this section, we establish our main results concerning the H^∞-calculus for the Laplacian on bounded domains. We begin by recalling the definition of the Laplacian in this setting. The relevant weighted Sobolev spaces with vanishing boundary conditions were introduced in Definition 6.3.10.

Definition 6.6.1. Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$ and let X be a UMD Banach space.

(i) Let $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and $\mathscr{O}$ a bounded $C^{1,\kappa}$-domain. The *Dirichlet Laplacian* Δ_{Dir} *on* $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ with $k \in \mathbb{N}_0$ is defined by

$$\Delta_{\mathrm{Dir}} u := \Delta u \quad \text{with} \quad D(\Delta_{\mathrm{Dir}}) := W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X).$$

(ii) Let $\gamma \in ((1-\kappa)p-1, p-1)$, $j \in \{0,1\}$ and $\mathscr{O}$ a bounded $C^{j+1,\kappa}$-domain. The *Neumann Laplacian* Δ_{Neu} *on* $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ is defined by

$$\Delta_{\mathrm{Neu}} u := \Delta u \quad \text{with} \quad D(\Delta_{\mathrm{Neu}}) := W^{k+j+2,p}_{\mathrm{Neu}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X).$$

(iii) Let $\gamma \in ((1-\kappa)p-1, p-1)$, $j \in \{0,1\}$ and $\mathscr{O}$ a bounded $C^{j+1,\kappa}$-domain. The *Neumann Laplacian* Δ_{Neu} *on the quotient space*

$$W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) / \{c\,\mathbf{1}_{\mathscr{O}} : c \in X\}$$

is defined by $\Delta_{\mathrm{Neu}} u := \Delta u$ with

$$D(\Delta_{\mathrm{Neu}}) := W^{k+j+2,p}_{\mathrm{Neu}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) / \{c\,\mathbf{1}_{\mathscr{O}} : c \in X\}.$$

We now state the main results of this chapter about the H^∞-calculus for the Laplacian on bounded domains. The proofs of the theorems below are given in Sections 6.6.2 and 6.6.3.

Theorem 6.6.2 (H^∞-calculus for $\lambda - \Delta_{\mathrm{Dir}}$ on domains)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$, $\sigma \in (0,\pi)$ and let X be a* UMD *Banach space. Moreover, assume that $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain. Let Δ_{Dir} on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ be the Dirichlet Laplacian as in Definition 6.6.1. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda > \widetilde{\lambda}$ the operator $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) \le \sigma$.*

Theorem 6.6.3 (H^∞-calculus for $\lambda - \Delta_{\mathrm{Neu}}$ on domains)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p-1, p-1)$, $j \in \{0,1\}$, $\sigma \in (0,\pi)$ and let X be a* UMD *Banach space. Moreover, assume that $\mathscr{O}$ is a bounded $C^{j+1,\kappa}$-domain. Let Δ_{Neu} on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ or $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) / \{c\,\mathbf{1}_{\mathscr{O}} : c \in X\}$ be the Neumann Laplacian as in Definition 6.6.1(ii) or (iii), respectively. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda > \widetilde{\lambda}$ the operator $\lambda - \Delta_{\mathrm{Neu}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) \le \sigma$.*

For $X = \mathbb{C}$ we obtain that the spectrum of the Laplacian is independent of the involved parameters. Hence, for the Dirichlet Laplacian we also obtain the H^∞-calculus with $\lambda = 0$ since zero is not contained in the spectrum.

Theorem 6.6.4. *Suppose that the assumptions of Theorem 6.6.2 hold with $X = \mathbb{C}$. Then the following assertions hold.*

(i) The spectrum $\sigma(-\Delta_{\mathrm{Dir}})$ is discrete, contained in $(0,\infty)$ and is independent of $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$.

(ii) Let $\widetilde{\lambda} > 0$ be the smallest eigenvalue of $-\Delta_{\mathrm{Dir}}$. For all $\lambda > -\widetilde{\lambda}$ the operator $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) = 0$.

The spectrum of the Neumann Laplacian on bounded domains contains the eigenvalue zero so we cannot allow for $\lambda = 0$ unless the constant functions are removed from the spaces.

Theorem 6.6.5. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in (0,1]$, $\gamma \in ((1-\kappa)p - 1, p-1)$ and $j \in \{0,1\}$. Moreover, assume that $\mathscr{O}$ is a bounded $C^{j+1,\kappa}$-domain. If Δ_{Neu} is the Neumann Laplacian on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ as in Definition 6.6.1(ii) with $X = \mathbb{C}$, then the following assertions hold.*

(i) The spectrum $\sigma(-\Delta_{\mathrm{Neu}})$ is discrete, contained in $[0,\infty)$ and is independent of $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in ((1-\kappa)p - 1, p-1)$ and $j \in \{0,1\}$.

(ii) For all $\lambda > 0$ the operator $\lambda - \Delta_{\mathrm{Neu}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) = 0$.

Moreover, if Δ_{Neu} is the Neumann Laplacian on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})/\{c\,\mathbf{1}_{\mathscr{O}} : c \in \mathbb{C}\}$ as in Definition 6.6.1(iii) with $X = \mathbb{C}$, then the following assertion holds.

(iii) Let $\widetilde{\lambda} > 0$ be the smallest eigenvalue of $-\Delta_{\mathrm{Neu}}$. For all $\lambda > -\widetilde{\lambda}$ the operator $\lambda - \Delta_{\mathrm{Neu}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) = 0$.

Remark 6.6.6.

(i) It is an open question whether Theorems 6.6.2 and 6.6.3 (in the case where Δ_{Neu} is defined as in Definition 6.6.1(iii)) with general UMD Banach spaces X also hold for $\lambda = 0$. In the following special cases, one can actually conclude the result of Theorems 6.6.2 and 6.6.3 with $\lambda = 0$.

- If X is a Hilbert space or isomorphic to a closed subspace of an L^p-space, then by redoing the proofs of [124, Proposition 2.1.2 & Theorem 2.1.9] for Sobolev spaces, one sees that the results in the scalar case with $\lambda = 0$ (Theorems 6.6.4 and 6.6.5) extend to the vector-valued case.
- If X is a UMD Banach space and $k = 0$, then using [124, Theorem 2.1.3] and that the semigroup corresponding to the Laplacian is positive and uniformly exponentially stable, we can obtain the bounded H^∞-calculus with $\lambda = 0$. The proof of this special case for the Dirichlet Laplacian is provided in Corollary 6.6.7 below. However, the proof does not extend to $k \geq 1$.

For the general case ($k \geq 0$ and X a UMD Banach space) we expect that one can show uniform exponential stability for the semigroup corresponding to the Laplacian via (weighted) kernel bounds for the scalar-valued case. Using a tensor extension and consistency, one could also obtain the required kernel bounds for the vector-valued case.

(ii) The p-independence of the spectra of the Laplacian on L^p-spaces is well studied. Moreover, in [64, 171] it is proved that on certain weighted L^p-spaces the spectrum is independent of the weight. However, the power weights $w^{\partial\mathscr{O}}_{\gamma}$ that we use do not fit into their settings. Instead, we will use compactness and consistency of the resolvent to obtain the spectral independence in Theorems 6.6.4 and 6.6.5.

6.6.1. An application to Riesz transforms

In this section, we discuss the boundedness of the Riesz transform associated with the Dirichlet Laplacian on the half-space and bounded domains. This can be derived as a consequence of the bounded H^∞-calculus and the interpolation theory developed in Chapter 3. For an elaborate study of Riesz transforms associated with the Laplacian on the half-space, the reader is referred to [86].

We start with an extension of the H^∞-calculus of $-\Delta_{\mathrm{Dir}}$ from scalar-valued Lebesgue spaces to vector-valued Lebesgue spaces, see also Remark 6.6.6. This extends the result in [190, Theorem 6.1 & Corollary 6.2].

Corollary 6.6.7 (H^∞-calculus for $-\Delta_{\mathrm{Dir}}$ on $L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$)**.** *Let $p \in (1,\infty)$, $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ be as in Definition 6.6.1. Then the operator $-\Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(-\Delta_{\mathrm{Dir}}) = 0$.*

Proof. We define the operators

$$\begin{aligned} \Delta_{\mathrm{Dir}}^{\mathbb{C}} &:= \Delta_{\mathrm{Dir}} \quad \text{on } L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}) \quad \text{and} \\ \Delta_{\mathrm{Dir}}^{X} &:= \Delta_{\mathrm{Dir}} \quad \text{on } L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \end{aligned}$$

as in Definition 6.6.1. Theorem 6.6.4 implies that $0 \in \rho(-\Delta_{\mathrm{Dir}}^{\mathbb{C}})$ and it follows from [126, Proposition K.2.3] that the analytic semigroup S_t generated by $\Delta_{\mathrm{Dir}}^{\mathbb{C}}$ is uniformly exponentially stable. Moreover, the resolvent $R(\lambda, \Delta_{\mathrm{Dir}}^{\mathbb{C}})$ is positive for $\lambda > 0$ (this follows from the L^2-case and consistency in Lemma 6.6.11) and [88, Theorem VI.1.8] yields that S_t is positive. Therefore, by [124, Theorem 2.1.3] the operator $S_t \otimes \mathrm{id}_X$ defined by

$$(S_t \otimes \mathrm{id}_X)(f \otimes x) := S_t f \otimes x, \qquad f \in L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}),\ x \in X,$$

extends to a bounded operator on $L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ with equal operator norm. It is straightforward to verify that $S_t \otimes \mathrm{id}_X$ is generated by Δ_{Dir}^X and that $R(\lambda, \Delta_{\mathrm{Dir}}^X)(f \otimes x) = (R(\lambda, \Delta_{\mathrm{Dir}}^{\mathbb{C}})f) \otimes x$ for $f \in L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$, $x \in X$ and $\lambda \in \rho(\Delta_{\mathrm{Dir}}^{\mathbb{C}}) \cap \rho(\Delta_{\mathrm{Dir}}^X)$. The semigroup $S_t \otimes \mathrm{id}_X$ is also uniformly exponentially stable, which shows that $-\Delta_{\mathrm{Dir}}^X$ is sectorial. Proposition 6.2.2 and Theorem 6.6.2 now give the desired result. □

We have the following result for the Riesz transform associated with the Dirichlet Laplacian.

Corollary 6.6.8 (Riesz transform associated with $-\Delta_{\mathrm{Dir}}$)**.** *Let $p \in (1,\infty)$, $\kappa \in [0,1]$ and let X be a* UMD *Banach space. Assume that either*

(i) $\mathscr{O} = \mathbb{R}^d_+$, $k = 0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ and X is a UMD *Banach space, or,*

(ii) $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain, $k = 0$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and X is a UMD *Banach space, or,*

(iii) $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain, $k \in \mathbb{N}_0$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and $X = \mathbb{C}$.

Let Δ_{Dir} on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ be as in Definition 6.4.1 or 6.6.1. Then

$$\|\nabla(-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}} f\|_{W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)} \leq C\|f\|_{W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)}, \qquad f \in W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X),$$

for some $C > 0$ which only depends on $p, k, \gamma, \mathscr{O}$ and X.

Proof. First, we claim that

$$(-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}} : W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) \to W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) \tag{6.6.1}$$

is bounded. Indeed, since

$$(-\Delta_{\mathrm{Dir}})^{-1} : W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) \to W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$$

is bounded (see Theorems 6.4.2, 6.6.4 and Corollary 6.6.7) and the identity operator is bounded on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$, it holds by Stein interpolation [268, Theorem 2.1] that

$$(-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}} : W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) \to [W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X), W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)]_{\frac{1}{2}}$$

is bounded. To verify the conditions for Stein interpolation, one uses that $-\Delta_{\mathrm{Dir}}$ has BIP, which follows again from the bounded H^∞-calculus in Theorem 6.4.2, Theorem 6.6.4 and Corollary 6.6.7. The claim (6.6.1) now follows from Proposition 6.3.12.

Therefore, (6.6.1), Proposition 6.3.12 and Proposition 6.2.1 (using that $-\Delta_{\mathrm{Dir}}$ has BIP), imply

$$\begin{aligned}
\|\nabla(-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}} f\|_{W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)} &\leq \|(-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}} f\|_{W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)} \\
&\eqsim \|(-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}} f\|_{[W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X), W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)]_{\frac{1}{2}}} \\
&\eqsim \|(-\Delta_{\mathrm{Dir}})^{-\frac{1}{2}} f\|_{D((-\Delta_{\mathrm{Dir}})^{\frac{1}{2}})} \lesssim \|f\|_{W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)}.
\end{aligned}$$

This completes the proof. □

Remark 6.6.9.

(i) Boundedness of the Riesz transforms on $L^p(\mathbb{R}^d, w; X)$ holds if and only if $w \in A_p(\mathbb{R}^d)$, see [105, Sections 7.4.3 & 7.4.4]. Corollary 6.6.8 also allows for weights outside the class of Muckenhoupt weights. On the other hand, we are restricted to power weights since the interpolation results from Proposition 6.3.12 are only available for this type of weights.

(ii) With the same proof as in Corollary 6.6.8 and using Theorems 6.6.2 and 6.6.3 it follows that the Riesz transforms associated with $\lambda - \Delta_{\mathrm{Dir}}$ and $\lambda - \Delta_{\mathrm{Neu}}$ are bounded on weighted vector-valued Sobolev spaces for λ large enough. Following the proof of Corollary 6.6.7, we could also obtain the bounded H^∞-calculus for $-\Delta_{\mathrm{Neu}}$ on $L^p(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma}; X)/\{c\,\mathbf{1}_{\mathscr{O}} : c \in X\}$.

(iii) In view of Remark 6.6.6(i), the condition in Corollary 6.6.8(iii) on the space X can be weakened to X being a Hilbert space or being isomorphic to a closed subspace of an L^p-space.

6.6.2. The proofs of Theorems 6.6.2 and 6.6.3

To transfer the H^∞-calculus on special domains (Section 6.5) to bounded domains, we employ a localisation procedure based on the decomposition of weighted Sobolev spaces as in Lemma 6.3.9. For this localisation of the H^∞-calculus, we need the following abstract lemma, which follows from lower order perturbation results.

Lemma 6.6.10 ([190, Lemma 6.11])**.** *Let A be a linear operator on a Banach space Y and let $\widetilde{A}$ be a sectorial operator on a Banach space $\widetilde{Y}$ with a bounded H^∞-calculus. Assume that there exist bounded linear mappings $\mathscr{I}\colon Y \to \widetilde{Y}$ and $\mathscr{P}\colon \widetilde{Y} \to Y$ satisfying*

(i) $\mathscr{P}\mathscr{I} = \mathrm{id}$,

(ii) $\mathscr{I} D(A) \subseteq D(\widetilde{A})$ *and* $\mathscr{P} D(\widetilde{A}) \subseteq D(A)$,

(iii) $(\mathscr{I} A - \widetilde{A}\mathscr{I})\mathscr{P}\colon D(\widetilde{A}) \to \widetilde{Y}$ *and* $\mathscr{I}(A\mathscr{P} - \mathscr{P}\widetilde{A})\colon D(\widetilde{A}) \to \widetilde{Y}$ *extend to bounded linear operators* $[\widetilde{Y}, D(\widetilde{A})]_\theta \to \widetilde{Y}$ *for some* $\theta \in (0,1)$.

Then A is a closed and densely defined operator and for every $\sigma > \omega_{H^\infty}(\widetilde{A})$ there exists a $\lambda > 0$ such that $\lambda + A$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda + A) \le \sigma$.

We now turn to the proofs of Theorems 6.6.2 and 6.6.3 concerning the H^∞-calculus on bounded domains.

Proof of Theorems 6.6.2 and 6.6.3. We start with the proof for the Dirichlet Laplacian. Let $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p - 1, 2p-1) \setminus \{p-1\}$ and let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Define $A := -\Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$. We show that the operator $\lambda - \Delta_{\mathrm{Dir}}$ has a bounded H^∞-calculus for λ sufficiently large.

If $\kappa = 0$, then take $(V_n)_{n=1}^N, (\mathscr{O}_n)_{n=1}^N, (\eta_n)_{n=0}^N$ from Lemma 6.3.9 such that for all $n \in \{1,\dots,N\}$ we have $[\mathscr{O}_n]_{C^1} < \delta$ where $\delta \in (0,1)$ is small enough such that Theorem 6.5.2 applies for every $\mathscr{O}_n$. If $\kappa \in (0,1]$, then let $\varepsilon \in (0,\kappa)$ be such that $\gamma > (1-(\kappa-\varepsilon))p - 1$. Take $(V_n)_{n=1}^N, (\mathscr{O}_n)_{n=1}^N, (\eta_n)_{n=0}^N$ from Lemma 6.3.9 such that for all $n \in \{1,\dots,N\}$ we have $[\mathscr{O}_n]_{C^{1,\kappa-\varepsilon}} < \delta$ where $\delta \in (0,1)$ is small enough such that Theorem 6.5.2 (applied with κ replaced by $\kappa - \varepsilon$) applies for every $\mathscr{O}_n$. We define the following operators

(i) $\widetilde{A} := \bigoplus_{n=0}^N \widetilde{A}_n$ on $\mathscr{W}^{k,p}_{\gamma+kp}$ as defined in (6.3.12), where

  a) $\widetilde{A}_0$ on $W^{k,p}(\mathbb{R}^d; X)$ with $D(\widetilde{A}_0) := W^{k+2,p}(\mathbb{R}^d; X)$ is given by $\widetilde{A}_0\widetilde{u} := -\Delta\widetilde{u}$,

  b) $\widetilde{A}_n$ on $W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}; X)$ with $D(\widetilde{A}_n) := W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}; X)$ is given by $\widetilde{A}_n\widetilde{u} := -\Delta_{\mathrm{Dir}}\widetilde{u}$ for $n \in \{1,\dots,N\}$,

(ii) $B : D(A) \to \mathscr{W}^{k,p}_{\gamma+kp}$ given by $Bu := ([-\Delta, \eta_n]u)_{n=0}^N$,

(iii) $C : D(\widetilde{A}) \to W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ given by $C\widetilde{u} := \sum_{n=0}^N [-\Delta, \eta_n]\widetilde{u}_n$.

Let $\lambda > 0$. By Lemma 5.2.1, Proposition 6.2.2 and Theorem 6.5.2 it holds that $\lambda - \widetilde{A}_n$ for any $n \in \{0,\dots,N\}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \widetilde{A}_n) = 0$. Thus $\lambda - \widetilde{A}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \widetilde{A}) = 0$ as well.

Let $\mathscr{P}$ and $\mathscr{I}$ be as defined in (6.3.13). It is straightforward to verify that the conditions (i) and (ii) from Lemma 6.6.10 hold. It remains to check condition (iii) in Lemma 6.6.10. From Proposition 6.3.12 we obtain

$$[W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}; X), D(\widetilde{A}_n)]_{\frac{1}{2}} = W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}; X) \quad \text{for } n \in \{1,\dots,N\},$$

and in combination with (see [124, Theorems 5.6.9 & 5.6.11])

$$[W^{k,p}(\mathbb{R}^d; X), D(\widetilde{A}_0)]_{\frac{1}{2}} = W^{k+1,p}(\mathbb{R}^d; X),$$

this yields

$$\begin{aligned}[\mathscr{W}^{k,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}} &= [W^{k,p}(\mathbb{R}^d; X), D(\widetilde{A}_0)]_{\frac{1}{2}} \oplus \bigoplus_{n=1}^{N} [W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}; X), D(\widetilde{A}_n)]_{\frac{1}{2}} \\ &= W^{k+1,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^{N} W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}; X).\end{aligned} \tag{6.6.2}$$

Note that

$$\mathscr{I}Au - \widetilde{A}\mathscr{I}u = -Bu, \quad u \in D(A), \quad \text{and} \quad A\mathscr{P}\widetilde{u} - \mathscr{P}\widetilde{A}\widetilde{u} = C\widetilde{u}, \quad \widetilde{u} \in D(\widetilde{A}),$$

and every commutator $[-\Delta, \eta_n]$ is a first-order partial differential operator with smooth and compactly supported coefficients. This and (6.6.2) yield that

$$\begin{aligned}\mathscr{I}A - \widetilde{A}\mathscr{I} &\colon W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) \to \mathscr{W}^{k,p}_{\gamma+kp} \quad \text{and} \\ \mathscr{P} &\colon [\mathscr{W}^{k,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}} \to W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)\end{aligned}$$

are bounded. Similarly, we obtain by (6.6.2) that

$$\begin{aligned}A\mathscr{P} - \mathscr{P}\widetilde{A} &\colon [\mathscr{W}^{k,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}} \to W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) \quad \text{and} \\ \mathscr{I} &\colon W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X) \to \mathscr{W}^{k,p}_{\gamma+kp}\end{aligned}$$

are bounded. This shows that $(\mathscr{I}A - \widetilde{A}\mathscr{I})\mathscr{P}$ and $\mathscr{I}(A\mathscr{P} - \mathscr{P}\widetilde{A})$ extend to bounded operators from $[\mathscr{W}^{k,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}}$ to $\mathscr{W}^{k,p}_{\gamma+kp}$. Applying Lemma 6.6.10 gives that for all $\sigma \in (0,\pi)$ there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda > \widetilde{\lambda}$ the operator $\lambda - \Delta_{\mathrm{Dir}}$ on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Dir}}) \le \sigma$.

The boundedness of the H^∞-calculus for the Neumann Laplacian on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ can be shown similarly as for the Dirichlet Laplacian using Theorem 6.5.3 and Proposition 6.3.13.

It remains to prove the boundedness of the H^∞-calculus for $\lambda - \Delta_{\mathrm{Neu}}$ on the quotient space $Y/K := W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)/\{c\mathbf{1}_{\mathscr{O}} : c \in X\}$. Fix $\sigma \in (0,\pi)$ and let λ be large enough such that $\lambda - \Delta_{\mathrm{Neu}}$ on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) \le \sigma$. Let $\omega \in (\sigma,\pi)$ and let $\varphi \in H^1(\Sigma_\omega) \cap H^\infty(\Sigma_\omega)$. For any $c \in K$ we have that $x \in Y/K$ can be represented as $x = y + c$ with $y \in Y$. Note that for $z \in \rho(\lambda - \Delta_{\mathrm{Neu}})$ the equation

$$zu - (\lambda - \Delta_{\mathrm{Neu}})u = c$$

has the unique solution $u = c/(z-\lambda)$. Therefore, by definition of the functional calculus and Cauchy's integral formula, we obtain

$$\begin{aligned}\varphi(\lambda - \Delta_{\mathrm{Neu}})c &= \frac{1}{2\pi \mathrm{i}} \int_{\partial\Sigma_\nu} \varphi(z) R(z, \lambda - \Delta_{\mathrm{Neu}}) c \, \mathrm{d}z \\ &= \frac{1}{2\pi \mathrm{i}} \int_{\partial\Sigma_\nu} \frac{\varphi(z)c}{z-\lambda} \, \mathrm{d}z = \varphi(\lambda)c \in K, \quad \nu \in (\sigma, \omega).\end{aligned} \tag{6.6.3}$$

By (6.6.3) and the bounded H^∞-calculus for $\lambda - \Delta_{\mathrm{Neu}}$ on Y, it follows that for $x \in Y/K$ and $c \in K$ we have

$$\begin{aligned}\|(\varphi(\lambda - \Delta_{\mathrm{Neu}})x) - \varphi(\lambda)c\|_Y &= \|(\varphi(\lambda - \Delta_{\mathrm{Neu}})(y+c)) - \varphi(\lambda)c\|_Y = \|\varphi(\lambda - \Delta_{\mathrm{Neu}})y\|_Y \\ &\lesssim \|\varphi\|_{H^\infty(\Sigma_\omega)} \|y\|_Y = \|\varphi\|_{H^\infty(\Sigma_\omega)} \|x - c\|_Y.\end{aligned}$$

Taking the infimum over $c \in K$ yields that $\|\varphi(\lambda - \Delta_{\mathrm{Neu}})x\|_{Y/K} \lesssim \|\varphi\|_{H^\infty(\Sigma_\omega)} \|x\|_{Y/K}$ for $x \in Y/K$, which proves the boundedness of the H^∞-calculus on Y/K with angle $\omega_{H^\infty}(\lambda - \Delta_{\mathrm{Neu}}) \le \sigma$. □

6.6.3. The proofs of Theorems 6.6.4 and 6.6.5

We continue with the proof of Theorems 6.6.4 and 6.6.5, which deal with the H^∞-calculus in the special case of $X = \mathbb{C}$. We start with some preliminary results about the consistency of resolvents.

Let X_0 and X_1 be two compatible Banach spaces and suppose that $B_0 \in \mathscr{L}(X_0)$ and $B_1 \in \mathscr{L}(X_1)$. Then we call the operators B_0 and B_1 *consistent* if

$$B_0 u = B_1 u \qquad \text{for all } u \in X_0 \cap X_1.$$

For $z \in \Sigma \subseteq \mathbb{C}$ the two families of operators $B_0(z) \in \mathscr{L}(X_0)$ and $B_1(z) \in \mathscr{L}(X_1)$ are called consistent if $B_0(z)$ and $B_1(z)$ are consistent for all $z \in \Sigma$.

We introduce the forms on the Hilbert spaces V (as dense subspace of $L^2(\mathscr{O})$) given by

$$\begin{aligned}a_{\mathrm{Dir}}(v_1, v_2) &:= \int_{\mathscr{O}} \nabla v_1 \cdot \overline{\nabla v_2} \, \mathrm{d}x, \qquad v_1, v_2 \in V = W_0^{1,2}(\mathscr{O}), \\ a_{\mathrm{Neu}}(v_1, v_2) &:= \int_{\mathscr{O}} \nabla v_1 \cdot \overline{\nabla v_2} \, \mathrm{d}x, \qquad v_1, v_2 \in V = W^{1,2}(\mathscr{O}).\end{aligned}$$

Associated with the forms a_{Dir} and a_{Neu} are the densely defined closed Laplace operators $-A_{\mathrm{Dir},2}$ and $-A_{\mathrm{Neu},2}$ on $L^2(\mathscr{O})$, respectively, see for instance [223, Chapter 12]. The domains of these operators are

$$\begin{aligned}D(A_{\mathrm{Dir},2}) &= \{f \in W_0^{1,2}(\mathscr{O}) \cap W^{2,2}_{\mathrm{loc}}(\mathscr{O}) : \Delta f \in L^2(\mathscr{O})\}, \\ D(A_{\mathrm{Neu},2}) &= \{f \in W^{1,2}(\mathscr{O}) \cap W^{2,2}_{\mathrm{loc}}(\mathscr{O}) : \Delta f \in L^2(\mathscr{O})\},\end{aligned}$$

see [223, Sections 12.3.b & 12.3.c]. A characterisation of the domains as a closed subspace of $W^{2,2}(\mathscr{O})$ requires more regularity of the domain (compared to the regularity we consider in Theorems 6.6.4 and 6.6.5), see [223, Sections 12.3.b & 12.3.c]. For instance, for the

Dirichlet Laplacian, C^2-regularity is required.

We have the following lemma on the consistency of the resolvents for the Dirichlet Laplacian.

Lemma 6.6.11. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Let*

$$A_{p,k,\gamma} := \Delta_{\mathrm{Dir}} \quad \text{on } W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \text{ with } D(A_{p,k,\gamma}) = W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$$

be as in Definition 6.6.1 and let

$$A_{\mathrm{Dir},2} = \Delta_{\mathrm{Dir}} \quad \text{on } L^2(\mathscr{O}) \text{ with } D(A_{\mathrm{Dir},2}) = \{f \in W^{1,2}_0(\mathscr{O}) \cap W^{2,2}_{\mathrm{loc}}(\mathscr{O}) : \Delta f \in L^2(\mathscr{O})\}$$

be as above. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda > \widetilde{\lambda}$ the resolvents $R(\lambda, A_{p,k,\gamma})$ and $R(\lambda, A_{\mathrm{Dir},2})$ are consistent.

Proof. Take $1 < q < \min\{p,2\}$ and $\mu \in (0, 2q-1) \setminus \{q-1\}$ such that

$$\mu > \frac{q(\gamma+1)}{p} - 1 > (1-\kappa)q - 1. \tag{6.6.4}$$

First, we claim that $L^p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma) \hookrightarrow L^q(\mathscr{O}, w^{\partial\mathscr{O}}_\mu)$. Indeed, for $u \in L^p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma)$ we have by Hölder's inequality that

$$\int_{\mathscr{O}} |u(x)|^q w^{\partial\mathscr{O}}_\mu(x)\,\mathrm{d}x \le \Big(\int_{\mathscr{O}} |u(x)|^p w^{\partial\mathscr{O}}_\gamma(x)\,\mathrm{d}x\Big)^{\frac{q}{p}} \Big(\int_{\mathscr{O}} w^{\partial\mathscr{O}}_{\frac{\mu p - q\gamma}{p-q}}(x)\,\mathrm{d}x\Big)^{\frac{p-q}{p}} < \infty.$$

The latter integral can be written as an integral over $\mathbb{R}^d_+$ (using localisation from Lemma 6.3.9 and the diffeomorphism from Lemma 2.6.5), hence the integral is finite since (6.6.4) implies $(\mu p - q\gamma)/(p-q) > -1$. This proves the claim.

To continue, we introduce the space

$$Z_{r,\nu} := \{f \in W^{1,r}_0(\mathscr{O}, w^{\partial\mathscr{O}}_\nu) \cap W^{2,r}_{\mathrm{loc}}(\mathscr{O}) : \Delta f \in L^r(\mathscr{O}, w^{\partial\mathscr{O}}_\nu)\} \quad \text{for } r \in (1,2],\ \nu > -1,$$

and note that $D(A_2) = Z_{2,0}$. Now, consider the equation

$$\lambda u - \Delta_{\mathrm{Dir}} u = f, \qquad f \in W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \cap L^2(\mathscr{O}). \tag{6.6.5}$$

By Theorem 6.6.2 (using that $\gamma > (1-\kappa)p - 1$) and [223, Section 12.3.b] there exist unique

$$u_0 \in W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \quad \text{and} \quad u_1 \in Z_{2,0}$$

solving (6.6.5) for λ sufficiently large. By Hardy's inequality (for bounded Lipschitz domains, see for instance [169, Section 8.8]) and the claim, we have

$$W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \hookrightarrow W^{2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma) \hookrightarrow W^{2,q}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_\mu).$$

Moreover, using $\mu > 0$, $q < 2$ and elliptic regularity (Theorem 6.6.2 using (6.6.4)), we have

$$Z_{2,0} \hookrightarrow Z_{q,\mu} = W^{2,q}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_\mu).$$

Note that the equation (6.6.5) with right-hand side $f \in L^q(\mathscr{O}, w^{\partial\mathscr{O}}_\mu)$ has a unique solution in $W^{2,q}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_\mu)$ by Theorem 6.6.2 (using (6.6.4)). It follows that $u_0 = u_1$, which proves that the resolvents of $A_{p,k,\gamma}$ and A_2 are consistent. □

For the Neumann Laplacian, we have the following result concerning the consistency of resolvents. Its proof is similar to the proof of Lemma 6.6.11.

Lemma 6.6.12. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p-1, p-1)$, $j \in \{0,1\}$ and let $\mathscr{O}$ be a bounded $C^{j+1,\kappa}$-domain. Let*

$$A_{p,k,j,\gamma} := \Delta_{\mathrm{Neu}} \quad \text{on } W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \text{ with } D(A_{p,k,j,\gamma}) = W^{k+j+2,p}_{\mathrm{Neu}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$$

be as in Definition 6.6.1(ii) and let

$$A_{\mathrm{Neu},2} = \Delta_{\mathrm{Neu}} \quad \text{on } L^2(\mathscr{O}) \text{ with } D(A_{\mathrm{Neu},2}) = \{f \in W^{1,2}(\mathscr{O}) \cap W^{2,2}_{\mathrm{loc}}(\mathscr{O}) : \Delta f \in L^2(\mathscr{O})\}$$

be as above. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda > \widetilde{\lambda}$ the resolvents $R(\lambda, A_{p,k,j,\gamma})$ and $R(\lambda, A_{\mathrm{Neu},2})$ are consistent.

We can now turn to the H^∞-calculus on scalar-valued spaces.

Proof of Theorems 6.6.4 and 6.6.5. We start with the proof of Theorem 6.6.4(i). Since the embedding $W^{1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \hookrightarrow L^p(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ is compact, see [116, Theorem 8.8], we have

$$D(\Delta_{\mathrm{Dir}}) = W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \hookrightarrow W^{k+1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \overset{\text{compact}}{\hookrightarrow} W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}).$$

Since $(\lambda - \Delta_{\mathrm{Dir}})^{-1}$ with $\lambda \in \rho(\Delta_{\mathrm{Dir}})$ exists (by Theorem 6.6.2), the compact embedding above implies that $(\lambda - \Delta_{\mathrm{Dir}})^{-1}$ is compact. Thus by the Riesz–Schauder theorem for compact operators, the resolvent operator $(\lambda - \Delta_{\mathrm{Dir}})^{-1}$ has a discrete countable spectrum $\{\sigma_j : j \in \mathbb{N}_0\}$, where $\sigma_j \neq 0$ are eigenvalues of $(\lambda - \Delta_{\mathrm{Dir}})^{-1}$. Moreover, zero is in the spectrum of $(\lambda - \Delta_{\mathrm{Dir}})^{-1}$ and is the only accumulation point of the spectrum. Therefore, by the spectral mapping theorem

$$\sigma(-\Delta_{\mathrm{Dir}}) = \{\lambda_j : \lambda_j = \sigma_j^{-1} - \lambda, j \in \mathbb{N}_0 \text{ with } \sigma_j \neq 0\}.$$

Next, we claim that the spectrum $\sigma(-\Delta_{\mathrm{Dir}})$ is independent of $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$. Let $A_{p,k,\gamma}$ and A_2 be as in Lemma 6.6.11. It suffices to show that $\sigma(-A_{p,k,\gamma}) = \sigma(-A_2)$. We proceed as in the proof of [19, Proposition 2.6]. Recall that $\sigma(-A_2)$ is discrete and only consists of a countable number of positive eigenvalues, see [223, Theorem 12.26]. By Lemma 6.6.11 and analytic continuation we find that $R(z,-A_2)$ and $R(z,-A_{p,k,\gamma})$ are consistent for all $z \in \rho(-A_2) \cap \rho(-A_{p,k,\gamma})$. Now, if $\lambda \in \rho(-A_2)$, then since $\sigma(-A_{p,k,\gamma})$ is discrete and countable it follows that there exists an $r > 0$ such that $\overline{B(\lambda,r)} \setminus \{\lambda\} \subseteq \rho(-A_2) \cap \rho(-A_{p,k,\gamma})$. Therefore, by consistency of the resolvents we obtain

$$\int_{\partial B(\lambda,r)} R(z,-A_{p,k,\gamma})\,\mathrm{d}z = \int_{\partial B(\lambda,r)} R(z,-A_2)\,\mathrm{d}z = 0,$$

and thus $\lambda \in \rho(-A_{p,k,\gamma})$. The other inclusion follows similarly. This proves that $\sigma(-A_{p,k,\gamma}) = \sigma(-A_2)$ and the claim follows.

Finally, using that $\sigma(-A_2)$ is discrete, $\sigma(-A_2) \subseteq [\widetilde{\lambda},\infty) \subseteq (0,\infty)$ with $\widetilde{\lambda} := \min\{\lambda_j : j \in \mathbb{N}_0\} > 0$ and the claim gives that $\sigma(-A_{p,k,\gamma})$ is discrete and $\sigma(-A_{p,k,\gamma}) \subseteq [\widetilde{\lambda},\infty) \subseteq (0,\infty)$. This completes the proof of Theorem 6.6.4(i).

We continue with the proof of Theorem 6.6.4(ii). From Theorem 6.6.2 we have that for fixed $\sigma \in (0,\pi)$ and λ sufficiently large, $\lambda - \Delta_{\mathrm{Dir}}$ is sectorial with $\omega(\lambda - \Delta_{\mathrm{Dir}}) \leq \sigma$. Combining this with the analyticity of $z \mapsto (z - \Delta_{\mathrm{Dir}})^{-1}$ on $\mathbb{C} \setminus (-\infty, -\widetilde{\lambda}]$ yields that for $\lambda > -\widetilde{\lambda}$ and $\sigma' > \sigma$ the operator $\lambda - \Delta_{\mathrm{Dir}}$ is sectorial with $\omega(\lambda - \Delta_{\mathrm{Dir}}) \leq \sigma'$. Therefore, Theorem 6.6.4(ii) follows from Proposition 6.2.2, Theorem 6.6.2 and the fact that $\sigma \in (0,\pi)$ is arbitrary.

The proof of Theorem 6.6.5 for the Neumann Laplacian is similar to the proof for the Dirichlet Laplacian above if we use Theorem 6.6.3 and Lemma 6.6.12. Note that for the Neumann Laplacian on $L^2(\mathscr{O})$, zero is an eigenvalue and the corresponding eigenspace consists of constant functions, see [223, Proposition 12.24 & Theorem 12.26]. Therefore, we obtain the bounded H^∞-calculus for $\lambda - \Delta_{\mathrm{Neu}}$ with $\lambda > 0$ on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$. In addition, on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})/\{c\,\mathbf{1}_{\mathscr{O}} : c \in X\}$ the eigenvalue zero is removed and we obtain the bounded H^∞-calculus for $\lambda - \Delta_{\mathrm{Neu}}$ with $\lambda > -\widetilde{\lambda}$, where $\widetilde{\lambda}$ is the first positive eigenvalue of $-\Delta_{\mathrm{Neu}}$. □

III

Applications to partial differential equations

7

The heat equation on rough domains

This chapter is based on the applications to the heat equation presented in:

N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on the half-space. *J. Funct. Anal.*, 289(8):110985, 2025.

N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on rough domains. *J. Differ. Equ.*, 454:113884, 2026.

R. Denk and F. Roodenburg. Trace theory for parabolic boundary value problems with rough boundary conditions. *J. Evol. Equ.*, 26(3):80, 2026.

In addition, this chapter is complemented with some unpublished results.

Abstract. *We prove well-posedness and higher-order regularity for some prototype linear heat equations in the setting of weighted Sobolev spaces. In particular, we apply the functional calculus for the Laplacian to obtain maximal regularity for the deterministic heat equation on the half-space and on rough domains. In addition, using trace theory for weighted Sobolev spaces, we can also allow for rough inhomogeneous boundary data. Finally, we study a linear stochastic heat equation on rough domains and derive stochastic maximal regularity for this equation without artificial compatibility conditions for the stochastic forcing.*

7.1. Introduction

Using the function space theory and functional calculus developed in the foregoing chapters, we examine well-posedness of some prototype linear heat equations. In particular, we consider

(i) the (deterministic) heat equation with inhomogeneous boundary conditions on rough domains,

(ii) the stochastic heat equation on rough domains without compatibility conditions for the stochastic forcing.

The deterministic heat equation

Let $\mathscr{O} \subset \mathbb{R}^d$ and consider the heat equation with Dirichlet boundary conditions

$$\begin{cases} \partial_t u - \Delta u = f & t > 0, x \in \mathscr{O}, \\ u|_{\partial\mathscr{O}} = g & t > 0, \\ u|_{t=0} = 0 & x \in \mathscr{O}. \end{cases} \tag{7.1.1}$$

We study this equation in the setting of weighted Sobolev spaces allowing for blow-up of the solution near the boundary. In the case of homogeneous boundary conditions $g = 0$, functional analytic properties of the Laplacian on $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ are studied in [190] and Chapters 5 and 6. In addition, we can apply the trace theory from Chapter 4 to deal with inhomogeneous boundary conditions. Our main result for (7.1.1) with inhomogeneous boundary conditions reads as follows. This is a special case of Theorem 7.4.4, where for simplicity we omit the temporal weights.

