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ALGEBRAIC PRESENTATIONS OF TYPE DEPENDENCY

BENEDIKT AHRENS [a], JACOPO EMMENEGGER [b], PAIGE RANDALL NORTH [c], AND EGBERT RIJKE [d]

[a] Delft University of Technology, Netherlands, and University of Birmingham, UK
e-mail address: B.P.Ahrens@tudelft.nl

[b] University of Genoa, Italy
e-mail address: jacopo.emmenegger@unige.it

[c] Utrecht University, Netherlands
e-mail address: p.r.north@uu.nl

[d] Johns Hopkins University, USA, and University of Ljubljana, Slovenia
e-mail address: erijke1@jh.edu

ABSTRACT. C-systems were defined by Cartmell as the algebraic structures that correspond exactly to generalised algebraic theories. B-systems were defined by Voevodsky in his quest to formulate and prove an initiality conjecture for type theories. They play a crucial role in Voevodsky's construction of a syntactic C-system from a term monad.

In this work, we construct an equivalence between the category of C-systems and the category of B-systems, thus proving a conjecture by Voevodsky. We construct this equivalence as the restriction of an equivalence between more general structures, called CE-systems and E-systems, respectively. To this end, we identify C-systems and B-systems as "stratified" CE-systems and E-systems, respectively; that is, systems whose contexts are built iteratively via context extension, starting from the empty context.

1. INTRODUCTION

In his unfinished and only partially published [Voe15, Voe23a, Voe16a, Voe16b, Voe17a] research programme on type theories, Voevodsky aimed to develop a mathematical theory of type theories, similar to the theory of groups or rings. In particular, he aimed to state and prove rigorously an "Initiality Conjecture" for type theories, in line with the initial semantics approach to the syntax of (programming) languages (cf. Section 1.1).

One aspect of this Initiality Conjecture is to construct, from the types and terms of a programming language, a "model", that is, a mathematical object (which is supposed to be an initial object in a category of models and their morphisms). To help with this endeavour

Key words and phrases: contextual categories, semantics of type theory, Martin-Löf type theory, B-systems, C-systems.

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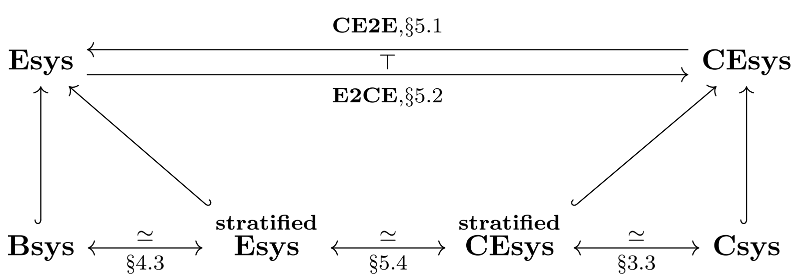
in the context of initial semantics for type theories, Voevodsky introduced the essentially-algebraic theory of *B-systems*. The models of this theory, he conjectured in [Voe14], are constructively equivalent to the well-known *C-systems* or *contextual categories* first introduced by Cartmell [Car86]. Furthermore, in his Templeton grant application [Voe16c], Voevodsky writes:

> The theory of B-systems is conjecturally equivalent to the theory of C-systems that were introduced by John Cartmell under the name “contextual categories” in [2],[3]. Proving this equivalence is among the first goals of the proposed research.

The precise role of B-systems in Voevodsky’s programme is described in [Voe16b]; we give an overview in Section 1.2 below.

In this present work, we construct an equivalence of categories between C-systems and B-systems, each equipped with a suitable notion of homomorphism. Our construction is entirely constructive, in the sense that it does not rely on the law of excluded middle or the axiom of choice.

C- and B-systems are “stratified”, in a sense that will be defined later (in Sections 3.3 and 4.3, respectively). In this work, we also introduce unstratified structures, under the name of E-system and CE-system, respectively. We construct an adjunction between these structures, and obtain the equivalence between B- and C-systems via an equivalence of suitable subcategories. The construction is summarized in the following diagram, in which maps are annotated with the respective section numbers where they are constructed:



The unstratified structures are of interest in their own right: they will serve, in a follow-up work, to relate C-systems and B-systems to other, well-established, unstratified categorical structures for the interpretation of type theories, such as categories with families [Dyb96] and natural models [Awo18], categories with attributes [Car78, Hof97],[1] and display map categories [Tay99, Nor19].

1.1. **Initial Semantics.** The “template” for initial semantics is as follows: One starts by defining a suitable notion of *signature*—an abstract specification device describing the (types and) terms of a language. To any signature, one then associates a category of *models* of that signature, in such a way that the[2] initial object in that category—if it exists—deserves to be called the *syntax generated by the signature*. Finally, one aims to construct such initial objects, or identify sufficient criteria for a signature to admit initial objects.

[1] Hofmann [Hof97, §§3.1, 3.2] also compares categories with families and categories with attributes in a set-theoretic setting, and a comparison between these notions in a univalent setting is given in [ALV18].

[2] We are working modulo isomorphism in a category.

A particularly simple example of initial semantics is the following: consider the category an object of which is given by a triple (X, x, s) where X is a set, $x \in X$, and $s : X \to X$. Then the initial object in that category is given by $(\mathbb{N}, 0, (+1))$, and the structure of being initial provides the well-known iteration principle: to define a map $\mathbb{N} \to X$, it suffices to specify $x \in X$ (the image of 0) and an endomap $s : X \to X$ (the recursive image of $(+1)$). That is, no explicit application of recursion or induction principles on $\mathbb{N}$ is required once it is established that $(\mathbb{N}, 0, (+1))$ is an initial object; instead, the initiality property provides an interface to these black-boxed principles.

For "simple" programming languages (e.g., for untyped or simply-typed lambda calculi), notions of signature, and initial semantics for such signatures, have been constructed; see, e.g., [LA24] for an overview.

For some specific dependently-typed languages, Streicher [Str91], and, more recently, De Boer, Brunerie, Lumsdaine, and Mörtberg [dBBLM], have constructed initial models. Voevodsky aimed at developing a general notion of signature for dependently-typed languages, and an initial semantics result for such signatures. In Section 1.2 we sketch Voevodsky's approach towards a theory of type theories, and the role of C- and B-systems therein.

Meanwhile, Uemura [Uem21, Section 5] has also developed a notion of signature for dependently-typed theories, and an initial semantics result for them.

1.2. **Voevodsky's approach towards a theory of type theories.** In this section, we sketch Voevodsky's plan for an initial semantics result for type theories. Voevodsky's Bonn lectures [Voe] served as the main source for this overview.

1.2.1. *Setting the scene.* In [Voe23a], Voevodsky opens with the following statement:

> *The first few steps in all approaches to the set-theoretic semantics of dependent type theories remain insufficiently understood.*

According to him, constructions and theorems about type theories are currently assumed by **analogy**. Instead, they should be proved by **specialization** of a general theorem.

Voevodsky aimed to build his theory on top of the notion of C-system, introduced by Cartmell [Car86] under the name of *contextual category*. Voevodsky calls a C-system equipped with extra operations corresponding to the inference rules of a type theory a **C-system model**—or just **model**—of type theory. To give semantics of type theory, Voevodsky aimed to build two C-system models: (i) one from the formulas and derivations of some type theory, and (ii) one from a category of abstract mathematical objects. Furthermore, one should construct an **interpretation** (a functor) from the first to the second.

Such an interpretation typically needs to be constructed by *recursion* over the derivations of the type theory. As explained in Section 1.1, the recursive pattern can be encapsulated in an initiality result; the methodology of initial semantics thus suggests the following approach:

(1) Show that the term model is initial in a suitable category.
(2) Then, any model yields automatically a (unique) interpretation from the term model.

Now, for the construction of the two desired models, syntactic and semantic, respectively, Voevodsky developed different methodologies. For the construction of *semantic* models, Voevodsky exhibited several constructions of C-systems from universe categories [Voe15]. He also sketched a strictification from categories with families to C-systems. For the construction of *syntactic* (or *term*) models, Voevodsky developed a framework outlined across several papers. We summarize the ingredients involved here:

(1) Restricted 2-sorted binding signatures (cf. [Voe23a, Section 1]) with sorts for terms and types are used as abstract specificiation devices for pretypes and preterms.
(2) From a restricted 2-sorted binding signature, a "term" monad $R : \mathbf{Set} \to \mathbf{Set}$ and a "type" module $LM : \mathbf{Set} \to \mathbf{Set}$ over R are constructed (cf. [Voe23a, Section 1]).
(3) Any monad R on **Set** gives rise to a C-system $C(R)$, corresponding to the mono-typed (or untyped) syntax of R, cf. [Voe23a, Section 4.2].
(4) The *presheaf extension* of $C(R)$ by the module LM over R, called $C(R)[LM]$, constitutes the C-system of pretypes and preterms—but without any typing relation yet, cf. [Voe23a, Section 4.2].
(5) Finally, Voevodsky's theory of *sub-C-systems and regular quotients of C-systems* [Voe16b] allows one to carve out C-systems of types and well-typed terms modulo a regular congruence relation.

In the following, we discuss some of these ingredients in slightly more detail, but without any rigorous definitions.

A "restricted 2-sorted binding signature" is a signature that specifies a 2-sorted language. We can think of these two sorts as a sort type of "types" and a sort term of "terms", respectively. The signatures are "restricted" in the sense that constructors can bind variables of sort term but not of sort type.

We do not dwell on the notion of signature, but refer instead to [Voe23a, Section 1] for details; here, we give an example of a language specified by such a signature.

Example 1.1. An example of a syntax generated by a 2-sorted binding signature is the syntax of the Calculus of Constructions, adapted from Streicher's *Semantics of Type Theory* [Str91]:

A, B	$::=$	$\Pi(A, x.B)$	Product of types
	$\mid$	$Prop$	Type of propositions
	$\mid$	$Proof(t)$	Type of proofs of proposition t
t, u	$::=$	x	Variable
	$\mid$	$\lambda(A, x.t)$	Function abstraction
	$\mid$	$App(A, x.B, t, u)$	Function application
	$\mid$	$\forall(A, x.t)$	Universal quant. over propositions t

This signature specifies a language with two sorts, the sort type of "types" and the sort term of "terms". It is restricted because there is no binding of variables of sort type, only of variables of sort term. Such a signature yields a monad $T : \mathbf{Set} \times \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}$,

$$(X, Y) \mapsto (\mathsf{type}(X, Y), \mathsf{term}(X, Y))$$

where $\mathsf{type}(X, Y)$ is the set of expressions of sort type with variables of kind type in X and of kind term in Y and similarly for $\mathsf{term}(X, Y)$. From such a monad on $\mathbf{Set} \times \mathbf{Set}$, Voevodsky [Voe23a] constructs, by fixing a set of type variables, a monad $R = \mathsf{term}$ on **Set**, and a module $LM = \mathsf{type}$ over R. Here, the action of the module LM is substitution of term expressions in type expressions. From R and LM, in turn, Voevodsky [Voe23a] constructs two C-systems, called $C(R)$ and $C(R)[LM]$, respectively. The C-system $C(R)$ corresponds to a mono-typed syntax of just terms—in detail:

(1) Objects are natural numbers (untyped contexts).
(2) Morphisms $m \to n$ are maps $[n] \to R([m])$, where $[k]$ is the standard finite set associated to $k \in \mathbb{N}$.

(3) The category thus obtained is the opposite of the Kleisli category on R restricted to natural numbers.[3]
(4) The morphism $\mathsf{p}_n : n+1 \to n$ is given by the composition $[n] \xrightarrow{\iota} [n+1] \xrightarrow{\eta} R([n+1])$.
(5) Given a morphism $f : m \to n$, that is, a function $f : [n] \to R([m])$, the pullback of $\mathsf{p}_n : n{+}1 \to n$ along f is the morphism $\mathsf{p}_m : m{+}1 \to m$. The morphism $m{+}1 \to n{+}1$ required to complete the pullback square is the morphism $q(f) : n+1 \to R([m+1])$ induced by the morphisms $n \xrightarrow{f} R([m]) \xrightarrow{R(\iota)} R([m+1])$ and $1 \xrightarrow{\iota} [m+1] \xrightarrow{\eta} R([m+1])$; intuitively, $q(f)$ extends the substitution f by one variable. See also [Voe23a, Lemma 4.2.2].

The C-system $C(R)[LM]$, in turn, looks as follows:
(1) $C(R)[LM]$ has, as contexts, finite sequences of types (with a suitable number of free variables).
(2) Pullback is given by substitution of terms in type expressions.
(3) There is no typing relationship yet: $C(R)[LM]$ is a C-system of pretypes and preterms.

In order to build, from $C(R)[LM]$, a C-system of types and well-formed terms, with the intended typing relation, Voevodsky devised (i) sub-C-systems (for eliminating ill-formed pretypes and preterms), (ii) quotients of C-systems (for considering terms and types modulo judgemental equality). To construct such subsystems and quotients, Voevodsky devised the theory of B-systems.

1.2.2. *B-systems for the construction of C-systems.* Intuitively, the idea is to use the C-system $C(R)[LM]$ to obtain the pretypes and preterms to formulate **judgements**:
- A statement $\Gamma \vdash$ is an element of
$$B(R, LM) := \coprod_{n\geq 0} \prod_{i=0}^{n-1} LM([i]) \tag{1.1}$$
- A statement $\Gamma \vdash t : T$ is an element of
$$\widetilde{B}(R, LM) := \coprod_{n\geq 0} \left(\prod_{i=0}^{n-1} LM([i]) \times R([n]) \times LM([n]) \right) \tag{1.2}$$

Voevodsky [Voe14] defines eight operations on B and $\widetilde{B}$, corresponding to structural rules of type theory. The resulting mathematical structure is captured by the notion of **B-system**, illustrated in more detail in Section 1.3 and studied in detail in Section 4.

Given a C-system C, we call $B(C)$ and $\widetilde{B}(C)$ the B-sets associated to C. Voevodsky [Voe16b] constructed a bijection between
(1) Sub-C-systems of a given C-system
(2) Subsets of $(B, \widetilde{B}(C))$ that are closed under the eight operations

and similar, but more complicated, for quotients. This bijection is used by Voevodsky to construct suitable C-systems; Voevodsky himself [Voe14] positions B-systems as follows:

> B-systems are algebras (models) of an essentially algebraic theory that is expected to be constructively equivalent to the essentially algebraic theory of C-systems which is, in turn, constructively equivalent to the theory of contextual categories. The theory of B-systems is closer in its form to the

[3] Put differently, it is the Kleisli category of the Jf-relative monad induced by the monad R, as indicated by the title of Voevodsky's article [Voe23a].

structures directly modeled by contexts and typing judgements of (dependent) type theories and further away from categories than contextual categories and C-systems.

This concludes our overview of the use of B-systems in Voevodsky's research program. In the remainder of the introduction, we provide more intuition for the notions of B-system and C-system, before giving rigorous definitions and constructions.

1.3. **Models of Type Theory.** When studying type theories mathematically, one question to answer is: what is the appropriate mathematical structure that captures the essential behaviour of type theories? Technically speaking: what are the objects in the category of models of a type theory?

Many different answers have been given to this question. The purpose of this section is to present the two contenders studied and compared in this work, and to relate them to other notions of "model".

