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ON THE MINIMUM DEGREE OF MINIMAL RAMSEY GRAPHS FOR CLIQUES VERSUS CYCLES*

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Abstract. A graph G is said to be q-Ramsey for a q-tuple of graphs $(H_1,\dots,H_q)$, denoted by $G \to_q (H_1,\dots,H_q)$, if every q-edge-coloring of G contains a monochromatic copy of H_i in color i for some $i \in [q]$. Let $s_q(H_1,\dots,H_q)$ denote the smallest minimum degree of G over all graphs G that are minimal q-Ramsey for $(H_1,\dots,H_q)$ (with respect to subgraph inclusion). The study of this parameter was initiated in 1976 by Burr, Erdős, and Lovász, who determined its value precisely for a pair of cliques. Over the past two decades the parameter s_q has been studied by several groups of authors, their main focus being on the symmetric case, where $H_i \cong H$ for all $i \in [q]$. The asymmetric case, in contrast, has received much less attention. In this paper, we make progress in this direction, studying asymmetric tuples consisting of cliques, cycles, and trees. We determine $s_2(H_1,H_2)$ when (H_1,H_2) is a pair of one clique and one tree, a pair of one clique and one cycle, and a pair of two different cycles. We also generalize our results to multiple colors and obtain bounds on $s_q(C_\ell,\dots,C_\ell,K_t,\dots,K_t)$ in terms of the size of the cliques t, the number of cycles, and the number of cliques. Our bounds are tight up to logarithmic factors when two of the three parameters are fixed.

Key words. Ramsey theory, minimum degree, cliques, cycles

MSC codes. 05C55, 05D10, 05D40

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1. Introduction.

A graph G is said to be *q-Ramsey* for a q-tuple of graphs $(H_1,\dots,H_q)$, denoted by $G \to_q (H_1,\dots,H_q)$, if, for every q-coloring of the edges of G, there exists a monochromatic copy of H_i in color i for some $i \in [q]$. In the symmetric case, when $H_i \cong H$ for all $i \in [q]$, we simply say that the graph G is *q-Ramsey* for H. It follows from Ramsey's theorem [34] that such a graph G exists for any choice of $(H_1,\dots,H_q)$. The most well-known object of study in this area is arguably the *Ramsey number* of a q-tuple of graphs $(H_1,\dots,H_q)$, denoted by $r_q(H_1,\dots,H_q)$ and defined as the smallest number of vertices in any graph that is q-Ramsey for $(H_1,\dots,H_q)$. Despite being studied intensively for many families of graphs, it has been determined for very few of them. The case where each H_i is isomorphic to a complete graph K_t is of particular interest. Early results by Erdős [13] and Erdős and Szekeres [15] establish that $2^{t/2} \leqslant r_2(K_t,K_t) \leqslant 4^t$. Despite being over seventy years old, these bounds have only been improved by subexponential factors: the best

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known lower bound is due to Spencer [42], while the best known upper bound was established very recently by Sah [37], improving on a previous result due to Conlon [12].

A natural generalization is to investigate other graph parameters. In their seminal paper [8], Burr, Erdős, and Lovász initiated the study of minimum degrees of Ramsey graphs. Observe that, given any graph G that is q-Ramsey for H, we can add an isolated vertex to G to obtain another graph G' that is also q-Ramsey for H, with minimum degree zero. To avoid such trivialities, we restrict our attention to graphs G that are minimal in the following sense. A graph G is said to be *q-Ramsey-minimal* for $(H_1,\dots,H_q)$ if G is q-Ramsey for $(H_1,\dots,H_q)$ but no proper subgraph of G is. We denote the family of all q-Ramsey-minimal graphs for $(H_1,\dots,H_q)$ by $\mathcal{M}_q(H_1,\dots,H_q)$. We are interested in studying the parameter $s_q(H_1,\dots,H_q)$, defined as the smallest minimum degree among all q-Ramsey-minimal graphs for $(H_1,\dots,H_q)$, that is, $s_q(H_1,\dots,H_q)=\min\{\delta(G):G\in\mathcal{M}_q(H_1,\dots,H_q)\}$, where $\delta(G)$ denotes the minimum degree of G. In the symmetric case, when $H_i\cong H$ for all $i\in[q]$, we simply write $s_q(H)$ instead of $s_q(H,\dots,H)$ (and use similar notation for $r_q(H)$ and $\mathcal{M}_q(H)$). It is not difficult to show that, for any tuple of graphs without isolated vertices, we have

$$\sum_{i=1}^{q}(\delta(H_i)-1)<s_q(H_1,\dots,H_q)\leqslant r_q(H_1,\dots,H_q)-1. \tag{1.1}$$

The proof for the symmetric case and when $q=2$ can be found in Fox and Lin [20, Theorem 3], and the argument easily extends to the more general inequalities.

Burr, Erdős, and Lovász [8] considered pairs of complete graphs and established that $s_2(K_t,K_k)=(t-1)(k-1)$. We want to remark that, in the symmetric case, there is a large gap between $s_2(K_t)$ and the exponential upper bound in (1.1). This surprising phenomenon tells us that, while every graph that is 2-Ramsey for K_t must have at least exponentially many vertices, there is such a graph G that contains a vertex of degree quadratic in t, and this vertex is essential for the Ramsey property of G.

Since the seminal article of Burr, Erdős, and Lovász [8], the parameter $s_2(H)$ has been studied for various graphs H. For example, Fox and Lin [20] showed that the lower bound in (1.1) is tight for complete bipartite graphs. Szabó, Zumstein, and Zürcher [43] extended this result to several other classes of bipartite graphs, including trees and even cycles, while Grinshpun [24] proved it for 3-connected bipartite graphs. Some nonbipartite cases were addressed as well, such as cliques with pendant edges [18], cliques with the edge set of a star removed [23], and odd cycles [6].

All these results address the symmetric case and, to the best of our knowledge, the result of Burr, Erdős, and Lovász concerning pairs of cliques is the only asymmetric case to date. It is then natural to consider pairs of graphs (K_t,H), where H is a very sparse graph such as a tree T_ℓ or a cycle C_ℓ (where ℓ is the number of vertices). These pairs have already been studied in Ramsey theory in the context of Ramsey numbers. A classical result by Chvátal [11] states that $r_2(K_t,T_\ell)=(t-1)(\ell-1)+1$. In fact, any red/blue-coloring witnessing the inequality $r_2(K_t,T_\ell)>(t-1)(\ell-1)$ is so special that we can easily deduce the following.

Proposition 1.1. *For all integers $t\geqslant 3$ and $\ell\geqslant 2$, we have $s_2(K_t,T_\ell)=t-1$.*

The Ramsey number $r_2(K_t,C_\ell)$ has received considerably more attention, as it shows different behavior depending on the magnitude of ℓ; after decades of effort by researchers, the study of these Ramsey numbers has culminated in several very recent breakthroughs. The case when $\ell = 3$ defaults to the notoriously difficult case of the asymmetric Ramsey number $r_2(K_t, K_3)$, which has been determined up to a factor of $4 + o(1)$ by Bohman and Keevash [4], Fiz Pontiveros, Griffiths, and Morris [17], and Shearer [39], following the earlier results by Ajtai, Komlós, and Szemerédi [1] and Kim [29] establishing that $r_2(K_t, K_3) = \Theta(t^2/\log t)$. At the other end of the spectrum, Keevash, Long, and Skokan [28] showed that $r_2(K_t, C_\ell) = (t-1)(\ell-1)+1$ for $\ell = \Omega(\log t/\log\log t)$ and that this bound on ℓ is best possible for the equality to hold. For a more detailed discussion on the history of $r_2(K_t, C_\ell)$ we refer the reader to [28]. We determine the value of $s_2(K_t, C_\ell)$ precisely, showing that, unlike the Ramsey number, our parameter of interest is independent of ℓ.

We also complete the study s_2 for pairs of graphs, each of which is a complete graph or a cycle, by determining $s_2(C_k, C_\ell)$. The study of the Ramsey number in this case was completed already in the 1970s by Rosta [36] and Faudree and Schelp [16], and also depends on the values of k and ℓ. The minimum degree s_2, however, is again independent of either cycle length.

THEOREM 1.2. *For all integers $t \geqslant 3$ and $k, \ell \geqslant 4$,*

(i) $s_2(C_k, C_\ell) = 3$.
(ii) $s_2(K_t, C_\ell) = 2(t-1)$.

Next, we venture into the multicolor setting. Boyadzhiyska, Clemens, and Gupta [6] showed that $s_q(C_\ell) = q+1$ for all $q \geqslant 2$ and $\ell \geqslant 4$. The only other case that has been studied deals with symmetric tuples of cliques, and no precise values are known for $s_q(K_t)$ for $q > 2$. Fox, Grinshpun et al. [19] showed that $s_q(K_t)$ is quadratic in q, up to a polylogarithmic factor, when the size of the clique is fixed. The polylogarithmic factor was settled to be $\Theta(\log q)$ when $t = 3$ by Guo and Warnke [25], following earlier work in [19]. In the other regime, when the number of colors is fixed, Hàn, Rödl, and Szabó [26] showed that $s_q(K_t)$ is quadratic in the clique size t up to a polylogarithmic factor. Bounds that are polynomial in both q and t are also known; see [19] and Bamberg, Bishnoi, and Lesgourgues [3].

In this paper, we investigate the parameter s_q in the case of multiple cliques and multiple cycles. For given integers $q, q_1, q_2 \geqslant 0$ with $q = q_1 + q_2$, $t \geqslant 3$, and $\ell \geqslant 4$, we define $\mathcal{T} = \mathcal{T}(q_1, q_2, \ell, t)$ to be the q-tuple consisting of q_1 cycles on ℓ vertices and q_2 cliques on t vertices, that is,

$$\mathcal{T}(q_1, q_2, \ell, t) = (\underbrace{C_\ell, \ldots, C_\ell}_{q_1 \text{ times}}, \underbrace{K_t, \ldots, K_t}_{q_2 \text{ times}}), \tag{1.2}$$

and let $s_q(\mathcal{T}(q_1, q_2, \ell, t))$ be the smallest minimum degree of a q-Ramsey-minimal graph for $\mathcal{T}(q_1, q_2, \ell, t)$. When the parameters are clear from the context, we will suppress them from the notation. Our main result in the multicolor setting is the following.

THEOREM 1.3. *For all $\ell \geqslant 4$, $t \geqslant 3$, and all $q, q_1, q_2 \geqslant 1$ such that $q_1 + q_2 = q$, we have*

$$s_{q_2}(K_t) + q_1 \leqslant s_q(\mathcal{T}(q_1, q_2, \ell, t)) \leqslant s_q(K_t). \tag{1.3}$$

Note that these upper and lower bounds are independent from the cycles' length ℓ. In fact, we prove a stronger statement in Lemma 4.2 from which it follows that $s_q(\mathcal{T})$ itself does not depend on ℓ. Using the known bounds for $s_q(K_t)$, we can deduce the following corollary.

Corollary 1.4.

(i) *For all $t \geqslant 4$ and $q_1 \geqslant 1$, there exist constants $c, C > 0$ such that, for all $\ell \geqslant 4$ and $q_2 \geqslant 1$, we have*

$$c\, q_2^2 \frac{\log q_2}{\log\log q_2} \leqslant s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, t)) \leqslant C q_2^2 (\log q_2)^{8(t-1)^2}.$$

(ii) *For all $q_1 \geqslant 1$ there exist constants $c, C > 0$ such that, for all $\ell \geqslant 4$ and $q_2 \geqslant 1$, we have*

$$c\, q_2^2 \log q_2 \leqslant s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, 3)) \leqslant C q_2^2 \log q_2.$$

(iii) *For all $q_1, q_2 \geqslant 1$, there exists a constant $C > 0$ such that, for all $\ell \geqslant 4$ and $t \geqslant 3$, we have*

$$(t-1)^2 \leqslant s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, t)) \leqslant C t^2 \log^2 t.$$

Thus, Theorem 1.3 is sufficient to determine $s_q(\mathcal{T}(q_1, q_2, \ell, t))$ in terms of q_2 and in terms of t when the other parameters are fixed. Similarly, the bounds in [3, 19] yield bounds on $s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, t))$ that are polynomial in both t and q.

When q_1 is large compared to the other parameters, the lower bound of (1.3) is linear in q_1, while the upper bound is essentially quadratic in q_1. In this case, using the already mentioned stronger statement of Lemma 4.2, we prove the following asymptotically optimal result.

