Stability estimates for the Vlasov-Poisson system in p-kinetic Wasserstein distances

Iacobelli, Mikaela; Junné, J.

DOI
10.1112/blms.13053

Publication date
2024

Document Version
Final published version

Published in
Bulletin of the London Mathematical Society

Citation (APA)
Iacobelli, M., & Junné, J. (2024). Stability estimates for the Vlasov-Poisson system in p-kinetic Wasserstein distances. *Bulletin of the London Mathematical Society*, *56*(7), 2250-2267. https://doi.org/10.1112/blms.13053

Received: 19 December 2022 | Revised: 14 March 2024 | Accepted: 15 March 2024
DOI: 10.1112/blms.13053

RESEARCH ARTICLE

Bulletin of the London Mathematical Society

Stability estimates for the Vlasov–Poisson system in p-kinetic Wasserstein distances

Mikaela Iacobelli[1] | **Jonathan Junné**[2]

[1]Department of Mathematics, ETH Zürich, Zurich, Switzerland
[2]Delft Institute of Applied Mathematics, TU Delft, Delft, The Netherlands

Correspondence
Jonathan Junné, Delft Institute of Applied Mathematics, TU Delft, Mekelweg 4, 2628 CD Delft, The Netherlands.
Email: j.junne@tudelft.nl

Funding information
Dutch Research Council (NWO), Grant/Award Number: OCENW.M20.251

Abstract
We extend Loeper's L^2-estimate (Theorem 2.9 in *J. Math. Pures Appl.* (9) **86** (2006), no. 1, 68–79) relating the force fields to the densities for the Vlasov–Poisson system to L^p, with $1 < p < +\infty$, based on the Helmholtz–Weyl decomposition. This allows us to generalize both the classical Loeper's 2-Wasserstein stability estimate (Theorem 1.2 in *J. Math. Pures Appl.* (9) **86** (2006), no. 1, 68–79) and the recent stability estimate by the first author relying on the newly introduced kinetic Wasserstein distance (Theorem 3.1 in *Arch Rational Mech. Anal.* **244** (2022), no. 1, 27–50) to kinetic Wasserstein distances of order $1 < p < +\infty$.

MSC 2020
35Q83 (primary), 82C40, 82D10, 35B35 (secondary)

1 | INTRODUCTION

1.1 | General overview

Monge–Kantorovich distances, also known as Wasserstein distances, are used extensively in kinetic theory, in particularly in the context of stability, convergence to equilibrium and mean-field limits. A first celebrated result for the 1-Monge–Kantorovich distance is due to Dobrushin [2, Theorem 1], who proved the well-posedness for Vlasov equations with $C^{1,1}$ potentials. An explanation of Dobrushin's stability estimate and its consequences on the mean-field limit for the Vlasov equation can be found in [5, chapter 1] and [11, chapter 3], and we refer to [7, section 3] for a survey on well-posedness for the Vlasov–Poisson system.

Regarding the 2-Wasserstein distance, Loeper proved [13, Theorem 1.2] a uniqueness criterion for solutions with bounded density based on a 2-Wasserstein distance stability estimate using both a link between the $\dot{H}^{-1}$-seminorm and the 2-Wasserstein distance, and the fact that the Coulomb kernel is generated by a potential solving the Poisson equation. In addition to the Vlasov–Poisson system, this criterion gives a new proof of uniqueness *à la* Yudovich for $2D$ Euler. Beyond bounded density, Loeper's uniqueness criterion has been extended for some suitable Orlicz spaces using the 1-Monge–Kantorovich distance by Miot [16, Theorem 1.1] and Miot and Holding [9, Theorem 1.1].

On the Torus, Loeper's criterion was improved by Han-Kwan and Iacobelli [8, Theorem 3.1] for the Vlasov–Poisson system, and more recently for the Vlasov–Poisson system with massless electrons by Griffin-Pickering, Iacobelli [6, Theorem 4.1].

The aim of this work is twofold. The first goal is to generalize Loeper's 2-Wasserstein distance stability estimate to p-Wasserstein distances for $1 < p < +\infty$. The second goal is to extend the recent stability estimate [10, Theorem 3.1] by the first author relying on the newly introduced kinetic Wasserstein distance [10, Theorem 3.1] to kinetic Wasserstein distances of order $1 < p < +\infty$.

1.2 | Definitions and main results

We first recall the classical Wasserstein distance (see [21, chapter 6]) on the product space $\mathcal{X} \times \mathbb{R}^d$, with $\mathcal{X}$ denoting in the sequel either the d-dimensional torus $\mathbb{T}^d$ or the Euclidean space $\mathbb{R}^d$:

Definition 1.1. Let μ, ν be two probability measures on $\mathcal{X} \times \mathbb{R}^d$. The *Wasserstein distance* of order p, with $p \geqslant 1$, between μ and ν is defined as

$$W_p(\mu, \nu) := \left(\inf_{\pi \in \Pi(\mu,\nu)} \int_{(\mathcal{X}\times\mathbb{R}^d)^2} |x-y|^p + |v-w|^p \, d\pi(x,v,y,w) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ is the set of *couplings*; that is, the set of probability measures with *marginals* μ and ν. A coupling is said to be *optimal* if it minimizes the Wasserstein distance.

We consider two solutions f_1, f_2 of the Vlasov–Poisson system on $\mathcal{X}$, with either gravitational or electrostatic interaction encoded by $\sigma = \pm 1$, namely,

$$\partial_t f + v \cdot \partial_x f - \nabla U \cdot \nabla_v f = 0, \quad \sigma \Delta U := \rho_f - 1, \quad \rho_f := \int_{\mathbb{R}^d} f \, dv \tag{1.1}$$

on the torus, and

$$\partial_t f + v \cdot \partial_x f - \nabla U \cdot \nabla_v f = 0, \quad \sigma \Delta U := \rho_f, \quad \rho_f := \int_{\mathbb{R}^d} f \, dv \tag{1.2}$$

on the whole space, with initial profiles $f_1(0), f_2(0)$, and respective flows $Z_1 := (X_1, V_1)$ and $Z_2 := (X_2, V_2)$ satisfying the system of characteristics

$$\dot{X} = V, \quad \dot{V} = -\nabla U, \quad X(0,x,v) = x, \quad V(0,x,v) = v.$$

The flows yield solutions $f_1(t) = Z_1(t,\cdot,\cdot)_\# f_1(0)$ and $f_2(t) = Z_2(t,\cdot,\cdot)_\# f_2(0)$ as pushforward of the initial data.

(1) A new L^p-estimate for the difference of force fields. Loeper estimates the L^2-norm [13, Theorem 2.9] of the difference of force fields with the Wasserstein distance between the densities. We extend the L^2-estimate to L^p for $1 < p < +\infty$ using the Helmholtz–Weyl decomposition of $L^p(\mathcal{X}) = G_p(\mathcal{X}) \oplus H_p(\mathcal{X})$ into its hydrodynamic space that we recall (see [3, chapter III]):

Definition 1.2. The *hydrodynamic spaces* are the closed subspaces of $L^p(\mathcal{X})$ defined as

$$G_p(\mathcal{X}) := \left\{ u \in L^p(\mathcal{X}); \quad u = \nabla w \text{ for some } w \in W^{1,p}_{\mathrm{loc}}(\mathcal{X}) \right\}$$

and

$$H_p(\mathcal{X}) := \overline{\left\{ u \in C^\infty_c(\mathcal{X}); \quad \operatorname{div} u = 0 \text{ on } \mathcal{X} \right\}}.$$

Remark 1.3. Note that this decomposition breaks down for $p = 1$ or $p = +\infty$ and does not hold for general domains in L^p. It is equivalent to the solvability of a Neumann problem (see [3, Lemma III.1.2]), while an orthogonal decomposition in L^2 is always possible, whatever the domain is.

