An Equation with a Time-periodic Damping Coefficient: stability diagram and an application

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Abstract

In this paper the second order differential equation with time-dependent damping coefficient

$$\ddot{x} + \epsilon \cos 2t\dot{x} + \lambda x = 0, \tag{0.1}$$

will be studied. In particular the coexistence of periodic solutions corresponding with the vanishing of domains of instability is investigated. This equation can be considered as a model equation for study of rain-wind induced vibrations of a special oscillator.

1 Introduction

In this paper we consider an inhomogeneous second order differential equation with time-dependent damping coefficient i.e.

$$\ddot{x} + (c + \epsilon \cos 2t)\dot{x} + (m^2 + \alpha)x + A \cos \omega t = 0 \tag{1.1}$$

where c, α, ϵ, A are small parameters and m, ω positive integers. A rather special property of equation (1.1) is that the coefficient of $\dot{x}$ is time dependent. For $m = 1$ and $A = 0$ some results especially related to the stability of the trivial solution can be found in [1]. Further for case $c = 0$ and $A = 0$ the equation (1.1) is a special case of Ince's equation (see [6], page 92 i.e. $a = 0$, $d = 0$ and $t \to t + \pi/4$). As is known, Ince's equation displays the phenomenon of coexistence of periodic solutions when m is an even integer. Coexistence implies that domains of instability disappear or in other words that an instability gap closes. The coexistence of periodic solutions of this equation will be studied in this paper. A new stability diagram is presented and the strained parameter is used to obtain approximations for the transition and the coexistence curves for small value of ϵ. Finally it is shown that (1.1) can be used as a model equation for the study of rain-wind induced vibrations of a special oscillator.

2 Coexistence of Time Periodic Solutions and the Stability Diagram

For the case $c = A = 0$ and replacing $m^2 + \alpha$ by λ, the equation (1.1) can be written as

$$\ddot{x} + (\epsilon \cos 2t)\dot{x} + \lambda x = 0. \tag{2.2}$$

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Transform x to the new variable y by

$$x = y \cdot e^{-\frac{1}{2}\int_0^t \epsilon \cos 2s ds} \tag{2.3}$$

to obtain a new equation of Hill's type:

$$\ddot{y} + (\lambda - \frac{1}{8}\epsilon^2 + \epsilon \sin 2t - \frac{1}{8}\epsilon^2 \cos 4t)y = 0. \tag{2.4}$$

The standard form of Hill's equation (in [6]) is

$$\ddot{y} + [\lambda + Q(t)]y = 0, \tag{2.5}$$

where λ is a parameter and Q is a real π-periodic function in t. Apparently (2.4) is of type (2.5) where $Q(t)$ depends additionally on a parameter ϵ. The determination of the value of λ for which the equation (2.5) has a π or 2π periodic solution can be related to the following theorem.
Theorem ([6] , page 11).
To every differential equation (2.5), there belong two monotonically increasing infinite sequences of real number $\lambda_o, \lambda_1, \lambda_2, \cdots$ and $\lambda'_1, \lambda'_2, \lambda'_3, \cdots$ such that (2.5) has a solution of period π if and only if $\lambda = \lambda_n, n = 0, 1, 2, \cdots$ and a solution of period 2π if and only if $\lambda = \lambda'_n, n = 1, 2, 3, \cdots$. λ_n and λ'_n satisfy the inequalities

$$\lambda_o < \lambda'_1 \le \lambda'_2 < \lambda_1 \le \lambda_2 < \lambda'_3 \le \lambda'_4 < \lambda_3 \le \lambda_4 < \cdots$$

and the relations

$$\lim_{n\to\infty} \lambda_n^{-1} = 0, \qquad \lim_{n\to\infty} (\lambda'_n)^{-1} = 0.$$

The solutions of (2.5) are stable [1] in the intervals

$$(\lambda_o, \lambda'_1), (\lambda'_2, \lambda_1), (\lambda_2, \lambda'_3), (\lambda'_4, \lambda_3), \cdots$$

At the endpoints of these intervals the solutions of (2.5) are, in general, unstable. The solutions of (2.5) are stable for $\lambda = \lambda_{2n+1}$ or $\lambda = \lambda_{2n+2}$ if and only if $\lambda_{2n+1} = \lambda_{2n+2}$, and they are stable for $\lambda = \lambda'_{2n+1}$ or $\lambda = \lambda'_{2n+2}$ if and only if $\lambda'_{2n+1} = \lambda'_{2n+2}$.

As described in [6], Hill's equation in general has only one periodic solution of period π or 2π. If the equation has two linearly independent solutions of period π or 2π, we say that two such solutions *coexist*. And then every solution of this equation can be expressed into linear combination of two periodic solutions in other words all solutions are bounded or they are stable. Thus the occurrence of coexisting periodic solutions is equivalent with the disappearance of intervals of instability. If for instance two linearly independent solutions of period π exist then the interval of instability $(\lambda_{2n+1}, \lambda_{2n+2})$ disappears, because $\lambda_{2n+1} = \lambda_{2n+2}$.
Further in [5] a special case of $Q(t)$ was studied, that is if $Q(t)$ in equation (2.5) has the form

$$Q(t) = \lambda_o + \dot{P}(t) + P^2(t) \tag{2.6}$$

where $P(t)$ is $\pi/2$-anti-periodic i.e. $P(t + \pi/2) = -P(t)$ then $\lambda_{2n+1} = \lambda_{2n+2}$ for all n.
Clearly equation (2.4) is of the form (2.6) with $P(t) = -\frac{1}{2}\epsilon \cos 2t$ and $\lambda_o = 0$, and $\cos 2t$ is $\pi/2$ anti-periodic. Thus coexistence in equation (2.4) exists for $\lambda = \lambda_{2n+1} = \lambda_{2n+2}$.

