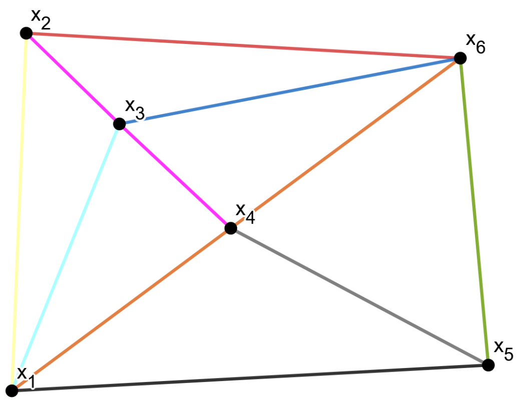
# Finding Structures within Large Point Sets

*Report for my research project as part of the Honours Programme*





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# Contents

# 1 Introduction

This report investigates two open problems in discrete geometry regarding how large subsets of sets of points need to be in order for certain structures to emerge.
First of all there is the Erdős-Szekeres convex polygon problem, also known as the Happy Ending problem. This problem is about how large point sets in general position have to be such that there are $n$ points in convex position. In the planar setting, this problem is nearly but not quite settled by Suk [16], who showed that the number of points required to have $n$ points in convex position is equal to $2^{n+o(n)}$. In 2022 Pohoata and Zakharov [13] had a breakthrough for this problem in higher dimensional setting, showing that point sets in dimension $d \geq 3$ differ substantially from point sets in dimension $d = 2$. In this setting only $2^{o(n)}$ points are required to ensure $n$ in convex position. This is very interesting, as it shows that the plane is a particular setting for point sets that is more restrictive than any other dimension.
The above result raises the question whether such a difference depending on the considered dimension can also be found in other problems concerning subsets of point sets. One problem that particularly sparked our interest is the Big-Line-Big-Clique Conjecture. This is a related problem that − instead of looking for large subsets in convex position − asks whether large enough point sets always have $l$ collinear points or $k$ pairwise visible points. As far as we know, this conjecture has not been studied yet in other dimensions than the plane. In this report we present a generalisation of the conjecture for any dimension $d \geq 3$. We show that there are a lot of similarities between the planar setting and the higher dimensional case. We were also intrigued by the graph theoretic background of the Big-Line-Big-Clique Conjecture and formulate a generalisation to ordered hypergraphs and generalised visibility graphs. We state a stronger version of the conjecture that would imply that the BLBC-Conjecture is true true. However, we found a counterexample to the stronger conjecture, leaving the original one open.
This report has two purposes:

- To provide insight in the methods that led to breakthroughs in the above mentioned problems.
- To show generalisations of the Big-Line-Big-Clique Conjecture in higher dimensions, as a hypergraph problem and as a visibility graph problem.

To achieve this goal, the report will follow a clear structure. First of all, the Happy Ending Problem will be explained and context will be given for all required notions. Then both the proof for the planar and the higher dimensional case are presented. This part comes closer to a literature review, illuminating the most important ideas of both proofs. Interestingly, quite some results are used in both proofs, with the main difference being that when not considering planar point sets, there is more 'space' to be used. This will lead to certain properties that only hold in higher dimensions. The similarities and differences between the two settings will be studied, showing why this distinction between dimension is so interesting.
Next, we shift our focus to the Big-Line-Big-Clique Conjecture. Again, the problem is stated within its context and relevant literature is discussed. Then we present our own findings regarding the conjecture. We again look into the similarities and differences of this problem in different dimensions. We also present a generalisation of the conjecture in terms of a (hyper)graph. This provides a link with Steiner Triple Systems and graphs constructed from $p$-arithmetic progressions in $\mathbb{F}_n^p$.
As already mentioned, the conjecture following this generalisation was too strong to say something about the original conjecture. But a lot of interesting ideas and questions arise from the research of the different versions of the conjecture. These are bundled and discussed in the conclusion and discussion, opening the door to future research.

## 2 The Happy Ending Problem

Let's first consider the Happy Ending Problem. This problem asks how large sets of points in general position need to be to ensure the presence of a subset of $n$ points in convex position. A set of points will also be called a *point set*. First of all, we will look at the Happy Ending problem in its original setting: in the plane. The best known result so far for this setting was given by Suk [16], where he showed that the Happy Ending Number grows as $2^{n+o(n)}$. Secondly we will also look at the Happy Ending problem in higher dimensions, which turns out to be significantly different than the planar setting. Interestingly, Pohoata and Zakharov [13] proved that in higher dimension, way fewer points are needed to guarantee the presence of $n$ points in convex position. But before we get ahead of ourselves, let's first look into how the problem was originally posed in the planar setting and how Suk came to his conclusion and then we will consider the higher dimensional case.

### 2.1 The Planar Setting

**Definition 1.** For $d \geq 2$, a point set $\mathcal{P} \subset \mathbb{R}^d$ with $|\mathcal{P}| \geq d+1$ is said to be in *general position* if no $d+1$ points from $\mathcal{P}$ lie on the same $(d-1)$-dimensional hyperplane.

**Definition 2.** A point set $\mathcal{P}$ is in *convex position* if the points from $\mathcal{P}$ represent the vertices of a convex polytope.

A useful small result (e.g. [4]) relates points in general position with points in convex position.

**Lemma 3.** *Let $\mathcal{P}$ be a finite point set in the plane in general position such that all 4-element subsets of $\mathcal{P}$ are in convex position. Then $\mathcal{P}$ is in convex position too.*

*Proof.* Suppose there is a point $p \in \mathcal{P}$ that lies in the interior of $\mathrm{conv}(\mathcal{P})$. Then if we triangulate $\mathrm{conv}(\mathcal{P})$ in an arbitrary way, there has to be a triangle such that $p$ lies in its interior. The vertices of this triangle combined with $p$ form a 4-element subset that is not in convex position. This forms a contradiction. □

The Happy Ending problem can be formulated as the search for an integer $ES(n)$: the smallest integer such that any set of $ES(n)$ points in the plane in general position contains $n$ points in convex position.
Already in 1935, Erdős and Szekeres [7] showed that this minimal integer $ES(n)$ exists for any $n \geq 3$. They gave two proofs of this existence. The first one is more intuitive and based on hypergraph Ramsey Theory. This proof is based on two observations. First of all, any point set $\mathcal{P}$ of five points such that no three points are on a line always contains four points in convex position. This can be seen by considering the convex hull of $\mathcal{P}$. If this convex hull contains at least four extreme points, we are done. Otherwise this hull is a triangle and using the fact that there are no three collinear points one can show that this triangle contains four points in convex position. This leads to the following theorem:

**Theorem 4.** *In any point set $\mathcal{P}$ such that $|\mathcal{P}| = R^{(4)}(n,5)$ and such that no three points are collinear, there must be $n$ points in convex position.*

*Proof.* Colour every 4-element subset $S \subset \mathcal{P}$ red if its points are in convex position and blue if not. By the above observation there has to be a red $K_n^{(4)}$, i.e., a set $T \subset \mathcal{P}$ of size $t$ such that all 4-element subsets of $T$ are in convex position. Then the set $T$ is in convex position too by Theorem 3. □

This yields the upper bound $ES(n) \leq R^{(4)}(n, 5)$, which is very poor. To find a better upper bound, they came up with a second theorem based on cups and caps.

**Definition 5.** Let $\mathcal{P}$ be a $k$-element point set in the plane in general position. Then $\mathcal{P}$ forms a *$k$-cup* if $\mathcal{P}$ is in convex position and its convex hull is bounded above by a single edge. Similarly, $\mathcal{P}$ forms a *$k$-cap* if $\mathcal{P}$ is in convex position and its convex hull is bounded below by a single edge.

An example of a cup and a cap is given in Figure 1.

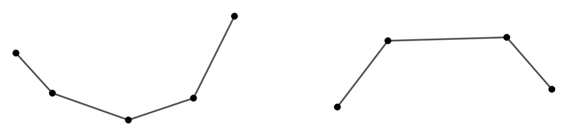

Figure 1: Example of a 5-cup and a 4-cap.

With this notion of cups and caps, they proved the following theorem.

**Theorem 6.** *Let $f(k, \ell)$ be the smallest integer $N$ such that any $N$-element point set in the plan contains a $k$-cup or a $\ell$-cap. Then*

$$f(k, l) = \binom{k + \ell - 4}{k - 2} + 1.$$

This leads to the upper bound

$$ES(n) \leq \binom{2n - 4}{n - 2} + 1 = 4^{n - o(n)}.$$

Later, in 1960 [8], they showed the upper bound $ES(n) \geq 2^{n-2} + 1$ and conjectured this to be sharp.
Making a leap in time, Suk [16] nearly settled this conjecture by proving that $ES(n) = 2^{n+o(n)}$. Specifically, he showed the following result.

**Theorem 7.** *For all $n \geq n_0$, where $n_0$ is a large absolute constant, $ES(n) \leq 2^{n+6n^{2/3}\log n}$.*

The proof of Theorem 7 relies on a well known theorem and two other results. First of all, there is Dilworth's theorem [5].

