**Technische Universiteit Delft**
**Faculteit Elektrotechniek, Wiskunde en Informatica**
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# Bepaaldheid van Oneindige Spellen

## (Engelse titel: Determinacy of Infinite Games)

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**"Bepaaldheid van Oneindige Spellen**
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## 0 Abstract

This paper introduces the notion of infinite games, i.e., games in which two players take turns playing moves ad infinitum, so that player I wins if the sequence of moves is in a predetermined payoff set. Theorems are then provided about whether a player in such games can have a winning strategy. The first theorem, Gale-Stewart, shows that games with an open or closed payoff set are determined, i.e., one of the players has a winning strategy. The second theorem, Martin, shows moreover that any game with a Borel payoff set is determined. Finally, the paper presents some results that follow from these theorems, for instance that the Continuum Hypothesis holds for all Borel sets. This paper only requires knowledge of very basic set theory and will clearly define any new or otherwise unfamiliar concepts.

# Contents

# 1 Introduction

In this paper we will investigate infinite games of perfect information between two players to see whether a winning strategy exists for one of the players. If such a strategy exists, then we call such a game determined. In particular we will investigate the determinacy of Gale-Stewart games. These games are of mathematical interest, because they can be used to show that all Borel sets have the perfect set property, i.e., the continuum hypothesis holds for them. The study of what 'well-behaved' sets have regularity properties such as the perfect set property, the Baire property, Lebesgue measurability, etc. is central to the field of Descriptive Set Theory.

Informally, a Gale-Stewart game is an infinite game where two players alternate playing moves from a given set $A$, where we say that the player who goes first, player I, wins if the sequence of moves lies in a predetermined payoff set $X$. Such a game can be represented in the following way:

$$\begin{array}{lccccc} \text{I} & a_0 & & a_2 & & \dots \\ \text{II} & & a_1 & & a_3 & \end{array}$$

A lot of work has been done on these games already. D. Gale and F.M. Stewart initially showed in 1953 that games with open or closed payoff sets are determined [4]. In the ensuing years it was shown that countable unions of closed sets and countable intersections of open sets are also determined. Almost 20 years later D.A. Martin proved that all Borel sets are determined [9]. Before Martin's proof however, H. Friedman had shown in 1971 that any proof of Borel determinacy in Zermelo-Fraenkel set theory requires the axiom of replacement to be used at least $\omega_1$ many times [3]. We will define $\omega_1$ in the next chapter.

We will look at proofs of the theorems by Gale-Stewart and Martin, and show how it follows that all Borel sets have the perfect set property. In order to do this however, we will need to establish what framework of set theory we use.

G. Cantor and R. Dedekind pioneered what is now often referred to as naive set theory, in which sets were defined as collections of definite, distinguishable objects. As the name might imply, this theory lead to many inconsistencies like Russel's paradox. The idea of the paradox is to consider the set of all sets that do not contain themselves $R = \{S : S \notin S\}$. Clearly, if $R$ contains itself, then it does not contain itself. However, if $R$ does not contain itself, then it does contain itself. As a result of such paradoxes, an effort was made in the early 1900s to create an axiomatic approach to set theory that would salvage the desired aspects of Cantor's theory. One of the axiomatic approaches that resulted from this effort was Zermelo-Fraenkel set theory, which has the following axioms:

1. Axiom of Extensionality
2. Axiom of Pairing
3. Axiom of Restricted Comprehension
4. Axiom of Union
5. Axiom of Power Set
6. Axiom of Infinity
7. Axiom Schema of Replacement
8. Axiom of Regularity
9. Axiom of Choice

It is common to use ZF to denote set theory that uses 1-8 and ZFC to denote ZF together with the Axiom of Choice (AC). ZFC is the theory in which Borel determinacy was proved, and is also the most widely accepted form of set theory. In the rest of the paper we will work in ZFC, unless clearly stated.

It should be mentioned that one could accept the Axiom of Determinacy (AD), which states that all Gale-Stewart games are determined. This has some interesting consequences, for instance that AC is inconsistent in ZF+AD. This follows from the fact that we can use AC to construct undetermined games.

The structure of the paper will be as follows. In chapter 2 some basic concepts from set theory are introduced, such as set descriptive trees, ordinal numbers and the Borel hierarchy. The notation and proofs are loosely based on those in *Classical Descriptive Set Theory* [7] and *Set Theory* [6]. Then in chapter 3, Gale-Stewart games are developed and some examples are given, proofs of Gale-Stewart and its equivalence to AC are given, and finally proofs of Borel determinacy and the perfect set property for Borel sets are given. The paper is concluded with a discussion on possible further research.

# 2 Basic Concepts from Set Theory

In this chapter we will look at some basic concepts from set theory. We will need these in the next chapter not only for the proofs, but also the precise definition of a Gale-Stewart game. First we will look at ordinal numbers, which will allow us to order sets and use that order to apply transfinite induction. We will also need ordinal numbers to define the Borel hierarchy, a useful classification of the different complexities of Borel sets. Lastly, we introduce trees from descriptive set theory, which are essential to almost everything in the next chapter.

## 2.1 Ordinal and Cardinal Numbers

In this section we develop a way to characterize the order and size of a set as an ordinal and cardinal number respectively. Since cardinal numbers will be described in terms of ordinal numbers we first look at ordinal numbers.

### 2.1.1 Ordinal Numbers

First, let us establish what an ordering is, before we define ordinal numbers.

**Definition 2.1.** *A binary relation $<$ on a set $A$ is a* ***linear ordering*** *on $A$ if for all $x, y, z \in A$ we have the following:*

1. *$x \not< x$,*
2. *If $x < y$ and $y < z$, then $x < z$,*
3. *Either $x < y$ or $y < x$ or $x = y$.*

*A linear ordering $<$ is a* ***well-ordering*** *if every nonempty $B \subset A$ has a least element, i.e., there is an $x \in B$ such that for all $y \in B$ either $x < y$ or $x = y$.*

We can now give the following two definitions.

**Definition 2.2.** *A set $T$ is* ***transitive*** *if for all $x \in T$, we have $x \subset T$.*

**Definition 2.3.** *A set $\alpha$ is an* ***ordinal number****, or ordinal for short, if it is transitive and well-ordered by $\in$. The class of all ordinals is denoted by Ord.*

It is common to denote ordinals by Greek letters and instead of writing $\alpha \in \beta$ we will write $\alpha < \beta$. The existence of ordinals, or most sets in general, is not completely trivial. The only axiom that guarantees us the existence of any set is the Axiom of Infinity, which states the following:

$$\exists I(\emptyset \in I \wedge \forall x \in I(x \cup \{x\} \in I)).$$

A set such as $I$ is called an **inductive set**. By using the Axiom of Separation we can then extract the ordinals $\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \dots$ which are usually denoted by $0, 1, 2, \dots$. We can also extract the first infinite ordinal $\{0, 1, 2, \dots\}$ which is usually denoted by $\omega$ or $\omega_0$. By repeated use of the Axiom of Power Set and the Axiom of Replacement we can reach larger and larger ordinals. The first uncountable ordinal is denoted by $\omega_1$.

We now give some powerful theorems back to back without proof. For the interested reader, proofs can be found in *Set Theory* [6].

**Theorem 2.1.** *Every well-ordered set is isomorphic to a unique ordinal. This ordinal is referred to as the order-type of the set.*

**Theorem 2.2** (Zermelo's Theorem)**.** *There exists a well-ordering for every set.*

**Theorem 2.3** (Transfinite Induction)**.** *Let $C$ be a class of ordinals such that,*

1. $0 \in C$,
2. *given $\alpha \in Ord$, if for all ordinals $\beta < \alpha$ we have $\beta \in C$, then $\alpha \in C$.*

*Then $C$ is the class of all ordinals Ord.*

Theorem 2.3 is especially powerful as it will allow us to use induction past the natural counting numbers, which we will need for our proof of Borel determinacy since there are Borel sets with complexity larger than $\omega$.

### 2.1.2 Cardinal Numbers

Cardinal numbers are a way to assign a number to the size of sets. We say that two sets have the same cardinality if there is a bijection between them. Since bijections composed with other bijections are again bijections, this gives us a way to partition all sets into equivalence classes of different cardinalities. These equivalence classes allow us to define cardinal numbers as their representatives.