Unfortunately it is not known how to calculate exactly the value of λ for which equation (2.4) has a periodic solution. However one can approximate the value of λ by the following

[1] All solutions of (2.5) are bounded

method [2].
We consider a Fourier series representation of the periodic solution:

$$y = \frac{a_o}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt). \tag{2.7}$$

Substituting (2.7) into (2.4) yields

$$\begin{aligned}
(\lambda - \tfrac{1}{8}\epsilon^2)\tfrac{a_o}{2} + \epsilon \tfrac{a_o}{2} \sin 2t - \tfrac{1}{16}\epsilon^2 a_o \cos 4t + \\
\sum_{n=1}^{\infty}[(\lambda - \tfrac{1}{8}\epsilon^2 - n^2)a_n \cos nt + \\
(\lambda - \tfrac{1}{8}\epsilon^2 - n^2)b_n \sin nt] + \\
\tfrac{1}{2}\epsilon \sum_{n=1}^{\infty}[a_n \sin(n+2)t - a_n \sin(n-2)t \\
-b_n \cos(n+2)t + b_n \cos(n-2)t] \\
-\tfrac{1}{16}\epsilon^2 \sum_{n=1}^{\infty}[a_n \cos(n+4)t + a_n \cos(n-4)t + \\
b_n \sin(n+4)t + b_n \sin(n-4)t] &= 0.
\end{aligned} \tag{2.8}$$

Equating the coefficients of sinus and cosines to zero we have a system of infinitely many equations for a_n and b_n. To get a π-periodic solution we put the odd indices in (2.7) equal to zero and for a 2π-periodic solution we put the even indices equal to zero. In this way we obtain two systems

$$\mathbf{A}(\lambda, \epsilon)\mathbf{v} = \mathbf{0}, \qquad \text{and} \qquad \mathbf{B}(\lambda, \epsilon)\mathbf{w} = \mathbf{0}$$

where $\mathbf{A}(\lambda, \epsilon), \mathbf{B}(\lambda, \epsilon)$ are square matrices of infinite dimension and $\mathbf{v}$ is an infinite column vector where the elements are coefficients of (2.7) with odd indices and $\mathbf{w}$ is an infinite column vector where the elements are coefficients of (2.7) with even indices. To have a non trivial solution the determinants of $\mathbf{A}$ and $\mathbf{B}$ must be equal zero. These determinants define the curves in the $\epsilon - \lambda$ plane on which periodic solutions exist. However it is not possible to compute this curves from the determinants as they are of infinite dimension. Hence we consider (2.7) and truncate the series up to 16 modes from which determinants of finite dimension follow. In this determinants we choose ϵ from the interval $(0, 24)$ arbitrary but fixed. Subsequently the determinants are evaluated yielding an algebraic equation for λ which can be solved numerically. Along this way a new stability diagram as depicted in fig 1b. is obtained. In similar way the famous stability diagram of the Mathieu equation:

$$\ddot{y} + (\lambda + \epsilon \cos(2t))y = 0 \tag{2.9}$$

is obtained and presented in fig 1a. . One can observe remarkable differences between the two diagrams. Especially the curves starting in $\lambda = 4n^2$, $n = 1, 2, 3, \cdots$ on which two periodic solutions coexists are of interest.

In case ϵ is small we can use the strained parameter method ,as described in [9], to approximate the value of λ for which the equation (2.4) has periodic solutions. In this method we assume that λ can be expanded as

$$m^2 + \epsilon\alpha_1 + \epsilon^2\alpha_2 + \epsilon^3\alpha_3 + \cdots \tag{2.10}$$

where m is an integer number and the solution of (2.4) is expanded as

$$a_o \cos mt + b_o \sin mt + \epsilon y_1(t) + \epsilon^2 y_2(t) + \epsilon^3 y_3(t) + \cdots. \tag{2.11}$$

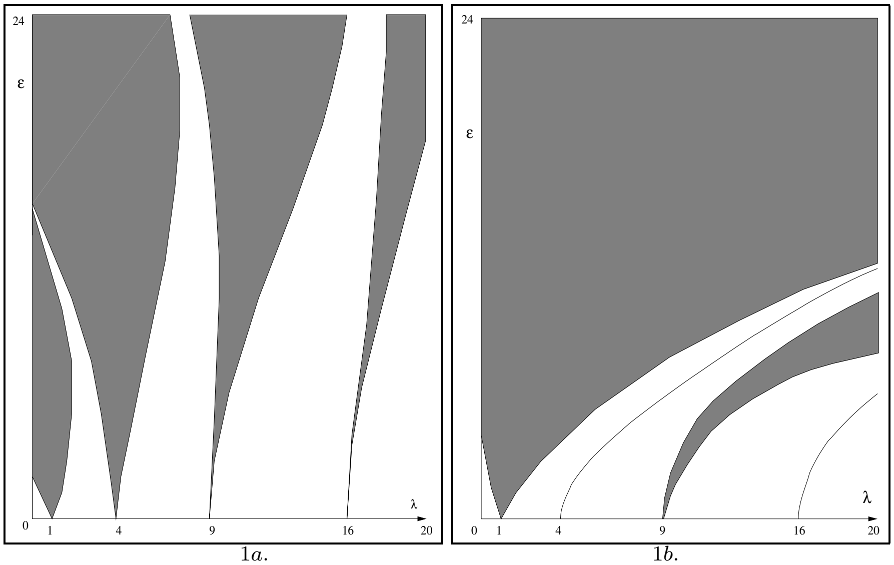


Figure 1: In the shaded regions the trivial solution is unstable. On the curves separating the white and shaded regions periodic solution exist. Figure 1a the Mathieu stability diagram. Figure 1b the new stability diagram.

Substituting (2.11) into (2.4) and eliminating the secular terms gives the values of $\alpha_i, i = 1, 2, 3, \cdots$. For instance $m = 1$ and $a_o = 1$ and $b_o = -1$ we obtain the value of λ up to order ϵ^8

$$\begin{aligned} \lambda_1' &= 1 - \tfrac{1}{2}\epsilon + \tfrac{3}{32}\epsilon^2 - \tfrac{3}{512}\epsilon^3 - \tfrac{3}{8192}\epsilon^4 + \tfrac{5}{141072}\epsilon^5 \\ &\quad - \tfrac{17}{4194304}\epsilon^6 - \tfrac{7}{134217728}\epsilon^7 - \tfrac{1}{16777216}\epsilon^8 + O(\epsilon^9). \end{aligned} \tag{2.12}$$

If $a_o = b_o = 1$, we get

$$\begin{aligned} \lambda_2' &= 1 + \tfrac{1}{2}\epsilon + \tfrac{3}{32}\epsilon^2 + \tfrac{3}{512}\epsilon^3 - \tfrac{3}{8192}\epsilon^4 - \tfrac{5}{141072}\epsilon^5 + \\ &\quad \tfrac{17}{4194304}\epsilon^6 + \tfrac{7}{134217728}\epsilon^7 - \tfrac{1}{16777216}\epsilon^8 + O(\epsilon^9). \end{aligned} \tag{2.13}$$

