The prime number theorem and the distribution of the zeros of the Riemann-zeta function

Bachelorproject Technische Wiskunde

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Foreword

Although most people actually don't know anything about advanced mathematics at all, some mathematical topics are considered as magic by the laity. Everyone has heard about the number π, but most people can't explain what the number actually means. Another mysterious mathematical subject are the prime numbers. Many people have heard about prime numbers, and probably quite many people will be able to explain what a prime number is. However, pupils don't learn much about prime numbers at high school. That's a shame, since some important properties of prime numbers are very lucrative.
In this report a proof of an important non-trivial mathematical theorem about the distribution of the prime numbers among the naturals, called the prime number theorem, will be discussed. Although the prime numbers form a subset of the natural numbers $\mathbb{N}$, most work we will do in the complex plane $\mathbb{C}$. This sounds quite strange, but the wonderful properties of holomorphic functions make the proof of the theorem very elegant. As the French mathematician Jacques Hadamard (not surprisingly one of the most important mathematicians for the prime number theorem) had said: "The shortest path between two truths in the real domain passes through the complex domain."
The Riemann-zeta function will play a significant role during our investigation, since we will prove that the prime number theorem only holds if the zeta function has no zeros on the line $\Re(s) = 1$. The function has raised a conjecture by Riemann, known as the Riemann-hypothesis. This conjecture has been studied for about 150 years, but neither a proof nor a counter-example has been found yet. The curious distribution of the zeros of the zeta function are therefore worth investigating. At the end of the project we will introduce the study of random matrices to describe this distribution.

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Preface

In the first part of the project, we will give a general proof of the prime number theorem. The Riemann-zeta function which is very important during the proof, will be extensively discussed, though the function still raises many questions we can't answer.

The Riemann-hypothesis, which gives a statement about the zeros of the zeta function, seems to hold according to accurate observations. However, a proof has not been found yet. Many theorems in number theory hold only if the hypothesis is true, so finding a complete proof will be very important.

In the second part of the project, we will try to find the relationship between the zeros of the zeta function and random matrices. Random matrix theory is a brand new subject in mathematics. The many applications of the theory (for example the zeros of the zeta function, and the description of energy levels of heavy nuclei) make the topic very popular at the moment, and indicate that there may be much more to discover! We will explain (but won't prove) that the distribution of eigenvalues of specific random matrices corresponds to the distribution of the critical zeros of the zeta function, assuming the Riemann-hypothesis.

The topics we discuss in this report are just a small tip of the iceberg, which seems to be enormous!

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Chapter 1

Prime numbers: a little bit of fundamental theory

1.1 An introduction

During the investigation, the following definition of a prime number will be used.

Definition 1.1.1 *A prime number is a natural number which has exactly two different integer divisors.*

Automatically, Definition 1.1.1 states that a prime number is only divisible by 1 and itself. Note that 1 is not a prime number according to this definition. Sometimes mathematicians argue about this. Later on it will be clear that it's convenient to define 1 as not prime.

The first prime numbers are $\{2,3,5,7,11,13,17,19,23,29,31,37,\dots\}$.

We could go on and on by writing the primes down, but we won't find a nice pattern in this collection, since its distribution is very erratic.

The following theorem is one of the most important theorems in arithmetics. We will use the notation $a|b$ if a is a divisor of b.

Lemma 1.1.2 *Every positive integer $n > 1$ has a prime factorization.*

Proof. We argue by contradiction. Let S be the set of natural numbers > 1 which do not have a prime number factorization. Let n be the smallest element of S. Because $n \in S$, n is not prime, so there exist integers $a > 1$ and $b > 1$ such that $n = ab$. Of course, $a < n$ and $b < n$. Therefore, $a \notin S$ and $b \notin S$, because n is the smallest element of S. This means that a and b have prime number factorizations. Then their producht $ab = n$ must have a prime number factorization, which means $n \notin S$. We have found a contradiction! S does not have a smallest element, which means that S is empty, as desired. □

Lemma 1.1.3 *There is no positive integer $n > 1$ that can be factorized by primes in two different ways.*

Proof. Again we argue by contradiction. Let $n > 1$ be an integer which has two prime factorizations

$$\begin{aligned} n &= p_1 p_2 \dots p_r \\ &= q_1 q_2 \dots q_s. \end{aligned}$$

p_1 is a divisor of n and therefore a divisor of $q_1 q_2 \dots q_s$. Because p_1 is prime, we must have that $p_1 | q_i$ for some i (We omit the proof of this fact here. A simple proof can be found in several books about elementary number theory, for example [1]. Because q_i is certainly prime, $p_1 = q_i$. If we proceed this way, we will find that the two factorizations are equal up to permutation. The prime factorization is therefore unique. $\square$

Theorem 1.1.4 *Every positive integer $n > 1$ has a unique prime factorization up to permutation.*

Proof. Combine Lemma 1.1.2 and Lemma 1.1.3. $\square$

Theorem 1.1.4 is the theorem that makes the prime numbers that special and important. In fact, we can regard the collection of prime numbers, which we will denote by $\mathbb{P}$, as the building blocks of the natural numbers $\mathbb{N}$.

In general, the prime factorization of a large number is very hard to find. Take two large prime numbers p_1, p_2. Finding the factorization of $n = p_1 p_2$ will take a lot of time, unless you are very lucky. This fact makes prime numbers that important; they are frequently used for encryption.
Unfortunately, a simple formula which gives the n^{th} prime doesn't exist. The distribution of the primes among the natural numbers is very erratic. The only thing we could say on forehand is that the density of the prime numbers among $\mathbb{N}$ will slowly decrease if we investigate larger numbers. This is a simple consequence of the fact that there are more candidates of factors for larger numbers.
However, we do want to find some results about the probability of finding a prime in a certain interval $[a, b]$, especially when a and b are large. If we want to encrypt documents by prime factorization, it would be congenial to know something about the chance of finding primes. The famous prime number theorem gives a solution to this problem. Before stating this theorem it's convenient to be familiar with a little bit of elementary prime number theory.

At first, let's wonder whether $\mathbb{P}$ has a finite number of elements or not. A first proof of the following theorem was found in Euclid's *Elements*, which was written in the 3^{rd} century B.C.

Theorem 1.1.5 *There are infinitely many prime numbers.*

Proof. We argue by contradiction. Suppose that there are only finitely many primes. Let $p_1, \dots, p_n$ denote all the primes. If we define

$$N = p_1 p_2 \dots p_n + 1,$$

we will certainly have that $N \notin \mathbb{P}$, because N is larger than all the primes. Theorem 1.1.4 ensures us that we can write N as a product of primes. However, by its definition it's obvious that N isn't divisible by any of the primes $p_1, \ldots, p_n$. This means that there must be other primes as well. We could repeat this argument over and over, getting the same contradiction every time. Therefore, we cannot have finitely many primes, completing the proof. $\square$

Now we are ready to investigate the prime number theorem. Its proof requires a lot of analysis. We will explain the theorem step by step, eventually leading to a complete proof.

Chapter 2

The link between number theory and complex analysis

At the end of the 18^{th} century, Gauss conjectured that

$$\pi(x) \sim \frac{x}{\log x} \quad \text{as} \quad x \to \infty, \tag{2.1}$$

where the number of primes less than or equal to x is denoted by $\pi(x)$. The former equation is known as the **prime number theorem**. It says that

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1. \tag{2.2}$$

Chebyshev proved that for sufficiently large x

$$A\frac{x}{\log x} \leq \pi(x) \leq B\frac{x}{\log x}, \tag{2.3}$$

where A and B are positive constants (with of course $A < B$).

Eventually, in 1896, Hadamard and de la Vallée Poussin independently gave a proof of (2.1), in which they had used complex analysis. About 50 years later, more elementary proofs of (2.1) have been found.

2.1 The Riemann-zeta function and Euler's Identity

It's quite amazing to see how we can use complex analysis to prove a pure number theoretical theorem. It was Euler who made this connection by finding an identity for the zeta function.

The zeta function is initially defined for $s > 1$ by the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{2.4}$$

By comparison with the integral $\int_1^\infty x^{-s}\,dx$, we see that this series converges for $s > 1$.

Theorem 2.1.1 *We can extend* (2.4) *to the half-plane* $\Re(s) > 1$ *in* $\mathbb{C}$. *The function* ζ *is holomorphic in this half-plane.*

Proof. Suppose $s = \sigma + it$, with $\sigma, t \in \mathbb{R}$. We have

$$|n^{-s}| = |e^{-s \log n}| = e^{-\sigma \log n} = n^{-\sigma}, \tag{2.5}$$

which follows from the fact that the modulus of a complex exponential equals the exponential of the real part of the complex number. Because $\Re(s) > 1$, we can choose $\delta > 0$ such that $\sigma > 1 + \delta > 1$. We now certainly have the inequality

$$\sum_{n=1}^{\infty} \frac{1}{n^{\sigma}} \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}}. \tag{2.6}$$

The right-hand side converges because $\delta > 0$. Therefore, the left-hand side converges uniformly on $\Re(s) > 1 + \delta > 1$. Besides, for every $n \geq 1, s \mapsto 1/n^s$ is a holomorphic function on the half-plane $\Re(s) > 1$. Therefore, the function ζ is holomorphic on the half-plane $\Re(s) > 1$. $\square$

One of the key steps during the investigation will be the determination of the zeros of the zeta function. For this we use Euler's famous identity:

Theorem 2.1.2 *For* $\Re(s) > 1$ *the zeta function can be expressed as an infinite product*

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - p^{-s}}.$$

Proof. At first we denote that we can write

$$\frac{1}{1 - p^{-s}}$$

as a convergent geometric power series

$$\sum_{n=0}^{\infty} \frac{1}{p^{ns}}.$$

If we take the product of these series over all primes, we will obtain for every n a term $\frac{1}{n^s}$, in which n is written as its unique prime factorization (Theorem 1.1.4 is therefore essential in this proof). A more precise argument goes as follows.

Take two arbitrary positive integers M and N with the only restriction that $M > N$. From Theorem 1.1.4 we know that we can write any positive integer $n \leq N$ uniquely as a product of primes. Each prime in this product must be less than or equal to N, because otherwise we will have $n > N$. Furthermore, each prime that occurs in the product must be repeated less than M times, otherwise we will get the same contradiction.

The statements above yield the following inequality:

$$\begin{aligned}\sum_{n=1}^{N} \frac{1}{n^s} &\leq \prod_{p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{Ms}}\right) \\ &\leq \prod_{p \leq N} \left(\frac{1}{1 - p^{-s}}\right) \\ &\leq \prod_{p \in \mathbb{P}} \left(\frac{1}{1 - p^{-s}}\right).\end{aligned}$$

Letting N tend to infinity gives us

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \leq \prod_{p \in \mathbb{P}} \left(\frac{1}{1 - p^{-s}}\right). \tag{2.7}$$

Now we will find the reverse inequality. Again we use Theorem 1.1.4, which yields

$$\prod_{p \leq N} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \ldots + \frac{1}{p^{Ms}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We can let M go to infinity to obtain

$$\prod_{p \leq N} \left(\frac{1}{1 - p^{-s}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Letting N tend to infinity finally gives us

$$\prod_{p \in \mathbb{P}} \left(\frac{1}{1 - p^{-s}}\right) \leq \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{2.8}$$

Together the inequalities (2.7) and (2.8) give the desired result. $\square$

With a little bit more effort, we can show that Theorem 2.1.2 is true for every s with $\Re(s) > 1$. We now have made the desired translation between number theory and complex analysis. Instead of looking at the very erratic behaviour of the distribution of the primes, we now can investigate an analytic function! Therefore, let's have a look at the zeta function in more detail.

2.2 Properties of the zeta function

The Riemann-zeta function is initially defined on the complex right half-plane $\Re(s) > 1$. The following theorem states that there exists a function $\zeta^*(t)$ which is meromorphic and equals the original zeta function on $\Re(s) > 1$.

Theorem 2.2.1 *The zeta function has a meromorphic continuation into $\mathbb{C}$, which only has a singularity (a single pole) at $s = 1$.*

For convenient reasons, from now on the notation ζ will be used for the meromorphic continuation of the original zeta function.
In order to prove the theorem above, we require a lot of analytic knowledge. Details can be found in the appendix, which is recommended for the interested reader.