Theorem 7.1.1. *Let $p, q \in (1,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1)$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ with $1 - \frac{1}{q} \neq \frac{\gamma+1}{2p}$. Furthermore, let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Then for all*

$$f \in L^q(\mathbb{R}_+; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})) \quad \text{and} \quad g \in F^{1-\frac{\gamma+1}{2p}}_{q,p}(\mathbb{R}_+; L^p(\partial\mathscr{O})) \cap L^q(\mathbb{R}_+; B^{2-\frac{\gamma+1}{p}}_{p,p}(\partial\mathscr{O})),$$

such that $g|_{t=0} = 0$ if $1 - \frac{1}{q} > \frac{\gamma+1}{2p}$, there exists a unique solution

$$u \in W^{1,q}(\mathbb{R}_+; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})) \cap L^q(\mathbb{R}_+; W^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$$

to (7.1.1). *Moreover, this solution u depends continuously on the data f and g.*

Theorem 7.1.1 is proved by combining maximal regularity for (7.1.1) with homogeneous boundary conditions with the trace result from Theorem 4.1.1. It generalises and unifies earlier results in [190, Section 7]. The main novelties of Theorem 7.1.1 are:

(i) *Rough domains*: larger values of γ allow for less regular domains. Furthermore, the required boundary regularity of $\mathscr{O}$ is independent of k, unlike the unweighted theory, where typically C^{k+2}-domains are necessary for localisation arguments (see, e.g., [67, 89, 166]). In particular, we note that in the unweighted setting $\gamma = 0$, Theorem 7.1.1 allows for a bounded $C^{1,\kappa}$-domain with $\kappa > 1 - \frac{1}{p}$. Even in the case $k = 0$, this improves the standard C^2-condition on the domain which is usually imposed in the existing literature.

(ii) *Rough boundary data*: the boundary regularity required of g is likewise independent of k. For large γ, one can treat a wider class of inhomogeneous boundary data than in the unweighted case, cf. (4.1.3).

Finally, using Theorem 7.1.1, we expect that well-posedness for the heat equation (7.1.1) with nonzero initial data can be obtained as well, see Remark 7.4.5.

The stochastic heat equation

One of the main motivations for studying the H^∞-functional calculus is that it implies (stochastic) maximal regularity for a stochastic linear evolution equation. Using the H^∞-functional calculus for the Dirichlet Laplacian, we can study the linear stochastic heat equation given by

$$\begin{aligned} \mathrm{d}u - \Delta_{\mathrm{Dir}} u \,\mathrm{d}t &= f \,\mathrm{d}t + g \,\mathrm{d}W(t), \qquad t \in [0,T], \\ u(0) &= 0, \end{aligned} \tag{7.1.2}$$

where W is a Brownian motion on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$.

A simplified version of the stochastic maximal regularity result reads as follows.

Theorem 7.1.2. *Let $p, q \in (2,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and $T \in (0,\infty)$. Moreover, assume that $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain. Let $-\Delta_{\mathrm{Dir}}$ be the Dirichlet Laplacian on $X_0 := W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ with domain $D(A) := W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$. Then for all progressively measurable*

$$f \in L^q(\Omega \times (0,T); X_0) \quad \text{and} \quad g \in L^q(\Omega \times (0,T); [X_0, D(A)]_{\frac{1}{2}})$$

there exists a unique strong solution u to (7.1.2) with $u|_{\partial\mathscr{O}} = 0$ and $u|_{t=0} = 0$ which satisfies

$$\|u\|_{L^q(\Omega\times(0,T);D(A))} \lesssim \|f\|_{L^q(\Omega\times(0,T);X_0)} + \|g\|_{L^q(\Omega\times(0,T);[X_0,D(A)]_{\frac{1}{2}})}.$$

By the trace theory developed in Chapter 3, it holds that

$$[X_0, D(A)]_{\frac{1}{2}} = \begin{cases} W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) & \text{if } \gamma \in (-1, p-1), \\ W^{k+1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) & \text{if } \gamma \in (p-1, 2p-1). \end{cases}$$

This means that for $\gamma \in (p-1, 2p-1)$ no additional compatibility conditions for the stochastic forcing g are needed.

Outline

The outline of this chapter is as follows. In Section 7.2 we introduce the notion of maximal regularity. In Section 7.3 we use the H^∞-calculus for the Laplacian from Chapter 5 to study the heat equation on the half-space. In addition, we derive some results on homogeneous weighted Sobolev spaces. In Section 7.4 we use the H^∞-calculus for the Laplacian from Chapter 6 to study the heat equation on rough domains. Finally, in Section 7.5 we consider stochastic maximal regularity for the stochastic heat equation.

7.2. Maximal regularity

We introduce the notion of (deterministic) maximal regularity for Cauchy problems. For a more elaborate study of this topic, we refer to, e.g., [67, 126, 174].

Let $T \in (0,\infty]$ and define the time interval $I := (0,T)$. Let A be a linear operator with domain $D(A)$ on a Banach space X. For $f \in L^1_{\mathrm{loc}}(\bar{I};X)$ consider the abstract Cauchy problem

$$\begin{aligned} \partial_t u(t) + Au(t) &= f(t), \qquad t \in I, \\ u(0) &= 0. \end{aligned} \tag{7.2.1}$$

We call a strongly measurable function $u : I \to X$ a *strong solution to* (7.2.1) if u takes values in $D(A)$ almost everywhere, $Au \in L^1_{\mathrm{loc}}(\bar{I};X)$ and u solves

$$u(t) + \int_0^t Au(s)\,\mathrm{d}s = \int_0^t f(s)\,\mathrm{d}s, \quad \text{for almost all } t \in I. \tag{7.2.2}$$

Moreover, for $q \in (1,\infty)$, $v \in A_q(I)$ and $f \in L^q(I,v;X)$ a strong solution u to (7.2.1) is called an $L^q(v)$*-solution* if $Au \in L^q(I,v;X)$. Here, v is a time weight that belongs to the class of *Muckenhoupt weights* $A_q(I)$, see Section 2.2.

Definition 7.2.1 (Maximal $L^q(v)$-regularity)**.** A linear operator A on a Banach space X has *maximal* $L^q(v)$*-regularity on* I if for all $f \in L^q(I,v;X)$ the Cauchy problem (7.2.1) has a unique $L^q(v)$-solution u on I and

$$\|Au\|_{L^q(I,v;X)} \leq C\|f\|_{L^q(I,v;X)},$$

where the constant C is independent of f.

In the following situations we can also obtain an estimate for the solution u instead of only Au.

Proposition 7.2.2 ([126, Corollary 17.2.9])**.** *Let A be an operator on a Banach space X that has maximal $L^q(v)$-regularity on I. Suppose that at least one of the following conditions holds.*

(i) $I = (0,T)$ *for some* $T \in (0,\infty)$,

(ii) $I = (0,T)$ *for some* $T \in (0,\infty]$ *and* $0 \in \rho(A)$.

Then for all $f \in L^q(I,v;X)$, the unique $L^q(v)$-solution u to (7.2.1) *satisfies*

$$\|u\|_{W^{1,q}(I,v;X)} + \|u\|_{L^q(I,v;D(A))} \leq C\|f\|_{L^q(I,v;X)},$$

where the constant C is independent of f. In the case that $0 \in \rho(A)$ the constant C is also independent of T.

7.2.1. Sufficient conditions for maximal regularity

Let X and Y be Banach spaces and let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space. Then $\varepsilon : \Omega \to \mathbb{C}$ is called a *Rademacher variable* if the random variable ε is uniformly distributed over $\{z \in \mathbb{C} : |z| = 1\}$. A *Rademacher sequence* $(\varepsilon_n)_{n\geq 1}$ is a sequence that consists of independent Rademacher variables ε_n. For $N \in \mathbb{N}_1$ we define the *random sequence space* $\ell^2_N(X)$ as the Banach space of finite sequences $(x_n)_{n=1}^N$ in X, endowed with the norm

$$\|(x_n)_{n=1}^N\|_{\ell^2_N(X)} := \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)}.$$

A family of bounded operators $\mathscr{T} \subseteq \mathscr{L}(X,Y)$ is called *R-bounded* if there exists a constant $C > 0$ such that for all $N \in \mathbb{N}_1$, $(T_n)_{n=1}^N \subseteq \mathscr{T}$ and $(x_n)_{n=1}^N \subseteq X$, we have

$$\Big\| \sum_{n=1}^N \varepsilon_n T_n x_n \Big\|_{L^2(\Omega;Y)} \leq C \Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_{L^2(\Omega;X)}.$$

In particular, we note that R-boundedness implies uniform boundedness, while the converse only holds under more geometric restrictions on the Banach spaces X and Y (for instance if X and Y are Hilbert spaces). Moreover, any singleton operator $\{T\} \subseteq \mathscr{L}(X,Y)$ is R-bounded.

Recall from Section 2.3 that for a sectorial operator A the set $\{zR(z,A) : z \in \mathbb{C} \setminus \overline{\Sigma}_\omega\}$ should be bounded for some $\omega \in (0,\pi)$. We now introduce the stronger notion of R-sectoriality.

Definition 7.2.3. A sectorial operator $(A, D(A))$ is called *R-sectorial* if for some $\sigma \in (\omega(A), \pi)$ the family $\{zR(z,A) : z \in \mathbb{C} \setminus \overline{\Sigma}_\sigma\}$ is R-bounded. The angle of R-sectoriality $\omega_R(A)$ is defined as the infimum over all possible σ.

The following theorem states that R-sectoriality on UMD Banach spaces is a sufficient condition for maximal regularity.

Theorem 7.2.4 ([126, Theorems 17.3.18 & 17.2.39])**.** *Let A be a linear operator on a* UMD *Banach space X. Suppose that at least one of the following conditions holds.*

(i) $I = (0,T)$ for some $T \in (0,\infty)$ and there exists a $\lambda \in \mathbb{R}$ such that $\lambda + A$ is R-sectorial with angle $\omega_R(\lambda + A) < \frac{\pi}{2}$,

(ii) $I = \mathbb{R}_+$ and A is R-sectorial with angle $\omega_R(A) < \frac{\pi}{2}$.

Then for all $q \in (1,\infty)$ and $v \in A_q(I)$ the operator A has maximal $L^q(v)$-regularity.

In fact, R-sectoriality provides even a necessary and sufficient condition for maximal L^q-regularity on $\mathbb{R}_+$, see [126, Theorem 17.3.1]. Moreover, a bounded H^∞-calculus implies R-sectoriality.

Proposition 7.2.5 ([125, Corollary 10.4.10])**.** *Let A be a sectorial operator on a* UMD *Banach space X. Additionally, suppose that A has a bounded H^∞-calculus. Then A is R-sectorial and $\omega_{H^\infty}(A) = \omega_R(A)$.*

The above proposition also holds under a weaker condition on the Banach space X, which is called the *triangular contraction property*. For more details, we refer to [125, Section 7.5].

The following sufficient condition for maximal regularity in terms of R-sectoriality holds: if X is a UMD Banach space and $\lambda + A$ is R-sectorial with angle $\omega_R(A) < \frac{\pi}{2}$ for some λ large enough, then A has maximal $L^q(v)$-regularity on $(0,T)$ for any $T < \infty$, $q \in (1,\infty)$ and $v \in A_q(I)$. If $\lambda = 0$, then we obtain maximal $L^q(v)$-regularity on $\mathbb{R}_+$ as well. This statement follows from combining the results in [126, Theorems 17.3.1(2), 17.2.26(1) & 17.2.39].

7.3. The heat equation on the half-space

We study elliptic maximal regularity of the Poisson equation and parabolic maximal regularity of the heat equation on the half-space. To this end, we apply the functional calculus results from Chapter 5, where we studied the Laplace operator on inhomogeneous Sobolev spaces. On the other hand, there is extensive literature on maximal regularity for elliptic and parabolic problems on homogeneous weighted Sobolev spaces, which is obtained via completely different techniques. We refer to [164, 165] for Dirichlet boundary conditions and [80, 81] for Neumann boundary conditions.

Using our regularity results on inhomogeneous spaces and a scaling argument, we derive maximal regularity on homogeneous weighted Sobolev spaces as well. Moreover, in certain cases our regularity results correspond to results earlier obtained in [80, 81, 164, 165].

This section is organised as follows. After introducing the homogeneous weighted Sobolev spaces, we will first discuss elliptic regularity for the Poisson equation on these spaces. Subsequently, we derive parabolic maximal regularity for the heat equation on both inhomogeneous and homogeneous spaces.

7.3.1. Homogeneous weighted Sobolev spaces

For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ and X a Banach space, we define the homogeneous space

$$\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X) := \Big\{ f \in \mathscr{D}'(\mathbb{R}^d_+; X) : \forall |\alpha| \leq k, \partial^\alpha f \in L^p(\mathbb{R}^d_+, w_{\gamma+|\alpha|p}; X) \Big\}$$

equipped with the canonical norm. In comparison to the existing literature, we have for $\gamma > -1$ by Hardy's inequality (Corollary 5.3.2)

$$W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \hookrightarrow \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X) = H^k_{p,\gamma+d}(\mathbb{R}^d_+; X),$$

where for $\theta \in \mathbb{R}$ the homogeneous spaces

$$H^k_{p,\theta}(\mathbb{R}^d_+; X) = \Big\{ f \in \mathscr{D}'(\mathbb{R}^d_+; X) : \forall |\alpha| \leq k, \partial^\alpha f \in L^p(\mathbb{R}^d_+, w_{\theta-d+|\alpha|p}; X) \Big\},$$

are as used in, for instance, [164, 165] with $X = \mathbb{C}$. For these homogeneous spaces, we recall the following density result.

Lemma 7.3.1. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ and let X be a Banach space. Then $C_c^\infty(\mathbb{R}^d_+; X)$ is dense in $\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$.*

Proof. This follows from [164, Theorem 1.19 & Corollary 2.3], where it should be noted that these results also hold in the vector-valued case. □

For $\gamma \in (-1, 2p-1) \setminus \{p-1\}$, we define the space with Dirichlet boundary conditions

$$\begin{aligned}&\dot{\mathbb{W}}^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\\ &\qquad := \Big\{ f \in \mathscr{D}'(\mathbb{R}^d_+; X) : \forall |\beta| \le 2, \partial^\beta f \in \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), \mathrm{Tr}(f) = 0 \Big\}\end{aligned}$$

equipped with the canonical norm. Similarly, for $\gamma \in (-1, p-1)$, we define the space with Neumann boundary conditions

$$\begin{aligned}&\dot{\mathbb{W}}^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)\\ &\qquad := \Big\{ f \in \mathscr{D}'(\mathbb{R}^d_+; X) : \forall |\beta| \le 2, \partial^\beta f \in \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), \mathrm{Tr}(\partial_1 f) = 0 \Big\}\end{aligned}$$

equipped with the canonical norm. Note that for the given ranges of γ we have that

$$\dot{\mathbb{W}}^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X), \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X) \hookrightarrow W^{2,p}(\mathbb{R}^d_+, w_\gamma; X), \tag{7.3.1}$$

so that the Dirichlet and Neumann traces are well defined. For Neumann boundary conditions, we will not consider the homogeneous counterpart of $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ with $\gamma \in (-1, p-1)$ to avoid weights w_γ with $\gamma < -1$.

Definition 7.3.2. Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$ and let X be a Banach space. The Dirichlet and Neumann Laplacian on homogeneous spaces are defined as follows.

(i) For $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ the *Dirichlet Laplacian* Δ_{Dir} *on* $\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is defined by

$$\Delta_{\mathrm{Dir}} u := \sum_{j=1}^d \partial_j^2 u \quad \text{with} \quad D(\Delta_{\mathrm{Dir}}) := \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

(ii) For $\gamma \in (-1, p-1)$ the *Neumann Laplacian* Δ_{Neu} *on* $\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is defined by

$$\Delta_{\mathrm{Neu}} u := \sum_{j=1}^d \partial_j^2 u \quad \text{with} \quad D(\Delta_{\mathrm{Neu}}) := \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X).$$

7.3.2. Elliptic maximal regularity for the Poisson equation

We recall that elliptic regularity on the inhomogeneous spaces $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ has already been studied in Sections 5.5.1, 5.5.2 and 5.6.2. By Proposition 5.5.4 and a scaling argument, we obtain the following regularity result for the Dirichlet Laplacian.

Corollary 7.3.3 (Homogeneous elliptic regularity for $-\Delta_{\mathrm{Dir}}$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$, $\omega \in (0,\pi)$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 7.3.2. Then for all $f \in \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and $\lambda \in \Sigma_{\pi-\omega}$, there exists a unique solution*

$$u \in \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$$

such that $\lambda u - \Delta_{\mathrm{Dir}} u = f$. Moreover, this solution satisfies

$$\sum_{|\beta|\le 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \le C \|f\|_{\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where the constant C only depends on p, k, γ, ω, d and X.

Proof. Note that the uniqueness follows from (7.3.1) and Corollary 5.4.7. For existence and the estimate we use a scaling argument. Let $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$, then Lemma 5.5.3 yields a smooth solution u to the equation. Let $r > 0$ and set $u_r(x) := u(rx)$ and $f_r(x) := f(rx)$. Then u_r satisfies $u_r(0,\cdot) = 0$ and the equation $r^2\lambda u_r - \Delta_{\mathrm{Dir}} u_r = r^2 f_r$, so by Proposition 5.5.4 we have for all $\ell \in \{0,\dots,k\}$ the estimate

$$\sum_{|\beta|\le 2} |r^2\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u_r\|_{W^{\ell,p}(\mathbb{R}^d_+, w_{\gamma+\ell p}; X)} \le C g_{\ell,\gamma}(r^2\lambda) \|r^2 f_r\|_{W^{\ell,p}(\mathbb{R}^d_+, w_{\gamma+\ell p}; X)},$$

where $g_{\ell,\gamma}$ is defined in (5.5.3). After the substitution $x \mapsto r^{-1}x$ we obtain

$$\begin{aligned}\sum_{|\beta|\le 2} \sum_{|\alpha|\le \ell} |\lambda|^{1-\frac{|\beta|}{2}} r^{|\alpha|-\ell} \|\partial^{\alpha+\beta} u\|_{L^p(\mathbb{R}^d_+, w_{\gamma+\ell p}; X)} \\ \le C g_{\ell,\gamma}(r^2\lambda) \sum_{|\alpha|\le \ell} r^{|\alpha|-\ell} \|\partial^\alpha f\|_{L^p(\mathbb{R}^d_+, w_{\gamma+\ell p}; X)}.\end{aligned}$$

Letting $r \to \infty$ and summing over $\ell \in \{0,\dots,k\}$ yields the estimate for $f \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+; X)$. Here it should be noted that

$$\lim_{r\to\infty} g_{\ell,\gamma}(r^2\lambda) = \lim_{y\to\infty} g_{\ell,\gamma}(y) = C,$$

where $C = 1$ or $C = 2$ depending on ℓ and γ. By Lemma 7.3.1 and a similar density argument as in the proof of Proposition 5.5.4, we obtain that for every $f \in \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ there exists an $u \in \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ solving the equation and the required estimate holds. □

In a similar fashion, we obtain the following elliptic regularity result for the Neumann Laplacian.

Corollary 7.3.4 (Homogeneous elliptic regularity for $-\Delta_{\mathrm{Neu}}$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, p-1)$, $\omega \in (0,\pi)$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 7.3.2. Then for all $f \in \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ and $\lambda \in \Sigma_{\pi-\omega}$, there exists a unique solution*

$$u \in \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Neu}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$$

to $\lambda u - \Delta_{\mathrm{Neu}} u = f$*. Moreover, this solution satisfies*

$$\sum_{|\beta|\leq 2} |\lambda|^{1-\frac{|\beta|}{2}} \|\partial^\beta u\|_{\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \leq C \|f\|_{\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)},$$

where the constant C *only depends on* p, k, γ, ω, d *and* X*.*

Proof. The proof is analogous to the proof of Corollary 7.3.3 using Proposition 5.5.6. □

Remark 7.3.5.

(i) *Second order Riesz transforms*: for $\lambda = 0$ we obtain the regularity estimates from Corollaries 7.3.3 and 7.3.4 as well. Indeed, if $u \in \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ is a solution to $-\Delta_{\mathrm{Dir}} u = f$, then u also satisfies the equation $\lambda u - \Delta_{\mathrm{Dir}} u = \lambda u + f$ with $\lambda > 0$. Therefore, by Corollary 7.3.3 we obtain

$$\begin{aligned}\sum_{|\beta|=2} \|\partial^\beta u\|_{\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} &\leq C \lim_{\lambda \downarrow 0} \|\lambda u + f\|_{\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)} \\ &= C \|f\|_{\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)}.\end{aligned}$$

The same proof works for the Neumann Laplacian.

(ii) The results from Corollaries 7.3.3 and 7.3.4 with $k = 0$ and $\gamma \in (-1, p-1)$ coincide with the inhomogeneous results in Corollaries 5.4.7 and 5.4.8. Moreover, these cases have already been obtained in [80, 81] for the scalar-valued case.

(iii) The regularity result in Corollary 7.3.3 with $k \in \mathbb{N}_0$, $\gamma \in (p-1, 2p-1)$ and $X = \mathbb{C}$ is already covered in [164, Theorem 4.1].

7.3.3. Maximal L^q-regularity for the heat equation

The following two corollaries on maximal regularity for the Dirichlet and Neumann Laplacian follow immediately using the functional calculus from Theorems 5.4.10, 5.7.1 and 5.7.2 together with Theorem 7.2.4 and Propositions 7.2.5 and 7.2.2.

Corollary 7.3.6 (Inhomogeneous maximal regularity for $-\Delta_{\mathrm{Dir}}$)**.** *Let* $p, q \in (1, \infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$ *and let* X *be a* UMD *Banach space. Let* Δ_{Dir} *on* $W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ *be as in Definition 5.4.1. Assume that either*

(i) $\gamma + kp \in (-1, 2p-1)$, $T \in (0, \infty]$ *and* $v \in A_q(I)$*, or,*

(ii) $\gamma + kp > 2p - 1$, $T \in (0, \infty)$ *and* $v \in A_q(I)$*.*

Then $-\Delta_{\mathrm{Dir}}$ *has maximal* $L^q(v)$*-regularity on* I*. In particular, for every* $T \in (0, \infty)$ *and* $f \in L^q(I, v; W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))$ *there exists a unique* $L^q(v)$*-solution* u *to* (7.2.1) *with* $A = -\Delta_{\mathrm{Dir}}$*, which satisfies*

$$\begin{aligned}&\|u\|_{W^{1,q}(I,v;W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))} + \|u\|_{L^q(I,v;W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}^d_+, w_{\gamma+kp}; X))} \\ &\qquad \leq C \|f\|_{L^q(I,v;W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))},\end{aligned} \tag{7.3.2}$$

where the constant C *only depends on* p, q, k, γ, v, T, d *and* X*.*

Corollary 7.3.7 (Inhomogeneous maximal regularity for $-\Delta_{\mathrm{Neu}}$)**.** *Let $p,q \in (1,\infty)$, $k \in \mathbb{N}_0 \cup \{-1\}$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 5.4.1. Assume that either*

(i) $\gamma + kp \in (-1, p-1)$, $T \in (0,\infty]$ and $v \in A_q(I)$, or,

(ii) $\gamma + kp > p-1$, $T \in (0,\infty)$ and $v \in A_q(I)$.

Then $-\Delta_{\mathrm{Neu}}$ has maximal $L^q(v)$-regularity on I. In particular, for every $T \in (0,\infty)$ and $f \in L^q(I,v;W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+kp};X))$ there exists a unique $L^q(v)$-solution u to (7.2.1) *with $A = -\Delta_{\mathrm{Neu}}$, which satisfies*

$$\begin{aligned}&\|u\|_{W^{1,q}(I,v;W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))} + \|u\|_{L^q(I,v;W^{k+3,p}_{\mathrm{Neu}}(\mathbb{R}^d_+,w_{\gamma+kp};X))}\\ &\quad\le C\|f\|_{L^q(I,v;W^{k+1,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))},\end{aligned}$$

where the constant C only depends on p,q,k,γ,v,T,d and X.

Remark 7.3.8.

(i) Corollaries 7.3.6 and 7.3.7 do not hold for $T=\infty$ in general. This follows from Dore's theorem (see [82] or [126, Theorem 17.2.15]) and Theorems 5.6.1(ii) and 5.6.2(ii), respectively. On the other hand, the shifted operators $\lambda - \Delta_{\mathrm{Dir}}$ and $\lambda - \Delta_{\mathrm{Neu}}$ with $\lambda > 0$ do have maximal $L^q(v)$-regularity on $\mathbb{R}_+$, see [126, Propositions 17.2.27 & 17.2.29].

(ii) Corollaries 7.3.6 and 7.3.7 only concern the Cauchy problem (7.2.1) with zero initial data. In addition, maximal regularity can be used to obtain existence and uniqueness for the Cauchy problem with non-zero initial data, see [96, Section 4.4] and [126, Section 17.2.b]. In particular, if $v = t^\eta$ is a power weight with $\eta \in (-1, q-1)$, then the space for the initial data is the real interpolation space

$$\big(W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X), D(A)\big)_{1-\frac{1+\eta}{q},q}$$

with $A = -\Delta_{\mathrm{Dir}}$ or $A = -\Delta_{\mathrm{Neu}}$, see [96, Example 4.16]. Characterisations of real interpolation spaces of weighted Sobolev spaces can be found in [262, Chapter 3].

In the following corollaries, we derive maximal regularity on homogeneous spaces.

Corollary 7.3.9 (Homogeneous maximal regularity for $-\Delta_{\mathrm{Dir}}$)**.** *Let $p,q \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,2p-1)\setminus\{p-1\}$, $v \in A_q(\mathbb{R}_+)$ and let X be a* UMD *Banach space. Let Δ_{Dir} on $\dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X)$ be as in Definition 7.3.2. Then $-\Delta_{\mathrm{Dir}}$ has maximal $L^q(v)$-regularity on $\mathbb{R}_+$. In particular, for every $f \in L^q(\mathbb{R}_+, v; \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))$ there exists a unique $L^q(v)$-solution u to* (7.2.1) *with $A = -\Delta_{\mathrm{Dir}}$, which satisfies*

$$\begin{aligned}&\|\partial_t u\|_{L^q(\mathbb{R}_+,v;\dot{W}^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))} + \sum_{|\beta|=2}\|\partial^\beta u\|_{L^q(\mathbb{R}_+,v;\dot{W}^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))}\\ &\quad\le C\|f\|_{L^q(\mathbb{R}_+,v;\dot{W}^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp};X))},\end{aligned} \tag{7.3.3}$$

where the constant C only depends on p,q,k,γ,v,d and X.

Proof. Let $f \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+^d; X)$, then $f \in L^q((0,T), v; W^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X))$ for all $T > 0$. Corollary 7.3.6 yields that for any $T > 0$ there exists a unique $L^q(v)$-solution u_T to (7.2.1) on $(0,T)$ satisfying the estimate (7.3.2). If $T_1 \le T$, then the restriction of u_T to $(0,T_1)$ coincides with u_{T_1}. This allows us to construct a solution $u : \mathbb{R}_+ \to W^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}_+^d, w_{\gamma+kp}; X) \hookrightarrow \dot{\mathbb{W}}^{k+2,p}_{\mathrm{Dir}}(\mathbb{R}_+^d, w_{\gamma+kp}; X)$ that coincides with u_T on $(0,T)$ for every $T > 0$. Moreover, u is a strong solution to (7.2.1) on $\mathbb{R}_+$ (with $-\Delta_{\mathrm{Dir}}$ considered as an operator on the homogeneous space $\dot{W}^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X)$).

To show the regularity of u, let $r > 0$ and set $u_r(t,x) := u(r^2 t, rx)$ and $f_r(t,x) := f(r^2 t, rx)$. Then u_r satisfies the equation $\partial_t u_r - \Delta_{\mathrm{Dir}} u_r = r^2 f_r$, so by (7.3.2) we have for all $\ell \in \{0,\dots,k\}$ the estimate

$$\begin{aligned}\|\partial_t u_r\|_{L^q((0,1),v(r^2\cdot);W^{\ell,p}(\mathbb{R}_+^d,w_{\gamma+\ell p};X))} &+ \|u_r\|_{L^q((0,1),v(r^2\cdot);W^{\ell+2,p}(\mathbb{R}_+^d,w_{\gamma+\ell p};X))}\\ &\le C\|r^2 f_r\|_{L^q((0,1),v(r^2\cdot);W^{\ell,p}(\mathbb{R}_+^d,w_{\gamma+\ell p};X))},\end{aligned}$$

since $v(r^2\cdot) \in A_q(\mathbb{R}_+)$ with the same A_q constant by [105, Proposition 7.1.5(1)]. After the substitutions $x \mapsto r^{-1}x$ and $t \mapsto r^{-2}t$, we obtain

$$\begin{aligned}\sum_{|\alpha|\le\ell} r^{|\alpha|-\ell}\|\partial_t\partial^\alpha u\|_{L^q((0,r^2),v;L^p(\mathbb{R}_+^d,w_{\gamma+\ell p};X))} &+ \sum_{|\alpha|\le\ell+2} r^{|\alpha|-\ell-2}\|\partial^\alpha u\|_{L^q((0,r^2),v;L^p(\mathbb{R}_+^d,w_{\gamma+\ell p};X))}\\ &\le C\sum_{|\alpha|\le\ell} r^{|\alpha|-\ell}\|\partial^\alpha f\|_{L^q((0,r^2),v;L^p(\mathbb{R}_+^d,w_{\gamma+\ell p};X))}.\end{aligned}$$

Letting $r \to \infty$ and summing over $\ell \in \{0,\dots,k\}$ yields the maximal regularity estimate (7.3.3). Thus u is an $L^q(v)$-solution and we claim that it is also unique on $\mathbb{R}_+$. Indeed, if $f = 0$, then for every $T > 0$ we have

$$\begin{aligned}\|u\|_{C([0,T];\dot{W}^{k,p}(\mathbb{R}_+^d,w_{\gamma+kp};X))} &\le \|Au\|_{L^1([0,T];\dot{W}^{k,p}(\mathbb{R}_+^d,w_{\gamma+kp};X))}\\ &\le \|Au\|_{L^q([0,T],v;\dot{W}^{k,p}(\mathbb{R}_+^d,w_{\gamma+kp};X))}\Big(\int_0^T |v^{-\frac{1}{q-1}}(t)|\,\mathrm{d}t\Big)^{\frac{q-1}{q}}\\ &\le 0,\end{aligned}$$

which follows from taking the supremum over $[0,T]$ in (7.2.2), Hölder's inequality and (7.3.3). This yields that $u = 0$ on $[0,T]$ for all $T > 0$. Therefore $u = 0$ on $\mathbb{R}_+$ as well and this proves the uniqueness.

Finally, from [190, Lemma 3.5] and Lemma 7.3.1, it follows that $C_c^\infty(\mathbb{R}_+ \times \mathbb{R}_+^d; X)$ is dense in $L^q(\mathbb{R}_+, v; \dot{W}^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X))$. Therefore, a density argument (see [126, Proposition 17.2.10] using that $-\Delta_{\mathrm{Dir}}$ is closed by Corollary 7.3.3) gives the result. □

Similarly, we have maximal regularity for the Neumann Laplacian on homogeneous spaces.

Corollary 7.3.10 (Homogeneous maximal regularity for $-\Delta_{\mathrm{Neu}}$)**.** *Let $p, q \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, p-1)$, $v \in A_q(\mathbb{R}_+)$ and let X be a* UMD *Banach space. Let Δ_{Neu} on $\dot{W}^{k,p}(\mathbb{R}_+^d, w_{\gamma+kp}; X)$ be as in Definition 7.3.2. Then $-\Delta_{\mathrm{Neu}}$ has maximal $L^q(v)$-regularity on $\mathbb{R}_+$. In particular,*

for every $f \in L^q(\mathbb{R}_+, v; \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))$ *there exists a unique* $L^q(v)$*-solution* u *to* (7.2.1) *with* $A = -\Delta_{\mathrm{Neu}}$*, which satisfies*

$$\|\partial_t u\|_{L^q(\mathbb{R}_+, v; \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))} + \sum_{|\beta|=2} \|\partial^\beta u\|_{L^q(\mathbb{R}_+, v; \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))} \\ \leq C \|f\|_{L^q(\mathbb{R}_+, v; \dot{W}^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}; X))},$$

where the constant C *only depends on* p, q, k, γ, v, d *and* X*.*

Proof. The proof is analogous to the proof of Corollary 7.3.9, using Corollary 7.3.7. □

We conclude with some remarks concerning Corollaries 7.3.9 and 7.3.10.

Remark 7.3.11.

(i) The maximal regularity estimate from Corollary 7.3.9 for $\gamma \in (p-1, 2p-1)$ and $v = 1$ is already obtained in [165, Theorem 3.2].

(ii) For $k = 0$ and $\gamma \in (-1, p-1)$ similar regularity results as those in Corollaries 7.3.9 and 7.3.10 are contained in [80, Theorem 6.4] and [81, Theorem 2.3] for the shifted operators $\lambda - \Delta_{\mathrm{Dir}}$ and $\lambda - \Delta_{\mathrm{Neu}}$ with $\lambda > 0$. The time weights for $k \geq 1$ appear to be new in the homogeneous setting. These time weights play an important role in nonlinear evolution equations, see, e.g., [80], [126, Chapter 18] and [234].

(iii) More general elliptic operators with time and space-dependent coefficients are treated in for example [79, Theorem 2.1], [80, Theorem 6.4] or [165, Theorem 3.2]. We only proved results for the Laplacian by taking limits in our maximal regularity estimates for inhomogeneous spaces.

7.4. The heat equation on rough domains

We continue with the study of maximal L^q-regularity for the heat equation on rough domains using the results from Chapter 6. The following two corollaries follow immediately using the functional calculus from Theorems 6.6.2, 6.6.3, 6.6.4, 6.6.5 together with Theorem 7.2.4 and Propositions 7.2.5 and 7.2.2.

Corollary 7.4.1 (Maximal regularity for $-\Delta_{\mathrm{Dir}}$)**.** *Assume that the conditions from Theorem 6.6.2 hold. In addition, let* $q \in (1, \infty)$*,* $T \in (0, \infty)$ *and* $v \in A_q(I)$*. Then* $-\Delta_{\mathrm{Dir}}$ *on* $W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ *has maximal* $L^q(v)$*-regularity on* I*, i.e., for all*

$$f \in L^q(I, v; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))$$

there exists a unique

$$u \in W^{1,q}(I, v; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)) \cap L^q(I, v; W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))$$

such that $\partial_t u - \Delta_{\mathrm{Dir}} u = f$ *with* $u(0) = 0$ *and*

$$\|u\|_{W^{1,q}(I,v;W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))} + \|u\|_{L^q(I,v;W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))} \lesssim \|f\|_{L^q(I,v;W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))},$$

where the constant only depends on p, q, k, γ, v, T, d *and* X*. Moreover, if* $X = \mathbb{C}$*, then the above statement holds for* $I = \mathbb{R}_+$ *as well.*

Corollary 7.4.2 (Maximal regularity for $-\Delta_{\mathrm{Neu}}$)**.** *Assume that the conditions from Theorem 6.6.3 hold. In addition, let $q \in (1,\infty)$, $T \in (0,\infty)$ and $v \in A_q(I)$. Then $-\Delta_{\mathrm{Neu}}$ on $W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)$ has maximal $L^q(v)$-regularity on I, i.e., for all*

$$f \in L^q(I, v; W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))$$

there exists a unique

$$u \in W^{1,q}(I, v; W^{k+j,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X)) \cap L^q(I, v; W^{k+2+j,p}_{\mathrm{Neu}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}; X))$$

such that $\partial_t u - \Delta_{\mathrm{Neu}} u = f$ with $u(0) = 0$ and

$$\|u\|_{W^{1,q}(I,v;W^{k+j,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X))} + \|u\|_{L^q(I,v;W^{k+2+j,p}_{\mathrm{Neu}}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X))} \lesssim \|f\|_{L^q(I,v;W^{k+j,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp};X))},$$

where the constant only depends on $p, q, k, \gamma, j, v, T, d$ and X. Moreover, the above statement also holds if we consider Δ_{Neu} on the spaces without constant functions as in Definition 6.6.1(iii). In this case, if additionally $X = \mathbb{C}$, then the statement also holds for $I = \mathbb{R}_+$.

Remark 7.4.3. Corollaries 7.4.1 and 7.4.2 concern the heat equation with zero initial data. Well-posedness for the heat equation with non-zero initial data can be obtained as a consequence, see [96, Section 4.4] and [126, Section 17.2.b].

We connect the above results to the existing literature about PDEs on homogeneous weighted Sobolev spaces, see [163, 164, 197]. For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\theta \in \mathbb{R}$ and $\mathscr{O} \subseteq \mathbb{R}^d$ a bounded C^1-domain, the homogeneous Sobolev spaces are given by

$$H^k_{p,\theta}(\mathscr{O}) = \Big\{ f \in \mathscr{D}'(\mathscr{O}) : \forall |\alpha| \le k, \partial^\alpha f \in L^p(\mathscr{O}, w^{\partial\mathscr{O}}_{\theta-d+|\alpha|p}) \Big\},$$

see for instance [197, Proposition 2.2]. Note that $L^p(\mathscr{O}, w^{\partial\mathscr{O}}_\gamma) = H^0_{p,\gamma+d}(\mathscr{O})$. In the setting for the Dirichlet Laplacian with $\gamma \in (p-1, 2p-1)$ we have the following relation between the involved homogeneous and inhomogeneous spaces:

$$W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) = H^k_{p,\gamma+d}(\mathscr{O}),$$
$$W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) = H^{k+2}_{p,\gamma+d-2p}(\mathscr{O}).$$

The first characterisation follows from the fact that $\mathscr{O}$ is bounded and Hardy's inequality using that $\gamma + kp > -1$. The second characterisation follows similarly using that $W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) = W^{k+2,p}_0(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ for $\gamma \in (p-1, 2p-1)$. Note that we have used that the domain is bounded, for unbounded domains the homogeneous and inhomogeneous spaces cannot be compared.

In [149], the authors use homogeneous spaces to study spatial regularity for boundary value problems with Dirichlet boundary conditions on bounded C^1-domains. There, the boundary condition is encoded implicitly within the function space. In contrast, our approach imposes boundary conditions explicitly, allowing greater flexibility – particularly when extending to more regular domains or handling smaller weight exponents and Neumann boundary conditions. In the homogeneous setting, some results for the

Neumann Laplacian on the half-space (in the special case $k = 0$) are contained in [80, 81], but a general study on bounded domains seems to be unavailable.

Finally, we remark that maximal L^q-regularity for the Dirichlet Laplacian on $L^p(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$ is also obtained in [157]. Here they treat bounded $C^{1,\kappa}$-domains with $\gamma \in ((1-\kappa)p-1, 2p-1)$ which corresponds to our result in Corollary 7.4.1 with $k = 0$.

To continue, we prove well-posedness and higher-order spatial regularity for the heat equation on rough domains with rough boundary data. This extends the results in Corollaries 7.4.1 and 7.4.2 (homogeneous boundary data), and [190, Section 7.3] (bounded C^2-domains and base regularity for only Dirichlet boundary conditions).