**Definition 2.4.** *Given a set $A$, let $\alpha$ be the smallest ordinal with a bijection to $A$, which exists by theorems 2.1 and 2.2. We say $\alpha$ is the* ***cardinality*** *of $A$ and we write $|A| = \alpha$. A* ***cardinal number*** *is an ordinal $\alpha$ where $|\alpha| = \alpha$.*

Given two sets $A$ and $B$ with the same cardinality we write $|A| = |B|$. If there is a one-to-one map from $A$ to $B$ we write $|A| \leq |B|$. To see that this notation makes sense we need the following theorem.

**Theorem 2.4** (Cantor-Bernstein)**.** *If $A$ and $B$ are sets such that $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

A proof can once again be found in *Set Theory* [6].

Lastly, we define the **alephs** as the infinite ordinals that are cardinals. The first aleph, $\aleph_0$, has order-type $\omega$. The order-types of subsequent alephs, $\aleph_\alpha$ with $\alpha \in Ord$, are denoted by $\omega_\alpha$. One might wonder what aleph can be assigned to the continuum, and Cantor hypothesized that it would be $\aleph_1$, so that we would have $2^{\aleph_0} = \aleph_1$. However, it has been shown that this hypothesis, referred to as the Continuum Hypothesis (CH) is independent from ZFC. That is, if we can prove it true or false within ZFC, then we have shown that mathematics is inconsistent and we would have $1 = 0$. The independence from ZFC was shown by K. Gödel and Paul J. Cohen. Gödel showed it was impossible to disprove CH [5], and Cohen showed that it was impossible to prove CH [1, 2].

## 2.2 Topological, Metric and Polish Spaces

### 2.2.1 Topological Spaces

A **topological space** is an ordered pair $(X, \mathcal{T})$, consisting of a set $X$ and a collection of subsets of $X$ denoted by $\mathcal{T}$ such that,

1. $\emptyset, X \in \mathcal{T}$,
2. $\mathcal{T}$ is closed under arbitrary unions,
3. $\mathcal{T}$ is closed under finite intersections.

A collection with such properties is called a **topology** on $X$. Furthermore, we call the members of $\mathcal{T}$ **open** and their complements **closed**. Consequently, the sets $\emptyset$ and $X$ are both open and closed in any topology. Sets that are both open and closed are called **clopen** sets. Since the topology $\mathcal{T}$ is often understood from context, we might informally refer to $X$ as a topological space.

Given topological spaces $X$ and $Y$, we call a map $f : X \to Y$ **continuous** if the preimage of any open set is again an open set. Continuous maps will be very important in the proof of Borel determinacy, because they preserve the complexity of sets under preimages. What complexity means here will be explained in the next section on the Borel hierarchy.

Lastly, a **basis** for a topology $\mathcal{T}$ is a collection $\mathcal{B} \subset \mathcal{T}$ such that every open set in $\mathcal{T}$ is the union of members of $\mathcal{B}$.

### 2.2.2 Metric Spaces

A **metric space** is an ordered pair $(X, d)$, consisting of a set $X$ and a function $d : X^2 \to [0, \infty)$ so that for all $x, y, z \in X$ we have the following:

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq d(x, z) + d(z, y)$.

A function with such properties is called a **metric**.

Given an $x \in X$ and $r \in [0, \infty)$ we define the **open ball** with center $x$ and radius $r$ as $B(x, r) = \{y \in X : d(x, y) < r\}$. It should be clear that the collection of all open balls in $X$ forms a basis for a topology on $X$, i.e., the metric $d$ induces a topology on $X$. It is however not true that every topology can be induced by a metric. Topological spaces whose topology can be induced by a metric are called **metrizable**.

### 2.2.3 Polish Spaces

Let $(X, \mathcal{T})$ be a topological space, we say that a subset $D \subset X$ is **dense** in $X$ if $X \cap U \neq \emptyset$ for all nonempty $U \in \mathcal{T}$. If it is possible to construct a countable dense subset in $X$, then we say that $X$ is **separable**.

If we let $(X, d)$ be a metric space, then we say that a sequence $(x_n)$ is **Cauchy** when $\lim_{n,m} d(x_n, x_m) = 0$. If all Cauchy sequences converge in $X$, then we say that $X$ is **complete**.

**Definition 2.5.** *A topological space $(X, \mathcal{T})$ is **Polish** if it is separable and completely metrizable, i.e., there is a metric $d$ such that $(X, d)$ is complete.*

In this paper we will be working in $\mathbb{R}$, which is a Polish space. However, it is not necessary for the proofs that we work in $\mathbb{R}$. In fact, Borel determinacy and the perfect set property for Borel sets both hold for any Polish space.

## 2.3 The Borel Hierarchy

Given a topology on a set $X$ we define the classes $\Sigma^0_\alpha$ and $\Pi^0_\alpha$ for all ordinals $0 < \alpha < \omega_1$. We first define the base case $\Sigma^0_1$ to be the class of all open sets in the topology. Next we recursively define all remaining classes as

$$\Pi^0_\alpha = \{A : X \setminus A \in \Sigma^0_\alpha\}$$
$$\Sigma^0_\alpha = \left\{ \bigcup_{n\in\mathbb{N}} A_n : A_n \in \Pi^0_{\beta_n} \text{ for } \beta_n < \alpha \right\} \qquad \text{for all } \alpha > 1$$

Given a set $A$, we say it is **Borel** if $A \in \Sigma^0_\alpha$ for some $\alpha < \omega_1$. The rank, or complexity, of a Borel set $A$ is then the smallest ordinal $\alpha$ for which $A \in \Sigma^0_\alpha$ or $A \in \Pi^0_\alpha$. We also give some special names to sets from specific classes; we call sets from $\Pi^0_1$ closed, sets in $\Sigma^0_2$ we call $F_\sigma$ and sets in $\Pi^0_2$ we call $G_\delta$.

Furthermore, if we assume that we are working in a metric space, we find a useful structure which we will call the Borel hierarchy shown below.

$$\begin{array}{cccccc} \Sigma^0_1 & \Sigma^0_2 & \dots & \Sigma^0_{\xi-1} & \Sigma^0_\xi & \dots \\ \Pi^0_1 & \Pi^0_2 & \dots & \Pi^0_{\xi-1} & \Pi^0_\xi & \dots \end{array}$$

Every class in the Borel hierarchy is contained in any other class to the right of it, that is for $\alpha < \beta < \omega_1$ we have the following inclusions:

$$\Sigma^0_\alpha \subset \Sigma^0_\beta, \quad \Sigma^0_\alpha \subset \Pi^0_\beta, \quad \Pi^0_\alpha \subset \Sigma^0_\beta, \quad \Pi^0_\alpha \subset \Pi^0_\beta.$$

We only need to show that $\Pi^0_1 \subset \Pi^0_2$, after which the remaining inclusions follow by induction on $\alpha$. We prove the following proposition:

**Proposition 2.1.** *Let $X$ be a metric space, then every closed set can be written as a countable intersection of open sets.*

*Proof.* Let $A \subset X$ be closed and let

$$A_n = \bigcup_{a\in A} B\left(a, \frac{1}{n}\right) \qquad \text{for all } n \in \mathbb{N}.$$

So the sets $A_n$ are unions of open balls. We now show that $A = \bigcap A_n$. Clearly $A \subset \bigcap A_n$. To prove the converse we take $x \notin A$ and since $A^C$ is open $\exists n \in \mathbb{N}$ such that $B(x, \frac{1}{n}) \subset A^C$, but then for all $a \in A$ we have $x \notin B(a, \frac{1}{n})$. This means $x \notin A_n$ and thus $x \notin \bigcap A_n$, hence $\bigcap A_n \subset A$. □

Another useful property of the hierarchy is that it is preserved under continuous preimages. This will be used in the proof of Borel determinacy later.

**Proposition 2.2.** *Let $f : X \to Y$ be a continuous function and $A \in \Sigma^0_\alpha$ in $Y$, then $f^{-1}(A) \in \Sigma^0_\alpha$ in $X$.*

*Proof.* We know by the definition of continuity that this hold for the open sets. If $\Sigma^0_\alpha$ is preserved under preimages, then because preimages are well behaved under complements, we also have that $\Pi^0_\alpha$ is preserved. Now assume that it holds for all $\beta < \alpha$, then if $A \in \Sigma^0_\alpha$ we can write $A = \bigcup A_n$ with $A_n \in \Pi^0_{\beta_n}$, where $\beta_n < \alpha$. We have $f^{-1}(A) = f^{-1}(\bigcup A_n) = \bigcup f^{-1}(A_n)$, but now it follows from our induction hypothesis that $f^{-1}(A_n) \in \Pi^0_{\beta_n}$ and thus $f^{-1}(A) \in \Sigma^0_\alpha$. □

### 2.4 Trees

This section will introduce trees from descriptive set theory. Before we can define such a tree, we need to introduce some basic concepts.