Theorem 1.5. *For all $\ell \geqslant 4$, $t \geqslant 3$, $q_2 \geqslant 1$, and $\varepsilon > 0$, there exists q_0 such that for all $q_1 \geqslant q_0$, we have*

$$s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, t)) \leqslant (1+\varepsilon) q_1.$$

Organization of the paper. In section 2, we introduce some of the key definitions and known results that will be necessary in the rest of the paper, and we state our main technical results, Theorems 2.8 and 2.9. Section 3 is dedicated to the proofs of the 2-color cases (Proposition 1.1 and Theorem 1.2). In section 4 we prove Theorems 1.3 and 1.5, assuming the existence of certain gadget graphs as guaranteed by Theorems 2.8 and 2.9. Finally, section 5 contains the proof of Theorems 2.8 and 2.9.

2. Preliminaries.

In this section, we introduce notation and key ideas that will be used throughout the article and state our main technical results, the existence of gadget graphs for a q-tuple of cycles and cliques (Theorems 2.8 and 2.9).

We use standard graph theoretic notation throughout the article. Given a hypergraph G, we write $v(G)$ for the size of its vertex set and $e(G)$ for the size of its edge set. We often identify a graph with its edge set. In particular, for two graphs G and H, we use $G - H$ to denote the graph on $V(G)$ with edge set $E(G) \setminus E(H)$. We say that a graph is *H-free* if it does not contain H as a (not necessarily induced) subgraph. The *distance* between two sets of vertices A and B in a graph is the length of a shortest path with one endpoint in A and one endpoint in B.

Unless otherwise specified, we use the term *coloring* to refer to an edge-coloring. If a coloring of a graph uses at most q colors, then we say that it is a q-coloring; unless otherwise specified, the color palette in a q-coloring is taken to be the set $[q] = \{1, \ldots, q\}$. When $q = 2$, we call the first color red and the second color blue. Given a q-coloring φ of a graph G and a subgraph $F \subseteq G$, we will write $\varphi_{|F}$ for the

q-coloring induced by φ on the edges of F. Given a q-tuple of graphs $(H_1,\dots,H_q)$, we say that a q-coloring φ of a graph G is $(H_1,\dots,H_q)$*-free* if, for all $i\in[q]$, the graph $\varphi^{-1}(\{i\})$ is H_i-free. When $H_i\cong H$ for all $i\in[q]$, we will simply say that φ is H*-free* when φ is $(H,\dots,H)$-free.

Given colorings φ_1 and φ_2 of G_1 and G_2, respectively, such that $\varphi_1(e)=\varphi_2(e)$ for all $e\in E(G_1)\cap E(G_2)$, we define the coloring $\varphi_1\cup\varphi_2$ on $G_1\cup G_2$ by setting

$$\varphi(e)=\begin{cases}\varphi_1(e) & \text{if } e\in E(G_1),\\ \varphi_2(e) & \text{if } e\in E(G_2).\end{cases}$$

Let $t\geqslant 3$, $\ell\geqslant 4$, and $q,q_1,q_2\geqslant 0$ be integers such that $q=q_1+q_2$. Recall that $\mathcal{T}=\mathcal{T}(q_1,q_2,\ell,t)$ denotes the q-tuple of cycles and cliques as defined in (1.2). For convenience, we will sometimes write $\mathcal{S}(C_\ell)$ for the color palette $\{1,\dots,q_1\}$ and refer to it as the *cycle-colors*; similarly, $\mathcal{S}(K_t)$ will denote the color palette $\{q_1+1,\dots,q\}$, referred to as the *clique-colors*.

2.1. Signal senders and determiners. For our constructions, we need gadget graphs similar to those introduced by Burr, Erdős, and Lovász [8] and Burr, Faudree, and Schelp [9]. Let $q\geqslant 2$ and $(H_1,\dots,H_q)$ be a q-tuple of graphs. We begin with the simpler of the two gadget graphs.

Definition 2.1 (Set-determiner). *Let $X\subseteq[q]$ be any subset of colors. An X-determiner for $(H_1,\dots,H_q)$ is a graph D with a distinguished edge d satisfying the following properties:*

- (D1) $D\nrightarrow_q (H_1,\dots,H_q)$.
- (D2) *For any $(H_1,\dots,H_q)$-free coloring φ of D, we have $\varphi(d)\in X$.*
- (D3) *For any color $c\in X$, there exists an $(H_1,\dots,H_q)$-free coloring φ of D such that $\varphi(d)=c$.*

The edge d is referred to as the signal edge of D.

In the special case where $X=\{c\}$, these gadgets are defined by Burr, Faudree, and Schelp in [9] and simply called *determiners*. It is not difficult to see that a $\{c\}$-determiner can only exist for a q-tuple $(H_1,\dots,H_q)$ if $H_c\ncong H_i$ for all $i\in[q]\setminus\{c\}$. Determiners are known to exist for all pairs (G,H) such that $G\ncong H$ and G and H are 3-connected (see Burr, Nešetřil, and Rödl [10]). More recently, they were shown to exist for pairs of the form (C_ℓ,H) by Siggers [40], where H is a 2-connected graph satisfying some additional properties.

While set-determiners allow us to pick which set the color of a certain edge should come from, in order to have control over the specific color pattern we see on a group of edges (e.g., which edges should have the same color), we also define the following more sophisticated gadgets.

Definition 2.2 (set-sender). *Let $X\subseteq[q]$ be any subset of colors. A negative (respectively, positive) X-sender for $(H_1,\dots,H_q)$ is a graph S with distinguished edges e and f, satisfying the following properties:*

- (S1) $S\nrightarrow_q (H_1,\dots,H_q)$.
- (S2) *For any $(H_1,\dots,H_q)$-free coloring φ of S, there exist colors $c_1,c_2\in X$ with $c_1\neq c_2$ (respectively, $c_1=c_2$) such that $\varphi(e)=c_1$ and $\varphi(f)=c_2$.*
- (S3) *For any colors $c_1,c_2\in X$ with $c_1\neq c_2$ (respectively, $c_1=c_2$), there exists an $(H_1,\dots,H_q)$-free coloring φ of S with $\varphi(e)=c_1$ and $\varphi(f)=c_2$.*

The edges e and f are referred to as the signal edges of S.

In the special case where $X=[q]$, these gadgets are introduced by Burr, Erdős, and Lovász [8] and called *signal senders*. In [8, 9], it was shown that positive and negative signal senders exist for pairs of complete graphs. Subsequently, it was proved that they exist for other graphs as well as for more colors; in particular, Rödl and Siggers [35] and Siggers [41] established their existence for any number of colors when $H_i \cong H$ for all $i \in [q]$ and H is either 3-connected or a cycle. In a later paper, Siggers [40] showed the existence of signal senders for some pairs of the form (C_ℓ, H).

In the symmetric case, when $H_i \cong H$ for all $i \in [q]$, we write set-senders for H to denote set-senders for $(H,\dots,H)$ and use similar notation for signal senders. Additionally, when $q=2$ we simplify the notation and write *red-determiners* (respectively *blue-determiners*) for {red}-determiners (respectively, {blue}-determiners).

Intuitively speaking, the utility of set-senders and set-determiners comes from the fact that these gadgets allow us to force specific color patterns on particular sets of edges. In our constructions, we usually start with a base graph G and add set-senders and set-determiners so that, in any $(H_1,\dots,H_q)$-free coloring of the resulting graph, we obtain a particular color pattern on the edges of G. More precisely, we will say that we *attach* a set-determiner D to an edge e of G to mean that we create a new copy $\widehat{D}$ of D such that e is the signal edge of $\widehat{D}$, and $\widehat{D}$ is otherwise vertex-disjoint from G. Similarly, we will say that we *connect* or *join* two edges e_1 and e_2 of G by a set-sender S to mean that we create a new copy $\widehat{S}$ of S such that e_1 and e_2 are the signal edges of $\widehat{S}$ (in an arbitrary fashion), and $\widehat{S}$ is otherwise vertex-disjoint from G.

In order for these constructions to be useful, we need to be able to control the new copies of $H_1,\dots,H_q$ that might be created in the process. In particular, since we usually use set-senders and set-determiners as black boxes, we would like to be able to obtain an $(H_1,\dots,H_q)$-free coloring of the entire graph by simply giving each of the building blocks an $(H_1,\dots,H_q)$-free coloring. This motivates the definition of a safe coloring given by Siggers in [40].

Definition 2.3 (safe coloring). *Let F be a graph, $A \subseteq F$ be a subgraph, and φ be an $(H_1,\dots,H_q)$-free q-coloring of F. We say that φ is safe at A if, for any graph G with $V(F)\cap V(G)=V(A)$ and $E(F)\cap E(G)=E(A)$, a q-coloring ψ of $F\cup G$ with $\psi_{|F}=\varphi$ is $(H_1,\dots,H_q)$-free if and only if $\psi_{|G}$ is $(H_1,\dots,H_q)$-free.*

We will call a set-sender (respectively, set-determiner) *safe* if the coloring guaranteed by property (S3) (respectively, (D3)) can be chosen to be safe at the signal edge(s).

As explained above, in the asymmetric setting, the work of [8, 9, 10] established the existence of signal senders and determiners for pairs of the form (H_1, H_2), where H_1 and H_2 are either 3-connected or isomorphic to K_3. These determiners can be shown to be safe following an argument similar to Remark 2.7. The only other result in this direction that we are aware of is due to Siggers [40], who used the ideas of Bollobás et al. [5] to prove the existence of safe signal senders and safe determiners for many pairs of the form (H, C_ℓ), where H is a 2-connected graph satisfying certain technical properties. The special cases that are relevant to our 2-color study in section 3 are given in the following lemma. While Lemma 2.4(ii) also follows from our more general Theorem 2.8, we briefly sketch Siggers's proof for both cases below, combining a few arguments from his paper.

Lemma 2.4 ([40]).

(i) *Let $k,\ell \geqslant 4$ be integers with $k<\ell$. Then there exist safe red-determiners and safe blue-determiners for (C_k, C_ℓ).*

(ii) *Let $\ell \geqslant 4$ and $t \geqslant 3$. Then there exist safe red-determiners and safe blue-determiners for (K_t, C_ℓ).*

Proof. We know that C_k and K_t are 2-connected, and, since $\ell > k$, these graphs contain no induced cycle of length at least $\ell + 1$. Therefore, by [40, Corollary 3.12], there exist safe red-determiners for (C_k, C_ℓ) and (K_t, C_ℓ).

Let C be a copy of C_k, and let e be any edge of C. Attach a copy of the safe red-determiner for (C_k, C_ℓ) from the previous paragraph to each edge of C except e, and let D be the resulting graph. Clearly, in any (C_k, C_ℓ)-free coloring of D, the edge e is blue. Furthermore, giving each copy of the red-determiner a safe (C_k, C_ℓ)-free coloring, as guaranteed by property (D3) and the safeness of that determiner, results in a (C_k, C_ℓ)-free coloring of D. The safeness of the red-determiner further ensures that this coloring is safe at the edge e. Therefore D is a safe blue-determiner for (C_k, C_ℓ), with signal edge e. A similar argument yields a safe blue-determiner for (K_t, C_ℓ). ∎

As explained in the introduction, in this paper we investigate the parameter s_q in the case of multiple cliques and multiple cycles. Our main technical result stated below proves the existence of some set-determiners for such tuples of graphs. In its proof, we need the following results concerning the existence of signal senders in the symmetric setting due to Siggers [41] and Rödl and Siggers [35], respectively.

LEMMA 2.5 ([41, Lemma 2.2]). *For any $\ell \geqslant 4$ and any number of colors $q \geqslant 2$, there exist positive and negative signal senders for the cycle C_ℓ that have girth ℓ and distance at least $\ell + 1$ between their signal edges.*

LEMMA 2.6 ([35, Lemma 2.2]). *For any graph H that is either 3-connected or isomorphic to K_3, any number of colors $q \geqslant 2$, and any integer $d \geqslant 1$, there exist positive and negative signal senders for H in which the signal edges are at distance at least d.*

Remark 2.7. We claim that the signal senders given by Lemmas 2.5 and 2.6 are safe. First, let S be a signal sender for C_ℓ with signal edges e, and let f and F be any graph such that $V(F) \cap V(S) = V(e) \cup V(f)$ and $E(F) \cap E(S) = \{e, f\}$. Let φ be a C_ℓ-free coloring of S, and let ψ be a coloring of $S \cup F$ extending φ. Suppose that $\psi_{|F}$ is also C_ℓ-free but ψ itself is not. This means that there exists a monochromatic copy C of C_ℓ containing an edge $vw \in E(S) \setminus E(F)$ and an edge $xy \in E(F) \setminus E(S)$. Since the signal edges of S are at distance at least $\ell + 1 \geqslant 2$, at least one vertex of vw, say v, is in $V(S) \setminus V(F)$. There are two internally vertex-disjoint paths between v and x in C, and since the edge xy is not in S, at least one of these paths leaves S. Let z_1 and z_2 be the first vertices of F appearing on each of these paths as we traverse them from v to x; note that, since $V(S) \cap V(F) = V(e) \cup V(f)$, we know that $z_1, z_2 \in V(e) \cup V(f)$. The path connecting z_1 and z_2 through v in C is entirely contained in S and has length at most $\ell - 1$. If z_1 and z_2 are contained in different signal edges, then there is a path of length at most ℓ between the two signal edges of S, which is a contradiction. Then z_1 and z_2 are contained in the same signal edge, say e. Since C is not fully contained in S, we know that C contains a vertex not contained in $V(S) \setminus V(f)$. Then the path connecting z_1 and z_2 in C, together with the edge e, forms a cycle of length less than ℓ that is fully contained in S, which is a contradiction. Hence, ψ must be a C_ℓ-free coloring. A similar argument shows that if H is 3-connected or isomorphic to K_3 and S is a signal sender as given by Lemma 2.6 with $d > v(H)$, then S is safe.