We only consider the case of Dirichlet boundary conditions, but the case of Neumann boundary conditions can be dealt with similarly. Consider the initial boundary value problem given by

$$\begin{cases} \partial_t u - \Delta u = f & (t,x) \in \mathbb{R}_+ \times \mathscr{O}, \\ \operatorname{Tr}^{\partial\mathscr{O}} u = g & (t,x) \in \mathbb{R}_+ \times \partial\mathscr{O}, \\ u(0) = 0 & x \in \mathscr{O}. \end{cases} \tag{7.4.1}$$

We study this problem with spatial regularity in weighted Sobolev spaces $W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}})$, where $k \in \mathbb{N}_0$ and

(i) $\mathscr{O}$ is a bounded C^1-domain if $\gamma \in (p-1, 2p-1)$, or,

(ii) $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain with $\kappa \in (0,1)$ if $\gamma \in ((1-\kappa)p-1, p-1)$.

For $p, q \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1, 2p-1) \setminus \{p-1\}$, $\mu \in (-1, q-1)$, $v_\mu(t) = |t|^\mu$ and $\mathscr{O}$ as above, we define

$$\begin{aligned} \mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) &:= L^q(\mathbb{R}_+, v_\mu; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})), \\ \mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) &:= W^{1,q}(\mathbb{R}_+, v_\mu; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})) \cap L^q(\mathbb{R}_+, v_\mu; W^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})), \\ \mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O}) &:= F^{1-\frac{\gamma+1}{2p}}_{q,p}(\mathbb{R}_+, v_\mu; L^p(\partial\mathscr{O})) \cap L^q(\mathbb{R}_+, v_\mu; B^{2-\frac{\gamma+1}{p}}_{p,p}(\partial\mathscr{O})). \end{aligned}$$

Moreover, we introduce

$$\begin{aligned} {}_0\mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O}) &:= \begin{cases} \mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O}) & \text{if } 1 - \frac{\mu+1}{q} < \frac{\gamma+1}{2p}, \\ \{g \in \mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O}) : g(0) = 0\} & \text{if } 1 - \frac{\mu+1}{q} > \frac{\gamma+1}{2p}, \end{cases} \\ {}_0\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) &:= \{u \in \mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) : u(0) = 0\}. \end{aligned}$$

The zeroth-order temporal traces in the above spaces are well defined, see Theorem 3.1.2.

We obtain the following maximal regularity result for (7.4.1).

Theorem 7.4.4. *Let $p, q \in (1,\infty)$, $k \in \mathbb{N}_0$, $\mu \in (-1, q-1)$, $\kappa \in [0,1)$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ with $1 - \frac{\mu+1}{q} \neq \frac{\gamma+1}{2p}$. Furthermore, let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Then for all*

$$f \in \mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \quad \text{and} \quad g \in {}_0\mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O}),$$

there exists a unique solution $u \in {}_0\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$ *to* (7.4.1). *Moreover, this solution satisfies*

$$\|u\|_{\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} \leq C\big(\|f\|_{\mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} + \|g\|_{\mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O})}\big),$$

where the constant $C > 0$ *only depends on* $p, q, k, \mu, \kappa, \gamma$, *and* d.

Proof. From Theorem 4.5.2 it follows that

$$\mathrm{Tr}^{\partial\mathscr{O}} : \mathbb{U}^k_{q,p}(\mathbb{R}, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \to \mathbb{G}_{q,p}(\mathbb{R}, v_\mu; \partial\mathscr{O})$$

is continuous and surjective. By using the zero extension operator from $\mathbb{R}_+$ to $\mathbb{R}$, we can argue similarly as in Step 2 of Theorem 4.4.1 to obtain that

$$\mathrm{Tr}^{\partial\mathscr{O}} : {}_0\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \to {}_0\mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O}) \tag{7.4.2}$$

is continuous and surjective. The corresponding extension operator is denoted by $\mathrm{ext}^{\partial\mathscr{O}}$. First consider

$$\begin{cases} \partial_t \widetilde{u} - \Delta \widetilde{u} = \widetilde{f} & (t,x) \in \mathbb{R}_+ \times \mathscr{O}, \\ \mathrm{Tr}^{\partial\mathscr{O}}\, \widetilde{u} = 0 & (t,x) \in \mathbb{R}_+ \times \partial\mathscr{O}, \\ \widetilde{u}(0) = 0 & x \in \mathscr{O}, \end{cases} \tag{7.4.3}$$

where $\widetilde{f} := f - (\partial_t - \Delta)\,\mathrm{ext}^{\partial\mathscr{O}}\, g \in \mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$. By Corollary 7.4.1, there exists a solution

$$\widetilde{u} \in W^{1,q}(\mathbb{R}_+, v_\mu; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})) \cap L^q(\mathbb{R}_+, v_\mu; W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$$

which satisfies the estimate

$$\begin{aligned} \|\widetilde{u}\|_{\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} &\lesssim \|f\|_{\mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} + \|(\partial_t - \Delta)\,\mathrm{ext}^{\partial\mathscr{O}}\, g\|_{\mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} \\ &\lesssim \|f\|_{\mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} + \|\mathrm{ext}^{\partial\mathscr{O}}\, g\|_{\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})}. \end{aligned} \tag{7.4.4}$$

Set $u := \widetilde{u} + \mathrm{ext}^{\partial\mathscr{O}}\, g$. Then u satisfies $\mathrm{Tr}^{\partial\mathscr{O}}\, u = \mathrm{Tr}^{\partial\mathscr{O}}\, \widetilde{u} + \mathrm{Tr}^{\partial\mathscr{O}}\, \mathrm{ext}^{\partial\mathscr{O}}\, g = g$ by (7.4.3) and properties of the extension operator. Furthermore, by (7.4.2) and (7.4.3), we have $u(0) = \widetilde{u}(0) + \mathrm{ext}^{\partial\mathscr{O}}\, g(0) = 0$. Hence, by the construction of $\widetilde{u}$ in (7.4.3), we have that u satisfies (7.4.1) and we have the estimate

$$\begin{aligned} \|u\|_{\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} &\lesssim \|f\|_{\mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} + \|\mathrm{ext}^{\partial\mathscr{O}}\, g\|_{\mathbb{U}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} \\ &\lesssim \|f\|_{\mathbb{F}^k_{q,p}(\mathbb{R}_+, v_\mu; \mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})} + \|g\|_{\mathbb{G}_{q,p}(\mathbb{R}_+, v_\mu; \partial\mathscr{O})}, \end{aligned}$$

using (7.4.4) and (7.4.2). Finally, the uniqueness of the solution u to (7.4.1) follows from Corollary 7.4.1. □

Remark 7.4.5.

(i) Well-posedness and regularity for the heat equation with non-zero initial data in a real interpolation space can be obtained as well, see [96, Section 4.4] or [126, Section 17.2.b]. In the unweighted setting, characterisations of the real interpolation spaces between Sobolev spaces are well known. However, in the weighted setting, characterisations of real interpolation spaces and temporal trace spaces are more difficult, see [190, Section 7.2].

(ii) The results are also valid in the UMD Banach space-valued case if $-\Delta_{\mathrm{Dir}}$ is replaced by $\lambda - \Delta_{\mathrm{Dir}}$ with $\lambda \geq \lambda_0$ for some $\lambda_0 \geq 0$ depending on the geometry of the Banach space. Hence, a version of Theorem 7.4.4 can be obtained in the vector-valued setting as well. For the special case $k = 0$, a vector-valued version of Theorem 7.4.4 can be obtained with $\lambda_0 = 0$, see Corollary 6.6.7.

(iii) Instead of the parabolic problem one could also consider the elliptic problem $\lambda u - \Delta u = f$ on $\mathscr{O}$ with $\lambda > 0$ and boundary conditions $u|_{\partial\mathscr{O}} = g$. If $g = 0$, then well-posedness and regularity (in the weighted setting on rough domains) follow from Theorem 6.1.1. Combining this with the trace results on rough domains from [140] yields well-posedness and regularity for the elliptic problem with inhomogeneous boundary conditions $g \neq 0$. We note that trace results similar to [140] are simply obtained from Theorem 3.1.2 and a localisation argument based on the Dahlberg–Kenig–Stein pullback in Section 2.6.

7.5. The stochastic heat equation

We introduce the notion of stochastic maximal regularity for linear stochastic equations. For a more elaborate study of this topic, we refer to [10, 11, 12] or the survey [13].

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbf{P})$ be a filtered probability space. Let $\mathscr{U}$ and $(W_{\mathscr{U}}(t))_{t\geq 0}$ be a separable Hilbert space and a cylindrical $\mathscr{F}$-Brownian motion on $\mathscr{U}$, respectively. Often the noise is modelled on the space $\mathscr{U} = \ell^2$.

Let X be a Banach space. By $\gamma(\mathscr{U}, X)$ we denote the space of γ-*radonifying operators*, see [125, Chapter 9]. In the case that X is a Hilbert space, the space of γ-radonifying operators coincides with the space of Hilbert–Schmidt operators $\mathscr{L}_2(\mathscr{U}; X)$.

Throughout the rest of this section, we assume the following standing assumptions: let X_0 and X_1 be UMD Banach spaces with type 2 such that $X_1 \hookrightarrow X_0$ and $A \in \mathscr{L}(X_1, X_0)$. Moreover, we write $X_{\frac{1}{2}} := [X_0, X_1]_{\frac{1}{2}}$.

Now, for $T \in (0, \infty)$ consider the equation

$$\begin{aligned} \mathrm{d}u + Au\,\mathrm{d}t &= f\,\mathrm{d}t + g\,\mathrm{d}W_{\mathscr{U}}(t), \qquad t \in [0, T], \\ u(0) &= 0, \end{aligned} \tag{7.5.1}$$

and we define the notion of stochastic maximal regularity for A. The equation (7.5.1) is understood in the Itô sense.

We start with the definition of a solution to (7.5.1). Let $\tau : \Omega \to [0, T]$ be a stopping time. Let f and g be progressively measurable such that $f \in L^1(0, \tau; X_0)$ and $g \in L^2(0, \tau;$

$\gamma(\mathscr{U}, X_{1/2}))$ almost surely. A progressively measurable process $u : [0,\tau] \times \Omega \to X_0$ is called a strong solution to (7.5.1) on $[0,\tau]$ if $u \in C([0,\tau]; X_0) \cap L^2(0,\tau; X_1)$ almost surely, and for all $t \in [0,\tau]$ we have almost surely

$$u(t) + \int_0^t Au(s)\,\mathrm{d}s = \int_0^t f(s)\,\mathrm{d}s + \int_0^t g(s)\,\mathrm{d}W_{\mathscr{U}}(s).$$

Definition 7.5.1. Let $q \in [2,\infty)$ and $\mu \in [0, q/2-1) \cup \{0\}$. We say $A \in \mathrm{SMR}_{q,\mu}$ if for all $T \in (0,\infty)$ there exists a constant C_T such that for every stopping time $\tau : \Omega \to [0,T]$, every progressively measurable $f \in L^q(\Omega; L^q(0,\tau,\nu_\mu; X_0))$ and $g \in L^q(\Omega; L^q(0,\tau,\nu_\mu; \gamma(\mathscr{U}, X_{1/2})))$, there exists a unique strong solution u to (7.5.1) on $[0,\tau]$ such that $u \in L^q(\Omega; L^q(0,\tau,\nu_\mu; X_1))$, and moreover the following estimate holds

$$\|u\|_{L^q(\Omega;L^q(0,\tau,\nu_\mu;X_1))} \leq C_T \|f\|_{L^q(\Omega;L^q(0,\tau,\nu_\mu;X_0))} + C_T \|g\|_{L^q(\Omega;L^q(0,\tau,\nu_\mu;\gamma(\mathscr{U},X_{1/2})))}.$$

In the deterministic situation, estimates for the time derivative u' can be obtained directly from the equation. In the stochastic case, this is not possible, and the time regularity needs to be studied separately.

Definition 7.5.2. We consider the case $q > 2$ and $q = 2$ separately.

- For $q \in (2,\infty)$ and $\mu \in [0, q/2-1)$, we say $A \in \mathrm{SMR}^\bullet_{q,\mu}$ if $A \in \mathrm{SMR}_{q,\mu}$ and for all $\theta \in (0,1/2)$ for all $T \in (0,\infty)$ there exists a constant $C_{T,\theta}$ such that for all $[0,b] \subseteq [0,T]$, for every progressively measurable

  $$f \in L^q(\Omega; L^q(0,b,\nu_\mu; X_0)) \quad \text{and} \quad g \in L^q(\Omega; L^q(0,b,\nu_\mu; \gamma(\mathscr{U}, X_{1/2}))),$$

  the strong solution u to (7.5.1) on $[0,b]$ satisfies

  $$\|u\|_{L^q(\Omega;H^{\theta,q}(0,b,\nu_\mu;X_{1-\theta}))} \leq C_{T,\theta} \|f\|_{L^q(\Omega;L^q(0,b,\nu_\mu;X_0))} + C_{T,\theta} \|g\|_{L^q(\Omega;L^q(0,b,\nu_\mu;\gamma(\mathscr{U},X_{1/2})))}.$$

- We say $A \in \mathrm{SMR}^\bullet_{2,0}$ if $A \in \mathrm{SMR}_{2,0}$ and for all $T \in (0,\infty)$ there exists a constant C_T such that for all $[0,b] \subseteq [0,T]$, for every progressively measurable $f \in L^2(\Omega; L^2(0,b; X_0))$ and $g \in L^2(\Omega; L^2(0,b; \gamma(\mathscr{U}, X_{1/2})))$, the strong solution u to (7.5.1) on $[0,b]$ satisfies

  $$\|u\|_{L^2(\Omega;C([0,b];X_{1/2}))} \leq C_T \|f\|_{L^2(\Omega;L^2(0,b;X_0))} + C_T \|g\|_{L^2(\Omega;L^2(0,b;\gamma(\mathscr{U},X_{1/2})))}.$$

In case $A \in \mathrm{SMR}^\bullet_{q,\mu}$ we will say that A has stochastic maximal $L^q(\nu_\mu)$-regularity.

Now, we present the cornerstone result in the theory of stochastic maximal regularity. The result provides a sufficient condition for stochastic maximal regularity in terms of the H^∞-calculus of A. The result for $\mu = 0$ was obtained in [219] and [221]. The weighted setting was obtained in [10].

Theorem 7.5.3. *Let $(\mathscr{O}, \Sigma, \nu)$ be a σ-finite measure space and let $p \in (2,\infty)$. Suppose that $X_0 = L^p(\mathscr{O})$ or X_0 is isomorphic to a closed subspace of $L^p(\mathscr{O})$. If there exists a $\lambda \geq 0$ such that $\lambda + A$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\lambda + A) < \pi/2$, then $A \in \mathrm{SMR}^\bullet_{q,\mu}$ for all $q \in (2,\infty)$ and $\mu \in [0, q/2-1)$. Moreover, if the above conditions hold for $p = 2$, then $A \in \mathrm{SMR}^\bullet_{2,0}$.*

As a direct consequence, we obtain the following well-posedness and regularity result for the stochastic heat equation. Again, we only consider the case of Dirichlet boundary conditions, but the case of Neumann boundary conditions can be treated similarly.

Corollary 7.5.4. *Let $p \in (2,\infty)$, $k \in \mathbb{N}_0$, $\kappa \in [0,1]$, $\gamma \in ((1-\kappa)p-1, 2p-1) \setminus \{p-1\}$ and let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Consider the Dirichlet Laplacian Δ_{Dir} on*

$$X_0 := W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \quad \text{with domain} \quad X_1 := D(\Delta_{\mathrm{Dir}}) = W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}).$$

Then $-\Delta_{\mathrm{Dir}} \in \mathrm{SMR}^{\bullet}_{q,\mu}$ for all $q \in (2,\infty)$ and $\mu \in [0, q/2-1)$. Moreover, if $p=2$, then $-\Delta_{\mathrm{Dir}} \in \mathrm{SMR}^{\bullet}_{2,0}$.

An important advantage of using the weighted setting with $\gamma \in (p-1, 2p-1)$ for stochastic equations is that we do *not* need any compatibility conditions for the stochastic forcing. Indeed, the spatial regularity of g in (7.5.1) is

$$X_{\frac{1}{2}} = [X_0, X_1]_{\frac{1}{2}} = \big[W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}), W^{k+2,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big]_{\frac{1}{2}} = W^{k+1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}).$$

for all $k \in \mathbb{N}_0$ and $\gamma \in (p-1, 2p-1)$, see Theorem 3.6.5. This is in contrast to the case $\gamma \in (-1, p-1)$ where the Dirichlet trace operator exists in $X_{\frac{1}{2}} = W^{k+1,p}_{\mathrm{Dir}}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})$. So for $\gamma \in (-1, p-1)$ one has to impose compatibility conditions on g at the boundary $\partial\mathscr{O}$, while for $\gamma \in (p-1, 2p-1)$ this is not necessary. Another advantage of the spatial weights with $\gamma \in (p-1, 2p-1)$ is that we can consider a rougher domain $\mathscr{O}$ with only C^1-regularity.

8

A structurally damped plate equation on rough domains

This chapter is based on:

R. Denk and F. Roodenburg. Higher-order regularity for a structurally damped plate equation on rough domains. *Math. Ann.*, 395(2):32, 2026.

Abstract. *We prove well-posedness and higher-order regularity for a linear structurally damped plate equation with inhomogeneous Dirichlet–Neumann boundary conditions on the half-space and on bounded domains. To this end, we study maximal regularity properties of the related first-order system on weighted Sobolev spaces of arbitrarily high smoothness. In particular, we consider Sobolev spaces with power weights that measure the distance to the boundary. This allows us to avoid unnatural compatibility conditions for the data and treat the plate equation with rough inhomogeneous boundary conditions on bounded $C^{1,\kappa}$-domains, where $\kappa \in (0,1)$ depends on the exponent of the spatial power weight, but is independent of the smoothness of the data.*

8.1. Introduction

In this chapter, we consider the linear structurally damped plate equation with inhomogeneous Dirichlet–Neumann boundary conditions given by

$$\begin{cases} \partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u = f & (t,x) \in (0,T)\times\mathscr{O}, \\ u = g_0 & (t,x) \in (0,T)\times\partial\mathscr{O}, \\ \partial_{\boldsymbol{n}} u = g_1 & (t,x) \in (0,T)\times\partial\mathscr{O}, \\ u|_{t=0} = (\partial_t u)|_{t=0} = 0 & x \in \mathscr{O}. \end{cases} \tag{8.1.1}$$

Here, $\rho > 0$ is a fixed parameter and $\partial_{\boldsymbol{n}}$ denotes the inner normal derivative. Furthermore, $T \in (0,\infty]$ is a time horizon and we consider the case of the half-space $\mathscr{O} = \mathbb{R}^d_+$ and the case of a bounded domain $\mathscr{O} \subseteq \mathbb{R}^d$.

The plate equation occurs as a linear model for vibrating stiff objects where the potential energy involves curvature-like terms which lead to the bi-Laplacian as the main 'elastic' operator, see, e.g., [180, Chapter 12] and [241]. The equation (8.1.1) relates to the Kirchhoff model of thin plates; for mathematical modelling of thin elastic plates, we also refer to [177, Chapter 2]. In the undamped situation (corresponding to $\rho = 0$ in (8.1.1)), there is no energy dissipation, and the equation has no smoothing effect. Apart from thermoelastic damping effects (as discussed in a semigroup context in, e.g., [178]), a standard damping model is given by structural damping as above. This model describes a situation where higher frequencies are more strongly damped than lower frequencies. In (8.1.1), the damping term has 'half of the order' of the leading elastic term. For mathematical models of internal damping mechanisms leading to structural damping of second-order, we also refer to [242, Section 4.2]. The generation of an analytic semigroup for the plate equation with structural damping was investigated, e.g., in [41] and [74].

The Dirichlet–Neumann boundary conditions in (8.1.1) are also called generalised Dirichlet conditions and are known in the context of the plate equation as clamped boundary conditions. We remark that these boundary conditions are mathematically more involved than the boundary conditions $u = \Delta u = 0$, which can be treated by operator-theoretic methods, see, e.g., [40, 42, 246].

In this chapter, we study well-posedness and regularity of (8.1.1) in a framework with weighted L^q-L^p-Sobolev spaces. Specifically, we consider spatial power weights of the form $w_\gamma^{\partial\mathscr{O}} := \operatorname{dist}(\cdot,\partial\mathscr{O})^\gamma$ for some suitable $\gamma > -1$. These weights are well suited to control boundary singularities of solutions and their derivatives as we have seen in previous chapters. Our main result for (8.1.1) on bounded domains reads as follows. This is a special case of Theorem 8.6.3, where for simplicity we omit the temporal weights. We recall that $F^s_{q,p}$ and $B^s_{p,p}$ denote the Triebel–Lizorkin and Besov spaces, respectively.

Theorem 8.1.1. *Let $p,q \in (1,\infty)$, $\kappa \in (0,1)$, $\gamma \in ((1-\kappa)p-1, p-1)$ and let $k \geq 2$ be an integer. Suppose that $1 - \frac{1}{q} \neq \frac{\gamma+1}{2p}$ and $\frac{1}{2} - \frac{1}{q} \neq \frac{\gamma+1}{2p}$. Furthermore, let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain.*

Then for all

$$f \in L^q(\mathbb{R}_+; W^{k-2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})),$$
$$g_0 \in F^{1-\frac{\gamma+1}{2p}}_{q,p}(\mathbb{R}_+; L^p(\partial\mathscr{O})) \cap L^q(\mathbb{R}_+, B^{2-\frac{\gamma+1}{p}}_{p,p}(\partial\mathscr{O})) \text{ with } g_0|_{t=0} = 0 \text{ if } 1-\tfrac{1}{q} > \tfrac{\gamma+1}{2p},$$
$$g_1 \in F^{\frac{1}{2}-\frac{\gamma+1}{2p}}_{q,p}(\mathbb{R}_+; L^p(\partial\mathscr{O})) \cap L^q(\mathbb{R}_+, B^{1-\frac{\gamma+1}{p}}_{p,p}(\partial\mathscr{O})) \text{ with } g_1|_{t=0} = 0 \text{ if } \tfrac{1}{2}-\tfrac{1}{q} > \tfrac{\gamma+1}{2p},$$

there exists a unique solution

$$u \in W^{2,q}(\mathbb{R}_+; W^{k-2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})) \cap L^q(\mathbb{R}_+; W^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}))$$

to (8.1.1). *Moreover, this solution u depends continuously on the data f , g_0 and g_1.*

To prove Theorem 8.1.1, we rewrite (8.1.1) as a first-order system for $v = (u, \partial_t u)^\top$, i.e.,

$$\partial_t v + Av = \begin{pmatrix} 0 \\ f \end{pmatrix}, \quad \text{with } A = \begin{pmatrix} 0 & -\mathrm{id} \\ \Delta^2 & -\rho\Delta \end{pmatrix}. \tag{8.1.2}$$

We establish results on sectoriality and R-sectoriality of the operator matrix A. With this operator-theoretic approach, in the first step, we obtain unique solvability and maximal regularity results for (8.1.1) with homogeneous boundary conditions $u = \partial_{\boldsymbol{n}} u = 0$. In a second step, we include inhomogeneous boundary data, using trace results for weighted spaces obtained in Chapter 4. The theory for temporal traces in the weighted setting is delicate and therefore we only consider zero initial values in (8.1.1).

The main novelties of Theorem 8.1.1 can be summarised as follows.

(i) *Fewer compatibility conditions for the data.* Due to the introduction of a spatial power weight, there is no need to include additional conditions of compatibility type on the right-hand side of (8.1.1). We discuss this in more detail below.

(ii) *Higher-order regularity.* We establish maximal regularity results in weighted Sobolev spaces of arbitrarily high smoothness.

(iii) *Rough domains.* The theory applies to bounded $C^{1,\kappa}$-domains, independent of the regularity order k. In this sense, one can speak of a rough domain.

(iv) *Rough boundary data.* We obtain higher-order regularity of the solution in weighted spaces, even if the boundary data is in the standard low-regularity trace spaces. In this sense, we can also handle rough boundary data.

It is remarkable that, due to the introduction of a spatial power weight, we can give a fairly complete well-posedness and regularity theory for (8.1.1) that avoids additional unnatural compatibility conditions as encountered in [74]. It was shown in [74] that the operator A in (8.1.2) is R-sectorial in the basic space $W^{2,p}_0(\mathbb{R}^d_+) \times L^p(\mathbb{R}^d_+)$, but *not* in the space $W^{2,p}(\mathbb{R}^d_+) \times L^p(\mathbb{R}^d_+)$. Therefore, one has to include the additional boundary conditions $h_1 = \partial_{\boldsymbol{n}} h_1 = 0$ on $\partial\mathbb{R}^d_+$ for the first component of the right-hand side of the equation $(\lambda - A)v = h$. It was shown in [66] that such compatibility conditions are in general necessary for the sectoriality in unweighted spaces. For higher-order regularity

results, even more compatibility conditions on the data have to be assumed, see [66, Theorem 3.6]. We can avoid any compatibility conditions on the data, independent of the smoothness order of the solution, by introducing weighted Sobolev spaces. We obtain R-sectoriality of A in the basic space

$$\mathbb{X}_\gamma^{k,p}(\mathbb{R}^d_+) := W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}) \times W^{k-2,p}(\mathbb{R}^d_+, w_{\gamma+kp}),$$

where $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, $k \in \mathbb{N}_0$ and $w_{\gamma+kp} = \operatorname{dist}(\cdot, \partial\mathbb{R}^d_+)^{\gamma+kp}$, see Theorem 8.5.7. In particular, setting $k = 2$, we can replace the basic space $W_0^{2,p}(\mathbb{R}^d_+) \times L^p(\mathbb{R}^d_+)$ from [74] by the weighted space $W^{2,p}(\mathbb{R}^d_+, w_{\gamma+2p}) \times L^p(\mathbb{R}^d_+, w_{\gamma+2p})$. The advantage of the weighted space $W^{2,p}(\mathbb{R}^d_+, w_{\gamma+2p})$ is that the Dirichlet and Neumann trace operators do not exist, in this way omitting the compatibility conditions.

We obtain R-sectoriality of A on $\mathbb{X}_\gamma^{k,p}(\mathbb{R}^d_+)$ by the following method. We start with R-sectoriality of A on the space $W_0^{2,p}(\mathbb{R}^d_+, w_\gamma) \times L^p(\mathbb{R}^d_+, w_\gamma)$ for $\gamma \in (-1, p-1)$, which can be obtained by extending the results in [74] to the weighted setting. Then we lower the smoothness of the spaces by extrapolating the operator in the sense of Amann's interpolation-extrapolation scales (see Section 8.2 for details on these spaces). For the extrapolated operator on $L^p(\mathbb{R}^d_+, w_\gamma) \times W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$, due to the low regularity, the boundary conditions are not seen, so there are no compatibility conditions needed. This proves R-sectoriality on $\mathbb{X}_\gamma^{0,p}(\mathbb{R}^d_+)$.

Afterwards, with an induction argument on k, we raise the smoothness and the weight with a similar technique as developed in Chapter 5 which essentially relies on Hardy's inequality (Lemma 2.2.2). In this way, we can get R-sectoriality in weighted spaces $\mathbb{X}_\gamma^{k,p}(\mathbb{R}^d_+)$ of arbitrary smoothness $k \in \mathbb{N}_0$ without artificial compatibility conditions.

Once the R-sectoriality is obtained on $\mathbb{R}^d_+$, a localisation argument allows us to treat the case of bounded domains. For the localisation, we make use of a special diffeomorphism known as the *Dahlberg–Kenig–Stein pullback*, which straightens the boundary and preserves the distance to the boundary. The blow-up of higher-order derivatives of this pullback near the boundary is compensated by the weights in the space. Therefore, we can establish the results for domains with $C^{1,\kappa}$-boundary, even for higher-order regularity. Note that even for the standard regularity $k = 2$, the classical localisation would require a domain of class C^4, as can be seen from [74, Section 5]. The Dahlberg–Kenig–Stein pullback was already used in Chapter 6 to obtain higher-order regularity for the Laplacian on rough domains.

The operator A in (8.1.2) is an example of a mixed-order system, also called a Douglis–Nirenberg system as considered in [6, 7]. In general, semigroup theory for mixed-order boundary value problems is not yet available. For a general mixed-order system in the whole space, the Newton polygon theory developed in, e.g., [69, 73, 101] leads to equivalent conditions for maximal regularity. For mixed-order boundary value problems, the Newton polygon approach is known only in special cases, see, e.g., [75] and for some recent results, see [224]. One of the difficulties for mixed-order systems lies in the fact that one cannot take a product of L^p-spaces as a basic space, due to the Douglis–Nirenberg order structure of the system. Instead, the basic space necessarily is a tuple of Sobolev

spaces, where now the question of compatibility conditions arises, similarly to the discussion above. We hope to provide a first step for a maximal regularity theory for mixed-order systems in weighted spaces, omitting compatibility conditions.

Outline

The chapter is organised as follows. In Section 8.2, we recall the necessary background on interpolation-extrapolation scales. Section 8.3 is devoted to weighted Sobolev spaces with power weights on the half-space and on rough domains. In Sections 8.4 and 8.5, we establish R-sectoriality of the elliptic operator associated with the damped plate equation on the half-space and on bounded domains, respectively. Finally, in Section 8.6, we apply these results to prove well-posedness and higher-order maximal regularity for the structurally damped plate equation with inhomogeneous boundary conditions.

8.2. Interpolation-extrapolation spaces

We recall the theory of interpolation-extrapolation scales. For a detailed study, we refer to [14, Chapter V] or [118, Section 6.3].

Let A be a sectorial operator on a Banach space X that satisfies $0 \in \rho(A)$. Hence, $(X, \|A^{-1}\cdot\|_X)$ is a normed space and we can define the extrapolated space

$$E_{-1,A} := (X, \|A^{-1}\cdot\|_X)^{\sim},$$

where $\sim$ denotes the completion of the space. Since $X \hookrightarrow E_{-1,A}$ and $\overline{D(A)} = X$, we have that $D(A) \ni x \mapsto Ax \in X$ extends to an isometric isomorphism $T_{-1,A}$ from X to $E_{-1,A}$. The operator A induces a closed linear operator $A_{-1} := T_{-1,A} A T_{-1,A}^{-1}$ which satisfies $A_{-1}|_{D(A)} = A$. Since (R-)sectoriality and boundedness of the H^∞-calculus are preserved under similarity transforms, we obtain the following result.

Proposition 8.2.1. *Let A be a sectorial operator on a Banach space X that satisfies $0 \in \rho(A)$. Then A_{-1} with $D(A_{-1}) = X$ is the closure of A in $E_{-1,A}$. Moreover, the following statements hold:*

(i) A_{-1} on $E_{-1,A}$ is sectorial with angle $\omega(A_{-1}) = \omega(A)$,

(ii) if A is R-sectorial on X, then so is A_{-1} on $E_{-1,A}$ and $\omega_R(A) = \omega_R(A_{-1})$,

(iii) if A has a bounded H^∞-calculus on X, then so does A_{-1} on $E_{-1,A}$ and $\omega_{H^\infty}(A) = \omega_{H^\infty}(A_{-1})$.

Proposition 8.2.1 shows that for $\alpha > 0$ the fractional powers $(A_{-1})^\alpha$ are well-defined closed operators on $E_{-1,A}$ (see, e.g., [126, Definition 15.2.2]). We denote the domain of $(A_{-1})^\alpha$ as

$$E_{-1+\alpha,A} := \big(D((A_{-1})^\alpha), \|(A_{-1})^\alpha \cdot\|_{E_{-1,A}}\big), \qquad \alpha \geq 0,$$

and note that $E_{0,A} = X$ by Proposition 8.2.1.

Let $\alpha \geq -1$ and let A_α be the realisation of A_{-1} on $E_{\alpha,A}$, i.e,

$$\begin{aligned} D(A_\alpha) &:= \{x \in E_{\alpha,A} : A_{-1}x \in E_{\alpha,A}\},\\ A_\alpha x &:= A_{-1}x \quad \text{for } x \in D(A_\alpha), \end{aligned}$$

and note that $A_0 = A$. The scale $(E_{\alpha,A}, A_\alpha)_{\alpha\geq -1}$ is called the *interpolation-extrapolation scale of A* and forms an interpolation scale with respect to complex interpolation.

Proposition 8.2.2 ([118, Theorem 6.6.9])**.** *Let A be a sectorial operator on a Banach space X that satisfies $0 \in \rho(A)$. Furthermore, assume that A has* BIP. *Then*

$$E_{\alpha(1-\theta)+\beta\theta,A} = \big[E_{\alpha,A}, E_{\beta,A}\big]_\theta, \qquad \alpha,\beta \geq -1, \theta \in (0,1).$$

Furthermore, the following characterisation of the extrapolated spaces holds.

Proposition 8.2.3 ([14, Theorem V.1.4.12])**.** *Let A be a sectorial operator on a reflexive Banach space X that satisfies $0 \in \rho(A)$, and let A' be the adjoint operator in X'. Then, for all $\theta \in (0,1)$ we have $E_{-\theta,A} = (E_{\theta,A'})'$ with respect to the duality induced by (X, X'), where*

$$E_{\theta,A'} = \big(D((A')^\theta), \|(A')^\theta \cdot \|_{X'}\big).$$

8.3. Weighted Sobolev spaces

Recall the definitions of the weighted Lebesgue and Sobolev spaces from Section 2.2. We introduce weighted Sobolev spaces with vanishing traces by means of taking the closure of test functions.

Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma > -1$ and let X be a Banach space. We define

$$\mathring{W}_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \overline{C_c^\infty(\mathscr{O}; X)}^{W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)}.$$

Alternative characterisations of this space in terms of traces are available, see [190] and Chapters 5 and 6. These characterisations are based on the mapping properties of the diffeomorphism in Lemma 2.6.5. We distinguish three cases: the half-space, special domains and bounded domains.

Trace characterisations on the half-space

For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and X a Banach space, we define

$$W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X) := \Big\{ f \in W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) : \mathrm{Tr}(\partial^\alpha f) = 0 \text{ if } k - |\alpha| > \tfrac{\gamma+1}{p} \Big\}. \tag{8.3.1}$$

All the traces in the above definition (8.3.1) are well defined. Indeed, this follows from the embedding

$$W^{1,p}(\mathbb{R}_+, w; X) \hookrightarrow C([0,\infty); X),$$

for $p \in (1,\infty)$, w such that $w^{-\frac{1}{p-1}} \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$ and X a Banach space, see [190, Lemma 3.1]. For details, we refer to Section 5.3.1.

Let $p \in (1,\infty)$ and X a Banach space. To be able to deal with negative weights, we define for $k \in \mathbb{N}_0$ and $\gamma \leq -1$ the space

$$W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X) := W^{k,p}(\mathbb{R}^d_+, w_\gamma; X),$$

which is an appropriate definition in view of [190, Lemma 3.1(2)]. For $k \in \mathbb{N}_1$ and $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$, weighted Sobolev spaces with negative smoothness are defined as

$$W^{-k,p}(\mathbb{R}^d_+, w_\gamma; X) := \big(W_0^{k,p'}(\mathbb{R}^d_+, w_{\gamma'}; X')\big)',$$

where $\gamma' = -\gamma/(p-1)$ and the duality is with respect to the unweighted pairing.

From [190, Proposition 3.8] (see also Proposition 6.3.1) the following trace characterisation follows.

Proposition 8.3.1 (Trace characterisation on $\mathbb{R}^d_+$)**.** *Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R} \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and let X be a Banach space. Then*

$$\mathring{W}_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X) = W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X).$$

Note that Proposition 8.3.1 means that $C_c^\infty(\mathbb{R}^d_+; X)$ is dense in $W_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X)$ for the given ranges of parameters. In particular, we also have

$$C_c^\infty(\mathbb{R}^d_+; X) \overset{\text{dense}}{\subseteq} W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \quad \text{for } \gamma > kp-1. \tag{8.3.2}$$

Trace characterisations on special domains

The diffeomorphism Φ in Lemma 2.6.5 will allow us to define weighted Sobolev spaces on special domains with vanishing traces. We first note that Φ gives rise to an isomorphism between weighted Sobolev spaces on special domains and the half-space.

Proposition 8.3.2. *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$, $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ and let X be a Banach space. Moreover, let $\mathscr{O}$ be a special $C_c^{\ell,\kappa}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \le 1$. Let $\Phi : \mathscr{O} \to \mathbb{R}^d_+$ be as in Lemma 2.6.5 and consider the change of coordinates mappings*

$$\Phi_* : W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to W^{k,p}(\mathbb{R}^d_+, w_\gamma; X) \qquad \text{for } \gamma > (k-(\ell+\kappa))_+ p - 1,$$
$$\Phi_* : \mathring{W}_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to \mathring{W}_0^{k,p}(\mathbb{R}^d_+, w_\gamma; X),$$

defined by $\Phi_* f := f \circ \Phi^{-1}$. Then Φ_* is an isomorphism of Banach spaces for which $(\Phi^{-1})_*$ acts as inverse.

Proof. The statement of the proposition follows from Proposition 6.3.4 and Remark 6.3.5. □

With the aid of the isomorphism in Proposition 8.3.2, we can now define traces of functions in weighted Sobolev spaces on special domains with low smoothness compared to the smoothness of the Sobolev space. We note that the standard localisation procedure in, e.g., [67, 89, 166] is not sufficient in our case since the regularity of the domain is too low, see Remark 6.3.8.

Definition 8.3.3. Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and X a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k-(\ell+\kappa))_+ p - 1$ and let $\mathscr{O}$ be a special $C_c^{\ell,\kappa}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \le 1$. Let Φ_* be the isomorphism from Proposition 8.3.2. We define

$$W_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \Big\{ f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \operatorname{Tr}(\partial^\alpha(\Phi_* f)) = 0 \text{ if } k - |\alpha| > \tfrac{\gamma+1}{p} \Big\}.$$

Note that the traces in the above definition are considered on the half-space and are thus well defined by Proposition 8.3.2. Moreover, provided that γ is large enough, Lemma 2.6.5 ensures that the definition of the space is consistent in the sense that viewing $\mathscr{O}$ as either a special $C_c^{\ell,\kappa}$-domain or a special C_c^1-domain yields the same space.

We now obtain a trace characterisation for the spaces $\mathring{W}_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ if $\mathscr{O}$ is a special domain.

Proposition 8.3.4 (Trace characterisation on special domains)**.** *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{j\,p - 1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k - (\ell+\kappa))_+ p - 1$ and let $\mathscr{O}$ be a special $C_c^{\ell,\kappa}$-domain with $[\mathscr{O}]_{C^{\ell,\kappa}} \le 1$. Let Φ_* be the isomorphism from Proposition 8.3.2. Then*

$$\mathring{W}_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) = W_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X).$$

Proof. The proof is the same as the proof of Proposition 6.3.7 using Proposition 8.3.2. □

Trace characterisations on bounded domains

To define traces on bounded domains, we employ a localisation procedure based on the following decomposition of weighted Sobolev spaces, see also [190, Section 2.2] and Lemma 6.3.9.