Let $A$ be a nonempty set and $n \in \mathbb{N}$. We define $A^n$ as the set of finite sequences from $A$ with domain $n$. We say a sequence with domain $n$ has a length of $n$. In case $n = 0$, we say $A^0 = \{\emptyset\}$, where $\emptyset$ represents the **empty sequence**. Similarly we define $A^{<\omega}$ and $A^\omega$ as the sets of all finite and infinite sequences from $A$ respectively.

Let $|s|$ denote the length of a finite sequence $s$. If $s \in A^n$ and $m \leq n$, then we define the **restriction** $s|m = (s_0, \ldots, s_{m-1})$. If $s, t \in A^{<\omega}$ and $s = t|m$ for some $m \leq |t|$, then we write $s \subset t$, moreover we say that $s$ is an **initial segment** of $t$ and that $t$ is an **extension** of $s$. Similarly, if $s \in A^{<\omega}$, $t \in A^\omega$ and $s = t|m$ for some $m \in \mathbb{N}$, then we also write $s \subset t$.

Now let $s, t \in A^{<\omega}$, then we define the **concatenation** of $s$ and $t$ as $s\hat{\ }t = (s_0, \ldots, s_{|s|}, t_0, \ldots, t_{|t|})$. If $|t| = 1$ we often write $s\hat{\ }t_0$ instead of $s\hat{\ }(t_0)$.

**Definition 2.6.** *$T \subset A^{<\omega}$ is a **tree** if for every sequence $t \in T$ and every initial segment $s \subset t$, we have that $s \in T$. In short, $T$ is closed under initial segments.*

We say that a tree without terminal nodes is **pruned**. In other words, every node has a proper extension. Trees can be visualized as seen in figures 1 and 2.

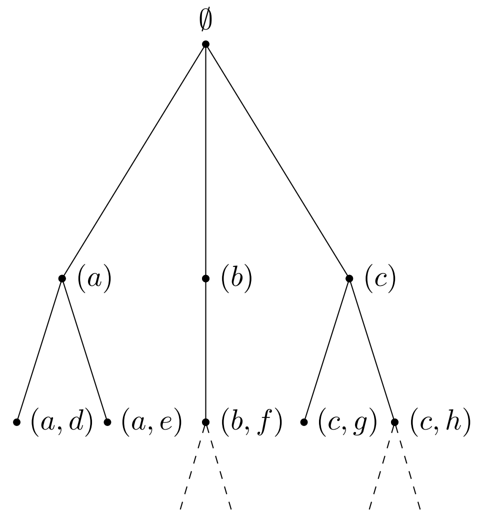


Figure 1: Visualization of an arbitrary tree in an unspecified space.

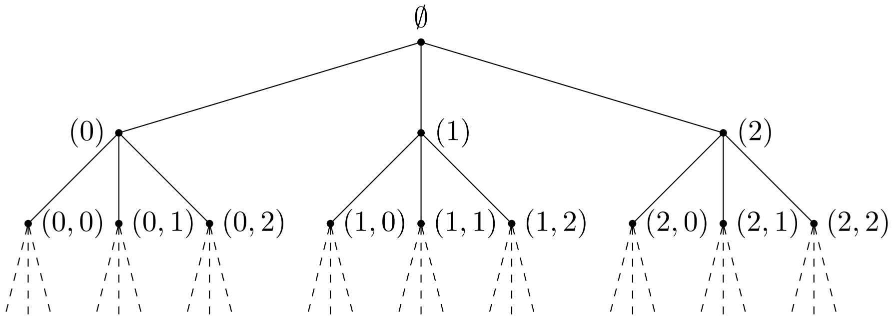


Figure 2: Visualization of the full ternary tree in $3^{<\omega}$.

Since we will be investigating infinite games, we will need notation for what happens at 'infinity' for these trees. Given a tree $T$, we let $[T]$ denote the **body** of $T$, where $[T]$ is the set of all infinite paths down from the root node $\emptyset$. More precisely, $x \in [T]$ if and only if $x|n \in T$ for all $n \in \mathbb{N}$.

For some technicalities in the proof of Borel determinacy we will need some extra notation. Given a tree $T$ and a finite sequence $s \in T$, we define the **tree generated by** $s$ as $T_s = \{t \in T : s \subset t \text{ or } t \subset s\}$.

We equip the body of a tree $T$ with the metric $d : [T]^2 \to [0, \infty)$ so that for $x, y \in [T]$ we have $d(x, y) = 2^{-n}$, where $n$ is the first index with $x|n \neq y|n$. The sets $N_s = \{x \in [T] : s \subset x\}$, with $s \in T$, then form a basis for the topology on $[T]$ induced by $d$.

# 3 Infinite Games

The previous sections have now properly equipped us to deal with the complexities of infinite games. This section will take a look at a particular type of games, called the Gale-Stewart games, of which some examples will be given. Next we will investigate the determinacy of these games. We will introduce Bernstein sets, which show that not all games in ZFC are determined. All hope is not lost however, as we prove the Gale-Stewart theorem, which shows all games with an open or closed payoff set are determined. Moreover, we give a proof of Martin's theorem, that shows we have determinacy for all games with a Borel payoff set.

## 3.1 Gale-Stewart Games

Let $A$ be a nonempty set, $T \subset A^{<\omega}$ a nonempty pruned tree on $A$ and $X \subset [T]$. Next, let player I and II take turns playing elements from $A$,

$$\begin{array}{llllll} \text{I} & a_0 & & a_2 & & \dots \\ \text{II} & & a_1 & & a_3 & \end{array}$$

so that $(a_0, \dots, a_n) \in T$ for all $n \in \mathbb{N}$. In order for this to be a game, we say that I wins if the sequence $(a_n) \in X$. We denote this game by $G(T, X)$, or $G(X)$ if $T$ is clear from context. In this game we view $A$ as a set of **moves**, $T$ as the tree of **legal positions** and $X$ as the **payoff set**.

We define a **strategy** for I as a map $\sigma : T \to A$, so that for all legal positions $p$ of even length, the extension $p\hat{\ }\sigma(p)$ is also legal. Similarly, we define a strategy for II on the legal positions of odd length. We say $\sigma$ is a **winning** strategy for I if every run of the game played according to this strategy ends up in $X$. If $\sigma(p)$ can take multiple values for some legal $p$, then we call $\sigma$ a **quasistrategy**. We define $T_\sigma$, the **tree generated by** $\sigma$, recursively as follows:

- $\emptyset \in T_\sigma$,
- if $(a_0, \dots, a_{2n-1}) \in T_\sigma$, then $(a_0, \dots, a_{2n-1}, \sigma((a_0, \dots, a_{2n-1})) \in T_\sigma$,
- if $(a_0, \dots, a_{2n}) \in T_\sigma$, then $(a_0, \dots, a_{2n}, x) \in T_\sigma$ for all legal $x \in A$.

In short, $T_\sigma$ restricts I to playing according to $\sigma$, but leaves II free to play any legal moves. It follows that $\sigma$ is winning when $[T_\sigma] \subset X$. Likewise, we say a strategy $\rho$ is winning for II when $[T_\rho] \subset X^C$. Clearly, we have either that I, II or neither has a winning strategy. If a winning strategy exists for either player, then we call the game $G(T, X)$ **determined**. More often than not $T$ is understood from context and we simply call the set $X$ determined.

Let us now look at some examples and try to identify winning strategies.

Consider the game where I and II take turns playing decimal digits, i.e., they pick moves from the set $10 = \{0, \dots, 9\}$. Let $f : 10^\omega \to [0, 1]$ be the map that takes a sequence $(a_n)$ to the real number $f((a_n)) = 0.a_0a_1a_2\dots$. Given a set $X \subset [0, 1]$, we call the game $G_d(X) = G(10^{<\omega}, f^{-1}(X))$ the digit game on $X$.