But for $m = 2$, by putting $a_o = 1$, $b_o = 0$ or $a_o = 0$, $b_o = 1$ one obtain the same result for λ that is

$$\lambda_1 = 4 + \tfrac{1}{6}\epsilon^2 - \tfrac{1}{3456}\epsilon^4 - \tfrac{1}{1244160}\epsilon^6 + \tfrac{11}{5733089280}\epsilon^8 + O(\epsilon^9). \tag{2.14}$$

For $m = 3$, $a_o = 1$, $b_o = 1$ we obtain

$$\begin{aligned} \lambda_3' &= 9 + \tfrac{9}{64}\epsilon^2 - \tfrac{3}{512}\epsilon^3 + \tfrac{9}{65536}\epsilon^4 + \tfrac{15}{524288}\epsilon^5 \\ &\quad - \tfrac{141}{33554432}\epsilon^6 - \tfrac{21}{536870912}\epsilon^7 + \tfrac{4101}{68719476736}\epsilon^8 + O(\epsilon^9), \end{aligned} \tag{2.15}$$

and for $a_o = 1, b_o = -1$ the result is

$$\begin{aligned} \lambda_4' &= 9 + \tfrac{9}{64}\epsilon^2 + \tfrac{3}{512}\epsilon^3 + \tfrac{9}{65536}\epsilon^4 - \tfrac{15}{524288}\epsilon^5 \\ &\quad - \tfrac{141}{33554432}\epsilon^6 + \tfrac{21}{536870912}\epsilon^7 + \tfrac{4101}{68719476736}\epsilon^8 + O(\epsilon^9). \end{aligned} \tag{2.16}$$

Finally for $m = 4$, the cases $a_o = 1, b_o = 0$ and $a_o = 0, b_o = 1$ have the same value of λ i.e.

$$\begin{aligned} \lambda_3 &= 16 + \tfrac{2}{15}\epsilon^2 + \tfrac{11}{108000}\epsilon^4 + \tfrac{1033}{1360800000}\epsilon^6 \\ &\quad - \tfrac{60703}{31352832000000}\epsilon^8 + O(\epsilon^9). \end{aligned} \tag{2.17}$$

Numerical	Analytical
$\lambda_1' = 0.587\mathbf{566498}$	$0.587\mathbf{555692}$
$\lambda_2' = 1.599\mathbf{209067}$	$1.599\mathbf{211767}$
$\lambda_1 = 4.166376513$	4.166376513
$\lambda_3' = 9.1349273\mathbf{78}$	$9.1349273\mathbf{83}$
$\lambda_4' = 9.14658899\mathbf{4}$	$9.14658899\mathbf{1}$
$\lambda_3 = 16.13343594$	16.13343594

Table 1: Comparison of the values of λ obtained with the numerical and the perturbation method for $\epsilon = 1$.

The approximations of λ_1' and λ_2' are given by (2.12) and (2.13) respectively. The approximation of λ_1 and λ_2 are the same and are given by (2.14). The expansions of λ_3' and λ_4' are given by (2.15) and (2.16) respectively, and finally the approximations of λ_3 and λ_4 are given by (2.17).

The analytical results as obtained above are compared with the numerical results as presented in fig 1b., for $\epsilon = 1$ in Table 1 . One can observe a striking resemblance.

The occurrence of the coexistence of periodic solutions in equation (2.4) depends on the periodicity of the coefficient of the damping term. As is known coexistence occurs when the coefficient of the damping term is $\pi/2$-anti periodic. So, if one perturbs the period then the coexistence does not occur anymore as is shown in the following example.
Consider the equation

$$\ddot{x} + (\epsilon \cos 2t + \epsilon b \cos t)\dot{x} + \lambda x = 0. \tag{2.18}$$

The period of the coefficient of the damping term is 2π if b is not equal zero, thus if one transform equation (2.18) in to Hill's type then this equation does not satisfy (2.6) i.e. $P(t + \pi/2) \neq -P(t)$ where $P(t) = -\frac{1}{2}(\epsilon \cos 2t + \epsilon b \cos t)$. So coexistence does not occur anymore, and the approximation of $\lambda_1', \lambda_2', \lambda_1, \lambda_2, \lambda_3', \lambda_4', \lambda_3$ and λ_4 (up to order $O(\epsilon^9)$) are given by

$$\begin{aligned}
\lambda_1' \quad = \quad & 1 - \tfrac{1}{2}\epsilon + (\tfrac{3}{32} + \tfrac{1}{6}b^2)\epsilon^2 - (\tfrac{3}{512} + \tfrac{1}{36}b^2)\epsilon^3 - (\tfrac{3}{8192} + \tfrac{7}{576}\, b^2 + \tfrac{1}{864}\, b^4)\epsilon^4 \\
& + (\tfrac{5}{131072} + \tfrac{11}{3072}\, b^2 + \tfrac{47}{13824}\, b^4)\epsilon^5 \\
& + (\tfrac{17}{4194304} - \tfrac{39}{573440}\, b^2 - \tfrac{653}{4976640}\, b^4 - \tfrac{1}{77760}\, b^6)\epsilon^6 \\
& - (\tfrac{7}{134217728} + \tfrac{187403}{2890137600}\, b^2 + \tfrac{430961}{1194393600}\, b^4 + \tfrac{3877}{18662400}\, b^6)\epsilon^7 \\
& + (-\tfrac{1}{16777216} + \tfrac{6431}{2055208960}\, b^2 + \tfrac{1259837}{17836277760}\, b^4 \\
& + \tfrac{10421}{627056640}\, b^6 + \tfrac{11}{89579520}\, b^8)\epsilon^8 + O(\epsilon^9)
\end{aligned} \tag{2.19}$$