**Theorem 8.** *Let $\mathcal{P}$ be a finite partially ordered set. The size of a maximum antichain equals the size of a minimum chain cover of $\mathcal{P}$.*

Secondly, there is a combinatorial reformulation of Theorem 6 based on transitive 2-colourings [9]. A transitive two colouring is a 2-colouring, say with colours red and blue, of all 3-element subsets of a set $\{1, \ldots, N\}$ , such that the following property holds. Consider $i_1 < i_2 < i_3 < i_4$. If the subsets $\{i_1, i_2, i_3\}$ and $\{i_2, i_3, i_4\}$ are red (blue), then $\{i_1, i_2, i_4\}$ and $\{i_1, i_3, i_4\}$ are also red (blue).

**Theorem 9.** *Let $g(k,\ell)$ denote the minimum integer $N$ such that, for every transitive 2-colouring on the 3-element subsets of $\{1,\dots,N\}$, there exists a red clique of size $k$ or a blue clique of size $\ell$. Then,*

$$g(k,\ell) = f(k,\ell) = \binom{k+\ell-4}{k-2} + 1.$$

The other important result is the Planar Positive Fraction Erdős-Szekeres Theorem which was formulated by Pór and Valtr in 2002 [15]. It requires some additional terminology.

**Definition 10.** Consider a $k+1$-cap ($k+1$-cup), $X = \{x_1,\dots,x_{k+1}\}$, where the points appear from left to right. The *support* of $X$ is the collection of open regions $\mathcal{C} = \{T_1,\dots,T_k\}$, where $T_i$ is the region outside of $\operatorname{conv}(X)$ bounded by the segment $\overline{x_i x_{i+1}}$ and by the lines $x_{i-1}x_i$ and $x_{i+1}x_{i+2}$. Note that the indices are taken modulo $k+1$.

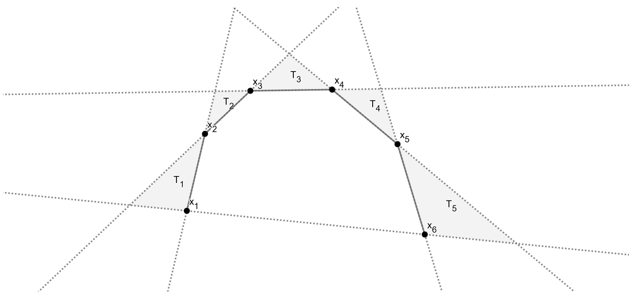


Figure 2: The regions $T_1,\dots,T_5$ of the support of the point set $X = \{x_1,\dots,x_6\}$.

An example of the support of a finite point set is given in Figure 2. The Planar Positive Fraction Theorem then becomes:

**Theorem 11.** *Let $k \geq 3$ and let $\mathcal{P} \subset \mathbb{R}^2$ be a finite planar point set in general position such that $|\mathcal{P}| \geq 2^{40k}$. Then there is a $k$-element subset $X$ of $\mathcal{P}$ such that $X$ is either a $k+1$-cap or a $k+1$-cup, and the regions $T_1,\dots,T_k$ from the support of $X$ satisfy $|T_i \cap \mathcal{P}| \geq \frac{|\mathcal{P}|}{2^{40k}}$. In particular, every $k$-tuple obtained by choosing one point from each $T_i \cap \mathcal{P}$, $i = 1,\dots k$ is in convex position.*

Suk's proof applies Theorem 11 to a planar point set $\mathcal{P}$ with $|\mathcal{P}| = \lfloor 2^{n+6n^{2/3}\log n} \rfloor$ and $n \geq n_0$, where $n_0$ is a large enough absolute constant. If we set $k = \lceil n^{2/3} \rceil$ and apply Theorem 11 with parameter $k+2$, we obtain a cup or a cap $X = \{x_1,\dots,x_{k+3}\} \subset \mathcal{P}$. Since $k$ is large, the regions $T_1,\dots,T_{k+2}$ of the support satisfy

$$|T_i \cap \mathcal{P}| \geq \frac{|\mathcal{P}|}{2^{40k}}.$$

Such two regions $T_i$ and $T_j$ are called *adjacent* if $i$ and $j$ are adjacent integers. Suk then defines the point sets $P_i = T_i \cap \mathcal{P}$ for all $i$ and the segments $B_i = \overline{x_{i-1}x_{i+2}}$. One could think of this $B_i$ as the line segment that connects the two $x_j$ that correspond to $T_{i-1}$ and $T_{i+i}$ that are not part of $T_i$. The crux of Suk's proof is that on these point sets $P_i$ we impose a partial order $\prec$, where $p \prec q$ if $p \neq q$ and $q \notin \operatorname{conv}(B_i \cup p)$. By Dilworth's Theorem, each $P_i$ either contains a chain of size at least $|P_i|^{1-\alpha}$ or an antichain of size at least $|P_i|^{\alpha}$ with respect to $\prec$, where we pick $\alpha = 3n^{-1/3}\log n$. Using additional arguments, both an chain and an antichain then form the basis for a convex point set of size $n$.
This proves that in two dimensions $ES(n) \leq 2^{n+o(n)}$.

## 2.2 Higher Dimensions

As already mentioned, the above result only holds for point sets in the plane. Curiously, when we consider point sets in general position in higher dimensions, way fewer points are needed to ensure the presence of $n$ points in convex position. To be able to consider the Happy Ending number in different dimensions, let $ES_d(n)$ be the smallest integer such that any set of $ES_d(n)$ points in $\mathbb{R}^d$ in general position contains $n$ points in convex position. One can relate $ES_d(n)$ to its counterpart in dimension $d-1$ by a projection argument. Consider a set of $ES_{d-1}(n)$ points in general position in $\mathbb{R}^d$ and project this point set injectively onto an arbitrary $(d-1)$-dimensional hyperplane. In this projection, we are guaranteed to find a subset in convex position. Lifting this subset back in the original configuration yields a point set in convex position. Hence,

$$ES_d(n) \leq ES_{d-1}(n) \leq \cdots \leq ES_2(n).$$

This shows that if we want to prove that for $d \geq 3$, $ES_d(n) = 2^{o(n)}$, it suffices to show that $ES_3(n) = 2^{o(n)}$. That is exactly what Pohoata and Zakharov [15] proved. Specifically, they proved the following theorem.

**Theorem 12.** *For $\epsilon > 0$, there exists $n_0(\epsilon)$ such that for every $n \geq n_0(\epsilon)$, the following holds: if $\mathcal{P} \subset \mathbb{R}^3$ is a set of points in general position with $|X| \geq 2^{\epsilon n}$, then $\mathcal{P}$ must always contain $n$ points in convex position. Hence,*

$$ES_3(n) = 2^{o(n)}.$$

Interestingly, to prove this statement, the ideas from the planar version are still used. In particular, Pohoata and Zakharov project point sets in $\mathbb{R}^3$ on $\mathbb{R}^2$ and lift them back to come to their result. The proof uses the notion of cups and caps, the Positive Fraction Theorem and new notions and statements for higher dimensions. A short outline of their proof is given next.
First of all, we need some definitions and related statements regarding points and point sets in $\mathbb{R}^3$.

**Definition 13.** Consider a projection $\pi : \mathbb{R}^3 \to \mathbb{R}^2 : (u_1, u_2, u_3) \mapsto (u_1, u_2)$ onto the first two coordinates. Let $\overline{ab}$ and $\overline{cd} \subset \mathbb{R}^3$ be two disjoint line segments whose projections $\pi(\overline{ab})$ and $\pi(\overline{cd})$ intersect at some point $x \in \mathbb{R}^2$. Then we say that $\overline{ab}$ *lies above (below)* $\overline{cd}$ if the third coordinate of the point $\overline{ab} \cap \pi^{-1}(x)$ is larger (smaller) than the third coordinate of the point $\overline{cd} \cap \pi^{-1}(x)$.

One can think of this definition of taking the intersection point of the projections in the $x, y$-plane as the reference point to decide which line segment lies above which. Perpendicular to the plane we go up to see which segment we first encounter. That segment lies below the other one.
When a point set in $\mathbb{R}^3$ is large enough, there is always a point set with a 'nice' structure in terms of segments above and below each other.

**Lemma 14.** *For any $k \geq 4$ there exists a number $AB(k)$ such that the following holds for any $N \geq AB(k)$. Let $x_1, \ldots, X_N \in \mathbb{R}^3$ be points in general position such that the projections $\pi(x_i), i = 1, \ldots, N$ of the $x_i$ onto the first two coordinates are consecutive vertices of a convex polygon in $\mathbb{R}^2$. Then there is a $k$-element set $S \subset [N]$ such that either for any indices $i < j < i' < j' \in S$ the segment $\overline{x_i x_j}$ is above $\overline{x_{i'} x_{j'}}$ or the segment $\overline{x_i x_j}$ is below $\overline{x_{i'} x_{j'}}$.*

This statement follows from Ramsey Theory. We colour all 4-element subsets $x_i, x_j, x_{i'}, x_{j'} \subset \{x_1, \ldots, x_N\}, i < j < i' < j'$ red if the segment $\overline{x_i x_j}$ is above $\overline{x_{i'} x_{j'}}$ and blue if the segment $\overline{x_i x_j}$ is below $\overline{x_{i'} x_{j'}}$. So any $AB(k) > R^{(4)}(k, k)$ satisfies the above lemma. Having obtained the existence of $AB(k)$, this value can be used in another statement.