**Example 3.1.** Consider the digit game on the payoff set $X$ given by

$$X = \left\{ (a_n) \;\middle|\; \sum_{i=0}^{n} a_i \equiv 0 \pmod{10} \text{ for all } n \in 2\mathbb{N} \right\}.$$

We show that this game is determined by constructing a winning strategy for I. Let a strategy $\sigma : 10^{<\omega} \to 10$ for I be given by

$$\sigma(p) = \begin{cases} 0 & p = \emptyset \\ 10 - p_{|p|-1} & p_{|p|-1} > 0 \\ 0 & p_{|p|-1} = 0 \end{cases}$$

so that $\sigma(p) + p_{|p|-1} \equiv 0 \pmod{10}$ for all possible positions $p$ played by II. By induction on $n$ it follows that $(a_n) \in X$ and that $\sigma$ is a winning strategy for I.

**Example 3.2.** Consider the digit game on a countable payoff set $X$. We will show that this game is determined, moreover, we will show that player II has a winning strategy. The proof is essentially Cantor's diagonal argument.

First assume $X$ is countably infinite, i.e., we can enumerate the elements from $X$. We write out any such enumeration as follows:

$$\begin{aligned} x^0 &= (x_0^0, \mathbf{x_1^0}, x_2^0, x_3^0, x_4^0, x_5^0, \dots) \\ x^1 &= (x_0^1, x_1^1, x_2^1, \mathbf{x_3^1}, x_4^1, x_5^1, \dots) \\ x^2 &= (x_0^2, x_1^2, x_2^2, x_3^2, x_4^2, \mathbf{x_5^2}, \dots) \\ x^3 &= (x_0^3, x_1^3, x_2^3, x_3^3, x_4^3, x_5^3, \dots) \\ x^4 &= (x_0^4, x_1^4, x_2^4, x_3^4, x_4^4, x_5^4, \dots) \\ &\vdots \end{aligned}$$

Let a strategy $\sigma : 10^{<\omega} \to 10$ for II be given by

$$\sigma((a_0, \dots, a_{2n})) = \begin{cases} x_{2n+1}^n + 1 & x_{2n+1}^n < 9 \\ x_{2n+1}^n - 1 & x_{2n+1}^n = 9 \end{cases}$$

If $(a_n)$ is a run of the game played according to $\sigma$, then it differs from $x^n$ in the $n$th digit played by II (highlighted above). Hence, $x^n \notin [T_\sigma]$ for all $n \in \mathbb{N}$, so the strategy $\sigma$ is winning for II. The same strategy can be used to win when $X$ is a finite set; we simply let $\sigma$ only play zeroes after II's $|X|$th turn.

Similarly, if we consider the digit game on a co-countable payoff set $Y$, then we can enumerate its complement by $(y^n)_{n\in\mathbb{N}}$ and construct a strategy for I so that any run of the game played by it differs from $y^n$ in the $n$th digit played by I. By the same arguments it must be a winning strategy for I.

### 3.2 Undetermined Games

It turns out not all games are determined, but to show this we actually need some pretty "ugly" sets. To this end we focus our attention on Bernstein sets.

**Theorem 3.1** (Bernstein). *There exists a set $B \subset \mathbb{R}$ such that for every uncountable closed subset $F \subset \mathbb{R}$, we have $B \cap F \neq \emptyset$ and $B^C \cap F \neq \emptyset$.*

Any set with the properties of theorem 3.1 is called a **Bernstein set**. If we have such a set $B$, then we can intersect it with $[0, 1]$ so that we can play the digit game on $B \cap [0, 1]$. We will show that such a game must be undetermined.

Before we do this however, we will need to prove theorem 3.1. For the construction we will require three lemmas and a theorem.

**Lemma 3.1.** *Let $A \subset \mathbb{R}$ be uncountable, then it has at least one limit point.*

*Proof.* Assume $A$ has no limit points, then for each point $x \in A$ we can construct an open interval $(l_x, r_x)$ with rational $l_x$ and $r_x$, so that $(l_x, r_x) \cap A = \{x\}$. However, this contradicts $A$ being uncountable, since $\mathbb{Q}^2$ has cardinality $\aleph_0$. □

**Lemma 3.2.** *There are $\mathfrak{c}$ many uncountable closed subsets of $\mathbb{R}$.*

*Proof.* Let $\mathcal{B}$ be a basis for the standard topology on $\mathbb{R}$ consisting of all open intervals with rational endpoints. Clearly there are at most $2^{\aleph_0} = \mathfrak{c}$ many open sets, hence at most $\mathfrak{c}$ many closed sets. However, there also at least $\mathfrak{c}$ many closed sets, e.g., the closed intervals $[x, x + 1]$ for all $x \in \mathbb{R}$. □

If $P \subset \mathbb{R}$ is a nonempty closed set that has no isolated points, then we call it a **perfect set**. We have the following lemma for perfect sets.

**Lemma 3.3.** *If $P$ is perfect then it has cardinality $\mathfrak{c}$.*

*Proof.* We look for a one-to-one map $f : 2^\omega \to C$, where $C \subset P$. If such a map exists, then $|P| \geq 2^{\aleph_0}$ and it follows that $P$ has cardinality $\mathfrak{c}$.

By induction we construct closed intervals $I_s$ such that for each $n$ and every finite sequence $s$ from $2 = \{0, 1\}$ of length $n$,

1. $P \cap I_s \neq \emptyset$ is perfect,
2. $\text{diam}(I_s) \leq \frac{1}{n}$,
3. $I_{s\hat{\ }0} \subset I_s$, $I_{s\hat{\ }1} \subset I_s$ and $I_{s\hat{\ }0} \cap I_{s\hat{\ }1} = \emptyset$.

We begin by claiming that if $x \in P$, then we can create an arbitrarily small closed interval $I$ that contains $x$ so that $P \cap I$ is perfect.

Let $\varepsilon > 0$ and consider the set $A = P \cap [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$. Either $A$ is perfect and we let $I = [x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$, or at least one of the endpoints of $[x - \frac{\varepsilon}{2}, x + \frac{\varepsilon}{2}]$ is isolated in $A$. Assume the left endpoint is isolated, then there exists $0 < \delta < \frac{\varepsilon}{2}$ such that the left endpoint of $I = [x - \delta, x + \frac{\varepsilon}{2}]$ is no longer in $P$. Likewise, we construct $I$ if the right, or both endpoints are isolated. Clearly, $P \cap I$ is perfect.

Let $x_\emptyset \in P$ be the closest point to 0, where we take $x$ positive if two points are closest. We use the claim to find $I_\emptyset$ containing $x_\emptyset$ with $\text{diam}(I_\emptyset) \leq 2$.

Assume that we have $I_s$ with (1)–(3) for all $s \in 2^n$. Let $a$ and $b$ in $P \cap I_s$ be the points closest to the left and right endpoints of $I_s$ respectively. We use the claim to find $J_{s^\frown 0}$ containing $a$ and $J_{s^\frown 1}$ containing $b$ such that for $i \in \{0, 1\}$,

$$\text{diam}(J_{s^\frown i}) \leq \min\left(\frac{1}{n+1}, b - a\right)$$

Let $i \in \{0, 1\}$ and let $I_{s^\frown i} = I_s \cap J_{s^\frown i}$. Clearly, $I_{s^\frown i}$ satisfies (2) and (3). Note that the endpoints of $I_{s^\frown i}$ are either endpoints of $I_s$ or $J_{s^\frown i}$. In either case these endpoints are not isolated, hence $I_{s^\frown i}$ satisfies (1).

Given $s \in 2^\omega$, the nested interval theorem implies that $P \cap \bigcap_{n \in \mathbb{N}} I_{s|n}$ consists of precisely one point $x_s$. Let $f(s) = x_s$ be our one-to-one map. □

**Theorem 3.2** (Cantor-Bendixson)**.** *If $F$ is an uncountable closed set, then $F = P \cup S$, where $P$ is perfect and $S$ is at most countable.*

*Proof.* For every $A \subset \mathbb{R}$ let $A'$ denote the set of all limit points of $A$. Clearly, $A'$ is closed and if $A$ is closed then $A' \subset A$. We let $F_0 = F$, $F_{\alpha+1} = F'_\alpha$ and $F_\alpha = \bigcap_{\beta<\alpha} F_\beta$ for all limit ordinals $\alpha > 0$. Since we remove points at each step, there must be a smallest ordinal $\theta$ such that $F_\alpha = F_\theta$ for all $\alpha \geq \theta$.