We are now ready to state our main technical result, proving the existence of safe $\mathcal{S}(C_\ell)$-determiners and safe $\mathcal{S}(K_t)$-determiners for q-tuples consisting of cycles and cliques, where we recall that $\mathcal{S}(C_\ell)$ and $\mathcal{S}(K_t)$ denote the cycle-colors $\{1,\dots,q_1\}$ and the clique-colors $\{q_1+1,\dots,q\}$, respectively.

THEOREM 2.8. *Let $\ell \geqslant 4$, $t \geqslant 3$, and $q_1, q_2 \geqslant 1$ be integers. Then there exist safe $\mathcal{S}(C_\ell)$-determiners and safe $\mathcal{S}(K_t)$-determiners for $\mathcal{T}(q_1,q_2,\ell,t)$.*

Most of section 5 is devoted to the proof of Theorem 2.8. In the same section, we also prove Theorem 2.9, showing that both safe $\mathcal{S}(C_\ell)$-senders and safe $\mathcal{S}(K_t)$-senders exist.

THEOREM 2.9. *Let $\ell \geqslant 4$, $t \geqslant 3$, and $q_1, q_2 \geqslant 1$ be integers. If $q_1 > 1$, then there exist safe positive and negative $\mathcal{S}(C_\ell)$-senders for $\mathcal{T}(q_1,q_2,\ell,t)$. If $q_2 > 1$, then there exist safe positive and negative $\mathcal{S}(K_t)$-senders for $\mathcal{T}(q_1,q_2,\ell,t)$.*

3. Two-color cases. Throughout this section the number of colors q is fixed to be 2, and we drop the color index q in the notation. In this section we determine $s(K_t,T_\ell)$, $s(C_k,C_\ell)$, and $s(K_t,C_\ell)$. We prove that the lower bound in (1.1) is tight for $s(K_t,T_\ell)$ and $s(C_k,C_\ell)$ but not for $s(K_t,C_\ell)$. In the latter two cases, we exemplify the power of the gadget graphs introduced in section 2. We begin with the case of one clique and one tree.

Proof of Proposition 1.1. Let $\ell \geqslant 2$ and $t \geqslant 3$. First, note that the inequality $s(K_t,T_\ell) > t-2$ follows directly from (1.1). For the upper bound, we construct a graph G of minimum degree $t-1$ as follows. Let $H \cong K_{(t-1)(\ell-1)}$, let $F \cong K_t$, and let v be a vertex of F. For each vertex u of $F-v$, create a copy H_u of H on a new set of vertices and identify u with an arbitrary vertex of H_u. Note that $d_G(v) = t-1$. We claim that $G \to (K_t,T_\ell)$ while $G - v \nrightarrow (K_t,T_\ell)$. For the former, suppose for a contradiction that φ is a (K_t,T_ℓ)-free red/blue-coloring of G. Then φ is (K_t,T_ℓ)-free on $H_u \cong K_{(t-1)(\ell-1)}$ for every vertex u in $F-v$. By [7, Lemma 9], there is a unique (K_t,T_ℓ)-free red/blue-coloring $\widetilde{\varphi}$ of H_u, in which the subgraph of blue edges of H is a collection of $(t-1)$ vertex-disjoint cliques, each of size $(\ell-1)$. In particular, in the coloring φ, every vertex u of $F-v$ is incident to a blue copy of $K_{\ell-1}$ in H_u. Therefore, every edge of F must be red, creating a monochromatic red copy of K_t, which is a contradiction. For the second claim, color the edges of $F-v$ red and use the (K_t,T_ℓ)-free coloring $\widetilde{\varphi}$ for every H_u. It is easy to see that this red/blue-coloring of $G-v$ is (K_t,T_ℓ)-free. Thus, any subgraph G' of G that is Ramsey-minimal for (K_t,T_ℓ) must contain v. This proves $s(K_t,T_\ell) \leqslant d_G(v) = t-1$. ∎

We now turn our attention to pairs of graphs involving cycles. It follows from (1.1) that $s(C_k,C_\ell) > 2$. For $k < \ell$, we now use the existence of safe determiners given by Lemma 2.4(i) to exhibit a Ramsey-minimal graph for (C_k,C_ℓ), with minimum degree three. Theorem 1.2(i) then follows by symmetry, since $s(C_k,C_\ell) = s(C_\ell,C_k)$.

PROPOSITION 3.1. *For any $4 \leqslant k < \ell$, we have*

$$s(C_k,C_\ell) \leqslant 3.$$

Proof. We construct an appropriate Ramsey-minimal graph. Start with an empty graph on three vertices $\{x,y,z\}$ and between any pair of these vertices add two paths, one of length $k-2$ and one of length $\ell-2$, so that all six paths are internally vertex-disjoint. Let D_r and D_b be safe red- and blue-determiners for (C_k,C_ℓ), respectively, as guaranteed by Lemma 2.4(i). Attach a copy of D_r to every edge contained in one

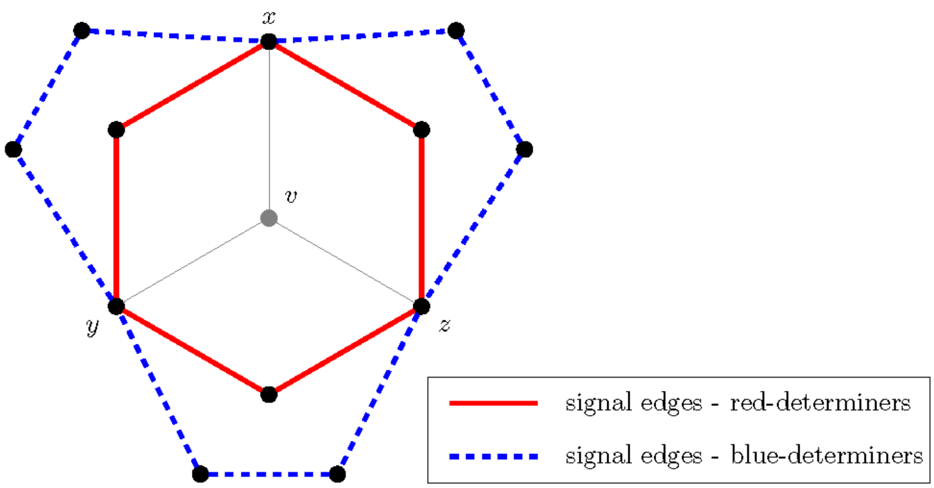

FIG. 1. *The graph F in the proof of Proposition* 3.1.

of the paths of length $k-2$ between x, y, and z and attach a copy of D_b to every edge contained in one of the paths of length $\ell-2$. Finally, add a new vertex v adjacent to x, y, and z, and call the resulting graph F. The construction is illustrated in Figure 1 for the case when $k=4$ and $\ell=5$, showing only the signal edges for each determiner and the edges incident to v. We will now show that $F \rightarrow (C_k, C_\ell)$ but $F - v \nrightarrow (C_k, C_\ell)$, implying that any subgraph G of F that is Ramsey-minimal for (C_k, C_ℓ) has to contain v, which in turn proves the proposition.

Consider an arbitrary red/blue-coloring of F. If any copy of D_r or D_b contains a red copy of C_k or a blue copy of C_ℓ, we are done. Otherwise, by property (D2) of D_r and D_b, the paths of length $k-2$ between the vertices x, y, and z must be all red, and the paths of length $\ell-2$ between those vertices must be all blue. By the pigeonhole principle, two of the edges incident to v must have the same color; these two edges, together with the corresponding red $(k-2)$-path or blue $(\ell-2)$-path, then form a red copy of C_k or a blue copy of C_ℓ.

For the second claim, consider $F-v$, and color each path of length $k-2$ between the vertices x, y, and z red and each path of length $\ell-2$ between those vertices blue. Since $k, \ell > 3$, it is easy to see that this partial coloring of $F-v$ is (C_k, C_ℓ)-free. By property (D3) of the copies of D_r and D_b, we can extend this coloring to the copies of D_r and D_b so that each determiner has a safe (C_k, C_ℓ)-free coloring. By the definition of safeness, this is a (C_k, C_ℓ)-free coloring of $F-v$. ∎

Note that the construction requires $k>3$. The case $k=3$ is covered by our next construction, dealing with cliques. To that end, we turn our attention to $s(K_t, C_\ell)$, proving Theorem 2(ii). The idea behind the upper bound construction is very similar to the previous one.

PROPOSITION 3.2. *For any integers $t \geqslant 3$ and $\ell \geqslant 4$, we have*

$$s(K_t, C_\ell) \leqslant 2(t-1).$$

Proof. Let $t \geqslant 3$ and $\ell \geqslant 4$. Using the safe determiners from Lemma 2.4(ii), we construct a graph G that is Ramsey-minimal for (K_t, C_ℓ) and satisfies $\delta(G) \leqslant 2(t-1)$.

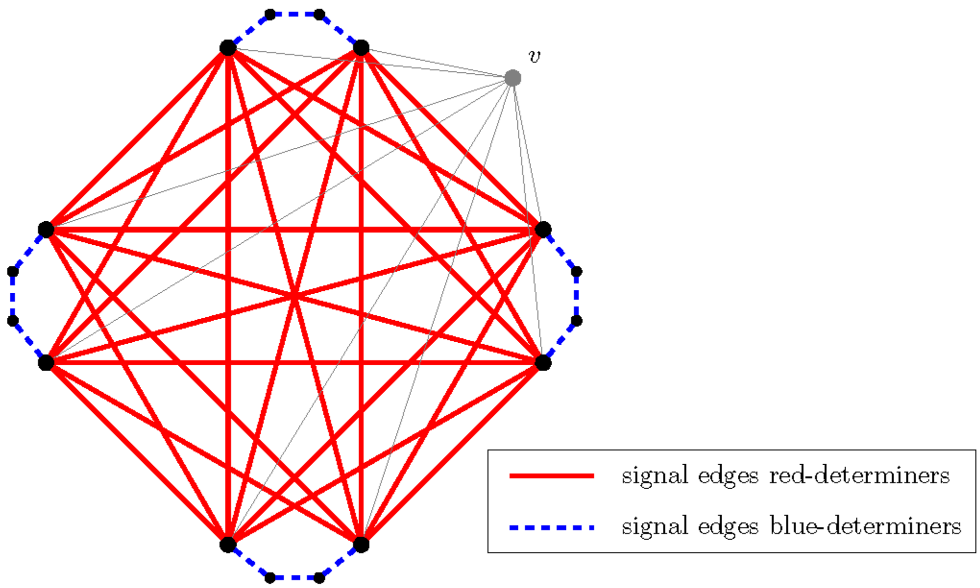

FIG. 2. *The graph F in the proof of Proposition* 3.2.

We start with the graph $T = K_{2,2,\ldots,2}$, the complete $(t-1)$-partite graph where each independent set contains two vertices. For any pair of vertices in the same class, add a path of length $\ell - 2$; as before, all these paths are vertex-disjoint. Let D_r and D_b be safe red- and blue-determiners, respectively, as guaranteed by Lemma 2.4(ii). Attach a copy of D_r to each edge of T and a copy of D_b to each edge belonging to one of the $t-1$ paths of length $\ell - 2$. Add a new vertex v adjacent to all vertices of T and call the resulting graph F. This construction is illustrated in Figure 2 for $t = 5$ and $\ell = 5$, showing only the signal edges for each determiner and the edges incident to v. As in the proof of Proposition 3.1, we will show that $F \to (K_t, C_\ell)$ but $F - v \nrightarrow (K_t, C_\ell)$.

To see the first claim, consider an arbitrary red/blue-coloring of F. If any copy of D_r or D_b contains a red copy of K_t or a blue copy of C_ℓ, then we are done. Hence, all determiners have (K_t, C_ℓ)-free colorings, forcing the edges of T to be all red and the edges in the $(\ell-2)$-paths connecting pairs of vertices from the same partite set of T to be blue. Now, if both edges between v and one of the vertex classes of T are blue, there is a blue copy of C_ℓ. Otherwise, there is a red edge from v to each of the $t-1$ partite sets of T, resulting in an all-red copy of K_t.