**Lemma 15.** *Let $\mathcal{P}$ be a point set in general position in $\mathbb{R}^3$ such that the projection $\pi(\mathcal{P}) \subset \mathbb{R}^2$ is in convex position. If $N \geq AB(k)$, then there are points $x_1, \ldots, x_k \in \mathcal{P}$ such that $\pi(x_1), \ldots, \pi(x_k)$ are consecutive vertices of a convex polygon and for any $1 \leq a < b < c < k$ the convex hulls of the sets $\{x_1, \ldots, x_{a-1}\} \cup \{x_b, \ldots, x_{c-1}\}$ and $\{x_a, \ldots, x_{b-1}\} \cup \{x_c, \ldots, x_k\}$.*

Secondly, we need the notion of 2-separability and related statement.

**Definition 16.** We say that a collection of point sets $P_1, \ldots, P_k \subset \mathbb{R}^3$ is *2-separated* if for any set of indices $i, j, i', j' \in [k]$ such that $\{i, j\} \cap \{i', j'\} = \emptyset$, we have

$$\operatorname{conv}(P_i \cup P_j) \cap \operatorname{conv}(P_{i'} \cup P_{j'}) = \emptyset.$$

**Lemma 17.** *Let $P_1, \ldots, P_k \subset \mathbb{R}^3$ be finite and pairwise disjoint point sets such that $|P_i| \geq 2^{k^3}$ for all $i$ and $P_1 \cup \cdots \cup P_k$ is in general position. Then there exist $Y_i \subset P_i$ such that $|Y_i| \geq 2^{-k^3}|P_i|$ for all $i$, and the collection $Y_1, \ldots, Y_k$ is 2-separated.*

Now to prove Theorem 12, fix $\epsilon > 0$ and let $\mathcal{P}$ be a point set in general position with $|\mathcal{P}| \geq 2^{\epsilon n}$. Th objective is to show that for $n$ large enough, $\mathcal{P}$ contains a subset in convex position.
Consider a projection $\kappa : \mathbb{R}^3 \to \mathbb{R}^2$ along a generic direction and let $Y = \kappa(\mathcal{P})$. We apply Theorem 11 to $Y$ with with parameter $k_0 = n^{1/4}$. Let $X$ be the corresponding $k_0 + 1$-cup or $k_0 + 1$-cap and $T_1, \ldots, T_{k_0}$ its support. Denote $Y_i = T_i \cap Y$ such that $|Y_i| > 2^{-40k_0}|\mathcal{P}|$ for each $i = 1, \ldots, k_0$ and finally, let $P_i'$ denote the preimage of $Y_i$ in $\mathcal{P}$.
Lemma 17 allows us to pick subset $P_i \subset P_i'$ such that $P_1, \ldots, P_{k_0}$ is 2-separated. Then

$$|P_i| \geq 2^{-k_0^3}|P_i| = 2^{-k_0^3}|Y_i| \geq 2^{-k_0^3 - 40k_0}|\mathcal{P}| \geq |\mathcal{P}|^{1-\delta(\epsilon)},$$

where $\delta(\epsilon) \to 0$ as $n \to \infty$.
From every $P_i$ we choose an arbitrary point $x_i$, where we can note that the projections $\kappa(x_1), \ldots, \kappa(x_{k_0})$ are consecutive vertices of a convex polygon. Now we let $k$ be the smallest integer such that $AB(k) < k_0$. We apply Lemma 15 to the points $x_1, \ldots, x_{k_0}$ to obtain the set of indices $\{i_1, \ldots, i_k\}$.
On this new set we apply the following statement.

**Lemma 18.** *Let $J = \{j_1, j_2, j_3\} \subset [k]$ with $j_1 < j_2 < j_3$. Then there exist unbounded polytopes $P_J^1, P_J^2$ with at most three edges such that for $j \in [k]$, we have*

$$X_j \subset \begin{cases} P_J^1 & \text{if } j \in [1, j_1), \cup (j_2, j_3] \\ P_J^2 & \text{if } j \in [j_1, j_2) \cup j_3, k] \end{cases}$$

*and such that sets $P_J^1, P_J^2, \operatorname{conv}(X_{j_2})$ are in convex position.*

Hence we fix $J = \{j_1 < j_2 < j_3\} \in \binom{[k]}{3}$ and define a partial order $\prec_J$ on $X_{j_2}$. For $x, x' \in X_{j2}$, $x \prec_J x'$ if $x \in \operatorname{conv}(\{x'\} \cup P_J^1$. Then by the above lemma, an $\prec_J$-antichain is a $P_J^1$-free set and a $\prec_J$-chain is a $P_J^2$-free set. Lastly, Pohoata and Zakharov use Dilworth's Theorem to show that in the case of a large chain or a large antichain, there is always a large point set in convex position, which is exactly what was wanted to be shown.

## 2.3 Conclusion on the Happy Ending Problem

This section considered the Happy Ending Problem in $\mathbb{R}^2$ and in $\mathbb{R}^3$, where we have seen that there is a clear difference between point sets in the plane and in higher dimensions. In the plane, we had

$$ES_2(n) = 2^{n+o(n)},$$

while for higher dimensions $d \geq 3$, we have

$$ES_d(n) = 2^{o(n)}.$$

The proofs of the two statements had a similar approach. In both proofs, a smaller cup or cap is found inside a large point set by the planar Positive Fraction Theorem. From this cup or cap, the point set is then split into the intersections with its support. In both cases, a partial order is defined inside these different sets and Dilworth's Theorem is used to find a large chain or antichain. These are then used to find large enough sets in convex position. The main intuitive difference between both cases is that in higher dimensions, there is 'more space' to find structure in the point sets. By projecting the point set on the plane, we can use the ideas and notions from Suk's proof and by lifting back to $\mathbb{R}^3$ we can use additional notions and machinery that wasn't available before, like line segments being above other segments and 2-separability. Hence this additional space in higher dimensions is in a sense less restrictive to find large subsets in convex position.
This difference in behaviour of point sets in the plane compared to higher dimensions has led us to consider a related problem: the Big-Line-Big-Clique Conjecture.

# 3 The Big-Line-Big-Clique Conjecture

## 3.1 The Planar Setting

Just as in the previous section, we will first consider the conjecture in the plane. As far as we know, this is the only setting in which the conjecture has been studied. Hence, we will first explain what the conjecture is and give a summary of what is known so far about it. Then, we will propose a reformulation of the conjecture to $\mathbb{R}^d$ for $d \geq 3$ and a generalisation of the conjecture to both graphs and hypergraphs.
To fully understand the Big-Line-Big-Clique Conjecture in the plane, we need the notion of pairwise visibility of points contained in a point set.

**Definition 19.** Let $\mathcal{P} \subset \mathbb{R}^2$ be a point set in the plane. We say that $x_i, x_j \in \mathcal{P}$ *see each other/are visible* if there does not exist some third point $x_k \in \mathcal{P}$ such that $x_k \in \overline{x_i x_j}$. We say that the points in a subset $S \subset \mathcal{P}$ *pairwise see each other* if for all $x_i, x_j \in S$, $x_i$ and $x_j$ see each other.

Figure 3 depicts a point set of six points in the plane. In this point set, points $x_1$ and $x_5$ are not visible, as $x_4$ is contained in the segment $\overline{x_1 x_5}$. Similarly, points $x_2$ and $x_4$ are not visible either. $x_6$ is visible with all other points.

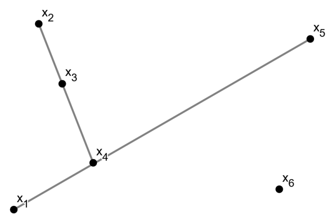


Figure 3: Example of a point set with pairwise visible and invisible points.

The Big-Line-Big-Clique Conjecture states that if a planar point set is large enough, it should always contain a lot of points that are pairwise visible, or a lot of points on a line. This conjecture was first made by Kára, Pór and Wood in 2005 [10] and can formally be formulated as follows.

**Conjecture 20.** *For every positive integers $\ell$ and $k$ there exists a number $n_{k,\ell}$ such that every point set $\mathcal{P}$ in the plane such that $|\mathcal{P}| \geq n_{k,\ell}$ contains $\ell$ collinear points or $k$ mutually visible points.*

So far, this conjecture has only been proven to be true for very small values of $\ell$ and $k$, namely for $\ell \leq 3$ and $k \leq 5$. Let us take a look at the arguments why the conjecture is true for these values.

- The case for $\ell \leq 3$ is very intuitive. Suppose we have a point set with no three points on a line. Then all points are mutually visible. So either a point set contains 3 points on a line or it contains $k$ points that are pairwise visible. Hence the conjecture is true for $\ell \leq 3$.
- For $k \leq 3$, the argument is also still quite simple. One could note that if there are no 3 mutually visible points, then all points need to be collinear. This can be formally proved

using induction on the number of points. For three points, they clearly are not mutually visible iff they are on a line.

Now suppose we have $n$ points such that no three points can mutually see each other and hence are on a line. Then if we want to add another point such that no three points become mutually visible, we need to add this point on the same line, because otherwise this point combined with any two other points that can see each other on the line become mutually visible. Hence the conjecture is true for $k \leq 3$.