If $F_\theta$ is nonempty, then by its definition it is perfect. To complete the proof, we show that $S = F \setminus F_\theta$ is countable which so that $F_\theta$ is nonempty. To this end we look for a one-to-one map $f : S \to \mathbb{N}$. If such a map exists, then $|S| \leq \aleph_0$.

Let $(I_n)_{n\in\mathbb{N}}$ be an enumeration of the intervals with rational endpoints. If $x \in S$, then there is a unique ordinal $\alpha$ such that $x \in F_\alpha \setminus F'_\alpha$. Let $f(x)$ be the least $n$ such that $\{x\} = F_\alpha \cap I_n$. This $f$ is one-to-one. □

The construction of a Bernstein set is now fairly simple. From Lemma 3.2 we know there are $\mathfrak{c}$ many uncountable closed subsets in $\mathbb{R}$. We can now use the Axiom of Choice to well-order the collection of all uncountable closed subsets and index them with ordinals up to $\mathfrak{c}$. Call these indexed sets $F_\alpha$ for $\alpha < \mathfrak{c}$.

We now use the Axiom of Choice to well-order $\mathbb{R}$ with well-ordering $\prec$. Let $x_0$ and $y_0$ be the first two elements of $F_0$ under $\prec$. Next we recursively define the points $x_\alpha$ and $y_\alpha$ as the first two elements of $F_\alpha \setminus \{x_\beta, y_\beta : \beta < \alpha\}$ for all $\alpha < \mathfrak{c}$. Note that $F_\alpha \setminus \{x_\beta, y_\beta : \beta < \alpha\}$ is not empty, because we have removed fewer than $\mathfrak{c}$ many points and $F_\alpha$ has $\mathfrak{c}$ many points by Lemma 3.2.

Now let $B = \{x_\alpha : \alpha < \mathfrak{c}\}$, then it is a Bernstein set. Indeed, B meets every uncountable closed set $F_\alpha$ in $x_\alpha$ and its complement meets every uncountable closed set $F_\alpha$ in $y_\alpha$. Similarly we could have taken $B' = \{y_\alpha : \alpha < \mathfrak{c}\}$.

All that is left to show is that we can indeed create a game that is not determined. To this end we look at the game $G(10^{<\omega}, B \cap [0, 1])$ where $B$ is a Bernstein set as given as above. We show that this game is not determined.

Assume that $\sigma$ is a winning strategy for player I and let $T_\sigma$ be the tree generated by $\sigma$. We observe by applying Cantor's diagonal argument that $[T_\sigma]$ is an uncountable closed set. The payoff set has the Bernstein structure, so $[T_\sigma] \cap (B \cap [0, 1])^C \neq \emptyset$ and thus $[T_\sigma] \not\subset B \cap [0, 1]$, but then $\sigma$ cannot be a winning strategy. At any stage of the game, II can still reach $(B \cap [0, 1])^C$.

Assume now that $\sigma$ is a winning strategy for II. A similar argument shows that $[T_\sigma] \cap (B \cap [0,1]) \neq \emptyset$ and thus $[T_\sigma] \not\subset (B \cap [0,1])^C$. In conclusion, neither player can have a winning strategy, hence we found a game that is not determined.

### 3.3 Determinacy for Closed and Open Sets

**Theorem 3.3** (Gale-Stewart)**.** *Let $T$ be a nonempty pruned tree on $A$. Let $X \subset [T]$ be closed or open in $[T]$. Then $G(T, X)$ is determined.*

*Proof.* To begin assume that $X$ is closed in $[T]$ and that player II has no winning strategy in $G(T, X)$. We will construct a winning strategy for player I.

Given a position $p = (a_0, a_1, \dots, a_{2n-1}) \in T$, we say it is **not losing** for I if II has no winning strategy in the game generated by $p$, more precisely, II has no winning strategy in the game $G(T_p, X_p)$ with $T_p = \{s : p\hat{\ }s \in T\}$ and $X_p = \{x : p\hat{\ }x \in X\}$. As a result, the empty sequence $\emptyset$ is not losing for I.

Next, we observe that if a position $p$ is not losing for I, then there must exist a play $a_{2n}$ with $(a_{2n}) \in T_p$ such that for all plays $a_{2n+1}$ with $(a_{2n}, a_{2n+1}) \in T_p$, the extended position $p\hat{\ }(a_{2n}, a_{2n+1})$ is not losing for I either.

We construct a strategy for I using not losing positions. The fact that $\emptyset$ is not losing together with the observation from before shows us the existence of plays $a_0, a_2, \dots$ for I in response to $a_1, a_3, \dots$ by II so that the corresponding positions are not losing for I. Lastly we invoke the Axiom of Choice to show the existence of a strategy for I that plays $a_0, a_2, \dots$ in response to II.

We show this strategy is winning for I. Let $(a_n)$ be a run of the game where I followed this strategy. Assume $(a_n) \notin X$, then because $X^C$ is open there exists a $k \in \mathbb{N}$ so that $N_{(a_0,\dots,a_{2k-1})} \bigcap [T] \subset X^C$, but then $(a_0, \dots, a_{2k-1})$ is losing for I because II wins with arbitrary legal plays, so by contradiction $(a_n) \in X$.

The case when $X$ is open is very similar. We assume that I has no winning strategy and construct a strategy for II using not losing positions for II. $\square$

The theorem turns out to be equivalent to the Axiom of Choice. The proof is inspired by a user called Lorenzo on Maths Stack Exchange [8].

**Theorem 3.4.** *The Gale-Stewart theorem on determinacy is equivalent to the Axiom of Choice (AC) in Zermelo-Fraenkel axiomatic set theory (ZF).*

*Proof.* The previous theorem already showed us that [AC $\implies$ Gale-Stewart], conversely we will now show that we also have [Gale-Stewart $\implies$ AC].

Let $Y$ be a collection of nonempty sets and let $A = Y \cup \bigcup Y$. Furthermore let $T$ be the nonempty pruned tree on $A$ with the following properties:

- $(a_0) \in T \iff a_0 \in Y$,
- $(a_0, a_1) \in T \iff a_1 \in a_0$ and $(a_0) \in T$,
- $(a_0, \dots, a_n) \in T$ and $n \geq 2 \iff a_n = a_1$ and $(a_0, \dots, a_{n-1}) \in T$.

Let the payoff set $X$ be the empty set. Clearly $X$ is clopen in $[T]$, so we can use the Gale-Stewart theorem for the game $G(T, X)$ to prove the existence of a winning strategy $\sigma$ for player II, since I cannot possibly win this game.

Let $y \in Y$, then the function $f : Y \to y$ defined by $f(y) = \sigma((y))$ fulfills the conditions of a choice function on $Y$, hence we have AC. $\square$

### 3.4 Determinacy for Borel Sets

In this final section we investigate the determinacy of Borel sets. The following theorem shows us that each set from the Borel hierarchy is determined.

**Theorem 3.5** (Martin)**.** *Let $T$ be a nonempty pruned tree on $A$ and let $X \subset [T]$ be Borel. Then $G(T, X)$ is determined.*

The proof that we will give for Borel determinacy will be based on a simplification of Martin's initial proof [10] and the proof by Kechris in [7].

The proof of theorem 3.5 hinges on the following definition of a covering.

**Definition 3.1.** *A* ***covering*** *of a pruned tree $T$ is a triple $(T', \pi, \varphi)$, such that*

*(i) $T'$ is a nonempty pruned tree.*

*(ii) $\pi : T' \to T$ is a monotone map ($s' \subset t'$ implies $\pi(s') \subset \pi(t')$) such that $|\pi(s')| = |s'|$ for all $s' \in T'$. Letting the length of $s'$ go to infinity gives rise to a continuous function $\pi : [T'] \to [T]$ which we denote by $\pi$ as well.*

*(iii) $\varphi : S(T') \to S(T)$ maps strategies for a player in $T'$ to strategies for the same player in $T$ ($S(T)$ denotes the set of strategies for games on $T$).*

*(iv) If $x \in [T]$ is reached by playing according to $\varphi(\sigma')$, then there is an $x' \in [T']$ that can be reached by playing according to $\sigma'$ such that $\pi(x') = x$.*

If $(T', \pi, \varphi)$ is a covering of $T$ and $X \subset [T]$, then by letting $X' = \pi^{-1}(X)$ we can associate the game $G(T', X')$ to the game $G(T, X)$. This auxiliary game can have a simpler payoff set if the covering is chosen carefully. We say the covering **unravels** $X$ when $X'$ is clopen in $[T']$. We have the following lemma.