For the second claim, color the edges of T red and the edges of the $(\ell-2)$-paths connecting vertices from the same vertex class of T blue. Then, using property (D3) of the copies of D_r and D_b, extend this coloring to all determiners so that each one receives a safe (K_t, C_ℓ)-free coloring. It is easy to see that this gives a (K_t, C_ℓ)-free coloring of the entire graph F. ∎

Note that this upper bound for $s(K_t, C_\ell)$ does not match the lower bound from (1.1), as the latter only implies $s(K_t, C_\ell) \geqslant t$. However, Proposition 3.4 will prove that our construction does yield the best possible upper bound. We will need an auxiliary lemma, which shows that if G is a graph on fewer than $2(t-1)$ vertices with no t-clique, then there must be at least one vertex common to all $(t-1)$-cliques.

LEMMA 3.3. *Let $t \geqslant 3$ be any integer, and let G be a graph on $n < 2(t-1)$ vertices with $K_{t-1} \subseteq G$. If*

$$\bigcap_{\substack{H \subseteq G \\ H \cong K_{t-1}}} V(H) = \emptyset,$$

then $K_t \subseteq G$.

Proof. We proceed by strong induction on t. It is easy to check that the statement is true for $t = 3$. Assume now that $t \geqslant 3$, and suppose the statement to be true up to t.

Let G be a graph on $n < 2t$ vertices, and let $\mathcal{F} = \{H_0, \ldots, H_m\}$ be a family of distinct t-cliques contained in G whose joint intersection is empty. Suppose, additionally, that this family is minimal, meaning that every subfamily has a nonempty intersection. Note that we may assume that $m \geqslant 1$.

Let $S = V(H_1) \cap \cdots \cap V(H_m)$ be the vertex set in the intersection of the t-cliques $H_1, \ldots, H_m$ (without considering H_0). By the minimality of the family $\mathcal{F}$, we know that $|S| > 0$. Further, since G has fewer than $2t$ vertices, it cannot contain two disjoint t-cliques. Therefore, as H_0 is a t-clique and S is another clique disjoint from H_0 in G, it follows that $|S| \leqslant t-1$. Write $|S| = t - j$ for some $0 < j < t$.

For $i \in [m]$, let $S_i = V(H_i) \setminus S$. Note that each S_i induces a j-clique. Each vertex in S_i is adjacent to all vertices in S. Therefore, since $|S| = t - j$, if we can find a $(j+1)$-clique in $G[\bigcup_{i=1}^m S_i]$, we will have found a $(t+1)$-clique in G. We consider two possible cases.

Case 1: Suppose that $\bigcup_{i=1}^m S_i$ has at least $2j$ elements. By definition, both $V(H_0)$ and $\bigcup_{i=1}^m S_i$ have an empty intersection with S, and therefore both are contained in the set $V(G) \setminus S$ whose size is less than $t+j$. Since $|V(H_0)| = t$ and $|\bigcup_{i=1}^m S_i| \geqslant 2j$, they must have at least $j+1$ vertices in common, forming a $(j+1)$-clique in $G[\bigcup_{i=1}^m S_i]$.

Case 2: Assume next that $\bigcup_{i=1}^m S_i$ has fewer than $2j$ elements. Then $G[\bigcup_{i=1}^m S_i]$ is a graph on fewer than $2j$ vertices containing a j-clique, namely $G[S_1]$. Since $j < t$ and $\bigcap_{i=1}^m S_i = \emptyset$, by the induction hypothesis, it follows that $G[\bigcup_{i=1}^m S_i]$ contains a $(j+1)$-clique. ◻

We are now ready to prove a lower bound on $s(K_t, C_\ell)$ using Lemma 3.3. Theorem 1.2(i) then follows immediately from Proposition 3.2.

PROPOSITION 3.4. *For any integers $t \geqslant 3$ and $\ell \geqslant 4$, we have*

$$s(K_t, C_\ell) \geqslant 2(t-1).$$

Proof. Suppose that G is a Ramsey-minimal graph for (K_t, C_ℓ), and let v be a vertex of degree at most $2(t-1) - 1$ in G, i.e., $|N(v)| < 2(t-1)$. By the minimality of G, there exists a red/blue-coloring φ of the edges of $G - v$ with no red copy of K_t and no blue copy of C_ℓ. If $G[N(v)]$ contains no red copy of K_{t-1}, then we can extend the coloring φ to G by coloring all edges incident to v red to obtain a (K_t, C_ℓ)-free coloring of G, which is a contradiction.

Therefore assume that we have at least one red copy of K_{t-1} in $G[N(v)]$. By Lemma 3.3, because $G[N(v)]$ has no red copy of K_t and $|N(v)| < 2(t-1)$, there exists at least one vertex u in the intersection of all red copies of K_{t-1} in $G[N(v)]$. Extend φ to G by coloring the edge uv blue and all other edges from v to $N(v) \setminus \{u\}$ red. This coloring does not create a red copy of K_t, and the unique blue edge incident to v cannot create a blue copy of C_ℓ, again contradicting the fact that G is Ramsey for (K_t, C_ℓ). ◻

Note that a straightforward generalization of Proposition 3.4 to the multicolor setting implies the following improvement on the lower bound in (1.1) for any tuple of the form $(K_t, H_2, \ldots, H_q)$, where $\delta(H_i) \geqslant 1$ for all $i \in \{2, \ldots, q\}$ and $\delta(H_i) > 1$ for at least one index $i \in \{2, \ldots, q\}$:

$$s_q(K_t, H_2, \ldots, H_q) > 2(t-2) + \sum_{i=2}^{q}(\delta(H_i) - 1).$$

4. Proofs of Theorems 1.3 and 1.5. As noted earlier, we defer the proofs of Theorems 2.8 and 2.9 to section 5. In this section, we assume their statements to be true and use them to prove our main results, Theorems 1.3 and 1.5. Recall that $\mathcal{T} = \mathcal{T}(q_1, q_2, \ell, t)$ denotes the q-tuple of cycles and cliques as defined in (1.2) and that $\mathcal{S}(C_\ell)$ and $\mathcal{S}(K_t)$ denote the cycle-colors $\{1, \ldots, q_1\}$ and clique-colors $\{q_1 + 1, \ldots, q\}$, respectively, while $\mathcal{S}$ denotes the full-color palette $\{1, \ldots, q\}$. The idea is to express our function $s_q(\mathcal{T})$ in a different way through a certain packing parameter. This idea was first formalized in [19] in their study of $s_q(K_t)$ in the multicolor setting, but, as the authors of [19] note, this idea is already implicit in [8].

4.1. Packing parameters. In this section we generalize the packing parameter defined in [19]. A *color pattern* on vertex set V is a collection of edge disjoint graphs $G_1, \ldots, G_m$ on the same vertex set V. A color pattern is H-free if every graph in it is H-free.

Definition 4.1. *Given positive integers $t \geqslant 2$ and $q_1, q_2 \geqslant 0$, let $P_{q_1,q_2}(t)$ be the smallest integer n such that there exists a color pattern $G_{q_1+1}, \ldots, G_{q_1+q_2}$ on vertex set $[n]$ such that*

(P1) *the graph G_j is K_{t+1}-free for every $j \in \mathcal{S}(K_t)$; and*

(P2) *for every vertex-coloring $\lambda : [n] \to \mathcal{S}$, we have that* (a) *two distinct vertices u and w receive the same cycle-color; or* (b) *there exists a clique-color $j \in \mathcal{S}(K_t)$ such that G_j contains a copy of K_t on the vertices of color j.*

For $q_1 = 0$, this parameter was introduced in [19], and for all $q_2 \geqslant 2$ and $t \geqslant 3$, Theorem 1.5 in [19] establishes that $s_{q_2}(K_t) = P_{0,q_2}(t-1)$. The following lemma generalizes this theorem and proves that $s_q(\mathcal{T})$ does not depend on ℓ.

Lemma 4.2. *For all integers $\ell \geqslant 4$, $t \geqslant 3$, and $q_1, q_2 \geqslant 0$, we have*

$$s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, t)) = P_{q_1,q_2}(t-1).$$

Proof. Set $q = q_1 + q_2$ and $\mathcal{T} = \mathcal{T}(q_1, q_2, \ell, t)$. We divide the proof into two claims.

Claim 4.3. $s_q(\mathcal{T}) \leqslant P_{q_1,q_2}(t-1)$.

Proof. As explained previously, in this proof we assume the existence of gadget graphs as guaranteed by Theorems 2.8 and 2.9. Let $n = P_{q_1,q_2}(t-1)$ and $G_{q_1+1}, \ldots, G_q$ be a color pattern on $[n]$ that satisfies (P1) and (P2). For every pair of distinct vertices $u, w \in [n]$ and every cycle-color $i \in \mathcal{S}(C_\ell)$, add a path $P_i(u, w)$ of length $\ell - 2$ between u and w such that the internal vertices of these paths are pairwise disjoint. Finally, add a new vertex v and connect it to each vertex in $[n]$. Call the resulting graph H.

Assume first that $q_1, q_2 > 1$. Now, let S_c^+ and S_c^- be safe positive and negative $\mathcal{S}(C_\ell)$-senders for $\mathcal{T}$, respectively, and let S_k^+ and S_k^- be safe positive and negative $\mathcal{S}(K_t)$-senders for $\mathcal{T}$; all of these gadgets exist by Theorem 2.9. Let $E = \{e_1, \ldots, e_q\}$ be a matching of size q. For each pair $i, j \in \mathcal{S}(C_\ell)$ of distinct cycle-colors, join the edges e_i and e_j by a copy of S_c^-. Similarly, for each pair $i, j \in \mathcal{S}(K_t)$ of distinct clique-colors, join the edges e_i and e_j by a copy of S_k^-. For every clique-color $i \in \mathcal{S}(K_t)$ and every edge $f \in E(G_i)$, join the edges e_i and f by a copy of S_k^+. Then for each $i \in \mathcal{S}(C_\ell)$ and for each edge $f \in P_i(u,w)$, join the edges e_i and f by a copy of S_c^+. Call the resulting graph G.

We will show that $G \to_q \mathcal{T}$ but $G - v \nrightarrow_q \mathcal{T}$. We begin with the latter. For this we define a $\mathcal{T}$-free coloring. For all $i \in \mathcal{S}(K_t)$, give all edges of G_i color i. For all $i \in \mathcal{S}(C_\ell)$ and every pair of distinct vertices $u, w \in [n]$, color the edges of $P_i(u,w)$ with color i. Finally, for all $i \in [q]$, give e_i color i. This coloring can now be extended to the set-senders so that each set-sender receives a safe $\mathcal{T}$-free coloring. Suppose there exists a monochromatic cycle in a cycle-color or monochromatic clique in a clique-color. By the safeness of the coloring of each set-sender, we know that such a monochromatic subgraph has to be contained in $H - v$. But $H - v$ contains no monochromatic copy K_t in a clique-color by property (P1) of the color pattern. By construction, it is not difficult to see that it also contains no monochromatic copy of C_ℓ in a cycle-color. Hence, this is a $\mathcal{T}$-free coloring of $G - v$, as claimed.

We now prove that $G \to_q \mathcal{T}$. For the sake of contradiction, let $\varphi : E(G) \to \mathcal{S}$ be a $\mathcal{T}$-free q-coloring of the edges of G. In any such coloring, property (S2) of the copies of S_c^- and S_k^- ensures that $\{\varphi(e_1), \ldots, \varphi(e_{q_1})\} = \mathcal{S}(C_\ell)$, while $\{\varphi(e_{q_1+1}), \ldots, \varphi(e_q)\} = \mathcal{S}(K_t)$. Without loss of generality, we may assume that for any $i \in \mathcal{S}$, we have $\varphi(e_i) = i$. Property (S2) of the copies of S_k^+ and S_c^+ further ensures that for any $i \in \mathcal{S}(K_t)$, each edge in G_i has color i, and for each pair of vertices $u, w \in [n]$ and each $j \in \mathcal{S}(C_\ell)$, the edges of $P_j(u,v)$ receive color j.

Consider now the edges from v to $N(v) = [n]$. These induce a natural vertex-coloring $\lambda : [n] \to \mathcal{S}$ defined by $\lambda(u) = \varphi(vu)$ for each $u \in [n]$. Then by property (P2), it follows that either there are two distinct vertices $u, w \in [n]$ such that $\lambda(u) = \lambda(w) = j$ for some $j \in \mathcal{S}(C_\ell)$, or there exists a clique-color $j \in \mathcal{S}(K_t)$ such that $G_j[\lambda^{-1}(\{j\})]$ contains a copy of K_{t-1}. In the former case, $P_j(u,w)$ forms a monochromatic copy of C_ℓ in color j together with v. In the latter case, the copy of K_{t-1} forms a monochromatic copy of K_t in color j together with v.