1.3.1. *Contextual categories and C-systems.* Contextual categories were defined, by Cartmell [Car86, §14], as a mathematical structure for the interpretation of generalized algebraic theories and of the judgements of Martin-Löf type theory. A contextual category comes with a tree structure, in particular, a partial ordering, on its objects; think of the objects of $\mathcal{C}$ as "contexts", and $\Gamma \leq \Delta$ stating that Γ can be obtained from Δ by truncation. Furthermore, there is a special class of morphisms, closed under pullback along arbitrary morphisms—thought of as substitution by that morphism. In his PhD dissertation [Car78, Section 2.4], Cartmell shows that the category of contextual categories and homomorphisms between them is equivalent to the category of generalized algebraic theories and (equivalence classes of) interpretations between them.

Voevodsky defined C-systems as equivalent to contextual categories: a C-system is a category coming, in particular, with a length function and a compatible "father" function on objects of the category, signifying truncation of contexts. Again, we have a class of morphisms closed under pullback along arbitrary morphisms. Voevodsky rejected the name "contextual category" for these mathematical object, for the reason that the extra structure on top of the underlying category cannot be transported along equivalence of categories and is thus not "categorical" in nature. As an example, consider the terminal category: it can be equipped with exactly one C-system structure. However, there is no C-system structure on any category with more than one, but finitely many, objects.

More recently, Cartmell [Car18] gave two Generalized Algebraic axiomatizations of contextual categories, one of which is using Voevodsky's s-operator [Voe16b, Definition 2.3] for pullbacks.

1.3.2. *B-systems.* Voevodsky's definition of B-systems [Voe14] is inspired by the presentation of type theories in terms of *inference rules.* Specifically, type theories "of Martin-Löf genus" are given by sets of five kinds of judgements:

Well-formed context:

$$\Gamma \vdash$$

Well-formed type in some context:

$$\Gamma \vdash A \ \mathsf{type}$$

Well-formed term of some type in some context:

$$\Gamma \vdash a : A$$

Equality of types:

$$\Gamma \vdash A \equiv B$$

Equality of terms:

$$\Gamma \vdash a \equiv b : A$$

Interpreting equality of types and terms as actual equality, and expressing $\Gamma \vdash A$ instead as $\Gamma, A \vdash$, lead Voevodsky to defining a B-system to consist of families of sets $(B_n)_{n\in\mathbb{N}}$ and $(\tilde{B}_n)_{n\in\mathbb{N}_{>0}}$, intuitively denoting, for any $n \in \mathbb{N}$, contexts of length n and terms in a context of length $n-1$, together with their types, respectively. Furthermore, any B-system has various operations on B and $\tilde{B}$, such as maps $\partial_n : \tilde{B}_{n+1} \to B_{n+1}$ specifying, intuitively, for each "term" $t \in \tilde{B}_{n+1}$, the context $\partial_n(t) \in B_{n+1}$ in which t lives.

Voevodsky's B-systems are very similar to the algebras of the theory *MetaGAT* defined by John Cartmell [Car14], and to the algebras of a monad studied by Richard Garner [Gar15]. The intention is that these are all equivalent notions of structure. Below, we will indicate more precise connections to Garner's work.

Compared to other semantics for dependent type theories, B-systems appear the closest to syntax. For this reason it seems easier to describe extensions of the structural rules by type constructors or modal operators for B-systems than it is for, say, C-systems (*i.e.* contextual categories). For the same reason, B-systems seem also more suitable than other semantics to describe notions of substructural dependent type theories and, more generally, variations on the syntax.

1.3.3. *Other Notions of Model.* There are many other mathematical structures for the interpretation of type theory. Here, we give some pointers to related literature.

Voevodsky sketched a relation between C-systems and categories with families in his Lectures in the Max Planck Institute in Bonn [Voe, Lecture 5], identifying C-systems as categories with families with a particular property. In the present work, we introduce and study unstratified categorical structures, in the form of E-systems and CE-systems, which we anticipate will be useful in giving a precise construction for Voevodsky's conjecture.

Categories with families, in turn, are related to categories with attributes (a.k.a. split type categories) in [Bla91] (in a categorical setting) and in [ALV18] (in the univalent setting). Composing these characterizations with the equivalence presented here provides a comparison between B-systems and other mathematical structures for type theory.

Garner [Gar15] studies and compares two structures related to Voevodsky's B-systems: Generalized Algebraic Theories (GATs) and algebras for a monad on the category of type-and-term structures (see also Examples 4.4 and 4.13).

Remark 1.2. Garner's and Cartmell's works, taken together, also point to another possible way to constructing an equivalence between C-systems and B-systems: Cartmell [Car78, Section 2.4] constructs an equivalence of categories between the category of contextual categories and homomorphisms between them, and the category of GATs and (equivalence classes of) interpretations between them. Garner [Gar15] constructs an equivalence of categories between the category of B-frames and the category of $\emptyset$-GATs (GATs without structural rules) (see also Example 4.4). Garner's equivalence looks like it could be "upgraded" to an equivalence between the category of B-systems and the category of GATs (see also Example 4.13). Constructing an equivalence between B- and C-systems in this way is, however, conceptually circular — if not in actuality, then at least in spirit. After all, B- and C-systems were studied by Voevodsky in the context of the Initiality Conjecture; one purpose of the Initiality Conjecture is to give a **specification of dependently-typed syntax**. We believe that such a specification is best given without recourse to such syntax itself.

1.4. **About the present work.** The main result of this paper is the construction of an equivalence of categories, between the category of C-systems and the category of B-systems. The existence of such an equivalence was conjectured by Voevodsky.

We construct this equivalence as a restriction of an equivalence between more general, unstratified structures introduced in this paper, called CE-systems and E-systems, respectively. While it is not *necessary* to pass via E-systems and CE-systems to construct an equivalence between B-systems and C-systems, it seems *desirable* to us for two reasons:

(1) The definitions and constructions are automatically more modular, isolating structure on either side that corresponds to each other.
(2) The study of unstratified structures is useful in connecting B-systems and C-systems to other unstratified structures, such as categories with families [Dyb96]. Work on constructing a suitable comparison is already underway.

This paper is organized as follows. In Section 2 we discuss some prerequisites that we build upon in later sections. In Section 3 we review the definition of C-systems given by Voevodsky in [Voe16b, Def. 2.1], itself an equivalent formulation of Cartmell's definition of contextual categories [Car86, §14]. Here we also introduce CE-systems. In Section 4, we give Voevodsky's definition of B-system [Voe14] and introduce E-systems. In Section 5 we construct our equivalence of categories between B-systems and C-systems.

1.4.1. *Foundations.* The work described in this result can be read to take place in intuitionistic set theory (IZF) or extensional type theory, i.e., a type theory with equality reflection. In particular, we do not make use of classical reasoning principles such as an axiom of choice or excluded middle. We consider in this work categories built from algebraic structures (which sometimes are themselves categories with structure, but see Section 1.4.2). Implicitly, we take these algebraic structures to be built from sets (or types) from a universe $\mathcal{U}_1$. The categories of such structures are hence categories built from sets (or types) of objects and morphisms of a universe $\mathcal{U}_2$. In the following, we leave the universe levels implicit.

1.4.2. *About our use of categories.* In this work, categories are used on two different levels.

Firstly, we use categories as algebraic structures, as the basis for C-systems and CE-systems. This use of categories is somewhat "accidental", and our constructions on these categories are not invariant under equivalence of categories. In particular, we liberally reason about equality of objects in such categories. Consequently, we avoid the unadorned word "category" for these structures, and call them *strict categories* instead. We denote by **Cat** the category of strict categories and functors between them.

Secondly, we use categories to compare different mathematical structures to each other, by considering a suitable category of such structures and their homomorphisms. Here, we never consider equality, but only isomorphism, of such mathematical structures; our reasoning on that level is entirely categorical. We reserve the word "category" for such uses of the concept.

We use different fonts for strict categories and categories, respectively: calligraphic font, such as $\mathcal{C}$, is used for strict categories; boldface, such as **Grph**, is used for categories. We use the same notation for arrows in strict categories and in categories. We write either $g \circ f$ or gf for the composition of $f : a \to b$ and $g : b \to c$.

1.5. **Version history.** We first reported on the construction of an equivalence of categories between B-systems and C-systems in a short summary paper [AENR23]. The present work expands on that previous work in the following ways:

(1) An expanded introduction summarizes the role of B-systems and C-systems in Voevodsky's unfinished theory of type theories.
(2) We consider here more general "unstratified" variants of B-systems and C-systems, called E-systems and CE-systems, respectively. We construct an equivalence of categories between E-systems and CE-systems.
(3) We then construct the equivalence between B-systems and C-systems as a restriction, to the respective subcategories of stratified objects, of the aforementioned equivalence.
(4) We give full details of all the constructions in this paper.

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2. Preliminaries: stratification of categories

In this section we collect definitions and results related to **stratification of strict categories** and morphisms between them. A stratification (see Definition 2.1) associates, to any object of a strict category a natural number, its "length", and to any length-decreasing morphism a factorization of this morphism into morphisms "of length 1". Such a stratification can equivalently be described as a rooted tree, see Section 2.2.

2.1. Stratification of strict categories.

Definition 2.1 (Stratified strict categories, stratified functors)**.** Let $\mathcal{C}$ be a strict category with terminal object 1. A **stratification** for $\mathcal{C}$ consists of a *stratification functor*

$$L : \mathcal{C} \to (\mathbb{N}, \geq)$$

such that

(1) $L(X) = 0$ if and only if X is the chosen terminal object 1,
(2) for any $f : X \to Y$ we have $L(X) = L(Y)$ if and only if $X = Y$ and $f = \mathrm{id}_X$, and

(3) every morphism $f : X \to Y$ in $\mathcal{C}$, where $L(X) = n + m + 1$ and $L(Y) = n$, has a unique factorization

$$X = X_{m+1} \xrightarrow{f_m} X_m \xrightarrow{f_{m-1}} \cdots \xrightarrow{f_1} X_1 \xrightarrow{f_0} X_0 = Y$$

where $L(X_i) = n + i$.

A functor $F : \mathcal{C} \to \mathcal{D}$ between strict categories with stratifications $L_{\mathcal{C}}$ and $L_{\mathcal{D}}$, respectively, is said to be **stratified** if $L_{\mathcal{C}} = L_{\mathcal{D}} \circ F$.

Remark 2.2. We emphasize that stratifications do not transport along equivalence of categories. For instance, there is a (necessarily unique, see Proposition 2.8) stratification on the terminal strict category, but no stratification on the strict chaotic category with two objects and a choice of terminal object.

Remark 2.3. Those readers familiar with Conduché functors might note that a stratified category $(\mathcal{C}, L)$ is equivalently a Conduché functor $L : \mathcal{C} \to \mathbb{N}$ with discrete fibers which takes the chosen terminal object of $\mathcal{C}$ to 0.

Definition 2.4. Let $\mathcal{C}$ be a category with a terminal object and $\ell\colon \mathrm{Ob}(C) \to \mathbb{N}$ a function. An arrow $f\colon X \to Y$ in $\mathcal{C}$ is **indecomposable** if $\ell(X) = \ell(Y) + 1$.

Remarks 2.5.

(1) In a stratified category, there is a unique terminal object 1. More generally, if there is an arrow $1 \to X$, then $X = 1$.
(2) The factorisation of an arrow $f\colon X \to Y$ such that $L(X) - L(Y) = m + 1 > 0$ in 2.1.3 consists of $m + 1$ indecomposable arrows.
(3) A stratified functor is determined by its action on indecomposable arrows.

Lemma 2.6. *Let $\mathcal{C}$ be a category with a terminal object 1. A function $\ell\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$ extends to a stratification $L\colon \mathcal{C} \to (\mathbb{N}, \geq)$ of $\mathcal{C}$ if and only if the following three conditions hold:*

(i) $\ell(1) = 0$,

(ii) for every object X and $k \leq \ell(X)$, the set

$$\coprod_{Y \,|\, \ell(Y)=k} \mathcal{C}(X, Y)$$

is a singleton, i.e. there is a unique arrow $x_k\colon X \to X_k$ such that $\ell(X_k) = k$, and

(iii) for every X and $k > \ell(X)$, the set

$$\coprod_{Y \,|\, \ell(Y)=k} \mathcal{C}(X, Y)$$

is empty, i.e. there are no arrows $X \to Y$ such that $\ell(X) < \ell(Y)$.

Proof. If $\mathcal{C}$ is stratified by L such that $L(X) = \ell(X)$, condition (i) follows from 2.1.1. To show condition (ii), note that every arrow $X \to 1$ factors uniquely into $l = \ell(X)$ indecomposable arrows

$$X \xrightarrow{x_l} X_{l-1} \xrightarrow{x_{l-1}} \cdots \xrightarrow{x_2} X_1 \xrightarrow{x_1} 1.$$

In particular, for every $n \leq \ell(X)$, the composite $x_{n+1} \cdots x_l\colon X \to X_n$ is such that $\ell(X_n) = n$. If $f\colon X \to Y$ is also such that $\ell(Y) = n$, then f factors into $l - n$ indecomposable arrows $(f_i)_{i=1}^{l-n}$. Let $y_1 \cdots y_n$ be the factorisation of $Y \to 1$ into indecomposable arrows. The composite $y_1 \cdots y_n f_0 \cdots f_{l-n-1}$ is a factorisation of $X \to 1$ into indecomposable arrows. It follows by uniqueness of such factorisations that

$$x_1 = y_1,\ x_2 = y_2,\ \ldots\ ,\ x_n = y_n,\ x_{n+1} = f_1,\ \ldots\ ,\ x_l = f_{l-n}.$$

In particular, $f = x_{n+1} \cdots x_l$ as required.

Since $(\mathbb{N}, \geq)$ is a poset, condition (*iii*) is equivalent to the fact that the function ℓ extends uniquely to a functor $L \colon \mathcal{C} \to (\mathbb{N}, \geq)$.

Suppose now that conditions (*i*–*iii*) above hold. In particular, the function ℓ extends to a functor L.

(1) Let X be such that $\ell(X) = 0$. Then $X \to 1$ and $\mathrm{id}_X \colon X \to X$ are both such that $\ell(X) = 0 = \ell(1)$. Hence $X = 1$ and the object X is terminal. Conversely, let X be terminal. Then there is $1 \to X$ and thus $0 \geq \ell(X)$.

(2) Let $f \colon X \to Y$ and suppose $\ell(X) = \ell(Y)$, then $Y = X$ and $f = \mathrm{id}_X$ by (*ii*) with $n = \ell(X)$.

(3) For every X such that $n + 1 = \ell(X) > 0$, let $\overline{X} \colon X \to X'$ be the unique arrow such that $\ell(X') = n$ given by (*ii*). For every $k \leq \ell(X)$, we have a composite x_k of k indecomposable arrows

$$X \xrightarrow{\overline{X}} X' \xrightarrow{\overline{X'}} \cdots \xrightarrow{\overline{X^{(k-2)}}} X^{(k-1)} \xrightarrow{\overline{X^{(k-1)}}} X^{(k)}, \qquad x_k \colon X \to X^{(k)} \tag{2.1}$$

where $\ell(X^{(k)}) = \ell(X) - k$, which is the unique arrow $X \to Y$ such that $\ell(Y) = \ell(X) - k$ by (*ii*). Let us show that (2.1) is also the unique factorisation of x_k into indecomposable arrows, for every $0 < k \leq \ell(X)$. We proceed by induction on n. If $n = 0$, then factorisations consist of only one indecomposable arrow and uniqueness follow from (*ii*). For $n > 0$, let $0 < k \leq n + 1$ and consider a factorisation $X \xrightarrow{g_0} Z_1 \xrightarrow{g_1} \cdots \xrightarrow{g_{k-2}} Z_{k-1} \xrightarrow{g_{k-1}} X^{(k)}$ of x_k into indecomposable arrows. Then $\ell(Z_1) = \ell(X) - 1 = \ell(X')$, and so $g_0 = \overline{X}$ by (*ii*). Again, $g_{k-1} \cdots g_1 = x'_{k-1} \colon X' \to (X')^{(k-1)}$ by (*ii*) and, by inductive hypothesis, $g_i = \overline{X^{(i)}}$ for $0 < i < k$. Therefore (2.1) is the unique factorisation of x_k into indecomposable arrows.