2.2.1 Zeros and poles

In the complex plane $\Re(s) > 1$, ζ has no zeros. This follows from the product formula for ζ and from the following lemma.

Lemma 2.2.2 *If $\sum_{n=1}^{\infty} |a_n| < \infty$, then the product $\prod_{n=1}^{\infty}(1 + a_n)$ converges, and it converges to 0 if and only if one of the factors is 0.*

A proof of this lemma can be found for example in [2], Proposition 5.3.1. In our case, we have $a_n = \sum_{m=1}^{\infty} p^{-ms}$, where m denotes the m^{th} prime number. $\sum_{n=1}^{\infty} |a_n|$ converges, so our product formula converges for $\Re(s) > 1$. Furthermore, it doesn't converge to 0 since none of the factors is 0. This means that ζ has no zeros on the half-plane $\Re(s) > 1$. Besides, ζ is analytic in this half-plane, so the function has no poles here as well.

In the appendix we found the following result, which we state here as a lemma:

Lemma 2.2.3

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{2.9}$$

The function ξ is holomorphic for $\Re(s) > 1$ and has an analytic continuation to $\mathbb{C}$ as a meromorphic function, with simple poles at $s = 0$ and $s = 1$. Furthermore, we have

$$\xi(s) = \xi(1 - s) \quad \text{for all } s \in \mathbb{C}. \tag{2.10}$$

Solving for ζ, we find

$$\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}. \tag{2.11}$$

The (analytic continuation of the) Gamma function has simple poles at $0, -1, -2, \ldots$, and has no zeros. As a consequence, $1/\Gamma(s/2)$ is entire, with simple zeros at $0, -2, -4, \ldots$, so the pole of $\xi(s)$ at $s = 0$ is cancelled by a zero of the denominator. Therefore, the only singularity of ζ is a simple pole at $s = 1$.

In order to investigate the zeros of ζ, we use the functional equation for ξ. Solving for ζ gives the following functional equation for ζ:

$$\zeta(s) = \pi^{s-1/2}\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\zeta(1 - s). \tag{2.12}$$

We can use (2.12) to investigate the zeros of ζ. For $\Re(s) > 1$ we've already found the desired results. We can use these results to investigate the zeros of ζ for $\Re(s) < 0$. We have the following for $\Re(s) < 0$:

1. $\zeta(1 - s)$ has no zeros because $\Re(1 - s) > 1$.
2. $\Gamma((1 - s)/2)$ has no zeros.
3. $1/\Gamma(s/2)$ has zeros at the negative even integers.

The results above imply that the zeros of ζ on the left half-plane $\Re(s) < 0$ are exactly at the negative even integers.

We summarize the results of the zeta function in Table 1.

Zeros	**Poles**
$-2, -4, -6, \ldots$	$s = 1$
Possibilities at $0 \leq \Re(s) \leq 1$	

Table 1.

The region $0 \leq \Re(s) \leq 1$ is often called the **critical strip**. From some mathematical theorems, it's known that its interior contains infinitely many zeros of the zeta function. Riemann conjectured that all these zeros lie on the line $\Re(s) = 1/2$. This statement is known as the **Riemann hypothesis**. This conjecture has neither been proved nor been disproved. Odlyzko studied in [3] and [4] the distribution of a set of many zeros of the zeta function, which led to the discovery of a curious pattern of the distribution of the zeros. In Chapter 4 we will come back to this topic.

It can be shown that proving the prime number theorem is equivalent to proving the fact that ζ has no zeros on the line $\Re(s) = 1$. We won't prove this equivalence here, since we won't need this to prove the prime number theorem. However, we will prove the fact that ζ has no zeros on $\Re(s) = 1$. At first we need some lemmas to be able to prove this.

Lemma 2.2.4 *For $\Re(s) > 1$ we have*

$$\log \zeta(s) = \sum_{n=1}^{\infty} c_n n^{-s} \tag{2.13}$$

for some $c_n \geq 0$.

Proof. Suppose that $s > 1$ (note that we choose $s \in \mathbb{R}$!). We use the power series expansion for the logarithm (use Taylor series to see this easily)

$$\log \frac{1}{1-x} = \sum_{n=1}^{\infty} \frac{x^n}{n}, \tag{2.14}$$

where x is real. This series converges for $0 \leq x < 1$. We now use the fact that the logarithm of a product equals the sum of the logarithms. Writing the zeta function as the product formula, we find

$$\log \zeta(s) = \log \prod_{p \in \mathbb{P}} \frac{1}{1-p^{-s}} = \sum_{p \in \mathbb{P}} \log \frac{1}{1-p^{-s}} = \sum_{p \in \mathbb{P}} \sum_{m \in \mathbb{N}} \frac{p^{-ms}}{m}. \tag{2.15}$$

The last equality holds because of (2.14) and since $0 \leq p^{-s} < 1$. Because all terms are positive, the order of summation doesn't matter, so we can write

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m} \tag{2.16}$$

for $s > 1$. But then this formula must hold for all $s \in \mathbb{C}$ with $\Re(s) > 1$ because of analytic continuation! The logarithm is well defined, since we've seen that ζ is holomorphic and doesn't vanish in the half-plane $\Re(s) > 1$, which is simply connected (for details of this argument, see for example [2], Theorem 3.6.2). If we choose $c_n = 1/m$ if $n = p^m$ and $c_n = 0$ otherwise, we've proved the lemma. □

To prove the fact that ζ has no zeros on $\Re(s) = 1$, we use the following lemma:

Lemma 2.2.5 *For $\sigma > 1$ and t real, we have*

$$\log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 0. \tag{2.17}$$

Proof.

$$\begin{aligned} & \log |\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \\ = \;& 3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)| \\ = \;& 3\Re(\log \zeta(\sigma)) + 4\Re(\log \zeta(\sigma + it)) + \Re(\log \zeta(\sigma + 2it)). \end{aligned} \tag{2.18}$$

We now can use (2.13) and the fact that

$$\begin{aligned} \Re(n^{-s}) &= \Re(e^{-(\sigma+it)\log n}) \\ &= e^{-\sigma \log n}\Re(e^{it \log n}) \\ &= e^{-\sigma \log n} \cos(t \log n) = n^{-\sigma} \cos(t \log n) \end{aligned}$$

to see that we can write (2.18) as

$$\sum_{n=1}^{\infty} c_n n^{-\sigma}(3 + 4\cos(t \log n) + \cos(2t \log n)). \tag{2.19}$$

We've already seen that $c_n \geq 0$. Moreover, $3 + 4\cos(t \log n) + \cos(2t \log n) \geq 0$, because of the equality

$$3 + 4\cos(x) + \cos(2x) = 2(1 + \cos(x))^2,$$

which completes the proof. □

Theorem 2.2.6 *ζ has no zeros on the line $\Re(s) = 1$.*

Proof. We recall that ζ has a simple pole at $s = 1$. We now argue by contradiction. Suppose that $\zeta(1 + it_0) = 0$ for some $t_0 \neq 0$. We know from Theorem 2.2.1 that ζ is holomorphic at $1 + it_0$. Therefore, there must exist a constant $C > 0$ such that

$$|\zeta(\sigma + it_0)|^4 \leq C(\sigma - 1)^4 \quad \text{as } \sigma \to 1.$$

Because $s = 1$ is a simple pole, we also have

$$|\zeta(\sigma)|^3 \leq D(\sigma - 1)^{-3} \quad \text{as } \sigma \to 1,$$

for some constant $D > 0$. At the points $\sigma + 2it_0$, ζ is holomorphic (recall that $t_0 \neq 0$). this means that $|\zeta(\sigma + 2it_0)| < M$ for some constant $M > 0$ as $\sigma \to 1$.

We now have the following result:

$$|\zeta^3(\sigma)\zeta^4(\sigma + it_0)\zeta(\sigma + 2it_0)| < MCD(\sigma - 1) \quad \text{as } \sigma \to 1. \tag{2.20}$$

This tends to 0. Since the value of a logarithm between 0 and 1 is negative, our assumption contradicts (2.17), which means that ζ has no zeros on $\Re(s) = 1$. □

2.2.2 Estimates for ζ and ζ'

Later on, we will require some knowledge about the growth of ζ and ζ', because the logarithmic derivative $\frac{d}{ds}\log(\zeta(s)) = \zeta'(s)/\zeta(s)$ will be important. At first we need to prove two other theorems.

Proposition 2.2.7 *There is a sequence of holomorphic functions $\{\delta_n\}_{n=1}^{\infty}$, that satisfy $|\delta_n(s)| \leq |s|/n^{\sigma+1}$, where $s = \sigma + it$, and such that*

$$\sum_{1 \leq n < N} \frac{1}{n^s} - \int_1^N \frac{dx}{x^s} = \sum_{1 \leq n < N} \delta_n(s), \tag{2.21}$$

where N is an integer > 1.

Proof. We set

$$\delta_n(s) = \int_n^{n+1} \left[\frac{1}{n^s} - \frac{1}{x^s}\right] dx. \tag{2.22}$$

We apply the mean-value theorem to $f(x) = x^{-s}$ and we obtain

$$\left|\frac{1}{n^s} - \frac{1}{x^s}\right| \leq \frac{|s|}{n^{\sigma+1}},$$

whenever $n \leq x \leq n + 1$. Therefore $|\delta_n(s)| \leq |s|/n^{\sigma+1}$. Since

$$\int_1^N \frac{dx}{x^s} = \sum_{1 \leq n < N} \int_n^{n+1} \frac{dx}{x^s},$$

the proposition holds. □

Proposition 2.2.8 *For $\Re(s) > 0$ we have*

$$\zeta(s) - \frac{1}{s-1} = H(s)$$

where $H(s) = \sum_{n=1}^{\infty} \delta_n(s)$ is holomorphic in the half-plane $\Re(s) > 0$.

Proof. First we assume that $\Re(s) > 1$. We let N tend to infinity in our equation (2.21). The left-hand side tends to $\sum_{n=1}^{\infty} \frac{1}{n^s} - \frac{1}{s-1}$. Since $|\delta_n(s)| \leq |s|/n^{\sigma+1}$, we have uniform convergence of the series $\sum \delta_n(s)$ in the half-plane $\Re(s) > 1$. The fact that the series $\sum n^{-s}$ converges to $\zeta(s)$, the assertion is true for $\Re(s) > 1$. The uniform convergence also shows that $\sum \delta_n(s)$ is holomorphic when $\Re(s) > 0$, which shows us that we can extend $\zeta(s)$ to this half-plane, and the identity continues to hold there. □

Theorem 2.2.9 *For each $|t| \geq 1, 0 < \epsilon < \sigma_0 \leq 1$, there exists a constant $c_\epsilon > 0$ such that*

1. $|\zeta(s)| \leq c_\epsilon |t|^{1-\sigma_0+\epsilon}$ *if* $\sigma_0 \leq \sigma$
2. $|\zeta'(s)| \leq c_\epsilon |t|^\epsilon$, *if* $1 \leq \sigma$,

where $s = \sigma + it$.