- The case for $k \in \{4, 5\}$ suddenly becomes way harder and requires an additional result. In 2010, Abel et al. [1] proved that for every integer $\ell \geq 2$, every finite point set of at least $ES\left(\frac{(2\ell-1)^{\ell}-1}{2\ell-2}\right)$ points in the plane either contains $\ell$ collinear points or a 5-hole, i.e., a pentagon with no points of the point set in the interior of its convex hull. In a 5-hole, there are always five pairwise visible points, so the conjecture is true for $k \leq 5$.

Lastly, we can give a lower bound on $n_{k,\ell}$ when it exists. If we consider the $d$-dimensional $(\ell - 1) \times \cdots \times (\ell - 1)$ integer lattice and project this set onto the plane in a generic direction, we obtain a planar point set with no $\ell$ collinear points and no $k = 2^d + 1$ pairwise visible points [10]. Hence $n_{k,\ell} > (\ell - 1)^{\log_2(k-1)}$ when it exists.
The fact that the result about 5-holes is required to only prove the case for $k \leq 5$ and that we don't know whether the conjecture is true for larger values shows that it is a hard conjecture to prove or disprove, even for small values.

### 3.1.1 Infinite Point Sets

However, when we consider infinite point sets in the plane, we can prove that the Big-Line-Big-Clique Conjecture does not hold. And not only does the conjecture not hold, it does not hold for very small values for $k$ and $\ell$. In 2010, Pór and Wood [14] proved the following theorem.

**Theorem 21.** *There exists a countably infinite point set with no 4 collinear points and no 3 pairwise visible points.*

The proof of this theorem requires the Sylvester-Gallai Theorem [2].

**Theorem 22.** *For every finite point set $\mathcal{P} \subset \mathbb{R}^2$ there exists either a line that passes through exactly two points or a line that passes through all points.*

This theorem enables us to prove Theorem 21. We start with three non-collinear points in the plane: $x_1, x_2, x_3$. We will step by step add another point to this point set without creating four collinear points or 3 pairwise visible points. Given points $x_1, \ldots, x_{n-1}$, we will pick $x_n$ as follows. By the Sylvester-Gallai theorem, there is a line through exactly two of these points. We choose the line $x_i x_j$ with $i < j$, where we first pick $j$ to be minimal and then $i$ to be minimal. We insert $x_n$ on the segment $\overline{x_i x_j}$ such that $\{x_i, x_j, x_n\}$ is the only collinear triple that contains $x_n$. This is possible, as there are only finitely many excluded locations by this condition, while there is of course an infinite choice of locations for $x_n$.
We keep repeating this to obtain a point set $\mathcal{P} = \{x_i : i \in \mathbb{N}\}$ which does not contain four collinear points by construction. And if $x_i$ and $x_k$ are visible with $i < k$, then $x_i$ and $x_k$ are collinear with some point $x_{i'}$. And since $i < k$, we have $x_k \in \overline{x_i x_{i'}}$ with $i' < k$. So now let's suppose that we have three points $x_i, x_j, x_k$ with $i < j < k$ that are pairwise visible. Then as we just showed, $x_k \in \overline{x_i x_{i'}}$ and $x_k \in \overline{x_j x_{j'}}$ where $i', j' < k$. But since $x_k$ is only one collinear triple amongst $x_1, \ldots, x_k$, we have $i = j'$ and $j = i'$. So in fact, $x_i, x_j, x_k$ are collinear and $x_i$ and $x_j$ are not visible. Hence $\mathcal{P}$ does not contain 3 pairwise visible points and we indeed have an infinite point set with no 4 collinear nor 3 pairwise visible points.

## 3.2 Higher Dimensions

Let us now consider the Big-Line-Big-Clique Conjecture in higher dimension. The reformulation of the conjecture is pretty similar to the planar case, but the set-up is a bit more involved.

### 3.2.1 Notation and Preliminary Results

**Definition 23.** In this work, a *hyperplane* in $\mathbb{R}^d$ refers to a $(d-1)$-dimensional affine subspace of $\mathbb{R}^d$. Two points in $\mathbb{R}^d$ are said to be *coplanar* if they are contained in the same hyperplane.

**Definition 24.** Let $d, m$ be positive integers and $\mathcal{P}$ a countable point set in $\mathbb{R}^d$. A subset $X \subseteq \mathcal{P}$ is called *$m$-wise visible in $\mathcal{P}$* if

$$\mathrm{conv}(Y) \cap \mathcal{P} = Y$$

for every size $\leq m$ subset $Y \subseteq X$.

Note that there does not need to be a connection between $m$ and $d$. For $d = 2$, a set that is 3-wise visible is in convex position by Theorem 3. And of course, if a set is $m+1$-wise visible, then it is $m$-wise visible too. Moreover, 2-wise visibility is the same as pairwise visibility.

**Definition 25.** A subset $\mathcal{P} \subseteq \mathbb{R}^d$ is *in spanning position* if every $d$ points of $\mathcal{P}$ span a hyperplane, and not all points of $\mathcal{P}$ are contained in one hyperplane.

**Definition 26.** A subset $\mathcal{P} \subseteq \mathbb{R}^d$ of size $\geq d+1$ is *$d$-general position* if no $(d-2)$-dimensional hyperplane contains $\geq d$ points of $\mathcal{P}$.

**Lemma 27.** *Suppose $\mathcal{P} \subseteq \mathbb{R}^d$ is such that every $(d-2)-$dimensional hyperplane contains at most $d-1$ points of $\mathcal{P}$, and not all points of $\mathcal{P}$ are contained in one hyperplane. Then $\mathcal{P}$ is in spanning position.*

*Proof.* The dimension of the affine subspace spanned by $d$ points $x_1, \dots, x_d \in \mathcal{P}$ is at most $d-1$, as it equals the maximum number of linearly independent vectors among the $d-1$ vectors $x_d - x_1, x_d - x_2, \dots, x_d - x_{d-1}$. On the other hand, if $x_1, \dots, x_d$ were to span a hyperplane $H$ of dimension less than $d-1$, then $H$ would contain $d$ points of $\mathcal{P}$, which is not allowed. Thus every $d$ points of $\mathcal{P}$ span a $(d-1)$-dimensional hyperplane, and the conclusion follows. $\square$

### 3.2.2 The Conjecture and its Analysis on Small Values

Let us reformulate the conjecture to higher dimensions.

**Conjecture 28.** *For every positive integer $d$ and for every positive integers $k$ and $\ell$, there exists a number $n_{d,k,\ell}$ such that every point set in $\mathbb{R}^d$ contains $\ell$ coplanar points, or a $d$-wise visible subset of size $k$.*

First of all, note that in this formulation the $d$-wise visibility depends on the dimension that we are working in. We can repeat the same analysis as in the planar case for smaller values.
The conjecture holds for $\ell \leq d+1$. The proof of this statement is similar to the planar case. Suppose we have a point set $\mathcal{P} \subset \mathbb{R}^d$, $d \geq 3$, such that there are no $d+1$ points in a $(d-1)$-dimensional hyperplane. Then by the result of Pohoata and Zakharov [13], as long as $\mathcal{P}$ is large enough, it always contains $k$ points in convex position. And in this case, there is a $d$-wise visible subset of $P$ of size $k$. To see why this is true, we assume we have a point set $\mathcal{P} \subseteq \mathbb{R}^d$, $d \geq 3$ and that $P$ contains a subset of $k$ elements in convex position. Let $X_1 := \{x_1, \dots, x_k\}$ be such a set. We may assume that $X_1$ contains other points in its convex hull, otherwise we are done. Moreover, no $d+1$ points lie in the same $(d-1)$-dimensional hyperplane. So none of the points

in the convex hull can lie in a hyperplane spanned by $d$ points in $\mathcal{P}$. This means that $X_1$ is $d$-wise visible. Indeed, suppose this is not the case. Then there are $d$ points such that an additional point of $\mathcal{P}$ lies in their convex hull. This is impossible.

Secondly, we look at the bound given by the result on 5-holes. As we could not find a result where the size of the hole depends on the dimension, we will reuse the result from Abel et al. [1] to conclude that the conjecture holds for $k \leq 5$. This can be done using induction on $d$. The base case for $d = 2$ is already given. Now suppose that for some positive integer $\ell$ we have a point set of size at least $ES\left(\frac{(2\ell-1)^{\ell}-1}{2\ell-2}\right)$. Then we can project the point set onto a $(d-1)$-dimensional hyperplane such that no two points are projected onto each other. By the induction hypothesis this point set in $\mathbb{R}^{d-1}$ either has $\ell$ coplanar points or a 5-hole, which in the higher dimensional case is a subset of five points with no other points in its convex hull. In the former case, if we lift these points back, we have $\ell$ coplanar points in $\mathbb{R}^d$ and in the latter case, since there are no points in the interior of the convex hull of the projection of the five points, these points form a 5-hole when lifted back in the higher dimension as well. So if we pick the 5-hole with the smallest volume, these points are $d$-wise visible. Hence the conjecture is true for $k \leq 5$.