Given an arrow $f \colon X \to Y$ such that $n = \ell(Y) < \ell(X) = m + n + 1$, it must be $Y = X^{(m+1)}$ and $f = x_{m+1}$ by (*ii*). It follows that f factors uniquely into $m + 1$ indecomposable arrows $\overline{X^{(m)}} \cdots \overline{X'}\,\overline{X}$. □

Remark 2.7. Condition (*ii*) in Lemma 2.6 is equivalent to requiring that, for every object X:

(*ii.a*) for every $n \leq \ell(X)$ there is at most one arrow $f \colon X \to Y$ such that $\ell(Y) = n$, and
(*ii.b*) there is an indecomposable arrow $\overline{X} \colon X \to X'$.

One direction is clear. For the converse it is enough to show that for every $n < \ell(X)$ there is $f \colon X \to Y$ such that $\ell(Y) = n$. Such an arrow is given as the composite of $\ell(X) - n$ indecomposable arrows as in Eq. (2.1) above.

Proposition 2.8. *Any category can be stratified in at most one way.*

Proof. Consider a category $\mathcal{C}$ with two stratifications $L, M : \mathcal{C} \to \mathbb{N}$. By Lemma 2.6.*ii*,

$$L^{-1}(n+1) = \big\{X \mid \exists\ Y \in L^{-1}(n) \text{ and an indecomposable} f : X \to Y\big\}$$

and similarly for $M^{-1}(n+1)$. Thus, if $L^{-1}(n) = M^{-1}(n)$, we find that $L^{-1}(n+1) = M^{-1}(n+1)$, and the claim follows by induction since $L^{-1}(0) = M^{-1}(0)$ by Definition 2.1.1. □

Uniqueness of stratification justifies the following definition:

Definition 2.9. We define $\mathbf{Cat_s}$ to be the subcategory of **Cat** consisting of stratified strict categories and stratified functors between them.

Lemma 2.10. *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor between stratified categories. The following are equivalent.*

(1) *The functor F is stratified.*
(2) *The functor F preserves terminal objects and indecomposable arrows.*

Proof. That 1 implies 2 is clear. The converse is by induction on the length of objects using that, for every $f: X \to Y$, $L_{\mathcal{C}}(X) = L_{\mathcal{C}}(Y) + 1$ implies $L_{\mathcal{D}}(F(X)) = L_{\mathcal{D}}(F(Y)) + 1$. □

Lemma 2.11. *Let $\mathcal{C}$ be a stratified category with stratification functor L. Then for every object X and every $f: Y \to X$,*

$$L_X(f) := L(Y) - L(X)$$

defines a stratification functor L_X for the slice $\mathcal{C}/X$.

Proof. The above clearly defines a functor $L_X: \mathcal{C}/X \to (\mathbb{N}, \geq)$ and conditions (1–3) in Definition 2.1 are easily verified. □

Corollary 2.12. *Let $F: \mathcal{C} \to \mathcal{D}$ be a stratified functor between stratified categories. For every object X in $\mathcal{C}$, the functor*

$$\mathcal{C}/X \xrightarrow{F/X} \mathcal{D}/FX$$

is stratified.

2.2. **Rooted trees.** In this section, we compare stratified categories to rooted trees. Rooted trees were used by Cartmell [Car86] to give his original definition of contextual categories.

Definition 2.13.

(1) We define a **rooted tree** T to be a family of sets $(T_n)_{n\in\mathbb{N}}$ indexed by the natural numbers such that T_0 is a singleton, together with functions $(t_n: T_{n+1} \to T_n)_{n\in\mathbb{N}}$ mapping a node to its parent. A homomorphism of rooted trees $f: T \to S$ is a family of functions $(f_n: T_n \to S_n)_{n\in\mathbb{N}}$ such that $f_n \circ t_n = s_n \circ f_{n+1}$ for every $n \in \mathbb{N}$. Let **RtTr** be the category of rooted trees and homomorphisms.
(2) Let **Grph** be the category of directed (multi)graphs and homomorphisms. We define the functor $\mathbf{G} : \mathbf{RtTr} \to \mathbf{Grph}$ as follows. For a rooted tree T, the directed graph $\mathbf{G}(T)$ has the set of vertices given by the disjoint union $\coprod_{n\in\mathbb{N}} T_n$, and there is an edge $(n+1, X) \to (n, t_n(X))$ for every $n \in \mathbb{N}$ and $X \in T_{n+1}$. It is straightforward to verify that each homomorphism $T \to T'$ of rooted trees gives rise to a homomorphism of graphs $\mathbf{G}(T) \to \mathbf{G}(T')$.
(3) Let $\mathbf{F} : \mathbf{Grph} \to \mathbf{Cat}$ be the well-known functor [Mac98, II.7] that takes a graph to the category freely generated by it.

We now show that the image of the composite

$$\mathbf{RtTr} \xrightarrow{\mathbf{G}} \mathbf{Grph} \xrightarrow{\mathbf{F}} \mathbf{Cat}$$

is the subcategory of *stratified strict categories* defined in Definition 2.9.

Proposition 2.14. *The functor* $\mathbf{FG}\colon \mathbf{RtTr} \to \mathbf{Cat}$ *lifts to an equivalence* $\mathbf{RtTr} \xrightarrow{\simeq} \mathbf{Cat_s}$.

Proof. First, observe that, for a rooted tree T, the free category $\mathbf{FG}(T)$ is stratified. We define $L : \mathbf{FG}(T) \to (\mathbb{N}, \geq)$ by sending an object (n, X) to n and a generating morphism $(n+1, X) \to (n, t_n(X))$ to $n+1 \geq n$. Given a morphism $f : S \to T$ of rooted trees, the functor $\mathbf{FG}(f) : \mathbf{FG}(S) \to \mathbf{FG}(T)$ is stratified by construction. Thus, the functor $\mathbf{FG} : \mathbf{RtTr} \to \mathbf{Cat}$ lifts to an functor $\mathbf{RtTr} \to \mathbf{Cat_s}$.

Next we define a functor $\mathbf{I}\colon \mathbf{Cat_s} \to \mathbf{RtTr}$. Consider a stratified category $(\mathcal{C}, L)$ and define a rooted tree $\mathbf{I}(\mathcal{C}, L)$ as follows. Let $\mathbf{I}(\mathcal{C}, L)_n := L^{-1}(n)$. By Lemma 2.6(*ii*), for every $X \in \mathbf{I}(\mathcal{C}, L)_n$ there is exactly one indecomposable arrow with domain X, say $X \to X'$. Then we define $t_n : \mathbf{I}(\mathcal{C}, L)_{n+1} \to \mathbf{I}(\mathcal{C}, L)_n$ by $t_n(X) := X'$. A stratified functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a homomorphism of rooted trees $\mathbf{I}(\mathcal{C}, L) \to \mathbf{I}(\mathcal{C}, M)$ since it commutes with the stratification functors and it preserves indecomposable arrows.

It is now straightforward to verify that $\mathbf{FG} \circ \mathbf{I} \cong 1_{\mathbf{Cat_s}}$ and $\mathbf{I} \circ \mathbf{FG} \cong 1_{\mathbf{RtTr}}$. ☐

3. The category of C-systems

This section is dedicated to the study of C-systems.

In Section 3.1 we review Voevodsky's definition of C-system, an equivalent formulation of Cartmell's contextual categories. We then give, in Section 3.2 our definition of CE-system, and identify, in Section 3.3, the category of C-systems as a subcategory of "stratified" objects in the category of CE-systems.

3.1. The category of C-systems.

John Cartmell [Car86, Section 14] defined *contextual categories* as mathematical structures for the interpretation of type theories. Vladimir Voevodsky [Voe16b, Definition 2.1] gave a slightly modified, but obviously equivalent definition, and coined them *C-systems*.

Definition 3.1 (C-system, [Voe16b, Def. 2.1])**.** A **C-system** consists of

(1) a strict category $\mathcal{C}$,
(2) a "length" function $\ell\colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$,
(3) a chosen object $1 \in \mathrm{Ob}(\mathcal{C})$,
(4) a function $\mathsf{ft}\colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{C})$,
(5) for any object $\Gamma \in \mathrm{Ob}(\mathcal{C})$ such that $\ell(\Gamma) > 0$, a morphism $\mathsf{p}_\Gamma\colon \Gamma \to \mathsf{ft}(\Gamma)$,
(6) for any $\Gamma \in \mathrm{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$ and any $f\colon \Delta \to \mathsf{ft}(\Gamma)$, an object $f^*\Gamma$ and a morphism $\mathsf{q}(f, \Gamma)\colon f^*\Gamma \to \Gamma$.

satisfying the following axioms:

i) $\ell^{-1}(0) = \{1\}$,
ii) for Γ with $\ell(\Gamma) > 0$, we have $\ell(\mathsf{ft}(\Gamma)) = \ell(\Gamma) - 1$,
iii) $\mathsf{ft}(1) = 1$,
iv) 1 is a final object,
v) for $\Gamma \in \mathrm{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$ and $f\colon \Delta \to \mathsf{ft}(\Gamma)$, one has $\ell(f^*\Gamma) > 0$, $\mathsf{ft}(f^*\Gamma) = \Delta$, and the square

$$\begin{array}{ccc} f^*\Gamma & \xrightarrow{\mathsf{q}(f,\Gamma)} & \Gamma \\ {\scriptstyle \mathsf{p}_{f^*(\Gamma)}}\downarrow & & \downarrow{\scriptstyle \mathsf{p}_\Gamma} \\ \Delta & \xrightarrow{f} & \mathsf{ft}(\Gamma) \end{array} \tag{3.1}$$

commutes and is a pullback square,

vi) for $\Gamma \in \mathrm{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$, we have $\left(\mathrm{id}_{\mathsf{ft}(\Gamma)}\right)^* \Gamma = \Gamma$ and $\mathsf{q}(\mathrm{id}_{\mathsf{ft}(\Gamma)}, \Gamma) = \mathrm{id}_\Gamma$, and

vii) for $\Gamma \in \mathrm{Ob}(\mathcal{C})$ with $\ell(\Gamma) > 0$, $g \colon \Delta \to \mathsf{ft}(\Gamma)$ and $f \colon E \to \Delta$, we have $(g \circ f)^*\Gamma = f^* g^* \Gamma$ and $\mathsf{q}(g \circ f, \Gamma) = \mathsf{q}(g, \Gamma) \circ \mathsf{q}(f, g^*\Gamma)$.

Intuitively, an object Γ of the category underlying a C-system can be thought of as a context of length $\ell(\Gamma)$. Types in context Γ are encoded by the projections p_Δ with $\mathsf{ft}(\Delta) = \Gamma$ (hence, in particular, $\ell(\Delta) = \ell(\Gamma) + 1$). Terms are not explicitly given; a term of type p_Δ (in context $\mathsf{ft}(\Delta)$) corresponds to a section to p_Δ. This is exactly how terms are defined in the E-system constructed from a CE-system in Construction 5.6.

In case the reader wonders whether the length function ℓ lifts to a stratification, in Corollary 3.9 we show that it does so on a suitable subcategory of $\mathcal{C}$.

Definition 3.2. A **morphism of C-systems** from $\mathbb{C}$ to $\mathbb{D}$ is a functor $F \colon \mathcal{C} \to \mathcal{D}$ between the underlying categories that strictly preserves the rest of the structure, that is:

i) $F(1_\mathbb{C}) = 1_\mathbb{D}$,

ii) $\ell_\mathbb{D} \circ \mathrm{Ob}(F) = \ell_\mathbb{C} \colon \mathrm{Ob}(\mathcal{C}) \to \mathbb{N}$,

iii) $\mathrm{Ob}(F) \circ \mathsf{ft}_\mathbb{C} = \mathsf{ft}_\mathbb{D} \circ \mathrm{Ob}(F) \colon \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$,

iv) $F\mathsf{p}_\Gamma = \mathsf{p}_{F\Gamma}$, for every $\Gamma \in \mathrm{Ob}(\mathcal{C})$,

v) $F(f^*\Gamma) = (Ff)^*(F\Gamma)$ and $F(\mathsf{q}(f, \Gamma)) = \mathsf{q}(Ff, F\Gamma)$, for every $\Gamma \in \mathrm{Ob}(\mathcal{C})$ such that $\ell_\mathbb{C}(\Gamma) > 0$ and $f \colon \Delta \to \mathsf{ft}(\Gamma)$.

Example 3.3 (C-systems and Lavwere theories)**.** Fiore and Voevodsky [FV20] construct an isomorphism of categories between the category of Lawvere theories and the category of *ℓ-bijective* C-systems, that is, of C-systems whose length function is a bijection. Intuitively, such a C-system can be seen as modelling an untyped (or single-sorted) language.

Example 3.4 (C-systems and contextual categories)**.** C-systems are equivalent to Cartmell's contextual categories. In his Ph.D. dissertation, Cartmell [Car78, Section 2.4] constructs an equivalence between the category of contextual categories and homomorphisms between them and the category of Generalized Algebraic Theories (GATs) and (equivalence classes of) interpretations between them. Hence C-systems are equivalent to GATs.

Example 3.5 (C-system from a universe category)**.** Any universe category gives rise to a C-system, via a construction by Voevodsky [Voe15, Construction 2.12]. A universe category is a category with a chosen terminal object and a *universe*, that is, a morphism $p : \tilde{\mathcal{U}} \to \mathcal{U}$ together with a choice of pullback of p along any morphism $X \to \mathcal{U}$. Roughly, the C-system constructed from a universe category has, as objects of length n, sequences of n morphisms $f_1, \ldots, f_n$ into $\mathcal{U}$ such that the domain of f_{i+1} is the chosen pullback of p along f_i. Such a sequence can be thought of as a sequence of (dependent) types $(A_1, A_2, \ldots, A_n)$ such that $A_1, \ldots A_i \vdash A_{i+1}$. Furthermore, any small C-system can be obtained via this construction; given a C-system $\mathbb{C}$, a universe can be constructed [Voe15, Construction 5.2] on the presheaf category $\hat{\mathbb{C}}$ such that the C-system obtained from that universe is isomorphic to the C-system $\mathbb{C}$. For a brief overview of these constructions, see [KL21, Section 1.3].

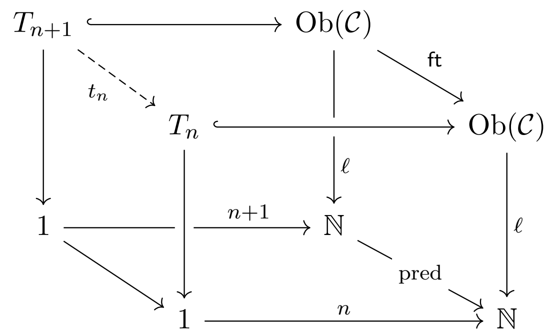
Voevodsky's simplicial model of univalent foundations [KL21] is built on top of a C-system obtained from a universe in the category of simplicial sets.

Problem 3.6. To construct a functor $\mathbf{C2RtTr} \colon \mathbf{Csys} \to \mathbf{RtTr}$.