$$
\begin{aligned}
\lambda_2' &= 1+\tfrac{1}{2}\epsilon+(\tfrac{3}{32}+\tfrac{1}{6}b^2)\epsilon^2+(\tfrac{3}{512}+\tfrac{1}{36}b^2)\epsilon^3-(\tfrac{3}{8192}+\tfrac{7}{576}\,b^2+\tfrac{1}{864}\,b^4)\epsilon^4\\
&+(\tfrac{5}{131072}+\tfrac{11}{3072}\,b^2+\tfrac{47}{13824}\,b^4)\epsilon^5\\
&+(\tfrac{17}{4194304}-\tfrac{39}{573440}\,b^2-\tfrac{653}{4976640}\,b^4-\tfrac{1}{77760}\,b^6)\epsilon^6\\
&-(\tfrac{7}{134217728}+\tfrac{187403}{2890137600}\,b^2+\tfrac{430961}{1194393600}\,b^4+\tfrac{3877}{18662400}\,b^6)\epsilon^7\\
&+(-\tfrac{1}{16777216}+\tfrac{6431}{2055208960}\,b^2+\tfrac{1259837}{17836277760}\,b^4+\tfrac{10421}{627056640}\,b^6\\
&+\tfrac{11}{89579520}\,b^8)\epsilon^8+O(\epsilon^9)
\end{aligned}
\tag{2.20}
$$

$$
\begin{aligned}
\lambda_1 &= 4+(\tfrac{1}{6}+\tfrac{2}{15}\,b^2)\epsilon^2-\tfrac{1}{36}b^2\epsilon^3+(-\tfrac{1}{3456}+\tfrac{1}{180}\,b^2+\tfrac{11}{27000}\,b^4)\epsilon^4\\
&+(-\tfrac{37}{64800}\,b^2-\tfrac{1}{1350}\,b^4)\epsilon^5\\
&-(\tfrac{1}{1244160}+\tfrac{79}{6531840}\,b^2+\tfrac{6397}{108864000}\,b^4+\tfrac{1033}{85050000}\,b^6)\epsilon^6\\
&+(\tfrac{1739}{232243200}\,b^2+\tfrac{7639}{51030000}\,b^4+\tfrac{409}{58320000}\,b^6)\epsilon^7\\
&+(\tfrac{11}{5733089280}-\tfrac{67}{470292480}\,b^2-\tfrac{19979}{261273600}\,b^4-\tfrac{864931}{48988800000}\,b^6\\
&-\tfrac{60703}{489888000000}\,b^8)\epsilon^8+O(\epsilon^9)
\end{aligned}
\tag{2.21}
$$

$$
\begin{aligned}
\lambda_2 &= 4+(\tfrac{1}{6}+\tfrac{2}{15}\,b^2)\epsilon^2+\tfrac{1}{36}b^2\epsilon^3+(-\tfrac{1}{3456}+\tfrac{1}{180}\,b^2+\tfrac{11}{27000}\,b^4)\epsilon^4\\
&+(-\tfrac{37}{64800}\,b^2-\tfrac{1}{1350}\,b^4)\epsilon^5\\
&-(\tfrac{1}{1244160}+\tfrac{79}{6531840}\,b^2+\tfrac{6397}{108864000}\,b^4-\tfrac{1033}{85050000}\,b^6)\epsilon^6\\
&-(\tfrac{1739}{232243200}\,b^2+\tfrac{7639}{51030000}\,b^4+\tfrac{409}{58320000}\,b^6)\epsilon^7\\
&+(\tfrac{11}{5733089280}-\tfrac{67}{470292480}\,b^2-\tfrac{19979}{261273600}\,b^4-\tfrac{864931}{48988800000}\,b^6\\
&-\tfrac{60703}{489888000000}\,b^8)\epsilon^8+O(\epsilon^9)
\end{aligned}
\tag{2.22}
$$

$$
\begin{aligned}
\lambda_3' &= 9+(\tfrac{9}{64}+\tfrac{9}{70}b^2)\epsilon^2-\tfrac{3}{512}\epsilon^3+(\tfrac{9}{65536}+\tfrac{9}{4480}\,b^2+\tfrac{279}{1372000}\,b^4)\epsilon^4\\
&+(\tfrac{15}{524288}-\tfrac{1311}{1254400}\,b^2-\tfrac{3}{12800}\,b^4)\epsilon^5\\
&+(-\tfrac{141}{33554432}+\tfrac{3207}{50462720}\,b^2+\tfrac{17789}{351232000}\,b^4+\tfrac{5953}{10084200000}\,b^6)\epsilon^6\\
&+(-\tfrac{21}{536870912}+\tfrac{19287}{1284505600}\,b^2-\tfrac{20945241}{786759680000}\,b^4-\tfrac{93}{31360000}\,b^6)\epsilon^7\\
&+(\tfrac{4101}{68719476736}-\tfrac{569953}{180858388480}\,b^2+\tfrac{6165641}{1186883174400}\,b^4+\tfrac{25654589}{28397107200000}\,b^6\\
&+\tfrac{171697}{316240512000000}\,b^8)\epsilon^8+O(\epsilon^9)
\end{aligned}
\tag{2.23}
$$

$$
\begin{aligned}
\lambda_4' \quad = \quad & 9+(\tfrac{9}{64}+\tfrac{9}{70}b^2)\epsilon^2+\tfrac{3}{512}\epsilon^3+(\tfrac{9}{65536}+\tfrac{9}{4480}\,b^2+\tfrac{279}{1372000}\,b^4)\epsilon^4 \\
& -(\tfrac{15}{524288}-\tfrac{1311}{1254400}\,b^2-\tfrac{3}{12800}\,b^4)\epsilon^5 \\
& +(-\tfrac{141}{33554432}+\tfrac{3207}{50462720}\,b^2+\tfrac{17789}{351232000}\,b^4+\tfrac{5953}{10084200000}\,b^6)\epsilon^6 \\
& -(-\tfrac{21}{536870912}+\tfrac{19287}{1284505600}\,b^2-\tfrac{20945241}{786759680000}\,b^4-\tfrac{93}{31360000}\,b^6)\epsilon^7 \\
& +(\tfrac{4101}{68719476736}-\tfrac{569953}{180858388480}\,b^2+\tfrac{6165641}{1186883174400}\,b^4+\tfrac{25654589}{28397107200000}\,b^6 \\
& +\tfrac{171697}{316240512000000}\,b^8)\epsilon^8+O(\epsilon^9)
\end{aligned} \tag{2.24}
$$