Lemma 8.3.5. *Let $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, and let $\mathscr{O} \subseteq \mathbb{R}^d$ be a bounded $C^{\ell,\kappa}$-domain. Then the following statements hold.*

(i) For all $\varepsilon \in (0,\kappa)$ there exists a finite open cover $(V_n)_{n=1}^N$ of $\partial\mathscr{O}$, together with special $C_c^{\ell,\kappa}$-domains $(\mathscr{O}_n)_{n=1}^N$ which satisfy $[\mathscr{O}_n]_{C^{\ell,\kappa-\varepsilon}} < \delta$, such that

$$\mathscr{O} \cap V_n = \mathscr{O}_n \cap V_n \qquad \text{and} \qquad \partial\mathscr{O} \cap V_n = \partial\mathscr{O}_n \cap V_n, \quad n \in \{1,\dots,N\}.$$

If $\kappa = 0$, then for any $\delta > 0$, the special C_c^ℓ-domains $(\mathscr{O}_n)_{n=1}^N$ can be chosen such that $[\mathscr{O}_n]_{C^\ell} < \delta$.

(ii) There exist $\eta_0 \in C_c^\infty(\mathscr{O})$ and $\eta_n \in C_c^\infty(V_n)$ for all $n \in \{1,\dots,N\}$ such that $\sum_{n=0}^N \eta_n^2 = 1$ on $\mathscr{O}$ (partition of unity).

(iii) For $p \in (1,\infty)$, $k \in \mathbb{N}_0$, $\gamma \in \mathbb{R}$ and X a Banach space, the space $W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$ has the direct sum decomposition

$$\mathscr{W}_\gamma^{k,p} := W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^N W^{k,p}(\mathscr{O}_n, w_\gamma^{\partial\mathscr{O}_n}; X). \tag{8.3.3}$$

Moreover, the mappings

$$\mathscr{I}\colon W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) \to \mathscr{W}_\gamma^{k,p} \quad \text{and} \quad \mathscr{P}\colon \mathscr{W}_\gamma^{k,p} \to W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X)$$

given by

$$\mathscr{I} f := (\eta_n f)_{n=0}^N \quad \text{and} \quad \mathscr{P}(f_n)_{n=0}^N := \sum_{n=0}^N \eta_n f_n, \tag{8.3.4}$$

are continuous and satisfy $\mathscr{P}\mathscr{I} = \mathrm{id}$. Thus, $\mathscr{P}$ is a retraction with coretraction $\mathscr{I}$.

In addition to (8.3.3), we define spaces incorporating zero boundary conditions as

$$\mathscr{W}_{0,\gamma}^{k,p} := W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^{N} W_0^{k,p}(\mathscr{O}_n, w_\gamma^{\partial\mathscr{O}_n}; X). \tag{8.3.5}$$

Using the above decomposition, we can define weighted Sobolev spaces on bounded domains with vanishing traces.

Definition 8.3.6. Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k-(\ell+\kappa))_+ p - 1$. Moreover, let $\mathscr{O}$ be a bounded $C^{\ell,\kappa}$-domain, let $(\mathscr{O}_n)_{n=1}^N$ be special $C_c^{\ell,\kappa}$-domains and let $\mathscr{I}$ be the coretraction from Lemma 8.3.5. We define

$$W_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) := \{f \in W^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) : \mathscr{I} f \in W^{k,p}(\mathbb{R}^d; X) \oplus \bigoplus_{n=1}^{N} W_0^{k,p}(\mathscr{O}_n, w_\gamma^{\partial\mathscr{O}_n}; X)\}.$$

This space is well defined by Lemma 8.3.5 and Definition 8.3.3. Moreover, the definitions are independent of the chosen covering of $\partial\mathscr{O}$ and the partition of unity in Lemma 8.3.5. Similar to Proposition 8.3.4, we obtain the following trace characterisation for bounded domains. For the proof of this trace characterisation, we refer to Proposition 6.3.11.

Proposition 8.3.7 (Trace characterisation on bounded domains)**.** *Let $p \in (1,\infty)$, $\ell \in \mathbb{N}_1$, $\kappa \in [0,1]$, $k \in \mathbb{N}_0$ and let X be a Banach space. Let $\gamma \in (-1,\infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ be such that $\gamma > (k-(\ell+\kappa))_+ p - 1$ and let $\mathscr{O}$ be a bounded $C^{\ell,\kappa}$-domain. Then*

$$\mathring{W}_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X) = W_0^{k,p}(\mathscr{O}, w_\gamma^{\partial\mathscr{O}}; X).$$

8.4. R-sectoriality for the elliptic operator on the half-space

In this section, we study the elliptic operator corresponding to the damped plate equation (8.1.1). This elliptic operator on $\mathbb{R}^d$, $\mathbb{R}^d_+$, and bounded domains is studied in [74], and these results will be the starting point for our analysis on weighted spaces. Therefore, we will first recall several results from [74] in a slightly more general setting with Muckenhoupt power weights.

On $\mathbb{R}^d$ the system (8.1.1) without boundary conditions can be written in the form

$$\begin{cases} \partial_t^2 u + \rho B^{\frac{1}{2}} \partial_t u + Bu = f & \text{for } t > 0, \\ u|_{t=0} = (\partial_t u)|_{t=0} = 0, \end{cases} \tag{8.4.1}$$

where $B := \Delta^2$ on $L^p(\mathbb{R}^d)$ with $D(B) := W^{4,p}(\mathbb{R}^d)$. Equation (8.4.1) is related to the quadratic operator pencil $V : W^{4,p}(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ given by

$$V(\lambda) := \lambda^2 + \lambda\rho B^{\frac{1}{2}} + B = (\alpha_+\lambda + B^{\frac{1}{2}})(\alpha_-\lambda + B^{\frac{1}{2}}),$$

where

$$\alpha_\pm = \begin{cases} \frac{\rho}{2} \pm \mathrm{i}\sqrt{1 - \frac{\rho^2}{4}} & \text{for } 0 < \rho < 2, \\ \frac{\rho}{2} \pm \sqrt{\frac{\rho^2}{4} - 1} & \text{for } \rho \geq 2. \end{cases}$$

The roots can be written as $\alpha_\pm = e^{\pm i\vartheta}$ for $0<\rho<2$ and $\alpha_\pm>0$ for $\rho\geq 2$, where the angle $\vartheta=\vartheta(\rho)$ is given by

$$\vartheta(\rho) := \begin{cases} \arctan\frac{2}{\rho}\sqrt{1-\frac{\rho^2}{4}} & \text{for } 0<\rho<2,\\ 0 & \text{for } \rho\geq 2. \end{cases}$$

Note that $\arg\alpha_\pm = \pm\vartheta(\rho)$ and $\vartheta(\rho)\uparrow\frac{\pi}{2}$ as $\rho\downarrow 0$.

By the classical theory of quadratic operator pencils and second-order Cauchy problems, the operator $V(\lambda)$ can be inverted to show maximal regularity for (8.4.1). We refer to [40, 42, 93, 246] and, in particular, see [93, Theorem 3.4] and [42, Theorem 4.1]. In [74], the related first-order system is studied which is obtained by setting $v := (u, \partial_t u)^\top$. Then we can write

$$\begin{cases} \partial_t v + Av = \begin{pmatrix} 0 \\ f \end{pmatrix} & \text{for } t>0,\\ v|_{t=0} = 0, \end{cases}$$

where, formally, the operator matrix A is given by

$$A := \begin{pmatrix} 0 & -\mathrm{id} \\ \Delta^2 & -\rho\Delta \end{pmatrix}, \quad \rho>0. \tag{8.4.2}$$

In addition, we occasionally study the closely related operator

$$A_\eta := \begin{pmatrix} 0 & -\mathrm{id} \\ \eta+\Delta^2 & -\rho\Delta \end{pmatrix}, \quad \eta\geq 0,\ \rho>0. \tag{8.4.3}$$

In the case of unbounded domains, the operator A_η with $\eta>0$ behaves slightly better than $A=A_0$, since $\eta+\Delta^2$ is invertible while Δ^2 is not.

Following [74, Section 2], the operator A on the full space can be defined via its symbol, i.e., $A := \mathscr{F}^{-1}A(\xi)\mathscr{F}$, where the matrix-valued symbol $A(\xi)$ is given by

$$A(\xi) := \begin{pmatrix} 0 & -1 \\ |\xi|^4 & \rho|\xi|^2 \end{pmatrix}, \qquad \xi\in\mathbb{R}^d.$$

For $p\in(1,\infty)$, $\gamma\in(-1,p-1)$, we define the Banach spaces

$$\mathbb{X} := W^{2,p}(\mathbb{R}^d, w_\gamma)\times L^p(\mathbb{R}^d, w_\gamma),$$
$$\mathbb{D} := W^{4,p}(\mathbb{R}^d, w_\gamma)\times W^{2,p}(\mathbb{R}^d, w_\gamma).$$

Recall that the notion of R-sectoriality is introduced in Section 7.2. We record the following version of [74, Proposition 2.4]. The proof of [74, Proposition 2.4] only relies on arguments involving Mihlin's multiplier theorem (for operator-valued symbols). Since Mihlin's multiplier theorem is also available on spaces with Muckenhoupt weights, the results can be extended. Furthermore, in Proposition 8.5.9 we will provide an extension of Proposition 8.4.1 concerning higher-order regularity.

Proposition 8.4.1 (Weighted version of [74, Proposition 2.4])**.** *Let $p\in(1,\infty)$, $\gamma\in(-1,p-1)$ and $\rho>0$. Let A be the realisation of the operator in* (8.4.2) *on $\mathbb{X}$ with domain $D(A):=\mathbb{D}$. Then, for all $\lambda>0$, the operator $\lambda+A$ is R-sectorial with angle $\omega_R(\lambda+A)=\vartheta(\rho)$.*

Proof. The proof is similar to that in [74, Lemma 2.1, Propositions 2.2 & 2.4] using weighted versions of Mihlin's multiplier theorem. For scalar-valued symbols, a weighted version of Mihlin's multiplier theorem can be found in, e.g., [214, Proposition 3.1]. For operator-valued symbols one has to apply [91, Theorem 5.12(b)] instead of [102, Corollary 3.3] to obtain R-boundedness. □

In the rest of this section, we consider the operator A on the half-space. First, in Section 8.4.1 we establish R-sectoriality of A on spaces with Muckenhoupt power weights. Subsequently, in Section 8.4.2, we lift the R-sectoriality to higher-order Sobolev spaces with power weights that do not belong to the class of Muckenhoupt weights.

8.4.1. R-sectoriality on spaces with Muckenhoupt power weights

For $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$, we define the Banach spaces

$$\mathbb{X}_+ := W_0^{2,p}(\mathbb{R}^d_+, w_\gamma) \times L^p(\mathbb{R}^d_+, w_\gamma),$$
$$\mathbb{D}_+ := \big(W^{4,p}(\mathbb{R}^d_+, w_\gamma) \cap W_0^{2,p}(\mathbb{R}^d_+, w_\gamma)\big) \times W_0^{2,p}(\mathbb{R}^d_+, w_\gamma).$$

Similar to [74, Theorem 4.4] we obtain R-sectoriality of A_η on spaces with Muckenhoupt power weights.

Theorem 8.4.2 (Weighted version of [74, Theorem 4.4])**.** *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, $\eta \geq 0$, $\rho > 0$ and $\sigma \in (\vartheta(\rho), \pi)$. Let A_η be the realisation of the operator in* (8.4.3) *on $\mathbb{X}_+$ with domain $D(A_\eta) := \mathbb{D}_+$. Then there exists a $\widetilde{\lambda} \geq 0$ such that for all $\lambda > \widetilde{\lambda}$, the operator $\lambda + A_\eta$ is R-sectorial with angle $\omega_R(\lambda + A_\eta) \leq \sigma$.*

Proof. It suffices to consider $\eta = 0$, since a lower-order perturbation result [126, Corollary 16.2.5] yields the case $\eta > 0$. For $\eta = 0$ the result follows from adapting the proof of [74, Theorem 4.4] to include weights. We briefly sketch how [74, Theorem 4.4] can be extended to the weighted setting. First, note that the representation formulas for the solution in [74, Lemmas 3.1 & 3.2] do not depend on the underlying space. Then a weighted version of [74, Proposition 3.3] can be proved using a weighted version of [67, Proposition 4.12] and verifying that the integral operator related to the kernel $(x_1 + y_1)^{-1}$ is bounded in $L^p(\mathbb{R}_+, w_\gamma)$ for $\gamma \in (-1, p-1)$. The proof of [67, Proposition 4.12] directly carries over to the weighted setting, and the boundedness of the integral operator follows from [17, Theorem 1], where the condition on the weight functions in this theorem is satisfied for power weights exactly for $\gamma \in (-1, p-1)$. Moreover, [74, Lemma 4.3] also holds on $L^p(\mathbb{R}^d_+, w_\gamma)$. This provides all the ingredients for the proof of [74, Theorem 4.4] including weights. □

The zero boundary conditions in the base space $\mathbb{X}_+$ are *necessary* for the result of Theorem 8.4.2 to be true, see [66] and [74, Proposition 3.4]. The existence of the zeroth and first-order trace of functions in $W^{2,p}(\mathbb{R}^d_+, w_\gamma)$ is problematic here. We aim to obtain R-sectoriality for A on spaces *without* the unnatural boundary conditions. The results in Chapters 5 and 6 indicate that this can be achieved on higher-order spaces with suitable power weights. The main advantage of these weighted spaces is that fewer traces exist than in the unweighted case. In particular, no traces exist for functions in the weighted

space $W^{2,p}(\mathbb{R}^d_+, w_{\gamma+2p})$ with $\gamma \in (-1, p-1)$. This allows us to obtain R-sectoriality for A on $W^{2,p}(\mathbb{R}^d_+, w_{\gamma+2p}) \times L^p(\mathbb{R}^d_+, w_{\gamma+2p})$ in Section 8.4.2.

To be able to deal with spaces with more general power weights outside the Muckenhoupt class, we first study R-sectoriality of A on spaces with negative smoothness in Theorem 8.4.7. This is done by extrapolating the R-sectoriality result for A on $W^{2,p}_0(\mathbb{R}^d_+, w_\gamma) \times L^p(\mathbb{R}^d_+, w_\gamma)$ to the space $L^p(\mathbb{R}^d_+, w_\gamma) \times W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$, similar to [70, Section 3] where the case $p = 2$ is considered for bounded domains.

For proving Theorem 8.4.7 we need some preliminaries. We start off with the study of Δ^2 in the strong and weak setting, where we have the following result that can be proved similarly to [67, Theorem 7.4].

Theorem 8.4.3. *Let $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$. Then Δ^2 on $L^p(\mathbb{R}^d_+, w_\gamma)$ with domain $D(\Delta^2) := W^{4,p}(\mathbb{R}^d_+, w_\gamma) \cap W^{2,p}_0(\mathbb{R}^d_+, w_\gamma)$ has a bounded H^∞-calculus with angle $\omega_{H^\infty}(\Delta^2) = 0$.*

Using the interpolation-extrapolation scale (Section 8.2), we can obtain results for Δ^2 in the weak setting. For this, we first determine the adjoint operator of Δ^2 on $L^p(\mathbb{R}^d_+, w_\gamma)$. We recall that we use the notation $\gamma' = -\frac{\gamma}{p-1}$.

Lemma 8.4.4. *Let $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$. Let $B_{p,\gamma} := \Delta^2$ on $L^p(\mathbb{R}^d_+, w_\gamma)$ with domain $D(B_{p,\gamma}) := W^{4,p}(\mathbb{R}^d_+, w_\gamma) \cap W^{2,p}_0(\mathbb{R}^d_+, w_\gamma)$. Then the adjoint operator is $B'_{p,\gamma} = B_{p',\gamma'}$.*

Proof. By integration by parts we obtain

$$\langle \Delta^2 f, g\rangle_{L^p(\mathbb{R}^d_+, w_\gamma)\times L^{p'}(\mathbb{R}^d_+, w_{\gamma'})} = \langle f, \Delta^2 g\rangle_{L^p(\mathbb{R}^d_+, w_\gamma)\times L^{p'}(\mathbb{R}^d_+, w_{\gamma'})}$$

for $f \in D(B_{p,\gamma})$ and $g \in D(B_{p',\gamma'})$. In particular, by definition of the domain of the adjoint (see for instance [138, Section III.5.5]) we obtain $B_{p,\gamma} \subseteq B'_{p',\gamma'}$ and $B_{p',\gamma'} \subseteq B'_{p,\gamma}$. For the other inclusion $B'_{p,\gamma} \subseteq B_{p',\gamma'}$ it suffices to show that $B_{p',\gamma'}$ is surjective and $B'_{p,\gamma}$ is injective, see [88, Exercise IV.1.21(5)]. The surjectivity of $B_{p',\gamma'}$ follows from Theorem 8.4.3. By reflexivity and Theorem 8.4.3 it also follows that $B''_{p,\gamma} = B_{p,\gamma}$ is surjective, hence $B'_{p',\gamma'}$ is injective. □

For Δ^2 in the weak setting, we now obtain the following regularity result.

Proposition 8.4.5. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, and $\eta > 0$. Then $\eta + \Delta^2$ on $W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$ with domain $D(\Delta^2) := W^{2,p}_0(\mathbb{R}^d_+, w_\gamma)$ has a bounded H^∞-calculus with angle $\omega_{H^\infty}(\eta+\Delta^2) = 0$.*

Proof. Let $B_{p,\gamma} = \Delta^2$ on $L^p(\mathbb{R}^d_+, w_\gamma)$ be as in Lemma 8.4.4 and let $B'_{p,\gamma} = B_{p',\gamma'}$ be its adjoint. Note that the bounded H^∞-calculus (Theorem 8.4.3) implies that $B_{p',\gamma'}$ on $L^{p'}(\mathbb{R}^d_+, w_{\gamma'})$ has bounded imaginary powers. Therefore, by [126, Corollary 15.3.10] and Lemma 8.4.4 we obtain

$$\begin{aligned} D((B'_{p,\gamma})^{\frac{1}{2}}) &= \big[L^{p'}(\mathbb{R}^d_+, w_{\gamma'}), D(B'_{p,\gamma})\big]_{\frac{1}{2}} \\ &= \big[L^{p'}(\mathbb{R}^d_+, w_{\gamma'}), W^{4,p'}(\mathbb{R}^d_+, w_{\gamma'}) \cap W^{2,p'}_0(\mathbb{R}^d_+, w_{\gamma'})\big]_{\frac{1}{2}} = W^{2,p'}_0(\mathbb{R}^d_+, w_{\gamma'}), \end{aligned} \tag{8.4.4}$$

where the last identification follows from Theorem 3.6.5 (see [251] for the unweighted case). Using the interpolation-extrapolation scale (see Proposition 8.2.3), (8.4.4) and duality, we obtain

$$E_{-\frac{1}{2},\eta+B_{p,\gamma}} = (E_{\frac{1}{2},(\eta+B_{p,\gamma})'})' = \big(D((B'_{p,\gamma})^{\frac{1}{2}})\big)' = W^{-2,p}(\mathbb{R}^d_+, w_\gamma). \tag{8.4.5}$$

From Theorem 8.4.3, Propositions 8.2.1 and 8.2.2, and (8.4.5), it follows now that $\eta+\Delta^2$ on $W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$ has a bounded H^∞-calculus with angle zero. Moreover, (8.4.4) (with p' and γ' replaced by p and γ, respectively) shows that the domain of $\eta+\Delta^2$ on $W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$ is given by $W^{2,p}_0(\mathbb{R}^d_+, w_\gamma)$. □

With the above results for Δ^2 at hand, we can now study the more sophisticated operator A_η from (8.4.3) for $\eta > 0$. Again, we make use of the interpolation-extrapolation scales to obtain properties of A_η on spaces with less smoothness than those in Theorem 8.4.2. We first determine the adjoint of A_η.

Lemma 8.4.6. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, $\eta > 0$ and $\rho > 0$. Let A_η be the realisation of the operator in* (8.4.3) *on $\mathbb{X}_+$ with domain $D(A_\eta) := \mathbb{D}_+$. Then the adjoint operator A'_η on $W^{-2,p'}(\mathbb{R}^d_+, w_{\gamma'}) \times L^{p'}(\mathbb{R}^d_+, w_{\gamma'})$ is given by*

$$A'_\eta = \begin{pmatrix} 0 & \eta+\Delta^2 \\ -\mathrm{id} & -\rho\Delta \end{pmatrix} \quad \text{with} \quad D(A'_\eta) = L^{p'}(\mathbb{R}^d_+, w_{\gamma'}) \times W^{2,p'}_0(\mathbb{R}^d_+, w_{\gamma'}).$$

Proof. For notational convenience we write $W^{k,p}_{(0)}(w_\gamma) := W^{k,p}_{(0)}(\mathbb{R}^d_+, w_\gamma)$ for $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and $k \in \mathbb{Z}$. Let $v = (v_1, v_2) \in L^{p'}(w_{\gamma'}) \times W^{2,p'}_0(w_{\gamma'}) \subseteq W^{-2,p'}(w_{\gamma'}) \times L^{p'}(w_{\gamma'})$ and $u = (u_1, u_2) \in D(A_\eta) = (W^{4,p}(w_\gamma) \cap W^{2,p}_0(w_\gamma)) \times W^{2,p}_0(w_\gamma)$. By integration by parts and the definition of the distributional derivative, we obtain

$$\begin{aligned}
\langle A_\eta u, v\rangle &= \langle -u_2, v_1\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} + \langle \eta u_1, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} + \langle \Delta^2 u_1, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} \\
&\quad + \langle -\rho\Delta u_2, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} \\
&= \langle u_2, -v_1\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} + \langle u_1, \eta v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} + \langle u_1, \Delta^2 v_2\rangle_{W^{2,p}_0(w_\gamma)\times W^{-2,p'}(w_{\gamma'})} \\
&\quad + \langle u_2, -\rho\Delta v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} \\
&= \langle u, \widetilde{A}_\eta v\rangle,
\end{aligned}$$

where

$$\widetilde{A}_\eta := \begin{pmatrix} 0 & \eta+\Delta^2 \\ -\mathrm{id} & -\rho\Delta \end{pmatrix} \quad \text{with} \quad D(\widetilde{A}_\eta) := L^{p'}(w_{\gamma'}) \times W^{2,p'}_0(w_{\gamma'}).$$

For $v \in D(\widetilde{A}_\eta)$ it holds that the mapping $(u \mapsto \langle A_\eta u, v\rangle) : D(A_\eta) \to \mathbb{C}$ is continuous with respect to the norm on $W^{2,p}_0(w_\gamma) \times L^p(w_\gamma)$. Hence, we have that $\widetilde{A}_\eta \subseteq A'_\eta$.

For the converse inclusion, we need to show that if $v \in D(A'_\eta)$, then $v \in D(\widetilde{A}_\eta)$. First, note that if $v \in W^{-2,p'}(w_{\gamma'}) \times L^{p'}(w_{\gamma'})$ and $u \in D(A_\eta)$, then we have

$$\begin{aligned}
\langle A_\eta u, v\rangle = \langle -u_2, v_1\rangle_{W^{2,p}_0(w_\gamma)\times W^{-2,p'}(w_{\gamma'})} &+ \langle (\eta+\Delta^2) u_1, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} \\
&+ \langle -\rho\Delta u_2, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})}.
\end{aligned} \tag{8.4.6}$$

Now, let $v \in D(A_\eta')$. Note that the mapping $(u \mapsto \langle A_\eta u, v\rangle) : D(A_\eta) \to \mathbb{C}$ can be extended to a linear and continuous mapping from $W_0^{2,p}(w_\gamma) \times L^p(w_\gamma)$ to $\mathbb{C}$. Moreover, the mapping

$$\varphi : W^{4,p}(w_\gamma) \cap W_0^{2,p}(w_\gamma) \to \mathbb{C}, \quad u_1 \mapsto \varphi(u_1) := \langle (\eta + \Delta^2) u_1, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} \tag{8.4.7}$$

is continuous with respect to the norm on $W^{2,p}(w_\gamma)$. Indeed, this follows from

$$\begin{aligned}\left|\langle (\eta + \Delta^2) u_1, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})}\right| &= |\langle A_\eta (u_1, 0), v\rangle| \\ &\le C \|(u_1, 0)\|_{W^{2,p}(w_\gamma)\times L^p(w_\gamma)} = C\|u_1\|_{W^{2,p}(w_\gamma)},\end{aligned}$$

for $u_1 \in W^{4,p}(w_\gamma) \times W_0^{2,p}(w_\gamma)$. Furthermore, from Proposition 8.4.5 and $\eta > 0$, we obtain that

$$\eta + \Delta^2 : W_0^{2,p}(w_\gamma) \to W^{-2,p}(w_\gamma) \tag{8.4.8}$$

is an isomorphism. Combining (8.4.7) and (8.4.8) gives

$$\left(\widetilde{u}_1 \mapsto \varphi\left((\eta + \Delta^2)^{-1}\widetilde{u}_1\right) = \langle \widetilde{u}_1, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})}\right) : L^p(w_\gamma) \to \mathbb{C}$$

is continuous as mapping from $(L^p(w_\gamma), \|\cdot\|_{W^{-2,p}(w_\gamma)})$ to $\mathbb{C}$. Since $L^p(w_\gamma)$ is dense in $W^{-2,p}(w_\gamma)$, there exists a unique continuous extension $\widetilde{\varphi} : W^{-2,p}(w_\gamma) \to \mathbb{C}$ which we identify with $\widetilde{\varphi} \in (W^{-2,p}(w_\gamma))' = W_0^{2,p'}(w_{\gamma'})$ and it holds that

$$\langle \widetilde{u}_1, \widetilde{\varphi}\rangle_{W^{-2,p}(w_\gamma)\times W_0^{2,p'}(w_{\gamma'})} = \langle \widetilde{u}_1, v_2\rangle_{W^{-2,p}(w_\gamma)\times W_0^{2,p'}(w_\gamma')}$$

for $\widetilde{u}_1 \in L^p(w_\gamma)$. We conclude that $v_2 = \widetilde{\varphi} \in W_0^{2,p'}(w_{\gamma'})$ and this implies that the last term on the right-hand side of (8.4.6), given by

$$\left(u_2 \mapsto \langle -\rho\Delta u_2, v_2\rangle_{L^p(w_\gamma)\times L^{p'}(w_{\gamma'})} = \langle -\rho\Delta u_2, v_2\rangle_{W^{-2,p}(w_\gamma)\times W_0^{2,p'}(w_{\gamma'})}\right) : W_0^{2,p}(w_\gamma) \to \mathbb{C},$$

is continuous on $L^p(w_\gamma)$. Since (8.4.6) needs to be continuous, by setting $u_1 = 0$ it follows that also the first term on the right-hand side of (8.4.6), given by

$$\left(u_2 \mapsto \langle u_2, v_1\rangle_{W_0^{2,p}(w_\gamma)\times W^{-2,p'}(w_{\gamma'})}\right) : W_0^{2,p}(w_\gamma) \to \mathbb{C}$$

can be extended continuously to $L^p(w_\gamma)$. This implies that $v_1 \in L^{p'}(w_{\gamma'})$ and thus we have $v \in D(\widetilde{A}_\eta)$. Therefore, $D(A_\eta') = D(\widetilde{A}_\eta)$ and $A_\eta' = \widetilde{A}_\eta$, which completes the proof. □

The above lemma allows us to obtain R-sectoriality for A_η on $L^p(\mathbb{R}^d_+, w_\gamma) \times W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$.

Theorem 8.4.7. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ $\eta \ge 0$, $\rho > 0$ and $\sigma \in (\vartheta(\rho), \pi)$. Let A_η be the realisation of the operator in* (8.4.3) *on $L^p(\mathbb{R}^d_+, w_\gamma) \times W^{-2,p}(\mathbb{R}^d_+, w_\gamma)$ with domain $D(A_\eta) := W_0^{2,p}(\mathbb{R}^d_+, w_\gamma) \times L^p(\mathbb{R}^d_+, w_\gamma)$. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda > \widetilde{\lambda}$, the operator $\lambda + A_\eta$ is R-sectorial with angle $\omega_R(\lambda + A_\eta) \le \sigma$.*

Proof. First consider $\eta > 0$. Take $\widetilde{\lambda}$ from Theorem 8.4.2 and let $\lambda > \widetilde{\lambda}$. From Theorem 8.4.2 and Proposition 8.2.1 we obtain R-sectoriality on the extrapolation space $E_{-1,\lambda+A_\eta}$, which by Proposition 8.2.3 and Lemma 8.4.6 (using $\eta > 0$) is

$$\begin{aligned} E_{-1,\lambda+A_\eta} &= (E_{1,(\lambda+A_\eta)'})' = \big(D(A'_\eta)\big)' \\ &= \big(L^{p'}(\mathbb{R}^d_+, w_{\gamma'}) \times W^{2,p'}_0(\mathbb{R}^d_+, w_{\gamma'})\big)' = L^p(\mathbb{R}^d_+, w_\gamma) \times W^{-2,p}(\mathbb{R}^d_+, w_\gamma). \end{aligned}$$

Again, by Proposition 8.2.1 the domain is given by $\mathbb{X}_+ = W^{2,p}_0(\mathbb{R}^d_+, w_\gamma) \times L^p(\mathbb{R}^d_+, w_\gamma)$.

The case $\eta = 0$ can be obtained with the lower-order perturbation result [126, Corollary 16.2.5] for R-sectoriality upon taking $\widetilde{\lambda}$ larger. □

8.4.2. R-sectoriality on spaces with more general power weights

In this section, we study the R-sectoriality of A on higher-order spaces with power weights that do not belong to the class of Muckenhoupt weights. For fixed $p \in (1,\infty)$ and $\gamma \in (-1, p-1)$, we define for $k \in \mathbb{N}_0$ the Banach spaces

$$\begin{aligned} \mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+) &:= W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}) \times W^{k-2,p}(\mathbb{R}^d_+, w_{\gamma+kp}), \\ \mathbb{D}^{k,p}_\gamma(\mathbb{R}^d_+) &:= W^{k+2,p}_0(\mathbb{R}^d_+, w_{\gamma+kp}) \times W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp}). \end{aligned} \tag{8.4.9}$$

In particular, note that no traces exist of functions in $\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d)$, see (8.3.1). Furthermore, only the zeroth- and first-order trace exist of functions in $\mathbb{D}^{k,p}_\gamma(\mathbb{R}^d_+)$, modelling the correct Dirichlet–Neumann boundary conditions of (8.1.1).

Theorem 8.4.8. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ $k \in \mathbb{N}_0$, $\rho > 0$ and $\sigma \in (\vartheta(\rho), \pi)$. Let A be the realisation of the operator in* (8.4.2) *on $\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+)$ with domain $D(A) := \mathbb{D}^{k,p}_\gamma(\mathbb{R}^d_+)$. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda_0 > \widetilde{\lambda}$, the operator $\lambda_0 + A$ is R-sectorial with angle $\omega_R(\lambda_0 + A) \le \sigma$.*

Before we prove Theorem 8.4.8, we collect a few preliminary results. We start with a density result.

Lemma 8.4.9. *Let $p \in (1,\infty)$, $k \in \mathbb{N}_1$, $\gamma \in (-1,\infty) \setminus \{jp - 1 : j \in \mathbb{N}_1\}$ and let X be a reflexive Banach space. Then $C^\infty_c(\mathbb{R}^d_+; X)$ is dense in $W^{-k,p}(\mathbb{R}^d_+, w_\gamma; X)$.*

Proof. From [190, Proposition 3.8] we have

$$C^\infty_c(\mathbb{R}^d_+; X') \overset{\text{dense}}{\hookrightarrow} W^{k,p'}_0(\mathbb{R}^d_+, w_{\gamma'}; X') \quad \text{and} \quad C^\infty_c(\mathbb{R}^d_+; X') \overset{\text{dense}}{\hookrightarrow} L^{p'}(\mathbb{R}^d_+, w_{\gamma'}; X'),$$

where we recall that $\gamma' = -\frac{\gamma}{p-1}$. Hence, $W^{k,p'}_0(\mathbb{R}^d_+, w_{\gamma'}; X')$ is dense in $L^{p'}(\mathbb{R}^d_+, w_{\gamma'}; X')$ as well. Since X is reflexive, it also holds that $W^{k,p'}_0(\mathbb{R}^d_+, w_{\gamma'}; X')$ is reflexive. Hence, it follows that

$$L^p(\mathbb{R}^d_+, w_\gamma; X) \overset{\text{dense}}{\hookrightarrow} \big(W^{k,p'}_0(\mathbb{R}^d_+, w_{\gamma'}; X')\big)' = W^{-k,p}(\mathbb{R}^d_+, w_\gamma; X),$$

and since $C^\infty_c(\mathbb{R}^d_+; X)$ is dense in $L^p(\mathbb{R}^d_+, w_\gamma; X)$, the result follows. □

We continue with some estimates on functions in weighted Sobolev spaces.

Lemma 8.4.10. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and $k \in \mathbb{N}_0$. Then, it holds that*

$$\|f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})} \le C\|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})}, \quad f \in W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}),$$
$$\|f\|_{W^{k-2,p}(\mathbb{R}^d_+, w_{\gamma+kp})} \le C\|f\|_{W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})}, \quad f \in W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}).$$

Proof. We only consider the second estimate for $k \in \{0,1\}$. All the other estimates follow immediately from Hardy's inequality (Lemma 2.2.2). For $k \in \{0,1\}$ it suffices by duality to prove

$$\|g\|_{W_0^{1-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'-p'})} \le C\|g\|_{W_0^{2-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'})}, \quad g \in W_0^{2-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'}),$$

which again follows from Hardy's inequality noting that $\gamma' = -\frac{\gamma}{p-1} \in (-1, p'-1)$. □

Let $\theta \in \mathbb{R}$ and define the pointwise multiplication operator

$$M^\theta : C_c^\infty(\mathbb{R}^d_+; X) \to C_c^\infty(\mathbb{R}^d_+; X) \quad \text{by} \quad M^\theta u(x) = x_1^\theta \cdot u(x), \qquad x \in \mathbb{R}^d_+.$$

By duality the operator extends to $M^\theta : \mathscr{D}'(\mathbb{R}^d_+; X) \to \mathscr{D}'(\mathbb{R}^d_+; X)$, and $M^{-\theta}$ acts as inverse for M^θ on $\mathscr{D}'(\mathbb{R}^d_+; X)$. Moreover, we write $M := M^1$. We have the next estimates involving the multiplication operator M on weighted Sobolev spaces.

Lemma 8.4.11. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$ and $k \in \mathbb{N}_0$. Then, it holds that*

$$\sum_{|\alpha|\le 1} \|M\partial^\alpha f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})} \eqsim \|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})}, \quad f \in W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}),$$
$$\sum_{|\alpha|\le 1} \|M\partial^\alpha f\|_{W_0^{k+2,p}(\mathbb{R}^d_+, w_{\gamma+kp})} \eqsim \|f\|_{W_0^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})}, \quad f \in W_0^{k+3,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}).$$

Moreover, for $|\alpha| \le 1$, we have the estimate

$$\|M\partial^\alpha f\|_{W^{k-2,p}(\mathbb{R}^d_+, w_{\gamma+kp})} \le C\|f\|_{W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})}, \quad f \in W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}).$$

Proof. Note that by Lemma 5.3.5 it follows that

$$M : W_0^{\ell,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}) \to W_0^{\ell,p}(\mathbb{R}^d_+, w_{\gamma+kp})$$

is an isomorphism for all $\ell \in \mathbb{N}_0$. Moreover, if $\ell = k$, then $M : W^{k,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}) \to W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ is an isomorphism since in this case no traces exist. Hence, we have

$$\|f\|_{W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})} \eqsim \sum_{|\alpha|\le 1} \|\partial^\alpha f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})} \eqsim \sum_{|\alpha|\le 1} \|M\partial^\alpha f\|_{W^{k,p}(\mathbb{R}^d_+, w_{\gamma+kp})},$$

for $f \in W^{k+1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})$. The second estimate follows similarly.

Using that $M : W^{\ell,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p}) \to W^{\ell,p}(\mathbb{R}^d_+, w_{\gamma+kp})$ is bounded for $\ell \in \mathbb{N}_0$ (Lemma 5.3.5), we obtain for $k \ge 2$ and $|\alpha| \le 1$ that

$$\|M\partial^\alpha f\|_{W^{k-2,p}(\mathbb{R}^d_+, w_{\gamma+kp})} \le C\|\partial^\alpha f\|_{W^{k-2,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})} \le C\|f\|_{W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})},$$

for $f \in W^{k-1,p}(\mathbb{R}^d_+, w_{\gamma+(k+1)p})$. It remains to prove the above estimate for $k \in \{0,1\}$. By duality, it suffices to prove

$$\|\partial^\alpha M g\|_{W_0^{1-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'-p'})} \le C\|g\|_{W_0^{2-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'})}, \quad g \in W_0^{2-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'}),$$

where by integration by parts $(-1)^{|\alpha|}\partial^\alpha M$ is the adjoint operator to $M\partial^\alpha$. Let $\alpha = (\alpha_1, \widetilde{\alpha}) \in \mathbb{N}_0 \times \mathbb{N}_0^{d-1}$ with $|\alpha| = 1$. Then, $\partial^\alpha M = M\partial^\alpha + \alpha_1$. Moreover, since

$$M : W_0^{\ell,p'}(\mathbb{R}^d_+, w_\mu) \to W_0^{\ell,p'}(\mathbb{R}^d_+, w_{\mu-p'})$$

is an isomorphism for all $\ell \in \mathbb{N}_0$ and $\mu \in \mathbb{R} \setminus \{jp' - 1 : j \in \mathbb{N}_1\}$, it follows that

$$\begin{aligned}
\|\partial^\alpha M g\|_{W_0^{1-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'-p'})} &\le \|M\partial^\alpha g\|_{W_0^{1-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'-p'})} + \alpha_1\|g\|_{W_0^{1-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'-p'})} \\
&\le C\big(\|\partial^\alpha g\|_{W_0^{1-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'})} + \alpha_1\|g\|_{W_0^{1-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'-p'})}\big) \\
&\le C\|g\|_{W_0^{2-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'})}, \quad g \in W_0^{2-k,p'}(\mathbb{R}^d_+, w_{\gamma'-kp'}),
\end{aligned}$$

where we have additionally applied Hardy's inequality (Lemma 2.2.2). □

Furthermore, in the proof of Theorem 8.4.8, we need an elliptic regularity result in the case that the right-hand side of the equation is smooth.