Construction 3.7 (for Problem 3.6)**.** Let $\mathbb{C} = (\mathcal{C}, 1, \ell, \mathsf{ft}, \mathsf{p}, \dots)$ be a C-system. The objects of $\mathcal{C}$ can be arranged into a rooted tree by defining

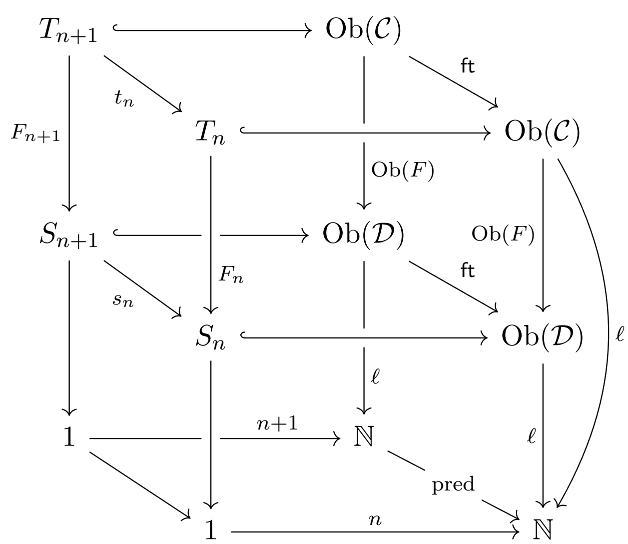
$$T_n := \{\Gamma \mid \ell(\Gamma) = n\} \qquad \text{and} \qquad t_n(\Gamma) := \mathsf{ft}(\Gamma) \in T_n, \text{ for } \Gamma \in T_{n+1}.$$

That is, the front square in the diagram of sets and functions



is a pullback for every $n \in \mathbb{N}$, and the function t_n is defined by its universal property as the right-hand square commutes by 3.1.*ii*. The set T_0 is a singleton by (i) in Definition 3.1.

A homomorphism of C-systems $F\colon \mathbb{C} \to \mathbb{D}$ restricts, for every $n \in \mathbb{N}$, to a function $F_n : T_n \to S_n$ between the fibres T_n and S_n of the length function of $\mathcal{C}$ and $\mathcal{D}$, respectively, by 3.2.*ii* as in the front part of the diagram below.



(3.2)

The upper-right square commutes by Definition 3.2.*iii*, thus the upper-left square commutes as well since the rest of the diagram commutes. Functoriality holds since each F_n is defined by a universal property. □

Lemma 3.8. *Let* $\mathbb{C}$ *be a C-system with underlying strict category* $\mathcal{C}$ *and let* $\mathsf{p}(\mathbb{C})$ *denote the wide subgraph of* $\mathcal{C}$ *on the canonical projections* p_Γ *for* Γ *in* $\mathcal{C}$. *Then* $\mathsf{p}(\mathbb{C})$ *is isomorphic to the graph* $\mathbf{G} \circ \mathbf{C2RtTr}(\mathbb{C})$ *naturally in* $\mathbb{C}$, *where* $\mathbf{G}\colon \mathbf{RtTr} \to \mathbf{Grph}$ *is from Definition 2.13.*

Proof. The vertices of $\mathbf{G}\circ\mathbf{C2RtTr}(\mathbb{C})$ are pairs $(\ell(\Gamma), \Gamma)$ and edges are of the form $(\ell(\Gamma), \Gamma) \to (\ell(\mathsf{ft}(\Gamma)), \mathsf{ft}(\Gamma))$ for $\ell(\Gamma) > 0$. In particular, every vertex $(n+1, \Gamma)$ has exactly one outgoing edge. The bijection between vertices then extends to an isomorphism between $\mathsf{p}(\mathbb{C})$ and $\mathbf{G} \circ \mathbf{C2RtTr}(\mathbb{C})$.

Every C-homomorphism $F\colon \mathbb{C} \to \mathbb{D}$ induces a morphism of graphs $\mathsf{p}(F)\colon \mathsf{p}(\mathcal{C}_{\mathbb{C}}) \to \mathsf{p}(\mathcal{C}_{\mathbb{D}})$ by 3.2.*iv*. Naturality then follows from 3.2*ii*. □

Corollary 3.9. *Let $\mathcal{F}$ be the free category on the graph $\mathsf{p}(\mathbb{C})$ on the canonical projections. Then the terminal object $\mathsf{1}$ of $\mathcal{C}$ is terminal in $\mathcal{F}$ and the function ℓ extends to a stratification functor on $\mathcal{F}$.*

Proof. By Lemma 3.8 there is an iso $\mathcal{F} = \mathbf{F}\mathsf{p}(\mathbb{C}) \cong \mathbf{F} \circ \mathbf{G} \circ \mathbf{C2RtTr}(\mathbb{C})$. The claim thus follows from Proposition 2.14. □

3.2. **The category of CE-systems.** In this section, we define CE-systems and their morphisms.

Definition 3.10. A **CE-system** consists of two strict category structures $\mathcal{F}$ and $\mathcal{C}$ on the same set of objects $\mathrm{Ob}(\mathcal{F}) = \mathrm{Ob}(\mathcal{C})$ and an identity-on-objects functor $I\colon \mathcal{F} \to \mathcal{C}$ between them, together with

(1) a chosen object $\mathbf{1}$ which is terminal in $\mathcal{F}$, and
(2) for any $f\colon \Delta \to \Gamma$ in $\mathcal{C}$ and any $A \in \mathcal{F}/\Gamma$, a functorial choice of a pullback square

$$\begin{array}{ccc} \Delta.f^*A & \xrightarrow{\pi_2(f,A)} & \Gamma.A \\ {\scriptstyle I(f^*A)}\downarrow & & \downarrow{\scriptstyle I(A)} \\ \Delta & \xrightarrow{f} & \Gamma \end{array}$$

such that $f^*A \in \mathcal{F}/\Delta$. Explicitly, the functoriality requirement is that
(a) For any $f\colon \Delta \to \Gamma$, one has

$$f^*(\mathrm{id}_\Gamma) = \mathrm{id}_\Delta \qquad \text{and} \qquad \pi_2\,(f, \mathrm{id}_\Gamma) = f.$$

(b) For any $A \in \mathcal{F}/\Gamma$, one has

$$(\mathrm{id}_\Gamma)^*A = A \qquad \text{and} \qquad \pi_2\,(\mathrm{id}_\Gamma, A) = \mathrm{id}_{\Gamma.A}$$

(c) For any $f\colon \Delta \to \Gamma$, $g\colon \Xi \to \Delta$ and $A \in \mathcal{F}/\Gamma$, one has

$$(f \circ g)^*A = g^*(f^*A) \qquad \text{and} \qquad \pi_2\,(f \circ g, A) = \pi_2\,(f, A) \circ \pi_2\,(g, f^*A)$$

(d) For any $P \in \mathcal{F}/\Gamma.A$ and $f\colon \Delta \to \Gamma$, one has

$$f^*(A.P) = f^*A \circ (\pi_2\,(f, A))^*P \qquad \text{and} \qquad \pi_2\,(f, A.P) = \pi_2\,(\pi_2\,(f, A)\,, P)$$

A CE-system is **rooted** if $I(\mathbf{1}) = \mathbf{1}$ is terminal in $\mathcal{C}$.

For any $f\colon \Delta \to \Gamma$ we write f^* for the induced functor $\mathcal{F}/\Gamma \to \mathcal{F}/\Delta$ and refer to the arrows in $\mathcal{F}$ as the **families** of the CE-system. We shall write arrows in $\mathcal{F}$ with a double head as in the above diagram.

We may write $I_{\mathbb{A}}\colon \mathcal{F}_{\mathbb{A}} \to \mathcal{C}_{\mathbb{A}}$ for the categories and functor underlying a CE-system $\mathbb{A}$, whenever we need to make the CE-system explicit.

We show in Section 3.3 that CE-systems generalize C-systems. To provide some intuition, we can think of the image of $\mathcal{F}$ in $\mathcal{C}$ as the subcategory of $\mathcal{C}$ spanned by the projections $\mathsf{p}_\Gamma\colon \Gamma \to \mathsf{ft}(\Gamma)$ of a C-system.

Example 3.11 (CE-system on finite sets)**.** Let $\mathbb{F}$ be the category whose objects are natural numbers, and whose morphisms $f\colon m \to n$ are functions $f\colon \mathsf{std}(m) \to \mathsf{std}(n)$ from the standard finite set of m elements to the standard finite set of n elements. Consider the identity-on-objects functor $[-]\colon (\mathbb{N}, \geq) \to \mathbb{F}^{\mathrm{op}}$ given, on $n + k \geq n$, by the opposite of the initial-segment inclusion, which we write $i_n^{n+k}\colon [n+k] \to [n]$.

We equip it with the structure of a CE-system as follows. The chosen pullback of a family $n + k \geq n$ and an arrow $f\colon [m] \to [n]$ in $\mathbb{F}^{\mathrm{op}}$ is

$$\begin{array}{ccc} [m+k] & \xrightarrow{\pi_2(f, n+k \geq n)} & [n+k] \\ \big\downarrow{\scriptstyle i_m^{m+k}} & & \big\downarrow{\scriptstyle i_n^{n+k}} \\ [m] & \xrightarrow{\quad f \quad} & [n] \end{array}$$

where the morphism $\pi_2(f, n+k \geq n)$ is the opposite of the arrow $[f, 1_k]\colon [n+k] \to [m+k]$ in $\mathbb{F}$ obtained from the universal property of the coproduct $[n+k]$. Functoriality follows immediately from the definitions.

This CE-system is, of course, rooted — as $[0]$ is terminal in $\mathbb{F}^{\mathrm{op}}$ — and stratified in the sense of Definition 3.16 — as initial-segment inclusions factor uniquely into arrows i_n^{n+1} which are indecomposable in the sense of Definition 2.4. Note also that the choice of pullback squares is forced by Remark 3.19.

We can think of this example as the category of renamings, that is, variable-for-variable substitutions, of a untyped (or uni-typed) theory; see, for instance, [FPT99, LA24].

Example 3.12. Categories with attributes [Car78], or type categories [Pit01], produce examples of CE-systems which are rooted but not stratified. A category with attributes consists of a category $\mathcal{C}$ with a terminal object 1 together with a set of "types" $T(X)$ for each object of $\mathcal{C}$ such that each $A \in T(X)$ is assigned an arrow p_A in $\mathcal{C}$ with codomain X. Moreover a strictly functorial choice of pullbacks of these arrows along any arrow in $\mathcal{C}$ is required. A CE-system is obtained by taking as $\mathcal{F}$ the free category on the arrows of the form p_A and as I the obvious functor into $\mathcal{C}$. Another CE-system is obtained by taking as $\mathcal{F}$ the subcategory of $\mathcal{C}$ spanned by the arrows of the form p_A. In this case the functor I is simply the inclusion.

Display map categories [Tay99] and clans [Joy17] also produce examples of rooted non-stratified CE-systems, as soon as the choice of pullbacks is strictly functorial. Recall that a display map category consists of a category $\mathcal{C}$ together with a class of arrows $\mathcal{D}$ such that pullbacks of arrows in $\mathcal{D}$ along any arrow in $\mathcal{C}$ exist and are again in $\mathcal{D}$. Clans also have a terminal object 1 and require $\mathcal{D}$ to be closed under composition and to contain all arrows towards 1. When the choice of pullbacks is strictly functorial, the wide subcategory of $\mathcal{C}$ on the arrows in $\mathcal{D}$ together with the inclusion $\mathcal{D} \to \mathcal{C}$ provides an example of a rooted, non-stratified CE-system.

Definition 3.13. Let $\mathbb{A}$ and $\mathbb{B}$ be two CE-systems. A **CE-homomorphism** $F\colon \mathbb{A} \to \mathbb{B}$ consists of a commutative square of functors

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{F}}} & \mathcal{F}_{\mathbb{B}} \\ {\scriptstyle I_{\mathbb{A}}}\big\downarrow & & \big\downarrow{\scriptstyle I_{\mathbb{B}}} \\ \mathcal{C}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{C}}} & \mathcal{C}_{\mathbb{B}} \end{array}$$

such that,

(1) $F_{\mathcal{F}}(1_{\mathbb{A}}) = 1_{\mathbb{B}}$, and
(2) for every $A \in \mathcal{F}_{\mathbb{A}}/\Gamma$ and $f\colon \Delta \to \Gamma$, it is

$$F_{\mathcal{F}}(f^*A) = (F_{\mathcal{C}}f)^*(F_{\mathcal{F}}A) \qquad \text{and} \qquad F_{\mathcal{C}}(\pi_2\,(f,A)) = \pi_2\,(F_{\mathcal{C}}f, F_{\mathcal{F}}A)\,.$$

Remark 3.14. If F is a CE-homomorphism between rooted CE-systems $\mathbb{A}$ and $\mathbb{B}$, then $F_{\mathcal{C}}(1_{\mathbb{A}}) = 1_{\mathbb{B}}$ and $F_{\mathcal{C}}$ preserves terminal objects in the usual categorical sense.

Definition 3.15. We write **CEsys** for the category of CE-systems and CE-system homomorphisms and **rCEsys** for its full subcategory on rooted CE-systems.

For the comparison of CE-systems with C-systems, the notion of stratification of a CE-system is needed:

Definition 3.16. A CE-system $\mathbb{A}$ is **stratified** if its category of families $\mathcal{F}$ is stratified in the sense of Definition 2.1 and, for every $f\colon \Delta \to \Gamma$ in $\mathcal{C}$, the functor

$$\mathcal{F}/\Gamma \xrightarrow{f^*} \mathcal{F}/\Delta$$

induced by the functorial choice of pullbacks is stratified with respect to the stratification induced on slices in Lemma 2.11.

A CE-homomorphism between stratified CE-systems is **stratified** if its component on families is a stratified functor.

Remark 3.17. It follows from Proposition 2.8 that CE-systems are stratified in at most one way.

Definition 3.18. We denote by $\mathbf{CEsys_s} \hookrightarrow \mathbf{CEsys}$ and $\mathbf{rCEsys_s} \hookrightarrow \mathbf{rCEsys}$ the respective subcategories spanned by *stratified* (rooted) CE-systems and *stratified* CE-homomorphisms between them.

Remark 3.19. In a stratified CE-system, for every $f\colon \Delta \to \Gamma$ in $\mathcal{C}$ and $A \in \mathcal{F}/\Gamma$ we have

$$L(\Delta.f^*A) = L(\Delta) + L(\Gamma.A) - L(\Gamma).$$

Lemma 3.20. *Let $\mathbb{A}$ and $\mathbb{B}$ be two stratified CE-system. A commuting square of functors*

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{F}}} & \mathcal{F}_{\mathbb{B}} \\ {\scriptstyle I_{\mathbb{A}}}\downarrow & & \downarrow{\scriptstyle I_{\mathbb{B}}} \\ \mathcal{C}_{\mathbb{A}} & \xrightarrow{F_{\mathcal{C}}} & \mathcal{C}_{\mathbb{B}} \end{array}$$

is a stratified CE-homomorphism $\mathbb{A} \to \mathbb{B}$ if and only if

(1) *$F_{\mathcal{F}}$ is a stratified functor, and*
(2) *for every indecomposable arrow $A \in \mathcal{F}_{\mathbb{A}}/\Gamma$ and every $f\colon \Delta \to \Gamma$, we have*

$$F_{\mathcal{F}}(f^*A) = (F_{\mathcal{C}}f)^*(F_{\mathcal{F}}A) \qquad \textit{and} \qquad F_{\mathcal{C}}(\pi_2\,(f,A)) = \pi_2\,(F_{\mathcal{C}}f, F_{\mathcal{F}}A)\,.$$

Proof. One direction is trivial. The other one is proved by induction on the length n of an arrow $A \in \mathcal{F}/\Gamma$. ∎

3.3. **Characterising C-systems as stratified CE-systems.** Recall from Corollary 3.9 that every C-system $\mathbb{C}$ has a stratified wide subcategory $\mathcal{F}$ of its underlying category $\mathcal{C}$. In this section, we show that the inclusion $\mathcal{F} \to \mathcal{C}$ has the structure of a stratified CE-system (Construction 3.22). Moreover, we prove that this correspondence is functorial (Construction 3.25) and, in fact, an equivalence between the category of C-systems and the category of stratified CE-systems (Theorem 3.31).