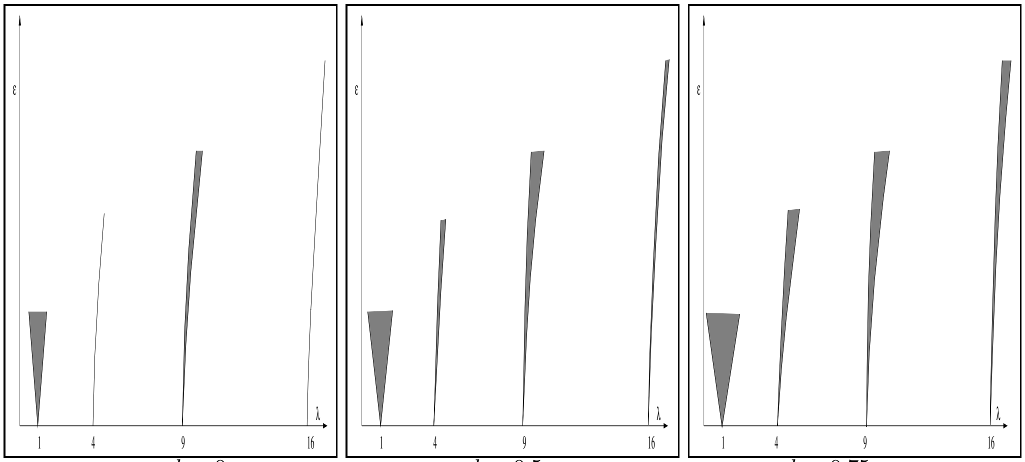
$$
\begin{aligned}
\lambda_3 \quad = \quad & 16+(\tfrac{2}{15}+\tfrac{8}{63}b^2)\epsilon^2+(\tfrac{11}{108000}+\tfrac{1}{945}\,b^2+\tfrac{59}{500094}\,b^4)\epsilon^4 \\
& -\tfrac{25}{127008}b^2\epsilon^5+(\tfrac{1033}{1360800000}+\tfrac{58031}{5837832000}\,b^2+\tfrac{19363}{1584297792}\,b^4+\tfrac{19561}{218336039460}\,b^6)\epsilon^6 \\
& -(\tfrac{1}{529200}\,b^2+\tfrac{61069}{6301184400}\,b^4+\tfrac{1}{1411200}\,b^6)\epsilon^7 \\
& +(-\tfrac{60703}{31352832000000}+\tfrac{10021589}{73556683200000}\,b^2+\tfrac{2034457}{41191742592000}\,b^4+\tfrac{7397773}{74313648339840}\,b^6 \\
& +\tfrac{41146789}{110921694798942720}\,b^8)\epsilon^8+O(\epsilon^9)
\end{aligned} \tag{2.25}
$$

$$
\begin{aligned}
\lambda_4 \quad = \quad & 16+(\tfrac{2}{15}+\tfrac{8}{63}b^2)\epsilon^2+(\tfrac{11}{108000}+\tfrac{1}{945}\,b^2+\tfrac{59}{500094}\,b^4)\epsilon^4 \\
& +\tfrac{25}{127008}b^2\epsilon^5+(\tfrac{1033}{1360800000}+\tfrac{58031}{5837832000}\,b^2+\tfrac{19363}{1584297792}\,b^4+\tfrac{19561}{218336039460}\,b^6)\epsilon^6 \\
& +(\tfrac{1}{529200}\,b^2+\tfrac{61069}{6301184400}\,b^4+\tfrac{1}{1411200}\,b^6)\epsilon^7 \\
& +(-\tfrac{60703}{31352832000000}+\tfrac{10021589}{73556683200000}\,b^2+\tfrac{2034457}{41191742592000}\,b^4+\tfrac{7397773}{74313648339840}\,b^6 \\
& +\tfrac{41146789}{110921694798942720}\,b^8)\epsilon^8+O(\epsilon^9)
\end{aligned} \tag{2.26}
$$

One can easily check that for $b \to 0$ (2.19)-(2.26) reduce to (2.12)-(2.17). It can be shown that for $b \neq 0$ the stability diagram of equation (2.18) has a similar geometry for ϵ small as the stability diagram of the Mathieu equation (see fig 1.). The areas of instability depend on the parameter b in the coefficient of the damping term. As depicted in fig 2. one may observe that when b goes to zero the areas of instability become narrower and finally when b equals zero the areas of instability vanish especially for $\lambda = 4n^2$, $n = 1, 2, 3, \cdots$. This phenomenon has been described for an equation which differs from the one presented here in [10, 11].

3 An application in the theory of rain-wind induced vibrations.

In this section an application is given of the results obtained above. The application is concerned with the rain-wind induced vibrations of a simple one degree of freedom system related to the dynamics of cable-stayed bridges. Firstly it will be shown how to model this problem in order to obtain a model equation of the form (1.1). Cable-stayed bridges are



$b = 0$ $\qquad$ $b = 0.5$ $\qquad$ $b = 0.75$

Figure 2: Stability diagram of equation (2.18) for various value of b. The shaded regions are areas of instability. When $b = 0$ the instability areas have disappeared for $\lambda = 4n^2$.

characterized by inclined stay cables connecting the bridge deck with one or more pylons. Usually the stay cables have a smooth polyurethane mantle and a cross section which is nearly circular. Under normal circumstances for such type of cables one would not expect galloping type of vibrations due to wind-forces. There are however exceptions: in the winter season ice accretion on the cable may induce aerodynamic instability resulting in vibrations with relatively large amplitudes. The instability mechanism for this type of vibrations is known and can be understood on the basis of quasi-steady modeling and analysis. In this analysis the so-called Den Hartog's criterion expressing a condition to have an unstable equilibrium state plays an important part. The other exception concerns vibrations excited by a wind-field containing raindrops. This phenomenon has probably been detected for the first time by Japanese researchers as can be derived from the papers by Matsumoto a.o. [7, 8]. As has been observed on scale models in wind-tunnels the raindrops that hit the inclined stay cable generate one or more rivulets on the surface of the cable. The presence of flowing water on the cable changes the cross section of the cable as experienced by the wind field. Accordingly the pressure distribution on the cable with respect to the direction of the (uniform) wind flow may became asymmetric, resulting in a lift force perpendicular to direction of the wind velocity.