Proof. We use Proposition 2.2.8 and choose a $|t| = 1/2$. We've found the estimate $|\delta_n(s)| \leq |s|/n^{\sigma+1}$. We can find another estimate by using (2.22). The fact that $|n^{-s}| = n^{-\sigma}$, and $|x^{-s}| \leq n^{-\sigma}$ if $x \geq n$, gives us the estimate $|\delta_n(s)| \leq 2/n^\sigma$. We combine these two estimates to find one new estimate:

$$|\delta_n(s)| = |\delta_n(s)|^\delta |\delta_n(s)|^{1-\delta} \leq \left(\frac{|s|}{n^{\sigma_0+1}}\right)^\delta \left(\frac{2}{n^{\sigma_0}}\right)^{1-\delta} \leq \frac{2|s|^\delta}{n^{\sigma_0+\delta}}.$$

for $0 \leq \delta \leq 1$. We choose $\delta = 1 - \sigma_0 + \epsilon$, which is justified for $\epsilon < \sigma_0$. Applying the identity of Proposition 2.2.8 together with $\sigma = \Re(s) \geq \sigma_0$ gives us

$$|\zeta(s)| \leq \left|\frac{1}{s-1}\right| + 2|s|^{1-\sigma_0+\epsilon} \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}.$$

The first term on the left-hand side is bounded since $|t| \geq 1$. The sum converges for every $\epsilon > 0$. We've proved the first estimate. We can prove the second estimate by using the Cauchy integral formula:

$$\zeta'(s) = \frac{1}{2\pi r} \int_0^{2\pi} \zeta(s + re^{i\theta}) e^{-i\theta} d\theta.$$

We choose $r = \epsilon < 1/2$, so that we integrate over a circle that lies in the half-plane $\Re(s) \geq 1 - \epsilon$. We now use the established estimate for ζ. This yields

$$\begin{aligned} |\zeta'(s)| &\leq \frac{1}{2\pi\epsilon} \int_0^{2\pi} \left|\zeta(s + \epsilon e^{i\theta})\right| d\theta. \\ &\leq \frac{1}{2\pi\epsilon} \int_0^{2\pi} c_\epsilon \left|t + \epsilon \sin\theta\right|^{1-\sigma_0+\epsilon} d\theta. \end{aligned}$$

We replace 2ϵ by ϵ which gives the desired estimate. □

Theorem 2.2.10 *For every $\epsilon > 0$ we can find a $c_\epsilon > 0$ such that $1/|\zeta(s)| \leq c_\epsilon |t|^\epsilon$ for $s = \sigma + it$ with $\sigma \geq 1$ and $|t| \geq 1$.*

Proof. From (2.17), it follows easily that for $\sigma > 1$ we have

$$|\zeta^3(\sigma)\zeta^4(\sigma + it)\zeta(\sigma + 2it)| \geq 1.$$

Besides, for $\sigma = 1$ the above holds as well, since $|\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \to \infty$ as $\sigma \to 1$.

Therefore, we have

$$|\zeta^4(\sigma + it)| \geq |\zeta^{-3}(\sigma)\zeta^{-1}(\sigma + 2it)|.$$

Using Theorem 2.2.9 we know that $|\zeta(\sigma + 2it)| \leq c_\epsilon |2t|^{1-\sigma_0+\epsilon}$, hence

$$|\zeta^4(\sigma + it)| \geq c|\zeta^{-3}(\sigma)||t|^{-\epsilon} \geq c'(\sigma - 1)^3|t|^{-\epsilon}$$

for all $\sigma \geq 1$ and $|t| \geq 1$. The last inequality follows from the fact that $\zeta(\sigma)$ is bounded on $[\sigma_1, \infty)$ for every $\sigma_1 > 1$ fixed. Besides, we've seen in Theorem 2.2.6 that $\zeta(\sigma) \leq D(\sigma - 1)^{-1}$ as $\sigma \to 1$.
It follows that

$$|\zeta(\sigma + it)| \geq c'(\sigma - 1)^{3/4}|t|^{-\epsilon/4} \tag{2.23}$$

If $\sigma - 1 \geq A|t|^{-5\epsilon}$ for some constant A, we find

$$|\zeta(\sigma + it)| \geq A'|t|^{-4\epsilon}.$$

If we replace 4ϵ by ϵ we find the desired result.

If $\sigma - 1 < A|t|^{-5\epsilon}$, the computations are a little bit more difficult. We choose $\sigma' > \sigma$ such that $\sigma' - 1 = A|t|^{-5\epsilon}$. From the triangle inequality it follows that

$$|\zeta(\sigma + it)| \geq |\zeta(\sigma' + it)| - |\zeta(\sigma' + it) - \zeta(\sigma + it)|. \tag{2.24}$$

We use the mean value theorem for the real and imaginary part of ζ, and applying the estimate for the derivative of ζ in Theorem 2.2.9 yields

$$|\zeta(\sigma' + it) - \zeta(\sigma + it)| \leq c''|\sigma' - \sigma||t|^\epsilon \leq c''|\sigma' - 1||t|^\epsilon$$

We substitute the last inequality and inequality (2.23) into the triangle inequality (2.24), which yields

$$|\zeta(\sigma + it)| \geq c'(\sigma' - 1)^{3/4}|t|^{-\epsilon/4} - c''(\sigma' - 1)|t|^\epsilon.$$

We choose $A = (c'/(2c''))^4$. We now have

$$c'(\sigma' - 1)^{3/4}|t|^{-\epsilon/4} = 2c''(\sigma' - 1)|t|^\epsilon,$$

because we've chosen $\sigma' - 1 = A|t|^{-5\epsilon}$. Eventually we find

$$|\zeta(\sigma + it)| \geq c''(\sigma' - 1)|t|^\epsilon = c''A|t|^{-4\epsilon}.$$

Replacing 4ϵ by ϵ completes the proof. □

Chapter 3

A proof of the prime number theorem

3.1 Auxiliary functions

We've already mentioned that the prime number theorem holds if and only if ζ has no zeros on $\Re(s) = 1$. We proved the last statement in the previous section! Now we have to make the right translation backwards from analysis to number theory.

Chebyshev wasn't able to prove the prime number theorem. However, he found some results which were a prelude to a complete proof. He used an auxiliary function ψ, which resembles the function π to a large extent. The function ψ is defined by

$$\psi(x) = \sum_{p^m \leq x} \log p. \tag{3.1}$$

p is a prime number and m is an arbitrary positive integer. We now use a slightly different notation. If we define

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

we can write

$$\psi(x) = \sum_{n \leq x} \Lambda(n) \tag{3.3}$$

Finally, we can use the fact that if $p^m \leq x$, then $m \leq \log x / \log p$. Therefore, we can write (3.1) as

$$\psi(x) = \sum_{p \leq x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p, \tag{3.4}$$

where $\lfloor u \rfloor$ denotes the greatest integer $\leq u$. The following theorem is a great result since we can use the function ψ instead of the function π to prove the prime number theorem. The first function is much easier to manipulate.

Theorem 3.1.1 *If $\psi(x) \sim x$ as $x \to \infty$, then $\pi(x) \sim x/\log x$ as $x \to \infty$.*

Proof. We assume that $\psi(x) \sim x$ as $x \to \infty$. This means that

$$\lim_{x\to\infty} \psi(x)/x = 1.$$

We note that

$$\psi(x) = \sum_{p\le x} \left\lfloor \frac{\log x}{\log p} \right\rfloor \log p \le \sum_{p\le x} \frac{\log x}{\log p} \log p = \pi(x)\log x.$$

Dividing both sides by x yields

$$\frac{\psi(x)}{x} \le \frac{\pi(x)\log x}{x}.$$

Our assumption now gives the inequality

$$1 \le \liminf_{x\to\infty} \pi(x)\frac{\log x}{x}. \tag{3.5}$$

We now have to find the other inequality. We use (3.1), choose $0 < \alpha < 1$ fixed and observe that

$$\begin{aligned}\psi(x) \ge \sum_{p\le x} \log p \ \ge \sum_{x^\alpha < p \le x} \log p \ge \log(x^\alpha) \sum_{x^\alpha < p\le x} 1 \\ = (\pi(x) - \pi(x^\alpha))\log x^\alpha\end{aligned}$$

This yields the following result:

$$\frac{\psi(x)}{x} + \alpha\pi(x^\alpha)\frac{\log x}{x} \ge \alpha\pi(x)\frac{\log x}{x}$$

We use the fact that $\pi(x^\alpha) \le x^\alpha, \alpha < 1$, and our assumption that $\psi(x) \sim x$. The second term on the left-hand side will therefore vanish as $x \to \infty$. We obtain

$$1 \ge \alpha \limsup_{x\to\infty} \pi(x)\frac{\log x}{x} \tag{3.6}$$

Because we could choose $\alpha < 1$ arbitrary, we've found the other inequality. Together, (3.5) and (3.6) give the desired result. □

The previous theorem states that the prime number theorem holds if $\psi(x) \sim x$. This means that we don't have to worry about the function π anymore, working with the auxiliary function ψ is more convenient.

However, it is even more convenient to work with another auxiliary function. We define the function ψ_1 as the antiderivative of ψ:

$$\psi_1(x) = \int_1^x \psi(u)\,du.$$

The following theorem says how we can use this auxiliary function.

Theorem 3.1.2 *If* $\psi_1(x) \sim x^2/2 \quad as \quad x \to \infty$*, then* $\psi(x) \sim x \quad as \quad x \to \infty$.

Proof. At first we notice that $\psi(x)$ is an increasing function, since $\log p > 0$ for every prime p. If we choose an arbitrary $0 < \alpha < 1$,

$$\frac{1}{(1-\alpha)x} \int_{\alpha x}^{x} \psi(u)\, du$$

gives the mean value of ψ on the interval $[\alpha x, x]$. Since ψ is increasing, we find the inequality

$$\frac{1}{(1-\alpha)x} \int_{\alpha x}^{x} \psi(u)\, du \leq \psi(x)$$

Using the definition of our auxiliary function ψ_1 yields

$$\frac{1}{(1-\alpha)x} (\psi_1(x) - \psi_1(\alpha x)) \leq \psi(x)$$

We divide both sides by x since we would like to find an expression for $\psi(x)/x$:

$$\frac{1}{1-\alpha} \left[\frac{\psi_1(x)}{x^2} - \frac{\psi_1(\alpha x)}{(\alpha x)^2}\, \alpha^2 \right] \leq \frac{\psi(x)}{x}$$

Now we use our assumption that $\psi_1(x) \sim x^2/2$ as $x \to \infty$. This yields

$$\liminf_{x\to\infty} \frac{\psi(x)}{x} \geq \frac{1}{1-\alpha} \left[\frac{1}{2} - \frac{1}{2}\alpha^2 \right] = \frac{1}{2}(1+\alpha)$$

Because this must hold for every $\alpha < 1$, we have proved that $\liminf_{x\to\infty} \psi(x)/x \geq 1$.

We can prove that $\limsup_{x\to\infty} \psi(x)/x \leq 1$ in exactly the same way by using a $\beta > 1$ and using the opposite inequalities. We leave this as an exercise for the reader. Finding the other inequality completes the proof of the fact that $\lim_{x\to\infty} \frac{\psi(x)}{x} = 1$. $\square$

3.1.1 The link between ζ and Chebyshev's auxiliary functions

From Theorem 3.1.1 and 3.1.2, it follows that we can prove the prime number theorem by proving the asymptotic relation for $\psi_1(x)$.

We can only use this auxiliary function if we find its relation with the zeta function. In Lemma 2.2.4 we've seen that we can write for $\Re(s) > 1$

$$\log \zeta(s) = \sum_{p,m} \frac{p^{-ms}}{m}.$$

If we differentiate this expression (we may do this because of uniform convergence) and change the minus signs, we get

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum_{m,p} p^{-ms} \log p = \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}. \tag{3.7}$$

We will need (3.7) in the following proposition.

Proposition 3.1.3 *For all $c > 1$*

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right) ds. \tag{3.8}$$

This expression relates ψ_1 with ζ.

Before we can prove this proposition, we will need the following lemma.

Lemma 3.1.4 *If $c > 0$, then*

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{a^s}{s(s+1)}\, ds = \begin{cases} 0 & \text{if } 0 < a \leq 1, \\ 1 - 1/a & \text{if } 1 \leq a. \end{cases} \tag{3.9}$$

This is an example of an inverse Mellin transform, which is closely related to the well-known Laplace transform.

Proof. At first we note that the integral converges, since $|a^s| = a^c$, with c fixed. We suppose that $1 \leq a$. We now can write $a = e^\beta$ for some $\beta \geq 0$. We define

$$f(s) = \frac{a^s}{s(s+1)} = \frac{e^{s\beta}}{s(s+1)}.$$

It's easy to see that we will find poles at $s = -1$ and at $s = 0$. Since both poles are simple, we can calculate the residues quite easily. We find

$$\begin{aligned} \text{res}\{f(s); -1\} &= \frac{a^{-1}}{-1} = -1/a \\ \text{res}\{f(s); 0\} &= \frac{a^0}{1} = 1. \end{aligned}$$

For a more detailed explanation of calculation of residues we refer to [5]. We now choose a positive oriented path $\Gamma(T)$ which is a semi-circle. It consists of the vertical segment $S(T)$ from $c - iT$ to $c + iT$, and of the half-circle $C(T)$, lying to the left of the vertical segment, with radius T and centered at c.