Problem 3.21. To construct a CE-system $\mathbf{CE}(\mathbb{C})$ from a C-system $\mathbb{C} = (\mathcal{C}, 1, \ell, \mathsf{ft}, \ldots)$.

Construction 3.22 (for Problem 3.21)**.** Recall from Lemma 3.8 that $\mathsf{p}(\mathbb{C})$ denotes the wide subgraph of $\mathcal{C}$ on the canonical projections p_Γ for Γ in $\mathcal{C}$ and let $\mathcal{F}$ be the free category on $\mathsf{p}(\mathbb{C})$. In particular, $\mathcal{F}$ has the same objects of $\mathcal{C}$ and the object 1 is terminal in $\mathcal{F}$ by Corollary 3.9. It follows that the inclusion $\mathsf{p}(\mathbb{C}) \hookrightarrow \mathcal{C}$ extends to an identity-on-objects functor $I\colon \mathcal{F} \to \mathcal{C}$ that maps a path of length $n > 0$ in $\mathsf{p}(\mathbb{C})$, *i.e.* a list of composable canonical projections

$$\Gamma \xrightarrow{\mathsf{p}_\Gamma} \mathsf{ft}(\Gamma) \xrightarrow{\mathsf{p}_{\mathsf{ft}(\Gamma)}} \cdots \xrightarrow{\mathsf{p}_{\mathsf{ft}^2(\Gamma)}} \mathsf{ft}^{n-1}(\Gamma) \xrightarrow{\mathsf{p}_{\mathsf{ft}^{n-1}(\Gamma)}} \mathsf{ft}^n(\Gamma). \tag{3.3}$$

to their composite in $\mathcal{C}$.

It remains to provide I with a suitable choice of pullback squares along an arbitrary arrow $f\colon \Delta \to \Gamma$ in $\mathcal{C}$. As an arrow $p\colon \Xi \to \Gamma$ in $\mathcal{F}$ is a path in $\mathsf{p}(\mathbb{C})$, we proceed by induction on the length n of the path p, proving also conditions (2b) and (2c) from Definition 3.10.

If $n = 0$, the path p is the identity on Γ and we take $f^*(\mathrm{id}_\Gamma) := \mathrm{id}_\Delta$ and $\pi_2(f, \mathrm{id}_\Gamma) := f$. This choice is clearly functorial in f and it trivially gives rise to a pullback square. It also ensures condition (2a).

For $n > 0$, it is $I(p) = \mathsf{p}_\Xi \circ I(p')$ where the length of p' is $n-1$. By inductive hypothesis we have $f^*p' \in \mathcal{F}/\Delta$ and a chosen pullback square of $I(p')$ along f, which is the lower square in the diagram below. The upper square is the canonical pullback square (3.1) given by the C-system structure.

$$\begin{array}{ccc}
(\pi_2(f,p'))^*\Xi & \xrightarrow{\mathsf{q}(\pi_2(f,p'),\Xi)} & \Xi \\
\downarrow{\scriptstyle \mathsf{p}_{(\pi_2(f,p'))^*\Xi}} & & \downarrow{\scriptstyle \mathsf{p}_\Xi} \\
\Delta.f^*(p') & \xrightarrow{\pi_2(f,p')} & \mathsf{ft}(\Xi) \\
\downarrow{\scriptstyle I(f^*p')} & & \downarrow{\scriptstyle I(p')} \\
\Delta & \xrightarrow{f} & \Gamma
\end{array} \tag{3.4}$$

Thus we define $\pi_2(f,p) := \mathsf{q}(\pi_2(f,p'),\Xi)$ and f^*p to be the concatenation of f^*p' with $\mathsf{p}_{\mathsf{p}_{(\pi_2(f,p'))^*\Xi}}$ so that $I(f^*p) = I(f^*p') \circ \mathsf{p}_{\mathsf{p}_{(\pi_2(f,p'))^*\Xi}}$. Functoriality in f of this choice of pullback squares follows from the fact that both the lower and upper pullback squares are functorial by inductive hypothesis and by assumption, respectively. In more details: given $g\colon \Theta \to \Delta$, the inductive hypothesis yields $(f \circ g)^*p' = g^*(f^*p')$ and $\pi_2(f \circ g, p') = \pi_2(f,p') \circ \pi_2(g, f^*p')$. It follows by 3.1.*vii* that

$$\pi_2(f \circ g, p')^*\Xi = \pi_2(g, f^*p')^*\big(\pi_2(f,p')^*\Xi\big)$$

and, in turn, that $(f \circ g)^* p = g^*(f^* p)$. The other component also follows from 3.1.*vii*:

$$\begin{aligned}\pi_2(f \circ g, p) &= \mathsf{q}(\pi_2(f \circ g, p'), \Xi) \\ &= \mathsf{q}(\pi_2(f, p'), \Xi) \circ \mathsf{q}(\pi_2(g, f^* p'), \pi_2(f, p')^* \Xi) \\ &= \pi_2(f, p) \circ \pi_2(g, f^* p).\end{aligned}$$

Finally, condition (2d) for a composite $q \circ p$ in $\mathcal{F}$ is proven by induction on the length of the path p. □

Lemma 3.23. *Let $\mathbb{C}$ be a C-system and $\mathcal{F}$ the category of families of* $\mathbf{CE}(\mathbb{C})$.

(1) *The indecomposable arrows in $\mathcal{F}$ are of the form p_Γ for some object Γ.*
(2) *The CE-system $\mathbf{CE}(\mathbb{C})$ is stratified and $L(\Gamma) = \ell(\Gamma)$, for every object Γ.*
(3) *The CE-system $\mathbf{CE}(\mathbb{C})$ is rooted.*

Proof.

1. Immediate from the description of arrows in $\mathcal{F}$ in (3.3) and 3.1.*ii*.
2. By Corollary 3.9, the category $\mathcal{F}$ is stratified and $L(\Gamma) = \ell(\Gamma)$. By Lemma 2.10, it is enough to show that the choice of pullback squares in Construction 3.22 preserves indecomposable arrows. But this follows immediately from the construction of pullbacks in (3.4) and (1) just shown.
3. The terminal object in $\mathcal{F}$ is terminal in $\mathcal{C}$ by assumption. □

Problem 3.24. To construct a functor $\mathbf{CE}\colon \mathbf{Csys} \to \mathbf{rCEsys_s}$ into rooted stratified CE-systems and stratified homomorphisms.

Construction 3.25 (for Problem 3.24)**.** The action of $\mathbf{CE}$ on objects is defined in Construction 3.22. Every morphism $F\colon \mathbb{C} \to \mathbb{D}$ of C-systems restricts to the graphs of canonical projections $\mathsf{p}(F)\colon \mathsf{p}(\mathbb{C}) \to \mathsf{p}(\mathbb{D})$ by conditions (i,iii,iv) in Definition 3.2 and induces, in turn, a functor between free categories $F_\mathcal{F}\colon \mathcal{F}_\mathbb{C} \to \mathcal{F}_\mathbb{D}$ whose action is determined by the action of F on indecomposable arrows. The square

$$\begin{array}{ccc} \mathcal{F}_\mathbb{C} & \xrightarrow{F_\mathcal{F}} & \mathcal{F}_\mathbb{D} \\ {\scriptstyle I_\mathbb{C}}\downarrow & & \downarrow{\scriptstyle I_\mathbb{D}} \\ \mathcal{C}_\mathbb{C} & \xrightarrow{F} & \mathcal{C}_\mathbb{D} \end{array}$$

commutes since it does so when precomposed by the unit $\mathsf{p}(\mathbb{C}) \to \mathcal{F}_\mathbb{C}$. The functor $F_\mathcal{F}$ is stratified by 3.2.*ii*. Lemma 3.20 then ensures that the pair $\mathbf{CE}(F) := (F_\mathcal{F}, F)$ lifts to a stratified CE-homomorphism as soon as it preserves pullbacks of indecomposable arrows. But this is precisely condition 3.2.*v*. Functoriality of $\mathbf{CE}$ follows since $F_\mathcal{F}$ is defined by a universal property. □

Problem 3.26. To construct a C-system $\mathbf{C}(\mathbb{A})$ from a stratified and rooted CE-system $\mathbb{A}$.

Construction 3.27 (for Problem 3.26)**.** Let $I\colon \mathcal{F} \to \mathcal{C}$ be the underlying functor of $\mathbb{A}$. The underlying category of $\mathbf{C}(\mathbb{A})$ is $\mathcal{C}$ and the length function ℓ is given by the action of the stratification functor L on objects. Since $\mathbb{A}$ is rooted, the chosen terminal object 1 in $\mathcal{F}$ is terminal in $\mathcal{C}$ too. Conditions (iv) and (i) are clearly met.

Given an object X with $n = L(X) > 0$, let $X \xrightarrow{x_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{x_1} 1$ be the factorisation of $X \to 1$ into n indecomposable arrows in $\mathcal{F}$. We define

$$\mathsf{ft}(1) := 1, \quad \mathsf{ft}(X) := X_{n-1} \quad \text{and} \quad \mathsf{p}_X := I(x_n). \tag{3.5}$$

Conditions (*ii*) and (*iii*) hold by construction.

Given also $f\colon Y \to \mathsf{ft}(X)$, let $Y \xrightarrow{y_n} Y_{n-1} \to \cdots \to Y_1 \xrightarrow{y_1} 1$ be the factorisation of $Y \to 1$ into indecomposable arrows and consider the pullback square below.

$$\begin{array}{ccc} Y.f^*x_n & \xrightarrow{\pi_2(f,x_n)} & X \\ {\scriptstyle I(f^*x_n)}\downarrow & & \downarrow{\scriptstyle I(x_n)} \\ Y & \xrightarrow{f} & \mathsf{ft}(X) \end{array} \tag{3.6}$$

It is $L(Y.f^*(x_n)) = L(Y) + 1$ by Remark 3.19, thus $Y.f^*x_n \xrightarrow{f^*x_n} Y \xrightarrow{y_n} Y_{n-1} \to \cdots \to Y_1 \xrightarrow{y_1} 1$ is the factorisation of $Y.f^*x_n \to 1$ into indecomposable arrows. It follows that $\mathsf{ft}(Y.f^*x_n) = Y$ and $\mathsf{p}_{f^*x_n} = I(f^*x_n)$. Condition (*v*) follows defining $f^*X := Y.f^*(x_n)$ and $\mathsf{q}(f,X) := \pi_2(f,x_n)$. Condition (*vi*) holds by 3.10.2b since $\mathsf{ft}(X).x_n = X$, and (*vii*) by 3.10.2c as below:

$$\begin{aligned} (f \circ g)^*X &= Z.(f \circ g)^*x_n = Z.(g^*(f^*x_n)) \\ &= g^*(f^*X) \\ \mathsf{q}(f \circ g, X) &= \pi_2(f \circ g, x_n) = \pi_2(f, x_n) \circ \pi_2(g, f^*x_n) \\ &= \mathsf{q}(f,X) \circ \mathsf{q}(g, f^*X). \end{aligned}$$

□

Lemma 3.28. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a stratified homomorphism of rooted stratified CE-systems. Then the underlying functor $F\colon \mathcal{C}_{\mathbb{A}} \to \mathcal{C}_{\mathbb{B}}$ is a homomorphism of C-systems $\mathbf{C}(F)\colon \mathbf{C}(\mathbb{A}) \to \mathbf{C}(\mathbb{B})$.*

Proof. We verify the conditions in Definition 3.2. (*i*) The functor F maps the chosen terminal object of $\mathbb{A}$ to the one of $\mathbb{B}$ by assumption. (*ii*) Since F is stratified, its action on objects commutes with the length functions. (*iii–iv*) The action on objects also preserves indecomposable arrows by Lemma 2.10, thus it commutes with the father functions and preserves canonical projections. (*v*) F maps chosen pullback squares in $\mathbb{A}$ to chosen ones in $\mathbb{B}$ by 3.13.2. In particular, it preserves the choice of pullbacks along indecomposable arrows. □

Definition 3.29. Let $\mathbf{C}\colon \mathbf{rCEsys_s} \to \mathbf{Csys}$ be the functor given by Construction 3.27 and Lemma 3.28.

Lemma 3.30. *For every C-system $\mathbb{C}$, the identity functor on the underlying strict category of $\mathbb{C}$ is an isomorphism $\mathbf{C}(\mathbf{CE}(\mathbb{C})) \cong \mathbb{C}$ of C-systems, naturally in $\mathbb{C}$.*

Proof. Let $\mathcal{C}$ be the underlying strict category of $\mathbb{C}$. To see that the identity functor $\mathrm{id}_{\mathcal{C}}$ is a $\mathbb{C}$-homomorphism note first that the category $\mathcal{C}$, its terminal object and the length function are the same in $\mathbf{C}(\mathbf{CE}(\mathbb{C}))$ and $\mathbb{C}$. Since indecomposable arrows in $\mathbf{CE}(\mathbb{C})$ coincide with the canonical projections p_Γ by Lemma 3.23, factorisations in $\mathbf{CE}(\mathbb{C})$ into indecomposable arrows are of the form in (3.3). It follows that the function ft and the canonical projections as defined in (3.5) are equal to the ones from $\mathbb{C}$. Since the choice of pullback squares in $\mathbf{CE}(\mathbb{C})$ is defined inductively by the choice along indecomposable arrows in (3.4), the choice of pullbacks along canonical projections in (3.6) coincides with the one in $\mathbb{C}$.

Naturality follows from the fact that $\mathbf{C}(\mathbf{CE}(F)) = F$ for every C-homomorphism F. □

Theorem 3.31. *The functor $\mathbf{CE}\colon \mathbf{Csys} \to \mathbf{rCEsys_s}$ from Construction 3.25 is an equivalence.*

Proof. By Lemma 3.30, it is enough to find, for every stratified rooted CE-system $\mathbb{A}$, an isomorphism $\mathbf{CE}(\mathbf{C}(\mathbb{A})) \cong \mathbb{A}$ natural in $\mathbb{A}$. Let $I \colon \mathcal{F} \to \mathcal{C}$ be the underlying functor of $\mathbb{A}$ and let $\mathcal{F}_{\mathrm{i}} := \mathsf{p}(\mathbf{C}(\mathbb{A}))$ be the subgraph of $\mathcal{F}$ on the indecomposable arrows. The CE-system $\mathbf{CE}(\mathbf{C}(\mathbb{A}))$ consists, in particular, of a functor $\widehat{I} \colon \widehat{\mathcal{F}} \to \mathcal{C}$, where $\widehat{\mathcal{F}}$ is the free category on $\mathcal{F}_{\mathrm{i}}$, and $\widehat{I}$ maps a list of composable indecomposable arrows to the composite of their images in $\mathcal{C}$ under I, by (3.5), (3.3) and Lemma 3.23.2.