It is of interest to remark that there is an important difference between the presence of ice accretion and rivulets as far as it concerns the dynamical behaviour. The ice accretion concerns an ice coating fixed to the surface of the cable whereas the rivulet concerns a flow of water on the surface of the cable where the position of the rivulet depends on the resulting wind velocity, the surface tension of the water and the adhesion between the water and polyurethane mantle of the cable.

For the interesting cases the thickness of the ice accretion is not uniform: the evolution process of ice accretion usually results in an ice coating involving a ridge of ice. The case with water rivulets can also be characterized by the presence of the ridge of water be it with the difference that this water ridge is not fixed to the surface of the cable. As long as the water ridge is present, it may be blown off if the wind-speed exceeds a critical value, one may assume that the position of the ridge varies in time. Subsequently one may assume that this time-dependence has a similar character as the motion of the cable i.e. if the cable oscillates harmonically then one may expect that the water ridge moves accordingly. The observation of this complicated system of an inclined cable, connecting a bridge deck and a pylon, with a moving rivulet leads to the following conclusion.

The inclination of the cable is relevant for having a rivulet. The rivulet however can be viewed as a moving ridge which may be modeled by a solid state. According this way

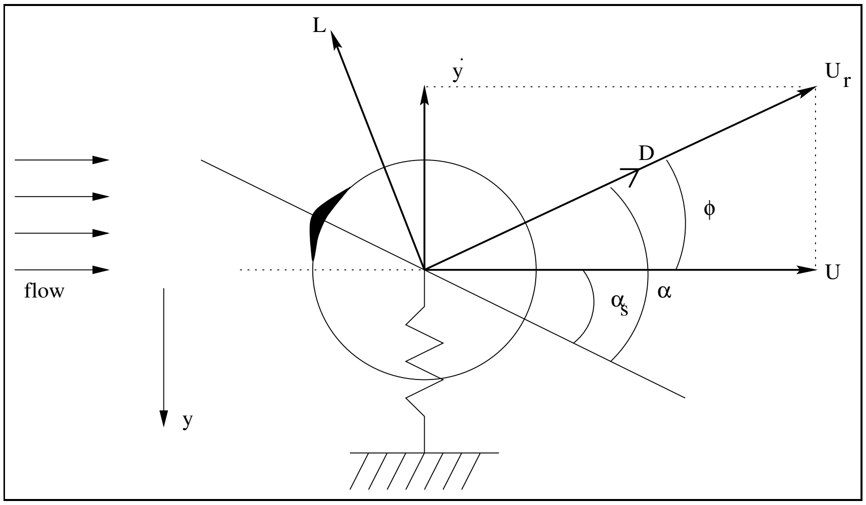


Figure 3: Cross-section of the cylinder-spring system, fluid flow with respect to the cylinder and wind forces on the cylinder

of modeling the inclination of the cable is no longer relevant. Hence we consider as a prototype of an oscillator a one degree of freedom system consisting of a horizontal rigid cable supported by springs with a solid state ridge moving with small amplitude oscillations. From the point of view of the type of equation of motion, we arrive at a second order differential equation with external forcing. A more detailed description of the modeling is presented in the following section.

3.1 The Model Equation for Rain-Wind Induced Vibrations of a Prototype Oscillator

The modeling principles we use are closely related to the quasi-steady approach as given in [3]. We consider a rigid cylinder with uniform cross-section supported by springs in a uniform rain-wind flow directed perpendicular to the axis of the cylinder. The oscillator is constructed in such a way that only vertical (one degree of freedom) oscillations are possible. The basic cross-section of the cylinder is circular, however on the surface of the cylinder there is a ridge able to carry out small amplitude oscillations. To model the rain-wind forces on the cylinder a quasi-steady approach is used; the type of oscillations which can be studied on the respective assumptions are known as galloping. A more detailed description of the quasi-steady approach can be found in [13]. The basic assumption of the quasi-steady approach is that at each moment in the dynamic situation the rain-wind force can be taken equal to the steady force exerted on the cylinder in static state. In the dynamic situation one should take into account that the flow-induced forces are based on the instantaneous flow velocity which is equal to the vector sum of flow velocity and the time varying vertical flow velocity induced by the (vertical) motion of the cylinder.
The steady rain-wind forces can be measured in a wind-tunnel and are expressed in the form of non-dimensional aerodynamic coefficients which depend on the angle of attack α. This angle, an essential variable for the description of the dynamic of the oscillator, is defined as the angle between the resultant flow velocity and an axis of reference fixed to the cylinder; measured positive in clockwise direction. The system we will study is more detail is sketched in fig 3.

The horizontal wind velocity is U and as the cylinder is supposed to move in the positive y direction, there is a virtual vertical wind velocity $-\dot{y}$. The drag force D is indicated in the direction of the resultant wind-velocity U_r, whereas the lift force L is perpendicular to D in anti clockwise direction. The ridge on the cylinder shaded indicated in fig 3. is able to carry out small amplitude oscillations. The aerodynamic force F_y in vertical direction can easily be derived from fig 3. :

$$F_y = -D\sin\phi - L\cos\phi \tag{3.27}$$

where ϕ is the angle between U_r and U, positive in clockwise direction, with $|\phi| \leq \pi/2$.
The drag and lift force are given by the empirical relations:

$$\begin{aligned} D &= \frac{1}{2}\rho\ d\ l\ U_r^2\ C_D(\alpha) \\ L &= \frac{1}{2}\rho\ d\ l\ U_r^2\ C_L(\alpha) \end{aligned} \tag{3.28}$$

where ρ is the density of air, d the diameter of the cylinder, l the length of the cylinder, $C_D(\alpha)$ and $C_L(\alpha)$ are the drag and lift coefficient curves respectively, determined by measurements in a wind-tunnel.