It follows that G is q-Ramsey for $\mathcal{T}$, while $G - v$ is not. So any q-Ramsey-minimal subgraph of G must contain the vertex v, and therefore $s_q(\mathcal{T}) \leqslant d_G(v) = n = P_{q_1,q_2}(t-1)$.

If $q_1 = 1$ and/or $q_2 = 1$, we use a safe $\mathcal{S}(C_\ell)$-determiner D_c instead of $\mathcal{S}(C_\ell)$-senders, and/or a safe $\mathcal{S}(K_t)$-determiner D_k instead of $\mathcal{S}(K_t)$-senders. These gadgets exist by Theorem 2.8. If $q_1 = 1$, for each $i \in \mathcal{S}(C_\ell)$ and for each edge $f \in P_i(u,w)$, we attach a copy of D_c to f. If $q_2 = 1$, for each $i \in \mathcal{S}(K_t)$ and every edge $f \in E(G_i)$, we attach a copy of D_k to f. The rest of the proof is identical to the case $q_1, q_2 > 1$, using corresponding properties of set-determiners. ∎

CLAIM 4.4. $s_q(\mathcal{T}) \geqslant P_{q_1,q_2}(t-1)$.

Proof. Towards a contradiction, assume that there exists a graph G with a vertex v of degree $n < P_{q_1,q_2}(t-1)$, such that G is q-Ramsey-minimal for $\mathcal{T}$. By minimality, there exists a $\mathcal{T}$-free q-coloring φ of the edges of $G - v$. This coloring induces a color pattern $G_{q_1+1}, \ldots, G_q$ on $N(v)$, corresponding to the colors $q_1+1, \ldots, q$, respectively, such that every G_j is K_t-free. Since $|N(v)| < P_{q_1,q_2}(t-1)$ and each G_j is K_t-free, by property (P2) there must exist a vertex-coloring $\lambda : N(v) \to \mathcal{S}$ such that no two vertices in $N(v)$ receive the same cycle-color, and there is no clique-color j such that $G_j[\lambda^{-1}(\{j\})]$ contains a copy of K_{t-1}. Now, we extend φ to all of G by setting $\varphi(uv) = \lambda(u)$ for each $u \in N(v)$.

By the properties of λ, this extended coloring has no monochromatic copy of C_ℓ in any color $j \in \mathcal{S}(C_\ell)$ and no monochromatic copy of K_t in any color $j \in \mathcal{S}(K_t)$, contradicting the fact that G is q-Ramsey for $\mathcal{T}$. ∎

4.2. Proof of Theorem 1.3. We are now ready to prove our first main result in the multicolor setting. We begin with the lower bound.

LEMMA 4.5. *For all $q_1, q_2 \geqslant 1$, $t \geqslant 3$, and $\ell \geqslant 4$, we have*

$$s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, t)) \geqslant s_{q_2}(K_t) + s_{q_1}(C_\ell) - 1 = s_{q_2}(K_t) + q_1. \tag{4.1}$$

Proof. Set $q = q_1 + q_2$ and $\mathcal{T} = \mathcal{T}(q_1, q_2, \ell, t)$, and suppose that G is a q-Ramsey-minimal graph for $\mathcal{T}$ containing a vertex v of degree at most $s_{q_2}(K_t) + s_{q_1}(C_\ell) - 2$. Let $\varphi : E(G-v) \to [q]$ be a $\mathcal{T}$-free q-coloring of $G-v$. Let G' be the subgraph of G containing all edges of $G-v$ with colors $q_1+1, \dots, q$ and any set of $\min\{s_{q_2}(K_t) - 1, \ \deg_G(v)\}$ edges of G incident to v. We know that $G' - v$ is not q_2-Ramsey for K_t, and since $\deg_{G'}(v) < s_{q_2}(K_t)$, it follows that G' itself cannot be q_2-Ramsey for K_t. Thus, we can recolor the edges of G' using colors $q_1+1, \dots, q$ so that there is no monochromatic copy of K_t inside. Now, we can apply the same argument to $G - G'$ to obtain a C_ℓ-free coloring of it with the colors $1, \dots, q_1$. These two colorings together yield a $\mathcal{T}$-free coloring of G, which is a contradiction. The last equality follows from the fact that $s_q(C_\ell) = q + 1$ [6]. ∎

From the proof of this lower bound it becomes clear that this is actually a generalization of the trivial lower bound given in (1.1). We now proceed with the upper bound. For this we take a slightly indirect approach: instead of working directly with the parameter s_q, we show a relation between the two packing parameters.

LEMMA 4.6. *For all $q_1, q_2 \geqslant 1$, $t \geqslant 3$, and $\ell \geqslant 4$, we have*

$$s_{q_1+q_2}(\mathcal{T}(q_1, q_2, \ell, t)) = P_{q_1,q_2}(t-1) \leqslant P_{0,q_1+q_2}(t-1) = s_{q_1+q_2}(K_t). \tag{4.2}$$

Proof. Again set $q = q_1 + q_2$ and let $n = P_{0,q}(t-1)$. Let $G_1, \dots, G_q$ be a color pattern on $[n]$, as guaranteed by Definition 4.1 of $P_{0,q}(t-1)$. Consider only the last q_2 graphs; we claim that this color pattern satisfies properties (P1) and (P2) from Definition 4.1 of $P_{q_1,q_2}(t-1)$. The first property is clear. Now let $\lambda : [n] \to \mathcal{S}$ be any coloring. Then we know that there is some $j \in \mathcal{S}$ such that G_j contains a monochromatic copy of K_{t-1} on the vertices of color j. Now, if $j > q_1$, then case (b) from property (P2) occurs. Otherwise, we have $j \leqslant q_1$, and thus there must be at least $t - 1 \geqslant 2$ vertices of color j, implying that case (a) from property (P2) happens. Hence, $P_{q_1,q_2}(t-1) \leqslant P_{0,q}(t-1)$, and the two equalities follow from Lemma 4.2 and the discussion that precedes it. ∎

4.3. Proof of Theorem 1.5. We now prove our second main result for multiple colors. In [19], it was shown that, for all $q \geqslant 2$ and $t \geqslant 3$, there exists a color pattern $G_1, \dots, G_q$ on the vertex set $[n]$, for some n, such that

(i) G_i is K_t-free for every $i \in [n]$, and
(ii) any subset of $[n]$ of size n/q contains a copy of K_{t-1} in each color.

The results in [19] include bounds on n in terms of q, which are unnecessary for our purpose. Theorem 1.5 follows from the next lemma and Lemma 4.2.

LEMMA 4.7. *Given $0 < \varepsilon < 1$ and integers $q_2 \geqslant 1$ and $t \geqslant 3$, there exists an integer $q_0 \geqslant 1$ such that for all $q_1 \geqslant q_0$, we have*

$$P_{q_1,q_2}(t-1) \leqslant (1+\varepsilon) q_1.$$

Proof. Let $0<\varepsilon<1$, $q_2\geqslant 1$, and $t\geqslant 3$ be fixed. For q_1 large enough, there exists a color pattern $G_1,\dots,G_{q^*}$ on $n\in[(1+\varepsilon/2)q_1,(1+\varepsilon)q_1]$ vertices, given by the result in [19], with q^* large enough compared to q_2.

Keeping only the first q_2 graphs in the color pattern, which we denote for convenience by $G_{q_1+1},\dots,G_{q_1+q_2}$, we claim that they satisfy properties (P1) and (P2). The first one is clear. For the second, consider a vertex coloring $\lambda:[n]\to[q]$, where $q=q_1+q_2$. Let $\mathcal{C}$ be its largest color class in $\mathcal{S}(K_t)$, with color c. If (a) does not hold, then by the pigeonhole principle the color class $\mathcal{C}$ has size at least $\frac{n-q_1}{q_2}$. Since q^* is large enough compared to q_2, and by choice of n, we have $\frac{n-q_1}{q_2}\geqslant\frac{n}{q^*}$. By property (ii) above, we know that there exists a copy of K_{t-1} in $G_c[\mathcal{C}]$. Therefore if (a) of (P2) does not hold, then (b) does, and $P_{q_1,q_2}(t-1)\leqslant n\leqslant(1+\varepsilon)q_1$. ◻

5. Existence of set-determiners and set-senders. In this section we construct set-determiners and set-senders for tuples of the form $(C_\ell,\dots,C_\ell,K_t,\dots,K_t)$, that is, we prove Theorems 2.8 and 2.9. Our set-senders will be constructed in several stages. Before diving into the proofs, we give a brief overview.

Throughout the rest of the section, assume that $\ell\geqslant 4$, $t\geqslant 3$, and $q,q_1,q_2\geqslant 1$ are fixed integers such that $q_1+q_2=q$ and recall that $\mathcal{T}=\mathcal{T}(q_1,q_2,\ell,t)$ denotes the q-tuple of cycles and cliques as defined in (1.2). First, we construct a graph Γ that is q-Ramsey for the tuple $\mathcal{T}$ and has certain special properties; for this, we generalize the ideas of Bollobás et al. [5] used to construct 2-Ramsey graphs for certain pairs of graphs, including (C_ℓ,K_t), to multiple colors. This graph Γ is built by sampling a random hypergraph, applying alterations to remove all short cycles from it, and then replacing every hyperedge by a large (depending only on t) clique. In order to prove the claimed properties of Γ, we use a number of results, all of which are fairly standard by now. Second, we modify Γ slightly and construct set-determiners for each of the color palettes $\mathcal{S}(C_\ell)$ and $\mathcal{S}(K_t)$. This is a generalization of a construction given by Siggers in [40] which was valid for certain pairs of the form (C_ℓ,H). Finally, since we need finer control over the color patterns that we force on given set of edges when $q_1>1$ or $q_2>1$, we build set-senders from our set-determiners. This final step is the main novelty in this section.

5.1. Preliminary results. We begin by collecting the different results that will be needed for the construction and proof of the claimed properties of the graph Γ.

Hypergraphs with few short cycles. First, we need to construct a uniform hypergraph with no short cycles that is, nevertheless, not too sparse. This is done using a standard construction due to Erdős and Hajnal [14], starting from a random hypergraph. We state the necessary results about random hypergraphs without proof, as these are standard applications of the probabilistic method. A *cycle of length s* in a hypergraph $\mathcal{H}$ is a sequence $e_1,v_1,e_2,v_2\dots,e_s,v_s$ of distinct hyperedges and vertices of $\mathcal{H}$ such that $v_i\in e_i\cap e_{i+1}$ for all $1\leqslant i<s$ and $v_s\in e_s\cap e_1$. Note in particular that two edges intersecting in more than one vertex form a cycle of length two in $\mathcal{H}$. The *girth* of a hypergraph $\mathcal{H}$ is the length of the shortest cycle in $\mathcal{H}$ (if no cycle exists, then by convention we say that the girth of $\mathcal{H}$ is infinity).

LEMMA 5.1. *Let $\ell,h\geqslant 2$ be fixed integers and let $p_h=An^{-(h-1)+1/(\ell-1)}$, where A is a constant. For an integer $n\geqslant 1$, let $\mathcal{H}_{n,p_h}$ be a random h-uniform hypergraph on $[n]$ in which each h-subset of $[n]$ is added as an edge with probability p_h, independently of all other h-subsets. Then, as $n\to\infty$, the following hold with high probability:*

(i) $e(\mathcal{H}_{n,p_h}) = (1+o(1))\binom{n}{h}p_h$.
(ii) *The number of cycles in* $\mathcal{H}_{n,p_h}$ *of length less than* ℓ *is* $o(e(\mathcal{H}_{n,p_h}))$.

Part (i) follows from an application of the Chernoff bound (see, for example, [32]), while part (ii) is shown using a first-moment argument.

Quantitative version of Ramsey's theorem. The following lemma is a simple consequence of Ramsey's theorem and is obtained by a straightforward averaging argument. Informally, it says that for any r-tuple of graphs $(H_1,\dots,H_r)$, if we r-color a sufficiently large complete graph, then we can find not just one monochromatic H_i in the correct color but many of them. The proof is a simple generalization of the one given, for example, in [33, Theorem 2].

Lemma 5.2 (quantitative version of Ramsey's theorem). *Let* $r \geqslant 1$ *and* $H_1,\dots,H_r$ *be graphs. Then there exist a real number* $c = c(H_1,\dots,H_r) > 0$ *and an integer* $k_0 = k_0(H_1,\dots,H_r) \geqslant 1$ *such that if* $k \geqslant k_0$ *and the edges of* K_k *are colored with* r *colors, then there exists an* $i \in [r]$ *such that there are at least* $ck^{v(H_i)}$ *monochromatic copies of* H_i *in color* i.