Lemma 8.4.12. *Let $\rho > 0$, $\sigma \in (\vartheta(\rho), \pi)$ and $\lambda_0 > \widetilde{\lambda}$ with $\widetilde{\lambda}$ as in Theorem 8.4.7. Then, for all $f \in C_c^\infty(\mathbb{R}^d_+) \times C_c^\infty(\mathbb{R}^d_+)$ and $\lambda \in \lambda_0 + \Sigma_{\pi-\sigma}$, there exists a unique $u = (u_1, u_2)^\top \in \mathscr{S}(\mathbb{R}^d_+) \times \mathscr{S}(\mathbb{R}^d_+)$ such that*

$$\lambda u + Au = f, \qquad u_1(0,\cdot) = (\partial_1 u_1)(0,\cdot) = 0. \tag{8.4.10}$$

Proof. We fix $\lambda > \widetilde{\lambda}$. Note that, in particular, we have $f = (f_1, f_2)^\top \in L^2(\mathbb{R}^d_+) \times W^{-2,2}(\mathbb{R}^d_+)$. Then, by Theorem 8.4.7, we know that for every $\lambda > \widetilde{\lambda}$, equation (8.4.10) has a unique solution $u = (u_1, u_2)^\top \in W_0^{2,2}(\mathbb{R}^d_+) \times L^2(\mathbb{R}^d_+)$. By the definition of A, this shows that u_1 is a solution of the scalar equation

$$\Delta^2 u_1 - \lambda\rho\Delta u_1 + \lambda^2 u_1 = f_2 + (\lambda - \rho\Delta) f_1, \qquad u_1(0,\cdot) = (\partial_1 u_1)(0,\cdot) = 0. \tag{8.4.11}$$

Let $L(D) := \Delta^2 - \lambda\rho\Delta + \lambda^2$. We extend $g := f_2 + (\lambda - \rho\Delta) f_1 \in C_c^\infty(\mathbb{R}^d_+)$ by zero to $\widetilde{g} \in C_c^\infty(\mathbb{R}^d)$ and consider the equation

$$L(D) v = \widetilde{g}$$

in $\mathbb{R}^d$. The operator $L(D)$ can be written as $\mathscr{F}^{-1} m \mathscr{F}$ with the symbol

$$m(\xi) := |\xi|^4 + \lambda\rho|\xi|^2 + \lambda^2, \qquad \xi \in \mathbb{R}^d.$$

Using $m(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d$, we obtain $v = P\widetilde{g} := \mathscr{F}^{-1}\frac{1}{m}\mathscr{F}\widetilde{g}$. As P is a pseudodifferential operator of order -4 (see, e.g., [1, Definition 3.1]), we obtain $v = P\widetilde{g} \in \mathscr{S}(\mathbb{R}^d)$ by [1, Theorem 3.6].

We use the ansatz $u_1 = v|_{\mathbb{R}^d_+} + w$, which yields for w the boundary value problem

$$L(D)w = 0, \quad (w, \partial_1 w)|_{\partial\mathbb{R}^d_+} = -(v, \partial_1 v)|_{\partial\mathbb{R}^d_+} \in \mathscr{S}(\mathbb{R}^{d-1}) \times \mathscr{S}(\mathbb{R}^{d-1}). \tag{8.4.12}$$

Such boundary value problems were studied in [123]. As the principal part of $L(D)$ is given by Δ^2, it is easily seen that the ellipticity conditions (E) and (LS) in [123, Section 2.5.2] are satisfied for any $\phi \in [0, \pi)$. By [123, Proposition 4.16], there exists a solution $w \in \mathscr{S}(\mathbb{R}^d_+)$ of (8.4.12), which is given by the application of a Poisson operator to the right-hand side.

By construction, $\widetilde{u}_1 := v|_{\mathbb{R}^d_+} + w \in \mathscr{S}(\mathbb{R}^d_+)$ solves (8.4.11). Setting $\widetilde{u}_2 = \lambda \widetilde{u}_1 - f_1 \in \mathscr{S}(\mathbb{R}^d_+)$ yields a solution $(\widetilde{u}_1, \widetilde{u}_2)^\top$ of (8.4.10). By unique solvability of (8.4.10), we obtain $u_j = \widetilde{u}_j \in \mathscr{S}(\mathbb{R}^d_+)$ for $j \in \{1,2\}$. □

Remark 8.4.13. Alternatively, the result of Lemma 8.4.12 can be derived from properties of the Boutet de Monvel calculus for pseudodifferential boundary value problems. The boundary value problem $\mathscr{A} := (L(D), (\mathrm{Tr}, \mathrm{Tr} \circ \partial_1))^\top$ belongs to the class $\mathscr{B}^{4,2}_{\mathrm{cl}}(\overline{\mathbb{R}^d_+})$ of classical operators in Boutet de Monvel's calculus of order 4 and type 2 in the sense of [249, Definition 5.1]. Therefore, by [249, Theorem 5.11], there exists a parametrix of the form $\mathscr{B} = (\widetilde{P}_+ + \widetilde{G}, \widetilde{K}) \in \mathscr{B}^{-4,0}_{\mathrm{cl}}(\overline{\mathbb{R}^d_+})$, i.e., $S := \mathscr{B}\mathscr{A} - \mathrm{id}$ is a regularizing Boutet de Monvel operator. Using (8.4.11) we write

$$u_1 = \mathscr{B}\mathscr{A} u_1 - S u_1 = (\widetilde{P}_+ + \widetilde{G})\big(f_2 + (\lambda - \rho\Delta) f_1\big) - S_1.$$

As $g := f_2 + (\lambda - \rho\Delta) f_1 \in C^\infty_{\mathrm{c}}(\mathbb{R}^d_+)$, we get $(\widetilde{P}_+ + \widetilde{G}) g \in \mathscr{S}(\mathbb{R}^d_+)$ by [249, Corollary 5.5]. Since $\mathscr{A}$ has constant coefficients it even belongs to the SG-calculus in the sense of [248, Definition 2.9], so we can even choose S to belong to the class of smoothing operators in the SG-calculus, see [248, Definition 2.18]. By the mapping properties of such operators [248, Theorem 4.5], we get $S u_1 \in \mathscr{S}(\mathbb{R}^d_+)$. Hence, $u_1 \in \mathscr{S}(\mathbb{R}^d_+)$ and $u_2 = \lambda u_1 - f_1 \in \mathscr{S}(\mathbb{R}^d_+)$.

We now turn to the proof of R-sectoriality of the operator A as stated in Theorem 8.4.8.

Proof of Theorem 8.4.8. For notational convenience we write $\mathbb{X}^{k,p}_\gamma := \mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+)$ and $\mathbb{D}^{k,p}_\gamma := \mathbb{D}^{k,p}_\gamma(\mathbb{R}^d_+)$ for $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, and $k \in \mathbb{Z}$. Moreover, for vectors we write $u = (u_1, u_2)$ and for sequences of vectors we use the notation $(u^{(n)})_{n=1}^N$ for some $N \in \mathbb{N}_1$.

Step 1: elliptic regularity. Let $\sigma > \vartheta(\rho)$ and let $\widetilde{\lambda}$ be as in Theorem 8.4.7. Let $\lambda_0 > \widetilde{\lambda}$ and $N \in \mathbb{N}_1$. We first prove the following elliptic regularity result: for all sequences $(\lambda_n)_{n=1}^N$ in $\lambda_0 + \Sigma_{\pi-\sigma}$ and $(f^{(n)})_{n=1}^N \in \ell^2_N(\mathbb{X}^{k,p}_\gamma)$, there exists a unique $(u^{(n)})_{n=1}^N \in \ell^2_N(\mathbb{D}^{k,p}_\gamma)$ such that

$$\lambda_n u^{(n)} + A u^{(n)} = f^{(n)}, \quad \mathrm{Tr}\, u_1^{(n)} = \mathrm{Tr}\, \partial_1 u_1^{(n)} = 0, \tag{8.4.13}$$

holds for each $n \in \{1, \dots, N\}$. Moreover, this solution satisfies the estimates

$$\Big\| \sum_{n=1}^N \varepsilon_n \lambda_n u^{(n)} \Big\|_{L^2(\Omega; \mathbb{X}^{k,p}_\gamma)} \lesssim \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega; \mathbb{X}^{k,p}_\gamma)}, \tag{8.4.14a}$$

$$\Big\| \sum_{n=1}^N \varepsilon_n u^{(n)} \Big\|_{L^2(\Omega; \mathbb{D}^{k,p}_\gamma)} \lesssim \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega; \mathbb{X}^{k,p}_\gamma)}. \tag{8.4.14b}$$

The case $k=0$ follows from Theorem 8.4.7. For $k\in\mathbb{N}_1$, the uniqueness of $(u^{(n)})_{n=1}^N$ in the above statement follows from the uniqueness in $\mathbb{X}_\gamma^{0,p}$ and Lemma 8.4.10. The proof of the existence and (8.4.14) goes by induction on $k\geq 0$. Now assume that (8.4.13) and (8.4.14) hold for some $k\in\mathbb{N}_0$. It remains to show existence of a solution to (8.4.13) and the estimates (8.4.14) for $k+1$.

Fix $N\in\mathbb{N}_1$ and let $n\in\{1,\dots,N\}$. Moreover, let $(f^{(n)})_{n=1}^N\in\varepsilon_N^2(C_c^\infty(\mathbb{R}^d_+)\times C_c^\infty(\mathbb{R}^d_+))$ and consider the equations (8.4.13) with $\lambda_n\in\lambda_0+\Sigma_{\pi-\sigma}$. For each $f^{(n)}\in C_c^\infty(\mathbb{R}^d_+)\times C_c^\infty(\mathbb{R}^d_+)$, the unique solution $u^{(n)}$ to (8.4.13) is in $\mathscr{S}(\mathbb{R}^d_+)\times\mathscr{S}(\mathbb{R}^d_+)$ and $u_1^{(n)}(0,\cdot)=(\partial_1 u_1^{(n)})(0,\cdot)=0$. This follows from Lemma 8.4.12.

For $|\alpha|\leq 1$ we define $v^{(n)}:=M\partial^\alpha u^{(n)}$. We claim that for $n\in\{1,\dots,N\}$ and $|\alpha|\leq 1$, $v^{(n)}$ satisfies the equation

$$\lambda_n v^{(n)}+Av^{(n)}=M\partial^\alpha f^{(n)}+\begin{pmatrix}0\\ 4\partial_1\partial^\alpha\Delta u_1^{(n)}-2\rho\partial_1\partial^\alpha u_2^{(n)}\end{pmatrix}\tag{8.4.15}$$

and the boundary conditions $\operatorname{Tr}v_1^{(n)}=\operatorname{Tr}\partial_1 v_1^{(n)}=0$. Indeed, note that we have the commutation relations

$$\begin{aligned}[\Delta,M\partial^\alpha]g&=2\partial_1\partial^\alpha g,\\ [\Delta^2,M\partial^\alpha]g&=4\partial_1\partial^\alpha\Delta g,\end{aligned}$$

for $g\in\mathscr{S}(\mathbb{R}^d_+)$. Therefore, we obtain

$$\begin{aligned}\lambda_n M\partial^\alpha u^{(n)}+AM\partial^\alpha u^{(n)}&=\begin{pmatrix}M\partial^\alpha(\lambda_n u_1^{(n)}-u_2^{(n)})\\ M\partial^\alpha(\Delta^2 u_1^{(n)}+\lambda_n u_2^{(n)}-\rho\Delta u_2^{(n)})\end{pmatrix}\\ &\quad+\begin{pmatrix}0\\ 4\partial_1\partial^\alpha\Delta u_1^{(n)}-2\rho\partial_1\partial^\alpha u_2^{(n)}\end{pmatrix}\end{aligned}$$

and appealing to (8.4.13) yields (8.4.15). Moreover, note that $\operatorname{Tr}M\partial^\alpha u_1^{(n)}=0$ and

$$\operatorname{Tr}\partial_1 M\partial^\alpha u_1^{(n)}=\operatorname{Tr}\big(M\partial_1\partial^\alpha u_1^{(n)}+\partial^\alpha u_1^{(n)}\big)=\operatorname{Tr}\partial^\alpha u_1^{(n)}=0,$$

where the last identity follows from $\operatorname{Tr}u_1^{(n)}=\operatorname{Tr}\partial_1 u_1^{(n)}=0$ (if $\alpha=0$ or $\alpha=(1,0,\dots,0)$) and $\operatorname{Tr}\partial^{\widetilde{\alpha}}u_1^{(n)}=\partial^{\widetilde{\alpha}}\operatorname{Tr}u_1^{(n)}=0$ if $\alpha=(0,\widetilde{\alpha})\in\mathbb{N}_0\times\mathbb{N}_0^{d-1}$ with $|\widetilde{\alpha}|=1$. This proves the claim.

From the induction hypothesis (8.4.14b) applied to (8.4.15) and (8.4.13), we obtain for $|\alpha|\leq 1$ that

$$\begin{aligned}\Big\|\sum_{n=1}^N\varepsilon_n v^{(n)}\Big\|_{L^2(\Omega;\mathbb{D}_\gamma^{k,p})}&\lesssim\Big\|\sum_{n=1}^N\varepsilon_n M\partial^\alpha f^{(n)}\Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k,p})}+\Big\|\sum_{n=1}^N\varepsilon_n u_1^{(n)}\Big\|_{L^2(\Omega;W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp}))}\\ &\quad+\Big\|\sum_{n=1}^N\varepsilon_n u_2^{(n)}\Big\|_{L^2(\Omega;W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp}))}\\ &\lesssim\Big\|\sum_{n=1}^N\varepsilon_n M\partial^\alpha f^{(n)}\Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k,p})}+\Big\|\sum_{n=1}^N\varepsilon_n f^{(n)}\Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k,p})}\\ &\lesssim\Big\|\sum_{n=1}^N\varepsilon_n f^{(n)}\Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})},\end{aligned}$$

where in the last estimate we have used Lemmas 8.4.10 and 8.4.11. Using Lemma 8.4.11 once more gives

$$\Big\| \sum_{n=1}^N \varepsilon_n u^{(n)} \Big\|_{L^2(\Omega;\mathbb{D}_\gamma^{k+1,p})} \lesssim \sum_{|\alpha|\leq 1} \Big\| \sum_{n=1}^N \varepsilon_n v^{(n)} \Big\|_{L^2(\Omega;\mathbb{D}_\gamma^{k,p})} \lesssim \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})}. \quad (8.4.16)$$

The equation $\lambda_n u^{(n)} + A u^{(n)} = f^{(n)}$ and (8.4.16) yield

$$\begin{aligned}
\Big\| \sum_{n=1}^N \varepsilon_n \lambda_n u^{(n)} \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})} &\lesssim \Big\| \sum_{n=1}^N \varepsilon_n A u^{(n)} \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})} + \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})} \\
&\lesssim \Big\| \sum_{n=1}^N \varepsilon_n u^{(n)} \Big\|_{L^2(\Omega;\mathbb{D}_\gamma^{k+1,p})} + \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})} \\
&\lesssim \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})},
\end{aligned}$$

which proves the desired estimates for $(f^{(n)})_{n=1}^N \in \varepsilon_N^2(C_c^\infty(\mathbb{R}_+^d) \times C_c^\infty(\mathbb{R}_+^d))$.

The general case follows by a density argument. First, note that $C_c^\infty(\mathbb{R}_+^d) \times C_c^\infty(\mathbb{R}_+^d)$ is dense in $\mathbb{X}_\gamma^{k+1,p}$ by Lemma 8.4.9 and (8.3.2) for negative and positive smoothness, respectively. Let $(f^{(n)})_{n=1}^N \in \varepsilon_N^2(\mathbb{X}_\gamma^{k+1,p})$, then for every $n \in \{1,\dots,N\}$ there exists a sequence $(f^{(n,m)})_{m\geq 1} \subseteq C_c^\infty(\mathbb{R}_+^d) \times C_c^\infty(\mathbb{R}_+^d)$ such that $f^{(n,m)} \to f^{(n)}$ in $\mathbb{X}_\gamma^{k+1,p}$ as $m \to \infty$. Moreover, $(f^{(n,m)})_{n=1}^N \to (f^{(n)})_{n=1}^N$ in $\varepsilon_N^2(\mathbb{X}_\gamma^{k+1,p})$ as $m \to \infty$. Every $(f^{(n,m)})_{n=1}^N$ defines a $(u^{(n,m)})_{n=1}^N \in \varepsilon_N^2(\mathbb{D}_\gamma^{k+1,p})$ such that $\lambda_n u^{(n,m)} + A u^{(n,m)} = f^{(n,m)}$ for all $n \in \{1,\dots,N\}$ and $m \geq 1$. In addition, by (8.4.16) we have

$$\Big\| \sum_{n=1}^N \varepsilon_n (u^{(n,m_1)} - u^{(n,m_2)}) \Big\|_{L^2(\Omega;\mathbb{D}_\gamma^{k+1,p})} \lesssim \Big\| \sum_{n=1}^N \varepsilon_n (f^{(n,m_1)} - f^{(n,m_2)}) \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})} \to 0$$

as $m_1, m_2 \to \infty$. Completeness of $\varepsilon_N^2(\mathbb{D}_\gamma^{k+1,p})$ implies that there exists a $(u^{(n)})_{n=1}^N \in \varepsilon_N^2(\mathbb{D}_\gamma^{k+1,p})$ such that $(u^{(n,m)})_{n=1}^N \to (u^{(n)})_{n=1}^N$ in $\varepsilon_N^2(\mathbb{D}_\gamma^{k+1,p})$ as $m \to \infty$. Moreover, we have

$$\Big\| \sum_{n=1}^N \varepsilon_n (\lambda_n + A)(u^{(n,m)} - u^{(n)}) \Big\|_{L^2(\Omega;\mathbb{X}_\gamma^{k+1,p})} \lesssim \Big\| \sum_{n=1}^N \varepsilon_n (u^{(n,m)} - u^{(n)}) \Big\|_{L^2(\Omega;\mathbb{D}_\gamma^{k+1,p})} \to 0,$$

as $m \to \infty$. This implies that

$$\big((\lambda_n + A) u^{(n)}\big)_{n=1}^N = \lim_{m\to\infty} \big((\lambda_n + A) u^{(n,m)}\big)_{n=1}^N = \lim_{m\to\infty} (f^{(n,m)})_{n=1}^N = (f^{(n)})_{n=1}^N.$$

Thus, for any $(f^{(n)})_{n=1}^N \in \varepsilon_N^2(\mathbb{X}_\gamma^{k+1,p})$ there exists a $(u^{(n)})_{n=1}^N \in \varepsilon_N^2(\mathbb{D}_\gamma^{k+1,p})$ such that $\lambda_n u^{(n)} + A u^{(n)} = f^{(n)}$ for every $n \in \{1,\dots,N\}$ and the required estimates hold. This finishes the induction.

Step 2: R-sectoriality. To show that $\lambda_0 + A$ is R-sectorial, we prove that for $\sigma > \vartheta(\rho)$ the set

$$\{(\lambda - \lambda_0)(\lambda + A)^{-1} : \lambda \in \lambda_0 + \Sigma_{\pi-\sigma}\}$$

is R-bounded. Note that Step 1 implies that $\lambda_0 + A : \mathbb{D}_\gamma^{k,p} \to \mathbb{X}_\gamma^{k,p}$ is a closed and surjective operator. Hence, it has dense range, and [125, Propositions 10.1.7(3) and 10.1.9] imply dense domain and injectivity.

Fix $N \in \mathbb{N}_1$ and let $n \in \{1, \dots, N\}$. By the Kahane contraction principle [125, Theorem 6.1.13] and Step 1, we obtain for $\lambda_n \in \lambda_0 + \Sigma_{\pi-\sigma}$

$$\begin{aligned}
&\Big\| \sum_{n=1}^N \varepsilon_n (\lambda_n - \lambda_0)(\lambda_n + A)^{-1} f^{(n)} \Big\|_{L^2(\Omega; \mathbb{X}_\gamma^{k,p})} \\
&\qquad \lesssim \max_{1 \le n \le N} \big| \tfrac{\lambda_n - \lambda_0}{\lambda_n} \big| \Big\| \sum_{n=1}^N \varepsilon_n \lambda_n (\lambda_n + A)^{-1} f^{(n)} \Big\|_{L^2(\Omega; \mathbb{X}_\gamma^{k,p})} \\
&\qquad \lesssim \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega; \mathbb{X}_\gamma^{k,p})},
\end{aligned}$$

where we have used that

$$|\lambda_n - \lambda_0| \le \begin{cases} |\lambda_n| & \text{if } \sigma \in [\frac{\pi}{2}, \pi) \\ \frac{|\lambda_n|}{\sin(\sigma)} & \text{if } \sigma \in (\vartheta(\rho), \frac{\pi}{2}] \end{cases}, \qquad \text{for } \lambda_n \in \lambda_0 + \Sigma_{\pi-\sigma}.$$

This shows that $\lambda_0 + A$ is R-sectorial with angle $\omega_R(\lambda_0 + A) \le \sigma$. □

8.5. R-sectoriality for the elliptic operator on domains

In this section, we apply perturbation and localisation techniques to transfer R-sectoriality of A on the half-space to bounded domains. Localisation techniques are quite standard in the literature, see, e.g., [67, 89, 166]. In the weighted setting, one can use a more advanced localisation procedure based on the diffeomorphism in Lemma 2.6.5. In this way, we can deal with domains that have low regularity and cannot be treated by applying standard localisation techniques.

We will use a similar strategy as in Chapter 6, where the H^∞-calculus on weighted Sobolev spaces for the Laplacian on bounded C^1-domains is derived. For the convenience of the reader, we will include the details of the perturbation and localisation argument. We proceed in three steps.

1. Use R-sectoriality for $\lambda_0 + A$ on $\mathbb{R}^d_+$ (Theorem 8.4.8) and known perturbation theory for R-sectoriality to obtain R-sectoriality for $\lambda_0 + A$ on special domains with λ_0 large enough.

2. Perform a localisation procedure to transfer R-sectoriality for $\lambda_0 + A$ on special domains to bounded domains.

3. Use discreteness of the spectrum of A on bounded domains to obtain R-sectoriality for the unshifted operator A.

These three steps are carried out in the subsequent Sections 8.5.1, 8.5.2 and 8.5.3.

8.5.1. Perturbation of R-sectoriality to special domains

Let $p \in (1,\infty)$, $k \in \mathbb{N}_0$, and $\gamma \in (-1, p-1)$. We define the following spaces on domains

$$\begin{aligned} \mathbb{X}^{k,p}_\gamma(\mathscr{O}) &:= W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \times W^{k-2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}), \\ \mathbb{D}^{k,p}_\gamma(\mathscr{O}) &:= W^{k+2,p}_0(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \times W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}), \end{aligned} \tag{8.5.1}$$

where the space with zero boundary conditions is as in Section 8.3 if $\mathscr{O}$ is a special or bounded domain. Note that this definition for $\mathscr{O} = \mathbb{R}^d_+$ coincides with (8.4.9).

We start with the perturbation of the R-sectoriality of A to special domains.

Theorem 8.5.1. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$, $\gamma \in ((1-\kappa)p-1, p-1)$, $k \geq 2$ an integer, $\rho > 0$, $\sigma \in (\vartheta(\rho), \pi)$, and let $\mathscr{O}$ be a special $C^{1,\kappa}_{\mathrm{c}}$-domain with $[\mathscr{O}]_{C^{1,\kappa}} \leq 1$. Let A be the realisation of the operator in* (8.4.2) *on $\mathbb{X}^{k,p}_\gamma(\mathscr{O})$ with domain $D(A) := \mathbb{D}^{k,p}_\gamma(\mathscr{O})$. Then there exist $\widetilde{\lambda} > 0$ and $\delta \in (0,1)$ such that, if $[\mathscr{O}]_{C^{1,\kappa}} < \delta$, then for all $\lambda_0 > \widetilde{\lambda}$, the operator $\lambda_0 + A$ is R-sectorial with angle $\omega_R(\lambda_0 + A) \leq \sigma$.*

Remark 8.5.2. The conditions that $\mathscr{O}$ is a special $C^{1,\kappa}_{\mathrm{c}}$-domain and $\gamma > (1-\kappa)p-1$ are only required to obtain the domain characterisation $D(A) = \mathbb{D}^{k,p}_\gamma(\mathscr{O})$, see Proposition 8.3.4. In fact, the rest of the proof as given in this section also works for special C^1_{c}-domains and $\gamma \in (-1, p-1)$. Therefore, Theorem 8.5.1 remains valid for special C^1_{c}-domains and A with domain

$$D(A) := \mathring{W}^{k+2,p}_0(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \times W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}).$$

Indeed, using these spaces, Φ_* is an isomorphism without any additional conditions on the smoothness of the domain or weight exponent, see Proposition 8.3.2.

To relate the operator A in (8.4.2) on special domains and the half-space, let h_1, h_2 and Φ be as in Lemma 2.6.5 defining a diffeomorphism between a special $C^{1,\kappa}_{\mathrm{c}}$-domain and the half-space. Recall that $\Phi_* f := f \circ \Phi^{-1}$ for $f \in L^1_{\mathrm{loc}}(\mathscr{O})$ and define $A^\Phi : W^{4,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \times W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \to W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \times L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ by

$$A^\Phi := \Phi_* \circ A \circ (\Phi^{-1})_* = A + B, \tag{8.5.2}$$

where A is the operator on $\mathbb{R}^d_+$ and B is a perturbation term. Since A consists of the Laplacian and the bi-Laplacian, we introduce the coordinate changes $\Delta^\Phi : W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \to L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ and $(\Delta^2)^\Phi : W^{4,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \to L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$, which are given by

$$\Delta^\Phi := \Phi_* \circ \Delta \circ (\Phi^{-1})_* \quad \text{and} \quad (\Delta^2)^\Phi := \Phi_* \circ \Delta^2 \circ (\Phi^{-1})_*. \tag{8.5.3}$$

More generally, if C is a linear differential operator of order $N \in \mathbb{N}_1$, we can define the coordinate change $C^\Phi := \Phi_* \circ C \circ (\Phi^{-1})_*$. We will compute the perturbation terms corresponding to this coordinate change if C has constant coefficients. To this end, we introduce the following class of differential operators.

Definition 8.5.3. For $N \in \mathbb{N}_1$, we call a differential operator P in $\mathbb{R}^d_+$ an *operator of special form of order N* if it is a linear combination of terms of the form $p_1(y) \cdot \ldots \cdot p_\ell(y)\partial^\mu$ with $\mu \in \mathbb{N}_0^d$, where each $p_i(y)$ is of the form

$$p_i(y) = \partial^{\nu_i} h_1(\Phi^{-1}(y)) \quad \text{or} \quad p_i(y) = \partial^{\nu_i} h_2(y), \qquad i \in \{1,\ldots,\ell\},\ y \in \mathbb{R}^d_+,$$

with $\nu_i \in \mathbb{N}_0^d \setminus \{0\}$ for $i \in \{1,\ldots,\ell\}$. Here, $\ell \in \{1,\ldots,N\}$ and $|\mu| \in \{1,\ldots,N\}$ are such that $|\mu| + \sum_{i=1}^{\ell} |\nu_i| = \ell + N$.

The following lemma provides the necessary properties of differential operators of special form and the coordinate change C^Φ.

Lemma 8.5.4. *The following statements hold.*

(i) If C is a linear differential operator of order $N \in \mathbb{N}_1$ with constant coefficients, then $P := C^\Phi - C$ is of special form of order N, where additionally all coefficients p_i have the form $p_i = (\partial^{\nu_i} h_1) \circ \Phi^{-1}$.

(ii) Let $\widetilde{P}$ be a differential operator of special form of order $N \in \mathbb{N}_1$. Then $\partial^\delta \widetilde{P}$ is a differential operator of special form of order $N + |\delta|$ for all $\delta \in \mathbb{N}_0^d$.

Proof. Proof of (i). By Lemma 2.6.5, the Jacobian of Φ is given by

$$(D\Phi)(x) = \begin{pmatrix} 1 - \partial_1 h_1(x) & -\partial_2 h_1(x) & \ldots & -\partial_d h_1(x) \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}.$$

From this and the chain rule, we see that

$$\partial_k(u \circ \Phi) = (\partial_k u) \circ \Phi - (\partial_k h_1)\big((\partial_1 u) \circ \Phi\big), \qquad k \in \{1,\ldots,d\}, \tag{8.5.4}$$

for sufficiently smooth $u\colon \mathbb{R}^d_+ \to \mathbb{C}$. We show by induction on N that for every differential operator C of order N with constant coefficients, the difference $(C(u \circ \Phi))(x) - (Cu)(\Phi(x))$ is a linear combination of terms of the form

$$\Big(\prod_{i=1}^{\ell} (\partial^{\nu_i} h_1)(x)\Big)(\partial^\mu u)(\Phi(x)), \qquad x \in \mathcal{O}, \tag{8.5.5}$$

where ℓ, ν_i, μ are as in Definition 8.5.3. From this, the statement in (i) follows by taking $x = \Phi^{-1}(y)$.

For $N = 1$, this follows immediately from (8.5.4). Now, let C be a differential operator of order $N+1$. By linearity, we may assume that $C = \partial^\alpha$ with $\alpha \in \mathbb{N}_0^d$, $|\alpha| = N+1$. Writing $\partial^\alpha = \partial_k \partial^{\alpha'}$ with $k \in \{1,\ldots,d\}$ and $\alpha' \in \mathbb{N}_0^d$, $|\alpha'| = N$, we get by induction and (8.5.4) that

$$\begin{aligned} \partial^\alpha(u \circ \Phi) - (\partial^\alpha u) \circ \Phi &= \partial_k \partial^{\alpha'}(u \circ \Phi) - (\partial^\alpha u) \circ \Phi \\ &= \partial_k\big((\partial^{\alpha'} u) \circ \Phi + Bu\big) - (\partial^\alpha u) \circ \Phi \\ &= (\partial_k \partial^{\alpha'} u) \circ \Phi - (\partial_k h_1)((\partial_1 \partial^{\alpha'} u) \circ \Phi) + \partial_k Bu - (\partial^\alpha u) \circ \Phi \\ &= -(\partial_k h_1)((\partial_1 \partial^{\alpha'} u) \circ \Phi) + \partial_k Bu, \end{aligned} \tag{8.5.6}$$

where Bu is a linear combination of terms of the form (8.5.5). The first term in (8.5.6) is obviously of special form (8.5.5) of order $N+1$. For the second term, we use the product rule to see that $\partial_k Bu$ is a linear combination of terms of the form

$$\partial_k\Big(\prod_{i=1}^{\ell}\partial^{\nu_i}h_1\Big)\big((\partial^\mu u)\circ\Phi\big) = \sum_{j=1}^{\ell}\Big(\prod_{i\neq j}\partial^{\nu_i}h_1\Big)(\partial_k\partial^{\nu_j}h_1)\big((\partial^\mu u)\circ\Phi\big) + \Big(\prod_{i=1}^{\ell}\partial^{\nu_i}h_1\Big)\big((\partial_k\partial^\mu u)\circ\Phi\big) - \partial_k h_1\Big(\prod_{i=1}^{\ell}\partial^{\nu_i}h_1\Big)\big((\partial_1\partial^\mu u)\circ\Phi\big),$$

where we have again used (8.5.4). From this we see that all terms are of special form (8.5.5) of order $N+1$, which finishes the proof of (i).

Proof of (ii). In the same way as above, we see that

$$\partial_k(h_1\circ\Phi^{-1}) = (\partial_k h_1)\circ\Phi^{-1} + (\partial_k h_2)\big((\partial_1 h_1)\circ\Phi^{-1}\big), \qquad k\in\{1,\dots,d\}. \tag{8.5.7}$$

To show (ii), it is sufficient to consider $|\delta|=1$, i.e., $\partial^\delta=\partial_k$ for some $k\in\{1,\dots,d\}$. By the product rule, we have

$$\partial_k\Big(\prod_{i=1}^{\ell}\widetilde{p}_i\Big)\partial^\mu = \Big(\prod_{i=1}^{\ell}\widetilde{p}_i\Big)\partial_k\partial^\mu + \sum_{i=1}^{\ell}\Big(\prod_{j\neq i}\widetilde{p}_j\Big)(\partial_k\widetilde{p}_i)\partial^\mu.$$

Here, each term on the right-hand side is an operator of special form of order $N+1$. This is obvious if $\widetilde{p}_i=\partial^{\nu_i}h_2$ and follows from (8.5.7) if $\widetilde{p}_i=(\partial^{\nu_i}h_1)\circ\Phi^{-1}$. □

As a special case of Lemma 8.5.4 we obtain the perturbation terms arising from the change of coordinates corresponding to the (bi-)Laplacian and A in (8.5.2) and (8.5.3).

Corollary 8.5.5. *Let Δ^Φ and $(\Delta^2)^\Phi$ be as in* (8.5.3). *Then for $j\in\{1,2\}$ it holds that*

$$(\Delta^j)^\Phi = \Delta^j + B_j,$$

where B_j is an operator of special form of order $2j$. Moreover, for A^Φ as in (8.5.2), *we have*

$$A^\Phi = A+B, \quad \text{with} \quad B := \begin{pmatrix} 0 & 0 \\ B_2 & -\rho B_1 \end{pmatrix}.$$

Proof. The statements on $(\Delta^j)^\Phi$ follow immediately from Lemma 8.5.4(i). The perturbation term B for A^Φ simply follows from the structure of A in (8.4.2). □

To apply perturbation theorems for R-sectoriality to B, we provide the necessary estimates on these perturbations.

Lemma 8.5.6. *Let $p\in(1,\infty)$, $\kappa\in(0,1)$, $\gamma\in((1-\kappa)p-1,p-1)$, $k\geq 2$ an integer, $\rho>0$, and let $\mathscr{O}$ be a special $C_{\mathrm{c}}^{1,\kappa}$-domain with $[\mathscr{O}]_{C^{1,\kappa}}\leq 1$. Then the perturbation term B from Corollary 8.5.5 satisfies the estimate*

$$\|Bu\|_{\mathbb{X}_\gamma^{k,p}(\mathbb{R}^d_+)} \leq C\cdot[\mathscr{O}]_{C^{1,\kappa}}\cdot\|u\|_{\mathbb{D}_\gamma^{k,p}(\mathbb{R}^d_+)}, \quad u\in\mathbb{D}_\gamma^{k,p}(\mathbb{R}^d_+),$$

where the constant $C>0$ only depends on p,k,κ,γ,ρ and d.

Proof. Let $u=(u_1,u_2)\in \mathbb{D}_\gamma^{k,p}(\mathbb{R}^d_+)=W_0^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp})\times W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp})$. By Corollary 8.5.5, it suffices to prove the estimates

$$\|B_2u_1\|_{W^{k-2,p}(\mathbb{R}^d_+,w_{\gamma+kp})}\le C\cdot[\mathscr{O}]_{C^{1,\kappa}}\cdot\|u_1\|_{W^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp})},\qquad u_1\in W_0^{k+2,p}(\mathbb{R}^d_+,w_{\gamma+kp}),$$
$$\|B_1u_2\|_{W^{k-2,p}(\mathbb{R}^d_+,w_{\gamma+kp})}\le C\cdot[\mathscr{O}]_{C^{1,\kappa}}\cdot\|u_2\|_{W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp})},\qquad u_2\in W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp}).$$

Let $j\in\{1,2\}$ and set $v_j:=u_{3-j}\in W_0^{k-2+2j,p}(\mathbb{R}^d_+,w_{\gamma+kp})$, where it should be noted that for $j=1$ we have $v_j=u_2\in W^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp})=W_0^{k,p}(\mathbb{R}^d_+,w_{\gamma+kp})$ by (8.3.1). Thus, we have to show that

$$\|B_jv_j\|_{W^{k-2,p}(\mathbb{R}^d_+,w_{\gamma+kp})}\lesssim[\mathscr{O}]_{C^{1,\kappa}}\cdot\|v_j\|_{W^{k-2+2j,p}(\mathbb{R}^d_+,w_{\gamma+kp})},\tag{8.5.8}$$

for $v_j\in W_0^{k-2+2j,p}(\mathbb{R}^d_+,w_{\gamma+kp})$.

Let $\alpha\in\mathbb{N}_0^d$ with $|\alpha|\le k-2$. By Corollary 8.5.5, we know that B_j is an operator of special form of order $2j$. Moreover, by Lemma 8.5.4(ii), the operator $\partial^\alpha B_j$ is of special form of order $2j+|\alpha|$. More precisely, $\partial^\alpha B_jv_j$ is a linear combination of terms of the form $\prod_{i=1}^\ell b_i\partial^\mu u$, where $b_i=(\partial^{\nu_i}h_1)\circ\Phi^{-1}$ or $b_i=\partial^{\nu_i}h_2$ with $\ell\in\{1,\dots,2j+|\alpha|\}$, $\nu_i\in\mathbb{N}_0^d\setminus\{0\}$, $|\mu|\in\{1,\dots,2j+|\alpha|\}$ satisfying

$$|\mu|+\sum_{i=1}^\ell|\nu_i|=\ell+2j+|\alpha|.\tag{8.5.9}$$

To estimate a factor of the form $b_i=(\partial^{\nu_i}h_1)\circ\Phi^{-1}$, we apply Lemma 2.6.5(ii) and (iv) to obtain the (non-optimal) estimate

$$|(\partial^{\nu_i}h_1)(\Phi^{-1}(y))|\lesssim\frac{[\mathscr{O}]_{C^{1,\kappa}}}{\operatorname{dist}(\Phi^{-1}(y),\partial\mathscr{O})^{|\nu_i|-1}}\lesssim\frac{[\mathscr{O}]_{C^{1,\kappa}}}{y_1^{|\nu_i|-1}},\qquad y\in\mathbb{R}^d_+.\tag{8.5.10}$$

For factors of the form $b_i=\partial^{\nu_i}h_2$, we use the same lemma to get

$$|(\partial^{\nu_i}h_2)(y)|\lesssim\frac{[\mathscr{O}]_{C^{1,\kappa}}}{y_1^{|\nu_i|-1}},\qquad y\in\mathbb{R}^d_+.\tag{8.5.11}$$

For a term of the form $\prod_{i=1}^\ell b_i\partial^\mu v_j$, we obtain, using (8.5.10), (8.5.11), the fact that $[\mathscr{O}]_{C^{1,\kappa}}\le1$, and Hardy's inequality (Lemma 2.2.2 applied $\sum_{i=1}^\ell(|\nu_i|-1)$ times),

$$\begin{aligned}\Big\|\prod_{i=1}^\ell b_i\partial^\mu v_j\Big\|_{L^p(\mathbb{R}^d_+,w_{\gamma+kp})}&\lesssim[\mathscr{O}]_{C^{1,\kappa}}\cdot\|\partial^\mu v_j\|_{L^p(\mathbb{R}^d_+,w_{\gamma+kp-\sum_{i=1}^\ell(|\nu_i|-1)p})}\\&\lesssim[\mathscr{O}]_{C^{1,\kappa}}\cdot\|v_j\|_{W^{|\mu|+\sum_{i=1}^\ell(|\nu_i|-1),p}(\mathbb{R}^d_+,w_{\gamma+kp})}\\&\lesssim[\mathscr{O}]_{C^{1,\kappa}}\cdot\|v_j\|_{W^{k-2+2j,p}(\mathbb{R}^d_+,w_{\gamma+kp})},\end{aligned}$$

where in the last estimate we have used (8.5.9) and $|\alpha|\le k-2$ to obtain

$$|\mu|+\sum_{i=1}^\ell(|\nu_i|-1)=|\mu|-\ell+\sum_{i=1}^\ell|\nu_i|\le k-2+2j.$$

As $\partial^\alpha B_jv_j$ is a linear combination of such terms, this proves the desired estimate (8.5.8) and finishes the proof. □

With the estimates on the perturbation terms at hand, we can now prove Theorem 8.5.1.