We can also generalise the argument that gave the bound $k \leq 3$ in the plane. Then we obtain that if we have a point set such that there are no $d+1$ points that can $d$-wise see each other, all points in the point set need to lie in the same $(d-1)$-dimensional hyperplane. This can be proven by showing that if we have a point set such that there are no $d+1$ points that can $d$-wise see each other and we want to add another point while still ensuring this condition, the new point set still has to lie in a $(d-1)$-dimensional hyperplane. First of all, suppose we have $d+1$ points that cannot $d$-wise see each other. Then clearly these points have to lie on the same hyperplane. Now suppose that we have $n-1$ points such that no $d+1$ of them are $d$-wise visible and assume these points lie in the same $(d-1)$-dimensional hyperplane. Note that this hyperplane does not have to be unique. Either all points lie in the same affine subspace of some dimension $< d-1$, and there are infinitely many $(d-1)$-dimensional hyperplanes passing through all points, or there is one unique $(d-1)$-dimensional hyperplane containing all points. In the case there is one unique hyperplane, any additional point that we want to add without having $d+1$ points that can $d$-wise see each other has to lie in that same hyperplane as it would otherwise create a subset that is $d$-wise visible. If all points are lying in a lower dimensional hyperplane, adding an additional point can never create $d+1$ points that are $d$-wise visible. Hence this point can be added anywhere. However, then all points still lie in the same $(d-1)$-dimensional hyperplane (which may now be unique). This shows that the conjecture is also true for $k \leq d+1$.

Hence, the ideas that were valid for the Big-Line-Big-Clique Conjecture in the plane lead also to useful results in higher dimensions. However, so far there have not been any meaningful breakthroughs that show a clear distinction between the planar version and the version in higher dimensions as there was in the Happy Ending Problem. Further research to investigate whether such a distinction exists is required. A first step in this direction could be to reduce the problem to the case in $\mathbb{R}^3$. Just as with the Happy Ending Problem, a projection argument shows that if the Big-Line-Big-Clique Conjecture is true in dimension $d$, the conjecture also holds for all dimensions larger than $d$, so:

$$n_{d,k,\ell} \leq n_{d-1,k,\ell} \leq \cdots \leq n_{2,k,\ell}.$$

To see why this is true, consider a point set of $n_{d-1,k,\ell}$ points in $\mathbb{R}^d$. We can project this set on a generic $(d-1)$-dimensional hyperplane. Then there are either $\ell$ points on a $(d-2)$-dimensional hyperplane or $(d-1)$-wise visible subset of $k$ points. Suppose the former is true. Then these $\ell$

points also lie on a (non-unique) $(d-1)$-dimensional hyperplane. So suppose we have $k$ $(d-1)$-wise visible points. Since there is no point 'blocking the view' in $d-1$ dimensions, there cannot be a point blocking the view in $d$ dimensions either. Hence $n_{d,k,\ell} \leq n_{d-1,k,\ell}$. This implies that any upper bound for the two-dimensional case yields an upper bound for any higher dimensional case.

#### 3.2.3 The Infinite Case

Just as in the planar case, saying something about finite point sets requires creativity and is not straightforward. So far, we don't know whether the conjecture in higher dimensions holds or not. However, just as in the plane, the conjecture does not hold for infinite point sets. Let us first consider the more intuitive case of $\mathbb{R}^3$ and then we will generalise the proof to any point set in $\mathbb{R}^d$, $d \geq 3$.

The proof of Pór and Wood regarding infinite planar points sets relied heavily on the Sylvester-Gallai Theorem. This theorem was specific for the plane, but Ball and Montserrat [3] have generalised the statement to higher dimensions. From their work, we will use the following observation:

**Observation 29.** *Let $\mathcal{P} \subseteq \mathbb{R}^d$ be a finite point set in spanning position. Then there exists a hyperplane that intersects $\mathcal{P}$ in precisely $d$ points.*

**Theorem 30.** *There is a countably infinite point set in $\mathbb{R}^3$ with no 5 coplanar points and no 4 3-wise visible points.*

*Proof.* Let $x_1, x_2, x_3, x_4$ be 4 non-coplanar points. Observe that then no 3 points are collinear. Throughout the construction we will construct a point set $\{x_1, x_2, \ldots, x_n\}$ such that the following properties are preserved.

1. No 3 points are collinear. This implies that every three points span a hyperplane.
2. Every point $x_i$ is the interior vertex of at most one coplanar quadruple, and precisely one for $i \geq 5$.
3. No 5 points are coplanar.

Clearly $\{x_1, x_2, x_3, x_4\}$ satisfy these properties. Let $n \geq 5$. Assume that we have constructed $\mathcal{P}_{n-1} := \{x_1, \ldots, x_{n-1}\}$ and that this point set satisfies properties 1−3. Given points $x_1, \ldots, x_{n-1}$, define $x_n$ as follows. As stated above, since no 3 points are collinear, any 3 points span a plan. By Observation 29, there is such a plane through exactly 3 points. Choose a plane $x_i x_j x_k$ with $i < j < k$ such that first $k$ is minimised, then $j$ is minimised and lastly $i$ is minimised. Choose $x_n$ in the interior of $\text{conv}\{x_i, x_j, x_k\}$ such that $x_i x_j x_k x_n$ is the only coplanar quadruple that contains $x_n$ and such that no 3 points of the point set become collinear. This is possible since there are only finitely many obstructions where the added point would become coplanar/collinear in an undesired way. We obtain a point set $\mathcal{P}_n := \{x_1, \ldots, x_n\}$.

By construction, $\mathcal{P}_n$ has no 3 collinear points, thus preserving property 1. To see that property 3 is preserved, suppose for a contradiction that there are 5 coplanar points among $\mathcal{P}_n$. Since they cannot form a subset of $\mathcal{P}_{n-1}$, $x_n$ must be one of them. But then $x_n$ is contained in more than one coplanar quadruple, contradicting the choice of $x_n$.

It remains to show that property 2 continues to hold for $\mathcal{P}_n$. By construction, $x_n$ is the interior point of precisely one coplanar quadruple among $\mathcal{P}_n$. Suppose for a contradiction that for some $i < n$, $x_i$ is the interior point of more than one quadruple among $\mathcal{P}_n$. By property 2, $x_i$ is the interior point of at most one coplanar quadruple among $\mathcal{P}_{n-1}$. Therefore every other coplanar

quadruple containing $x_i$ must also contain $x_n$. However, by construction of $x_n$, the interior point of this quadruple must be $x_n$, so $x_i$ cannot be the interior point; contradiction.
We conclude that $\mathcal{P}_n$ satisfies properties $1-3$, as desired. Repeating this, we obtain a countably infinite point set $\mathcal{P} := \bigcup_{n\in\mathbb{N}} \mathcal{P}_n = \{x_i : i \in \mathbb{N}\}$. It is not true in general that a property that holds for $\mathcal{P}_n$ for every finite $n$ also holds for $\mathcal{P}$. However, observe that properties $1-3$ are such that $\mathcal{P}$ still obeys them, since if any of them were not satisfied then this would be witnessed by $\mathcal{P}_n$ for some $n$. We are now in a position to analyse visibility in $\mathcal{P}$.
Note that if $x_i x_j x_k$ are 3-wise visible with $i < j < k$, then $x_i, x_j, x_k$ are coplanar with some point $x_n$. Since $i < j < k$, we have $x_k \in \operatorname{conv}\{x_i, x_j, x_n\}$. This is because since the three points are triplewise visible, one of the points must have been inserted in the interior of a convex hull of three points. If this wouldn't be the case, there would be a point added to the interior of $\operatorname{conv}\{x_i, x_j, x_k\}$, contradicting the fact that they are triplewise visible. And since $k$ is the largest integer of the three, $x_k$ has to be the point that has been inserted.
Now assume that 4 points $x_i, x_j, x_k, x_l$ are triplewise visible with $i < j < k < l$. As noted above, $x_l \in \operatorname{conv}\{x_i, x_j, x_m\}$ and $x_l \in \operatorname{conv}\{x_i, x_k, x_n\}$, for some $m, n < l$. Since $x_l$ is the interior vertex of only one coplanar quadruple amongst $x_1, \ldots, x_l$, we have $x_m = x_k$ and $x_n = x_j$. Thus $x_i, x_j, x_k, x_l$ are coplanar and $x_i, x_j, x_k$ are not visible. This contradiction proves that no 4 points are 3-wise visible. $\square$

Now that we have established that the Big-Line-Big-Clique Conjecture does not hold for infinite point sets in $\mathbb{R}^3$, we can make the theorem more abstract and generalise it to infinite point sets in $\mathbb{R}^d, d \geq 3$. The construction is exactly the same, just harder to picture if anything.

**Theorem 31.** *For every positive integer $d$, there exists a countably infinite point set in $\mathbb{R}^d$ with no $d+2$ coplanar points and no $d$-wise visible set of size $d+1$.*

*Proof.* Let $x_1, \ldots, x_{d+1}$ be $d+1$ points in spanning position that are not all in one hyperplane. Throughout the construction we will again construct a point set such that three properties are preserved.

1. Every $(d-2)$-dimensional hyperplane contains at most $d-1$ points. Since we have started with not all points contained in one hyperplane, this means that all points are in spanning position.

2. Every point $x_i$ is the interior vertex of at most one convex hull of exactly $d$ points that lie in the same hyperplane, and exactly one for $i \geq d+2$.