In our setting; that is either on the torus or on the Euclidean space, the Helmholtz–Weyl decomposition holds [3, Theorems III.1.1 and III.1.2];

Theorem 1.4 (Helmholtz–Weyl decomposition). *The Helmholtz–Weyl decomposition holds for $L^p(\mathcal{X})$, for any $1 < p < +\infty$; that is,*

$$L^p(\mathcal{X}) = G_p(\mathcal{X}) \oplus H_p(\mathcal{X}).$$

Moreover, when $p = 2$, this decomposition is orthogonal.

The validity of the Helmholtz–Weyl decomposition implies the existence of an Helmholtz–Weyl bounded linear projection operator (see [3, Remark III.1.1])

$$P_p : L^p(\mathcal{X}) \to H_p(\mathcal{X})$$

with range $H_p(\mathcal{X})$ and with $G_p(\mathcal{X})$ as null space. More precisely, there is a constant $C_{P_p} > 0$ that only depends on p and $\mathcal{X}$ such that for all $u \in L^p(\mathcal{X})$, it holds

$$\|P_p(u)\|_{L^p(\mathcal{X})} \leqslant C_{P_p} \|u\|_{L^p(\mathcal{X})}. \tag{1.3}$$

Using optimal transport techniques, Loeper manages to link the strong dual homogeneous Sobolev norm and Wasserstein distances between densities, and we recall those notions:

Definition 1.5. Let $1 < p < +\infty$. The *homogeneous Sobolev space* is the space

$$\dot{W}^{1,p}(\mathcal{X}) := \left\{ [g]; \quad g \in W^{1,p}_{\mathrm{loc}}(\mathcal{X}), \quad \nabla g \in L^p(\mathcal{X}) \right\},$$

where $[\cdot] := \{\cdot + c;\ c \in \mathbb{R}\}$ denotes the equivalence class of functions up to a constant, together with the norm

$$\|[g]\|_{\dot{W}^{1,p}(\mathcal{X})} := \|\nabla g\|_{L^p(\mathcal{X})}.$$

This is a Banach space for which the equivalence classes of test functions

$$\dot{D}(\mathcal{X}) := \{[\phi];\quad \phi \in C_c^\infty(\mathcal{X})\}$$

are dense in it (see [17, Theorem 2.1]).

Definition 1.6. We define the *dual homogeneous Sobolev space* $\dot{W}^{-1,p}(\mathcal{X})$ to be the topological dual of $\dot{W}^{1,p'}(\mathcal{X})$ equipped with the *strong dual homogeneous Sobolev norm*. For a function h with $\int h = 0$, by density,

$$\|h\|_{\dot{W}^{-1,p}(\mathcal{X})} := \sup\left\{\int_{\mathcal{X}} h\,[g]\,dx;\quad g \in \dot{W}^{1,p'}(\mathcal{X}),\quad \|[g]\|_{\dot{W}^{1,p'}(\mathcal{X})} \leqslant 1\right\}$$
$$= \sup\left\{\int_{\mathcal{X}} h\,[\phi]\,dx;\quad \phi \in \dot{D}(\mathcal{X}),\quad \|[\phi]\|_{\dot{W}^{1,p'}(\mathcal{X})\leqslant 1}\right\}.$$

First, we extend this connection for densities to L^p. Using the machinery of Helmholtz–Weyl decomposition, we generalize [13, Lemma 2.10] into the following:

Lemma 1.7. *Let $\rho_1, \rho_2 \in L^\infty(\mathcal{X})$ be two probability measures, and let U_i satisfy $\sigma\Delta U_i = \rho_i$ for $\mathcal{X} = \mathbb{R}^d$, or $\sigma\Delta U_i = \rho_i - 1$ for $\mathcal{X} = \mathbb{T}^d$, with $i = 1, 2, \sigma = \pm 1$, in the distributional sense. Let $1 < p < +\infty$. Then there is a constant $C_{\mathrm{HW}} > 0$ that only depends on p and $\mathcal{X}$ such that*

$$\|\nabla U_1 - \nabla U_2\|_{L^p(\mathcal{X})} \leqslant C_{\mathrm{HW}}\|\rho_1 - \rho_2\|_{\dot{W}^{-1,p}(\mathcal{X})}. \tag{1.4}$$

Second, we adapt Loeper's argument of the L^2-estimate [13, Theorem 2.9](see also [20, Proposition 1.1] in bounded convex domains) relating negative homogeneous Sobolev norms to Wasserstein distances with this new link on force fields to get the new L^p-estimate allowing us to generalize stability estimates;

Proposition 1.8. *Let $\rho_1, \rho_2 \in L^\infty(\mathcal{X})$ be two probability measures, and let U_i satisfy $\sigma\Delta U_i = \rho_i$ for $\mathcal{X} = \mathbb{R}^d$, or $\sigma\Delta U_i = \rho_i - 1$ for $\mathcal{X} = \mathbb{T}^d$, with $i = 1, 2, \sigma = \pm 1$, in the distributional sense. Let $1 < p < +\infty$. Then there is a constant $C_{\mathrm{HW}} > 0$ that only depends on p and $\mathcal{X}$ such that*

$$\|\nabla U_1 - \nabla U_2\|_{L^p(\mathcal{X})} \leqslant C_{\mathrm{HW}} \max\left\{\|\rho_1\|_{L^\infty(\mathcal{X})}, \|\rho_2\|_{L^\infty(\mathcal{X})}\right\}^{1/p'} W_p(\rho_1, \rho_2). \tag{1.5}$$

(2) Loeper's stability estimate in W_p. Loeper noted [13, Lemma 3.6] that both the Wasserstein distance of order two of the solutions and of the associated densities are bounded by a flow quantity Q given by

$$Q(t) := \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|X_1(t,x,v) - X_2(t,x,v)\right|^2 + \left|V_1(t,x,v) - V_2(t,x,v)\right|^2 df^0(x,v),$$

and the bounds read as

$$W_2^2(f_1(t), f_2(t)) \leqslant Q(t), \quad W_2^2(\rho_{f_1}(t), \rho_{f_2}(t)) \leqslant Q(t). \tag{1.6}$$

Loeper uses the quantity $Q(t)$ together with the L^2-estimate on the force fields to prove the stability estimate [13, Theorem 1.2] leading to the uniqueness of weak solutions. By modifying the quantity $Q(t)$ to

$$\begin{aligned} Q_p(t) &:= \int_{(\mathcal{X}\times\mathbb{R}^d)^2} |X_1(t,x,v) - X_2(t,y,w)|^p + |V_1(t,x,v) - V_2(t,y,w)|^p \, d\pi_0(x,v,y,w) \\ &= \int_{(\mathcal{X}\times\mathbb{R}^d)^2} |x-y|^p + |v-w|^p \, d\pi_t(x,v,y,w), \end{aligned}$$

where $\pi_t \in \Pi(f_1(t), f_2(t))$ and π_0 is an optimal W_p coupling (see [6, section 4] for a construction of π_t), we are able to generalise Loeper's stability estimate [13, Theorem 1.2], and [8, Theorem 3.1] both on the torus $\mathbb{T}^d$ and on the whole space $\mathbb{R}^d$, to any Wasserstein distance of order p, with $1 < p < +\infty$;

Theorem 1.9. *Let f_1, f_2 be two weak solutions to the Vlasov–Poisson system on $\mathcal{X}$ with respective densities*

$$\rho_{f_1} := \int_{\mathbb{R}^d} f_1 \, dv, \quad \rho_{f_2} := \int_{\mathbb{R}^d} f_2 \, dv.$$