We have to choose $T > c + 1$ to make sure that 0 and -1 are contained in the interior of our contour $\Gamma(T)$. By Cauchy's residue formula we then have

$$\frac{1}{2\pi i}\int_{\Gamma(T)} f(s)\, ds = 1 - 1/a. \tag{3.10}$$

We are interested in the integral

$$\int_{S(T)} f(s)\, ds$$

as T tends to infinity. Therefore, we investigate the value of the integral

$$\int_{C(T)} f(s)\, ds$$

as T tends to infinity. If $s = \sigma + it \in C(T)$, then for T large we have

$$|s(s+1)| \geq (1/2)T^2.$$

Moreover, we have the estimate $|e^{\beta s}| = e^{\beta\sigma} \leq e^{\beta c}$, since $\sigma \leq c$ and $\beta \geq 0$. The results above yield

$$\left| \int_{C(T)} f(s)\, ds \right| \leq \frac{C}{T^2} 2\pi T \to 0 \quad \text{as} \quad T \to \infty.$$

Therefore, we have

$$\int_{S(T)} f(s)\, ds = \int_{\Gamma(T)} f(s)\, ds = 1 - 1/a$$

for $a \geq 1$ as T tends to infinity.

We leave the case $0 < a \leq 1$ as an exercise for the reader. Use the same arguments as above and use the half-circle lying to the *right* of the line $\Re(s) = c$. Note that there aren't any poles in the interior of this contour! □

Proof of Proposition 3.1.3. At first we notice that we can write (3.3) also as

$$\psi(u) = \sum_{n=1}^{\infty} \Lambda(n) f_n(u),$$

with

$$f_n(u) = \begin{cases} 1 & \text{if} \quad n \leq u; \\ 0 & \text{otherwise.} \end{cases}$$

Because $\int_0^1 \psi(u)\, du = 0$, we can write

$$\psi_1(x) = \int_0^x \psi(u)\, du$$

Because of uniform convergence we can now change the order of summation and integration, which yields

$$\begin{aligned} \psi_1(x) &= \sum_{n=1}^{\infty} \int_0^x \Lambda(n) f_n(u)\, du \\ &= \sum_{n=1}^{\infty} \Lambda(n) \int_0^x f_n(u)\, du \\ &= \sum_{n=1}^{x} \Lambda(n) \int_n^x du, \end{aligned}$$

so we have

$$\psi_1(x) = \sum_{n \leq x} \Lambda(n)(x - n) \tag{3.11}$$

as a final result.

We now use equation (3.7) and lemma 3.1.4 to see that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds = x\sum_{n=1}^{\infty}\Lambda(n)\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{(x/n)^s}{s(s+1)}\,ds \quad (3.12)$$

$$= \quad x\sum_{n=1}^{x}\Lambda(n)\left(1-\frac{n}{x}\right) \quad (3.13)$$

$$= \quad \psi_1(x), \quad (3.14)$$

where the last step follows from (3.11). □

Now we have all the ingredients to prove the prime number theorem. According to the theorems 3.1.1 and 3.1.2, we have to prove the following theorem:

Theorem 3.1.5

$$\psi_1(x) \sim x^2/2 \quad \text{as} \quad x \to \infty \quad (3.15)$$

In order to prove this theorem, we need our formula from Proposition 3.1.3:

$$\psi_1(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds \quad (3.16)$$

for $c > 1$. We would like to change the line of integration to $\Re(s) = 1$ in the integral above, since we then would have the factor x^2 in our integrand. However, we have a pole for $\zeta(s)$ at $s = 1$, so we must be careful.

At first we deform the path of integration from $c - i\infty$ to $c + i\infty$ to the path $\gamma(T)$ as shown in Figure 1. The vertical segments of $\gamma(T)$ consist of $T \le t < \infty$ and $-\infty < t \le -T$. Let $F(s)$ denote the integrand:

$$F(s) = \frac{x^{s+1}}{s(s+1)}\left(-\frac{\zeta'(s)}{\zeta(s)}\right).$$

We may use Cauchy's theorem to see that

Theorem 3.1.6

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)\,ds = \frac{1}{2\pi i}\int_{\gamma(T)}F(s)\,ds. \quad (3.17)$$

Proof. From the theorems 2.2.9 and 2.2.10, we know that $|\zeta'(s)/\zeta(s)| \le A|t|^\eta$ for any fixed $\eta > 0$, whenever $s = \sigma + it, \sigma \ge 1$, and $|t| \ge 1$. Hence it follows that $|F(s)| \le A'|t|^{-2+\eta}$ for some constant $A' > 0$, in the two infinite rectangles bounded by the line $(c - i\infty, c + i\infty)$ and $\gamma(T)$. F is holomorphic in that region, and its decrease is rapid enough to establish the assertion. □

We now pass from the contour $\gamma(T)$ to the contour $\gamma(T, \delta)$. For fixed T we choose $\delta > 0$ small enough so that ζ has no zeros in the box

$$\{s = \sigma + it, 1 - \delta \le \sigma \le 1, |t| \le T\}.$$

We always can choose such a δ, since we know that ζ does not vanish on the line $\sigma = 1$. By Proposition 2.2.8, we know that $\zeta(s) = 1/(s-1) + H(s)$, where

H is holomorphic near $s = 1$. Hence $-\zeta'(s)/\zeta(s) = 1/(s-1) + h(s)$, where h is holomorphic near $s = 1$. This means that $F(s)$ has a simple pole at $s = 1$ and the residue of F at $s = 1$ equals $x^2/2$. We obtain

$$\frac{1}{2\pi i}\int_{\gamma(T)} F(s)\,ds = \frac{x^2}{2} + \frac{1}{2\pi i}\int_{\gamma(T,\delta)} F(s)\,ds. \tag{3.18}$$

We decompose the contour $\gamma(T,\delta)$ as $\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5$. We will estimate each integral $\int_{\gamma_j} F(s)\,ds$ for $j = 1, 2, 3, 4, 5$ (see Figure 1).

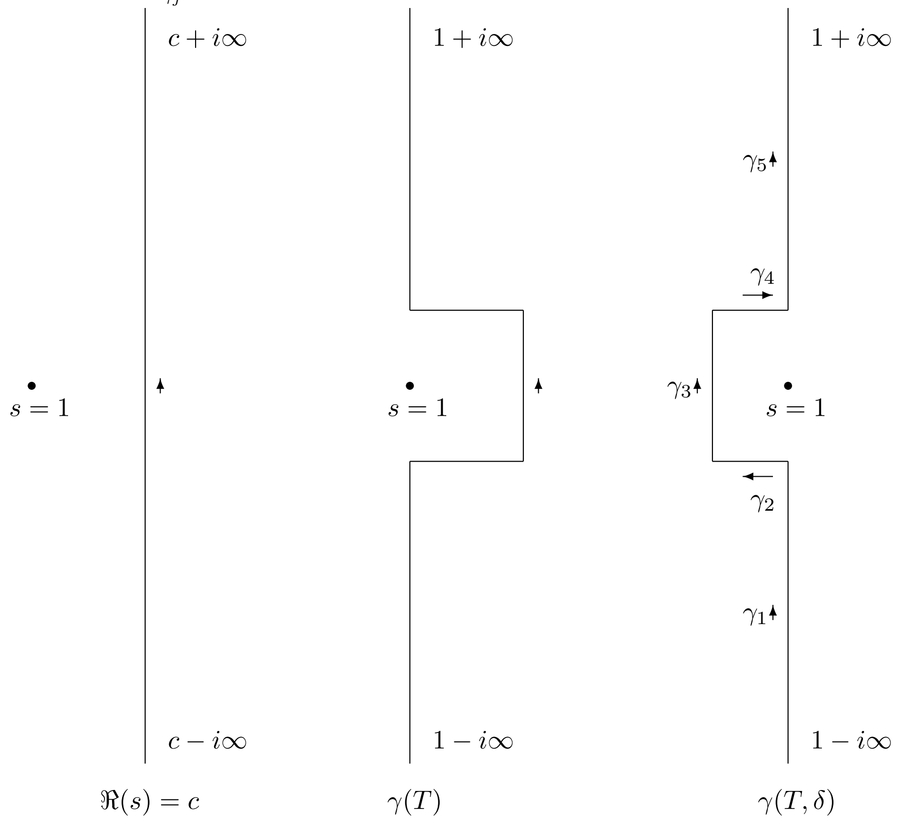

Figure 1. The paths of integration for the three different stages.

Theorem 3.1.7 *There exists T so large such that*

$$\left|\int_{\gamma_1} F(s)\,ds\right| \le \frac{\epsilon}{2}x^2 \quad \textit{and} \quad \left|\int_{\gamma_5} F(s)\,ds\right| \le \frac{\epsilon}{2}x^2.$$

Proof. For $s \in \gamma_1, \gamma_5$ we have

$$|x^{1+s}| = x^{1+\sigma} = x^2.$$

We estimate $|\zeta'(s)/\zeta(s)| \le A|t|^{1/2}$ by the established estimates from section 2.2.2, so we have

$$\left|\int_{\gamma_1} F(s)\,ds\right| \le Cx^2\int_T^\infty \frac{|t|^{1/2}}{t^2}\,dt.$$

.

We know that the integral converges, so we can choose a T sufficiently large such that the right hand side $\leq \epsilon x^2/2$. The argument for the integral over γ_5 is the same. $\square$

We now look at the segment γ_3.

Theorem 3.1.8 *For fixed T we can choose δ sufficiently small, such that*

$$\left| \int_{\gamma_3} F(s)\, ds \right| \leq C_T x^{2-\delta}.$$

Proof. On γ_3 we have

$$|x^{1+s}| = x^{1+1-\delta} = x^{2-\delta},$$

so we can make the estimate

$$\left| \int_{\gamma_3} F(s)\, ds \right| \leq x^{2-\delta} \int_{\gamma_3} \frac{1}{s(s+1)} \left| \frac{\zeta'(s)}{\zeta(s)} \right| ds,$$

which implies that there exists a constant C_T such that the inequality holds. $\square$

We now look at the horizontal segments γ_2 and γ_4. We can estimate the integrals by

$$\left| \int_{\gamma_2} F(s)\, ds \right| \leq C'_T \int_{1-\delta}^{1} x^{1+\sigma}\, d\sigma \leq C'_T \frac{x^2}{\log x}.$$

We have the same estimation for γ_4.

We now use (3.18) and the estimations above to see that

$$\left| \psi_1(x) - \frac{x^2}{2} \right| \leq \epsilon x^2 + D_T x^{2-\delta} + D'_T \frac{x^2}{\log x},$$

where D and D' are new constants.

Dividing by $x^2/2$ yields

$$\left| \frac{2\psi_1(x)}{x^2} - 1 \right| \leq 2\epsilon + 2D_T x^{-\delta} + 2D'_T \frac{1}{\log x}.$$

Therefore, for sufficiently large x we have

$$\left| \frac{2\psi_1(x)}{x^2} - 1 \right| \leq 4\epsilon.$$

This concludes the proof that

$$\psi_1(x) \sim x^2/2 \quad \text{as} \quad x \to \infty. \quad \square$$

With the last theorem, we've completed the proof of the prime number theorem!

Chapter 4

Distribution of the zeros of the Riemann-zeta function

As we already said briefly, the critical zeros of the Riemann-zeta function are the zeros which lie in the infinite strip $0 \leq \Re(s) \leq 1$. One of the most famous mathematical conjectures involve these zeros. Riemann conjectured that all the critical zeros lie on the line $\Re(s) = 1/2$. This is the so-called Riemann-hypothesis. Because there are infinitely many critical zeros, we can't check them one by one to see whether Riemann was right or not. However, accurate calculations show that the Riemann-hypothesis seems to be true, though a proof would be an important result. Proving the Riemann hypothesis is one of the seven challenges the Clay Mathematics Institute in Massachussetts has put up a reward of \$1.000.000. Its importance is caused by the fact that many theorems in number theory only hold if the Riemann-hypothesis is true. A counterexample of the conjecture (which one does not expect) would therefore have immense consequences.

4.1 Random Matrices

There have been many investigations to see how the critical zeros are distributed in the course of a search for the Riemann Hypothesis. This has led to some amazing results. By investigating the distribution of the zeros of ζ on $\Re(s) = 1/2$, some amazing patterns have been discovered (which only hold if the Riemann hypothesis is true!). Montgomery published a paper in 1973 in which he conjectured a specific distribution of the (rescaled) critical zeros of ζ (see [8]). Odlyzko confirmed this conjecture with extraordinary accuracy by computing the spacing distribution of the zeros of ζ. Eventually in 1995, Rudnick and Sarnak ([9]) gave a proof of Montgomery's conjecture up to some technical restrictions. The reason why this hard-earned result is that special, is caused by the fact that this kind of distribution also corresponds to the energy levels of heavy nuclei, a result that Wigner obtained in the 1950s! Random matrix theory was introduced by theoretical physicists that time to study this subject. Now it won't be a surprise that there is a link between random matrix theory and the distribution of our critical zeros as well.