Colorful sparse regularity lemma. One of the tools required for showing that Γ is q-Ramsey for the tuple $\mathcal{T}$ is a version of Szemerédi's celebrated regularity lemma [44]. More specifically, we will need the colorful sparse version of the lemma as given, for example, in [31] (see also [27, Lemma 3.1]). Before giving the precise statement in Lemma 5.4, we again need several definitions.

Definition 5.3. *Let* G *be a graph on* n *vertices, and let* $0 < \eta \leqslant 1$ *and* $0 < p \leqslant 1$. *Also let* U *and* W *be disjoint subsets of* $V(G)$. *The* p*-density of the pair* (U,W) *is defined to be*

$$d_{G,p}(U,W) = \frac{e_G(U,W)}{p|U||W|},$$

where $e_G(U,W)$ *denotes the number of edges in* G *with one endpoint in* U *and one endpoint in* W.

The pair (U,W) *is said to be* (ε,p)*-regular if for all* $U' \subseteq U$ *and* $W' \subseteq W$ *with* $|U'| \geqslant \varepsilon|U|$ *and* $|W'| \geqslant \varepsilon|W|$, *we have*

$$|d_{G,p}(U',W') - d_{G,p}(U,W)| \leqslant \varepsilon.$$

If (U,W) *is* (ε,p)*-regular with* $p = \frac{e_G(U,W)}{|U||W|}$, *then we say that* (U,W) *is* (ε)*-regular for short. A partition* $P = (V_1,\dots,V_k)$ *of* $V(G)$ *is an equipartition if* $|V_i| \in \{\lfloor \frac{v(G)}{k} \rfloor, \lceil \frac{v(G)}{k} \rceil\}$ *for all* $i \in [k]$. *An equipartition is said to be an* (ε,p)*-regular partition if all but at most* $\varepsilon\binom{k}{2}$ *pairs* (V_i,V_j) *are* (ε,p)*-regular.*

A graph G *is said to be* (η,D,p)*-upper uniform if for all disjoint* $U,W \subseteq V(G)$ *with* $|U|,|W| \geqslant \eta v(G)$, *we have* $d_{G,p}(U,W) \leqslant D$.

We are now ready to state the version of the regularity lemma that we are going to use.

Lemma 5.4 (colorful sparse regularity lemma). *Let* $\varepsilon > 0$ *and* $D > 1$ *be fixed reals, and let* $k_0 \geqslant 1$ *and* $r \geqslant 1$ *be integers. Then there exist constants* $\eta = \eta(\varepsilon,k_0,D,r)$ *and* $K_0 = K_0(\varepsilon,k_0,D,r)$ *for which the following holds: If* $0 \leqslant p \leqslant 1$ *and* $G_1,\dots,G_r$ *are* (η,D,p)*-upper uniform graphs on vertex set* $[n]$, *then there is an equipartition* $(V_1,\dots,V_k)$ *of* $[n]$ *for some* $k_0 \leqslant k \leqslant K_0$ *such that all but at most* $\varepsilon\binom{k}{2}$ *of the pairs* (V_i,V_j) *are* (ε,p)*-regular in* G_s *for all* $s \in [r]$.

We will also need the following additional technical lemma, which can be found, for example, in [22, Lemma 4.3].

LEMMA 5.5. *Given $0 < \varepsilon < 1/6$, there exists a constant $\beta > 0$ such that the following holds. For any graph $F = (V_1 \cup V_2, E)$, where the pair (V_1, V_2) is (ε)-regular in F, and for all M satisfying $\beta v(F) \leqslant M \leqslant e(F)$, there exists a subgraph $F' = (V_1 \cup V_2, E')$ with $|E'| = M$ and such that (V_1, V_2) is (2ε)-regular in F'.*

Enumeration lemma for C_ℓ-free graphs. Let $m, M \geqslant 1$ and $\ell \geqslant 4$ be integers, and let $\varepsilon > 0$ be a real number. Let $V_1, \ldots, V_\ell$ be disjoint sets, each of size m. Let $\mathcal{G}(\ell, m, (V_i)_{i=1}^{\ell}, M, \varepsilon)$ denote the collection of graphs G such that

- $V(G) = V_1 \cup \cdots \cup V_\ell$, where $|V_i| = m$ for each $i \in [\ell]$;
- each V_i is an independent set in G;
- the pair (V_i, V_{i+1}) is $(\varepsilon, \frac{M}{m^2})$-regular in G with $e_G(V_i, V_{i+1}) = M$ for all $i \in [\ell]$[1]; and
- there are no edges between any other pair (V_i, V_j).

In other words, the graphs in $\mathcal{G}(\ell, m, (V_i)_{i=1}^{\ell}, M, \varepsilon)$ are blowups of the cycle C_ℓ in which each vertex v_i of C_ℓ is blown up to an independent set V_i of size m and such that each edge $v_i v_{i+1}$ of C_ℓ corresponds to an $(\varepsilon, \frac{M}{m^2})$-regular pair (V_i, V_{i+1}). Let $\mathcal{F}(\ell, m, (V_i)_{i=1}^{\ell}, M, \varepsilon)$ denote the set of graphs in $\mathcal{G}(\ell, m, (V_i)_{i=1}^{\ell}, M, \varepsilon)$ that do not contain C_ℓ as a subgraph.

The following enumeration lemma was shown by Gerke et al. [21, Theorem 5.2]; it is a special case of a well-known conjecture by Kohayakawa, Łuczak, and Rödl [30] (the so-called KŁR conjecture), which was famously resolved in the general case using the container method [2, 38].

LEMMA 5.6 (counting lemma). *For any real number $\alpha > 0$ and integer $\ell \geqslant 4$, there are constants $\varepsilon_0 = \varepsilon_0(\ell, \alpha) > 0, C_0 = C_0(\ell, \alpha) > 0$, and $m_0 = m_0(\ell, \alpha) \geqslant 1$ such that for all $m \geqslant m_0$, $0 < \varepsilon \leqslant \varepsilon_0$, and $M \geqslant C_0 m^{1+1/(\ell-1)}$, we have*

$$|\mathcal{F}(\ell, m, (V_i)_{i=1}^{\ell}, M, \varepsilon)| \leqslant \alpha^M \binom{m^2}{M}^{\ell}.$$

5.2. Construction of a special graph Γ. For the rest of the section, assume that n is a sufficiently large integer with respect to ℓ, t, q, q_1, and q_2; in all asymptotic estimates in this section, we assume that n tends to infinity. We begin by fixing some constants. Let $h = r_{q_2}(K_t)$; it is not difficult to check that K_h is minimal q_2-Ramsey for K_t. Let

$$k_0 = k_0(\underbrace{C_\ell, \ldots, C_\ell}_{q_1 \text{ times}}, K_h, K_2), \quad c = c(\underbrace{C_\ell, \ldots, C_\ell}_{q_1 \text{ times}}, K_h, K_2)$$

be the constants given by Lemma 5.2. We next set

$$\rho = \frac{c}{2q_1}, \quad \alpha = \frac{\rho^\ell}{e^{\ell+1}}, \quad D = 3h^2.$$

Let $\varepsilon_0 = \varepsilon_0(\ell, \alpha), m_0 = m_0(\ell, \alpha)$, and $C_0 = C_0(\ell, \alpha)$ be the constants given by Lemma 5.6, and set

$$\varepsilon = \min\{\rho\varepsilon_0/2, \rho/10\}, \quad C = \max\{C_0, 1\}.$$

[1]For convenience, we define $V_{\ell+1} = V_1$.

Further, let

$$\eta = \eta(\varepsilon, k_0, D, q_1), \quad K_0 = K_0(\varepsilon, k_0, D, q_1), \quad \beta = \beta(\varepsilon/\rho)$$

be the constants from Lemmas 5.4 and 5.5. Finally, define

$$A = \max\{(h+1)e^{-h}, \rho^{-1}K_0^{1-1/(\ell-1)}C\},\ p_h = An^{-(h-1)+1/(\ell-1)},\ p_e = An^{-1+1/(\ell-1)}.$$

Let $\mathcal{H}$ be a hypergraph on $[n]$ sampled from $\mathcal{H}_{n,p_h}$ as in Lemma 5.1. Let $\mathcal{G}$ be the hypergraph obtained from $\mathcal{H}$ after the removal of one hyperedge from each cycle of length less than ℓ. Then $\mathcal{G}$ contains no cycles of length less than ℓ; by Lemma 5.1 (i) and (ii), we also know that $e(\mathcal{G}) = (1+o(1))\binom{n}{h}p_h$.

Let Γ be the graph on $[n]$ obtained by embedding a copy of K_h into every hyperedge of $\mathcal{G}$; i.e., Γ is the graph on $[n]$ in which two vertices are adjacent if and only if they are contained in a common hyperedge of the hypergraph $\mathcal{G}$. The main difference between this construction and the one given in [5] is that in order to deal with multiple colors, instead of placing just a copy of our target graph K_t in each hyperedge of $\mathcal{G}$, we place a Ramsey graph for it. For a given graph F and a subgraph $\Gamma' \subseteq \Gamma$, we call a copy F' of F in Γ' a *hyperedge copy* if the vertex set of F' is contained within a single hyperedge of $\mathcal{G}$. All remaining copies of F in Γ' are referred to as *nonhyperedge copies*. In addition, we call a subgraph $\Gamma' \subseteq \Gamma$ *transversal* if there exists a bijection $f : E(\Gamma') \to E(\mathcal{G})$ such that $e \subseteq f(e)$ for all $e \in E(\Gamma')$; that is, Γ' is transversal if it contains exactly one edge from each hyperedge copy of K_h in Γ.

Before showing that with high probability $\Gamma \to_q \mathcal{T}(q_1, q_2, \ell, t)$ in Theorem 5.8, we discuss some properties of the graph Γ in Lemma 5.7. The proofs of parts (a), (b), and (d) are essentially the same as those given in [5]. The proof of (c) is now also standard in light of the recently resolved KŁR conjecture; as we believe that our version (using more modern results) can be generalized more easily to other tuples of graphs, we include the details in Appendix A.

LEMMA 5.7. *The graph Γ satisfies each of the following properties with high probability:*

(a) *If F is a 2-connected graph with no induced cycles of length ℓ or longer, then every copy of F in Γ is a hyperedge copy; in particular, every copy of K_h, K_t, and $C_{\ell'}$ for any $\ell' < \ell$ in Γ is a hyperedge copy.*

(b) *Γ is (η, D, p_e)-upper uniform.*

(c) *Let m be an integer satisfying $\frac{n}{K_0} \leqslant m \leqslant \frac{n}{k_0}$, let $(V_1, \ldots, V_\ell)$ be any ℓ-tuple of disjoint subsets of $V(\Gamma)$ such that $|V_i| = m$ for all $i \in [\ell]$, and let $\Gamma' \subseteq \Gamma$ be transversal. If the pairs (V_i, V_{i+1}) are (ε, p_e)-regular in Γ' with p_e-density at least ρ for all $i \in [\ell]$, then $\Gamma'[V_1 \cup \cdots \cup V_\ell]$ contains a copy of C_ℓ.*

(d) *Let m be an integer satisfying $\frac{n}{\log n} \leqslant m \leqslant \frac{n}{h}$, and let $(W_1, \ldots, W_h)$ be an h-tuple of pairwise disjoint subsets of $V(\Gamma)$ with $|W_i| = m$ for all $i \in [h]$. Then there are at least $\frac{1}{4}m^h p_h$ distinct copies of K_h contained in the multipartite subgraph of Γ spanned by $W_1 \cup \cdots \cup W_h$.*

We are now ready to show the main result of this section.

THEOREM 5.8. *With high probability, $\Gamma \to_q \mathcal{T}$.*

Proof. We condition on Γ having all of the properties given in Lemma 5.7. For convenience, we may assume also that $\frac{n}{k}$ is an integer for all $k_0 \leqslant k \leqslant K_0$. Consider an arbitrary q-coloring φ of the graph Γ. If any copy of K_h receives only colors in $\mathcal{S}(K_t)$, then we are done since $h = r_{q_2}(K_t)$. So suppose that each such copy has at least one edge whose color comes from $\mathcal{S}(C_\ell)$. Let Γ' be a graph on $V(\Gamma) = [n]$ obtained by taking exactly one edge that has a cycle-color from each hyperedge copy of K_h in Γ; note that Γ' is a transversal subgraph. We claim that Γ' contains a copy of C_ℓ in some cycle-color.