Let $1 < p < +\infty$, and set

$$A(t) := \|\rho_{f_2}(t)\|_{L^\infty(\mathcal{X})} + \|\rho_{f_1}(t)\|_{L^\infty(\mathcal{X})}^{1/p} \max\left\{ \|\rho_{f_1}(t)\|_{L^\infty(\mathcal{X})}, \|\rho_{f_2}(t)\|_{L^\infty(\mathcal{X})} \right\}^{1/p'}, \tag{1.7}$$

which is assumed to be in $L^1([0,T))$ for some $T > 0$. Then there is a constant $C_{\mathrm{L}} > 0$ that only depends on p and d such that if $W_p^p(f_1(0), f_2(0))$ is sufficiently small so that $W_p^p(f_1(0), f_2(0)) \leqslant (4\sqrt{d}/e)^p$ and

$$\left| \log\left(\frac{W_p^p(f_1(0), f_2(0))}{\left(4\sqrt{d}\right)^p} \right) \right| \geqslant p \exp\left(-C_{\mathrm{L}} \int_0^T A(s)\, ds \right) \tag{1.8}$$

then

$$W_p^p(f_1(t), f_2(t)) \leqslant (4\sqrt{d})^p \exp\left\{ \log\left(\frac{W_p^p(f_1(0), f_2(0))}{\left(4\sqrt{d}\right)^p} \right) \exp\left(C_{\mathrm{L}} \int_0^t A(s)\, ds \right) \right\}. \tag{1.9}$$

(3) An improved W_p stability estimate via kinetic Wasserstein distance. Due to the anisotropy between position and momentum variables, we use an adapted Wasserstein distance designed for kinetic problems taking this into account as introduced in [10, section 4]:

Definition 1.10. Let μ, ν be two probability measures on $\mathcal{X} \times \mathbb{R}^d$. The *kinetic Wasserstein distance* of order p, with $p \geqslant 1$, between μ and ν is defined as

$$W_{\lambda,p}(\mu,\nu) := \left(\inf_{\pi \in \Pi(\mu,\nu)} D_p(\pi,\lambda) \right)^{1/p},$$

where $D_p(\pi,\lambda)$ is the unique number s such that

$$s - \lambda(s) \int_{(\mathcal{X}\times\mathbb{R}^d)^2} |x-y|^p \, d\pi(x,y,v,w) = \int_{(\mathcal{X}\times\mathbb{R}^d)^2} |v-w|^p \, d\pi(x,y,v,w),$$

with $\lambda : \mathbb{R}^+ \to \mathbb{R}^+$ a decreasing function.

We consider the quantity $D_p(t)$ for π_t and $\lambda(t) = \left|\log D_p(t)\right|^{p/2}$ (see [10, Lemma 3.7] for the proof of existence) given by

$$\begin{aligned} D_p(t) &:= \frac{1}{p} \int_{(\mathcal{X}\times\mathbb{R}^d)^2} \lambda(t) \left|X_1(t,x,v) - X_2(t,y,w)\right|^p + \left|V_1(t,x,v) - V_2(t,y,w)\right|^p d\pi_0(x,v,y,w) \\ &= \frac{1}{p} \int_{(\mathcal{X}\times\mathbb{R}^d)^2} \lambda(t)|x-y|^p + |v-w|^p \, d\pi_t(x,v,y,w). \end{aligned}$$

This quantity also compares to the usual Wasserstein distance W_p as does $Q_p(t)$, and this allows us to generalize the recent Iacobelli's stability estimate [10, Theorem 3.1] to the following:

Theorem 1.11. *Let f_1, f_2 be two weak solutions to the Vlasov–Poisson system on $\mathcal{X}$ (1.2) with respective densities*

$$\rho_{f_1} := \int_{\mathbb{R}^d} f_1 \, dv, \quad \rho_{f_2} := \int_{\mathbb{R}^d} f_2 \, dv.$$

Let $1 < p < +\infty$, and set

$$A(t) := \|\rho_{f_2}(t)\|_{L^\infty(\mathcal{X})} + \|\rho_{f_1}(t)\|_{L^\infty(\mathcal{X})}^{1/p} \max\left\{ \|\rho_{f_1}(t)\|_{L^\infty(\mathcal{X})}, \|\rho_{f_2}(t)\|_{L^\infty(\mathcal{X})} \right\}^{1/p'}, \tag{1.10}$$

which is assumed to be in $L^1([0,T))$ for some $T > 0$. Then there is a universal constant $c_0 > 0$ and a constant $C_{\mathrm{KW}} > 0$ that depends only on p and d such that if $W_p^p(f_1(0), f_2(0))$ is sufficiently small so that $W_p^p(f_1(0), f_2(0)) \leqslant pc_0$ and

$$\sqrt{\left|\log\left\{ W_p^p(f_1(0),f_2(0)) \left|\log\left(\frac{1}{p} W_p^p(f_1(0),f_2(0))\right)\right| \right\}\right|} \geqslant C_{\mathrm{KW}} \int_0^T A(s)\, ds + 1, \tag{1.11}$$

then

$$\begin{aligned} &W_p^p(f_1(t), f_2(t)) \\ &\leqslant p \exp\left\{ -\left(\sqrt{\left|\log\left\{ W_p^p(f_1(0),f_2(0)) \left|\log\left(\frac{1}{p} W_p^p(f_1(0),f_2(0))\right)\right| \right\}\right|} - C_{\mathrm{KW}} \int_0^t A(s)\, ds \right)^2 \right\}. \end{aligned} \tag{1.12}$$

The improvement of this stability estimate (1.12) of Theorem 1.11 via p-kinetic Wasserstein distance compared to Loeper's stability estimate in W_p (1.9) of Theorem 1.9 lies in the order of magnitude of the time interval in which the two solutions are close to each other in Wasserstein distance. Indeed, if $W_p^p(f_1(0), f_2(0)) = \delta \ll 1$, then Loeper's stability estimate yields $W_p^p(f_1(t), f_2(t)) \lesssim 1$ for $t \in [0, \log(|\log \delta|)]$ while the kinetic stability estimates yields a better control of the time interval; $W_p^p(f_1(t), f_2(t)) \lesssim 1$ for $t \in [0, \sqrt{|\log \delta|}]$.

2 | A NEW L^P-ESTIMATE VIA THE HELMHOLTZ–WEYL DECOMPOSITION FOR $1 < P < +\infty$

2.1 | Proof of the L^p-estimate

Proof of Lemma 1.7. Let $[\phi] \in \dot{D}(\mathcal{X})$ be a quotient test function. Note that $\int \rho_1 - \rho_2 = 0$ as both ρ_1 and ρ_2 are probability measures, an integration by parts yields

$$\int_{\mathcal{X}} [\phi](\rho_1 - \rho_2)\, dx = \int_{\mathcal{X}} \phi(\rho_1 - \rho_2)\, dx = -\sigma \int_{\mathcal{X}} \nabla\phi \cdot (\nabla U_1 - \nabla U_2)\, dx.$$

First, we consider the torus case $\mathcal{X} = \mathbb{T}^d$: We use the Helmholtz–Weyl decomposition given by Theorem 1.4 to write any $\mathbb{R}^d$-valued test function $\Phi \in C_c^\infty(\mathbb{T}^d)$ as $\Phi = \nabla\phi + g$, where $\nabla\phi \in G_{p'}(\mathbb{T}^d)$ and $g \in H_{p'}(\mathbb{T}^d)$ with $1/p + 1/p' = 1$. By definition, there is a divergence-free sequence of test functions $(g_k)_{k\in\mathbb{N}}$ whose $L^{p'}$-limit is g. By continuity of the force fields (see [8, Lemma 3.2]), $\nabla U_1 - \nabla U_2 \in L^\infty(\mathbb{T}^d)$, and in particular $\nabla U_1 - \nabla U_2 \in L^p(\mathbb{T}^d)$. An integration by parts yields

$$\begin{aligned}
\|\nabla U_1 - \nabla U_2\|_{L^p(\mathbb{T}^d)} &= \sup_{\|\Phi\|_{L^{p'}(\mathbb{T}^d)}\leqslant 1} \left\{ \int_{\mathbb{T}^d} \Phi \cdot (\nabla U_1 - \nabla U_2)\, dx \right\} \\
&= \sup_{\|\Phi\|_{L^{p'}(\mathbb{T}^d)}\leqslant 1} \left\{ \int_{\mathbb{T}^d} \nabla\phi \cdot (\nabla U_1 - \nabla U_2)\, dx - \lim_{k\to\infty} \int_{\mathbb{T}^d} \operatorname{div} g_k (U_1 - U_2)\, dx \right\} \\
&= \sup_{\phi;\, \|\Phi\|_{L^{p'}(\mathbb{T}^d)}\leqslant 1} \left\{ \int_{\mathbb{T}^d} \nabla\phi \cdot (\nabla U_1 - \nabla U_2)\, dx \right\}.
\end{aligned}$$