The inclusion $\mathcal{F}_{\mathrm{i}} \hookrightarrow \mathcal{F}$ induces an identity-on-objects functor $\mathsf{comp} \colon \widehat{\mathcal{F}} \to \mathcal{F}$. Conversely, the factorisation into indecomposable arrows (3) in $\mathbb{A}$ yields an identity-on-objects functor $\mathsf{fact} \colon \mathcal{F} \to \widehat{\mathcal{F}}$, which is a (strict) inverse of comp. Since I is a functor, the squares

$$\begin{array}{ccc} \widehat{\mathcal{F}} & \xrightarrow{\mathsf{comp}} & \mathcal{F} \\ {\scriptstyle \widehat{I}}\downarrow & & \downarrow{\scriptstyle I} \\ \mathcal{C} & \xrightarrow{\mathrm{id}_{\mathcal{C}}} & \mathcal{C} \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{F} & \xrightarrow{\mathsf{fact}} & \widehat{\mathcal{F}} \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle \widehat{I}} \\ \mathcal{C} & \xrightarrow{\mathrm{id}_{\mathcal{C}}} & \mathcal{C} \end{array} \tag{3.7}$$

commute. Since both functors comp and fact are identities on objects and on indecomposable arrows, the squares above are stratified CE-homomorphisms $\mathbf{CE}(\mathbf{C}(\mathbb{A})) \to \mathbb{A}$ and $\mathbb{A} \to \mathbf{CE}(\mathbf{C}(\mathbb{A}))$, respectively, by Lemma 3.20. Therefore $\mathbf{CE}(\mathbf{C}(\mathbb{A})) \cong \mathbb{A}$.

To see that this isomorphism is natural in $\mathbb{A}$, note that $(\mathsf{comp}, \mathrm{id}_{\mathcal{C}})$ is natural in $\mathbb{A}$ since comp is equivalently defined as the composite of the counit of the free-forgetful adjunction at $\mathcal{F}$ with the image under the left adjoint of the graph inclusion $\mathcal{F}_{\mathrm{i}} \hookrightarrow \mathcal{F}$. □

4. The category of B-systems

In this section, we study Voevodsky's B-systems.

In Section 4.1 we review the definition of B-systems and their homomorphisms. In Section 4.2 we introduce the notion of E-system and their homomorphisms. Intuitively, E-systems model type theory with strict Σ-types, see Section 4.2.5. Finally, in Section 4.3 we construct an equivalence between the category of B-systems and the subcategory of "stratified" E-systems.

In order to simplify the construction of such equivalence, we structure the definitions in the next sections in three steps. In the case of B-systems, for example, we first introduce some piece of structure on sets consisting of functions, which we refer to as *pre*-B-systems, see Definition 4.7. Then we define morphisms between these structures, also called *pre*-homomorphisms, and finally we define B-systems as those pre-B-systems whose structure functions are themselves pre-homomorphisms. Homomorphisms are then just pre-homomorphisms between B-systems. We shall follow the same pattern when introducing each of the structures that give rise to an E-system, in Sections 4.2.1 to 4.2.3, and when defining E-systems in Section 4.2.4.

4.1. The category of B-systems.

In this section, we review the definition of Voevodsky's B-systems [Voe14]. We shall rephrase his definition in order to introduce a few auxiliary intermediate structures which we will use in later constructions. An explicit comparison is in Remark 4.12.

Definition 4.1. A **B-frame** $\mathbb{B}$ is a diagram of sets of the following form:

$$\begin{array}{ccccccc} & & \tilde{B}_1 & & \tilde{B}_2 & & \\ & & \swarrow_{\partial} & & \swarrow_{\partial} & & \\ \{*\} \cong B_0 & \xleftarrow[\mathsf{ft}]{} & B_1 & \xleftarrow[\mathsf{ft}]{} & B_2 & \xleftarrow[\mathsf{ft}]{} & \cdots \end{array}$$

In other words, a B-frame consists of:

(1) for all $n \in \mathbb{N}$ two sets B_n and $\tilde{B}_{n+1}$.
(2) for all $n \in \mathbb{N}$ functions of the form

$$\mathsf{ft}_n : B_{n+1} \to B_n$$
$$\partial_n : \tilde{B}_{n+1} \to B_{n+1}.$$

called **father functions** and **boundary functions**, respectively.
(3) B_0 is a singleton.

For $m, n \in \mathbb{N}$, we denote the composition $\mathsf{ft}_n \circ \cdots \circ \mathsf{ft}_{n+m} : B_{n+m+1} \to B_n$ by ft_n^m.

A **homomorphism** $H : \mathbb{B} \to \mathbb{A}$ **of B-frames** is a natural transformation of B-frames, i.e., it consists of maps $H_n : B_n \to A_n$ and $\tilde{H}_{n+1} : \tilde{B}_{n+1} \to \tilde{A}_{n+1}$ such that

$$\mathsf{ft}(H(X)) = H(\mathsf{ft}(X))$$
$$\partial(\tilde{H}(x)) = H(\partial(x))$$

for any $X \in B_n$ and $x \in \tilde{B}_{n+1}$. The category of B-frames is denoted by **Bfr**.

As we already did in the above definition, we shall often omit the subscripts from father and boundary functions and from homomorphisms of B-frames, since these can be easily inferred from the context (often from their argument).

We should remark that we are abusing notation when denoting the pieces of structure of B-frames and, later on, of B-systems. Regarding a B-frame as a diagram on the "comb category"

$$\begin{array}{ccccccc} & & & \tilde{1} & & \tilde{2} & \\ & & \swarrow & & \swarrow & & \\ 0 & \longleftarrow & 1 & \longleftarrow & 2 & \longleftarrow & \cdots \end{array}$$

we denote the value of a B-frame at n by changing the font from blackboard bold to roman, and the value at $\tilde{n}$ by also adding a tilde. This (can) only apply to the blackboard bold letter: for instance, the values of the slice B-frame (defined in Definition 4.5) $\mathbb{B}/X$ are denoted $(B/X)_m$ and $(\tilde{B}/X)_{m+1}$.

We shall consider specific B-frames (and B-systems) only as individual examples, and we do not need to give names to them. Otherwise, we shall only deal with generic B-frames (and B-systems), denoted by capital blackboard letters. Therefore we do not expect this abuse of notation to create any inconvenience.

To provide some intuition for B-frames, we look back at the introduction, where we constructed, implicitly, a B-frame from a module over a monad.

Example 4.2. Recall from Section 1.2.2 the two sets $B(R, LM)$ (see Eq. (1.1)) and $\widetilde{B}(R, LM)$ (see Eq. (1.2)). From these sets, we obtain a B-frame with the following sets of families (note the shift in the indexing of $\tilde{B}$),

$$B_n := B(R, LM)_n := \prod_{i=0}^{n-1} LM([i])$$
$$\tilde{B}_{n+1} := \widetilde{B}(R, LM)_n := \prod_{i=0}^{n-1} LM([i]) \times R([n]) \times LM([n])$$

and the obvious maps for ft and ∂. We call this B-frame the *B-frame generated by a module LM over a monad R*. We write elements of B_n as $A_0, \ldots, A_{n-1} \vdash A_n$, and elements of $\widetilde{B}_{n+1}$ as $A_0, \ldots, A_n \vdash t : A_{n+1}$, where $t \in R([n])$.

More generally, the elements of B_{n+1} of a B-frame can be thought of as a pair of a context of length n, and a type in that context. Hence, the elements of B_1 are the types in the empty context. Just like with C-systems, there is no explicit structure to denote types in a given context. An element $t \in \tilde{B}_{n+1}$ is then a term, and the context and type t lives in is given by $\partial_{n+1}(t)$.

Example 4.3. Recall from Example 3.11 that $\mathsf{std}(n)$ denotes the set $\{0, \ldots, n-1\}$. We shall consider the B-frame defined, for each $n \in \mathbb{N}$, by $B_n := \{n\}$ and $\tilde{B}_{n+1} := \mathsf{std}(n)$.

Example 4.4. B-frames are the same as Garner's "type-and-term structures" [Gar15, Def. 8]. Garner [Gar15, Prop. 13] constructs an equivalence between the category of type-and-term structures and the category of $\emptyset$-GATs, that is, of Generalized Algebraic Theories [Car86] without weakening, projection, and substitution rules, and interpretations between them.

We now define more structure on B-frames which represents operations on syntax.

The first operation could be called "slicing"; given a B-frame $\mathbb{B}$ and a "context" $X \in B_n$ in that B-frame, we construct the slice of $\mathbb{B}$ over X:

Definition 4.5. For every B-frame $\mathbb{B}$ and any $X \in B_n$, there is a B-frame $\mathbb{B}/X$ given by

$$(B/X)_m := \{Y \in B_{n+m} \mid \mathsf{ft}^m(Y) = X\}$$
$$(\tilde{B}/X)_{m+1} := \{y \in \tilde{B}_{n+m+1} \mid \mathsf{ft}^{m+1}(\partial(y)) = X\}.$$

Also, for any homomorphism $H : \mathbb{B} \to \mathbb{A}$ of B-frames and any $X \in B_n$, there is a homomorphism $H/X : \mathbb{B}/X \to \mathbb{A}/H(X)$ defined in the obvious way.

Note that for $X \in B_n$ and $Y \in B_{n+m}$ such that $\mathsf{ft}^m(Y) = X$, we have an isomorphism $(\mathbb{B}/X)/Y \cong B/Y$ of B-frames, constructed in the obvious way, which is natural in the sense that for any homomorphism $H : \mathbb{B} \to \mathbb{A}$ of B-frames, the square

$$\begin{array}{ccc} (\mathbb{B}/X)/Y & \xrightarrow{\cong} & \mathbb{B}/Y \\ {\scriptstyle (H/X)/Y}\downarrow & & \downarrow{\scriptstyle H/Y} \\ (\mathbb{A}/H(X))/H(Y) & \xrightarrow{\cong} & \mathbb{A}/H(Y) \end{array}$$

commutes.

Definition 4.6. Every B-frame $\mathbb{B}$ has an underlying rooted tree given by the sets B_n and the functions $\mathsf{ft}_n \colon B_{n+1} \to B_n$, for $n \in \mathbb{N}$. Similarly, a homomorphism of B-frames is in particular a homomorphism of rooted trees. Thus we define

$$\mathbf{Bfr} \xrightarrow{\mathbf{R}} \mathbf{RtTr}$$

to be the forgetful functor from B-frames to rooted trees.

We will now consider different type-theoretic structures on B-frames, specifically substitution, weakening, and projection. Garner considers similar structures in terms of algebras of suitable monads on the category of B-frames a. k. a. type-and-term structures. We have not established a precise relationship (e.g., an equivalence) between our structures and the ones obtained by Garner as the algebras for his monads.

Definition 4.7. (1) A **substitution structure** on a B-frame $\mathbb{B}$ is a collection of homomorphisms

$$S_x : \mathbb{B}/\partial(x) \to \mathbb{B}/\mathsf{ft}(\partial(x))$$

for all $x \in \tilde{B}_{n+1}$ and all $n \in \mathbb{N}$.

(2) A **weakening structure** on a B-frame $\mathbb{B}$ is a collection of homomorphisms

$$W_X : \mathbb{B}/\mathsf{ft}(X) \to \mathbb{B}/X$$

for all $X \in B_{n+1}$ and all $n \in \mathbb{N}$.

(3) The **structure of generic elements** on a B-frame $\mathbb{B}$ equipped with weakening structure W is a collection of functions

$$\delta_n : B_{n+1} \to \tilde{B}_{n+2}$$

such that $\partial(\delta_n(X)) = W_X(X)$ for any $X \in B_{n+1}$.

A **pre-B-system** $\mathbb{B}$ is a B-frame equipped with weakening structure, substitution structure, and generic elements.

We shall often omit the subscript from the functions δ_n, since it can be easily inferred from the context.

Example 4.8. Consider the B-frame generated by a module LM over a monad R of Example 4.2. Given an element $x \in \tilde{B}_{n+1}$, and hence in particular, a term $t \in R([n])$, we obtain a substitution map $S_x : \mathbb{B}/\partial(x) \to \mathbb{B}/\mathsf{ft}(\partial(x))$ that substitutes the term t for the "last" free variable in any element of $\mathbb{B}$ lying "over" $\partial(x)$. For instance, taking x to be $A_0 \vdash t_1 : A_1$, the substitution S_x maps the element $A_0, A_1 \vdash s : A_2$ to $A_0 \vdash s[t_1] : A_2[t_1]$.

For weakening, consider $X \in B_{1+1}$ to be a context $A_0 \vdash A_1$. The weakening W_X maps any context of the form $A_0, A'_1, \ldots, A'_n \vdash A'_{n+1}$ to the weakened context $A_0, A_1, A'_1, \ldots, A'_n \vdash A'_{n+1}$, and similar for elements in $\widetilde{B}$.

For the generic element, consider, for instance, a context $X = A_0 \vdash A_1$ in B_2. This context induces the generic element $A_0, A_1 \vdash \mathsf{var}(1) : A_1$, where $\eta(1) \in R([2])$ is the "de Bruijn" variable 1 bound by A_1 in the context, and considered as a term by being wrapped in an application of the monadic unit η of the monad R (the inclusion of variables into terms). We have

$$\partial(A_0, A_1 \vdash \mathsf{var}(1) : A_1) \;=\; A_0, A_1 \vdash A_1 \;=\; W_{A_0 \vdash A_1}(A_0 \vdash A_1).$$

Example 4.9. Recall the B-frame of finite sets defined in Example 4.3. Here we construct structures of substitution, weakening and generic elements on it.

Note first that its slice on the (unique) element n in B_n is such that

$$(\tilde{B}/n)_{m+1} = \tilde{B}_{n+m+1} = \mathsf{std}(n+m).$$

It follows that a substitution structure must consist of a family of functions $S_{x,j} \colon \mathsf{std}(n+1+j) \to \mathsf{std}(n+j)$, for $n, j \in \mathbb{N}$ and $x \in \tilde{B}_{n+1}$. We define $S_{x,j} := s_x + \mathrm{id}_j$, where s_x is the function $[\mathrm{id}_n, x] \colon \mathsf{std}(n+1) \to \mathsf{std}(n)$ given by the universal property of the coproduct

$\mathsf{std}(n+1)$. In other words, $S_{x,j}$ lists all elements in $\mathsf{std}(n+j)$ repeating the element $x \in \mathsf{std}(n)$ in position $n+1$. In particular, it fixes the first n elements, and decreases the last j by 1.

Similarly, a weakening structure must consist of a family of functions $W_{n,j} \colon \mathsf{std}(n+j) \to \mathsf{std}(n+1+j)$. We define $W_{n,j}$ to be the function $i_n + \mathrm{id}_j$, where $i_n \colon \mathsf{std}(n) \to \mathsf{std}(n+1)$ is the initial-segment inclusion. In other words, it lists all elements in $\mathsf{std}(n+1+j)$ except for n. Equivalently, it fixes the first n elements, and increases the remaining j by 1.

Finally, the structure of generic elements is given by an element $\delta_n \in \tilde{B}_{n+2} = \mathsf{std}(n+1)$ for every $n \in \mathbb{N}$, which we define to be its maximum, that is, $\delta_n := n$.