From fig 3. it follows that :

$$\begin{aligned} \sin\phi &= \dot{y}/U_r \\ \cos\phi &= U/U_r \\ \alpha &= \alpha_s + \arctan(\dot{y}/U) \end{aligned} \tag{3.29}$$

The equation of motion of the oscillator readily becomes :

$$m\ddot{y} + c_y\dot{y} + k_y y = F_y, \tag{3.30}$$

where m is the mass of the cylinder, $c_y > 0$ the structural damping coefficient of the oscillator, $k_y > 0$ the spring constant.
By using (3.28) and (3.29) we obtain for F_y:

$$F_y = -\frac{1}{2}\rho\ d\ l\ \sqrt{U^2 + \dot{y}^2}\ (C_D(\alpha)\dot{y} + C_L(\alpha)U) \tag{3.31}$$

Setting $\omega_y^2 = k_y/m$, $\tau = \omega_y t$ and $z = \omega_y y/U$ equation (3.30) becomes:

$$\begin{aligned} \ddot{z} + 2\beta\dot{z} + z &= -K\ \sqrt{1 + \dot{z}^2}\ (C_D(\alpha)\dot{z} + C_L(\alpha)) \\ \alpha &= \alpha_s + \arctan(\dot{z}) \end{aligned} \tag{3.32}$$

where $2\beta = c_y/m\omega_y$ and $K = \rho\ d\ lU/2m\omega_y$ are non-dimensional parameters, and $\dot{z}$ now stands for differentiation with respect to τ.
We study the case where the drag and lift coefficient curve can be approximated by a constant and a cubic polynomial respectively:

$$\begin{aligned} C_D(\alpha) &= C_{D_o} \\ C_L(\alpha) &= C_{L_1}(\alpha - \alpha_o) + C_{L_3}(\alpha - \alpha_o)^3, \end{aligned} \tag{3.33}$$

where $C_{D_o} > 0$ and for the interesting cases $C_{L_1} < 0$ and $C_{L_3} > 0$. By using $\alpha = \alpha_s + \arctan\dot{z}$ we obtain for $C_L(\alpha)$:

$$C_L(\alpha) = C_{L_1}(\alpha_s - \alpha_o + \arctan\dot{z}) + C_{L_3}(\alpha_s - \alpha_o + \arctan\dot{z})^3 \tag{3.34}$$

The cases that $\alpha_s = \alpha_o$ and $\alpha_s \neq \alpha_o$ where α_s and α_o are (time independent) parameters have been studied in [3]. Here we study the case that the position of the (water) ridge varies with time:

$$\alpha_s - \alpha_o = f(t) = f(\tau/\omega_y) \tag{3.35}$$

Substitution of (3.34) and (3.35) in (3.32) and expanding the right hand side with respect to $\dot{z}$ in the neighbourhood of $\dot{z} = 0$ yields:

$$\begin{aligned} \ddot{z} + z \;=\; & -K[C_{L_1} f(t) + C_{L_3} f^3(t) + \\ & (C_{D_o} + C_{L_1} + 2\beta/K + 3C_{L_3} f^2(t))\ \dot{z} + \\ & (\frac{1}{2}C_{L_1} f(t) + \frac{1}{2}C_{L_3} f^3(t) + 3C_{L_3} f(t))\ \dot{z}^2 + \\ & (\frac{1}{6}C_{L_1} + C_{L_3} + \frac{1}{2}C_{D_o} + \frac{1}{2}C_{L_3} f^2(t))\ \dot{z}^3] + 0(\dot{z}^4) \end{aligned} \tag{3.36}$$

Inspection of this equation shows that for $f(t) \equiv 0$ one obtains:

$$\ddot{z} + z = K[-(C_{D_o} + C_{L_1} + 2\beta/K)\ \dot{z} - (\frac{1}{6}C_{L_1} + C_{L_3} + \frac{1}{2}C_{D_o})\ \dot{z}^3]. \tag{3.37}$$

When the following conditions hold :

$$\begin{aligned} C_{D_o} + C_{L_1} + 2\beta/K \;&<\; 0 \text{ (Den Hartog's Criterion)} \\ \frac{1}{6}C_{L_1} + C_{L_3} + \frac{1}{2}C_{D_o} \;&>\; 0 \end{aligned} \tag{3.38}$$

the equation can be reduced to the Rayleigh equation, which has, as is well-known, a unique periodic solution (limit-cycle). The linearized version of equation (3.37) has apart from $z \equiv 0$ only unbounded solutions if Den Hartog's criterion applies. Linearization of equation (3.36) however leads to an equation which may have periodic solutions and is hence of interest to study in more detail.

The linearized version of (3.36) can be written as :

$$\begin{aligned} \ddot{z} + K(C_{D_o} + C_{L_1} + 2\beta/K + 3C_{L_3} f^2(t))\ \dot{z} + z \quad & + \\ K(C_{L_1} f(t) + C_{L_3} f^3(t)) \quad & = \quad 0. \end{aligned} \tag{3.39}$$

We consider the case that $f(t) = A\cos\omega t = A\cos(\frac{\omega}{\omega_y}\tau) = A\cos\Omega\tau$ where $\Omega = \frac{\omega}{\omega_y}$ with

$$\begin{aligned} f^2(t) \;&=\; \frac{1}{2}A^2(1 + \cos 2\Omega\tau) \qquad \text{and} \\ f^3(t) \;&=\; \frac{3}{4}A^3(\cos\Omega\tau + \frac{1}{3}\cos 3\Omega\tau) \end{aligned}$$

(3.39) becomes:

$$\begin{aligned} \ddot{z} + (KA_o + KA_1\cos 2\Omega\tau)\ \dot{z} + z \quad & + \\ KA_2\cos\Omega\tau + KA_3\cos 3\Omega\tau \quad & = \quad 0 \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} A_o \;&=\; C_{D_o} + C_{L_1} + 2\beta/K + \frac{3}{2}C_{L_3}A^2, \\ A_1 \;&=\; \frac{3}{2}C_{L_3}A^2, \\ A_2 \;&=\; C_{L_1}A + \frac{3}{4}C_{L_3}A^3, \\ A_3 \;&=\; \frac{1}{4}C_{L_3}A^3. \end{aligned}$$

For the oscillator we study the interesting case $\Omega = 1 + \epsilon\eta$ where $|\epsilon| \ll 1$. By setting $(1 + \epsilon\eta)\tau = \theta$ (3.40) becomes:

$$\begin{aligned} (1 + \epsilon\eta)^2\ \ddot{z} + (1 + \epsilon\eta)(KA_o + KA_1\cos 2\theta)\ \dot{z} + z \quad & + \\ KA_2\cos\theta + KA_3\cos 3\theta \quad & = \quad 0 \end{aligned} \tag{3.41}$$

where a dot now stands for differentiation with respect to θ.
Let the coefficients $KA_i \;\; i = 0, 1, 2, 3$ be of $O(\epsilon)$. Then (3.41) can be written as:

$$\begin{aligned} \ddot{z} + (KA_o + KA_1 \cos 2\theta)\ \dot{z} + (1 - 2\epsilon\eta)z \quad & + \\ KA_2 \cos\theta + KA_3 \cos 3\theta + O(\epsilon^2) \quad & = \quad 0. \end{aligned} \tag{3.42}$$