**Lemma 3.4.** *If the covering $(T', \pi, \varphi)$ unravels $X$, then $G(T, X)$ is determined.*

*Proof.* $X'$ is clopen, so by Gale-Stewart the game $G(T', X')$ is determined. Let $\sigma'$ be a winning strategy for the player that has one. We show $\varphi(\sigma')$ is winning for the same player in $G(T, X)$. Assume $x \in [T]$ is reached by playing according to $\varphi(\sigma')$, then by (iv) we find $x'$ according to $\sigma'$ so that $\pi(x') = x$. Because $\sigma'$ is winning we must have $x' \in \pi^{-1}(X)$ and thus $x = \pi(x') \in X$. □

A key observation is that we can chain these coverings recursively, i.e., we can keep covering a game until at some point we can unravel everything. We will need a stronger form of covering to make an induction argument work.

**Definition 3.2.** *A covering $(T', \pi, \varphi)$ of $T$ is a $k$-covering if*

*(i) $\varphi(\sigma')$ restricted to positions of length $\leq n$ depends only on $\sigma'$ restricted to positions of length $\leq n$ for all $n \in \mathbb{N}_0$. Informally $\varphi(\sigma')|n = \varphi(\sigma'|n)$.*

*(ii) $T'|2k = T|2k$, where $T|2k = \{s \in T : |s| = 2k\}$.*

*(iii) $\pi$ is the identity on $T'|2k$.*

We require two more lemmas to carry out an induction argument. Lemma 3.5 gives us the base case and 3.6 will be used in the induction step.

**Lemma 3.5.** *If $T$ is a nonempty pruned tree on $A$ and $X \subset [T]$ is closed, then for each $k \in \mathbb{N}$ there exists a $k$-covering of $T$ that unravels $X$.*

**Lemma 3.6.** *Fix $k \in \mathbb{N}$ and let $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ be a $(k+i)$-covering of $T_i$ for each $i \in \mathbb{N}_0$. Then there is a pruned tree $T_\infty$ and maps $\pi_{\infty,i}$, $\varphi_{\infty,i}$ for each $i \in \mathbb{N}_0$ such that $(T_\infty, \pi_{\infty,i}, \varphi_{\infty,i})$ is a $(k+i)$-covering of $T_i$ and $\pi_{\infty,i} = \pi_{i+1} \circ \pi_{\infty,i+1}$ and $\varphi_{\infty,i} = \varphi_{i+1} \circ \varphi_{\infty,i+1}$.*

In the interest of clarity, we first prove theorem 3.5 given these two lemmas.

*Proof (Theorem 3.5).* From lemma 3.5 we find that for each $A \in \Pi^0_1$ there is a $k$-covering that unravels it. Note that any covering that unravels $A$ must also unravel its complement $A^C$. As a result there is a $k$-covering that unravels $A \in \Pi^0_\xi$ if and only if it also unravels $A^C \in \Sigma^0_\xi$. This gives us the base case for induction because every set in $\Sigma^0_1$ has a $k$-covering that unravels it.

Now let $\alpha < \omega_1$ and assume that for all ordinals $\beta < \alpha$ and every $k \in \mathbb{N}$ the sets in $\Sigma^0_\beta$ can be unraveled by a $k$-covering. Let $X \in \Sigma^0_\alpha$, then it is of the form $X = \bigcup_{i\in\mathbb{N}} X_i$, where each $X_i \in \Pi^0_{\beta_i}$ for some $\beta_i < \alpha$. It follows from the induction hypothesis that there is a $k$-covering $(T_1, \pi_1, \varphi_1)$ of $T_0 = T$ that unravels $X_0$. We can now use proposition 2.2 to find that $\pi_1^{-1}(X_i) \in \Pi^0_{\beta_i}$ for all $i \in \mathbb{N}$. In other words we have unraveled $X_0$ while not making the other $X_i$ more complex. We now recursively apply the same reasoning to find a $(k+i)$-covering $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$ of $T_i$ that unravels $\pi_i^{-1} \circ \pi_{i-1}^{-1} \circ \cdots \circ \pi_1^{-1}(X_i)$ for all $i \in \mathbb{N}$.

Next, we let $(T_\infty, \pi_{\infty,i}, \varphi_{\infty,i})$ be as in lemma 3.6. It follows from using the identities $\pi_{\infty,i} = \pi_{i+1} \circ \pi_{\infty,i+1}$ and $\varphi_{\infty,i} = \varphi_{i+1} \circ \varphi_{\infty,i+1}$ that the covering $(T_\infty, \pi_{\infty,0}, \varphi_{\infty,0})$ unravels every $X_i$. Indeed, for a given $X_i$ the set $\pi_{\infty,0}^{-1}(X_i) = \pi_{\infty,i+1}^{-1} \circ \pi_{i+1}^{-1} \circ \cdots \circ \pi_1^{-1}(X_i)$ is clopen. Thus $\pi_{\infty,0}^{-1}(X) = \bigcup_{i\in\mathbb{N}} \pi_{\infty,0}^{-1}(X_i)$ is open in $[T_\infty]$. We can now use the base case to find a $k$-covering $(T', \pi, \varphi)$ of $T_\infty$ that unravels $\pi_{\infty,0}^{-1}(X)$. Finally we chain these coverings to find the covering $(T', \pi_{\infty,0} \circ \pi, \varphi_{\infty,0} \circ \varphi)$ of $T$ that unravels $X$. Lemma 3.4 finishes the proof. □

The proof should give motivation for why we need $k$-coverings and why we want a clopen set from an unraveling and not just an open or closed set.

We will now prove both lemmas.

*Proof (Lemma 3.5).* Let $T$ be a nonempty pruned tree, $k \geq 1$ a fixed integer and $X \subset [T]$ a closed set. We set out to construct a $k$-covering by letting the players play an auxiliary game based on $G(T, X)$. The game we consider is this:

$$\begin{array}{lcccccccccc} \text{I} & a_0 & & \dots & & a_{2k-2} & & (a_{2k}, T_I) & & a_{2k+2} & & \dots \\ \text{II} & & a_1 & & \dots & & a_{2k-1} & & (a_{2k+1}, T_{II}) & & a_{2k+3} & \end{array}$$

In the first $2k$ moves both players are allowed to play any move that would be legal in $G(T, X)$, but in the moves beyond they have to follow different rules:

In move $2k$ player I has to play a tuple consisting of a legal move in $G(T, X)$ and a subtree $T_I$ of $T$. We require that $T_I$ is a I-imposed subtree of $T_{(a_0,\dots,a_{2k})}$, i.e., it must be a tree generated by a quasistrategy in $T_{(a_0,\ldots,a_{2k})}$. More explicitly, $T_I$ follows the played game up to $a_{2k}$, after which I's quasistrategy is followed.

Next, player II has to play a tuple $(a_{2k+1}, T_{II})$, where $(a_0, \ldots, a_{2k+1}) \in T_{II}$. For her choice of $T_{II}$ she has two options:

1. $T_{II}$ is a II-imposed subtree of $T_I$ such that $[T_{II}] \subset X$, where we define a II-imposed subtree similar to a I-imposed subtree.

2. $T_{II}$ is of the form $\{s \in T : s \subset u \text{ or } u \subset s\}$ for some $u \in T_I$ such that $(a_0, \ldots, a_{2k}) \subset u$ and $[T_{II}] \cap X = \emptyset$.

Finally, all moves $a_i$ with $i \geq 2k+2$ must be legal in $T_{II}$. Figure 3.4 gives an example of choices for $T_I$ (blue) and $T_{II}$ (red) that could be played if $k = 1$.