For each $s \in \mathcal{S}(C_\ell)$, let G_s be the subgraph of Γ' on vertex set $[n]$ consisting of all edges that have color s under φ. By Lemma 5.7 (b), we know that Γ is (η, D, p_e)-upper uniform, and hence G_s is (η, D, p_e)-upper uniform for all $s \in \mathcal{S}(C_\ell)$. So by Lemma 5.4, there exists an equipartition $(V_1, \dots, V_k)$ of $[n]$ in which all but at most $\varepsilon\binom{k}{2}$ pairs (V_i, V_j) are (ε, p_e)-regular in every G_s for $s \in \mathcal{S}(C_\ell)$. Let $m = \frac{n}{k}$; by our choice of k_0, K_0, and n, we know that m is an integer and that $\frac{n}{K_0} \leqslant \frac{n}{k} = m \leqslant \frac{n}{k_0}$.

Let K_k be the complete graph on vertex set $\{V_1, \dots, V_k\}$. Consider the following (q_1+2)-coloring of the edges of K_k with the color palette $\{c_1, \dots, c_{q_1+2}\}$. If the pair (V_i, V_j) is (ε, p_e)-regular in all G_s for $s \in \mathcal{S}(C_\ell)$ and has p_e-density at least ρ in some G_s, give the edge between V_i and V_j in K_k color c_s (breaking ties arbitrarily). If the pair (V_i, V_j) is (ε, p_e)-regular in G_s for all $s \in \mathcal{S}(C_\ell)$ but its p_e-density is less than ρ in every such G_s, then color the edge between V_i and V_j in K_k with color c_{q_1+1}. Finally, if (V_i, V_j) is not (ε, p_e)-regular in G_s for some $s \in \mathcal{S}(C_\ell)$, let the edge between V_i and V_j in K_k have color c_{q_1+2}.

By the fact that $k \geqslant k_0$ and our choice of k_0 (from Lemma 5.2), we know that at least one of the following must occur:

(a) For some $s \in [q_1]$, there are at least ck^ℓ copies of C_ℓ in color c_s.

(b) There are at least ck^h copies of K_h that are monochromatic in color c_{q_1+1}.

(c) There are at least ck^2 edges of color c_{q_1+2}.

If (a) occurs for some color $c_s \in [q_1]$, the fact that $ck^\ell \geqslant ck_0^\ell > 0$, together with property (c) in Lemma 5.7, implies that there is a copy of C_ℓ in Γ' in color s. It remains to show that neither of the other cases can occur.

First, consider option (c). We know that there are at most $\varepsilon\binom{k}{2}$ pairs (V_i, V_j) that are not (ε, p_e)-regular in G_s for some $s \in \mathcal{S}(C_\ell)$, and we have

$$\varepsilon\binom{k}{2} \leqslant \frac{1}{10}\rho\binom{k}{2} \leqslant \frac{1}{10}c\binom{k}{2} < ck^2,$$

where the first two inequalities follow by the definitions of ε and ρ. Hence, option (c) is indeed impossible.

We now prove that option (b) cannot occur. Suppose it does. We estimate the number of edges of Γ' corresponding to pairs of color c_{q_1+1} in two different ways. First, note that if there is an edge of color c_{q_1+1} between vertices V_i and V_j, then the (ε, p_e)-regular pair (V_i, V_j) has p_e-density at most ρ in G_s for each $s \in \mathcal{S}(C_\ell)$. Hence, in total, the pair (V_i, V_j) has p_e-density at most $q_1\rho$ in Γ'. Hence, the number of edges in Γ' between pairs (V_i, V_j) corresponding to color c_{q_1+1} is at most

$$\binom{k}{2}q_1\rho p_e m^2 = \binom{k}{2}q_1\rho p_e\left(\frac{n}{k}\right)^2 < \frac{1}{2}q_1\rho A n^{1+1/(\ell-1)} = \frac{c}{4}An^{1+1/(\ell-1)}. \tag{5.1}$$

Now, since option (b) occurs, we have at least ck^h copies of K_h that are monochromatic in color c_{q_1+1} in K_k. Denote these by $K_h^1, K_h^2, \dots, K_h^x$, where $x = \lceil ck^h \rceil$. The vertex set $V(K_h^i)$ of each such copy gives an h-partite subgraph $J_i \subseteq \Gamma$ induced by the sets V_j corresponding to the vertices of K_h^i. As each partite set of J_i has size $m \geqslant \frac{n}{K_0} \geqslant \frac{n}{\log n}$, Lemma 5.7(d) guarantees that J_i contains a family $\mathcal{H}_i$ of at least $\frac{1}{4}m^h p_h$ distinct hyperedge copies of K_h for every $i \in [x]$. As each hyperedge copy in $\mathcal{H}_i$ intersects each partite set of K_h^i, it is immediate that $\mathcal{H}_i \cap \mathcal{H}_j \neq \emptyset$ for $i \neq j$. Hence, there exist $|\bigcup_{i\in[x]}\mathcal{H}_i| \geqslant \frac{1}{4}ck^hm^hp_h$ copies of K_h in Γ. Since every copy of K_h in Γ is a hyperedge copy and no two hyperedge copies share an edge, we find that Γ' has at least

$$\frac{1}{4}ck^hm^hp_h \geqslant ck^h\frac{1}{4}\left(\frac{n}{k}\right)^h An^{-h+1+1/(\ell-1)} = \frac{c}{4}An^{1+1/(\ell-1)} \tag{5.2}$$

edges corresponding to pairs (V_i, V_j) in color c_{q_1+1}, contradicting (5.1). ∎

5.3. Construction of set-determiners. This section uses ideas from [40] to prove Theorem 2.8. Recall that $\mathcal{S}(C_\ell)$ and $\mathcal{S}(K_t)$ denote the cycle-colors $\{1,\dots,q_1\}$ and clique-colors $\{q_1+1,\dots,q\}$, respectively. By construction and by Lemma 5.7, we know that Γ satisfies the following properties:

(i) Every copy of K_t in Γ is a hyperedge copy.
(ii) Every copy of $C_{\ell'}$ for $\ell' < \ell$ is a hyperedge copy.
(iii) Each edge of Γ belongs to a unique copy of K_h.

Now, let $G\subseteq\Gamma$ be a minimal q-Ramsey graph for the q-tuple $\mathcal{T}(q_1,q_2,\ell,t)$; it is not difficult to see that G satisfies properties (i) and (ii) given above. In fact, we have a good understanding of what G needs to look like, as given in the following lemma. Naturally, the lemma also establishes that G satisfies property (iii) above.

LEMMA 5.9. *The graph G is the union of hyperedge copies of K_h, that is, every edge of G belongs to a hyperedge copy of K_h in G.*

Proof. Suppose there is an edge e that does not belong to a copy of K_h in G. We know that e *does* belong to a copy of K_h in $\Gamma\supseteq G$; let H denote this copy of K_h in Γ, and let F denote the set of edges on $V(H)$ that are in Γ but not in G. Notice that $\emptyset\subsetneq F\subsetneq E(H)$ by our assumption.

By the minimality of G, we know that $G-H$ has a $\mathcal{T}$-free q-coloring φ. Additionally, since K_h is minimal q_2-Ramsey for K_t, the graph $H-F$ has a K_t-free q_2-coloring $\varphi' : E(H-F)\to\mathcal{S}(K_t)$. We now define a q-coloring $\widetilde{\varphi}$ of G by setting $\widetilde{\varphi}=\varphi\cup\varphi'$.

We claim that $\widetilde{\varphi}$ is a $\mathcal{T}$-free q-coloring of G. Indeed, since φ is a $\mathcal{T}$-free coloring of $G-H$, there are no monochromatic cycles in any cycle-color, and since in the coloring of $H-F$ we add no further edges in these colors, we know that there are no monochromatic copies of C_ℓ in any cycle-color in all of G. Furthermore, since there are no nonhyperedge copies of K_t in G and neither φ nor φ' contains a monochromatic copy of K_t in any color in $\mathcal{S}(K_t)$, we know that there are also no monochromatic copies of K_t in any clique-color in all of G. Hence, $\widetilde{\varphi}$ is a $\mathcal{T}$-free q-coloring of G, contradicting the fact that $G\to_q\mathcal{T}$. ∎

Now, let e be a fixed edge of G, and let H be the copy of K_h in G containing e. Let D be the graph obtained from G by removing all edges of H except for e, that is, $D=G-(H-e)$. We now claim that D is an $\mathcal{S}(K_t)$-determiner for the tuple $\mathcal{T}$. This construction generalizes the one presented by Siggers [40].

LEMMA 5.10. *The graph D is a safe $\mathcal{S}(K_t)$-determiner for the tuple $\mathcal{T}$ with signal edge e.*

Proof. We first show property (D2). For a contradiction, suppose ψ is a $\mathcal{T}$-free coloring of D in which $\psi(e)\in\mathcal{S}(C_\ell)$. Then, by an argument similar to the one used in Lemma 5.9, by putting together this $\mathcal{T}$-free coloring of D and a K_t-free q_2-coloring of $H-e$ (with colors in $\mathcal{S}(K_t)$) we obtain a $\mathcal{T}$-free coloring of G, which contradicts the fact that $G\to_q\mathcal{T}$.

To see properties (D1) and (D3), note that D is a proper subgraph of G, so D has a $\mathcal{T}$-free q-coloring φ. Further, by permuting the clique-colors in φ appropriately, we can obtain a $\mathcal{T}$-free coloring of D in which the edge e has any color in $\mathcal{S}(K_t)$.

It remains to show that φ is safe at $\{e\}$. Let F be any graph such that $V(D) \cap V(F) = V(e)$ and $E(D) \cap E(F) = \{e\}$. Let φ' be a $\mathcal{T}$-free q-coloring of F that agrees with φ on the edge e. We claim that the coloring $\widetilde{\varphi}$, given by $\widetilde{\varphi} = \varphi \cup \varphi'$, is a $\mathcal{T}$-free q-coloring of $D \cup F$. We know that the restrictions of $\widetilde{\varphi}$ to both D and F are $\mathcal{T}$-free; it remains to show that there are no monochromatic cliques or cycles in the appropriate colors intersecting both $V(D) - e$ and $V(F) - e$.

First, it is not difficult to see that there can be no such copy of K_t. For $t = 3$, this is clear. If $t \geqslant 4$ and there is a t-clique K intersecting both $D - e$ and $F - e$, then we can disconnect K by removing the vertices of e, which is impossible. Suppose there is such a copy C of C_ℓ. Note first that C must contain both vertices of e because C_ℓ is 2-connected. Now, let v be a vertex of C contained in $V(D) - e$, and let w be a vertex of C contained in $V(F) - e$. Now, there are no nonhyperedge cycles of length less than ℓ in D, so every cycle containing e in D has length at least ℓ. Hence, the vertices v and w cannot be contained in a cycle of length ℓ with both endpoints of e, and therefore C cannot exist. Thus the coloring $\widetilde{\varphi}$ is $\mathcal{T}$-free, implying that φ is safe. This completes the verification of the safeness property. ∎

Now we construct a safe $\mathcal{S}(C_\ell)$-determiner D' by taking a copy H of K_h, fixing one edge f and attaching copies of the $\mathcal{S}(K_t)$-determiner D constructed above to all remaining edges of H. This again generalizes a construction of Siggers [40].

LEMMA 5.11. *The graph D' is a safe $\mathcal{S}(C_\ell)$-determiner for the tuple $\mathcal{T}$ with signal edge f.*

Proof. We again begin with property (D2). Take an arbitrary $\mathcal{T}$-free coloring of D'. This coloring induces a $\mathcal{T}$-free coloring on each copy of D, so, by property (D2) of D, all edges of $H - f$ have colors in $\mathcal{S}(K_t)$. If f has one of these colors too, then H is fully colored with colors in $\mathcal{S}(K_t)$. Since H is q_2-Ramsey for K_t, there exists a monochromatic copy of K_t in H, contradicting the fact that the coloring φ is $\mathcal{T}$-free. So the color of f must be in the set $\mathcal{S}(C_\ell)$.

We show properties (D1) and (D3) next. By minimality, we know that $H - f$ is not q_2-Ramsey for K_t, and hence it has a K_t-free coloring ψ from the palette $\mathcal{S}(K_t)$. Let φ be a q-coloring extending ψ in which each copy of the determiner D has a safe $\mathcal{T}$-free coloring and the edge f has an arbitrary color from $\mathcal{S}(C_\ell)$; this coloring $\widetilde{\varphi}$ exists by property (D3) of D. Since the coloring of each copy of D is safe, and since H has a $\mathcal{T}$-free q-coloring, the coloring φ of D' is also $\mathcal{T}$-free.