4.1.1 Montgomery's conjecture

The mathematician Montgomery had been working for some years on the problem of the critical zeros of ζ at the Institute for Advanced Study in Princeton, in the early 1970s. He assumed the Riemann hypothesis to be true, and rescaled the imaginary parts $0 < \gamma_1 < \gamma_2 < \ldots$ of the critical zeros $1/2 + i\gamma$ of $\zeta(s)$

$$\gamma_j \to \tilde{\gamma}_j = \frac{\gamma_j \log \gamma_j}{2\pi},$$

such that the mean spacing between adjacent zeros is 1:

$$\frac{\#j \geq 1 : \tilde{\gamma}_j < T}{T} \to 1 \quad \text{as} \quad T \to \infty.$$

This follows by a classical result of Von Mangoldt, assuming the Riemann Hypothesis. For convenience, we will only look at the zeros with positive imaginary parts, i.e. $\gamma_j > 0$. However, for negative γ_j, we can almost proceed in the same way.

Let's define the pair correlation of the zeros to be

$$R(a,b) \equiv \lim_{N\to\infty} \frac{1}{N} \#\{\text{pairs}(j_1, j_2) : 1 \leq j_1, j_2 \leq N, \tilde{\gamma}_{j_1} - \tilde{\gamma}_{j_2} \in (a,b)\}$$

for any interval (a,b), with $a, b > 0$. Montgomery found the result

$$R(a,b) = \int_a^b \left(1 - \left(\frac{\sin 2\pi u}{2\pi u}\right)^2\right) du. \tag{4.1}$$

He wasn't able to prove this pair correlation function. It goes beyond the scope of this report to explain how Montgomery had found his result, but the result itself is quite astonishing.

At first, Montgomery wasn't aware of that. This changed when he met the physicist and mathematician Freeman Dyson. He couldn't attend a lecture Montgomery had given about his work, but he was interested and astounded him by asking whether he found his final result (4.1)! It seemed that Dyson had already expected that the zeros of ζ behave like the eigenvalues of a specific random matrix! Dyson was one of the pioneers on random matrix theory, and the pair correlation function (4.1) was already known to be the pair correlation function of eigenvalues of a random matrix chosen from the so-called *Gaussian Unitary Ensemble*. From that moment on, random matrix theory became even more important.

4.1.2 Random Matrix Theory

Since many natural phenomena involve randomness, Wigner tried to model the energy levels of heavy nuclei by using $N \times N$ matrices where the entries were independently chosen from a probability distribution p. For specific matrices (for example real symmetric or complex Hermitian) and specific probability distributions, astonishing things happen. If $N \to \infty$, the behavior of the eigenvalues of such a matrix is often well approximated by the behavior we will find if we average over all matrices. Besides, this behavior corresponds to the energy levels of heavy nuclei.

Let's consider the collection of $N \times N$ real symmetric matrices, with the entries independently chosen from a fixed probability distribution p on $\mathbb{R}$. For such a matrix A, we have $A = A^T$, so $a_{ij} = a_{ji}$ for $1 \leq i, j \leq N$. The probability density of observing A is therefore

$$\text{Prob}(A)dA = \prod_{1\leq i\leq j\leq N} p(a_{ij})da_{ij}.$$

Because p is a probability density, it integrates to 1. This yields

$$\int \text{Prob}(A)dA = \prod_{1\leq i\leq j\leq N} \int_{a_{ij}=-\infty}^{\infty} p(a_{ij})da_{ij} = 1.$$

Also, we want to put some restrictions on the probability density function p. It may not be too spread out. We will rescale the eigenvalues and we only can say something about them if they are not too spread out as well. Therefore we study p satisfying

$$\begin{aligned}
\forall x : p(x) &\geq 0 \quad \text{(since we must have a probability density)}\\
\int_{-\infty}^{\infty} p(x)\,dx &= 1 \quad \text{(since we must have a probability density)}\\
\int_{-\infty}^{\infty} |x|^k p(x)\,dx &< \infty \quad \text{(for every non-negative integer } k\text{)}.
\end{aligned}$$

If we want to say something about the spacings between eigenvalues, the eigenvalues must be necessarily real. The following theorem guarantees this.

Definition 4.1.1 *A Hermitian matrix is a square matrix which is equal to its own conjugate transpose.*

Note that the diagonal entries of a Hermitian matrix are always real. Besides, a symmetric matrix is a Hermitian matrix!

Theorem 4.1.2 *Every Hermitian matrix has only real eigenvalues.*

Proof. According to the spectral theorem, we can diagonalize a Hermitian matrix by a unitary matrix. The resulting diagonal matrix has only real entries. Since the eigenvalues of the Hermitian matrix equal the diagonal entries of this matrix, its eigenvalues are real. □

The theorem above is very important for us. If we investigate the collection of Hermitian matrices, we will certainly find real eigenvalues! Another important theorem is the *Eigenvalue Trace Formula*. This theorem relates the eigenvalues of a matrix to the entries of that matrix:

Theorem 4.1.3 *For any non-negative integer k, if A is an $N \times N$ matrix with eigenvalues $\lambda_i(A)$, then*

$$\text{Trace}(A^k) = \sum_{i=1}^{N} \lambda_i(A)^k \tag{4.2}$$

Proof. We use the Jordan decomposition for A, writing $A = PJP^{-1}$, with J an upper triangular matrix with the eigenvalues of A on the diagonal. Since $\text{tr}(AB) = \text{tr}(BA)$, the result follows for $k = 1$. Now using $A^k = PJ^kP^{-1}$, the result follows for every non-negative integer k. $\square$

We've said earlier that we should scale the eigenvalues of our $N \times N$ matrices, because we don't want the eigenvalues to be large if we increase N. It's convenient to choose the entries a_{ij} of our matrix A randomly and independently from a fixed probability distribution p with mean 0 and variance 1. For real symmetric A, we have

$$\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij} a_{ji} = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2.$$

We expect each a_{ij}^2 to be of size 1, by assumption on p. Therefore, we expect

$$\text{Trace}(A^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \sim N^2 \cdot 1.$$

Using the Eigenvalue Trace Formula yields

$$\sum_{i=1}^{N} \lambda_i(A)^2 \sim N^2,$$

and using the average of $\lambda_i(A)^2$ gives us eventually

$$|\text{Ave}(\lambda_i(A))| \sim \sqrt{N}. \tag{4.3}$$

Equation (4.3) tells us that we should scale the eigenvalues of an $N \times N$ real symmetric matrix by $c\sqrt{N}$, with c a constant. Choosing $c = 2$ is convenient, but we omit the details. This means that we will work with normalized eigenvalues $\tilde{\lambda}_i(A) = \frac{\lambda_i(A)}{2\sqrt{N}}$.

For every $N \times N$ matrix A, we want to attach a probability measure which is the eigenvalue probability distribution. We want to have a probability distribution $\mu_{A,N}(x)$ for which

$$\int_a^b d\mu_{A,N}(x) = \frac{\#\{i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\}}{N}, \tag{4.4}$$

i.e. $\int_a^b d\mu_{A,N}(x)$ is the percentage of normalized eigenvalues in $[a, b]$. If we define

$$\mu_{A,N} = \frac{1}{N} \sum_{i=1}^{N} \delta \left(x - \frac{\lambda_i(A)}{2\sqrt{N}} \right), \tag{4.5}$$

we have the right probability distribution.

Later on, we require that the moments of $\mu_{A,N}$ are finite. The following theorem guarantees that.

Theorem 4.1.4 *Let $M_{N,k}(A)$ denote the k^{th} moment of $\mu_{A,N}$. Then the following identity holds:*

$$M_{N,k}(A) = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}. \tag{4.6}$$

Proof. We use the Eigenvalue Trace Formula from Theorem 4.1.3 to see that

$$\begin{aligned} M_{n,k}(A) &= \int x^k d\mu_{A,N}(x) \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_{-\infty}^{\infty} x^k \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) dx \\ &= \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i(A)^k}{(2\sqrt{N})^k} \\ &= \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}. \quad \square \end{aligned}$$

We know that the percentage of the normalized eigenvalues of an $N \times N$ symmetric matrix lying in an interval $[a, b]$ is given by

$$\int_a^b d\mu_{A,N}(x).$$

We choose the entries of our matrix A at random, so we would like to know how this percentage behaves as we vary A.

Theorem 4.1.5 ***(Semi-Circle Law)***. *Consider the ensemble of $N \times N$ real symmetric matrices with entries independently chosen from a fixed probability density $p(x)$ with mean 0 and variance 1. If p has finite moments, $\mu_{A,N}$ converges to the semi-circle density $\frac{2}{\pi}\sqrt{1-x^2}$, for almost all A, as $N \to \infty$.*

We can prove this theorem by calculating the k^{th}-moments of each $\mu_{A,N}(x)$. If we let $M_{n,k}$ be the average of the moments $M_{N,k}(A)$ over all A, we can show that $M_{n,k}$ converges to the k^{th} moment of the semi-circle as $N \to \infty$.

We've already mentioned that the distribution of the critical zeros of ζ corresponds in a way to spacings between adjacent normalized eigenvalues of some random matrix. We therefore investigate this spacing.

We will consider the space of 2×2 real symmetric matrices:

$$\left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} : a_{11}, a_{12}, a_{22} \in \mathbb{R} \right\}.$$

We focus on 2×2 matrices, since for general N, the computations will be too complicated. We choose our entries from some probability density p. We desire the following properties:

1. The entries of our matrix A should be independent: $P(A) = P_{11}(a_{11})P_{12}(a_{12})P_{22}(a_{22})$.
2. The probability of a transformation A should not depend on the chosen basis: For any orthogonal transformation Q, we must have $P(QAQ^T) = P(A)$.

The last two restrictions force our probability distributions to be Gaussians. We therefore call the above model the GOE, the Gaussian Orthogonal Ensemble.

Using fundamental linear algebra and Taylor series, we can write an orthogonal matrix Q as

$$Q = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} = \begin{pmatrix} 1+O(\epsilon^2) & -\epsilon+O(\epsilon^3) \\ \epsilon+O(\epsilon^3) & 1+O(\epsilon^2) \end{pmatrix}. \tag{4.7}$$

For a symmetric matrix A we must have $P(QAQ^T) = P(A)$. Using (4.7), we find

$$QAQ^T = \begin{pmatrix} a_{11} - 2\epsilon a_{12} + O(\epsilon^2) & a_{12} - \epsilon(a_{22}-a_{11}) + O(\epsilon^2) \\ a_{12} - \epsilon(a_{22}-a_{11}) + O(\epsilon^2) & a_{22} + 2\epsilon a_{12} + O(\epsilon^2) \end{pmatrix}.$$

We now use Taylor expansion (where we assume that P_{11}, P_{12}, P_{22} are smooth enough) to find

$$\begin{aligned} P_{11}(a_{11} - 2\epsilon a_{12} + O(\epsilon^2)) &= P_{11}(a_{11}) - 2\epsilon a_{12}\frac{dP_{11}}{da_{11}} + O(\epsilon^2) \\ P_{12}(a_{12} - \epsilon(a_{22}-a_{11}) + O(\epsilon^2)) &= P_{12}(a_{12}) - \epsilon(a_{22}-a_{11})\frac{dP_{12}}{da_{12}} + O(\epsilon^2) \\ P_{22}(a_{22} + 2\epsilon a_{12} + O(\epsilon^2)) &= P_{22}(a_{22}) + 2\epsilon a_{12}\frac{dP_{22}}{da_{22}} + O(\epsilon^2). \end{aligned}$$