Taking advantage of the fact that finite sets are finite coproducts, and slightly abusing notation, we find it convenient to write

for the functions $S_{x,j} \colon \mathsf{std}(n+1+j) \to \mathsf{std}(n+j)$ and $W_{n,j} \colon \mathsf{std}(n+j) \to \mathsf{std}(n+1+j)$, respectively.

Definition 4.10.

(1) Consider two B-frames $\mathbb{B}$ and $\mathbb{A}$, both equipped with substitution structure. A homomorphism $H : \mathbb{B} \to \mathbb{A}$ of B-frames is said to **preserve the substitution structure** if the diagram

$$\begin{array}{ccc} \mathbb{B}/\partial(x) & \xrightarrow{H/\partial(x)} & \mathbb{A}/\partial(H(X)) \\ {\scriptstyle S_x}\downarrow & & \downarrow{\scriptstyle S_{\tilde{H}(x)}} \\ \mathbb{B}/\mathsf{ft}(\partial(x)) & \xrightarrow[H/\mathsf{ft}(\partial(x))]{} & \mathbb{A}/\mathsf{ft}(\partial(H(X))) \end{array}$$

of B-frame homomorphisms commutes for every $x \in \tilde{B}_{n+1}$ and every $n \in \mathbb{N}$.

(2) Consider two B-frames $\mathbb{B}$ and $\mathbb{A}$, both equipped with weakening structure. A homomorphism $H : \mathbb{B} \to \mathbb{A}$ of B-frames is said to **preserve the weakening structure** if the diagram

$$\begin{array}{ccc} \mathbb{B}/X & \xrightarrow{H/X} & \mathbb{A}/H(X) \\ {\scriptstyle W_X}\uparrow & & \uparrow{\scriptstyle W_{H(X)}} \\ \mathbb{B}/\mathsf{ft}(X) & \xrightarrow[H/\mathsf{ft}(X)]{} & \mathbb{A}/\mathsf{ft}(H(X)) \end{array}$$

of B-frame homomorphisms commutes for all $X \in B_n$ and all $n \in \mathbb{N}$.

(3) Consider two B-frames $\mathbb{B}$ and $\mathbb{A}$, both equipped with weakening structure, and both equipped with generic elements. A B-frame homomorphism $H : \mathbb{B} \to \mathbb{A}$ is said to **preserve the generic elements** if

$$\tilde{H}(\delta(X)) = \delta(H(X))$$

for any $X \in B_{n+1}$ and any $n \in \mathbb{N}$.

A **pre-B-homomorphism** $H : \mathbb{B} \to \mathbb{A}$ is a homomorphism of pre-B-systems preserving the weakening structure, substitution structure and the generic elements.

Definition 4.11. A **B-system** is a pre-B-system for which the following conditions hold:
(1) Every S_x is a pre-B-homomorphism.
(2) Every W_X is a pre-B-homomorphism.
(3) For every $x \in \tilde{B}_{n+1}$ one has $S_x \circ W_{\partial(x)} = \mathrm{id}_{\mathbb{B}/\mathsf{ft}(\partial(x))}$.
(4) For every $x \in \tilde{B}_{n+1}$ one has $S_x(\delta(\partial(x))) = x$.
(5) For every $X \in B_{n+1}$ one has $S_{\delta(X)} \circ W_X/X = \mathrm{id}_{\mathbb{B}/X}$.

B-homomorphisms are pre-B-homomorphisms between B-systems. We denote the category of B-systems by **Bsys**.

The idea is that first substitution and weakening preserve all the structure of a (pre-)B-system. The third axiom asserts that substitution in weakened type families is constant. Furthermore, the generic elements should behave like internal identity morphisms. Axioms 4 and 5 are akin to two of the well-known monadic laws of substitution.

Remark 4.12. Here we provide an explicit comparison of Definition 4.11 with Voevodsky's definition of B-system in the arXiv version of [Voe14].

Conditions 1–3 in [Voe14, Def. 2.1] and condition 1 in [Voe14, Def. 2.5] define a B-frame $\mathbb{B}$. The functions T and $\widetilde{T}$ from condition 4 in [Voe14, Def. 2.1] together with conditions 2 and 3 in [Voe14, Def. 2.5] define a weakening structure W on $\mathbb{B}$. The functions S and $\widetilde{S}$ from condition 4 in [Voe14, Def. 2.1] together with conditions 4 and 5 in [Voe14, Def. 2.5] define a substitution structure S on $\mathbb{B}$. The function δ in [Voe14, Def. 2.1] together with the condition in [Voe14, Def. 2.6] defines a structure of generic elements on $\mathbb{B}$ with weakening structure W. Therefore what we call a pre-B-system is a unital B0-system in [Voe14].

Consider now the conditions in [Voe14, Def. 3.1 and 3.2]. The TT-condition amounts to saying that every W_X preserves the weakening structure, the ST-condition amounts to saying that every W_X preserves the substitution structure, and the δT-condition amounts to saying that every W_X preserves the generic elements. Therefore condition (2) in Definition 4.11 unfolds to [Voe14, 3.1.1, 3.1.4, 3.2.1]. Similarly, the SS-condition amounts to saying that every S_x preserves the substitution structure, the TS-condition amounts to saying that every S_x preserves the weakening structure, and the δS-condition amounts to saying that every S_x preserves the generic elements. Therefore condition (1) in Definition 4.11 unfolds to [Voe14, 3.1.2, 3.1.3, 3.2.2]. Finally, conditions (3), (4), and (5) in Definition 4.11 unfold to conditions 3.1.5, 3.2.3, and 3.2.4, respectively, in [Voe14].

Example 4.13 (B-frames with structure and D-GATs)**.** Garner [Gar15] constructs an equivalence between the category of GATs and a category of algebras for a monad on B-frames. We expect B-systems to be equivalent to Garner's algebras.

Lemma 4.14. *The forgetful functor* **Bsys** $\to$ **Bfr** *is faithful.*

Proof. This functor faithful because its action on morphisms only forgets a property. □

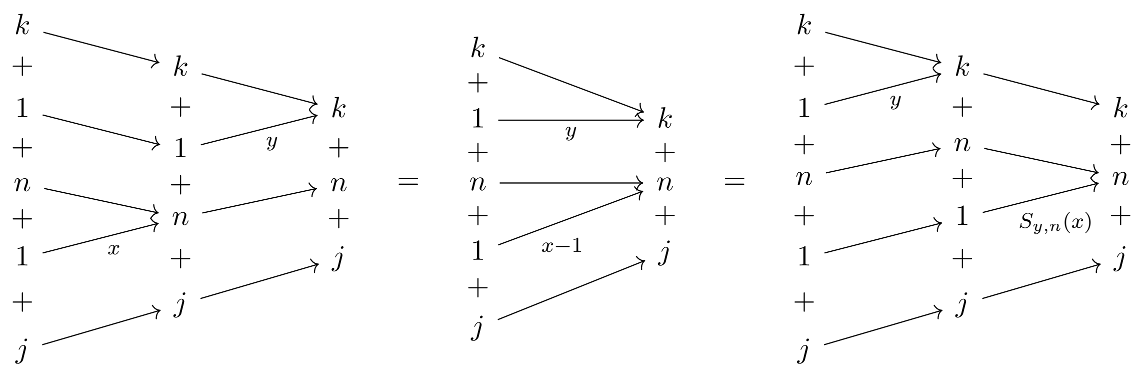
Example 4.15. The structures given in Example 4.9 make the B-frame defined in Example 4.3 into a B-system as follows. This B-system and the category of renamings from Example 3.11 correspond to each other under the equivalence between B-systems and C-systems in Theorem 5.45, in the sense that each of them is isomorphic to the image of the other (under the correct functor). This B-system can thus be regarded as the B-system of renamings of an untyped theory.

Consider first homomorphism of B-frames $S_y\colon \mathbb{B}/(k+1) \to \mathbb{B}/k$, for $k \in \mathbb{N}$ and $y \in \mathsf{std}(k)$. The homomorphism S_y preserves the substitution structure if, for every $n \in \mathbb{N}$,

$x \in \mathsf{std}(k+1+n)$ and $j \in \mathbb{N}$, the square

$$\begin{array}{ccc} \mathsf{std}(k+1+n+1+j) & \xrightarrow{S_{y,n+1+j}} & \mathsf{std}(k+n+1+j) \\ {\scriptstyle S_{x,j}}\downarrow & & \downarrow{\scriptstyle S_{S_{y,n}(x),j}} \\ \mathsf{std}(k+1+n+j) & \xrightarrow{S_{y,n+j}} & \mathsf{std}(k+n+j) \end{array}$$

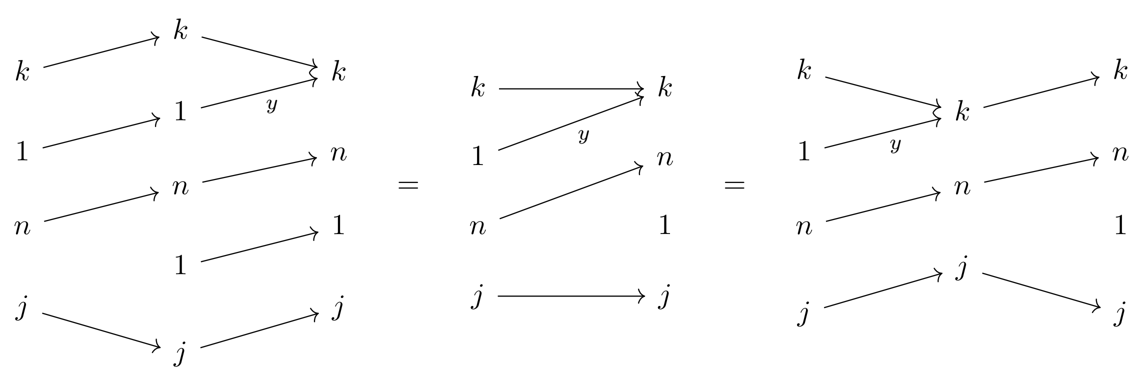
commutes. This can be readily verified in the three cases $x < k$, $x = k$ or $k < x < n+1+k$. For example, in the last case $S_{y,n}(x) = x - 1$ and



The homomorphism S_y preserves the weakening structure if for every $n, j \in \mathbb{N}$, the square

$$\begin{array}{ccc} \mathsf{std}(k+1+n+1+j) & \xrightarrow{S_{y,n+1+j}} & \mathsf{std}(k+n+1+j) \\ {\scriptstyle W_{k+1+n,j}}\uparrow & & \uparrow{\scriptstyle W_{k+n,j}} \\ \mathsf{std}(k+1+n+j) & \xrightarrow{S_{y,n+j}} & \mathsf{std}(k+n+j) \end{array}$$

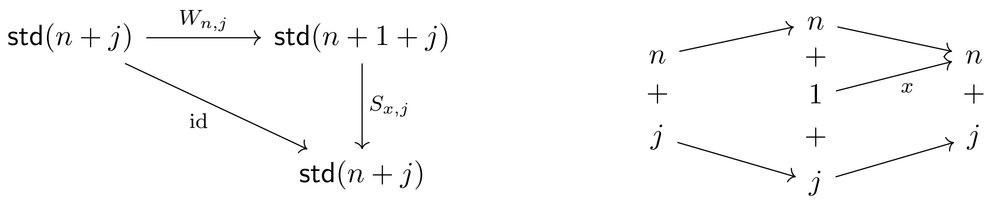
commutes. This is indeed the case:



Finally, for every $n \in \mathbb{N}$, the function $S_{y,n+1} \colon \mathsf{std}(k+1+n+1) \to \mathsf{std}(k+n+1)$ preserves the maximum. It follows that S_y preserves the generic elements.

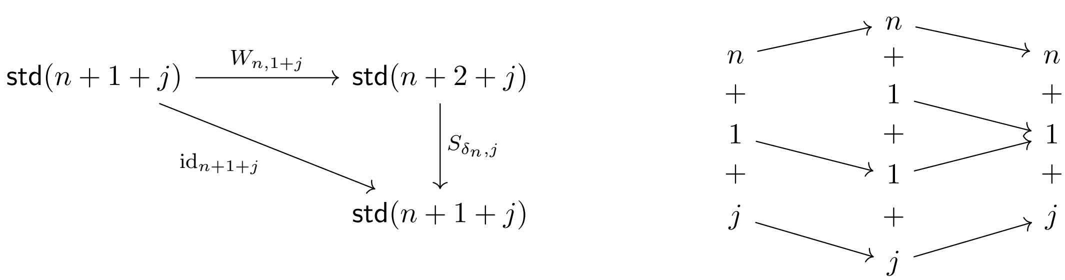
We have shown that S_y is a pre-B-homomorphism. We leave the verification that $W_n \colon \mathbb{B}/n \to \mathbb{B}/(n+1)$ is a pre-B-homomorphism to the reader and consider instead the remaining three conditions of Definition 4.11.

Condition 3 amounts to the commutativity of the left-hand diagram below, for every $n, j \in \mathbb{N}$ and $x \in \mathsf{std}(n)$. Its commutativity is shown in the right-hand diagram.



Condition 4 holds since $S_{x,0}(\delta_n) = S_{x,0}(n) = x$, for every $n \in \mathbb{N}$ and $x \in \mathsf{std}(n)$.

Condition 5 amounts to the commutativity of the left-hand diagram below, for every $n, j \in \mathbb{N}$. Its commutativity is shown in the right-hand diagram.



Example 4.16 (B-systems and Generalized Algebraic Theories)**.** Continuing Example 4.13, any Generalized Algebraic Theory (in Garner's taxonomy also known as $\{w, p, s\}$-GATs [Gar15, Definition 4]) gives rise to a B-system. The axioms of Definition 4.11 follow mostly from the definition of substitution and the congruence rules that substitution satisfies.

Composing Cartmell's equivalence of categories between contextual categories and GATs with our equivalence between B-systems and C-systems constructed in Section 5.4, we later can establish a more precise relationship between B-systems and GATs, in the form of an equivalence of categories.

4.2. **The category of E-systems.**

In Section 4.3 we will show how for any B-frame we get a category $\mathcal{F}$ with objects (n, X) where $X \in B_n$. As we saw in Definition 4.6, the family of sets B_n induces a tree, with objects (n, X), and $\mathcal{F}$ is the free category generated by this tree. The sets $\tilde{B}_{n+1}$ then induce a family of sets of terms indexed by the morphisms of $\mathcal{F}$. In particular for a morphism $(n+1, X) \to (n, \mathsf{ft}(X))$ we get a set of terms $\partial^{-1}(X)$.

In this section we will define the structure of a type theory directly on $\mathcal{F}$ of the kind that one gets when turning a B-system into a category. Such systems are called E-systems, and in Section 4.3 we will show that the category of B-systems is equivalent to a subcategory of E-systems. Thus, E-systems can be seen as a generalisation of B-systems.

Just like B-systems (and different from C-systems), E-systems have an explicit structure for "terms". Indeed, the first step towards the definition of E-system is that of a "term structure":

Definition 4.17. A **category with term structure** is a category $\mathcal{F}$ equipped with a family of sets $(T(A))_{A \in \mathrm{Mor}(\mathcal{F})}$ indexed by the morphisms $\mathrm{Mor}(\mathcal{F})$ of $\mathcal{F}$. Given two categories $\mathcal{F}$ and $\mathcal{D}$ with term structure, a **functor with term structure** from $\mathcal{F}$ to $\mathcal{D}$ is a functor $F : \mathcal{F} \to \mathcal{D}$ equipped with a family of functions $T(A) \to T(F(A))$ for every morphism A in $\mathcal{F}$.

Any B-frame, and hence any B-system, generates a category with term structure; details will be given in Construction 4.69.