Finally, to see the safeness of φ, let F be a graph such that $V(D') \cap V(F) = V(f)$ and $E(D') \cap E(F) = \{f\}$. If F is given a $\mathcal{T}$-free q-coloring φ' that agrees with φ on f, then the coloring $\widetilde{\varphi} = \varphi \cup \varphi'$ is a $\mathcal{T}$-free q-coloring of $D' \cup F$. Indeed, since each copy of D is safe, and since the only edge of H that has color in $\mathcal{S}(C_\ell)$ is f, we know that there can be no monochromatic copy of C_ℓ in $D' \cup F$ using a cycle-color in $\widetilde{\varphi}$. Similarly, since we cannot disconnect K_t by removing at most two vertices, we know that there can be no copy of K_t intersecting both $V(D') - f$ and $V(F) - f$, and hence there can be no monochromatic copy of K_t in a clique-color in $\widetilde{\varphi}$. Hence, $\widetilde{\varphi}$ is a $\mathcal{T}$-free q-coloring, and thus φ is a safe coloring of D'. ∎

5.4. Construction of set-senders. So far, we have constructed an $\mathcal{S}(K_t)$-determiner D and an $\mathcal{S}(C_\ell)$-determiner D', generalizing ideas from [5, 40]. We now take the constructions a step further and use our set-determiners to build set-senders for these sets of colors when $q_1 > 1$ or $q_2 > 1$, proving Theorem 2.9.

If $q_1 > 1$, let S be a safe negative (respectively, positive) signal sender for C_ℓ with q_1 colors, as guaranteed by Lemma 2.5 and Remark 2.7; let e and f denote its signal edges. Let R be a graph obtained from S by attaching a copy of D' to every edge of S.

LEMMA 5.12. *If S is a negative (respectively, positive) signal sender for C_ℓ with signal edges e and f as above, then R is a safe negative (respectively, positive) $\mathcal{S}(C_\ell)$-sender for $\mathcal{T}$ with signal edges e and f.*

Proof. Assume S is a negative signal sender for C_ℓ in q_1 colors; the other case is similar. We first show properties (S1) and (S3). Let $c_1, c_2 \in \mathcal{S}(C_\ell)$ be distinct. We know that $S \nrightarrow_{q_1} C_\ell$, so S has a safe C_ℓ-free coloring from the set $\mathcal{S}(C_\ell)$, and by property (S3) of S, we can ensure that e and f receive colors c_1 and c_2, respectively. Now, since the signal edge of each copy of D' has color in $\mathcal{S}(C_\ell)$, by property (S3) of D', this coloring of S can be extended to each copy of D' so that each copy of D' has a safe $\mathcal{T}$-free q-coloring. The coloring of each copy of D' is safe, so the q-coloring defined on R is $\mathcal{T}$-free. To see the safeness of this coloring, notice that the coloring of each copy D' is safe at its signal edge and that the coloring of S, containing only colors from $\mathcal{S}(C_\ell)$, is safe at $\{e, f\}$. Property (S2) of R follows immediately from properties (S2) and (D2) of S and D'. ☐

Finally, if $q_2 > 1$, we build $\mathcal{S}(K_t)$-senders for $\mathcal{T}$. Let S' be a safe negative (respectively, positive) signal sender for K_t with q_2 colors taken as $\mathcal{S}(K_t)$, as guaranteed by Lemma 2.6 and Remark 2.7; let e and f denote its signal edges. Let R' be a graph obtained from S' by attaching a copy of D to every edge of S'. We omit the proof that R' is a set-sender for K_t, as it is essentially the same as that of Lemma 5.12.

LEMMA 5.13. *If S' is a negative (respectively, positive) signal sender for K_t with signal edges e and f, then R' is a safe negative (respectively, positive) $\mathcal{S}(K_t)$-sender R' for $\mathcal{T}$ with signal edges e and f.*

6. Concluding remarks. In this paper, we initiated the study of the parameter s_q in the asymmetric setting for tuples consisting of cliques and cycles. The upper and lower bounds we obtain are strongly dependent on the existing bounds for the symmetric parameter $s_q(K_t)$. As noted by the authors of [19], the study of $s_q(K_t)$ appears to be tightly connected to the Erdős–Rogers function, implying that any improvements on our current results would probably be nontrivial. We refer the reader to [19, section 5] for a more detailed discussion on the relationship between $s_q(K_t)$ and the Erdős–Rogers function.

It would be desirable to study other asymmetric cases of the problem, and a natural place to start is to consider pairs of graphs for which safe determiners are known to exist (including all pairs of 3-connected graphs and the pairs considered by Siggers in [40]).

The multicolor asymmetric setting offers even more room for study, as the existence of gadget graphs is an open problem even in some very natural cases. Our method allows us to construct set-determiners and set-senders for tuples of the form $(C_\ell, \dots, C_\ell, K_s, K_t)$. However, we are not aware of a way to build gadget graphs for asymmetric q-tuples of cliques with $q > 2$. Since studying Ramsey graphs for cliques is a central theme in Ramsey theory, we believe that resolving the following problem would be of interest.

PROBLEM 6.1. *Construct signal senders for asymmetric q-tuples* $(K_{t_1}, \dots, K_{t_q})$.

The natural first instances to attack, which might also shed some light on the general case, are tuples of the form $(K_t, \dots, K_t, K_k)$ or (K_t, K_s, K_k). Once we have the necessary tools, it would be very interesting to investigate the parameter s_q for such tuples.

It would also be desirable to determine whether the upper bound in Theorem 1.3 holds in other cases. In particular, it was conjectured by Fox et al. [19] that $s_q(K_{t-1}) \leqslant s_q(K_t)$ for $q > 3$. Perhaps the following asymmetric version would be more approachable.

PROBLEM 6.2. *Show that*

$$s_q(\underbrace{K_{t-1}, \dots, K_{t-1}}_{q_1+1 \text{ times}}, \underbrace{K_t, \dots, K_t}_{q_2-1 \text{ times}}) \leqslant s_q(\underbrace{K_{t-1}, \dots, K_{t-1}}_{q_1 \text{ times}}, \underbrace{K_t, \dots, K_t}_{q_2 \text{ times}}).$$

Appendix A. Proof of Lemma 5.7 (c). We now give the proof of Lemma 5.7 (c). The proof is similar to the proof of Proposition 9 in [5], but we use modern results related to the KŁR conjecture.

Proof of Lemma 5.7 *(c).* Let m satisfy $\frac{n}{K_0} \leqslant m \leqslant \frac{n}{k_0}$; we can write $p_e = Bm^{-1+1/(\ell-1)}$, where $B = A\left(\frac{n}{m}\right)^{-1+1/(\ell-1)}$. Notice that B satisfies $AK_0^{-1+1/(\ell-1)} \leqslant B \leqslant Ak_0^{-1+1/(\ell-1)}$.

Let $(V_1, \dots, V_\ell)$ and Γ' be as given. Suppose that the pairs (V_i, V_{i+1}) for $i \in [\ell]$ are (ε, p_e)-regular with p_e-density at least ρ in Γ'. Then we have $e_{\Gamma'}(V_i, V_{i+1}) \geqslant \rho p_e m^2$ for all $i \in [\ell]$. Let M be an integer satisfying

$$\rho p_e m^2 \leqslant M \leqslant \min_{i \in [\ell]} e_{\Gamma'}(V_i, V_{i+1}).$$

Notice that this integer M satisfies

$$\begin{aligned} M &\geqslant \rho p_e m^2 = \rho B m^{1+1/(\ell-1)} \geqslant \rho A K_0^{-1+1/(\ell-1)} m^{1+1/(\ell-1)} \\ &\geqslant C m^{1+1/(\ell-1)} \geqslant 2\beta m = \beta \left| V_i \cup V_{i+1} \right|, \end{aligned}$$

since $A \geqslant K_0^{1-1/(\ell-1)} C/\rho$, and since n, and hence m, is taken to be sufficiently large.

Consider the pair (V_1, V_2), and let $d = \frac{e_{\Gamma'}(V_1, V_2)}{m^2}$; then we have $d \geqslant \rho p_e$, and thus $p_e \leqslant \frac{d}{\rho}$. By definition, it then follows that the pair (V_1, V_2) is $(\frac{\varepsilon}{\rho}, d)$-regular or simply $(\frac{\varepsilon}{\rho})$-regular. By Lemma 5.5, there is a subset $E_{1,2} \subseteq E_{\Gamma'}(V_1, V_2)$ such that $|E_{1,2}| = M$ and that the pair (V_1, V_2) is $(\frac{2\varepsilon}{\rho})$-regular in $(V_1 \cup V_2, E_{1,2})$. Repeating this argument for all pairs of the form (V_i, V_{i+1}), we find that $\Gamma'[V_1 \cup \cdots \cup V_\ell]$ contains at least one graph in $\mathcal{G}(\ell, m, (V_i)_{i=1}^{\ell}, M, \frac{2\varepsilon}{\rho})$.

Our goal now is to show that with high probability, there is no collection of subsets $(V_i)_{i=1}^{\ell}$, and subgraph $\Gamma' \subseteq \Gamma$ as given in the statement such that $\Gamma'[V_1 \cup \cdots \cup V_\ell]$ contains a subgraph belonging to $\mathcal{F}(\ell, m, (V_i)_{i=1}^{\ell}, M, \frac{2\varepsilon}{\rho})$. Again, let the ℓ-tuple $(V_1, \dots, V_\ell)$ be fixed. If $F \in \mathcal{F}(\ell, m, (V_i)_{i=1}^{\ell}, M, \frac{2\varepsilon}{\rho})$ has edges $e_1, \dots, e_{M\ell}$ and there exists a transversal Γ' such that $F \subseteq \Gamma'[V_1 \cup \cdots \cup V_\ell]$, there must exist distinct hyperedges $\mathcal{E}_1, \dots, \mathcal{E}_{M\ell} \in E(\mathcal{H}_{n,p_h})$ such that $e_i \subseteq \mathcal{E}_i$ for all $i \in [M\ell]$. Therefore

$$\mathbb{P}\left[\exists \text{ transversal } \Gamma' : \ F \subseteq \Gamma'[V_1 \cup \cdots \cup V_\ell]\right] \leqslant \left(\binom{n-2}{h-2} p_h\right)^{M\ell}$$
$$\leqslant \left((n-2)^{h-2} A n^{-(h-1)+1/(\ell-1)}\right)^{M\ell}$$
$$\leqslant \left(A n^{-1+1/(\ell-1)}\right)^{M\ell} = p_e^{M\ell}. \quad \text{(A.1)}$$

Note that when n is sufficiently large, we have $m \geqslant m_0$. By choice of $\varepsilon \leqslant \rho\varepsilon_0/2$, applying Lemma 5.6 and the union bound, we obtain

$$\mathbb{P}\left[\exists \text{ transversal } \Gamma', F \in \mathcal{F}\left(\ell, m, (V_i)_{i=1}^{\ell}, M, \frac{2\varepsilon}{\rho}\right) : F \subseteq \Gamma'[V_1 \cup \cdots \cup V_\ell]\right]$$
$$\leqslant \alpha^M \binom{m^2}{M}^{\ell} p_e^{\ell M} \leqslant \alpha^M \left(\frac{m^2 e}{M}\right)^{\ell M} p_e^{\ell M} \leqslant \alpha^M \left(\frac{e}{\rho}\right)^{\ell M} = e^{-M},$$

where the last inequality follows from the fact that $M \geqslant \rho p_e m^2$, and the final step follows by the choice of α.

This implies that for any fixed integers m and M and any collection of disjoint subsets $V_1, \ldots, V_\ell$ of $[n]$, each of size m, the probability that there exists a transversal Γ' such that $\Gamma'[V_1 \cup \cdots \cup V_\ell]$ contains some graph in $\mathcal{F}(\ell, m, (V_i)_{i=1}^{\ell}, M, \frac{2\varepsilon}{\rho})$ is at most e^{-M}.

Now, for any choice of $\frac{n}{K_0} \leqslant m \leqslant \frac{n}{k_0}$ and $Cm^{1+1/(\ell-1)} \leqslant M \leqslant m^2 \leqslant n^2$, there are at most $n^{m\ell}$ choices for the sets $V_1, \ldots, V_\ell$. Summing over the possible choices for the sets $V_1, \ldots, V_\ell$ and the possible choices for m and M, we find that the probability that (c) fails is bounded from above by the probability that there exist m, M, $(V_i)_{i=1}^{\ell}$, and Γ' such that $\Gamma'[V_1 \cup \cdots \cup V_\ell]$ contains a member of $\mathcal{F}(\ell, m, (V_i)_{i=1}^{\ell}, M, \frac{2\varepsilon}{\rho})$, which is at most

$$\sum_m \sum_M n^{m\ell} e^{-M} \leqslant \sum_m \sum_M \exp(-Cm^{1+1/(\ell-1)} + m\ell \log n)$$
$$\leqslant \sum_m \sum_M \exp\left(-C\left(\frac{n}{K_0}\right)^{1+1/(\ell-1)} + \frac{n}{k_0}\ell \log n\right)$$
$$\leqslant n^3 \exp\left(-C\left(\frac{n}{K_0}\right)^{1+1/(\ell-1)} + \frac{n}{k_0}\ell \log n\right) = o(1).$$

□

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