Since the probability of a matrix is the product of the probabilities of its entries, we have

$$\begin{aligned} P(QAQ^T) &= P_{11}(a_{11})P_{12}(a_{12})P_{22}(a_{22}) \\ &- \Bigg[2a_{12}P_{12}(a_{12})P_{22}(a_{22})\frac{dP_{11}}{da_{11}} + (a_{22}-a_{11})P_{11}(a_{11})P_{22}(a_{22})\frac{dP_{12}}{da_{12}} \\ &- 2a_{12}P_{11}(a_{11})P_{12}(a_{12})\frac{dP_{22}}{da_{22}}\Bigg]\epsilon + O(\epsilon^2). \end{aligned}$$

We know that $P(A) = P_{11}(a_{11})P_{12}(a_{12})P_{22}(a_{22})$, and this yields

$$\begin{aligned} &\frac{P(A) - P(QAQ^T)}{P_{11}(a_{11})P_{12}(a_{12})P_{22}(a_{22})} \\ &= \left[\frac{2a_{12}}{P_{11}(a_{11})}\frac{dP_{11}}{da_{11}} + \frac{a_{22}-a_{11}}{P_{12}(a_{12})}\frac{dP_{12}}{da_{12}} - \frac{2a_{12}}{P_{22}(a_{22})}\frac{dP_{22}}{da_{22}}\right]\epsilon \\ &\quad + O\left(\frac{\epsilon^2}{P_{11}(a_{11})P_{12}(a_{12})P_{22}(a_{22})}\right). \end{aligned}$$

Since $P(A) = P(QAQ^T)$, the coefficient of ϵ must vanish! Therefore

$$\frac{2a_{12}}{P_{11}(a_{11})}\frac{dP_{11}}{da_{11}} + \frac{a_{22}-a_{11}}{P_{12}(a_{12})}\frac{dP_{12}}{da_{12}} - \frac{2a_{12}}{P_{22}(a_{22})}\frac{dP_{22}}{da_{22}} = 0.$$

We can rewrite this as

$$\frac{1}{a_{12}P_{12}(a_{12})}\frac{dP_{12}}{da_{12}} = -\frac{2}{a_{22}-a_{11}}\left(\frac{1}{P_{11}(a_{11})}\frac{dP_{11}}{da_{11}} - \frac{1}{P_{22}(a_{22})}\frac{dP_{22}}{da_{22}}\right). \tag{4.8}$$

The left hand side of equation (4.8) is a function of a_{12}, while the right hand side is a function of only a_{11} and a_{22}. This forces each side to equal a constant $-C$. We now have

$$\frac{dP_{12}}{da_{12}} = -Ca_{12}P_{12}(a_{12}), \tag{4.9}$$

which has as solution

$$P_{12}(a_{12}) = \sqrt{\frac{C}{2\pi}}e^{-\frac{Ca_{12}^2}{2}}. \tag{4.10}$$

We have chosen the constant such that P_{12} is a probability density function for $C > 0$. We can proceed in the same way to obtain probability density functions for the entries a_{11} and a_{22}. From (4.8), it follows that

$$-\frac{2}{a_{22}-a_{11}}\left(\frac{1}{P_{11}(a_{11})}\frac{dP_{11}}{da_{11}} - \frac{1}{P_{22}(a_{22})}\frac{dP_{22}}{da_{22}}\right) = -C. \tag{4.11}$$

Separating variables yields

$$\frac{2}{P_{11}(a_{11})}\frac{dP_{11}}{da_{11}} + Ca_{11} = \frac{2}{P_{22}(a_{22})}\frac{dP_{22}}{da_{22}} + Ca_{22}. \tag{4.12}$$

Arguing as before, both sides must equal a constant CD, since both sides depend on another variable. We obtain

$$2\frac{dP_{11}}{da_{11}} = -C(a_{11}-D)P_{11}(a_{11}). \tag{4.13}$$

The solution of this ordinary differential equation is

$$P_{11}(a_{11}) = \sqrt{\frac{C}{4\pi}}e^{-\frac{C(a_{11}-D)^2}{4}}, \tag{4.14}$$

and similarly we find

$$P_{22}(a_{22}) = \sqrt{\frac{C}{4\pi}}e^{-\frac{C(a_{22}-D)^2}{4}}. \tag{4.15}$$

We may set $D = 0$, since we want our probability densities to have mean 0, which is convenient. We now have obtained the following result:

$$\begin{aligned} P_{11}(a_{11}) &= \sqrt{\frac{C}{4\pi}}e^{-\frac{Ca_{11}^2}{4}} \\ P_{12}(a_{12}) &= \sqrt{\frac{C}{2\pi}}e^{-\frac{Ca_{12}^2}{2}} \\ P_{22}(a_{22}) &= \sqrt{\frac{C}{4\pi}}e^{-\frac{Ca_{22}^2}{4}} \end{aligned} \tag{4.16}$$

This means that we've found the probability density function for our matrix A:

$$\begin{aligned} P(A) &= P_{11}(a_{11})P_{12}(a_{12})P_{22}(a_{22}) \\ &= \frac{C\sqrt{C}}{4\pi\sqrt{2\pi}}e^{-\frac{C}{4}(a_{11}^2+2a_{12}^2+a_{22}^2)} \\ &= \frac{C\sqrt{C}}{4\pi\sqrt{2\pi}}e^{-\frac{C}{4}\text{Trace}(A^2)}. \end{aligned} \tag{4.17}$$

We now have shown

Theorem 4.1.6 *For 2×2 real symmetric matrices satisfying the GOE assumptions, the entries are chosen independently from Gaussians, where the probability of a matrix is proportional to $e^{\frac{C}{4}\,Trace(A^2)}$.*

We will now investigate the distribution of eigenvalues of 2×2 real symmetric matrices.

4.1.3 Distribution of the eigenvalues (2×2 GOE)

If we choose a 2×2 real symmetric matrix from the GOE ensemble, how likely is it to have two eigenvalues close? We denote our matrix by

$$A = \begin{pmatrix} x & y \\ y & z \end{pmatrix}.$$

By the Spectral Theorem, we know that there is an orthogonal matrix Q such that $QAQ^T = \Lambda$ is diagonal:

$$Q^T \begin{pmatrix} x & y \\ y & z \end{pmatrix} Q = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \Lambda,$$

with λ_1 and λ_2 the eigenvalues of A. We may assume $\lambda_1 \geq \lambda_2$. To find the eigenvalues, we solve the characteristic equation $\det(A - \lambda I) = 0$:

$$\lambda^2 - \lambda(z + x) + xz - y^2 = 0,$$

which yields the solutions

$$\begin{aligned} \lambda_1 &= \frac{x+z}{2} + \sqrt{\left(\frac{x-z}{2}\right)^2 + y^2} \\ \lambda_2 &= \frac{x+z}{2} - \sqrt{\left(\frac{x-z}{2}\right)^2 + y^2}. \end{aligned} \tag{4.18}$$

The eigenvectors of A are

$$\vec{v_1} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \quad \vec{v_2} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

for some $\theta \in [0, 2\pi]$, because the eigenvectors of a real symmetric matrix are perpendicular, and we can always normalize eigenvectors. This means that our matrix Q is of the form

$$Q = Q(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

an arbitrary 2×2 rotation.

To understand the distribution of the spacing between the eigenvalues, we first need to change variables. According to (4.17), the probability density function for observing a matrix A is

$$p(x,y,z) = \frac{C\sqrt{C}}{4\pi\sqrt{2\pi}} e^{-C\text{Trace}(A^2)} = \frac{2C'\sqrt{C'}}{\pi\sqrt{2\pi}} e^{-C'\text{Trace}(A^2)}$$

If we diagonalize A, we obtain

$$A = Q^T \Lambda Q$$

for a diagonal Λ and an orthogonal Q, as before. In this new coordinate system, our parameters are λ_1, λ_2 and θ, instead of x, y and z. We will determine the probability density function $\tilde{p}(\lambda_1, \lambda_2, \theta)$ in this coordinate system.

Since we are not interested in the coordinate system itself, θ depends on the orientation of the coordinate axes, which is not important to us. Therefore, we integrate out the θ dependence, which yields the joint probability density function for λ_1 and λ_2.

$$\begin{aligned} A &= Q^T \Lambda Q \\ &= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= \begin{pmatrix} \cos^2(\theta)\lambda_1 + \sin^2(\theta)\lambda_2 & -\cos(\theta)\sin(\theta)\cdot(\lambda_1 - \lambda_2) \\ -\cos(\theta)\sin(\theta)\cdot(\lambda_1 - \lambda_2) & \sin^2(\theta)\lambda_1 + \cos^2(\theta)\lambda_2 \end{pmatrix} \\ &= \begin{pmatrix} x & y \\ y & z \end{pmatrix}. \end{aligned} \tag{4.19}$$

We see that the change of variable transformation is linear in the eigenvalues. We use the *Jacobian* J for the transformation from (x, y, z) to $(\lambda_1, \lambda_2, \theta)$:

$$p(x,y,z) = |\det(J)|\tilde{p}(\lambda_1, \lambda_2, \theta).$$

We use the eigenvalue trace formula (4.1.3) to see that $\text{Trace}(A^2) = \lambda_1^2 + \lambda_2^2$, so the probability density of observing a matrix with eigenvalues λ_1 and λ_2 is

$$\tilde{p}(\lambda_1, \lambda_2, \theta) = \frac{2C'\sqrt{C'}}{\pi\sqrt{2\pi}} e^{-C'(\lambda_1^2+\lambda_2^2)}.$$

We see that our probability density is indeed independent of the angle θ! We may rescale such that $C' = 1$ in the formula above. This yields

$$p(x,y,z)dxdydz = |\det(J)| \frac{2}{\pi\sqrt{2\pi}} e^{-(\lambda_1^2+\lambda_2^2)} d\lambda_1 d\lambda_2 d\theta, \tag{4.20}$$

where J is the Jacobian

$$J = \begin{pmatrix} \frac{\partial x}{\partial \lambda_1} & \frac{\partial y}{\partial \lambda_1} & \frac{\partial z}{\partial \lambda_1} \\ \frac{\partial x}{\partial \lambda_2} & \frac{\partial y}{\partial \lambda_2} & \frac{\partial z}{\partial \lambda_2} \\ \frac{\partial x}{\partial \theta} & \frac{\partial y}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{pmatrix}. \tag{4.21}$$

We can determine J by differentiating (4.19). We can pull the factor $(\lambda_1 - \lambda_2)$ out of the matrix, since this factor appears in each entry of the bottom row. This yields a 3×3 matrix which only depends on θ (you may try this). We obtain

$$\det(J) = g(\theta) \cdot (\lambda_1 - \lambda_2), \tag{4.22}$$

with $g(\theta)$ the absolute value of the determinant of the matrix obtained above. The probability of finding $\lambda_1 - \lambda_2 = 0$ is zero. Since we've chosen $\lambda_1 \geq \lambda_2$, we don't need to worry about the absolute value.

We can substitute (4.22) into (4.20) and find

$$p(x, y, z)dxdydz = \frac{2g(\theta)}{\pi\sqrt{2\pi}}(\lambda_1 - \lambda_2)e^{-(\lambda_1^2+\lambda_2^2)}d\lambda_1 d\lambda_2 d\theta. \tag{4.23}$$

As we briefly discussed earlier, we want to obtain the joint probability distribution of λ_1 and λ_2, so we integrate out θ:

$$\begin{aligned} \tilde{p}(\lambda_1, \lambda_2) &= \frac{2}{\pi\sqrt{2\pi}} \int (\lambda_1 - \lambda_2)e^{-(\lambda_1^2+\lambda_2^2)} g(\theta) d\theta \\ &= C(\lambda_1 - \lambda_2)e^{-(\lambda_1^2+\lambda_2^2)}, \end{aligned} \tag{4.24}$$

where C is a new (computable) constant depending on $g(\theta)$ (since $\tilde{p}(\lambda_1, \lambda_2)$ is a probability distribution, C is uniquely determined).