As the projection operator $P_{p'} : \Phi \mapsto g$ is bounded from $L^{p'}(\mathbb{T}^d)$ to $H_{p'}(\mathbb{T}^d)$, we have that

$$\|\nabla\phi\|_{L^{p'}(\mathbb{T}^d)} = \|g - \Phi\|_{L^{p'}(\mathbb{T}^d)} \leqslant \left(1 + C_{P_{p'}}\right) \|\Phi\|_{L^{p'}(\mathbb{T}^d)},$$

where $C_{P_{p'}}$ is the constant from (1.3), and we set $C_{\mathrm{HW}} := 1 + C_{P_{p'}}$. Consider the larger set

$$\left\{ \|\nabla\phi\|_{L^{p'}(\mathbb{T}^d)}/C_{\mathrm{HW}} \leqslant 1 \right\} \supset \left\{ \phi;\ \|\nabla\phi\|_{L^{p'}(\mathbb{T}^d)}/C_{\mathrm{HW}} \leqslant \|\Phi\|_{L^{p'}(\mathbb{T}^d)} \leqslant 1 \right\}$$

that does not depend on Φ anymore. By replacing the supremum over this set, we obtain

$$\|\nabla U_1 - \nabla U_2\|_{L^p(\mathbb{T}^d)} \leqslant C_{\mathrm{HW}} \sup_{\|\nabla\phi\|_{L^{p'}(\mathbb{T}^d)}\leqslant 1} \left\{ \int_{\mathbb{T}^d} \nabla\phi \cdot (\nabla U_1 - \nabla U_2)\,dx \right\}$$

$$= C_{\mathrm{HW}} \sup_{\|\nabla[\phi]\|_{L^{p'}(\mathbb{T}^d)}\leqslant 1} \left\{ \int_{\mathbb{T}^d} [\phi](\rho_1 - \rho_2)\,dx \right\}.$$

We conclude by density of quotient test functions $\dot{D}(\mathbb{T}^d)$ in $\dot{W}^{1,p}(\mathbb{T}^d)$ and by the definition of the strong dual homogeneous Sobolev norm.

Second, we consider the whole space case $\mathcal{X} = \mathbb{R}^d$: Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a test function and set $1/p + 1/p' = 1$. We have that $U_1 - U_2 = \sigma G_d * (\rho_1 - \rho_2)$ almost everywhere, where G_d is the fundamental solution of the Laplace equation. Then, by symmetry of the convolution,

$$\|\partial_{x_j} U_1 - \partial_{x_j} U_2\|_{L^p(\mathbb{R}^d)} = \sup_{\|\varphi\|_{L^{p'}(\mathbb{R}^d)}\leqslant 1} \left\{ \int_{\mathbb{R}^d} \varphi\, \partial_{x_j} G_d * (\rho_1 - \rho_2)\,dx \right\}$$

$$= \sup_{\|\varphi\|_{L^{p'}(\mathbb{R}^d)}\leqslant 1} \left\{ \int_{\mathbb{R}^d} (\rho_1 - \rho_2)\partial_{x_j} G_d * \varphi\,dx \right\}.$$

We denote $\phi = \partial_{x_j} G_d * \varphi$, and Calderon–Zygmund's inequality [3, Theorem II.11.4] yields

$$\|\nabla\phi\|_{L^{p'}(\mathbb{R}^d)} \leqslant C_{\mathrm{HW}} \|\varphi\|_{L^{p'}(\mathbb{R}^d)}$$

for some constant $C_{\mathrm{HW}} > 0$ which only depends on d and p, so that the supremum can be replaced by the larger set

$$\left\{ \phi \in \dot{W}^{1,p'}(\mathbb{R}^d);\ \|\nabla\phi\|_{L^{p'}(\mathbb{R}^d)}/C_{\mathrm{HW}} \leqslant 1 \right\} \supset \left\{ \varphi;\ \|\partial_{x_j} G_d * \varphi\|_{L^{p'}(\mathbb{R}^d)}/C_{\mathrm{HW}} \leqslant \|\varphi\|_{L^{p'}(\mathbb{R}^d)} \leqslant 1 \right\}$$

independent of φ. We obtain

$$\|\partial_{x_j} U_1 - \partial_{x_j} U_2\|_{L^{p'}(\mathbb{R}^d)} \leqslant C_{\mathrm{HW}} \sup_{\|\nabla\phi\|_{L^{p'}(\mathbb{R}^d)}\leqslant 1} \left\{ \int_{\mathbb{R}^d} \phi(\rho_1 - \rho_2)\,dx \right\}$$

$$= C_{\mathrm{HW}} \sup_{\|\nabla[\phi]\|_{L^{p'}(\mathbb{R}^d)}\leqslant 1} \left\{ \int_{\mathbb{R}^d} [\phi](\rho_1 - \rho_2)\,dx \right\},$$

and we conclude by definition of the strong dual homogeneous Sobolev norm. □

Before proving our new L^p-estimate, we first state the existence of an optimal transport map adapted to our context;

Theorem 2.1 (Gangbo–McCann [4, Theorem 1.2]). *Let ρ_1, ρ_2 be two probability measures on $\mathcal{X}$ that are absolutely continuous with respect to the Lebesgue measure. Then*

$$W_p(\rho_1, \rho_2) = \left(\inf_{T_\# \rho_1 = \rho_2} \left\{ \int_{\mathcal{X}} |x - T(x)|^p\, d\rho_1(x) \right\} \right)^{1/p},$$

where the infimum runs over all measurable mappings $T : \mathcal{X} \to \mathcal{X}$ that push forward ρ_1 onto ρ_2. Moreover, the infimum is reached by a $\rho_1(dx)$-almost surely unique mapping T, and there is a $|\cdot|^p$-convex function ψ such that $T = \mathrm{Id}_{\mathcal{X}} - \left(\nabla |\cdot|^p\right)^{-1} \circ \nabla\psi$, where we denote (∇h^*) by $(\nabla h)^{-1}$ for a function h with h^* its Legendre transform.

Proof of Proposition 1.8. Let us denote by

$$\rho_\theta := [(\theta - 1)T + (2 - \theta)\,\mathrm{Id}_{\mathcal{X}}]_\# \rho_1$$

the interpolant measure between ρ_1 and ρ_2, where T is the optimal transport map of Theorem 2.1. Let $\phi \in C_c^\infty(\mathcal{X})$ be a test function. By the properties of pushforward of measures, it follows immediately that

$$\int_{\mathcal{X}} \phi(x)\, d\rho_\theta(x) = \int_{\mathcal{X}} \phi((\theta - 1)T(x) + (2 - \theta)x)\, d\rho_1(x).$$

Lebesgue's dominated convergence theorem yields

$$\frac{d}{d\theta} \int_{\mathcal{X}} \phi(x)\, d\rho_\theta(x) = \int_{\mathcal{X}} \nabla\phi((\theta - 1)T(x) + (2 - \theta)x) \cdot (T(x) - x)\, d\rho_1(x).$$

Now, by using Hölder inequality with respect to the measure ρ_1, we get

$$\begin{aligned} &\frac{d}{d\theta} \int_{\mathcal{X}} \phi(x)\rho_\theta(x)\, dx \\ &\leqslant \left(\int_{\mathcal{X}} |\nabla\phi((\theta - 1)T(x) + (2 - \theta)x)|^{p'}\, d\rho_1(x) \right)^{1/p'} \left(\int_{\mathcal{X}} |x - T(x)|^p\, d\rho_1(x) \right)^{1/p} \\ &= \left(\int_{\mathcal{X}} |\nabla\phi(x)|^{p'}\, d\rho_\theta(x) \right)^{1/p'} \left(\int_{\mathcal{X}} |x - T(x)|^p\, d\rho_1(x) \right)^{1/p}. \end{aligned}$$