Proof of Theorem 8.5.1. Let $\mathscr{O}$ be a special $C_c^{1,\kappa}$-domain and let h_1, h_2 and Φ be as in Lemma 2.6.5. Recall that we introduced $A^\Phi : W^{4,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \times W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \to W^{2,1}_{\mathrm{loc}}(\mathbb{R}^d_+) \times L^1_{\mathrm{loc}}(\mathbb{R}^d_+)$ given by

$$A^\Phi := \Phi_* \circ A \circ (\Phi^{-1})_* = A + B,$$

with B as in Corollary 8.5.5. Consider the realisation of A^Φ in $\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+)$ with domain $D(A^\Phi) := \mathbb{D}^{k,p}_\gamma(\mathbb{R}^d_+)$. Due to the isomorphisms in Proposition 8.3.2, the trace characterisation in Proposition 8.3.4 and standard properties of R-sectoriality, the desired statements in Theorem 8.5.1 for A on $\mathbb{X}^{k,p}_\gamma(\mathscr{O})$ are equivalent to the corresponding statements for A^Φ on $\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+)$. To prove that $\lambda_0 + A^\Phi$ on $\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+)$ is R-sectorial for λ_0 large enough and $[\mathscr{O}]_{C^{1,\kappa}}$ small enough, we apply the perturbation result [126, Theorem 16.2.4]. Indeed, by Lemma 8.5.6 and Theorem 8.4.8, we have for all $u \in \mathbb{D}^{k,p}_\gamma(\mathbb{R}^d_+)$ that

$$\|Bu\|_{\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+)} \lesssim [\mathscr{O}]_{C^{1,\kappa}} \cdot \|u\|_{\mathbb{D}^{k,p}_\gamma(\mathbb{R}^d_+)} \eqsim [\mathscr{O}]_{C^{1,\kappa}} \cdot \|(\lambda_0 + A)u\|_{\mathbb{X}^{k,p}_\gamma(\mathbb{R}^d_+)}.$$

Therefore, applying [126, Theorem 16.2.4] gives the desired result. □

8.5.2. Localisation of R-sectoriality to bounded domains

To obtain R-sectoriality on bounded domains, it remains to apply a localisation procedure based on lower-order perturbations.

Theorem 8.5.7. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$, $\gamma \in ((1-\kappa)p - 1, p-1)$, $k \geq 2$ an integer, $\rho > 0$, $\sigma \in (\vartheta(\rho), \pi)$ and let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Let A be the realisation of the operator in* (8.4.2) *on $\mathbb{X}^{k,p}_\gamma(\mathscr{O})$ with domain $D(A) := \mathbb{D}^{k,p}_\gamma(\mathscr{O})$. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda_0 > \widetilde{\lambda}$, the operator $\lambda_0 + A$ is R-sectorial with angle $\omega_R(\lambda_0 + A) \leq \sigma$.*

To transfer R-sectoriality from special domains to bounded domains, we use the following abstract lemma based on lower-order perturbations. A version of the lemma below for the H^∞-calculus can be found in [190, Lemma 6.11].

Lemma 8.5.8. *Let A be a linear operator on a Banach space Y, and let $\widetilde{A}$ be an R-sectorial operator on a Banach space $\widetilde{Y}$. Assume that there exist bounded linear mappings $\mathscr{I} \colon Y \to \widetilde{Y}$ and $\mathscr{P} \colon \widetilde{Y} \to Y$ satisfying*

(i) $\mathscr{P}\mathscr{I} = \mathrm{id}$,

(ii) $\mathscr{I} D(A) \subseteq D(\widetilde{A})$ *and* $\mathscr{P} D(\widetilde{A}) \subseteq D(A)$,

(iii) $(\mathscr{I} A - \widetilde{A}\mathscr{I})\mathscr{P} \colon D(\widetilde{A}) \to \widetilde{Y}$ *and* $\mathscr{I}(A\mathscr{P} - \mathscr{P}\widetilde{A}) \colon D(\widetilde{A}) \to \widetilde{Y}$ *extend to bounded linear operators $[\widetilde{Y}, D(\widetilde{A})]_\theta \to \widetilde{Y}$ for some $\theta \in (0,1)$.*

Then A is a closed and densely defined operator and for every $\sigma > \omega_R(\widetilde{A})$ there exists a $\lambda > 0$ such that $\lambda + A$ is R-sectorial with $\omega_R(\lambda + A) \leq \sigma$.

Proof. The proof is similar to the proof of [190, Lemma 6.11] if one uses the perturbation theorem [126, Corollary 16.2.5] for R-sectoriality. □

To apply the localisation procedure from Lemma 8.5.8, we use the decomposition of weighted Sobolev spaces as in Lemma 8.3.5. This requires the R-sectoriality of A on special domains (Theorem 8.5.1) and on the full space $\mathbb{R}^d$. From Proposition 8.4.1 and a lifting argument, we obtain higher-order regularity for A on $\mathbb{R}^d$.

Proposition 8.5.9. *Let $p \in (1,\infty)$, $\gamma \in (-1, p-1)$, $k \geq 2$ an integer, $\rho > 0$, $\sigma \in (\vartheta(\rho), \pi)$ and let A be the realisation of the operator in* (8.4.2) *on $W^{k,p}(\mathbb{R}^d, w_\gamma) \times W^{k-2,p}(\mathbb{R}^d, w_\gamma)$ with domain $D(A) := W^{k+2,p}(\mathbb{R}^d, w_\gamma) \times W^{k,p}(\mathbb{R}^d, w_\gamma)$. Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda_0 > \widetilde{\lambda}$, the operator $\lambda_0 + A$ is R-sectorial with angle $\omega_R(\lambda_0 + A) \leq \sigma$.*

Proof. Recall that the Bessel potential operator is defined by

$$J_s f := (1-\Delta)^{\frac{s}{2}} f = \mathscr{F}^{-1}((1+|\cdot|^2)^{\frac{s}{2}} \mathscr{F} f), \qquad f \in \mathscr{S}'(\mathbb{R}^d),\ s \in \mathbb{R}.$$

Then, $J_{\ell-\ell_0}$ is an isometry from $W^{\ell,p}(\mathbb{R}^d, w_\gamma)$ to $W^{\ell_0,p}(\mathbb{R}^d, w_\gamma)$ for all $p \in (1,\infty)$ and $\ell, \ell_0 \in \mathbb{N}_0$ with $\ell \geq \ell_0$. Indeed, for the Bessel potential scale this follows from the definition and by [214, Proposition 3.2] it also holds for Sobolev spaces.

Let $f \in \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$, $\lambda_0 > 0$ and $\lambda \in \lambda_0 + \Sigma_{\pi-\sigma}$. Consider the equation $(\lambda + A)u = f$ which has the solution representation

$$u(x) = \mathscr{F}^{-1}\big(\xi \mapsto A^{-1}(\xi,\lambda)(\mathscr{F} f)(\xi)\big)(x)$$

where, following the notation of [74, Section 2], we have that

$$A(\xi,\lambda) := \begin{pmatrix} \lambda & -1 \\ |\xi|^4 & \lambda + \rho|\xi|^2 \end{pmatrix}$$

is invertible and its inverse (see [74, Equation (2.3)]) is given by

$$A^{-1}(\xi,\lambda) = \frac{1}{(\alpha_+\lambda + |\xi|^2)(\alpha_-\lambda + |\xi|^2)} \begin{pmatrix} \lambda + \rho|\xi|^2 & 1 \\ -|\xi|^4 & \lambda \end{pmatrix}.$$

Now, the solution to $(\lambda + A)\widetilde{u} = J_{k-2} f \in \mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d)$ is given by

$$\begin{aligned} \widetilde{u}(x) &= \mathscr{F}^{-1}\big(\xi \mapsto A^{-1}(\xi,\lambda)(\mathscr{F} J_{k-2} f)(\xi)\big)(x) \\ &= \mathscr{F}^{-1}\big(\xi \mapsto (1+|\xi|^2)^{\frac{k-2}{2}} A^{-1}(\xi,\lambda)(\mathscr{F} f)(\xi)\big)(x) = (J_{k-2} u)(x). \end{aligned}$$

This proves that $R(\lambda, -A) J_{k-2} f = J_{k-2} R(\lambda, -A) f$, which is equivalent to

$$R(z, \lambda_0 + A) J_{k-2} f = J_{k-2} R(z, \lambda_0 + A) f, \quad \text{for } z \in \mathbb{C} \setminus \overline{\Sigma}_\sigma. \tag{8.5.12}$$

We can now lift the R-sectoriality to higher-order spaces. Fix $N \in \mathbb{N}_1$, let $(\lambda_n)_{n=1}^N$ be a sequence in $\mathbb{C} \setminus \overline{\Sigma}_\sigma$ and let $(f^{(n)})_{n=1}^N \in \varepsilon_N^2(\mathscr{S}(\mathbb{R}^d) \times \mathscr{S}(\mathbb{R}^d))$. By Proposition 8.4.1 and the definition of R-sectoriality, we have the estimate

$$\begin{aligned} &\Big\| \sum_{n=1}^N \varepsilon_n \lambda_n R(\lambda_n, \lambda_0 + A) f^{(n)} \Big\|_{L^2(\Omega; W^{2,p}(\mathbb{R}^d, w_\gamma) \times L^p(\mathbb{R}^d, w_\gamma))} \\ &\qquad \lesssim \Big\| \sum_{n=1}^N \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega; W^{2,p}(\mathbb{R}^d, w_\gamma) \times L^p(\mathbb{R}^d, w_\gamma))}. \end{aligned} \tag{8.5.13}$$

From (8.5.12) and (8.5.13) it follows that

$$\begin{aligned}
&\Big\| \sum_{n=1}^{N} \varepsilon_n \lambda_n R(\lambda_n, \lambda_0 + A) f^{(n)} \Big\|_{L^2(\Omega; W^{k,p}(\mathbb{R}^d, w_\gamma) \times W^{k-2,p}(\mathbb{R}^d, w_\gamma))} \\
&\quad \eqsim \Big\| \sum_{n=1}^{N} \varepsilon_n \lambda_n R(\lambda_n, \lambda_0 + A) J_{k-2} f^{(n)} \Big\|_{L^2(\Omega; W^{2,p}(\mathbb{R}^d, w_\gamma) \times L^p(\mathbb{R}^d, w_\gamma))} \\
&\quad \lesssim \Big\| \sum_{n=1}^{N} \varepsilon_n J_{k-2} f^{(n)} \Big\|_{L^2(\Omega; W^{2,p}(\mathbb{R}^d, w_\gamma) \times L^p(\mathbb{R}^d, w_\gamma))} \\
&\quad \eqsim \Big\| \sum_{n=1}^{N} \varepsilon_n f^{(n)} \Big\|_{L^2(\Omega; W^{k,p}(\mathbb{R}^d, w_\gamma) \times W^{k-2,p}(\mathbb{R}^d, w_\gamma))}.
\end{aligned}$$

By a density argument, this proves the required R-sectoriality estimate. □

We can turn to the proof of our main result concerning R-sectoriality on domains.

Proof of Theorem 8.5.7. The result will be proved using Lemma 8.5.8 together with Theorem 8.5.1 and Proposition 8.5.9.

Let $\varepsilon \in (0, \kappa)$ be such that $\gamma > (1 - (\kappa - \varepsilon))p - 1$. Take $(V_n)_{n=1}^N, (\mathscr{O}_n)_{n=1}^N, (\eta_n)_{n=0}^N$ from Lemma 8.3.5 such that for all $n \in \{1, \dots, N\}$ we have $[\mathscr{O}_n]_{C^{1,\kappa-\varepsilon}} < \delta$ where $\delta \in (0,1)$ is small enough such that Theorem 8.5.1 (applied with κ replaced by $\kappa - \varepsilon$) applies for every $\mathscr{O}_n$. Note that this is possible due to the discussion below Definition 2.6.1. We define the following operators, where the spaces are as defined in (8.3.3) and (8.3.5).

(i) $\widetilde{A} := \bigoplus_{n=0}^N \widetilde{A}_n$ on $\mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}$ with $D(\widetilde{A}) = \mathscr{W}^{k+2,p}_{0,\gamma+kp} \times \mathscr{W}^{k,p}_{\gamma+kp}$, where

a) $\widetilde{A}_0$ on $W^{k,p}(\mathbb{R}^d) \times W^{k-2,p}(\mathbb{R}^d)$ with $D(\widetilde{A}_0) = W^{k+2,p}(\mathbb{R}^d) \times W^{k,p}(\mathbb{R}^d)$ is given by $\widetilde{A}_0 \widetilde{u} = A\widetilde{u}$,

b) $\widetilde{A}_n$ on $W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}) \times W^{k-2,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp})$ with $D(\widetilde{A}_n) = W^{k+2,p}_0(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}) \times W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp})$ is given by $\widetilde{A}_n \widetilde{u} = A\widetilde{u}$ for all $n \in \{1, \dots, N\}$.

(ii) $B : D(A) = \mathbb{D}^{k,p}_\gamma(\mathscr{O}) \to \mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}$ given by

$$Bu = B\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \big([\Delta^2, \eta_n] u_1 - \rho[\Delta, \eta_n] u_2\big) \end{pmatrix}_{n=0}^N.$$

(iii) $C : D(\widetilde{A}) \to \mathbb{X}^{k,p}_\gamma(\mathscr{O})$ given by

$$C\widetilde{u} = C\begin{pmatrix} \widetilde{u}_1 \\ \widetilde{u}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ \sum_{n=0}^N \big([\Delta^2, \eta_n] \widetilde{u}_{1,n} - \rho[\Delta, \eta_n] \widetilde{u}_{2,n}\big) \end{pmatrix}.$$

For $\lambda_0 > 0$ large enough, $\lambda_0 + \widetilde{A}_n$ for all $n \in \{0, \dots, N\}$ is R-sectorial with $\omega_R(\lambda_0 + \widetilde{A}_n) \leq \sigma$ by Proposition 8.5.9 and Theorem 8.5.1. Hence, for $\lambda_0 > 0$ large enough, also $\lambda_0 + \widetilde{A}$ is R-sectorial with $\omega_R(\lambda_0 + \widetilde{A}) \leq \sigma$.

Let $\mathscr{P}$ and $\mathscr{I}$ be as in (8.3.4). It is straightforward to verify that the conditions (i) and (ii) from Lemma 8.5.8 hold. It remains to check condition (iii) in Lemma 8.5.8. We first characterise the complex interpolation space

$$[\mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}} = [\mathscr{W}^{k,p}_{\gamma+kp}, \mathscr{W}^{k+2,p}_{0,\gamma+kp}]_{\frac{1}{2}} \times [\mathscr{W}^{k-2,p}_{\gamma+kp}, \mathscr{W}^{k,p}_{\gamma+kp}]_{\frac{1}{2}}.$$

Recall that $(\mathscr{O}_n)_{n=1}^N$ are special $C^{1,\kappa}_{\mathrm{c}}$-domains. Hence, using Propositions 8.3.2 and 8.3.4 to reduce to $\mathbb{R}^d_+$ and applying Proposition 3.6.2, it follows that

$$[W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}), W^{k+2,p}_0(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp})]_{\frac{1}{2}} = W^{k+1,p}_0(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}). \tag{8.5.14}$$

From (8.5.14) and interpolation on $\mathbb{R}^d$ (see, e.g., [124, Theorems 5.6.9 & 5.6.11]), we obtain

$$\begin{aligned}
&[\mathscr{W}^{k,p}_{\gamma+kp}, \mathscr{W}^{k+2,p}_{0,\gamma+kp}]_{\frac{1}{2}} \\
&\quad = [W^{k,p}(\mathbb{R}^d), W^{k+2,p}(\mathbb{R}^d)]_{\frac{1}{2}} \oplus \bigoplus_{n=1}^N [W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}), W^{k+2,p}_0(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp})]_{\frac{1}{2}} \\
&\quad = W^{k+1,p}(\mathbb{R}^d) \oplus \bigoplus_{n=1}^N W^{k+1,p}_0(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}) = \mathscr{W}^{k+1,p}_{0,\gamma+kp}.
\end{aligned}$$

Similarly, since $\Phi_* : W^{k_0,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma_0}) \to W^{k_0,p}(\mathbb{R}^d_+, w_{\gamma_0})$ is an isomorphism for all $k_0 \in \mathbb{N}_0$ and all $\gamma_0 \in ((k_0-1)p-1, \infty) \setminus \{jp-1 : j \in \mathbb{N}_1\}$ (see Proposition 8.3.2), it follows from Proposition 3.6.6 that

$$[W^{k-2,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}), W^{k,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp})]_{\frac{1}{2}} = W^{k-1,p}(\mathscr{O}_n, w^{\partial\mathscr{O}_n}_{\gamma+kp}).$$

Hence, we obtain $[\mathscr{W}^{k-2,p}_{\gamma+kp}, \mathscr{W}^{k,p}_{\gamma+kp}]_{\frac{1}{2}} = \mathscr{W}^{k-1,p}_{\gamma+kp}$ and thus we have proved that

$$[\mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}} = \mathscr{W}^{k+1,p}_{0,\gamma+kp} \times \mathscr{W}^{k-1,p}_{\gamma+kp}. \tag{8.5.15}$$

To continue, note that

$$\mathscr{I}Au - \widetilde{A}\mathscr{I}u = -Bu, \quad u \in D(A), \quad \text{and} \quad A\mathscr{P}\widetilde{u} - \mathscr{P}\widetilde{A}\widetilde{u} = C\widetilde{u}, \quad \widetilde{u} \in D(\widetilde{A}).$$

Furthermore, the commutators $[\Delta, \eta_n]$ and $[\Delta^2, \eta_n]$ are first and third-order partial differential operators with smooth and compactly supported coefficients, respectively. This and (8.5.15) yield that the operators

$$\begin{aligned}
\mathscr{I}A - \widetilde{A}\mathscr{I} &: W^{k+1,p}_0(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \times W^{k-1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \to \mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}, \\
\mathscr{P} &: [\mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}} \to W^{k+1,p}_0(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \times W^{k-1,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})
\end{aligned}$$

are bounded. Similarly, it can be shown that the operators

$$\begin{aligned}
A\mathscr{P} - \mathscr{P}\widetilde{A} &: [\mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}, D(\widetilde{A})]_{\frac{1}{2}} \to W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \times W^{k-2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}), \\
\mathscr{I} &: W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \times W^{k-2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}) \to \mathscr{W}^{k,p}_{\gamma+kp} \times \mathscr{W}^{k-2,p}_{\gamma+kp}
\end{aligned}$$

are bounded. This shows that $(\mathscr{I}A-\widetilde{A}\mathscr{I})\mathscr{P}$ and $\mathscr{I}(A\mathscr{P}-\mathscr{P}\widetilde{A})$ extend to bounded operators from $[\mathscr{W}^{k,p}_{\gamma+kp}\times\mathscr{W}^{k-2,p}_{\gamma+kp},D(\widetilde{A})]_{\frac{1}{2}}$ to $\mathscr{W}^{k,p}_{\gamma+kp}\times\mathscr{W}^{k-2,p}_{\gamma+kp}$. Applying Lemma 8.5.8 gives that for all $\sigma\in(\vartheta(\rho),\pi)$ there exists a $\widetilde{\lambda}>0$ such that for all $\lambda_0>\widetilde{\lambda}$ the operator λ_0+A on $\mathbb{X}^{k,p}_{\gamma}(\mathscr{O})$ is R-sectorial with $\omega_R(\lambda_0+A)\le\sigma$. □

8.5.3. R-sectoriality for the unshifted operator

In this section, we prove the following improved version of Theorem 8.5.7, which, in particular, yields R-sectoriality for the operator A without a shift.

Theorem 8.5.10. *Let $p\in(1,\infty)$, $\kappa\in(0,1)$, $\gamma\in((1-\kappa)p-1,p-1)$, $k\ge2$ an integer, $\rho>0$, and let $\mathscr{O}$ be a bounded $C^{1,\kappa}$-domain. Let A be the realisation of the operator in* (8.4.2) *on $\mathbb{X}^{k,p}_{\gamma}(\mathscr{O})$ with domain $D(A):=\mathbb{D}^{k,p}_{\gamma}(\mathscr{O})$. Then the following assertions hold.*

(i) The spectrum $\sigma(A)$ is a discrete set of eigenvalues which is contained in the open right half-plane. Furthermore, $\sigma(A)$ is independent of $p\in(1,\infty)$, $\gamma\in((1-\kappa)p-1,p-1)$, and $k\ge2$.

(ii) There exists a $\widetilde{\lambda}>0$ and a $\sigma<\frac{\pi}{2}$ such that for all $\lambda_0>-\widetilde{\lambda}$, the operator λ_0+A is R-sectorial with $\omega_R(\lambda_0+A)\le\sigma$.

Essentially, the R-sectoriality result in Theorem 8.5.10(ii) follows from Theorem 8.5.7 and Theorem 8.5.10(i). However, since we are dealing with weighted spaces and domains with low regularity, we cannot argue immediately as in the proof of [74, Theorem 5.1]. To prove Theorem 8.5.10(i), we consider the operator A on $\mathbb{X}^{k,p}_{\gamma}(\mathscr{O})$ and on $W^{2,2}_0(\mathscr{O})\times L^2(\mathscr{O})$. Then it suffices to show that the spectra of these two operators are equal and that the spectrum of A on $W^{2,2}_0(\mathscr{O})\times L^2(\mathscr{O})$ has the desired properties.

We start with the analysis of the operator in the unweighted L^2-setting. For sufficiently smooth domains, the following result is an easy consequence of classical elliptic regularity. As the domain $\mathscr{O}$ is of class $C^{1,\kappa}$ only, we have to argue slightly differently.

Lemma 8.5.11. *Let $\rho>0$ and let $\mathscr{O}$ be a bounded C^1-domain. Let $\mathscr{A}$ be the realisation of the operator in* (8.4.2) *on $W^{2,2}_0(\mathscr{O})\times L^2(\mathscr{O})$ with domain*

$$D(\mathscr{A}):=\big\{u\in W^{2,2}_0(\mathscr{O})\times L^2(\mathscr{O}):\mathscr{A}u\in W^{2,2}_0(\mathscr{O})\times L^2(\mathscr{O})\big\}.$$

Then the following assertions hold.

(i) The operator $\mathscr{A}$ is densely defined and closed.

(ii) Let $s\in(0,\frac{1}{2})$. Then we have the higher regularity $u_1\in W^{4,2}_{\mathrm{loc}}(\mathscr{O})$ for each $u=(u_1,u_2)^\top\in D(\mathscr{A})$ as well as the continuous embedding

$$D(\mathscr{A})\subseteq\big(H^{2+s,2}(\mathscr{O})\cap W^{2,2}_0(\mathscr{O})\big)\times W^{2,2}_0(\mathscr{O}),\tag{8.5.16}$$

where $H^{2+s,2}(\mathscr{O})$ stands for the Bessel potential space.

(iii) The spectrum $\sigma(\mathscr{A})$ is a discrete set of eigenvalues which is contained in the open right half-plane.

Proof. Proof of (i). As $C_c^\infty(\mathscr{O}) \times C_c^\infty(\mathscr{O}) \subseteq D(\mathscr{A})$, the operator $\mathscr{A}$ is densely defined. To show closedness, let $(u^{(k)})_{k\in\mathbb{N}} = ((u_1^{(k)}, u_2^{(k)})^\top)_{k\in\mathbb{N}} \subseteq D(\mathscr{A})$ be a sequence with $u^{(k)} \to u$ and $\mathscr{A}u^{(k)} \to v$ as $k \to \infty$ in $W_0^{2,2}(\mathscr{O}) \times L^2(\mathscr{O})$. By the definition of $\mathscr{A}$, this implies $u_2^{(k)} \to -v_1$ in $W_0^{2,2}(\mathscr{O})$. As we also have $u_2^{(k)} \to u_2$ in $L^2(\mathscr{O})$, the uniqueness of the limit in $L^2(\mathscr{O})$ yields $u_2 = -v_1 \in W_0^{2,2}(\mathscr{O})$.

Similarly, we have the convergence $\Delta^2 u_1^{(k)} - \rho\Delta u_2^{(k)} \to v_2$ in $L^2(\mathscr{O})$ which implies

$$\Delta^2 u_1^{(k)} \to v_2 + \rho\Delta u_2 \quad \text{in } L^2(\mathscr{O}), \tag{8.5.17}$$

using that $u_2^{(k)} \to -v_1 = u_2$ in $W_0^{2,2}(\mathscr{O})$. By the continuous embedding $L^2(\mathscr{O}) \subseteq W^{-2,2}(\mathscr{O})$, the convergence in (8.5.17) also holds in $W^{-2,2}(\mathscr{O})$. On the other hand, from $u_1^{(k)} \to u_1$ in $W_0^{2,2}(\mathscr{O})$ and the continuity of $\Delta^2\colon W_0^{2,2}(\mathscr{O}) \to W^{-2,2}(\mathscr{O})$, we also get

$$\Delta^2 u_1^{(k)} \to \Delta^2 u_1 \quad \text{in } W^{-2,2}(\mathscr{O}).$$

Again by the uniqueness of the limit, we see that $\Delta^2 u_1 = v_2 + \rho\Delta u_2$. Therefore, $u = (u_1, u_2)^\top \in D(\mathscr{A})$ and $\mathscr{A}u = (-u_2, \Delta^2 u_1 - \rho\Delta u_2)^\top = (v_1, v_2)^\top$, which shows that the operator $\mathscr{A}$ is closed.

Proof of (ii). Let $u = (u_1, u_2)^\top \in D(\mathscr{A})$. Then we obtain $u_1 \in W_{\mathrm{loc}}^{4,2}(\mathscr{O})$ by a similar argument as in the proof of [223, Lemma 11.36]. Setting $f = (f_1, f_2)^\top := \mathscr{A}u$, we get

$$\Delta^2 u_1 = \rho\Delta u_2 + f_2 = -\rho\Delta f_1 + f_2 \in L^2(\mathscr{O}).$$

Let $s \in (0, \frac{1}{2})$. By [244, Theorem 6], the function u_1 belongs to $H^{2+s,2}(\mathscr{O})$, and we have the estimate

$$\|u_1\|_{H^{s+2,2}(\mathscr{O})} \le C\big\| -\rho\Delta f_1 + f_2\big\|_{H^{s-2,2}(\mathscr{O})} \le C\big\| -\rho\Delta f_1 + f_2\big\|_{L^2(\mathscr{O})} \le C\|f\|_{W^{2,2}(\mathscr{O})\times L^2(\mathscr{O})}.$$

From this and $\|u_2\|_{W^{2,2}(\mathscr{O})} = \|f_1\|_{W^{2,2}(\mathscr{O})}$, we obtain the continuous embedding (8.5.16), where we endow $D(\mathscr{A})$ with the graph norm.

Proof of (iii). By the Rellich–Kondrachov theorem, the space on the right-hand side of (8.5.16) is compactly embedded into $W_0^{2,2}(\mathscr{O}) \times L^2(\mathscr{O})$. Therefore, the operator $\mathscr{A}$ has compact resolvent and, consequently, its spectrum is discrete and consists of eigenvalues of finite algebraic multiplicity. Therefore, to show the statement on the spectrum in (iii), it is sufficient to show that for each λ with $\operatorname{Re}\lambda \ge 0$, the operator $\mathscr{A} + \lambda$ is injective.

Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \ge 0$. On $W_0^{2,2}(\mathscr{O}) \times L^2(\mathscr{O})$, define the inner product

$$\langle v, w\rangle := (v_1, w_1)_{L^2(\mathscr{O})} + (\Delta v_1, \Delta w_1)_{L^2(\mathscr{O})} + (v_2, w_2)_{L^2(\mathscr{O})}, \quad v, w \in W_0^{2,2}(\mathscr{O}) \times L^2(\mathscr{O}).$$

For $u = (u_1, u_2)^\top \in D(\mathscr{A})$ with $(\mathscr{A} + \lambda)u = 0$, we obtain by integration by parts and using the boundary conditions,

$$0 = \operatorname{Re}\langle(\mathscr{A} + \lambda)u, u\rangle = (\operatorname{Re}\lambda)\big(\|\Delta u_1\|_{L^2(\mathscr{O})}^2 + \|u_2\|_{L^2(\mathscr{O})}^2\big) + \rho\|\nabla u_2\|_{L^2(\mathscr{O};\mathbb{C}^d)}^2.$$

As $\operatorname{Re}\lambda \geq 0$, this yields $\nabla u_2 = 0$. Hence, since $u_2 \in W_0^{2,2}(\mathcal{O})$, we obtain $u_2 = 0$. Furthermore, the second component of the equation $(\mathscr{A}+\lambda)u = 0$ implies that, $\Delta^2 u_1 = \rho\Delta u_2 - \lambda u_2 = 0$. Since $u_1 \in W_0^{2,2}(\mathcal{O})$, this implies, using integration by parts again, that $0 = (\Delta^2 u_1, u_1)_{L^2(\mathcal{O})} = \|\Delta u_1\|^2_{L^2(\mathcal{O})}$ and therefore $u_1 = 0$. So we see that $u = 0$, which shows that the operator $\mathscr{A}+\lambda$ is injective. □

In order to show that the spectrum of A on $\mathbb{X}_\gamma^{k,p}(\mathcal{O})$ is independent of p, γ and k, we need the following lemma about the consistency of resolvents.

Lemma 8.5.12. *Let $p \in (1,\infty)$, $\kappa \in (0,1)$, $\gamma \in ((1-\kappa)p-1, p-1)$, $k \geq 2$ an integer and let $\mathcal{O}$ be a bounded $C^{1,\kappa}$-domain. Let*

$$A_{p,k,\gamma} := A \quad \text{on } \mathbb{X}_\gamma^{k,p}(\mathcal{O}) \text{ with } D(A_{p,k,\gamma}) := \mathbb{D}_\gamma^{k,p}(\mathcal{O})$$

be as in (8.4.2). *Let $\mathscr{A}$ be the operator from Lemma 8.5.11, i.e., the realisation of the operator in* (8.4.2) *on $W_0^{2,2}(\mathcal{O}) \times L^2(\mathcal{O})$ with domain*

$$D(\mathscr{A}) := \big\{u \in W_0^{2,2}(\mathcal{O}) \times L^2(\mathcal{O}) : \mathscr{A}u \in W_0^{2,2}(\mathcal{O}) \times L^2(\mathcal{O})\big\}.$$

Then there exists a $\widetilde{\lambda} > 0$ such that for all $\lambda > \widetilde{\lambda}$ the resolvents $R(\lambda, -A_{p,k,\gamma})$ and $R(\lambda, -\mathscr{A})$ are consistent.

Proof. Let $f \in \mathbb{X}_\gamma^{k,p}(\mathcal{O}) \cap \big(W_0^{2,2}(\mathcal{O}) \times L^2(\mathcal{O})\big)$. We have to show that, for sufficiently large λ, the resolvents $R(\lambda, -A_{p,k,\gamma})$ and $R(\lambda, -\mathscr{A})$ exist and are equal when applied to f. For this, we first show an embedding result.

Take $1 < q < \min\{p, 2\}$ and $\mu \in (2q-1, 3q-1)$ such that

$$\mu > \frac{q(\gamma+2p+1)}{p} - 1 > (3-\kappa)q - 1. \tag{8.5.18}$$

Using that $\gamma \in ((1-\kappa)p-1, p-1)$, we can always find a μ such that (8.5.18) holds. First, we claim that

$$L^p(\mathcal{O}, w_{\gamma+2p}^{\partial\mathcal{O}}) \hookrightarrow L^q(\mathcal{O}, w_\mu^{\partial\mathcal{O}}). \tag{8.5.19}$$

Indeed, for $u \in L^p(\mathcal{O}, w_{\gamma+2p}^{\partial\mathcal{O}})$ we have by Hölder's inequality that

$$\int_{\mathcal{O}} |u(x)|^q w_\mu^{\partial\mathcal{O}}(x)\,\mathrm{d}x \leq \Big(\int_{\mathcal{O}} |u(x)|^p w_{\gamma+2p}^{\partial\mathcal{O}}(x)\,\mathrm{d}x\Big)^{\frac{q}{p}} \Big(\int_{\mathcal{O}} w_{\frac{\mu p - q(\gamma+2p)}{p-q}}^{\partial\mathcal{O}}(x)\,\mathrm{d}x\Big)^{\frac{p-q}{q}} < \infty.$$

The latter integral can be written as an integral over $\mathbb{R}^d_+$ (using the localisation from Lemma 8.3.5 and the diffeomorphism from Lemma 2.6.5), hence the integral is finite since (8.5.18) implies $(\mu p - q(\gamma+2p))/(p-q) > -1$. This proves the claim.

We introduce for $r \in (1,\infty)$ and $\nu > -1$ the space

$$Z_{r,\nu} := \big\{u \in W_0^{2,r}(\mathcal{O}, w_\nu^{\partial\mathcal{O}}) \times L^r(\mathcal{O}, w_\nu^{\partial\mathcal{O}}) : \mathscr{A}u \in W_0^{2,r}(\mathcal{O}, w_\nu^{\partial\mathcal{O}}) \times L^r(\mathcal{O}, w_\nu^{\partial\mathcal{O}})\big\}.$$

Note that $D(\mathscr{A}) = Z_{2,0}$. For λ large enough, consider the equation

$$\lambda\widetilde{w} + A\widetilde{w} = f, \qquad f \in \mathbb{X}_\gamma^{k,p}(\mathcal{O}) \cap \big(W_0^{2,2}(\mathcal{O}) \times L^2(\mathcal{O})\big). \tag{8.5.20}$$

Then by Theorem 8.5.7, there exists a unique solution $\widetilde{w_0} \in \mathbb{D}_\gamma^{k,p}(\mathscr{O})$. Hardy's inequality (for Lipschitz domains, see e.g., [169, Theorem 8.8]) and (8.5.19) yield

$$\begin{aligned}\widetilde{w_0} \in \mathbb{D}_\gamma^{k,p}(\mathscr{O}) \hookrightarrow \mathbb{D}_\gamma^{2,p}(\mathscr{O}) = W_0^{4,p}(\mathscr{O}, w_{\gamma+2p}^{\partial\mathscr{O}}) \times W^{2,p}(\mathscr{O}, w_{\gamma+2p}^{\partial\mathscr{O}}) \\ \hookrightarrow W_0^{4,q}(\mathscr{O}, w_\mu^{\partial\mathscr{O}}) \times W^{2,q}(\mathscr{O}, w_\mu^{\partial\mathscr{O}}).\end{aligned}$$

Furthermore, for each $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$, we have $\lambda \in \rho(-\mathscr{A})$ by Lemma 8.5.11(iii). As $\mathscr{A}$ is closed by Lemma 8.5.11(i), this is equivalent to the bijectivity of the operator $\lambda + \mathscr{A}$. Therefore, there also exists a unique solution $\widetilde{w_1} \in Z_{2,0}$ of (8.5.20). Using $\mu > 0$, $q < 2$ and the elliptic regularity (Theorem 8.5.7 using (8.5.18)), we find

$$\widetilde{w_1} \in Z_{2,0} \hookrightarrow Z_{q,\mu} = W_0^{4,q}(\mathscr{O}, w_\mu^{\partial\mathscr{O}}) \times W^{2,q}(\mathscr{O}, w_\mu^{\partial\mathscr{O}}).$$

Note that (8.5.20) with right-hand side $f \in L^q(\mathscr{O}, w_\mu^{\partial\mathscr{O}})$ has a unique solution $\widetilde{w}$ by Theorem 8.5.7 (again using (8.5.18)). It follows that $\widetilde{w_0} = \widetilde{w_1}$, which proves that the resolvents of $-A_{p,k,\gamma}$ and $-\mathscr{A}$ are consistent. □

Using the above lemmas about the spectrum of $\mathscr{A}$ and the consistency, we can now prove Theorem 8.5.10.

Proof of Theorem 8.5.10. Proof of (i). Since the embedding $W^{1,p}(\mathscr{O}, w_{\gamma_0}^{\partial\mathscr{O}}) \hookrightarrow L^p(\mathscr{O}, w_{\gamma_0}^{\partial\mathscr{O}})$ is compact for all $p \in (1,\infty)$ and $\gamma_0 > -1$ (see [116, Theorem 8.8]), it follows that the embedding $\mathbb{D}_\gamma^{k,p}(\mathscr{O}) \hookrightarrow \mathbb{X}_\gamma^{k,p}(\mathscr{O})$ is compact as well. Hence, the resolvent operator $R(\lambda, A)$ with $\lambda \in \rho(A)$ is compact. By the Riesz–Schauder theorem for compact operators, the resolvent operator $R(\lambda, A)$ has a discrete countable spectrum $\{\sigma_j : j \in \mathbb{N}_0\}$, where $\sigma_j \neq 0$ are eigenvalues. Furthermore, zero is contained in the spectrum of $R(\lambda, A)$ and is the only accumulation point of the spectrum. Therefore, the spectral mapping theorem implies that $\sigma(A)$ is a discrete set of eigenvalues.

To continue, we prove that the spectrum $\sigma(A)$ is independent of p, k and γ. Let $A_{p,k,\gamma}$ and $\mathscr{A}$ be as in Lemma 8.5.12. It suffices to prove that $\sigma(A_{p,k,\gamma}) = \sigma(\mathscr{A})$. Note that $\sigma(\mathscr{A})$ is a discrete set of eigenvalues contained in the open right half-plane by Lemma 8.5.11. By Lemma 8.5.12 and analytic continuation it holds that $R(z, A_{p,k,\gamma})$ and $R(z, \mathscr{A})$ are consistent for all $z \in \rho(A_{p,k,\gamma}) \cap \rho(\mathscr{A})$. If $\mu \in \rho(\mathscr{A})$, then since $\sigma(A_{p,k,\gamma})$ is discrete and countable, there exists an $r > 0$ such that $\overline{B(\mu, r)} \setminus \{\mu\} \subseteq \rho(A_{p,k,\gamma}) \cap \rho(\mathscr{A})$. Therefore, by consistency of the resolvents, we find

$$\int_{\partial B(\mu,r)} R(z, A_{p,k,\gamma})\, \mathrm{d}z = \int_{\partial B(\mu,r)} R(z, \mathscr{A})\, \mathrm{d}z = 0,$$

which implies that $\mu \in \rho(A_{p,k,\gamma})$. The other inclusion follows similarly. This proves that $\sigma(A_{p,k,\gamma}) = \sigma(\mathscr{A})$ and thus $\sigma(A_{p,k,\gamma})$ is independent of p, k and γ.

Using the properties of $\sigma(\mathscr{A})$ from Lemma 8.5.11 now yields that $\sigma(A_{p,k,\gamma})$ is a discrete set of eigenvalues contained in the open right half-plane.

Proof of (ii). By Theorem 8.5.7, it holds for λ_0 large enough that $\lambda_0 + A$ is R-sectorial with $\omega_R(\lambda_0 + A) < \frac{\pi}{2}$. From this and the fact that the spectrum of A is discrete, we see that for each $r > 0$ only finitely many eigenvalues λ with $\operatorname{Re}\lambda < r$ exist. As the imaginary

axis belongs to $\rho(A)$ by (i), this yields $\min_{\lambda\in\sigma(A)} \operatorname{Re}\lambda > 0$. We set $\widetilde{\lambda} := \frac{1}{2}\min_{\lambda\in\sigma(A)} \operatorname{Re}\lambda$. Using the analyticity of the resolvent, it follows that there exists some $\sigma < \frac{\pi}{2}$ such that for $\lambda_0 > -\widetilde{\lambda}$, the operator $\lambda_0 + A$ is R-sectorial with $\omega_R(\lambda_0 + A) \le \sigma$. □

8.6. Maximal regularity for the damped plate equation

We prove our main results on maximal regularity for the damped plate equation on the half-space and on bounded domains.

8.6.1. Damped plate equation with homogeneous boundary conditions

We first study the damped plate equation with homogeneous boundary conditions, i.e.,

$$\begin{cases} \partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u = f & (t,x) \in (0,T)\times\mathscr{O}, \\ u = \partial_{\boldsymbol{n}} u = 0 & (t,x) \in (0,T)\times\partial\mathscr{O}, \\ u|_{t=0} = (\partial_t u)|_{t=0} = 0 & x \in \mathscr{O}. \end{cases} \tag{8.6.1}$$

Throughout, $\rho > 0$ is fixed, we write $I := (0,T)$ for some $T \in (0,\infty]$ and $\boldsymbol{n}$ denotes the inward pointing unit normal vector at $\partial\mathscr{O}$. Furthermore, for $q \in (1,\infty)$ and $\mu \in (-1, q-1)$, we denote by $v_\mu(t) := |t|^\mu$ a temporal weight.