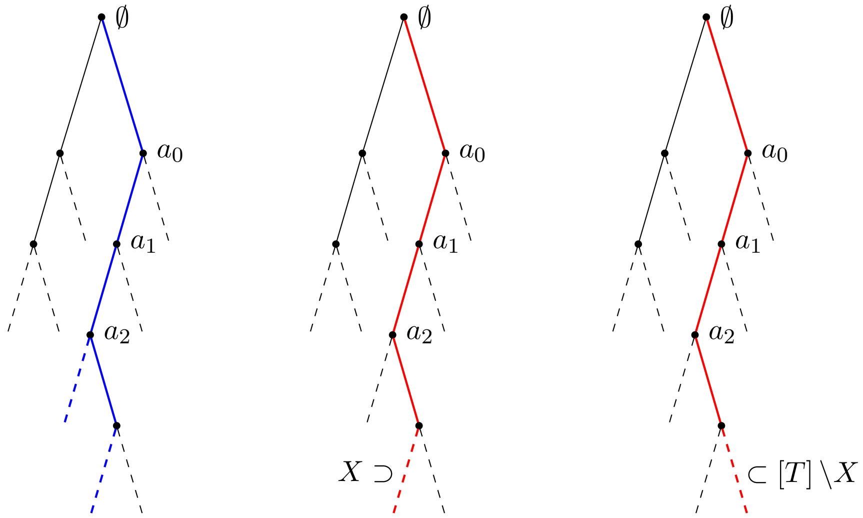


Figure 3: Let $k = 1$. From left to right we have a choice of $T_I$, an answer $T_{II}$ using option 1, and an answer $T_{II}$ using option 2.

We let the legal positions of this game define the tree $T'$. It is clear that both players can play legal moves at any stage of the game since they are always playing in a pruned (sub)tree, hence $T'$ is pruned. We let $\pi$ be the map that is the identity on all positions up to length $2k$, after which we define $\pi$ as:

$$\pi((a_0, \ldots, (a_{2k}, T_I), (a_{2k+1}, T_{II}), a_{2k+2}, \ldots)) = (a_0, \ldots, a_{2k}, a_{2k+1}, a_{2k+2}, \ldots)$$

It follows from this particular choice of $\pi$ that $x' \in \pi^{-1}(X)$ if and only if II chose to play option 1. But then $\pi^{-1}(X)$ is closed because $\pi$ is continuous and $X$ is closed, and furthermore it is open because $\pi^{-1}(X)$ is the body of a union of open sets in $T'$ (the trees where II chose option 1), hence $\pi^{-1}(X)$ is clopen.

We have everything we need for a $k$-covering except a map $\varphi$ that takes strategies in $T'$ to strategies in $T$. We need to make sure we have (i) from definition 3.2. We will separately construct $\varphi$ if $\sigma'$ is a strategy for I or II.

Assume that $\sigma'$ is a strategy for I. On the first $2k$ moves let $\varphi(\sigma')$ agree with $\sigma'$ such that if $\sigma'((\ldots, a_{2k-1}))=(a_{2k}, T_I)$, then $\varphi(\sigma')((\ldots, a_{2k-1})) = a_{2k}$.

On the remaining positions we will need to consider an extra game $G(T_I, [T_I] \setminus X)$. Since $[T_I] \setminus X$ is open in $T_I$, this game is determined by the Gale-Stewart theorem. This leads to two cases:

*Case 1.* Player I has a winning strategy in $G(T_I, [T_I] \setminus X)$.

We let $\varphi(\sigma')$ follow this winning strategy until at some point a $u$ is reached that satisfies the properties of option 2. Let $T_{II}$ be the subtree given by $u$ from option 2. Since $(a_0, \ldots, (a_{2k+1}, T_{II}))$ is a legal position in $T'$, we can let $\varphi(\sigma')$ simply follow $\sigma'$ from then on.

*Case 2.* Player II has a winning strategy in $G(T_I, [T_I] \setminus X)$.

Let $T_{II}$ be the II-imposed subtree of $T_I$ of positions that are not lost for II in this game. Clearly, all winning strategies for II must be in $T_{II}$. Again, we have that $(a_0, \ldots, (a_{2k+1}, T_{II}))$ is a legal position in $T'$. We let $\varphi(\sigma')$ follow $\sigma'$ until at some point II plays a $u \notin T_{II}$, which could happen immediately at $(a_0, \ldots, a_{2k+1})$, after which we let $\varphi(\sigma')$ be given by $u$ in case 1.

Assume now that $\sigma'$ is a strategy for II. On the first $2k$ moves we simply let $\varphi(\sigma')$ agree with $\sigma'$. For the remaining positions we need to introduce one more game on $T_{(a_0,\ldots,a_{2k})}$. We construct the following sets:

$$\begin{aligned} A = &\{u \in T_{(a_0,\ldots,a_{2k})} : \text{ there exists a } T'_I \text{ so that if I plays } (a_{2k}, T'_I), \\ &\quad \text{then } \sigma' \text{ calls for II to play } (a_{2k+1}, \{s \in T : s \subset u \text{ or } u \subset s\})\} \\ B = &\left[T_{(a_0,\ldots,a_{2k})}\right] \setminus \{x \in \left[T_{(a_0,\ldots,a_{2k})}\right] : \text{ there exists a } u \text{ such that } u \subset x \text{ and } u \in A\} \end{aligned}$$

Note that $B$ is closed in $\left[T_{(a_0,\ldots,a_{2k})}\right]$. Now consider the game $G(T_{(a_0,\ldots,a_{2k})}, B)$. This game is determined by Gale-Stewart, so that we again have two cases:

*Case 1.* Player II has a winning strategy in $G(T_{(a_0,\ldots,a_{2k})}, B)$.

We let $\varphi(\sigma')$ follow this winning strategy until at some point a $u \in A$ is reached. From the construction of $A$ it is clear that this $u$ satisfies the properties of option 2. Let $T_{II}$ be the subtree given by $u$ from option 2. Since $(a_0, \ldots, (a_{2k+1}, T_{II}))$ is a legal position in $T'$, we can let $\varphi(\sigma')$ simply follow $\sigma'$ from then on. This is the same argument as in case 1 for strategies for I.

*Case 2.* Player I has a winning strategy in $G(T_{(a_0,\ldots,a_{2k})}, B)$.

Let $T_I$ be the I-imposed subtree of $T$ of positions that are not lost for I in this game. Clearly $\sigma'$ cannot call for II to take option 2 because that would be a loss in $T_I$. We let $\varphi(\sigma')$ follow $\sigma'$ until at some point I plays a $u \notin T_I$, after which we let $\varphi(\sigma')$ be given by $u$ in case 1. This is again very similar to before.

It should be clear from the construction that $\varphi$ has (i) from definition 3.5. □

*Proof (Lemma 3.6).* Let $T_\infty$ be defined in the following way: a finite sequence $s$ is in $T_\infty$ if and only if it is in all $T_i$ where $|s| \leq 2(k+i)$. Then because the $T_i$ themselves are pruned it follows that $T_\infty$ is also pruned and moreover $T_\infty|2(k+i) = T_i|2(k+i)$ for every $i \in \mathbb{N}_0$.

Next we define $\pi_{\infty,i}$. Let $j \geq i$ be the smallest integer such that $|s| \leq 2(k+j)$. We then let $\pi_{\infty,i}(s) = \pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_j(s)$, where an empty product is seen as the identity, more specifically $\pi_{\infty,i}(s) = s$ when $|s| \leq 2(k+i)$.

$$\begin{array}{ccccccc}
T_0 & \xleftarrow{\pi_1} & T_1 & \xleftarrow{\pi_2} & T_2 & \dashleftarrow & \cdots \\
\uparrow{\scriptstyle \pi_{\infty,0}} & & \uparrow{\scriptstyle \pi_{\infty,1}} & & \uparrow{\scriptstyle \pi_{\infty,2}} & & \\
T_\infty & & T_\infty & & T_\infty & &
\end{array}$$

Figure 4: Commutative diagram for the maps $\pi_i$ and $\pi_{\infty,i}$.

Lastly, we define $\varphi_{\infty,i}$. Given a strategy $\sigma_\infty$ in $T_\infty$, let the partial strategy $\varphi_{\infty,i}(\sigma_\infty)|2(k+i)$ be given by $\sigma_\infty|2(k+i)$. We let the remaining partial strategies $\varphi_{\infty,i}(\sigma_\infty)|2(k+j)$ with $j > i$ be given by $\varphi_{i+1} \circ \varphi_{i+2} \circ \cdots \circ \varphi_j(\sigma_\infty|2(k+j))$.

$$\begin{array}{ccccccc}
S(T_0) & \xleftarrow{\varphi_1} & S(T_1) & \xleftarrow{\varphi_2} & S(T_2) & \dashleftarrow & \cdots \\
\uparrow{\scriptstyle \varphi_{\infty,0}} & & \uparrow{\scriptstyle \varphi_{\infty,1}} & & \uparrow{\scriptstyle \varphi_{\infty,2}} & & \\
S(T_\infty) & & S(T_\infty) & & S(T_\infty) & &
\end{array}$$

Figure 5: Commutative diagram for the maps $\varphi_i$ and $\varphi_{\infty,i}$.