The second term in the product is exactly $W_p(\rho_1, \rho_2)$ by Theorem 2.1. For the first one, thanks to [18, Remark 8], the L^∞-norm of the interpolant is controlled by the one of the two measures;

$$\|\rho_\theta\|_{L^\infty(\mathcal{X})} \leqslant \max\left\{ \|\rho_1\|_{L^\infty(\mathcal{X})}, \|\rho_2\|_{L^\infty(\mathcal{X})} \right\}.$$

Therefore,

$$\frac{d}{d\theta} \int_{\mathcal{X}} \phi(x)\rho_\theta(x)\, dx \leqslant \max\left\{ \|\rho_1\|_{L^\infty(\mathcal{X})}, \|\rho_2\|_{L^\infty(\mathcal{X})} \right\}^{1/p'} \left(\int_{\mathcal{X}} |\nabla\phi(x)|^{p'}\, dx \right)^{1/p'} W_p(\rho_1, \rho_2).$$

Combining the above estimate with the fact that $\int \rho_2 - \rho_1 = 0$ and Fubini's theorem yields

$$\begin{aligned} \int_{\mathcal{X}} [\phi](x)(\rho_2(x) - \rho_1(x))\, dx &= \int_{\mathcal{X}} \phi(x)(\rho_2(x) - \rho_1(x))\, dx \\ &= \int_1^2 \left(\frac{d}{d\theta} \int_{\mathcal{X}} \phi(x)\rho_\theta(x)\, dx \right) d\theta \\ &\leqslant \max\left\{ \|\rho_1\|_{L^\infty(\mathcal{X})}, \|\rho_2\|_{L^\infty(\mathcal{X})} \right\}^{1/p'} \left(\int_{\mathcal{X}} |\nabla[\phi](x)|^{p'}\, dx \right)^{1/p'} \\ &\quad \times W_p(\rho_1, \rho_2). \end{aligned}$$

By restricting to quotient test functions $[\phi]$ such that $\|\nabla[\phi]\|_{L^{p'}(\mathcal{X})} \leqslant 1$, we get the strong dual homogeneous norm so that

$$\|\rho_1 - \rho_2\|_{\dot{W}^{-1,p}(\mathcal{X})} \leqslant \max\left\{\|\rho_1\|_{L^\infty(\mathcal{X})}, \|\rho_2\|_{L^\infty(\mathcal{X})}\right\}^{1/p'} W_p(\rho_1, \rho_2),$$

and we conclude by Lemma 1.7. □

Remark 2.2. Loeper uses extensively that the optimal transport map T is convex to rely on the gas internal energy theory developed by McCann (see [15, section 2]) to estimate the L^∞-norm of the interpolant. Here, we only have $|\cdot|^p$-convexity instead, while still the L^∞-estimate on the interpolant is valid as showed, for instance, by Santambrogio [18, Remark 8]. Loeper gives also an alternative proof [12, Proposition 3.1] using the Benamou–Brenier formula [1, Proposition 1.1]. The interpolant measure satisfies the continuity equation

$$\partial_\theta \rho_\theta + \operatorname{div}_x(\rho_\theta v_\theta) = 0$$

for a vector field v_θ related to the Wasserstein distance through

$$\int_{\mathcal{X}} |v_\theta(x)|^2 \, d\rho_\theta(x) = W_2^2(\rho_1, \rho_2).$$

Differentiating both sides of Poisson's equation gives

$$\Delta \partial_\theta U_\theta = -\partial_\theta \rho_\theta = \operatorname{div}_x(\rho_\theta v_\theta),$$

and integrating by parts against $\partial_\theta U_\theta$ itself as test function yields

$$\int_{\mathcal{X}} |\partial_\theta \nabla U_\theta|^2 \, dx = \int_{\mathcal{X}} \rho_\theta v_\theta \cdot \partial_\theta \nabla U_\theta \, dx,$$

so that

$$\|\partial_\theta \nabla U_\theta\|_{L^2(\mathcal{X})} \leqslant \|\rho_\theta\|_{L^\infty(\mathcal{X})}^{1/2} W_2(\rho_1, \rho_2) \leqslant \max\left\{\|\rho_1\|_{L^\infty(\mathcal{X})}, \|\rho_2\|_{L^\infty(\mathcal{X})}\right\}^{1/2} W_2(\rho_1, \rho_2),$$

and the conclusion follows after integrating over $\theta \in [1, 2]$.

Even though there is a W_p version of Benamou–Brenier formula [19, Theorem 5.28], there is no analog test function that allows to mimic this proof for L^p.

3 | STABILITY ESTIMATES REVISITED FOR WASSERSTEIN-LIKE DISTANCES

3.1 | Loeper's estimate revisited

The proof of Loeper's stability estimate in W_p on the torus $\mathbb{T}^d$ is similar to [8, Theorem 3.1] using the new L^p-estimate (1.5) from Proposition 1.8. It relies on the modified quantity (see [6, section 4])

$$Q_p(t) := \int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} \left|X_1(t,x,v) - X_2(t,y,w)\right|^p + \left|V_1(t,x,v) - V_2(t,y,w)\right|^p \, d\pi_0(x,v,y,w)$$
$$= \int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} |x-y|^p + |v-w|^p \, d\pi_t(x,v,y,w),$$

where $\pi_t \in \Pi(f_1(t), f_2(t))$ and π_0 is an optimal coupling that satisfies the marginal property;

$$\int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} \phi(x,v,y,w)\, d\pi_t(x,v,y,w)$$
$$= \int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} \phi(Z_1(t,x,v), Z_2(t,y,w))\, d\pi_0(x,v,y,w), \quad \forall \phi \in C_b((\mathbb{T}^d\times\mathbb{R}^d)^2). \quad (3.1)$$

The last ingredients are the following estimates analog to (1.6) relying on the definition of Wasserstein distance (see [13, Lemma 3.6]):

$$W_p^p(f_1(t), f_2(t)) \leqslant Q_p(t), \quad W_p^p(\rho_{f_1}(t), \rho_{f_2}(t)) \leqslant Q_p(t). \quad (3.2)$$

3.2 | Kinetic Wasserstein distance revisited

We prove the recent Iacobelli's stability estimate in W_p both on the torus and on the whole space adapting the proof of [10, Theorem 3.1].

The proof relies on the modified quantity from the kinetic Wasserstein distance (see [10, section 4])

$$D_p(t) := \frac{1}{p}\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \lambda(t)\left|X_1(t,x,v) - X_2(t,y,w)\right|^p + \left|V_1(t,x,v) - V_2(t,y,w)\right|^p \, d\pi_0(x,v,y,w)$$
$$= \frac{1}{p}\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \lambda(t)|x-y|^p + |v-w|^p \, d\pi_t(x,v,y,w),$$

where $\lambda(t) = \left|\log D_p(t)\right|^{p/2}$, whose specific choice of comes from optimization considerations that will become apparent in the proof. One is able to bound

$$\dot{D}_p(t) \lesssim \frac{\dot{\lambda}(t)}{\lambda} D_p(t) + D_p(t)\left(\lambda^{1/p}(t) + \lambda^{-1/p}(t)\log\left(\frac{pD_p(t)}{\lambda(t)}\right)\right),$$

which can be rewritten as

$$\dot{D}_p(t) \lesssim D_p(t)\left[\lambda^{1/p}(t) + \lambda^{-1/p}(t)\log\left(D_p(t)\right)\right]$$

recalling that λ is a decreasing function and assuming that $\left|\log(pD_p(t)/\lambda(t))\right| \lesssim \left|\log D_p(t)\right|$ in some regime. The term inside the square brackets is now optimized considering $D_p(t)$ as a function of $\lambda(t)$.