We recall that the spaces $\mathbb{X}_\gamma^{k,p}(\mathscr{O})$ and $\mathbb{D}_\gamma^{k,p}(\mathscr{O})$ are as defined in (8.5.1). Furthermore, we introduce the following short-hand notation. Let $p, q \in (1,\infty)$, $\gamma \in (-1, p-1)$, $\mu \in (-1, q-1)$, $k \ge 2$ an integer and $T \in (0,\infty]$. We define the space for the data f as

$$\mathbb{F}_{q,p}^{k-2}(I, v_\mu; \mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}) := L^q(I, v_\mu; W^{k-2,p}(\mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}})).$$

For the solution u we define the spaces

$$\begin{aligned} \mathbb{U}_{q,p}^{k-2}(I, v_\mu; \mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}) &:= W^{2,q}(I, v_\mu; W^{k-2,p}(\mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}})) \cap L^q(I, v_\mu; W^{k+2,p}(\mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}})), \\ {}_0\mathbb{U}_{q,p}^{k-2}(I, v_\mu; \mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}) &:= \big\{u \in \mathbb{U}_{q,p}^{k-2}(I, v_\mu; \mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}) : u|_{t=0} = (\partial_t u)|_{t=0} = 0\big\}. \end{aligned}$$

Similarly, we define the spaces with zero boundary conditions at $\partial\mathscr{O}$ as

$$\begin{aligned} \mathbb{U}_{q,p,0}^{k-2}(I, v_\mu; \mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}) &:= W^{2,q}(I, v_\mu; W^{k-2,p}(\mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}})) \cap L^q(I, v_\mu; W_0^{k+2,p}(\mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}})), \\ {}_0\mathbb{U}_{q,p,0}^{k-2}(I, v_\mu; \mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}) &:= \big\{u \in \mathbb{U}_{q,p,0}^{k-2}(I, v_\mu; \mathscr{O}, w_{\gamma+kp}^{\partial\mathscr{O}}) : u|_{t=0} = (\partial_t u)|_{t=0} = 0\big\}. \end{aligned}$$

Using R-sectoriality of the operator matrix A obtained in Sections 8.4 and 8.5 for the half-space and bounded domains, respectively, we can establish maximal regularity for the damped plate equation.

Theorem 8.6.1. *Let $p, q \in (1,\infty)$, $\gamma \in (-1, p-1)$, $\mu \in (-1, q-1)$, $k \ge 2$ an integer and $\rho > 0$. Assume that either*

(i) $\mathscr{O} = \mathbb{R}^d_+$ and $T \in (0,\infty)$, or,

(ii) $\kappa \in (0,1)$, $\gamma > (1-\kappa)p - 1$, $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain and $T \in (0,\infty]$.

Then the following assertions hold.

(i) The realisation of the operator A in (8.4.2) *on $\mathbb{X}_\gamma^{k,p}(\mathscr{O})$ with domain $\mathbb{D}_\gamma^{k,p}(\mathscr{O})$ has maximal $L^q(\nu_\mu)$-regularity on I. Furthermore, $-A$ generates an analytic C_0-semigroup on $\mathbb{X}_\gamma^{k,p}(\mathscr{O})$. Moreover, if $\mathscr{O}$ is a bounded domain, then this semigroup is uniformly exponentially stable.*

(ii) For all $f \in \mathbb{F}_{q,p}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})$ there exists a unique solution

$$u \in {}_0\mathbb{U}_{q,p,0}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})$$

to (8.6.1). *Moreover, this solution satisfies*

$$\|u\|_{\mathbb{U}_{q,p}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})} \le C\|f\|_{\mathbb{F}_{q,p}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})},$$

where the constant $C>0$ only depends on p,q,γ,μ,k,ρ,d and T. If $\mathscr{O}$ is a bounded domain, then the constant C is independent of T.

Proof. Theorem 8.4.8 (for $\mathscr{O}=\mathbb{R}^d_+$) and Theorem 8.5.10(ii) (for $\mathscr{O}$ a bounded domain) together with [126, Theorems 17.3.1(2), 17.2.26(1) & 17.2.39] (using that $\mathbb{C}$ is a UMD Banach space and $\vartheta(\rho)<\frac{\pi}{2}$) give maximal $L^q(\nu_\mu)$-regularity of A on I. Furthermore, [126, Theorem 17.2.15] implies that $-A$ generates an analytic C_0-semigroup. If $\mathscr{O}$ is a bounded domain, then $0\in\rho(A)$ by Theorem 8.5.10(i), hence [126, Proposition 17.2.8 & Corollary 17.2.25] imply that this semigroup is uniformly exponentially stable. This proves (i). For (ii) it follows from (i) and [126, Propositions 17.2.7 & 17.2.8] that there exists a unique solution $\widetilde{u}=(\widetilde{u}_1,\widetilde{u}_2)^\top \in W^{1,q}(I,\nu_\mu;\mathbb{X}_\gamma^{k,p}(\mathscr{O}))\cap L^q(I,\nu_\mu;\mathbb{D}_\gamma^{k,p}(\mathscr{O}))$ to the first-order Cauchy problem

$$\partial_t\widetilde{u}+A\widetilde{u}=\begin{pmatrix}0\\ f\end{pmatrix},\qquad \widetilde{u}|_{t=0}=0. \tag{8.6.2}$$

Furthermore, this solution satisfies

$$\|\widetilde{u}\|_{W^{1,q}(I,\nu_\mu;\mathbb{X}_\gamma^{k,p}(\mathscr{O}))}+\|\widetilde{u}\|_{L^q(I,\nu_\mu;\mathbb{D}_\gamma^{k,p}(\mathscr{O}))}\le C\|f\|_{L^q(I,\nu_\mu;W^{k-2,p}(\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}}))}. \tag{8.6.3}$$

Next, by setting $u:=\widetilde{u}_1$ we obtain that $\partial_t u=\widetilde{u}_2$ by the first component in (8.6.2). Hence, u solves (8.6.1). Moreover, it holds that $u\in {}_0\mathbb{U}_{q,p,0}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})$ and

$$\begin{aligned}\|u\|_{\mathbb{U}_{q,p}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})} &\lesssim \sum_{j=0}^{1}\|\partial_t^j u\|_{W^{1,q}(I,\nu_\mu;W^{k-2,p}(\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}}))}+\|u\|_{L^q(I,\nu_\mu;W^{k+2,p}(\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}}))}\\ &\lesssim \|f\|_{\mathbb{F}_{q,p}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})},\end{aligned}$$

where we have used that $(\widetilde{u}_1,\widetilde{u}_2)=(u,\partial_t u)$ and (8.6.3).

Conversely, if $u\in {}_0\mathbb{U}_{q,p,0}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})$ solves (8.6.1), then $\widetilde{u}:=(u,\partial_t u)^\top$ belongs to $W^{1,q}(I,\nu_\mu;\mathbb{X}_\gamma^{k,p}(\mathscr{O}))\cap L^q(I,\nu_\mu;\mathbb{D}_\gamma^{k,p}(\mathscr{O}))$ and solves (8.6.2). Indeed, this follows from the embedding

$$\mathbb{U}_{q,p,0}^{k-2}(I,\nu_\mu;\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})\hookrightarrow W^{1,q}(I,\nu_\mu;W^{k,p}(\mathscr{O},w_{\gamma+kp}^{\partial\mathscr{O}})). \tag{8.6.4}$$

To prove (8.6.4), note that by [189, Proposition 5.5], [190, Theorem 3.18] (which also holds on I by a restriction argument) and Proposition 3.6.2 (using $k \geq 2$ and a localisation argument based on Proposition 8.3.2 and Lemma 8.3.5), we obtain

$$\begin{aligned} W^{2,q}\big(I, v_\mu; W^{k-2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big) &\cap L^q\big(I, v_\mu; W_0^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big) \\ &\hookrightarrow W^{1,q}\big(I, v_\mu; [W^{k-2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp}), W_0^{k+2,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})]_{\frac{1}{2}}\big) \\ &= W^{1,q}\big(I, v_\mu; W^{k,p}(\mathscr{O}, w^{\partial\mathscr{O}}_{\gamma+kp})\big). \end{aligned}$$

Thus, the solution u to (8.6.1) is unique and this completes the proof of (ii). □

8.6.2. Damped plate equation with inhomogeneous boundary conditions

We turn to maximal regularity for the damped plate equation with inhomogeneous boundary conditions. Consider

$$\begin{cases} \partial_t^2 u + \Delta^2 u - \rho\Delta\partial_t u = f & (t,x) \in (0,T)\times\mathscr{O}, \\ u = g_0 & (t,x) \in (0,T)\times\partial\mathscr{O}, \\ \partial_{\boldsymbol{n}} u = g_1 & (t,x) \in (0,T)\times\partial\mathscr{O}, \\ u|_{t=0} = (\partial_t u)|_{t=0} = 0 & x \in \mathscr{O}. \end{cases} \tag{8.6.5}$$

Again, $\rho > 0$ is fixed, we write $I := (0,T)$ for some $T \in (0,\infty]$ and $\boldsymbol{n}$ denotes the inward pointing unit normal vector at $\partial\mathscr{O}$. The functions g_0 and g_1 denote the boundary data. As is known from the standard unweighted theory for boundary value problems [68], the optimal space for the boundary data is an intersection space of Triebel–Lizorkin spaces in time and Besov spaces in space. In Chapter 4 the trace theory has been extended to the weighted setting with rough domains. The theory for the temporal trace in the weighted setting is more delicate than in the classical unweighted setting, see, e.g., [190, Section 7]. Therefore, we will not consider non-zero initial data.

For the definition of weighted Triebel–Lizorkin spaces, we refer to, e.g., [211, Section 3.1]. Similar to Chapter 4 (and, e.g., [262, Definition 3.6.1] in the case of smooth domains), we define spaces on $\partial\mathscr{O}$ as follows. Let $\mathscr{O}$ be a bounded C^1-domain and fix $(V_n)_{n=1}^N$, $(\mathscr{O}_n)_{n=1}^N$ and $(\eta_n)_{n=0}^N$ from Lemma 8.3.5. For $n \in \{1,\dots,N\}$, let $\Phi_n : \mathscr{O}_n \to \mathbb{R}^d_+$ be the Dahlberg–Kenig–Stein pullback from Lemma 2.6.5. Recall that for any f defined on $\overline{\mathscr{O}_n}$, the change of coordinates mapping is $(\Phi_n)_* f := f \circ \Phi_n^{-1}$. Furthermore, for any g defined on $\partial\mathscr{O}_n$, we write $(\Phi_n(0,\cdot))_* g = g(\Phi_n^{-1}(0,\cdot))$ which defines a function on $\mathbb{R}^{d-1}$.

Let $p \in (1,\infty)$ and $s > 0$. Let $L^p(\partial\mathscr{O})$ be the Lebesgue space with respect to the surface measure. We define

$$B^s_{p,q}(\partial\mathscr{O}) := \big\{ g \in L^p(\partial\mathscr{O}) : (\Phi_n(0,\cdot))_* \eta_n^2 g \in B^s_{p,q}(\mathbb{R}^{d-1}) \text{ for all } n \in \{1,\dots,N\} \big\},$$

equipped with the norm

$$\|g\|_{B^s_{p,q}(\partial\mathscr{O})} := \sum_{n=1}^N \|(\Phi_n(0,\cdot))_* \eta_n^2 g\|_{B^s_{p,q}(\mathbb{R}^{d-1})}, \qquad g \in B^s_{p,q}(\partial\mathscr{O}).$$

Note that this definition is independent of the partition of unity and the diffeomorphisms up to equivalent norms.

In addition to the notation in Section 8.6.1, we introduce the following short-hand notation for the spaces for the boundary data. Let $j \in \{0,1\}$, $p,q \in (1,\infty)$, $\gamma \in (-1,p-1)$, $\mu \in (-1,q-1)$, $k \geq 2$ an integer and $t \in (0,\infty]$. We define

$$\begin{aligned}
\mathbb{G}^{j,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O}) &:= F^{1-\frac{j}{2}-\frac{\gamma+1}{2p}}_{q,p}(I,v_\mu;L^p(\partial\mathscr{O})) \cap L^q(I,v_\mu;B^{2-j-\frac{\gamma+1}{p}}_{p,p}(\partial\mathscr{O})),\\
{}_0\mathbb{G}^{j,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O}) &:= \begin{cases} \mathbb{G}^{j,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O}) & \text{if } 1-\frac{j}{2}-\frac{\gamma+1}{2p} < \frac{\mu+1}{q},\\ \{g \in \mathbb{G}^{j,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O}) : g|_{t=0}=0\} & \text{if } 1-\frac{j}{2}-\frac{\gamma+1}{2p} > \frac{\mu+1}{q}. \end{cases}
\end{aligned}$$

The temporal traces in the above spaces are well defined, see Theorem 3.1.2. Furthermore, it should be noted that the first-order temporal trace in $\mathbb{G}^{j,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O})$ only exists if

$$1-\frac{j}{2}-\frac{\gamma+1}{2p} > 1+\frac{\mu+1}{q}.$$

Using that $j \in \{0,1\}$, $\gamma \in (-1,p-1)$ and $\mu \in (-1,q-1)$, we see that $\frac{j}{2}+\frac{\gamma+1}{2p} > 0 > -\frac{\mu+1}{q}$. Hence, for the given parameter ranges the first-order temporal trace never exists and, therefore, no compatibility conditions on $\partial_t g_0$ and $\partial_t g_1$ are needed. This is in contrast to the unweighted setting, see [74, Theorem 4.6 & 5.2] and, in particular, [74, Equation (4.10)].

For $\mathscr{O}$ a bounded C^1-domain and $u \in C^1(\overline{\mathscr{O}})$, we define the spatial trace as

$$\overline{\mathrm{Tr}}^{\partial\mathscr{O}}_1 := (u|_{\partial\mathscr{O}}, \boldsymbol{n}\cdot(\nabla u)|_{\partial\mathscr{O}}),$$

where $\boldsymbol{n}$ denotes the inner unit normal vector at $\partial\mathscr{O}$.

The following theorem gives the optimal space for the boundary data in the weighted setting. It should be noted that, due to the spatial weights, the trace space is *independent* of the smoothness parameter k. Also note that more smoothness of the domain is required: $C^{1,\kappa}$, while for homogeneous boundary conditions we could allow for C^1-domains.

Theorem 8.6.2 (Spatial trace operator)**.** *Let $p,q \in (1,\infty)$, $\gamma \in (-1,p-1)$, $\mu \in (-1,q-1)$, $k \geq 2$ an integer and $T \in (0,\infty]$. Furthermore, assume that either*

(i) $\mathscr{O} = \mathbb{R}^d_+$, or,

(ii) $\kappa \in (0,1)$, $\gamma > (1-\kappa)p-1$ and $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain.

Then the trace operator

$$\overline{\mathrm{Tr}}^{\partial\mathscr{O}}_1 : {}_0\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}) \to \prod_{j=0}^{1} {}_0\mathbb{G}^{j,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O})$$

is continuous and surjective.

Proof. If $I=\mathbb{R}$, then the result follows from Theorems 4.4.1 and 4.5.2 applied with k, k_1, ℓ and γ replaced by $k-2$, 4, 2 and $\gamma+2p$, respectively. By using the zero extension operator from $\mathbb{R}_+$ to $\mathbb{R}$, we can argue similarly to Step 2 of the proof of Theorem 4.4.1 to obtain the result for $I=\mathbb{R}_+$. The result for $I=(0,T)$ with $T<\infty$ follows by using an extension operator from I to $\mathbb{R}$. Such an extension operator can be obtained as in [11, Proposition 2.5]. □

We can now prove the main theorem of this chapter concerning well-posedness and regularity of the plate equation (8.6.5) with inhomogeneous boundary conditions.

Theorem 8.6.3. *Let $p,q\in(1,\infty)$, $\gamma\in(-1,p-1)$, $\mu\in(-1,q-1)$, $k\geq 2$ an integer and $\rho>0$. Suppose that $1-\frac{\gamma+1}{2p}\neq\frac{\mu+1}{q}$ and $\frac{1}{2}-\frac{\gamma+1}{2p}\neq\frac{\mu+1}{q}$. Moreover, assume that either*

(i) $\mathscr{O}=\mathbb{R}^d_+$ and $T\in(0,\infty)$, or,

(ii) $\kappa\in(0,1)$, $\gamma>(1-\kappa)p-1$, $\mathscr{O}$ is a bounded $C^{1,\kappa}$-domain and $T\in(0,\infty]$.

Then for all

$$f\in\mathbb{F}^{k-2}_{q,p}(I,\nu_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}),\quad g_0\in{}_0\mathbb{G}^{0,\gamma}_{q,p}(I,\nu_\mu;\partial\mathscr{O})\quad\text{and}\quad g_1\in{}_0\mathbb{G}^{1,\gamma}_{q,p}(I,\nu_\mu;\partial\mathscr{O}),$$

there exists a unique solution $u\in{}_0\mathbb{U}^{k-2}_{q,p}(I,\nu_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})$ to (8.6.5)*. Moreover, this solution satisfies*

$$\|u\|_{\mathbb{U}^{k-2}_{q,p}(I,\nu_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}\leq C\big(\|f\|_{\mathbb{F}^{k-2}_{q,p}(I,\nu_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}+\|g_0\|_{\mathbb{G}^{0,\gamma}_{q,p}(I,\nu_\mu;\partial\mathscr{O})}+\|g_1\|_{\mathbb{G}^{1,\gamma}_{q,p}(I,\nu_\mu;\partial\mathscr{O})}\big),$$

where the constant $C>0$ only depends on p,q,γ,μ,k,ρ,d and T. If $\mathscr{O}$ is a bounded domain, then the constant C is independent of T.

Proof. From Theorem 8.6.2 we obtain a continuous extension operator

$$\overline{\mathrm{ext}}^{\partial\mathscr{O}}_1:\prod_{j=0}^{1}{}_0\mathbb{G}^{j,\gamma}_{q,p}(I,\nu_\mu;\partial\mathscr{O})\to{}_0\mathbb{U}^{k-2}_{q,p}(I,\nu_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}).\tag{8.6.6}$$

First, consider the damped plate equation

$$\begin{cases}\partial_t^2\widetilde{u}+\Delta^2\widetilde{u}-\rho\Delta\partial_t\widetilde{u}=\widetilde{f} & (t,x)\in(0,T)\times\mathscr{O},\\ \widetilde{u}=\partial_{\boldsymbol{n}}\widetilde{u}=0 & (t,x)\in(0,T)\times\partial\mathscr{O},\\ \widetilde{u}|_{t=0}=(\partial_t\widetilde{u})|_{t=0}=0 & x\in\mathscr{O}.\end{cases}\tag{8.6.7}$$

where

$$\widetilde{f}:=f-\big(\partial_t^2+\Delta^2-\rho\Delta\partial_t\big)\overline{\mathrm{ext}}^{\partial\mathscr{O}}_1(g_0,g_1).$$

By Theorem 8.6.1 there exists a solution

$$\widetilde{u}\in W^{2,q}(I,\nu_\mu;W^{k-2,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))\cap L^q(I,\nu_\mu;W^{k+2,p}_0(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))\hookrightarrow\mathbb{U}^{k-2}_{q,p}(I,\nu_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})$$

to (8.6.7) which satisfies the estimate

$$\begin{aligned}
&\|\widetilde{u}\|_{\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}\\
&\quad\lesssim \|f\|_{\mathbb{F}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})} + \big\|\big(\partial_t^2+\Delta^2-\rho\Delta\partial_t\big)\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1)\big\|_{\mathbb{F}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})} \qquad (8.6.8)\\
&\quad\lesssim \|f\|_{\mathbb{F}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})} + \|\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1)\|_{\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})},
\end{aligned}$$

where for the last estimate we have also used the embedding

$$\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}) \hookrightarrow W^{1,q}(I,v_\mu;W^{k,p}(\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp}))$$

This embedding can be proved similarly to (8.6.4), using the localisation procedure in Proposition 8.3.2 and Lemma 8.3.5. Note that we may apply Proposition 8.3.2 for spaces without zero boundary conditions, since $\gamma+kp>(k+1-\kappa)p-1$.

We now set $u:=\widetilde{u}+\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1)$. Then, u satisfies

$$\overline{\operatorname{Tr}}_1^{\partial\mathscr{O}}u=\overline{\operatorname{Tr}}_1^{\partial\mathscr{O}}\widetilde{u}+\overline{\operatorname{Tr}}_1^{\partial\mathscr{O}}\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1)=(g_0,g_1),$$

by (8.6.7) and properties of the extension operator. Furthermore, by (8.6.6) and (8.6.7), we have

$$\begin{aligned}
u|_{t=0}&=\widetilde{u}|_{t=0}+\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1)|_{t=0}=0,\\
(\partial_t u)|_{t=0}&=(\partial_t\widetilde{u})|_{t=0}+(\partial_t\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1))|_{t=0}=0.
\end{aligned}$$

So, by construction of $\widetilde{u}$, we have that u satisfies (8.6.5) and we have the estimate

$$\begin{aligned}
\|u\|_{\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}&\lesssim \|\widetilde{u}\|_{\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}+\|\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1)\|_{\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}\\
&\lesssim \|f\|_{\mathbb{F}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}+\|\overline{\operatorname{ext}}_1^{\partial\mathscr{O}}(g_0,g_1)\|_{\mathbb{U}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}\\
&\lesssim \|f\|_{\mathbb{F}^{k-2}_{q,p}(I,v_\mu;\mathscr{O},w^{\partial\mathscr{O}}_{\gamma+kp})}+\|g_0\|_{\mathbb{G}^{0,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O})}+\|g_1\|_{\mathbb{G}^{1,\gamma}_{q,p}(I,v_\mu;\partial\mathscr{O})},
\end{aligned}$$

using (8.6.8) and (8.6.6). Finally, the uniqueness of the solution u to (8.6.5) follows from Theorem 8.6.1. □

Bibliography

[1] H. Abels. *Pseudodifferential and singular integral operators*. De Gruyter Graduate Lectures. De Gruyter, Berlin, 2012.

[2] H. Abels, M. Krbec and K. Schumacher. On the trace space of a Sobolev space with a radial weight. *J. Funct. Spaces Appl.*, 6(3):259–276, 2008.

[3] P. Acquistapace and B. Terreni. A unified approach to abstract linear nonautonomous parabolic equations. *Rend. Sem. Mat. Univ. Padova*, 78:47–107, 1987.

[4] R. Adams and J. Fournier. *Sobolev spaces*, volume 140 of *Pure and Applied Mathematics*. Elsevier, second edition, 2003.

[5] S. Agmon. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.*, 15:119–147, 1962.

[6] S. Agmon, A. Douglis and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.*, 12:623–727, 1959.

[7] S. Agmon, A. Douglis and L. Nirenberg. Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. II. *Comm. Pure Appl. Math.*, 17:35–92, 1964.

[8] A. Agresti, A. Blessing and E. Luongo. Global well-posedness of 2D Navier-Stokes with Dirichlet boundary fractional noise. *Nonlinearity*, 38(7):075023, 2025.

[9] A. Agresti and E. Luongo. Global well-posedness and interior regularity of 2D Navier–Stokes equations with stochastic boundary conditions. *Math. Ann.*:1–40, 2024.

[10] A. Agresti and M. Veraar. Stability properties of stochastic maximal L^p-regularity. *J. Math. Anal. Appl.*, 482(2):123553, 2020.

[11] A. Agresti and M. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces Part I. Stochastic maximal regularity and local existence. *Nonlinearity*, 35(8):4100–4210, 2022.

[12] A. Agresti and M. Veraar. Nonlinear parabolic stochastic evolution equations in critical spaces part II: Blow-up criteria and instantaneous regularization. *J. Evol. Equ.*, 22(2):56, 2022.

[13] A. Agresti and M. Veraar. Nonlinear SPDEs and maximal regularity: an extended survey. *Nonlinear Differ. Equ. Appl.*, 32(6):123, 2025.

[14] H. Amann. *Linear and quasilinear parabolic problems. Vol. I: abstract linear theory*, volume 89 of *Monographs in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 1995.

[15] H. Amann. *Anisotropic function spaces and maximal regularity for parabolic problems. Part 1*, volume 6 of *Jindřich Nečas Center for Mathematical Modeling Lecture Notes*. Matfyzpress, Prague, 2009.

[16] H. Amann. *Linear and quasilinear parabolic problems. Vol. II: function spaces*, volume 106 of *Monographs in Mathematics*. Birkhäuser/Springer, Cham, 2019.

[17] K. Andersen. Weighted inequalities for the Stieltjes transformation and Hilbert's double series. *Proc. Roy. Soc. Edinburgh Sect. A*, 86(1-2):75–84, 1980.

[18] S. Angenent. Nonlinear analytic semiflows. *Proc. Roy. Soc. Edinburgh Sect. A*, 115(1-2):91–107, 1990.

[19] W. Arendt. Gaussian estimates and interpolation of the spectrum in L^p. *Differ. Integral Equ.*, 7(5-6):1153–1168, 1994.

[20] W. Arendt. Semigroups and evolution equations: functional calculus, regularity and kernel estimates. In C. Dafermos and E. Feireisl, editors. Volume 1, Handb. Differ. Equ. Pages 1–85. Elsevier, 2002.

[21] W. Arendt, M. Bernhard and M. Kreuter. Elliptic problems and holomorphic functions in Banach spaces. English. *Ill. J. Math.*, 64(3):331–347, 2020.

[22] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh and P. Tchamitchian. The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$. *Ann. of Math. (2)*, 156(2):633–654, 2002.

[23] P. Auscher and J. Martell. Weighted norm inequalities, off-diagonal estimates and elliptic operators. III. Harmonic analysis of elliptic operators. *J. Funct. Anal.*, 241(2):703–746, 2006.

[24] P. Auscher and J. Martell. Weighted norm inequalities, off-diagonal estimates and elliptic operators. I. General operator theory and weights. *Adv. Math.*, 212(1):225–276, 2007.

[25] S. Bechtel and M. Egert. Interpolation theory for Sobolev functions with partially vanishing trace on irregular open sets. *J. Fourier Anal. Appl.*, 25(5):2733–2781, 2019.

[26] S. Bechtel, M. Egert and R. Haller-Dintelmann. The Kato square root problem on locally uniform domains. *Adv. Math.*, 375:107410, 2020.

[27] J. Bergh and J. Löfström. *Interpolation spaces*, volume 223 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1976.

[28] M. Berkolaĭko. Traces of generalized spaces of differentiable functions with mixed norm. *Dokl. Akad. Nauk SSSR*, 277(2):270–274, 1984.

[29] M. Berkolaĭko. Theorems on traces on coordinate subspaces for some spaces of differentiable functions with anisotropic mixed norm. *Dokl. Akad. Nauk SSSR*, 282(5):1042–1046, 1985.

[30] M. Berkolaĭko. Traces of functions in generalized Sobolev spaces with a mixed norm on an arbitrary coordinate subspace. I. *Trudy Inst. Mat. Sib. Otd. AN SSSR*, 7:30–44, 1987.

[31] M. Berkolaĭko. Traces of functions in generalized Sobolev spaces with a mixed norm on an arbitrary coordinate subspace. II. *Trudy Inst. Mat. Sib. Otd. AN SSSR*, 9:34–41, 1987.

[32] M. Borsuk and V. Kondratiev. *Elliptic boundary value problems of second order in piecewise smooth domains*, volume 69 of *North-Holland Mathematical Library*. Elsevier Science B.V., Amsterdam, 2006.

[33] G. Bourdaud and Y. Meyer. Fonctions qui opèrent sur les espaces de Sobolev. *J. Funct. Anal.*, 97(2):351–360, 1991.

[34] M. Bravin, M. Gnann, H. Knüpfer, N. Masmoudi, F. Roodenburg and J. Sauer. Well-posedness and regularity of the heat equation with Robin boundary conditions in the two-dimensional wedge. *Comm. Partial Differential Equations*, 50(9):1099–1134, 2025.

[35] M. Bravin, M. Gnann, H. Knüpfer, N. Masmoudi, F. Roodenburg and J. Sauer. Well-posedness of the Stokes equations on a wedge with Navier-slip boundary conditions. To appear in J. Math. Soc. Japan., 2026.

[36] Y. S. Bugrov. Function spaces with mixed norm. *Math. USSR Izv.*, 5:1145–1167, 1971.

[37] H. Bui. Weighted Besov and Triebel spaces: interpolation by the real method. *Hiroshima Math. J.*, 12(3):581–605, 1982.

[38] M. Bulíček, L. Diening and S. Schwarzacher. Existence, uniqueness and optimal regularity results for very weak solutions to nonlinear elliptic systems. *Anal. PDE*, 9(5):1115–1151, 2016.

[39] V. Burenkov. *Sobolev spaces on domains*, volume 137 of *Teubner Texts in Mathematics*. B.G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1998.

[40] A. Carvalho, J. Cholewa and T. Dłotko. Strongly damped wave problems: bootstrapping and regularity of solutions. *J. Differ. Equ.*, 244(9):2310–2333, 2008.

[41] S. Chen and R. Triggiani. Proof of extensions of two conjectures on structural damping for elastic systems. *Pacific J. Math.*, 136(1):15–55, 1989.

[42] R. Chill and S. Srivastava. L^p-maximal regularity for second order Cauchy problems. *Math. Z.*, 251(4):751–781, 2005.

[43] P. Ciarlet and P.-A. Raviart. Interpolation theory over curved elements, with applications to finite element methods. *Comput. Methods Appl. Mech. Eng.*, 1(2):217–249, 1972.

[44] P. Cioica-Licht. An L_p-theory for the stochastic heat equation on angular domains in $\mathbb{R}^2$ with mixed weights. arXiv:2003.03782, 2020.

[45] P. Cioica-Licht, K.-H. Kim, K. Lee and F. Lindner. An L_p-estimate for the stochastic heat equation on an angular domain in $\mathbb{R}^2$. *Stoch. Partial Differ. Equ. Anal. Comput.*, 6(1):45–72, 2018.

[46] P. Cioica-Licht, K. Kim and K. Lee. On the regularity of the stochastic heat equation on polygonal domains in $\mathbb{R}^2$. *J. Differ. Equ.*, 267(11):6447–6479, 2019.

[47] P. Cioica-Licht, E. Lorist and P. Werner. H^∞-calculus for the Dirichlet Laplacian on conical domains. arXiv:2509.09519, 2025.

[48] P. Cioica-Licht, C. Schneider and M. Weimar. Sobolev spaces with mixed weights and the Poisson equation on angular domains. To appear in Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 2025.

[49] P. Clément and G. Simonett. Maximal regularity in continuous interpolation spaces and quasilinear parabolic equations. *J. Evol. Equ.*, 1(1):39–67, 2001.

[50] G. Constantine and T. Savits. A multivariate Faà di Bruno formula with applications. *Trans. Amer. Math. Soc.*, 348(2):503–520, 1996.

[51] T. Coulhon and D. Lamberton. Régularité L^p pour les équations d'évolution. In *Séminaire d'Analyse Fonctionelle 1984/1985*. Volume 26, Publ. Math. Univ. Paris VII, pages 155–165. Univ. Paris VII, Paris, 1986.

[52] M. Cowling, I. Doust, A. McIntosh and A. Yagi. Banach space operators with a bounded H^∞ functional calculus. *J. Austral. Math. Soc. Ser. A*, 60(1):51–89, 1996.

[53] M. Cwikel and A. Einav. Interpolation of weighted Sobolev spaces. *J. Funct. Anal.*, 277(7):2381–2441, 2019.

[54] G. Da Prato. Regularity results of a convolution stochastic integral and applications to parabolic stochastic equations in a Hilbert space. *Confer. Sem. Mat. Univ. Bari*, (182):17, 1982.

[55] G. Da Prato and P. Grisvard. Sommes d'opérateurs linéaires et équations différentielles opérationnelles. *J. Math. Pures Appl. (9)*, 54(3):305–387, 1975.

[56] G. Da Prato and J. Zabczyk. Evolution equations with white-noise boundary conditions. *Stochastics and Stochastic Reports*, 42(3-4):167–182, 1993.

[57] G. Da Prato and J. Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1996.

[58] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 152 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, second edition, 2014.

[59] B. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977.

[60] B. Dahlberg. L^q-estimates for Green potentials in Lipschitz domains. *Math. Scand.*, 44(1):149–170, 1979.

[61] B. Dahlberg. On the Poisson integral for Lipschitz and C^1-domains. *Studia Math.*, 66(1):13–24, 1979.

[62] B. Dahlberg. Poisson semigroups and singular integrals. *Proc. Amer. Math. Soc.*, 97(1):41–48, 1986.

[63] B. Dahlberg, C. Kenig, J. Pipher and G. Verchota. Area integral estimates for higher order elliptic equations and systems. *Ann. Inst. Fourier*, 47(5):1425–1461, 1997.

[64] E. Davies. L^p spectral independence and L^1 analyticity. *J. Lond. Math. Soc.*, 52(1):177–184, 1995.

[65] R. Denk, G. Dore, M. Hieber, J. Prüss and A. Venni. New thoughts on old results of R.T. Seeley. *Math. Ann.*, 328(4):545–583, 2004.

[66] R. Denk and M. Dreher. Resolvent estimates for elliptic systems in function spaces of higher regularity. *Electron. J. Differ. Equ.*, 2011(109):1–12, 2011.

[67] R. Denk, M. Hieber and J. Prüss. $\mathscr{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788), 2003.

[68] R. Denk, M. Hieber and J. Prüss. Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data. *Math. Z.*, 257(1):193–224, 2007.

[69] R. Denk and M. Kaip. *General parabolic mixed order systems in L_p and applications*, volume 239 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer, 2013.

[70] R. Denk and F. Kammerlander. Exponential stability for a coupled system of damped-undamped plate equations. *IMA J. Appl. Math.*, 83(2):302–322, 2018.

[71] R. Denk and F. Roodenburg. Higher-order regularity for a structurally damped plate equation on rough domains. *Math. Ann.*, 395(2):32, 2026.

[72] R. Denk and F. Roodenburg. Trace theory for parabolic boundary value problems with rough boundary conditions. *J. Evol. Equ.*, 26(3):80, 2026.

[73] R. Denk, J. Saal and J. Seiler. Inhomogeneous symbols, the Newton polygon, and maximal L^p-regularity. *Russ. J. Math. Phys.*, 15(2):171–191, 2008.

[74] R. Denk and R. Schnaubelt. A structurally damped plate equation with Dirichlet-Neumann boundary conditions. *J. Differ. Equ.*, 259(4):1323–1353, 2015.

[75] R. Denk and L. Volevich. Elliptic boundary value problems with large parameter for mixed order systems. In *Partial differential equations*. Volume 206, Amer. Math. Soc. Transl. Ser. 2, pages 29–64. Amer. Math. Soc., Providence, RI, 2002.

[76] W. Desch and S.-O. Londen. An L_p-theory for stochastic integral equations. *J. Evol. Equ.*, 11(2):287–317, 2011.

[77] H. Dong and C. Gallarati. Higher-order elliptic and parabolic equations with VMO assumptions and general boundary conditions. *J. Funct. Anal.*, 274(7):1993–2038, 2018.

[78] H. Dong and D. Kim. Elliptic equations in divergence form with partially BMO coefficients. *Arch. Ration. Mech. Anal.*, 196(1):25–70, 2010.

[79] H. Dong and D. Kim. Elliptic and parabolic equations with measurable coefficients in weighted Sobolev spaces. *Adv. Math.*, 274:681–735, 2015.

[80] H. Dong and D. Kim. On L_p-estimates for elliptic and parabolic equations with A_p weights. *Trans. Amer. Math. Soc.*, 370:5081–5130, 2018.

[81] H. Dong, D. Kim and H. Zhang. Neumann problem for non-divergence elliptic and parabolic equations with BMO_x coefficients in weighted Sobolev spaces. *Discrete Continuous Dyn. Syst.*, 36(9):4895–4914, 2016.

[82] G. Dore. Maximal regularity in L^p spaces for an abstract Cauchy problem. *Adv. Differential Equations*, 5(1-3):293–322, 2000.

[83] G. Dore and A. Venni. On the closedness of the sum of two closed operators. *Math. Z.*, 196(2):189–201, 1987.

[84] D. Drihem. Complex interpolation of function spaces with general weights. *Comment. Math. Univ. Carol.*, 64(3):289–320, 2023.

[85] K. Du. $W^{2,p}$-solutions of parabolic SPDEs in general domains. *Stoch. Process. Appl.*, 130(1):1–19, 2020.

[86] X. Duong, I. Holmes, J. Li, B. Wick and D. Yang. Two weight commutators in the Dirichlet and Neumann Laplacian settings. *J. Funct. Anal.*, 276(4):1007–1060, 2019.

[87] M. Egert. *On Kato's conjecture and mixed boundary conditions*. PhD thesis, Sierke Verlag, Göttingen, Technische Universität Darmstadt, 2015.

[88] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[89] L. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, second edition, 2010.

[90] E. Fabes, M. Jodeit Jr. and N. Rivière. Potential techniques for boundary value problems on C^1-domains. *Acta Math.*, 141(3-4):165–186, 1978.

[91] S. Fackler, T. Hytönen and N. Lindemulder. Weighted estimates for operator-valued Fourier multipliers. *Collect. Math.*, 71(3):511–548, 2020.

[92] W. Farkas, J. Johnsen and W. Sickel. Traces of anisotropic Besov-Lizorkin-Triebel spaces—a complete treatment of the borderline cases. *Math. Bohem.*, 125(1):1–37, 2000.

[93] A. Favini and E. Obrecht. Conditions for parabolicity of second order abstract differential equations. *Differ. Integral Equ.*, 4(5):1005–1022, 1991.

[94] J. Franke. On the spaces F^s_{pq} of Triebel-Lizorkin type: pointwise multipliers and spaces on domains. *Math. Nachr.*, 125:29–68, 1986.

[95] C. Gallarati and M. Veraar. Evolution families and maximal regularity for systems of parabolic equations. *Adv. Differential Equations*, 22(3-4):169–190, 2017.

[96] C. Gallarati and M. Veraar. Maximal regularity for non-autonomous equations with measurable dependence on time. *Potential Anal.*, 46(3):527–567, 2017.

[97] F. Gazzola, H.-C. Grunau and G. Sweers. *Polyharmonic boundary value problems*, volume 1991 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010.

[98] G. Geymonat. Su alcuni problemi ai limiti per i sistemi lineari ellitici secondo Petrowsky. *Matematiche (Catania)*, 20:211–253, 1965.

[99] Y. Giga. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Differ. Equ.*, 62(2):186–212, 1986.

[100] D. Gilbarg and L. Hörmander. Intermediate Schauder estimates. *Arch. Rational Mech. Anal.*, 74:297–318, 1980.

[101] S. Gindikin and L. Volevich. *The method of Newton's polyhedron in the theory of partial differential equations*, volume 86 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1992. Translated from the Russian manuscript by V. M. Volosov.

[102] M. Girardi and L. Weis. Criteria for R-boundedness of operator families. In *Evolution equations*. Volume 234, Lecture Notes in Pure and Appl. Math. Pages 203–221. Dekker, New York, 2003.

[103] B. Goldys and S. Peszat. Linear parabolic equation with Dirichlet white noise boundary conditions. *J. Differ. Equ.*, 362:382–437, 2023.