3. No $d+2$ points lie in the same hyperplane.

Clearly $\{x_1, \ldots, x_{d+1}\}$ satisfy these properties. Let $n \geq d+2$. Assume that we have constructed $\mathcal{P}_{n-1} := \{x_1, \ldots, x_{n-1}\}$ and that this point set satisfies properties $1-3$. Given points $x_1, \ldots, x_{n-1}$, define $x_n$ as follows. Any $d$ points span a hyperplane. By Observation 29 there is a such a hyperplane through exactly $d$ points. Choose a plane $x_{i_1} \ldots x_{i_d}$ with $i_1 < \cdots < i_d$ where first $i_d$ is minimised, then $i_{d-1}$ etc. Choose $x_n$ in the interior of $\operatorname{conv}\{x_{i_1} \ldots x_{i_d}\}$ such that $x_n$ is the interior vertex of only one convex hull of exactly $d$ points and such that every $(d-2)$-dimensional hyperplane contains at most $d-1$ points. This is possible since there are only finitely many obstructions where the added point would end up in a hyperplane where we don't want it to be. We obtain a point set $\mathcal{P}_n := \{x_1, \ldots, x_n\}$.
By construction, every $(d-2)$-dimensional hyperplane in $\mathcal{P}_n$ has at most $d-1$ points, thus preserving property 1. To see that property 3 is preserved, suppose for a contradiction that there are $d+2$ points in some hyperplane among $\mathcal{P}_n$. Since they cannot form a subset of $\mathcal{P}_{n-1}$,

$x_n$ must be one of them. But then $x_n$ is contained in more than one convex hull of exactly $d$ points, contradicting the choice of $x_n$.
It again remains to show that property 2 continues to hold for $\mathcal{P}_n$. By construction, $x_n$ is the interior point of precisely one convex hull of exactly $d$ points among $\mathcal{P}_n$. Suppose for a contradiction that for some $i < n$, $x_i$ is the interior point of more than one such convex hull among $\mathcal{P}_n$. By property 2, $x_i$ is the interior point of at most one such convex hull among $\mathcal{P}_{n-1}$. Therefore every other convex hull of $d$ points containing $x_i$ must also contain $x_n$. However, by construction of $x_n$, the interior point of this convex hull must be $x_n$, so $x_i$ cannot be the interior point; contradiction.
We conclude that $\mathcal{P}_n$ satisfies properties $1-3$, as desired. Repeating this, we obtain a countably infinite point set $\mathcal{P} := \bigcup_{n\in\mathbb{N}} \mathcal{P}_n = \{x_i : i \in \mathbb{N}\}$. By the same arguments as before, $\mathcal{P}$ obeys properties 1-3 too.
If $x_{i_1}, \dots, x_{i_d}$ are $d$-wise visible, then $x_{i_1}, \dots, x_{i_d}$ lie in a hyperplane with some point $x_k$. Since $x_{i_1} < \dots < x_{i_d}$, $x_{i_d} \in \operatorname{int}\operatorname{conv}\{x_{i_1}, \dots, x_{i_{d-1}}, x_k\}$.
Now let us assume that $d+1$ points $x_{i_1}, \dots, x_{i_{d+1}}$ are $d$-wise visible. Then,

$$\begin{aligned} x_{i_{d+1}} &\in \operatorname{int}\operatorname{conv}\{x_{i_1}, \dots, x_{i_{d-1}}, x_a\} \\ &\in \operatorname{int}\operatorname{conv}\{x_{i_1}, \dots, x_{i_{d-2}}, x_{i_d}, x_b\} \end{aligned}$$

For some $a, b < i_{d+1}$. In fact $x_{i_{d+1}}$ is in the interior of $d-2$ other convex hulls. But since $x_{i_{d+1}}$ is in only one interior of a convex hull of exactly $d$ points amongst $x_1, \dots, x_{i_d}$, $x_a = x_{i_d}$ and $x_b = x_{i_{d-1}}$. Thus $x_{i_1}, \dots, x_{i_{d+1}}$ lie in the same hyperplane and are not $d$-wise visible. $\square$

## 3.3 Generalisation to Hypergraphs and Graphs

The Big-Line-Big-Clique Conjecture was originally posed in a paper regarding the chromatic number of visibility graphs, i.e. graphs where the vertex set is the point set and two vertices are connected by an edge iff they are pairwise visible [10]. The authors posed the Big-Line-Big-Clique Conjecture as a step to prove the following conjecture.

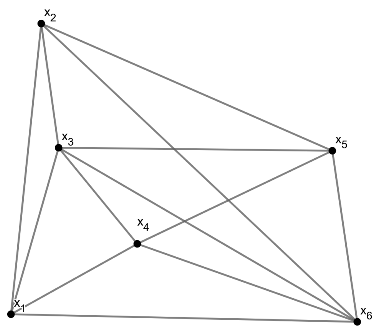


Figure 4: Visibility graph corresponding to the point set in Figure 3

**Conjecture 32.** *Let $\mathcal{V}(\mathcal{P})$ denote the visibility graph of a point set $\mathcal{P}$. Then there is a function $f$ such that $\chi(\mathcal{V}(\mathcal{P})) \leq f(\omega(\mathcal{V}(\mathcal{P})))$ for all $\mathcal{P}$.*

Even though this conjecture was disproven by Pfender in 2008 [11], this relation with visibility graphs has spiked our interest and we decided to look at possible (hyper)graphs reformulations of the conjecture. We present both a formulation in terms of linear and locally ordered hypergraphs and generalised visibility graphs.

### 3.3.1 Locally Ordered Linear Hypergraphs

Our reformulation of point sets to hyperplanes is based a linear ordering on the hyperedges. This order will enable us to tell us something on the pairwise visibility between points. Formally, we define the following hypergraphs.

**Definition 33.** A *locally ordered linear hypergraph* is a hypergraph $(V, E)$, where $V \subseteq \mathbb{N}$ is a set of vertices, and $E$ is a collection of subsets of $V$, such that

- every hyperedge $e \in E$ is equipped with a strict linear order $\sigma_e$, and
- every two distinct $e_1, e_2 \in E$ intersect in at most one vertex, and
- every two distinct $u, v \in V$ are contained in some (unique) hyperedge $e_{uv} \in E$.

Note that different edges $e_1, e_2 \in E$ may be equipped with different orders. The link between the order and visibility still needs to be defined.

**Definition 34.** Given a hypergraph as above, we say that two vertices $u, v \in V$ *see each other* if they are consecutive in the order of their hyperedge $e_{uv}$.

Such a partially ordered linear hypergraph is more general than a planar point set. Every planar point set can be formulated as a hypergraph, but not every hypergraph can be embedded as points in the plane. An example of a hypergraph with corresponding planar point set is given in Figure 5. For an example of a hypergraph that can not be drawn in the plane one could think of the Fano plane with an arbitrary order.
This formulation leads to a stronger conjecture than the Big-Line-Big-Clique Conjecture.

**Conjecture 35.** *For every $k, \ell \geq 2$, there exists a constant $f(k, \ell)$ such that for every locally ordered linear hypergraph $(V, E)$ with $\infty > |V| \geq f(k, \ell)$, there either exists a hyperedge $e \in E$ of size $\geq \ell$, or there exists a subset of $k$ vertices that pairwise see each other.*

One can see that this conjecture is stronger than the Big-Line-Big-Clique Conjecture by noting that the BLBC Conjecture is a special case of the above conjecture where $V$ is a point set in $\mathbb{R}^2$, and the lines through those points are identified with ordered hyperedges. Since each line has two possible orders we can just choose one arbitrarily.
The question can also be raised whether the second condition in the definition of the hypergraphs is actually necessary. The reason why we have included this is because if it is not imposed that two distinct hyperedges intersect in at most one vertex, there is a fairly simple counterexample to Conjecture 35. Consider vertices $y, x_1, \ldots, x_n$ and include every ordered hyperedge of the form $x_i y x_j$, for all $1 \leq i < j \leq n$. Every hyperedge has size 3. The 'visibility graph' obtained by including every pair of vertices that are consecutive in some ordered hyper-edge is isomorphic to a star centered in $y$, which has clique number two. Hence the largest number of pairwise visible points is 2. Taking $n$ arbitrarily large we get a family of counterexamples. Note that the second condition of Definition 33 is not satisfied by this example, so this is not yet a counterexample to Conjecture 35.

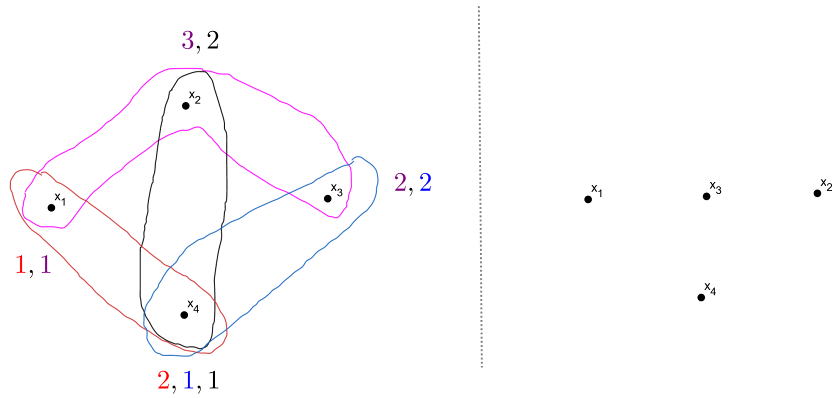


Figure 5: Example of a point set in the plane and a corresponding hypergraph. The colours in the hypergraph depict the hyperedges and the labels next to the vertex show their value in the order corresponding to the hyperedge with that specific colour.