We recall the Log-Lipschitz estimate on the force fields [8, Lemma 3.2], see also [6, Lemma 3.3] and [14, Lemma 8.1]:

Lemma 3.1. *Let U_i satisfy $\sigma\Delta U_i = \rho_{f_i} - 1,\ i = 1,2, \sigma = \pm 1$, on $\mathbb{T}^d$ in the distributional sense. Then there is a constant $C > 0$ such that for all $x, y \in \mathbb{T}^d,\ i = 1,2$, it holds*

$$\left|\nabla U_i(x) - \nabla U_i(y)\right| \leqslant C|x-y| \log\left(\frac{4\sqrt{d}}{|x-y|}\right) \|\rho_{f_i} - 1\|_{L^\infty(\mathbb{T}^d)}. \tag{3.3}$$

Lemma 3.2. *Let U_i satisfy $\sigma\Delta U_i = \rho_{f_i},\ i = 1,2, \sigma = \pm 1$, on $\mathbb{R}^d$ in the distributional sense. Then there is a constant $C_d > 0$ such that*

$$\|\nabla U_i\|_{L^\infty(\mathbb{R}^d)} \leqslant C_d\Big(\|\rho_{f_i}\|_{L^1(\mathbb{R}^d)} + \|\rho_{f_i}\|_{L^\infty(\mathbb{R}^d)}\Big),$$

and for all $x, y \in \mathbb{R}^d$ with $|x-y| < 1/e,\ i = 1,2$, it holds

$$\left|\nabla U_i(x) - \nabla U_i(y)\right| \leqslant C_d|x-y| \log\left(\frac{1}{|x-y|}\right)\Big(\|\rho_{f_i}\|_{L^1(\mathbb{R}^d)} + \|\rho_{f_i}\|_{L^\infty(\mathbb{R}^d)}\Big). \tag{3.4}$$

In particular, for all $x, y \in \mathbb{R}^d$,

$$\left|\nabla U_i(x) - \nabla U_i(y)\right| \leqslant C_d|x-y|(1 + \log^-(|x-y|))\Big(\|\rho_{f_i}\|_{L^1(\mathbb{R}^d)} + \|\rho_{f_i}\|_{L^\infty(\mathbb{R}^d)}\Big), \tag{3.5}$$

with $\log^-(s) := \max\{-\log(s), 0\}$.

Proof of Theorem 1.11.

$$\begin{aligned}\dot{D}_p(t) = &\frac{1}{p}\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \dot{\lambda}(t)\left|X_1 - X_2\right|^p d\pi_0 \\ &+ \int_{(\mathcal{X}\times\mathbb{R}^d)^2} \lambda(t)\left|X_1 - X_2\right|^{p-2}(X_1 - X_2)\cdot(V_1 - V_2)\, d\pi_0 \\ &+ \int_{(\mathcal{X}\times\mathbb{R}^d)^2} \left|V_1 - V_2\right|^{p-2}(V_1 - V_2)\cdot(\nabla_x U_2(t,X_2) - \nabla_x U_1(t,X_1))\, d\pi_0.\end{aligned}$$

The last two terms are estimated using Hölder's inequality with respect to the measure π_0, and we have

$$\begin{aligned}\dot{D}_p(t) \leqslant &\frac{1}{p}\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \dot{\lambda}(t)\left|X_1 - X_2\right|^p d\pi_0 \\ &+ \lambda^{1/p}(t)\big(pD_p(t)\big) + \big(pD_p(t)\big)^{1/p'}\left(\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \left|\nabla_x U_2(t,X_2) - \nabla_x U_1(t,X_1)\right|^p d\pi_0\right)^{1/p}. \end{aligned} \tag{3.6}$$

Recall the separation of the difference of force fields;

$$\left|\nabla_x U_2(t,X_2) - \nabla_x U_1(t,X_1)\right| \leqslant \left|\nabla_x U_2(t,X_2) - \nabla_x U_2(t,X_1)\right| + \left|\nabla_x U_1(t,X_1) - \nabla_x U_2(t,X_2)\right|,$$

whence

$$\left(\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \left|\nabla_x U_2(t,X_2) - \nabla_x U_1(t,X_1)\right|^p \, d\pi_0\right)^{1/p} \leqslant T_1(t) + T_2(t), \tag{3.7}$$

where

$$T_1(t) := \left(\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \left|\nabla_x U_2(t,X_2) - \nabla_x U_2(t,X_1)\right|^p \, d\pi_0\right)^{1/p},$$
$$T_2(t) := \left(\int_{(\mathcal{X}\times\mathbb{R}^d)^2} \left|\nabla_x U_1(t,X_1) - \nabla_x U_2(t,X_1)\right|^p \, d\pi_0\right)^{1/p}. \tag{3.8}$$

First, consider the torus case $\mathcal{X} = \mathbb{T}^d$: We estimate T_1 (3.8) using the nondecreasing concave function on $[0, +\infty)$ given by

$$\Phi_p(s) := \begin{cases} s\log^p\left(\frac{(4\sqrt{d})^p}{s}\right) & \text{if } s \leqslant (4\sqrt{d}/e)^p, \\ \left(\frac{4p\sqrt{d}}{e}\right)^p & \text{if } s > (4\sqrt{d}/e)^p, \end{cases}$$

together with the Log-Lipschitz estimate (3.3) from Lemma 3.1 and (1.10) to get[†]

$$T_1(t) \leqslant CA(t)\left(\int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} \Phi_p\left(|X_1 - X_2|^p\right) d\pi_0\right)^{1/p}$$

provided that $|X_1 - X_2|^p \leqslant (4\sqrt{d}/e)^p$, but this is always the case because the distance between points in the torus cannot exceed $\sqrt{d}$. Thus, by Jensen's inequality, we have

$$T_1(t) \leqslant CA(t)\left[\Phi_p\left(\int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} |X_1 - X_2|^p \, d\pi_0\right)\right]^{1/p} \leqslant CA(t)\left[\Phi_p\left(\frac{pD_p(t)}{\lambda(t)}\right)\right]^{1/p}.$$

Now, the considered regime becomes

$$\frac{pD_p(t)}{\lambda(t)} \leqslant \left(4\sqrt{d}/e\right)^p, \tag{3.9}$$

so that

$$T_1(t) \leqslant CA(t)\left(\frac{pD_p(t)}{\lambda(t)}\right)^{1/p}\left[\left|\log\left(\frac{pD_p(t)}{\lambda(t)}\right)\right| + p\log\left(4\sqrt{d}\right)\right]. \tag{3.10}$$

[†] Note that, as $\rho_{f_i}(t) \geqslant 0$ and $\|\rho_{f_i}(t)\|_{L^\infty(\mathbb{T}^d)} \geqslant 1$, then $\|\rho_{f_i}(t) - 1\|_{L^\infty(\mathbb{T}^d)} \leqslant \|\rho_{f_i}(t)\|_{L^\infty(\mathbb{T}^d)} \leqslant A(t)$ for i = 1, 2.

We replace $\lambda(t) = \left|\log D_p(t)\right|^{p/2}$, and consider yet another regime, now dictated by

$$D_p(t) \leqslant \frac{1}{e}, \tag{3.11}$$

so that $\left|\log D_p(t)\right| \geqslant 1$. Note that this regime is compatible with the regime (3.9) needed for the function Φ_p in the sense that if $pD_p(t) \leqslant (4\sqrt{d}/e)^p$ holds, then $pD_p(t)/\left|\log D_p(t)\right|^{p/2} \leqslant pD_p(t) \leqslant (4\sqrt{d}/e)^p$, and as $p/e \leqslant (4\sqrt{d}/e)^p$, we can only consider the regime (3.11). We estimate the logarithms in (3.10) in this new regime (3.11) using an elementary inequality valid within this regime;

$$\left|\log\left(\frac{pD_p(t)}{\left|\log D_p(t)\right|^{p/2}}\right)\right| \leqslant 2\left(1 + \log p + \frac{p}{2}\right)\left|\log D_p(t)\right|, \tag{3.12}$$

and set $C_p := 2(1 + \log p + p/2)$. Hence, (3.10) becomes

$$T_1(t) \leqslant CA(t)\left(\frac{pD_p(t)}{\lambda(t)}\right)^{1/p}\left[C_p\left|\log D_p(t)\right| + p\log\left(4\sqrt{d}\right)\right]. \tag{3.13}$$