In order to see that $\pi_{\infty,i}$ and $\varphi_{\infty,i}$ indeed define a covering we need to check property (iv) of definition 3.1. Let $\sigma_\infty$ be a strategy in $T_\infty$ and let $x_i \in [T_i]$ be a run of the game given by $\varphi_{\infty,i}(\sigma_\infty)$. Next, let $x_{i+1} \in [T_{i+1}]$, $x_{i+2} \in [T_{i+2}]$,... be given by (iv) for the coverings $(T_{i+1}, \pi_{i+1}, \varphi_{i+1})$, $(T_{i+2}, \pi_{i+2}, \varphi_{i+2})$,.... Note that for all $j \geq i$, the map $\pi_{j+1}$ is the identity on finite sequences $s$ where $|s| \leq 2(k+j)$. It follows that the sequence $(x_i, x_{i+1}, \dots)$ converges to a limit sequence $x_\infty$ given by $x_\infty|2(k+j) = x_j|2(k+j)$ for $j \geq i$. We then have $\pi_{\infty,i}(x_\infty|2(k+j)) = \pi_{i+1} \circ \pi_{i+2} \circ \cdots \circ \pi_j(x_\infty|2(k+j)) = x_i|2(k+j)$, and by letting $j$ go to infinity we find $\pi_{\infty,i}(x_\infty) = x_i$. One last observation we need is that $\varphi_{j+1}(\varphi_{\infty,j+1}(\sigma_\infty)) = \varphi_{\infty,j}(\sigma_\infty)$ for all $j \geq i$. As a result $x_j$ is actually a run of the game in $T_j$ consistent with $\varphi_{\infty,j}(\sigma_\infty)$. Since $\varphi_{\infty,j}(\sigma_\infty)|2(k+j) = \sigma_\infty|2(k+j)$, we can let $j$ go to infinity to find that $x_\infty$ is a run in $T_\infty$ consistent with $\sigma_\infty$. $\square$

This theorem now enables us to prove that all Borel sets enjoy the perfect set property, i.e., every Borel set is either countable or it has the same cardinality as the continuum. The proof will involve *-games in $\mathbb{R}$, which look as follows:

$$\begin{array}{llllll}
\text{I} & (U_0^0, U_1^0) & & (U_0^1, U_1^1) & & \cdots \\
\text{II} & & i_0 & & i_1 &
\end{array}$$

Note that a *-game is simply a special type of Gale-Stewart game, where the objects being played are pairs of sets for I, and binary digits for II.

$U_0^n$ and $U_1^n$ are open sets from a predetermined countable basis $\mathcal{B}$ of nonempty open sets in $\mathbb{R}$ so that $\operatorname{diam}(U_0^n) < 2^{-n}$ and $\operatorname{diam}(U_1^n) < 2^{-n}$ for all $n \in \mathbb{N}_0$. Furthermore we require that $i_n \in \{0, 1\}$, $\overline{U_0^n} \cap \overline{U_1^n} = \emptyset$, and $\overline{U_0^{n+1} \cup U_1^{n+1}} \subset U_{i_n}^n$ for all $n \in \mathbb{N}_0$. By Cantor's intersection theorem we can define the point $x \in \mathbb{R}$

by $\bigcap_{n\in\mathbb{N}_0} \overline{U^n_{i_n}} = \{x\}$. Given a set $X \subset \mathbb{R}$ we then say that the $*$-game $G^*(X)$ is won by I if $x \in X$.

So in this game I chooses two basic open sets with a diameter smaller than 1, so that their closures are disjoint. II then chooses one of these sets to continue in. I again chooses two basic open sets like before, but with diameter smaller than 1/2 and so that their closure lies within the set II chose in the previous turn. The game then continues similarly ad infinitum.

We will prove the following theorem about $*$-games.

**Theorem 3.6.** *Let $X \subset \mathbb{R}$, then*

1. *I has a winning strategy in $G^*(X)$ if and only if $X$ contains a Cantor set.*

2. *II has a winning strategy in $G^*(X)$ if and only if $X$ is countable.*

*Proof. (1)* Assume I plays a winning strategy. We construct a one-to-one map $f : 2^\omega \to X$. Given a sequence $s \in 2^\omega$ let $f(s) = x$ when $\bigcap_{n\in\mathbb{N}_0} \overline{U^n_{s|n}} = \{x\}$.

Conversely, if $C \subset X$ is a Cantor set, then we can construct a winning strategy for I. Simply let I pick $(U^n_0, U^n_1)$ according to the rules so that $U^n_0 \cap C \neq \emptyset$ and $U^n_1 \cap C \neq \emptyset$ for all $n \in \mathbb{N}_0$. This is possible because $C$ is a perfect set, which we implicitly showed in lemma 3.3. Clearly, this strategy is winning for I.

*(2)* Assume that $\sigma$ is a winning strategy for II. We call a position $p$ of even length good for $x \in X$ if it has been reached by playing according to $\sigma$ and $x$ is in the last set that II chose, i.e., $x \in U^{|p|-1}_{i_{|p|-1}}$. We consider the empty sequence to be good for $x$. Clearly there is a position that has no proper extension that is good for $x$, otherwise there exists a winning strategy for I that would reach $x$.

Assume now that $p$ is a position that is good for $x$, but has no proper extension that is also good for $x$. We define the following set:

$$X_p = \{y \in U^{n-1}_{i_{n-1}} : \text{If I plays a legal } (U^n_0, U^n_1) \text{ and } \sigma \text{ plays } i \text{ next, then } y \notin U^n_i\}.$$

Clearly $x \in X_p$, and since $x \in X$ was arbitrary we have $X \subset \bigcup_{p\in T_\sigma} X_p$. Note that $X_p$ is a singleton, for otherwise if distinct $y$ and $y'$ are in $X_p$, player I could play $(U, U')$ so that $y \in U$ and $y' \in U'$, contradicting the construction of $X_p$. Since $T_\sigma$ is countable, we conclude that $X$ must be countable as well.

Conversely, if $X$ is countable then a winning strategy for II would simply be to play according to Cantor's diagonalization argument, i.e., if $(x_0, x_1, \dots)$ is an enumeration for $X$, then in II's $n$th turn she chooses $i$ so that $x_n \notin U^n_i$. □

The map that takes $\bigcap_{n\in\mathbb{N}_0} \overline{U^n_{i_n}} = \{x\}$ to $x$ is continuous and one-to-one, hence if $X \in \mathbb{R}$ is Borel, then $X$ must be countable or have a perfect subset.

**Corollary 3.1.** *Let $X \subset \mathbb{R}$ be a Borel set, then either $|X| \leq \aleph_0$ or $|X| = \mathfrak{c}$.*

# 4 Discussion

We have investigated determinacy for the infinite Gale-Stewart games and found that even though we have determinacy for all Borel sets, there exist problematic sets such as the Bernstein sets that lead to undetermined games. As a consequence of Borel determinacy we found that all Borel sets have the perfect set property, and hence do not disprove the Continuum Hypothesis (CH). This should be expected since it was shown that CH is independent of ZFC, i.e., it can neither be proven true or false unless mathematics is inconsistent. K. Gödel showed it was impossible to disprove CH [5], and Paul J. Cohen showed that it was impossible to prove CH [1, 2].

It should be noted that we only looked at games played in $\mathbb{R}$, but in general one can look at games in any **Polish** space. A Polish space is a completely metrizable space that is separable. Note that the metric that makes the space complete is not necessarily the standard metric. With a few changes to this paper it can be shown that Borel determinacy holds in arbitrary Polish spaces.

Further research can be done on games with more players wherein each player has their own payoff set, particularly those in which multiple players cooperate, i.e., if one player in a team wins, then the entire team wins. For some games it is trivial to show determinacy if the union of a team's payoff sets is Borel. Consider for instance the 4 player game where players 1 and 3 form a team so that the union of their payoff sets is Borel. This game can be seen as a regular 2-player game, so by Martin's theorem it is determined. It becomes harder to say something about games where the two teams do not alternate playing moves.

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