#### 3.3.2 Generalised Visibility Graphs

This section considers a reformulation using graphs instead of hypergraphs. For such a reformulation we introduce the notion of a generalised visibility graph.

**Definition 36.** A *generalised visibility graph* is a graph $G = G(V, E; P)$ with vertex set $V$, edge set $E$, and $P$ a collection of subsets of $V$, such that

- every $p \in P$ induces a subgraph $G[p]$ that is isomorphic to a path (hence for short we will call $p$ a path), and
- every two distinct paths $p_1, p_2 \in P$ intersect in at most one vertex, and
- every two distinct $u, v \in V$ are contained in some (unique) path $p_{uv} \in P$.

Moreover, we say that $u, v \in V$ *see each other* iff $uv \in E(G)$.

Note that we did *not* require that $P$ contains *all* induced subpaths of $G$. In fact, in most cases the above definition implies that only a few of the induced paths of $G$ can be contained in $P$. The same example point set as Figure 5 can be depicted with a generalised visibility graph too, see Figure 6, where the different colours of the edges visualise the different paths.
The conjecture then becomes:

**Conjecture 37.** *For every $k, \ell \geq 2$, there exists a constant $f(k, \ell)$ such that for every generalised visibility graph $G(V, E; P)$ with $\infty > |V| \geq f(k, \ell)$, either some induced path $p \in P$ of $G$ has size $\geq \ell$, or $G$ has a clique of size $\geq k$.*

Just as with the hypergraphs, Conjecture 37 becomes false if we omit the second condition (every two paths intersect in at most one vertex) in the definition of generalised visibility graphs. This is again by the star-construction that provided a counterexample in the hypergraph case. The fact that this counterexample works in both formulation shouldn't come as a big surprise. In fact, the notion of a generalised visibility graph is equivalent to the hypergraphs with a local linear order, but now we have replaced the hyperedge by paths, where we follow the order of

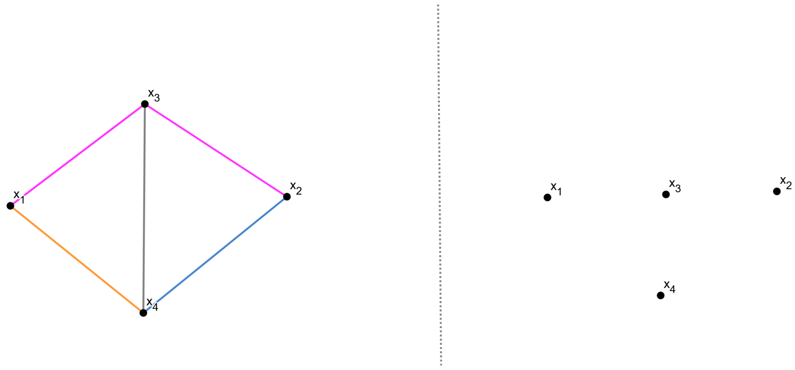


Figure 6: Example of a point set in the plane and a corresponding generalised visibility graph. The different colours of the edges show the different paths.

that specific hyperedge. One can choose whichever formulation they prefer. We came up with this second formulation because in some cases it was easier for us to consider graphs rather than hypergraphs, especially to find counterexamples to the conjecture. And indeed, using this formulation we managed to come up with a counterexample to Conjecture 37. The next section will explain in more detail how we came to our counterexample.

#### 3.3.3 The Conjecture on Generalised Visibility Graphs is Wrong

The reason why we shifted focus from hypergraphs to graphs is because we wanted to use Steiner Triple Systems as possible counterexamples to Conjecture 37 and we found it easier to consider them as graphs with paths rather than hypergraphs with orders. A Partial Steiner Triple System (PSTS) is a collection of 3-element subsets of $\{1, \ldots, m\}$ such that every two elements of $[m]$ are contained in at most one such subset. A Steiner Triple System (STS) is a PSTS such that every pair of vertices is included in precisely one 3-element set. Both STS and PSTS can equivalently be defined in terms of hypergraphs, e.g. a Steiner Triple System is a 3-uniform hypergraph with vertices $\{1, \ldots, m\}$, such that every pair of vertices is included in precisely one hyperedge. A Steiner Triple System $s$ can be used to construct a generalised visibility graph $G$ by taking the vertex set of $s$, and for every hyperedge $\{a, b, c\}$ of $s$, including precisely two of the pairs $\{ab, ac, bc\}$ as an edge of $G$.
In an attempt to obtain a counterexample to Conjecture 37, we first studied the following explicit example of an STS: take $\mathbb{F}_3^n$ as the vertex set, and let the hyperedges given by the 3-element subsets of $\mathbb{F}_3^n$ that form a 3-term arithmetic progression. If we then let $\mathcal{G}(\mathbb{F}_3^n)$ denote the class of graphs with vertex set $\mathbb{F}_3^n$ that are obtainable by selecting two edges on each 3-term AP, then this is immediately a class of generalised visibility graphs with no more than 3 points on a line. So if we can find a graph in $\mathcal{G}(\mathbb{F}_3^n)$ with small clique size, we have a counterexample. However, this turned out to be a difficult task. In fact it is false: we can show that the Steiner Triple System arising from $\mathbb{F}_3^n$ is such that all associated generalised visibility graphs have clique number $\Omega(\log n)$. To show that we need some definitions. Let $f(n)$ denote the minimum value of the clique number $\omega(G)$, minimized over all graphs $G \in \mathcal{G}(\mathbb{F}_3^n)$. Moreover, we define $\omega_N$ as

follows:

$$\omega_N := \max_{\substack{PSTS\ s\\ \text{on N elements}}} \min_{\substack{G\ \text{obtainable}\\ \text{from s}}} \omega(G),$$

where we say that graph $G$ is obtainable from a Partial Steiner Triple System if $V(G)$ is equal to the vertex set of $s$ and $E(G)$ is obtained by choosing two edges on every hyperedge in $s$. And now we are interested in the behaviour of $\omega_N$. Specifically, we will show that $\omega_N = \Omega(\log\log N)$. While this was already shown by Dudek et al. [6] by considering random Partial Steiner Triple Systems, we now also have an explicit Steiner Triple System (namely $\mathbb{F}_3^n$ with its 3-AP's) which certifies this lower bound. To show this lower bound, we prove the next lemma.

**Lemma 38.** *There exists a constant $C > 0$ such that $f(n) \geq C \cdot \log n$, for every $n \in \mathbb{N}$.*

To see why this lemma implies the lower bound, it is practical to define another variable $h(s)$:

$$h(s) := \min_{\substack{G\ \text{obtainable}\\ \text{from s}}} \omega(G),$$

so that

$$\omega_N = \max_{\substack{PSTS\ s\\ \text{on N elements}}} h(s).$$

Since every graph in the considered collection contains $3^n$ vertices, we have

$$\omega_{3^n} \geq h\left(s(\mathbb{F}_3^n)\right) = f(n) \geq C \cdot \log n.$$

This implies that indeed $\omega_N = \Omega(\log\log N)$. Note that this actually only proves the equality if $N$ is a power of three. However, this implies it for the other values of $N \geq 3$ as well since $\omega_{N_1} \leq \omega_{N_2}$ for every $N_1 \leq N_2$. The latter is true because removing any hyperedge or vertex (and every hyperedge containing such a removed vertex) from a PSTS preserves the property of being a PSTS.
Thus it remains to prove Lemma 38 and for that we need a special case of Corollary 3.2 in [12]:

**Lemma 39.** *If $S \subset \mathbb{F}_3^n$ and $|S| \geq 3^{(1-\delta)n}$ for $\delta \leq 0.07$ then $S$ contains at least $3^{(2-14\delta)n}$ 3-term arithmetic progressions.*

Now we are ready to prove the lemma.

*Proof of Lemma 38.* Let $G \in \mathcal{G}(\mathbb{F}_3^n)$ be a fixed graph. Let $S_1 = \mathbb{F}_3^n$ denote the vertex set of $G$. We choose a sequence of subsets $S_1 \supset S_2 \supset \ldots \supset S_k$ by repeating the following for $i = 1, 2, 3, \ldots$ until $S_i$ is empty. Choose a vertex $v_i \in S_i$ which maximises the number of neighbours in $S_i$. If there are no such neighbours we stop (so that the index of the final set is $k = i$), and otherwise we set $S_{i+1} = N_G(v_i) \cap S_i$. At the end of the process, the vertices $v_1, \ldots, v_k$ form a clique of size $k$. The lemma follows if we can show that $k \geq \log_{13}(0.07n)$.

To do so, we show by induction that $|S_i| \geq 3^{(1-\delta_i)n} > 0$ and $\delta_i \leq 0.07$, for all $1 \leq i \leq \log_{13}(0.07n)$. Here the constants $\delta_i$ are defined recursively by $\delta_1 = 0$ and $\delta_{i+1} = 13\delta_i + \frac{1}{n}$.