We can use (4.24) to study the spacings between the eigenvalues. We want to calculate the probability that $\lambda_1 - \lambda_2 \in [\lambda, \lambda + \Delta\lambda]$. We obtain, by substituting $\lambda_1 = \lambda_2 + \lambda + O(\Delta\lambda)$:

$$\begin{aligned} &\text{Prob}\,(\lambda_1 - \lambda_2 \in [\lambda, \lambda + \Delta\lambda]) \\ &= \int_{\lambda_2=-\infty}^{\infty} \int_{\lambda_1=\lambda_2+\lambda}^{\lambda_2+\lambda+\Delta\lambda} \tilde{p}(\lambda_1, \lambda_2) d\lambda_1 d\lambda_2 \\ &= \int_{\lambda_2=-\infty}^{\infty} \int_{\lambda_1=\lambda_2+\lambda}^{\lambda_2+\lambda+\Delta\lambda} C(\lambda_1 - \lambda_2)e^{-(\lambda_1^2+\lambda_2^2)} d\lambda_1 d\lambda_2 \\ &= C\int_{\lambda_2=-\infty}^{\infty} \int_{\lambda_1=\lambda_2+\lambda}^{\lambda_2+\lambda+\Delta\lambda} (\lambda + O(\Delta\lambda))e^{-((\lambda_2+\lambda+O(\Delta\lambda))^2+\lambda_2^2)} d\lambda_1 d\lambda_2. \end{aligned} \tag{4.25}$$

The λ_1-integral is over a region of size $\Delta\lambda$, so we obtain

$$\text{Prob}\,(\lambda_1 - \lambda_2 \in [\lambda, \lambda + \Delta\lambda])$$

$$
\begin{aligned}
&= C\int_{\lambda_2=-\infty}^{\infty} \lambda e^{-(\lambda^2+2\lambda_2^2+2\lambda\lambda_2)}\Delta\lambda d\lambda_2 + O((\Delta\lambda)^2) \\
&= C\lambda e^{-\lambda^2}\Delta\lambda \int_{\lambda_2=-\infty}^{\infty} e^{-2(\lambda_2^2+\lambda\lambda_2)}d\lambda_2 + O((\Delta\lambda)^2) \\
&= C\lambda e^{-\lambda^2}\Delta\lambda \int_{\lambda_2=-\infty}^{\infty} e^{-2(\lambda_2^2+\lambda\lambda_2+\frac{\lambda^2}{4})}e^{\frac{\lambda^2}{2}}d\lambda_2 + O((\Delta\lambda)^2) \\
&= C\lambda e^{-\frac{\lambda^2}{2}}\Delta\lambda \int_{\lambda_2=-\infty}^{\infty} e^{-2(\lambda_2+\frac{\lambda}{2})^2}d\lambda_2 + O((\Delta\lambda)^2) \\
&= C\lambda e^{-\frac{\lambda^2}{2}}\Delta\lambda \int_{\lambda_2=-\infty}^{\infty} e^{-2\lambda_2^2}d\lambda_2 + O((\Delta\lambda)^2) \\
&= C'\lambda e^{-\frac{\lambda^2}{2}}\Delta\lambda + O((\Delta\lambda)^2).
\end{aligned} \tag{4.26}
$$

C' is a new constant obtained by multiplying C by the λ_2-integration. We don't need to calculate this integration, since we can compute C' directly, because we must have a probability distribution! Letting $\Delta\lambda \to 0$ yields the following theorem.

Theorem 4.1.7 *If we denote the difference in eigenvalues with λ, we obtain the probability density $p_{GOE,2}(\lambda) = C'\lambda e^{-\frac{\lambda^2}{2}}$.*

4.1.4 Generalization to $N \times N$ GOE

In the 2×2 case, a real symmetric matrix has 3 degrees of freedom, written either as (a_{11}, a_{12}, a_{22}) or $(\lambda_1, \lambda_2, \theta)$. For a general $N \times N$ real symmetric matrix, it's easy to see that there are $\frac{N(N+1)}{2}$ independent parameters. For such a matrix A, we may write $A = Q^T \Lambda Q$, where Λ is the diagonal matrix with entries $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N$, and Q orthogonal. This follows from the Spectral Theorem.

The standard basis to work with are the upper diagonal entries of our matrix A. However, as we did in the 2×2 case, we want to switch to another coordinate system, since we are interested in the *eigenvalues* of A.

Theorem 4.1.8 *Another coordinate system for $N \times N$ real symmetric matrices is given by $(\lambda_1, \ldots \lambda_N, \theta_1, \ldots, \theta_n)$. We have $n = \frac{N(N-1)}{2}$.*

Proof. We write $A = Q^T \Lambda Q$. Λ is an $N \times N$ diagonal matrix with entries $\lambda_1, \ldots, \lambda_N$. The columns of our orthogonal matrix Q are mutually perpendicular, and each column has length 1. This means that we may choose $N-1$ entries of the first column. The last entry is forced since the length must be 1. The second column also must have length 1, and it must be perpendicular to the first column. Therefore we have $N-2$ free entries for the second column. We continue this way, and we notice that column i has $N-i$ free entries. Therefore, the number of free entries in an orthogonal matrix is

$$
\sum_{i=1}^{N}(N-i) = N^2 - \frac{N(N+1)}{2} = \frac{N(N-1)}{2}.
$$

The $N \times N$ case is much more difficult than the 2×2 case, since we find characteristic polynomials of degree N. We cannot proceed in the same way, since we're not able to solve general polynomials of degree 5 and higher. However, we can prove

Theorem 4.1.9 *As with 2×2 real symmetric matrices, for $N \times N$ real symmetric matrices the GOE conditions force the probabilities of the entries to be Gaussians:*

$$\begin{aligned} P_{ii}(a_{ii}) &= \sqrt{\frac{C}{4\pi}} e^{-\frac{Ca_{ii}^2}{4}} \\ P_{ij}(a_{ij}) &= \sqrt{\frac{C}{2\pi}} e^{-\frac{Ca_{ij}^2}{2}} \end{aligned} \tag{4.27}$$

The probability of a matrix A is the product of the probabilities of its N diagonal entries and its $\frac{N(N-1)}{2}$ upper diagonal entries, so we have

$$p(A) = 2^{-\frac{N}{2}} \left(\frac{C}{2\pi}\right)^{\frac{N(N+1)}{4}} e^{-\frac{C}{4}\mathrm{Trace}(A^2)}. \tag{4.28}$$

As in the 2×2 case, we want to integrate out the θ variable, since we want to find the joint distribution of the eigenvalues. Therefore, we write $A = Q^T \Lambda Q$ with Λ is diagonal with entries $\lambda_1 \geq \ldots \geq \lambda_N$ and Q is an orthogonal matrix with entries specified by the parameters $\theta_1, \ldots, \theta_n$.

Again, we change variables

$$(a_{11}, \ldots, a_{1N}, a_{22}, \ldots, a_{NN}) \longleftrightarrow (\lambda_1, \ldots, \lambda_N, \theta_1, \ldots, \theta_n).$$

We choose $a(1) = a_{11}, a(2) = a_{12}, \ldots, a(N) = a_{1N}, a(N+1) = a_{22}, \ldots,$ $a(\frac{N(N+1)}{2}) = a_{NN}$. As in the $2\times$ case, we define the Jacobian J, with entries

$$J_{ij} = \begin{cases} \frac{\partial a(j)}{\partial \lambda_i} & \text{if } \; j \leq N \\ \frac{\partial a(j)}{\partial \theta_{i-N}} & \text{if } \; N < j \leq \frac{N(N+1)}{2}. \end{cases}$$

Because we're changing variables, we have to compute $\det J$. We obtain this by following the same procedure as with the 2×2 case, where we've used (4.22). In the $N \times N$ case, we will find the same result:

$$|\det J| = g(\theta_1, \ldots, \theta_n) \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j). \tag{4.29}$$

Notice that we use the Vandermonde determinant:

$$\det \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ \lambda_1^{N-1} & \ldots & \lambda_N^{N-1} \end{pmatrix} = \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j).$$

A similar argument as in the 2×2 case now yields

Theorem 4.1.10

$$p(\lambda_1, \ldots, \lambda_N) = C' \prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j) e^{-(\lambda_1^2 + \ldots + \lambda_N^2)}. \tag{4.30}$$

By using the probability distribution for the eigenvalues above, we can analyze the distribution of the spacings between adjacent eigenvalues like we did in the 2×2 case. The answer is very close to the answer we've found from the 2×2 case in Theorem 4.1.7, but finding the answer goes beyond the scope of this research project. For details see [6].

4.1.5 The Gaussian Unitary Ensemble

In the previous section, we've found the joint probability distribution for the eigenvalues of $N \times N$ real symmetric matrices! Although it's too difficult to treat the subject here, we can use this distribution function to determine the spacing distribution between the eigenvalues, as we did in the 2×2 case.

We've investigated the *Gaussian Orthogonal Ensemble* to gain knowledge of random matrix theory. We could ask the same questions about the *Gaussian Unitary Ensemble*, where we investigate complex Hermitian matrices. For such a matrix A, we again want to have the following:

1. The entries of the matrix should be independent (note that we have more free (real) entries since now we should choose our entries from $\mathbb{C}$ instead of $\mathbb{R}$).
2. For any *unitary* transformation U, $P(UAU^*) = P(A)$.

U^* denotes the complex conjugate of U. Note that the restrictions above are the same restrictions as our restrictions for real symmetric matrices, but now we work in $\mathbb{C}$. Again we can compute the probability distributions for our entries of the matrix. However, we have to work in a different way, since we have more free parameters for our random matrices. Besides, our unitary matrix U has three degrees of freedom in the 2×2 case, whereas in the GOE-case, our orthogonal matrix Q has only one. Unfortunately, the required computations for the unitary case are too difficult. For the interested reader we refer to [6] and [7]. From these computations, we can conclude that the probability distributions must again be Gaussians.

It's also possible to find the eigenvalue distribution and the distribution of the spacings of the eigenvalues, as we've seen in the GOE-case. For the $N \times N$ GUE case, computations seem to indicate that

$$\lim_{N\to\infty} \#\{\delta_j = \tilde{\gamma}_{j+1} - \tilde{\gamma}_j, 1 \leq j \leq N : \delta_j \in (a,b)\} \to \int_a^b p(x)\,dx, \qquad (4.31)$$

where p is the distribution function of normalized spacings of eigenvalues of large random matrices from GUE! $\tilde{\gamma}_j$ denotes the j^{th} rescaled zero. Up to some technical restrictions, these results have been verified by Rudnick and Sarnak (see [9]). With (4.31), the link between random matrix theory and the zeros of the Riemann-zeta function has been made.

Appendix A

The Gamma function

In order to justify the analytic continuation of ζ to the complex plane, we need the gamma function Γ.
The gamma function is initially defined for $s > 0$ by

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt. \tag{A-1}$$

For each positive s, the integral is well defined, since near $t = 0$, the function t^{s-1} is integrable. For t large, the integral converges because of the exponential decay of the integrand.

The following proposition states that we can also use this definition in a complex half-plane, without any problems.

Theorem A.1 *We can extend the gamma function to an analytic function in the half-plane $\Re(s) > 0$. The function is still given there by (A-1).*

Proof. We need to show that the integral defines a holomorphic function in every strip

$$S_{\delta,M} = \{\delta < \Re(s) < M\},$$

with $0 < \delta < M < \infty$. If we denote $\Re(s)$ by σ, then $|e^{-t}t^{s-1}| = e^{-t}t^{\sigma-1}$. Therefore, we know that the integral

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt$$

converges for each $s \in S_{\delta,M}$. Note that we use

$$\int_0^\infty e^{-t} t^{s-1}\, dt = \lim_{\epsilon \to 0} \int_\epsilon^{1/\epsilon} e^{-t} t^{s-1}\, dt.$$

We define

$$F_\epsilon(s) = \int_\epsilon^{1/\epsilon} e^{-t} t^{s-1}\, dt.$$

The integrand is holomorphic in s for each t, and it's a continuous function. Therefore, $F_\epsilon(s)$ is holomorphic in the strip $S_{\delta,M}$ (for details of this argument, see [2], Theorem 2.5.4).

We now observe that

$$|\Gamma(s) - F_\epsilon(s)| \leq \int_0^\epsilon e^{-t} t^{\sigma-1}\, dt + \int_{1/\epsilon}^\infty e^{-t} t^{\sigma-1}\, dt.$$

Both integrals above converge uniformly to 0 as $\epsilon \to 0$. This means that F_ϵ converges uniformly to Γ on the strip $S_{\delta,M}$. This means that Γ is holomorphic in every strip $S_{\delta,M}$. $\square$

Our function Γ is now defined on the right half-plane $\Re(s) > 0$. However, we can go further by proving that there exists a meromorphic function definined on $\mathbb{C}$ that equals Γ on the half-plane $\Re(s) > 0$. To prove this assertion, we use the following lemma, which is a fundamental property of Γ.