We move to the estimation of T_2 (3.8). The L^p-estimate (1.5) from Proposition 1.8 yields

$$T_2(t) \leqslant C_{\mathrm{HW}} A(t) W_p(\rho_{f_1}(t), \rho_{f_2}(t)) \tag{3.14}$$

recalling (1.10). As $(X_1(t), X_2(t))_{\#}\pi_0$ has marginals $\rho_{f_1}(t)$ and $\rho_{f_2}(t)$, we can estimate the Wasserstein distance between the densities by D_p (see [13, Lemma 3.6]). More precisely, by the definition of the Wasserstein distance, we have

$$\begin{aligned}
W_p^p(\rho_{f_1}(t), \rho_{f_2}(t)) &= \inf_{\gamma \in \Pi(\rho_{f_1}(t), \rho_{f_2}(t))} \int_{\mathbb{T}^d \times \mathbb{T}^d} |x - y|^p \, d\gamma(x, y) \\
&\leqslant \int_{\mathbb{T}^d \times \mathbb{T}^d} |x - y|^p \, d\left[(X_1(t), X_2(t))_{\#}\pi_0\right](x, y) \\
&= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \left|X_1(t, x, v) - X_2(t, y, w)\right|^p \, d\pi_0(x, v, y, w) \leqslant \frac{pD_p(t)}{\lambda(t)}.
\end{aligned}$$

We replace $\lambda(t) = \left|\log D_p(t)\right|^{p/2}$ and the above estimate in (3.14) to get

$$T_2(t) \leqslant C_{\mathrm{HW}} A(t)\left(\frac{pD_p(t)}{\left|\log D_p(t)\right|^{p/2}}\right)^{1/p}. \tag{3.15}$$

Putting altogether estimates (3.6, 3.7, 3.13, 3.15) gives in the considered regime (3.11) that

$$\dot{D}_p(t) \leqslant \frac{1}{p}\int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} \frac{-\dot{D}_p(t)}{D_p(t)}\left|\log D_p(t)\right|^{p/2-1}\left|X_1-X_2\right|^p\,d\pi_0$$
$$+\left(pD_p(t)\right)\left[\sqrt{\left|\log D_p(t)\right|}+CA(t)\left(C_p\sqrt{\left|\log D_p(t)\right|}(t)+\frac{p\log\left(4\sqrt{d}\right)}{\sqrt{\left|\log D_p(t)\right|}}\right)+\frac{C_{\mathrm{HW}}A(t)}{\sqrt{\left|\log D_p(t)\right|}}\right].$$

If $\dot{D}_p \leqslant 0$, then we do not do anything. Otherwise, then the first term in the above estimate is negative, and therefore

$$\dot{D}_p(t) \leqslant \left(pD_p(t)\right)\left[\sqrt{\left|\log D_p(t)\right|}+CA(t)\left(C_p\sqrt{\left|\log D_p(t)\right|}(t)+\frac{p\log\left(4\sqrt{d}\right)}{\sqrt{\left|\log D_p(t)\right|}}\right)+\frac{C_{\mathrm{HW}}A(t)}{\sqrt{\left|\log D_p(t)\right|}}\right].$$

As the right-hand side is nonnegative, independently of the sign of $\dot{D}_p$, this bound is always valid in the regime, and using that $\left|\log D_p(t)\right| \geqslant 1$, together with $A(t)\geqslant 1$, we get

$$\dot{D}_p(t) \leqslant \tilde{C}_{\mathrm{KW}}A(t)D_p(t)\sqrt{\left|\log D_p(t)\right|},$$

where $\tilde{C}_{\mathrm{KW}} := p\times\left[1+C\times\left(C_p+p\log\left(4\sqrt{d}\right)\right)+C_{\mathrm{HW}}\right]$. Therefore,

$$D_p(t) \leqslant \exp\left\{-\left(\sqrt{\left|\log D_p(0)\right|}-C_{\mathrm{KW}}\int_0^t A(s)\,ds\right)^2\right\}, \tag{3.16}$$

where $C_{\mathrm{KW}} := \tilde{C}_{\mathrm{KW}}/2$ depends only on p and d. This implies in particular that (3.11) holds if

$$\sqrt{\left|\log D_p(0)\right|} \geqslant C_{\mathrm{KW}}\int_0^T A(s)\,ds+1. \tag{3.17}$$

It remains to compare D_p to the Wasserstein distance between the solutions in the regime (3.11). By the definition of the Wasserstein distance, as $(Z_1(t),Z_2(t))_{\#}\pi_0$ has marginals $f_1(t)$ and $f_2(t)$, we have that

$$\begin{aligned}
W_p^p(f_1(t),f_2(t)) &= \inf_{\gamma\in\Pi(f_1(t),f_2(t))}\int_{(\mathbb{T}^d\times\mathbb{R}^d)^2}|x-y|^p\,d\gamma(x,v,y,w)\\
&\leqslant \int_{(\mathbb{T}^d\times\mathbb{R}^d)^2}|x-y|^p+|v-w|^p\,d\left[(Z_1(t),Z_2(t))_{\#}\pi_0\right](x,v,y,w)\\
&= \int_{(\mathbb{T}^d\times\mathbb{R}^d)^2}\left|X_1-X_2\right|^p+\left|V_1-V_2\right|^p\,d\pi_0 \leqslant pD_p(t).
\end{aligned}$$

For the initial Wasserstein distance, as π_0 is optimal, we get

$$D_p(0) \leqslant \frac{1}{p}\left|\log D_p(0)\right| \int_{(\mathbb{T}^d\times\mathbb{R}^d)^2} |x-y|^p + |v-w|^p \, d\pi_0 = \frac{1}{p}\left|\log D_p(0)\right| W_p^p(f_1(0), f_2(0)),$$

which we rewrite as

$$\frac{D_p(0)}{\left|\log D_p(0)\right|} \leqslant \frac{1}{p} W_p^p(f_1(0), f_2(0)).$$

Note that, near the origin, the inverse of the function $s \mapsto s/\left|\log s\right|$ behaves like $\tau \mapsto \tau\left|\log \tau\right|$. In particular, there is a universal constant $c_0 > 0$ such that

$$\frac{s}{|\log s|} \leqslant \tau \quad \text{for some } 0 \leqslant \tau \leqslant c_0 \quad \Rightarrow \quad s \leqslant p\tau|\log \tau|.$$

Hence, for sufficiently small initial distance such that $W_p^p(f_1(0), f_2(0)) \leqslant pc_0$, then

$$D_p(0) \leqslant W_p^p(f_1(0), f_2(0))\left|\log\left(\frac{1}{p} W_p^p(f_1(0), f_2(0))\right)\right|.$$

Combining these bounds with (3.16), and recalling (3.17), this implies

$$\begin{aligned}
&W_p^p(f_1(t), f_2(t)) \\
&\leqslant p \exp\left\{-\left(\sqrt{\left|\log\left\{W_p^p(f_1(0), f_2(0))\left|\log\left(\frac{1}{p} W_p^p(f_1(0), f_2(0))\right)\right|\right\}\right|} - C_{\mathrm{KW}} \int_0^t A(s)\, ds\right)^2\right\}
\end{aligned}$$

provided $W_p^p(f_1(0), f_2(0)) \leqslant pc_0$ and

$$\sqrt{\left|\log\left\{W_p^p(f_1(0), f_2(0))\left|\log\left(\frac{1}{p} W_p^p(f_1(0), f_2(0))\right)\right|\right\}\right|} \geqslant C_{\mathrm{KW}} \int_0^T A(s)\, ds + 1.$$

We conclude the proof of the torus case by [10, Lemma 3.7 and Remark 3.8].