[104] D. Goodair and F. Roodenburg. Global well-posedness for the 2D stochastic hypoviscous Navier–Stokes equations. To appear in Oberwolfach Seminars, 2026.

[105] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[106] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, third edition, 2014.

[107] J. Griepentrog, K. Gröger, H.-C. Kaiser and J. Rehberg. Interpolation for function spaces related to mixed boundary value problems. *Math. Nachr.*, 241:110–120, 2002.

[108] P. Grisvard. Espaces intermédiaires entre espaces de Sobolev avec poids. *Ann. Sc. Norm. Super. Pisa, Sci. Fis. Mat., III. Ser.*, 17(3):255–296, 1963.

[109] P. Grisvard. Caractérisation de quelques espaces d'interpolation. *Arch. Rational Mech. Anal.*, 25:40–63, 1967.

[110] P. Grisvard. Équations différentielles abstraites. *Ann. Sci. École Norm. Sup. (4)*, 2:311–395, 1969.

[111] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[112] G. Grubb. Weakly semibounded boundary problems and sesquilinear forms. *Ann. Inst. Fourier (Grenoble)*, 23(4):145–194, 1973.

[113] G. Grubb. Properties of normal boundary problems for elliptic even-order systems. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 1:1–61, 1974.

[114] D. Guidetti. On interpolation with boundary conditions. *Math. Z.*, 207(3):439–460, 1991.

[115] P. Gurka and B. Opic. Continuous and compact imbeddings of weighted Sobolev spaces. I. *Czechoslovak Math. J.*, 38(113)(4):730–744, 1988.

[116] P. Gurka and B. Opic. Continuous and compact imbeddings of weighted Sobolev spaces. II. *Czechoslov. Math. J.*, 39(1):78–94, 1989.

[117] P. Gurka and B. Opic. Continuous and compact imbeddings of weighted Sobolev spaces. III. *Czechoslovak Math. J.*, 41(116)(2):317–341, 1991.

[118] M. Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[119] Y.-S. Han and Y. Meyer. A characterization of Hilbert spaces and the vector-valued Littlewood–Paley theorem. *Methods Appl. Anal.*, 3(2):228–234, 1996.

[120] D. Haroske and H.-J. Schmeisser. On trace spaces of function spaces with a radial weight: the atomic approach. *Complex Var. Elliptic Equ.*, 55(8-10):875–896, 2010.

[121] M. Hintermüller, K. Papafitsoros and C. Rautenberg. Variable step mollifiers and applications. *Integral Equ. Oper. Theory*, 92(6):1–34, 2020.

[122] L. Hornung. Quasilinear parabolic stochastic evolution equations via maximal L^p-regularity. *Potential Anal.*, 50(2):279–326, 2019.

[123] F. Hummel and N. Lindemulder. Elliptic and parabolic boundary value problems in weighted function spaces. *Potential Anal.*, 57(4):601–669, 2022.

[124] T. Hytönen, J. van Neerven, M. Veraar and L. Weis. *Analysis in Banach spaces. Volume I: Martingales and Littlewood-Paley Theory*, volume 63 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2016.

[125] T. Hytönen, J. van Neerven, M. Veraar and L. Weis. *Analysis in Banach spaces. Volume II: Probabilistic methods and operator theory*, volume 67 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2017.

[126] T. Hytönen, J. van Neerven, M. Veraar and L. Weis. *Analysis in Banach spaces. Volume III: Harmonic Analysis and Spectral Theory*, volume 76 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer, 2024.

[127] S. Janson, P. Nilsson and J. Peetre. Notes on Wolff's note on interpolation spaces. *Proc. London Math. Soc. (3)*, 48(2):283–299, 1984.

[128] D. Jerison and C. Kenig. The Neumann problem on Lipschitz domains. *Bull. Amer. Math. Soc. (N.S.)*, 4(2):203–207, 1981.

[129] D. Jerison and C. Kenig. Boundary value problems on Lipschitz domains. In *Studies in partial differential equations*. Volume 23, MAA Stud. Math. Pages 1–68. Math. Assoc. America, Washington, DC, 1982.

[130] D. Jerison and C. Kenig. The functional calculus for the Laplacian on Lipschitz domains. *Journ. Équ. dériv. partielles*:1–10, 1989.

[131] D. Jerison and C. Kenig. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal.*, 130(1):161–219, 1995.

[132] J. Johnsen and W. Sickel. On the trace problem for Lizorkin–Triebel spaces with mixed norms. *Math. Nachr.*, 281(5):669–696, 2008.

[133] R. Kaczmarek and A. Kałamajska. Density results and trace operator in weighted Sobolev spaces defined on the half-line, equipped with power weights. *J. Approx. Theory*, 291:105896, 2023.

[134] N. Kalton, P. Kunstmann and L. Weis. Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators. *Math. Ann.*, 336(4):747–801, 2006.

[135] N. Kalton and G. Lancien. A solution to the problem of L^p-maximal regularity. *Math. Z.*, 235(3):559–568, 2000.

[136] N. Kalton, E. Lorist and L. Weis. Euclidean structures and operator theory in Banach spaces. *Mem. Amer. Math. Soc.*, 288(1433), 2023.

[137] N. Kalton and L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.*, 321(2):319–345, 2001.

[138] T. Kato. *Perturbation theory for linear operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1996.

[139] C. Kenig and J. Pipher. The Dirichlet problem for elliptic equations with drift terms. *Publ. Mat.*, 45(1):199–217, 2001.

[140] D. Kim. Trace theorems for Sobolev-Slobodeckij spaces with or without weights. *J. Funct. Spaces Appl.*, 5(3):243–268, 2007.

[141] I. Kim and K.-H. Kim. A regularity theory for quasi-linear stochastic PDEs in weighted Sobolev spaces. *Stochastic Process. Appl.*, 128(2):622–643, 2018.

[142] K.-H. Kim. On L_p-theory of stochastic partial differential equations of divergence form in C^1 domains. *Probab. Theory Relat. Fields*, 130(4):473–492, 2004.

[143] K.-H. Kim. On stochastic partial differential equations with variable coefficients in C^1 domains. *Stoch. Process. Their Appl.*, 112(2):261–283, 2004.

[144] K.-H. Kim. $L_q(L_p)$-theory of parabolic PDEs with variable coefficients. *Bull. Korean Math. Soc.*, 45(1):169–190, 2008.

[145] K.-H. Kim. An L_p-theory of SPDEs on Lipschitz domains. *Potential Anal.*, 29(3):303–326, 2008.

[146] K.-H. Kim. An L_p-theory of stochastic PDEs of divergence form on Lipschitz domains. *J. Theoret. Probab.*, 22(1):220–238, 2009.

[147] K.-H. Kim. A weighted Sobolev space theory of parabolic stochastic PDEs on non-smooth domains. *J. Theoret. Probab.*, 27(1):107–136, 2014.

[148] K.-H. Kim. Erratum to "An L_p-theory of stochastic PDEs of divergence form on Lipschitz domains". *J. Theoret. Probab.*, 30:395–396, 2017.

[149] K.-H. Kim and N. Krylov. On the Sobolev space theory of parabolic and elliptic equations in C^1 domains. *SIAM J. Math. Anal.*, 36(2):618–642, 2004.

[150] K.-H. Kim and L. Lee. A weighted L_p-theory for parabolic PDEs with BMO coefficients on C^1-domains. *J. Differ. Equ.*, 254(2):368–407, 2013.

[151] M. Köhne, J. Prüss and M. Wilke. On quasilinear parabolic evolution equations in weighted L_p-spaces. *J. Evol. Equ.*, 10(2):443–463, 2010.

[152] V. Kondratiev. Boundary-value problems for elliptic equations in conical regions. *Dokl. Akad. Nauk SSSR*, 153:27–29, 1963.

[153] V. Kondratiev. Boundary value problems for elliptic equations in domains with conical or angular points. *Trudy Moskov. Mat. Obšč.*, 16:209–292, 1967.

[154] V. Kondratiev and O. Oleĭnik. Boundary value problems for partial differential equations in nonsmooth domains. *Uspekhi Mat. Nauk*, 38(2(230)):3–76, 1983.

[155] V. Kozlov, V. Maz'ya and J. Rossmann. *Elliptic boundary value problems in domains with point singularities*, volume 52 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1997.

[156] V. Kozlov, V. Maz'ya and J. Rossmann. *Spectral problems associated with corner singularities of solutions to elliptic equations*, volume 85 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2001.

[157] V. Kozlov and A. Nazarov. The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients in a wedge. *Math. Nachr.*, 287(10):1142–1165, 2014.

[158] V. Kozlov and A. Nazarov. Oblique derivative problem for non-divergence parabolic equations with time-discontinuous coefficients in a wedge. *J. Math. Anal. Appl.*, 435(1):210–228, 2016.

[159] S. Krantz and H. Parks. *The implicit function theorem*. Modern Birkhäuser classics. Birkhäuser, 2002.

[160] N. Krylov. A W_2^n-theory of the Dirichlet problem for SPDEs in general smooth domains. *Probab. Theory Relat. Fields*, 98:389–421, 1994.

[161] N. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*, volume 12 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1996.

[162] N. Krylov. An analytic approach to SPDEs. In *Stochastic partial differential equations: six perspectives*. Volume 64, Math. Surveys Monogr. Pages 185–242. Amer. Math. Soc., Providence, RI, 1999.

[163] N. Krylov. Some properties of weighted Sobolev spaces in $\mathbb{R}^d_+$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 28(4):675–693, 1999.

[164] N. Krylov. Weighted Sobolev spaces and Laplace's equation and the heat equations in a half space. *Commun. Partial Differ. Equ.*, 24(9-10):1611–1653, 1999.

[165] N. Krylov. The heat equation in $L_q((0,T),L_p)$-spaces with weights. *SIAM J. Math. Anal.*, 32(5):1117–1141, 2001.

[166] N. Krylov. *Lectures on elliptic and parabolic equations in Sobolev spaces*, volume 96 of *Graduate Studies in Mathematics*. American Mathematical Society, 2008.

[167] N. Krylov and S. Lototsky. A Sobolev space theory of SPDEs with constant coefficients in a half space. *SIAM J. Math. Anal.*, 31(1):19–33, 1999.

[168] N. Krylov and B. Rozovskiĭ. Stochastic evolution equations. In *Current problems in mathematics, Vol. 14 (Russian)*, Itogi Nauki i Tekhniki, pages 71–147, 256. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1979.

[169] A. Kufner. *Weighted Sobolev spaces*. John Wiley & Sons, 1985.

[170] A. Kufner and B. Opic. How to define reasonably weighted Sobolev spaces. *Comment. Math. Univ. Carol.*, 25(3):537–554, 1984.

[171] P. Kunstmann. Heat kernel estimates and L^p spectral independence of elliptic operators. *Bull. London Math. Soc.*, 31(3):345–353, 1999.

[172] P. Kunstmann. A new interpolation approach to spaces of Triebel-Lizorkin type. *Ill. J. Math.*, 59(1):1–19, 2015.

[173] P. Kunstmann and L. Weis. Perturbation theorems for maximal L_p-regularity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 30(2):415–435, 2001.

[174] P. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*. Volume 1855, Lecture Notes in Math. Pages 65–311. Springer, Berlin, 2004.

[175] P. Kunstmann and L. Weis. New criteria for the H^∞-calculus and the Stokes operator on bounded Lipschitz domains. *J. Evol. Equ.*, 17(1):387–409, 2017.

[176] O. Ladyženskaja, V. Solonnikov and N. Ural'ceva. *Linear and quasilinear equations of parabolic type*, volume Vol. 23 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1968. Translated from the Russian by S. Smith.

[177] J. Lagnese. *Boundary stabilization of thin plates*, volume 10 of *SIAM Studies in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1989.

[178] I. Lasiecka and R. Triggiani. Analyticity, and lack thereof, of thermo-elastic semigroups. In *Control and partial differential equations (Marseille-Luminy, 1997)*. Volume 4, ESAIM Proc. Pages 199–222. Soc. Math. Appl. Indust., Paris, 1998.

[179] J. LeCrone, J. Prüss and M. Wilke. On quasilinear parabolic evolution equations in weighted L_p-spaces II. *J. Evol. Equ.*, 14(3):509–533, 2014.

[180] R. Leis. *Initial-boundary value problems in mathematical physics*. B.G. Teubner, Stuttgart; John Wiley & Sons, Ltd., Chichester, 1986.

[181] J. Li, R. Rahm and B. Wick. A_p Weights and quantitative estimates in the Schrödinger setting. *Math. Z.*, 293(1-2):259–283, 2019.

[182] G. Lieberman. Regularized distance and its applications. *Pacific J. Math.*, 117(2):329–352, 1985.

[183] N. Lindemulder. *Parabolic initial-boundary value problems with inhomogeneous data: a maximal weighted L^q-L^p-regularity approach*. Master's thesis, Utrecht University, 2014.

[184] N. Lindemulder. Maximal regularity with weights for parabolic problems with inhomogeneous boundary conditions. *J. Evol. Equ.*, 20:59–108, 2020.

[185] N. Lindemulder. An intersection representation for a class of anisotropic vector-valued function spaces. *J. Approx. Theory*, 264:105519, 2021.

[186] N. Lindemulder and E. Lorist. A discrete framework for the interpolation of Banach spaces. *Adv. Math.*, 440:109506, 2024.

[187] N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on the half-space. *J. Funct. Anal.*, 289(8):110985, 2025.

[188] N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on rough domains. *J. Differ. Equ.*, 454:113884, 2026.

[189] N. Lindemulder, M. Meyries and M. Veraar. Complex interpolation with Dirichlet boundary conditions on the half line. *Math. Nachr.*, 291(16):2435–2456, 2018.

[190] N. Lindemulder and M. Veraar. The heat equation with rough boundary conditions and holomorphic functional calculus. *J. Differ. Equ.*, 269(7):5832–5899, 2020.

[191] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, Paris; Gauthier-Villars, Paris, 1969.

[192] J.-L. Lions and E. Magenes. *Non-homogeneous boundary value problems and applications. Vol. I*, volume 181 of *Die Grundlehren der mathematischen Wissenschaften*. Springer-Verlag, New York-Heidelberg, 1972.

[193] W. Liu and M. Röckner. *Stochastic partial differential equations: an introduction*. Universitext. Springer, Cham, 2015.

[194] J. Löfström. Interpolation of weighted spaces of differentiable functions on $\mathbb{R}^d$. *Ann. Mat. Pura Appl.*, 132:189–214, 1982.

[195] J. Löfström. Interpolation of boundary value problems of Neumann type on smooth domains. *J. London Math. Soc. (2)*, 46(3):499–516, 1992.

[196] E. Lorist and M. Veraar. Singular stochastic integral operators. *Anal. PDE*, 14(5):1443–1507, 2021.

[197] S. Lototsky. Sobolev spaces with weights in domains and boundary value problems for degenerate elliptic equations. *Methods Appl. Anal.*, 7(1):195–204, 2000.

[198] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*, volume 16 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Verlag, Basel, 1995.

[199] A. Lunardi. *Interpolation theory*. Appunti. Scuola Normale Superiore di Pisa (Nuova Serie). [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]. Edizioni della Normale, Pisa, second edition, 2009.

[200] J. Martell. Sharp maximal functions associated with approximations of the identity in spaces of homogeneous type and applications. *Studia Math.*, 161(2):113–145, 2004.

[201] V. Maz'ya. *Sobolev spaces with applications to elliptic partial differential equations*, volume 342 of *Grundlehren der Mathematischen Wissenschaften*. Springer, Heidelberg, 2011.

[202] V. Maz'ya and J. Rossmann. *Elliptic equations in polyhedral domains*, volume 162 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2010.

[203] V. Maz'ya and T. Shaposhnikova. *Theory of Sobolev multipliers*, volume 337 of *Grundlehren der Mathematischen Wissenschaften*. Springer Berlin, Heidelberg, 2009.

[204] A. McIntosh. Operators which have an H_∞ functional calculus. In *Miniconference on operator theory and partial differential equations (North Ryde, 1986)*. Volume 14, Proc. Centre Math. Anal. Austral. Nat. Univ. Pages 210–231. Austral. Nat. Univ., Canberra, 1986.

[205] A. McIntosh. The square root problem for elliptic operators a survey. In H. Fujita, T. Ikebe and S. Kuroda, editors, *Functional-Analytic Methods for Partial Differential Equations*, pages 122–140. Springer Berlin Heidelberg, 1990.

[206] A. McIntosh and S. Monniaux. Hodge-Dirac, Hodge-Laplacian and Hodge-Stokes operators in L^p spaces on Lipschitz domains. *Rev. Mat. Iberoam.*, 34(4):1711–1753, 2018.

[207] A. McIntosh and A. Yagi. Operators of type ω without a bounded H_∞ functional calculus. In *Miniconference on Operators in Analysis (Sydney, 1989)*. Volume 24, Proc. Centre Math. Anal. Austral. Nat. Univ. Pages 159–172. Austral. Nat. Univ., Canberra, 1990.

[208] R. McOwen and P. Topalov. Spatial asymptotics and equilibria of heat flow on $\mathbb{R}^d$. *J. Math. Anal. Appl.*, 518(1):126668, 2023.

[209] M. Meyries and R. Schnaubelt. Interpolation, embeddings and traces of anisotropic fractional Sobolev spaces with temporal weights. *J. Funct. Anal.*, 262(3):1200–1229, 2012.

[210] M. Meyries and R. Schnaubelt. Maximal regularity with temporal weights for parabolic problems with inhomogeneous boundary conditions. *Math. Nachr.*, 285(8-9):1032–1051, 2012.

[211] M. Meyries and M. Veraar. Sharp embedding results for spaces of smooth functions with power weights. *Studia Math.*, 208(3):257–293, 2012.

[212] M. Meyries and M. Veraar. Characterization of a class of embeddings for function spaces with Muckenhoupt weights. *Arch. Math.*, 103(5):435–449, 2014.

[213] M. Meyries and M. Veraar. Traces and embeddings of anisotropic function spaces. *Math. Ann.*, 360(3-4):571–606, 2014.

[214] M. Meyries and M. Veraar. Pointwise multiplication on vector-valued function spaces with power weights. *J. Fourier Anal. Appl.*, 21(1):95–136, 2015.

[215] B. Muckenhoupt. Weighted norm inequalities for the Hardy maximal function. *Trans. Amer. Math. Soc.*, 165:207–226, 1972.

[216] T. Nau. *L^p-theory of cylindrical boundary value problems*. Springer Spektrum, Wiesbaden, 2012.

[217] S. Nazarov and B. Plamenevsky. *Elliptic problems in domains with piecewise smooth boundaries*. De Gruyter, 1994.

[218] J. Nečas. *Direct methods in the theory of elliptic equations*. Springer Monographs in Mathematics. Springer, Heidelberg, 2012.

[219] J. van Neerven, M. Veraar and L. Weis. Stochastic maximal L^p-regularity. *Ann. Probab.*, 40(2):788–812, 2012.

[220] J. van Neerven, M. Veraar and L. Weis. Stochastic integration in UMD Banach spaces. *Ann. Probab.*, 35(4):1438–1478, 2007.

[221] J. van Neerven, M. Veraar and L. Weis. Stochastic integration in Banach spaces - a survey. In *Stochastic analysis: a series of lectures*. Volume 68, Progr. Probab. Pages 297–332. Birkhäuser/Springer, Basel, 2015.

[222] J. van Neerven, M. Veraar and L. Weis. Maximal L^p-regularity for stochastic evolution equations. *SIAM J. Math. Anal.*, 44(3):1372–1414, 2012.

[223] J. van Neerven. *Functional Analysis*, volume 201 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2022.

[224] G. Neuttiens and J. Sauer. The time-periodic Cahn-Hilliard-Gurtin system on the half space as a mixed-order system with general boundary conditions. arXiv:2512.23582, 2025.

[225] S. Nikol'skij. Inequalities for entire functions of finite degree and their application in the theory of differentiable functions of several variables. *Proc. Steklov Inst.*, 38:244–278, 1951.

[226] T. Ogawa and S. Shimizu. Maximal L^1-regularity for parabolic boundary value problems with inhomogeneous data in the half-space. *Proc. Japan Acad. Ser. A Math. Sci.*, 96(7):57–62, 2020.

[227] T. Ogawa and S. Shimizu. Maximal L^1-regularity for parabolic initial-boundary value problems with inhomogeneous data. *J. Evol. Equ.*, 22(2):30, 2022.

[228] E. Otárola and A. Salgado. The Poisson and Stokes problems on weighted spaces in Lipschitz domains and under singular forcing. *J. Math. Anal. Appl.*, 471(1-2):599–612, 2019.

[229] E. Pardoux. Équations aux dérivées partielles stochastiques de type monotone. In *Séminaire sur les Équations aux Dérivées Partielles (1974–1975), III*, Exp. No. 2, 10. Collège de France, Paris, 1975.

[230] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[231] P. Portal and M. Veraar. Stochastic maximal regularity for rough time-dependent problems. *Stoch. Partial Differ. Equ. Anal. Comput.*, 7(4):541–597, 2019.

[232] J. Prüss and R. Schnaubelt. Solvability and maximal regularity of parabolic evolution equations with coefficients continuous in time. *J. Math. Anal. Appl.*, 256(2):405–430, 2001.

[233] J. Prüss and G. Simonett. Maximal regularity for evolution equations in weighted L_p-spaces. *Arch. Math. (Basel)*, 82(5):415–431, 2004.

[234] J. Prüss and G. Simonett. *Moving interfaces and quasilinear parabolic evolution equations*, volume 105 of *Monographs in Mathematics*. Birkhäuser/Springer, 2016.

[235] J. Prüss, G. Simonett and M. Wilke. Critical spaces for quasilinear parabolic evolution equations and applications. *J. Differ. Equ.*, 264(3):2028–2074, 2018.

[236] J. Prüss and H. Sohr. On operators with bounded imaginary powers in Banach spaces. *Math. Z.*, 203(3):429–452, 1990.

[237] F. Roodenburg. Complex interpolation of power-weighted Sobolev spaces with boundary conditions. To appear in Stud. Math., 2026.

[238] F. Roodenburg. Optimal semigroup estimates and functional calculus for the Laplacian on weighted Sobolev spaces. arXiv:2608.13314, 2026.

[239] J. Rozendaal and M. Veraar. Sharp growth rates for semigroups using resolvent bounds. *J. Evol. Equ.*, 18:1721–1744, 2018.

[240] J. Rubio de Francia, F. Ruiz and J. Torrea. Calderón-Zygmund theory for operator-valued kernels. *Adv. in Math.*, 62(1):7–48, 1986.

[241] D. Russell. Mathematical models for the elastic beam and their control-theoretic implications. In *Semigroups, theory and applications, Vol. II (Trieste, 1984)*. Volume 152, Pitman Res. Notes Math. Ser. Pages 177–216. Longman Sci. Tech., Harlow, 1986.

[242] D. Russell. *On mathematical models for the elastic beam with frequency-proportional damping*. In *Control and estimation in distributed parameter systems*. Society for Industrial and Applied Mathematics, 1992. Chapter 4, pages 125–169.

[243] V. Rychkov. On restrictions and extensions of the Besov and Triebel–Lizorkin spaces with respect to Lipschitz domains. *J. London Math. Soc.*, 60(1):237–257, 1999.

[244] G. Savaré. Regularity results for elliptic equations in Lipschitz domains. *J. Funct. Anal.*, 152(1):176–201, 1998.

[245] B. Scharf, H.-J. Schmeißer and W. Sickel. Traces of vector-valued Sobolev spaces. *Math. Nachr.*, 285(8-9):1082–1106, 2012.

[246] R. Schnaubelt and M. Veraar. Structurally damped plate and wave equations with random point force in arbitrary space dimensions. *Differ. Integral Equ.*, 23(9-10):957–988, 2010.

[247] C. Schneider. Trace operators in Besov and Triebel-Lizorkin spaces. *Z. Anal. Anwend.*, 29(3):275–302, 2010.

[248] E. Schrohe. Fréchet algebra techniques for boundary value problems on noncompact manifolds: Fredholm criteria and functional calculus via spectral invariance. *Math. Nachr.*, 199:145–185, 1999.

[249] E. Schrohe. A short introduction to Boutet de Monvel's calculus. In *Approaches to singular analysis (Berlin, 1999)*. Volume 125, Oper. Theory Adv. Appl. Pages 85–116. Birkhäuser, Basel, 2001.

[250] R. Seeley. Norms and domains of the complex powers A_B^z. *Am. J. Math.*, 93(2):299–309, 1971.

[251] R. Seeley. Interpolation in L^p with boundary conditions. *Studia Math.*, 44(1):47–60, 1972.

[252] W. Sickel, L. Skrzypczak and J. Vybíral. Complex interpolation of weighted Besov and Lizorkin-Triebel spaces. *Acta Math. Sin.*, 30(8):1297–1323, 2014.

[253] L. de Simon. Un'applicazione della teoria degli integrali singolari allo studio delle equazioni differenziali lineari astratte del primo ordine. *Rend. Sem. Mat. Univ. Padova*, 34:205–223, 1964.

[254] I. Šneĭberg. The solvability of linear equations in interpolational families of Banach spaces. *Dokl. Akad. Nauk SSSR*, 212:57–59, 1973.

[255] I. Šneĭberg. Spectral properties of linear operators in interpolation families of Banach spaces. *Mat. Issled.*, 9(2):214–229, 1974.

[256] P. Sobolevskiĭ. Coerciveness inequalities for abstract parabolic equations. *Dokl. Akad. Nauk SSSR*, 157:52–55, 1964.

[257] V. Solonnikov. On boundary value problems for linear parabolic systems of differential equations of general form. *Trudy Mat. Inst. Steklov.*, 83:3–163, 1965.

[258] E. Stein. Interpolation of linear operators. *Trans. Amer. Math. Soc.*, 83:482–492, 1956.

[259] E. Stein and G. Weiss. Interpolation of operators with change of measures. *Trans. Amer. Math. Soc.*, 87:159–172, 1958.

[260] H. Tanabe. *Equations of evolution*, volume 6 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, Mass.-London, 1979.

[261] P. Tolksdorf. On the L^p-theory of the Navier-Stokes equations on three-dimensional bounded Lipschitz domains. *Math. Ann.*, 371(1-2):445–460, 2018.

[262] H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[263] H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1983.

[264] H. Triebel. *The structure of functions*, volume 97 of *Monographs in Mathematics*. Birkhäuser/Springer Basel, 2001.

[265] B. Turesson. *Nonlinear potential theory and weighted Sobolev spaces*, volume 1736 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[266] M. Veraar. *Stochastic integration in Banach spaces and applications to parabolic evolution equations*. PhD thesis, Delft University of Technology, 2006.

[267] G. Verchota. Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. *J. Funct. Anal.*, 59(3):572–611, 1984.

[268] J. Voigt. Abstract Stein interpolation. *Math. Nachr.*, 157:197–199, 1992.

[269] P. Weidemaier. Existence results in L_p-L_q spaces for second order parabolic equations with inhomogeneous Dirichlet boundary conditions. In *Progress in partial differential equations, Vol. 2*. Volume 384, Pitman Res. Notes Math. Ser. Pages 189–200. Longman, Harlow, 1998.

[270] P. Weidemaier. Maximal regularity for parabolic equations with inhomogeneous boundary conditions in Sobolev spaces with mixed L_p-norm. *Electron. Res. Announc. Amer. Math. Soc.*, 8:47–51, 2002.

[271] P. Weidemaier. Lizorkin-Triebel spaces of vector-valued functions and sharp trace theory for functions in Sobolev spaces with a mixed L_p-norm in parabolic problems. *Mat. Sb.*, 196(6):3–16, 2005.

[272] L. Weis. A new approach to maximal L_p-regularity. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*. Volume 215, Lecture Notes in Pure and Appl. Math. Pages 195–214. Dekker, New York, 2001.

[273] L. Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.*, 319(4):735–758, 2001.

[274] L. Weis. The H^∞ holomorphic functional calculus for sectorial operators - a survey. In *Partial differential equations and functional analysis*. Volume 168, Oper. Theory Adv. Appl. Pages 263–294. Birkhäuser, Basel, 2006.

[275] J. Wloka. *Partial differential equations*. Cambridge University Press, 1987.

[276] T. Wolff. A note on interpolation spaces. In F. Ricci and G. Weiss, editors, *Harmonic Analysis*, pages 199–204. Springer Berlin Heidelberg, 1982.

[277] I. Wood. Maximal L^p-regularity for the Laplacian on Lipschitz domains. *Math. Z.*, 255(4):855–875, 2007.

[278] A. Yagi. *Abstract parabolic evolution equations and their applications*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2010.

Summary

In this dissertation, we investigate well-posedness and maximal regularity of boundary value problems on rough domains and with singular data. To this end, the problems are formulated in the setting of weighted Sobolev spaces, where the weights are powers of the distance to the boundary. We develop the necessary mathematical framework to treat boundary value problems in this weighted setting, including the theoretical study of weighted function spaces, the abstract operator-theoretic study of the Laplace operator, and the application of these tools to prototype boundary value problems.

In Part I of this dissertation, we establish the required function space theory by characterising the trace and complex interpolation spaces of weighted vector-valued function spaces. Specifically, we develop a general higher-order trace theory for Besov, Triebel–Lizorkin, Bessel potential and Sobolev spaces in the weighted setting, crucially allowing for power weights that fall outside the class of Muckenhoupt A_p weights. From these trace theorems, we subsequently derive characterisations of the complex interpolation spaces for weighted Sobolev spaces subject to boundary conditions.

In addition, we develop a trace theory for intersections of weighted Sobolev spaces. By analysing boundary traces within the setting of anisotropic Triebel–Lizorkin spaces, we can identify the optimal trace space of an intersection of weighted Sobolev spaces. This provides the machinery required to handle parabolic boundary value problems with rough inhomogeneous boundary data.

With the underlying function space theory established, we continue with the operator-theoretic analysis of the Laplace operator in Part II of this dissertation. We prove that the Laplacian with Dirichlet or Neumann boundary conditions admits a bounded H^∞-functional calculus on the half-space in the setting of weighted Sobolev spaces. In addition, we study the corresponding Dirichlet and Neumann heat semigroup and it is shown that these semigroups, in contrast to the unweighted L^p-case, have polynomial growth.

By a localisation procedure, the H^∞-calculus results for the Laplacian are then generalised to bounded domains. For the localisation procedure, we employ a version of the Dahlberg–Kenig–Stein pullback, which is a boundary-straightening diffeomorphism specifically tailored to the weighted setting. In particular, we obtain that the Dirichlet Laplacian has a bounded H^∞-calculus on bounded $C^{1,\kappa}$-domains, where $\kappa \in [0,1]$ depends on the exponent of the spatial power weight, but is independent of the smoothness of the underlying Banach space. The analysis reveals a crucial trade-off: lower boundary regularity can be compensated for by enlarging the exponent of the spatial power weight.

In Part III of this dissertation, the trace theory and functional calculus developed in Parts I & II are applied to prove maximal L^q-regularity for several evolution equations. We

first examine the deterministic and stochastic heat equations on rough domains. A profound consequence of the weighted setting is that we can avoid unnatural compatibility conditions for the data, which are in general required in the unweighted case.

Finally, the weighted framework is applied to a linear structurally damped plate equation with inhomogeneous Dirichlet–Neumann boundary conditions. To this end, we study maximal regularity properties of the related first-order system on weighted Sobolev spaces of arbitrarily high smoothness. In this way, we obtain a fairly complete well-posedness and higher-order regularity theory for the damped plate equation on rough domains, while avoiding unnatural compatibility conditions for the data.

Samenvatting

In dit proefschrift onderzoeken we de goedgesteldheid en maximale regulariteit van randwaardeproblemen op ruwe domeinen en met singuliere data. Daartoe worden de problemen geformuleerd in de context van gewogen Sobolevruimten, waarbij de gewichten machten zijn van de afstand tot de rand. We ontwikkelen de benodigde wiskundige theorie om randwaardeproblemen in deze gewogen context te behandelen, waaronder de theoretische studie van gewogen functieruimten, de abstracte operatortheoretische studie van de Laplace-operator, en de toepassing van deze theorie op prototypische randwaardeproblemen.

In Deel I van dit proefschrift behandelen we functieruimtetheorie en karakteriseren we de spoor- en complexe interpolatieruimten van gewogen vectorwaardige functieruimten. We ontwikkelen een algemene hogere-orde spoortheorie voor Besov-, Triebel–Lizorkin-, Bessel-potentiaal- en Sobolevruimten in de gewogen setting, waarbij het cruciaal is dat machtsgewichten worden toegestaan die buiten de klasse van Muckenhoupt A_p-gewichten vallen. Uit deze spoorstellingen leiden we vervolgens karakteriseringen af van de complexe interpolatieruimten voor gewogen Sobolevruimten onderworpen aan randvoorwaarden.

Daarnaast ontwikkelen we een spoortheorie voor doorsneden van gewogen Sobolevruimten. Door sporen op de rand te analyseren binnen de setting van anisotrope Triebel–Lizorkinruimten, kunnen we de optimale spoorruimte identificeren van een doorsnede van gewogen Sobolevruimten. Dit geeft de benodigde theorie om parabolische randwaardeproblemen met ruwe inhomogene randdata te behandelen.

Nadat de onderliggende functieruimtetheorie is behandeld, gaan we in Deel II van dit proefschrift verder met de operatortheoretische analyse van de Laplace-operator. We bewijzen dat de Laplaciaan met Dirichlet- of Neumann-randvoorwaarden een begrensde H^∞-calculus toelaat op de halfruimte in de context van gewogen Sobolevruimten. Daarnaast bestuderen we de corresponderende Dirichlet- en Neumann-warmtehalfgroepen en we laten zien dat deze halfgroepen, in tegenstelling tot het ongewogen L^p-geval, polynomiale groei vertonen.

Middels een lokalisatieprocedure worden de resultaten voor de H^∞-calculus van de Laplaciaan vervolgens gegeneraliseerd naar begrensde domeinen. Voor de lokalisatieprocedure gebruiken we een variant van de Dahlberg–Kenig–Stein-pullback; dit is een diffeomorfisme waarmee de rand lokaal rechtgemaakt kan worden en dat specifiek is afgestemd op de gewogen setting. In het bijzonder verkrijgen we dat de Dirichlet-Laplaciaan een begrensde H^∞-calculus heeft op begrensde $C^{1,\kappa}$-domeinen, waarbij $\kappa \in [0,1]$ afhangt van de exponent van het ruimtelijke machtsgewicht, maar onafhankelijk is van de gladheid van de onderliggende Banachruimte. De analyse onthult een cruciale wisselwerking:

lagere randregulariteit kan worden gecompenseerd door de exponent van het ruimtelijke machtsgewicht te vergroten.

In Deel III van dit proefschrift worden de spoortheorie en functionaalcalculus, die ontwikkeld zijn in Delen I en II, toegepast om maximale L^q-regulariteit te bewijzen voor verscheidene evolutievergelijkingen. We onderzoeken eerst de deterministische en stochastische warmtevergelijkingen op ruwe domeinen. Een diepgaand gevolg van de gewogen setting is dat we onnatuurlijke compatibiliteitsvoorwaarden voor de data, die in het algemeen vereist zijn in het ongewogen geval, kunnen vermijden.

Tot slot wordt deze gewogen methode toegepast op een lineaire structureel gedempte plaatvergelijking met inhomogene Dirichlet–Neumann-randvoorwaarden. Hiertoe bestuderen we maximale regulariteitseigenschappen van het gerelateerde eerste-ordesysteem op gewogen Sobolevruimten van willekeurig hoge gladheid. Op deze manier verkrijgen we een relatief complete theorie voor de goedgesteldheid en hogere-orde regulariteit voor de gedempte plaatvergelijking op ruwe domeinen, terwijl we de artificiële compatibiliteitsvoorwaarden voor de data volledig omzeilen.

Acknowledgements

I would like to express my sincere gratitude to everyone who contributed in whatever way to the completion of this dissertation. Foremost, I want to thank my supervisors, Emiel and Mark, for their guidance, support and careful proofreading. I appreciate our engaging discussions and the freedom you gave me to pursue the research topics I enjoyed. In addition, I thank Robert for the invitation to stay and work in Konstanz for several months. I really enjoyed the time in Konstanz and special thanks also go to Birgit and Margit for their warm hospitality and tremendous help during my stay.

Furthermore, I would like to thank all my coauthors of ongoing and finished projects: Antonio, Daniel, Emiel, Hans, Jonas, Katharina, Manuel, Marco, Mark, Nader, Nick, Robert and Tobias. Collaborating with you has been an absolute pleasure, whether through long days of hard work together or only brief online discussions.

I wish to thank all my colleagues for four enjoyable years together, and in particular my office mates: Chenxi, Esmée, Foivos, Jesse and Milan in Delft, and Lukas in Konstanz.

Last but not least, I want to thank my family and friends for their support. Most importantly, thanks and love go to Anouk, and of course, our floofy cats Loky and Pluk.

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List of Publications

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9. F. Roodenburg. Optimal semigroup estimates and functional calculus for the Laplacian on weighted Sobolev spaces. arXiv:2608.13314, 2026.

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8. D. Goodair and F. Roodenburg. Global well-posedness for the 2D stochastic hypoviscous Navier–Stokes equations. To appear in Oberwolfach Seminars, 2026.

7. R. Denk and F. Roodenburg. Trace theory for parabolic boundary value problems with rough boundary conditions. *J. Evol. Equ.*, 26(3):80, 2026.

6. R. Denk and F. Roodenburg. Higher-order regularity for a structurally damped plate equation on rough domains. *Math. Ann.*, 395(2):32, 2026.

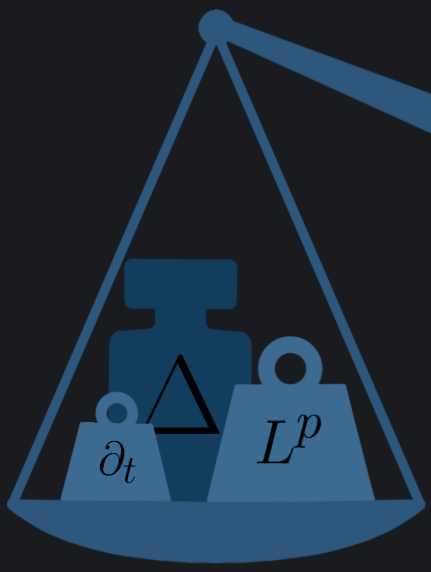
5. F. Roodenburg. Complex interpolation of power-weighted Sobolev spaces with boundary conditions. To appear in Stud. Math., 2026.

4. M. Bravin, M. Gnann, H. Knüpfer, N. Masmoudi, F. Roodenburg and J. Sauer. Well-posedness of the Stokes equations on a wedge with Navier-slip boundary conditions. To appear in J. Math. Soc. Japan., 2026.

3. N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on rough domains. *J. Differ. Equ.*, 454:113884, 2026.

2. M. Bravin, M. Gnann, H. Knüpfer, N. Masmoudi, F. Roodenburg and J. Sauer. Well-posedness and regularity of the heat equation with Robin boundary conditions in the two-dimensional wedge. *Comm. Partial Differential Equations*, 50(9):1099–1134, 2025.

1. N. Lindemulder, E. Lorist, F. Roodenburg and M. Veraar. Functional calculus on weighted Sobolev spaces for the Laplacian on the half-space. *J. Funct. Anal.*, 289(8):110985, 2025.



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