Lemma A.2 *For* $\Re(s) > 0$,

$$\Gamma(s+1) = s\Gamma(s). \tag{A-2}$$

Therefore, $\Gamma(n+1) = n!$ *for* $n = 0, 1, 2, \ldots$.

Proof. We use integrating by parts in the finite integral F_ϵ from the previous proposition:

$$\int_\epsilon^{1/\epsilon} \frac{d}{dt}(e^{-t} t^s)\, dt = -\int_\epsilon^{1/\epsilon} e^{-t} t^s\, dt + s \int_\epsilon^{1/\epsilon} e^{-t} t^{s-1}\, dt.$$

If we let ϵ tend to 0, the left-hand side vanishes, since $e^{-t} t^s \to 0$ as t tends to 0 or ∞. Thus we have found the functional equation $\Gamma(s+1) = s\Gamma(s)$.

Furthermore,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = [e^{-t}]_0^\infty = 1.$$

We can apply (A-2) successively to find that $\Gamma(n+1) = n!$. $\square$

We now have the necessary ingredients to prove the following theorem.

Theorem A.3 *The function* Γ *has an analytic continuation to a meromorphic function on* $\mathbb{C}$, *whose singularities are simple poles at the negative integers* $s = 0, -1, \ldots$. *The residue of* Γ *at* $s = -n$ *is* $(-1)^n/n!$.

Proof. We extend Γ to each half-plane $\Re(s) > -m$, where $m \geq 1$ is an integer. For $\Re(s) > -1$, we define

$$F_1(s) = \frac{\Gamma(s+1)}{s}.$$

$\Gamma(s+1)$ is holomorphic in $\Re(s) > -1$, so F_1 is meromorphic in that half-plane. Since $\Gamma(1) = 1$, F_1 has a simple pole at $s = 0$, with residue 1. Furthermore, by the previous lemma we have

$$F_1(s) = \frac{\Gamma(s+1)}{s} = \Gamma(s)$$

if $\Re(s) > 0$. This means that F_1 extends Γ to a meromorphic function on the half-plane $\Re(s) > -1$.

We can repeat the argument above over and over, by defining a meromorphic function F_m for $\Re(s) > -m$, which equals Γ on $\Re(s) > 0$. To be more precise, for $\Re(s) > -m$, with m an integer ≥ 1, we define

$$F_m(s) = \frac{\Gamma(s+m)}{(s+m-1)(s+m-2)\ldots s}.$$

On the half plane $\Re(s) > -m$, the function F_m is meromorphic, and has simple poles at $s = 0, -1, \ldots, -m+1$ with residues

$$\begin{aligned} \mathrm{res}_{s=-n} F_m(s) &= \frac{\Gamma(-n+m)}{(m-1-n)!(-1)(-2)\ldots(-n)} \\ &= \frac{(m-n-1)!}{(m-1-n)!(-1)(-2)\ldots(-n)} \\ &= \frac{(-1)^n}{n!}. \end{aligned}$$

F_m coincides with Γ on $\Re(s) > 0$, so we have found the analytic continuation. $\square$

We now have all the information we need about the *poles* of Γ. We now will look at its zeros.

Lemma A.4 *For* $0 < a < 1$,

$$\int_0^\infty \frac{v^{a-1}}{1+v}\,dv = \frac{\pi}{\sin \pi a}$$

We won't give a full proof of this lemma here. The identity can be found by making the change of variables $v = e^x$ and by using contour integration. We use this lemma to prove the following.

Theorem A.5 ***(Reflection formula)***. *For all* $s \in \mathbb{C}$, *we have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \tag{A-3}$$

Proof. We first observe that $\Gamma(1-s)$ has simple poles at the positive integers, while $\Gamma(s)$ has simple poles at $s = 0, -1, \ldots$. This means that $\Gamma(s)\Gamma(1-s)$ is a meromorphic function on $\mathbb{C}$ with simple poles at all the integers. This property is shared by $\pi/\sin \pi s$. We will now prove the identity for $0 < s < 1$, since it then holds on all of $\mathbb{C}$ by analytic continuation. For $0 < s < 1$ we may write

$$\Gamma(1-s) = \int_0^\infty e^{-u} u^{-s}\,du = t\int_0^\infty e^{-vt}(vt)^{-s}\,dv,$$

where we made the change of variables $vt = u$ for $t > 0$. We now use the following steps:

$$\begin{aligned} \Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t} t^{s-1}\Gamma(1-s)\,dt \\ &= \int_0^\infty e^{-t} t^{s-1}\left(t\int_0^\infty e^{-vt}(vt)^{-s}\,dv\right)dt \end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty e^{-t[1+v]} v^{-s}\, dvdt \\
&= \int_0^\infty \frac{v^{-s}}{1+v}\, dv \\
&= \frac{\pi}{\sin \pi(1-s)} \\
&= \frac{\pi}{\sin \pi s},
\end{aligned}
$$

so we've proved the theorem. □

We now write (A-3) as

$$\frac{1}{\Gamma(s)} = \Gamma(1-s)\frac{\sin \pi s}{\pi}.$$

The simple poles of $\Gamma(1-s)$ are canceled by the simple zeros of $\sin \pi s$. Therefore $1/\Gamma$ is an entire function with simple zeros at $s = 0, -1, -2, \ldots$.

Appendix B

The functional equation for ζ and analytic continuation on the complex plane

During our investigation to prove the prime number theorem, we've used the analytic continuation of ζ to a meromorphic function in $\mathbb{C}$. Here we will justify this analytic continuation.

The proof of the analytic continuation is based on a relation between ζ, Γ and the *theta function*, which is defined for real $t > 0$ by

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}.$$

We recall that a function f defined on $\mathbb{R}$ is of *moderate decrease* if f is continuous and there exists a constant $A > 0$ such that

$$|f(x)| \leq \frac{A}{1+x^2} \quad \text{for all} \quad x \in \mathbb{R}.$$

The function $f(t) = e^{-\pi n^2 t}$ is of moderate decrease for every $n \in \mathbb{Z}$, so we may use the Poisson summation formula, which we recall in the following proposition which we won't prove.

Proposition B.1 ***(Poisson summation formula).*** *Suppose that f is a function with the properties*

1. *The function f is holomorphic in a horizontal strip*

$$S_a = \{z \in \mathbb{C} : |\Im(z)| < a\}$$

for some a.

2. *There exists a constant $A > 0$ such that*

$$|f(x+iy)| \leq \frac{A}{1+x^2}$$

for all $x \in \mathbb{R}$ and $|y| < a$ for some a.

Then

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where $\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ix\xi}\,dx$, *the Fourier transform of* f.

We use the Poisson summation formula to find a functional equation for θ. At first, we recall that the function $e^{-\pi x^2}$ equals its own Fourier transform:

$$\int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi ix\xi}\,dx = e^{-\pi\xi^2}.$$

We fix $t > 0$ and $a \in \mathbb{R}$, and apply the change of variables $x \mapsto t^{1/2}(x+a)$ in the integral above, to show that the Fourier transform of the function

$$f(x) = e^{-\pi t(x+a)^2}$$

is

$$\hat{f}(\xi) = t^{-1/2}e^{-\pi\xi^2/t}e^{2\pi ia\xi}.$$

We may apply the Poisson equation formula to the pair f and $\hat{f}$ above, and we obtain the relation

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-1/2}e^{-\pi n^2/t}e^{2\pi ina}.$$

If we choose $a = 0$, we find the *modular inversion formula*

$$\theta(t) = t^{-1/2}\theta(1/t) \quad \text{for} \quad t > 0, \tag{B-1}$$

a functional equation for θ which we will need to find the analytic continuation for ζ.

Since

$$\lim_{t\to 0^+} t^{-1/2}\theta(1/t) \leq Ct^{-1/2},$$

we obtain the estimation

$$\theta(t) \leq Ct^{-1/2} \quad \text{as} \quad t \to 0 \tag{B-2}$$

for some $C > 0$. Furthermore, since

$$\sum_{n=1}^{\infty} e^{-\pi n^2 t} \leq \sum_{n=1}^{\infty} e^{-\pi nt} = \frac{e^{-\pi t}}{1-e^{-\pi t}}$$

for all $t > 0$, we find

$$|\theta(t) - 1| \leq Ce^{-\pi t} \tag{B-3}$$

for some $C > 0$ and all $t \geq 1$.

We are now ready to prove the following theorem, which we will use to justify the analytic continuation.

Theorem B.2 *If $\Re(s) > 1$, then*

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \frac{1}{2}\int_0^\infty u^{(s/2)-1}[\theta(u) - 1]\, du.$$

Proof. We use the observation that

$$\int_0^\infty e^{-\pi n^2 u} u^{(s/2)-1}\, du = \pi^{-s/2}\Gamma(s/2)n^{-s}, \quad \text{if} \quad n \geq 1. \tag{B-4}$$

We've found this result by making the change of variables $u = t/\pi n^2$ in the integral above. We recall that

$$\frac{\theta(u) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 u}.$$

We substitute this result in the integral and the estimates (B-2) and (B-3) justify an interchange of summation and integration:

$$\begin{aligned}
\frac{1}{2}\int_0^\infty u^{(s/2)-1}[\theta(u) - 1]\, du &= \sum_{n=1}^{\infty}\int_0^\infty u^{(s/2)-1}e^{-\pi n^2 u}\, du \\
&= \pi^{-s/2}\Gamma(s/2)\sum_{n=1}^{\infty} n^{-s} \\
&= \pi^{-s/2}\Gamma(s/2)\zeta(s). \quad \Box
\end{aligned}$$

For convenient reasons, we define the *xi function* for $\Re(s) > 1$ by

$$\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{B-5}$$

Theorem B.3 *The function ξ is holomorphic for $\Re(s) > 1$ and has an analytic continuation to all of $\mathbb{C}$ as a meromorphic function with simple poles at $s = 0$ and $s = 1$. Moreover,*

$$\xi(s) = \xi(1 - s) \quad \textit{for all} \quad s \in \mathbb{C}.$$

Proof. We use the functional equation for θ:

$$\sum_{n=-\infty}^{\infty} e^{-\pi n^2 u} = u^{-1/2}\sum_{n=-\infty}^{\infty} e^{-\pi n^2/u}, \quad u > 0.$$

Let $\chi(u) = [\theta(u) - 1]/2$. Using the functional equation gives us

$$\chi(u) = u^{-1/2}\chi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}.$$

We now use theorem B.2 for $\Re(s) > 1$, which yields the following result:

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty u^{(s/2)-1}\chi(u)\, du$$

$$
\begin{aligned}
&= \int_0^1 u^{(s/2)-1}\chi(u)\,du + \int_1^\infty u^{(s/2)-1}\chi(u)\,du \\
&= \int_0^1 u^{(s/2)-1}\left[u^{-1/2}\chi(1/u) + \frac{1}{2u^{1/2}} - \frac{1}{2}\right] du + \\
&\quad + \int_1^\infty u^{(s/2)-1}\chi(u)\,du \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (u^{(-s/2)-1/2} + u^{(s/2)-1})\chi(u)\,du
\end{aligned}
$$

for $\Re(s) > 1$. Thus

$$
\xi(s) = \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty (u^{(-s/2)-1/2} + u^{(s/2)-1})\chi(u)\,du.
$$

The function χ has exponential decay at infinity by (B-3), so the integral above defines an entire function in s. Therefore ξ has an analytic continuation to $\mathbb{C}$ with simple poles at $s = 0$ and $s = 1$. Besides, if we replace s by $1-s$ in the equation above, we find the same equation. We conclude that $\xi(s) = \xi(1-s)$ as was to be shown. $\square$

Finally, we can justify the analytic continuation for the zeta function:

Theorem B.4 *The zeta function has a meromorphic continuation into the entire complex plane, whose only singularity is a simple pole at $s = 1$.*

Proof. We look at (B-5) which provides the meromorphic continuation of ζ:

$$
\zeta(s) = \pi^{s/2}\frac{\xi(s)}{\Gamma(s/2)}.
$$

$1/\Gamma(s/2)$ is entire with simple zeros at $0, -2, -4, \ldots$, so the simple pole of $\xi(s)$ at the origin is cancelled by the corresponding zero of $1/\Gamma(s/2)$. Therefore, the only singularity of ζ is a simple pole at $s = 1$. $\square$

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