If one neglects the $O(\epsilon^2)$ term then the only difference between equation (3.42) and equation (1.1)(for $m = 1$) is the term $KA_3 \cos 3\theta$. This term can be regarded as a forcing term, but as the frequency is three times greater than the natural frequency it is not relevant for the $O(\epsilon)$ approximation. Putting $KA_o = a_o\epsilon$, $KA_1 = a_1\epsilon$, $KA_2 = a_2\epsilon$, and $KA_3 = a_3\epsilon$ and neglecting $O(\epsilon^2)$ of (3.42) one obtains:

$$\ddot{z} + \epsilon(a_o + a_1 \cos 2\theta)\ \dot{z} + (1 - 2\epsilon\eta)z + a_2 \cos\theta + a_3 \cos 3\theta \quad = \quad 0. \tag{3.43}$$

Transforming (3.43) by new variables y_1 and y_2 i.e.

$$\begin{aligned} z \quad & = \quad y_1 \cos\theta + y_2 \sin\theta \\ \dot{z} \quad & = \quad -y_1 \sin\theta + y_2 \cos\theta, \end{aligned} \tag{3.44}$$

one obtains by averaging:

$$\begin{pmatrix} \dot{\bar{y}}_1 \\ \dot{\bar{y}}_2 \end{pmatrix} = \epsilon \begin{pmatrix} -\frac{1}{2}a_o + \frac{1}{4}a_1 & -\eta \\ \eta & -\frac{1}{2}a_o - \frac{1}{4}a_1 \end{pmatrix} \begin{pmatrix} \bar{y}_1 \\ \bar{y}_2 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ -\frac{1}{2}a_1 \end{pmatrix}. \tag{3.45}$$

The critical point of (3.45) is

$$\left(\frac{\frac{1}{2}\eta a_2}{\frac{1}{4}a_o^2 - \frac{1}{16}a_1^2 + \eta^2}, \quad \frac{\frac{1}{2}a_2(-\frac{1}{2}a_o + \frac{1}{4}a_1)}{\frac{1}{4}a_o^2 - \frac{1}{16}a_1^2 + \eta^2} \right).$$

If the determinant of the coefficient matrix of (3.45) is not equal to zero then (3.43) has a periodic solution and its stability depends on the eigenvalues of the coefficient matrix . The eigenvalues of the coefficient matrix are

$$\frac{1}{2}\epsilon \left(-a_o \pm \sqrt{\frac{1}{4}a_1^2 - 4\eta^2} \right).$$

By equating these eigenvalues to zero one obtains the transition curves between the stable and unstable regions $a_o = \sqrt{\frac{1}{4}a_1^2 - 4\eta^2}$ in the $a_1 - \eta$ plane. In case $a_o = 0$ the stability diagram is depicted in fig 4a. and if $a_o > 0$ then the instability area separates from η-axis with a distance $2a_o$ as shown in fig 4b. In similar way the instability tongues are separated from the η-axis for $\lambda = (2n+1)^2 \;\; n = 0, 1, 2, \cdots$. Additionally it can be shown that for $\lambda = 4n^2$ the curves of coexistence of periodic solutions disappear for $a_o > 0$

4 Conclusion

In this paper a linear second order equation with time-periodic damping coefficient is investigated. It is shown that the equation can be used as a model for study of rain-wind induced vibrations of a simple oscillator. The equation is a special case of Ince's equation. It is known that this equation displays coexistence, corresponding with curves in the stability diagram on which two linearly independent periodic solutions exist. These curves can be considered as a limiting case of the closure of instability gaps. Although this phenomenon has been described in [6] in qualitative sense, little quantitative results such as stability diagrams have been obtained. A new remarkable stability diagram is presented in fig 1b; for small values of ϵ the (transition) curves are additionally given by (truncated) power

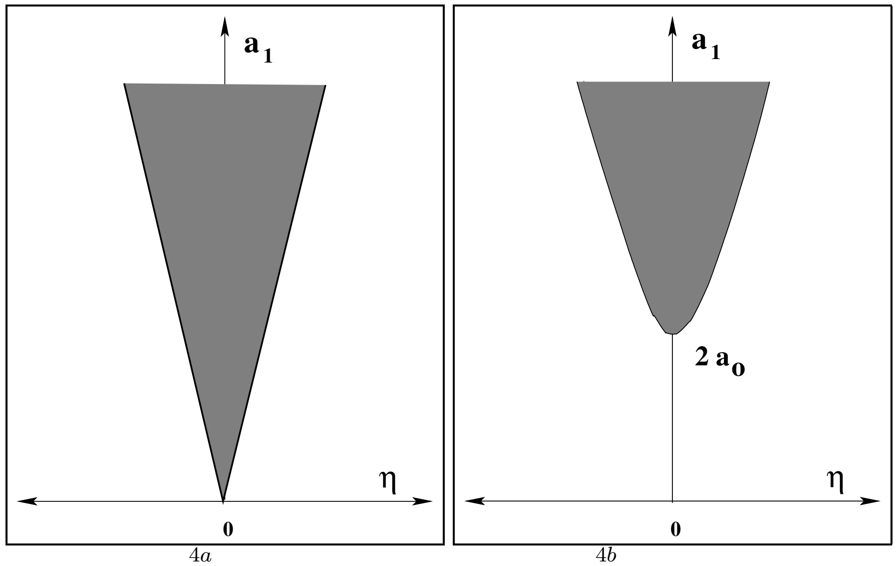


Figure 4: Separation of the instability tongue: fig 4a: $a_o = 0$ fig 4b: $a_o > 0$

series in ϵ. As far as it concerns the application it seems that only one application in [11] has been published yet. The application presented here seems to be new and is of practical relevance. Problems with a time-varying damping coefficient play a role in the dynamics of rain-wind induced vibrations of elastic structures and are hence of considerable interest. To evaluate the respective model equation laboratory experiments could be set up with a time varying electromagnetic damping device. Finally one may conclude that now the way looks open to investigate more complicated oscillators including the ones with two or more degrees of freedom.

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