Second, we consider the whole space case $\mathcal{X} = \mathbb{R}^d$: The only difference lies in the the separation of force fields. We have to estimate T_1 and T_2 defined in (3.8). We split T_1 in two integrals;

$$\begin{aligned}
T_1(t)^p &= \left(\int_{|X_1-X_2|<1/e} d\pi_0 + \int_{|X_1-X_2|\geqslant 1/e} d\pi_0\right)\left[\left|\nabla_x U_2(t, X_2) - \nabla_x U_2(t, X_1)\right|^p\right] \\
&:= I_1(t) + I_2(t).
\end{aligned}$$

On one hand, for I_1, using the Log-Lipschitz estimate (3.4) from Lemma 3.2 and (1.10), we get

$$\begin{aligned} I_1(t) &\leqslant C_d^p A^p(t) \int_{|X_1-X_2|<1/e} |X_1 - X_2|^p \log^p\left(\frac{1}{|X_1 - X_2|^p}\right) d\pi_0 \\ &\leqslant C_d^p A^p(t) \int_{|X_1-X_2|<1/e} \Phi_p(|X_1 - X_2|^p)\, d\pi_0. \end{aligned}$$

Applying Jensen's inequality, we have

$$I_1(t) \leqslant C_d^p A^p(t)\Phi_p\left(\int_{|X_1-X_2|<1/e} |X_1 - X_2|^p\, d\pi_0\right) \leqslant C_d^p A^p(t)\Phi_p\left(\frac{pD_p(t)}{\lambda(t)}\right).$$

On the other hand, for I_2, the estimate (3.5) from Lemma 3.2 yields

$$I_2(t) \leqslant C_d^p A^p(t) \int_{|X_1-X_2|\geqslant 1/e} |X_1 - X_2|^p\, d\pi_0 \leqslant C_d^p A^p(t)\left(\frac{pD_p(t)}{\lambda(t)}\right).$$

Again, we impose the regime $D_p(t) \leqslant 1/e$, with $\lambda(t) = \left|\log D_p(t)\right|^{p/2}$, so that

$$T_1(t) \leqslant (I_1(t) + I_2(t))^{1/p} \leqslant 2^{1/p} C_d A(t)\left(\frac{pD_p(t)}{\lambda(t)}\right)^{1/p} \left|\log\left(\frac{pD_p(t)}{\lambda(t)}\right) + p\log\left(4\sqrt{d}\right)\right|$$

becomes

$$T_1(t) \leqslant 2^{1/p} C_d A(t)\left(\frac{pD_p(t)}{\lambda(t)}\right)^{1/p} \left[C_p\left|\log D_p(t)\right| + p\log\left(4\sqrt{d}\right)\right]$$

after using the elementary inequality (3.12) valid within the considered regime. The estimation of T_2 (3.8) is again a direct consequence of the L^p-estimate (1.5) from Proposition 1.8;

$$T_2(t) \leqslant C_{\mathrm{HW}} A(t) W_p(\rho_{f_1}(t), \rho_{f_2}(t)).$$

From now on, the proof is the same as in the torus case $\mathcal{X} = \mathbb{T}^d$ with

$$C_{\mathrm{KW}} := p \times \left[1 + 2^{1/p} C_d \times \left(C_p + p\log\left(4\sqrt{d}\right)\right) + C_{\mathrm{HW}}\right]. \qquad \square$$

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their useful and detailed comments, which improved the presentation of the paper. The second author also acknowledges partial financial support from the Dutch Research Council (NWO): project number OCENW.M20.251.

JOURNAL INFORMATION

The *Bulletin of the London Mathematical Society* is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and

mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

ORCID

Jonathan Junné https://orcid.org/0000-0001-7901-7718

REFERENCES

1. J. Benamou and Y. Brenier, *A computational fluid mechanics solution to the Monge–Kantorovich mass transfer problem*, Numer. Math. **84** (2000), no. 3, 375–393, DOI 10.1007/s002110050002.
2. R. L. Dobrushin, *Vlasov equations*, Funktsional Anal.i Prilozhen. **13** (1979), 48–58.
3. G. P. Galdi, *An introduction to the mathematical theory of the Navier–Stokes equations*, Springer Monographs in Mathematics, Springer New York, New York, NY, 2011, DOI 10.1007/978-0-387-09620-9.
4. W. Gangbo and R. J. McCann, *The geometry of optimal transportation*, Acta Math. **177** (1996), no. 2, 113–161, DOI 10.1007/BF02392620. https://projecteuclid.org/euclid.acta/1485890981.
5. F. Golse, *On the dynamics of large particle systems in the mean field limit*, A. Muntean, J. Rademacher, and A. Zagaris, (eds.), Macroscopic and large scale phenomena: coarse graining, Mean field limits and ergodicity, Lect. Notes Appl. Math. Mech., vol. 3, Springer, Berlin, 2016, pp. 1–144.
6. M. Griffin-Pickering and M. Iacobelli, *Global well-posedness for the Vlasov–Poisson system with massless electrons in the 3-dimensional torus*, Commun. Partial Differ. Equ. **46** (2021), no. 10, 1892–1939, DOI 10.1080/03605302.2021.1913750
7. M. Griffin-Pickering and M. Iacobelli, *Recent developments on the well-posedness theory for Vlasov-type equations*, C. Bernardin, F. Golse, P. Gonçalves, V. Ricci, A. J. Soares (eds.), From Particle Systems to Partial Differential Equations, Springer International Publishing, Cham, 2021, pp. 301–319.
8. D. Han-Kwan and M. Iacobelli, *Quasineutral limit for Vlasov–Poisson via Wasserstein stability estimates in higher dimension*, J. Differential Equations **263** (2017), no. 1, 1–25.
9. T. Holding and E. Miot, *Uniqueness and stability for the Vlasov–Poisson system with spatial density in Orlicz spaces*, Mathematical analysis in fluid mechanics—selected recent results, Contemp. Math., vol. 710, Amer. Math. Soc., Providence, RI, 2018, pp. 145–162, DOI 10.1090/conm/710/14368.
10. M. Iacobelli, *A new perspective on Wasserstein distances for kinetic problems*, Arch Rational Mech Anal. **244** (2022), no. 1, 27–50, DOI 10.1007/s00205-021-01705-9.
11. P.-E. Jabin, *A review of the mean field limits for Vlasov equations*, Kinet. Relat. Models **7** (2014), no. 4, 661–711, DOI 10.3934/krm.2014.7.661.
12. G. Loeper, Uniqueness of the solution to the Vlasov–Poisson system with bounded density, arXiv:math/0504140, 2005.
13. G. Loeper, *Uniqueness of the solution to the Vlasov–Poisson system with bounded density*, J. Math. Pures Appl. (9) **86** (2006), no. 1, 68–79.
14. A. J. Majda and A. L. Bertozzi, *Vorticity and incompressible flow*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
15. R. J. McCann, *A convexity principle for interacting gases*, Adv. Math. **128** (1997), no. 1, 153–179, DOI 10.1006/aima.1997.1634.
16. E. Miot, *A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system*, Comm. Math. Phys. **346** (2016), no. 2, 469–482, DOI 10.1007/s00220-016-2707-7.
17. C. Ortner and E. Suli, *A note on linear elliptic systems on* $\mathbb{R}^d$, arXiv:1202.3970[math], 2012.
18. F. Santambrogio, *Absolute continuity and summability of transport densities: simpler proofs and new estimates*, Calc. Var. **36** (2009), no. 3, 343–354, DOI 10.1007/s00526-009-0231-8.
19. F. Santambrogio, *Optimal transport for applied mathematicians*, Progress in nonlinear differential equations and their applications, vol. 87, Springer International Publishing, Berlin, 2015, DOI 10.1007/978-3-319-20828-2.
20. F. Santambrogio, *Sharp Wasserstein estimates for integral sampling and Lorentz summability of transport densities*, J. Funct. Anal. **284** (2023), no. 4, 109783, DOI 10.1016/j.jfa.2022.109783.
21. C. Villani, *Optimal transport: Old and new*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338 Springer, Berlin, 2009, pp. xxii+973, DOI 10.1007/978-3-540-71050-9.