Observe that the induction hypothesis holds for $i = 0$. Assuming it is true for all $j \leq i$, we now prove it is true for $i + 1$. By taking a subset if necessary, we may assume without loss of generality that $|S_i| = \lceil 3^{(1-\delta_i)\cdot n} \rceil$. Let $F_i$ be the collection of 3−AP's that are contained in $S_i$. By the induction hypothesis and Lemma 39 we have $|F_i| \geq 3^{(2-14\delta_i)n} > 0$. Recall that no two $3 - AP's$ intersect in more than one vertex. Since furthermore every 3-AP induces a subgraph (on three vertices) with at least one edge, it follows that the number of edges in the

induced subgraph $G[S_i]$ is at least $\frac{|F_i|}{3}$. Note that the average degree of this subgraph is given by $\frac{1}{|S_i|}\sum_{v\in S_i}\deg(v)$. By the Handshaking Lemma, this equals $\frac{2|E|}{|S_i|} \geq \frac{2|F_i|}{3|S_i|}$. Hence, we can select a vertex $v_i \in S_i$ with at least $\frac{2|F_i|}{3|S_i|}$ neighbours in $S_i$. If we roughly estimate this, the latter is $\geq \frac{2\cdot 3^{(2-14\delta_i)n}}{3\cdot 3^{(1-\delta_i)n}} > \frac{1}{3}\cdot 3^{(1-13\delta_i)n}$. Thus $|S_{i+1}| \geq 3^{(1-13\delta_i-1/n)n} = 3^{(1-\delta_{i+1})n}$.
It remains to demonstrate that also $\delta_{i+1} \leq 0.07$. For that we expand the recursive definition, obtaining $\delta_{i+1} = \frac{1}{n}\sum_{j=1}^{i+1} 13^{j-1} = \frac{(13^{i+1}-1)}{12n} < \frac{13^{i+1}}{n}$, which is $\leq 0.07$ since by assumption $i+1 \leq \log_{13}(0.07n)$. □

In conclusion, the generalised visibility graphs arising from the STS associated with $\mathbb{F}_3^n$ exhibit arbitrarily large cliques as $n \to \infty$, and hence do not yield a counterexample to Conjecture 37. So this approach doesn't seem to yield the wanted result. If anything, it is an argument why the conjecture could be true.
Therefore we resorted to another idea – using another STS – that did lead to a counterexample.

**Theorem 40.** *For every odd $n \geq 1$, there exists a generalised visibility graph on $3n$ vertices where every path has length 3 and that is complete tripartite and hence has clique number 3.*

*Proof.* Let $n$ be an odd positive integer and partition $[3n]$ into

$$A = \{a_1, \dots, a_n\}$$
$$B = \{b_1, \dots, b_n\}$$
$$C = \{c_1, \dots, c_n\}$$

We first construct a hypergraph from which we will then construct a generalised visibility graph. The hyperedges are defined by

$$\{\{a_i, a_j, b_k\} : i \neq j \text{ and } 2k = i + j \mod n)\} \quad \cup \tag{1}$$
$$\{\{b_i, b_j, c_k\} : i \neq j \text{ and } 2k = i + j \mod n)\} \quad \cup \tag{2}$$
$$\{\{c_i, c_j, a_k\} : i \neq j \text{ and } 2k = i + j \mod n)\} \quad \cup \tag{3}$$
$$\{\{a_i, b_i, c_i\} : i \in [n]\} \tag{4}$$

Note that if $i + j = 2k \mod n$, then either $i = j = k$ or $i, j, k$ are pairwise distinct.
Now every pair of vertices is in a unique hyperedge and none of the sets $A, B, C$ contains a hyperedge. So we can replace every hyperedge by a path in such a way that we obtain a graph for which $A, B, C$ are independent sets.

□

So it turns out that Conjectures 35 & 37 are false. Unfortunately, the fact that these conjectures are false does not say anything about the Big-Line-Big-Clique Conjecture. When trying to embed this counterexample in the plane for $n = 3$, we can, without loss of generality, start by embedding the points $a_1, a_2$ and $a_3$. There are two possible options: they form a triangle or they are collinear. Let's start with the former case. Including the edges of type (1) and keeping in mind that $A$ should form an independent set, the points in $B$ should form a smaller triangle inside the triangle formed by $A$. With the same reasoning for the edges of type (2), $C$ should form a smaller triangle inside $A$, forming the configuration drawn in Figure 7. But now we need the points from $A$ to block $C$ from being pairwise visible. This is impossible.
So now let's assume that the $a_i$ are collinear, without loss of generality the line is $a_1a_2a_3$. Then by the edge of type (1), $b_3$ is inserted on the line segment $\overline{a_1a_2}$ and $b_1$ is inserted on $\overline{a_2a_3}$. $b_2$ can be embedded wherever. Edge type (2) requires us to place $c_1$ on $\overline{b_2b_3}$ and $c_3$ on $\overline{b_1b_2}$, yielding

the configuration in Figure 8. Now we want to place $c_2$ such that $a_3$ is inside the segment $\overline{c_1c_2}$ and at the same time $a_1$ is inside the segment $\overline{c_2c_3}$. This is impossible too. Hence there is no way of embedding this generalised visibility graph in the plane.

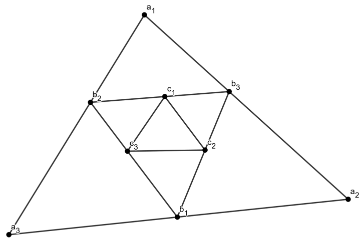


Figure 7: Embedding of the point set in the plane where we started with $A$ forming a triangle. In this case it is impossible to satisfy the edges of type (3).

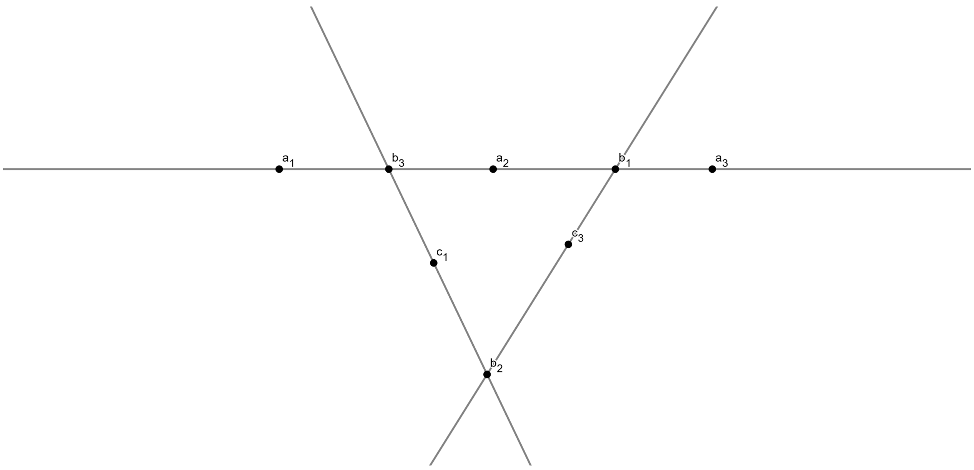


Figure 8: Embedding of the point set in the plane where we started with the points from $A$ being collinear. In this case it is impossible to satisfy the edges of type (3) too.

# 4 Conclusion and Discussion

There is an interesting distinction between point sets in the plane and in any other dimension. That is the main motif of this research. This report first presented the Happy Ending Problem where this distinction was made very clear. Suk [16] proved that for point sets in the plane,

$$ES_2(n) = 2^{n+o(n)},$$

while Pohoata and Zakharov [13] showed that

$$ES_d(n) = 2^{o(n)}$$

for $d \geq 3$. We have looked into the main similarities and differences in both proofs. In both cases, the planar Positive Fraction Theorem and Dilworth's Theorem was used. Yet the higher dimensional case enabled us to fully use the additional space to define new notions like segments *being above* other segments and *2-separability*.
We were interested whether this strength of higher dimensions also translated to other problems. Given its broad range of possible generalisations, we were specifically interested in the Big-Line-Big-Clique Conjecture. This broad range came from the fact that it could be stated for higher dimensions, but also to graphs and hypergraphs. We fully embraced this by looking into all three generalisations.
First, we considered the case of higher dimensions. We extended the idea of pairwise visibility to $d$-wise visibility. However, we didn't exploit the opening of a new dimension to the fullest yet. Apart from verifying the existing bounds from the planar case for higher dimensions, we were not able to find a clear distinction yet between the dimensions, as is there is for the Happy Ending Problem. This leads to a lot of possible and exciting further research directions to answer the following question.

**Question 1.** *Does the behaviour of $n_{d,k\ell}$ change depending on $d$?*

The hope is that adequate techniques for $d = 3$ allow to prove the Conjecture in that dimension. Whether those techniques are also up for the task in the planar case would give a good indication whether such a distinction exists.
We have also looked into a generalisation as ordered hypergraphs and generalised visibility graphs. This generalisation was driven by the original statement of the conjecture, which also considered graphs. We reformulated the conjecture to this setting and gave a counterexample, proving that this formulation was too strong. However, this counterexample was not capable of saying something to the planar case. It seems like a stretch, but it would be great if this counterexample could be adapted such that we can embed in the plane.

**Question 2.** *Can the counterexample for the conjecture regarding visibility graphs be adapted such that it can be embedded in the plane?*

And besides trying to use the counterexample to say something about the conjecture in the planar setting, there is another incentive for giving the proposed graphs in that section a more detailed look, especially into their chromatic number.

**Question 3.** *Can we say something about the chromatic number of generalised visibility graphs?*

This question is driven by the fact that the very first conjecture posed by Kára et al. considered the chromatic number of graphs, while we have for now only considered the clique number. And even though this original conjecture was disproved, we believe it would be interesting to look further into the chromatic number of these graphs.

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