The identity functor with term structure $\mathrm{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F}$ is the identity functor on $\mathcal{F}$ equipped with the identity functions $T(A) \to T(A)$ indexed by the morphisms A in $\mathcal{F}$. Similarly, the composition $G \circ F$ of two functors F and G with term structure is defined to be the composition of the underlying functors, equipped with the composites

$$T(A) \longrightarrow T(F(A)) \longrightarrow T(G(F(A))).$$

Definition 4.18. Let $\mathcal{F}$ be a strict category with term structure and Γ an object of $\mathcal{F}$. The **slice term structure** on the strict slice category $\mathcal{F}/\Gamma$ is given by $T_{\mathcal{F}/\Gamma}(A) = T_{\mathcal{F}}(A)$.

Remark 4.19. Every functor $F : \mathcal{F} \to \mathcal{F}'$ with term structure gives rise to a functor with term structure $F/X : \mathcal{F}/X \to \mathcal{F}'/F(X)$.

In order to illustrate the additional structure that we shall consider on a category with term structure, we introduce the following example.

Example 4.20. Consider the the poset $(\mathbb{N}, \geq)$. We write $(n, k)\colon n + k \geq n$ for arrows in $(\mathbb{N}, \geq)$. Let $\mathcal{N}$ be the category with term structure which consists of the poset $(\mathbb{N}, \geq)$ and the term structure given by $T(n, k) := \mathbf{Set}([k], [n])$, *i.e.* the set of functions from the standard set with k elements to the standard set with n elements.

This category with term structure, equipped with the additional structure described in this section, corresponds (up to isomorphism) to the B-system of renaming from Example 4.15 under the equivalence between B-systems and stratified E-sytems in Theorem 4.90.

Example 4.21. Consider a category $\mathcal{C}$ with a terminal object together with a class of arrows $\mathcal{F}$ such that pullbacks along arrows in $\mathcal{F}$ exist in $\mathcal{C}$, $\mathcal{F}$ is closed under composition and pullback, and it contains all isomorphisms and arrows towards a terminal object. This is a type-theoretic fibration category in the sense of [Shu15], or a clan in the sense [Joy17], or a display map category [Tay99] which models Σ-types in the sense of [Nor19].

If we also denote by $\mathcal{F}$ the wide subcategory of $\mathcal{C}$ on the arrows that occur in $\mathcal{F}$, then we can equip $\mathcal{F}$ with a term structure T by requiring $T(A)$ to be the set of sections of A, that is, those arrows $x\colon \Gamma \to \Gamma.A$ in $\mathcal{C}$ such that $A \circ x = \mathrm{id}_\Gamma$.

4.2.1. *Substitution systems.* Given a (strict) category $\mathcal{F}$, an object $\Gamma \in \mathcal{F}$ and an object $A \in \mathcal{F}/\Gamma$, we will write $\Gamma.A$ for the domain of A. In other words, A is a morphism $\Gamma.A \to \Gamma$.

Definition 4.22. A **pre-substitution structure** on a strict category with term structure $\mathcal{F}$ consists of a functor with term structure $S_x : \mathcal{F}/\Gamma.A \to \mathcal{F}/\Gamma$ for every $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$, such that $S_x(\mathrm{id}_{\Gamma.A}) = \mathrm{id}_\Gamma$.

A **pre-substitution system** is a strict category with term structure together with a pre-substitution structure.

Definition 4.23. A **pre-substitution homomorphism** $F : \mathcal{F} \to \mathcal{D}$ is a functor with term structure for which the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A & \xrightarrow{F/\Gamma.A} & \mathcal{D}/F(\Gamma.A) \\ {\scriptstyle S_x}\downarrow & & \downarrow{\scriptstyle S_{F(x)}} \\ \mathcal{F}/\Gamma & \xrightarrow[F/\Gamma]{} & \mathcal{D}/F(\Gamma) \end{array}$$

commutes for every $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$.

Definition 4.24. Let $\mathcal{F}$ be a pre-substitution system and Γ an object of $\mathcal{F}$. The **slice pre-substitution structure** on the strict slice category with term structure $\mathcal{F}/\Gamma$ from Definition 4.18 is given by $S(\mathcal{F}/\Gamma)_x = S(\mathcal{F})_x$, for every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$ and $x \in T_{\mathcal{F}}(P)$.

Definition 4.25. A **substitution system** is a pre-substitution system for which each S_x is a pre-substitution homomorphism. A **substitution homomorphism** is a pre-substitution homomorphism between substitution systems.

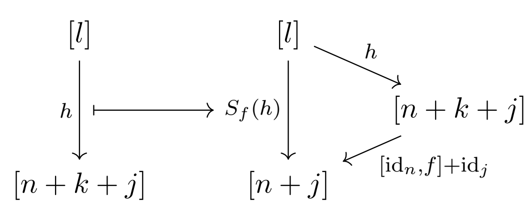
Corollary 4.26. *For any object Γ of a substitution system $\mathcal{F}$, the slice pre-substitution system $\mathcal{F}/\Gamma$ from Definition 4.24 is a substitution system, called the **slice substitution system** on Γ.*

Remark 4.27. The condition that every S_x is a substitution homomorphism, asserts that the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A.P.Q & \xrightarrow{S_x/P.Q} & \mathcal{F}/\Gamma.S_x(P).S_x(Q) \\ {\scriptstyle S_y}\big\downarrow & & \big\downarrow{\scriptstyle S_{S_x(y)}} \\ \mathcal{F}/\Gamma.A.P & \xrightarrow[S_x/P]{} & \mathcal{F}/\Gamma.S_x(P) \end{array}$$

commutes for every $y \in T(Q)$.

Example 4.28. We can equip the category with term structure $\mathcal{N}$ from Example 4.20 with a substitution structure as follows. Consider the functor $-k\colon \mathbb{N}/(n+k) \to \mathbb{N}/n$ that maps $(n+k+j, l)$ to $(n+j, l)$. It preserves terminal objects since an arrow (m, i) is an identity if and only if $i = 0$. Given $(n, k)\colon n+k \geq n$ and a function $f\colon [k] \to [n]$, define $S_f\colon \mathcal{N}/(n+k) \to \mathcal{N}/n$ as the functor $-k$ together with functions $T(n+k+j, l) \to T(n+j, l)$ defined by postcomposition



where $[\mathrm{id}_n, f]$ is the function given by the universal property of the coproduct $[n] \leftarrow [n+k] \to [k]$ in **Set**, and similarly for $[\mathrm{id}_n, f] + \mathrm{id}_j$.

The fact that S_f is a pre-substitution homomorphism follows from the fact that post-composition distributes on $[-,-]$ as shown below: given $g\colon [l] \to [n+k+j]$, then

$$S_f/(n+k, j) \circ S_g = S_{S_f(g)} \circ S_f/(n+k, j+l)$$

since

$$\begin{aligned} ([\mathrm{id}_n, f] + \mathrm{id}_j)\,[\mathrm{id}_{n+k+j}, g] &= [[\mathrm{id}_n, f] + \mathrm{id}_j, S_f(g)] \\ &= [\mathrm{id}_{n+j}, S_f(g)]\,([\mathrm{id}_n, f] + \mathrm{id}_{j+l})\,. \end{aligned}$$

Example 4.29. Given a clan $(\mathcal{C}, \mathcal{F})$, consider the induced category with term structure from Example 4.21, where $T(A)$ is the set of all sections of $A \in \mathcal{F}/\Gamma$.

Say that a choice of pullbacks of arrows in $\mathcal{F}$ is locally functorial if, for every $f\colon \Delta \to \Gamma$ in $\mathcal{C}$, we have $f^*(\mathrm{id}_\Gamma) = \mathrm{id}_\Delta$, $f^{\mathrm{id}_\Gamma} = f$, and, for every composable A, P in $\mathcal{F}$, we also have

$f^*(A \circ P) = f^*A \circ (f^A)^*P$ and $f^{A\circ P} = (f^A)^P$, where f^*A and f^A are the first and second leg, respectively, of the chosen pullback of A along f.

Every choice of pullbacks of arrows in $\mathcal{F}$ uniquely determines a choice of pullbacks of sections of arrows in $\mathcal{F}$. It follows that the category with term structure from Example 4.21 can be equipped with a pre-substitution structure, by setting $S_x :\equiv x^*$ for every $x \in T(A)$. This pre-substitution structure gives rise to a substitution system if the choice of pullbacks is functorial, *i.e.* such that $(f \circ g)^*A = g^*(f^*A)$ and $(f \circ g)^A = f^A \circ g^{f^*A}$. Note that every choice of pullbacks can be made normal, *i.e.* such that $\mathrm{id}_\Gamma^* A = A$ and $\mathrm{id}_\Gamma^A = \mathrm{id}_{\Gamma.A}$ (this holds true more generally for every cleavage on a Grothendieck fibration).

The following is not an intended example, but rather a surprising one.

Example 4.30. Consider a group G. A term structure on G consists of a set $T(g)$ for every element g of G.

A pre-substitution structure on G consists of a functor with term structure $S_x : G/\bullet \to G/\bullet$ (where $\bullet$ denotes the only object in G viewed as a category) for every $g \in G$ and every $x \in T(g)$ such that $S_x(\mathrm{id}_\bullet) = \mathrm{id}_\bullet$. One can show that such functors $S_x : G/\bullet \to G/\bullet$ correspond to functions $G \to G$ which preserve the identity, so a pre-substitution structure amounts to functions $S_x : G \to G$ for every $g \in G$, $x \in T(g)$ preserving the identity together with functions $S_x : T(h) \to T(S_x(h))$ for every $g, h \in G$, $x \in T(g)$.

A substitution structure T on G is a pre-substitution structure S as described above such that the following diagrams commute for all $g, h, k \in G$, $x \in T(g)$, and $y \in T(h)$.

$$
\begin{array}{ccc}
G & \xrightarrow{S_x} & G \\
\downarrow{\scriptstyle S_y} & & \downarrow{\scriptstyle S_{S_x(y)}} \\
G & \xrightarrow{S_x} & G
\end{array}
\qquad\qquad
\begin{array}{ccc}
T(k) & \xrightarrow{S_x} & T(S_x k) \\
\downarrow{\scriptstyle S_y} & & \downarrow{\scriptstyle S_{S_x(y)}} \\
T(S_y k) & \xrightarrow{S_x} & T(S_x S_y k)
\end{array}
$$

Now for a particular example, suppose that each $T(g)$ is $\mathrm{Aut}(G)$, that each $S_x : G \to G$ is just the automorphism x, and that each $S_x : T(h) \to T(S_x h)$ takes $y \in T(h)$ to xyx^{-1}. Then we find indeed that the first diagram commutes since $(xyx^{-1})x = xy$ for all $x \in S_x = \mathrm{Aut}(G)$ and all $y \in S_y = \mathrm{Aut}(G)$. The second diagram commutes since $(xyx^{-1})xzx^{-1}(xyx^{-1})^{-1} = xyzy^{-1}x^{-1}$ for all $x \in T(g) = \mathrm{Aut}(G)$, $y \in T(h) = \mathrm{Aut}(G)$, and $z \in T(k) = \mathrm{Aut}(G)$.

4.2.2. *Weakening systems.*

Definition 4.31. Consider a category $\mathcal{F}$ with term structure T. A **pre-weakening structure** on $\mathcal{F}$ is a family of functors with term structure $W_A : \mathcal{F}/\Gamma \to \mathcal{F}/\Gamma.A$ indexed by the morphisms $A : \Gamma.A \to \Gamma$ in $\mathcal{F}$ such that

(1) $W_{\mathrm{id}_\Gamma} = \mathrm{id}_{\mathcal{F}/\Gamma}$ for every object $\Gamma \in \mathcal{F}$.
(2) $W_{A\circ P} = W_P \circ W_A$ for every $P \in \mathcal{F}/\Gamma.A$ and $A \in \mathcal{F}/\Gamma$.
(3) W_A strictly preserves the final object, i.e., $W_A(\mathrm{id}_\Gamma) = \mathrm{id}_{\Gamma.A}$.

A **pre-weakening system** is a strict category with term structure equipped with a pre-weakening structure.

Definition 4.32. A **pre-weakening homomorphism** $F : \mathcal{F} \to \mathcal{D}$ between pre-weakening systems is a functor $F : \mathcal{F} \to \mathcal{D}$ with term structure such that the square

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A & \xrightarrow{F/\Gamma.A} & \mathcal{D}/F(\Gamma.A) \\ {\scriptstyle W_A}\uparrow & & \uparrow{\scriptstyle W_{F(A)}} \\ \mathcal{F}/\Gamma & \xrightarrow[F/\Gamma]{} & \mathcal{D}/F(\Gamma) \end{array}$$

of functors with term structure commutes for any $A \in \mathcal{F}/\Gamma$.

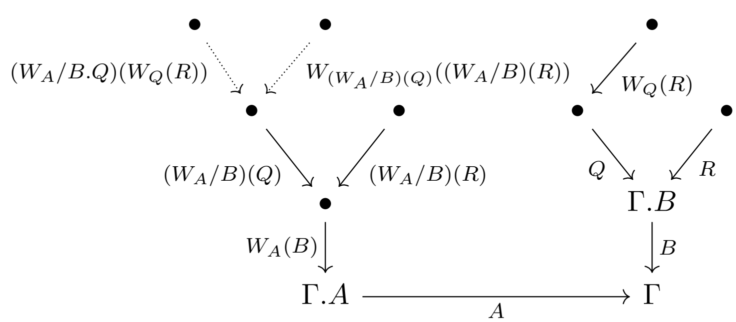
Definition 4.33. Let $\mathcal{F}$ be a pre-weakening system and Γ an object of $\mathcal{F}$. The **slice pre-weakening system** on the strict slice category with term structure from Definition 4.18 $\mathcal{F}/\Gamma$ is given by $W(\mathcal{F}/\Gamma)_P = W(\mathcal{F})_P$ for every $P \in \mathcal{F}/\Gamma.A$ and $A \in \mathcal{F}/\Gamma$.

Definition 4.34. A **weakening system** is a pre-weakening system $\mathcal{F}$ such that W_A is a pre-weakening homomorphism for every morphism A in $\mathcal{F}$. A **weakening homomorphism** is a pre-weakening homomorphism between weakening systems.

Remark 4.35. The condition that every W_A is a pre-weakening homomorphism implies that the square

$$\begin{array}{ccc} \mathcal{F}/\Gamma.B.Q & \xrightarrow{W_A/B.Q} & \mathcal{F}/\Gamma.A.W_A(B.Q) \\ {\scriptstyle W_Q}\uparrow & & \uparrow{\scriptstyle W_{W_A(Q)}} \\ \mathcal{F}/\Gamma.B & \xrightarrow[W_A/B]{} & \mathcal{F}/\Gamma.A.W_A(B) \end{array}$$

commutes for each $A, B \in \mathcal{F}/\Gamma$ and $Q \in \mathcal{F}/\Gamma.B$. On objects, this property asserts that for any $k \in \mathcal{F}/\mathrm{E}$, the dotted arrows in the diagram



are equal.

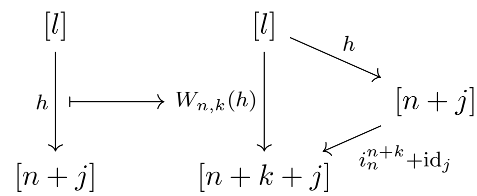
A useful special case of this property is where $B = \mathrm{id}_\Gamma$. Thus, if W is a weakening system, then the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.C & \xrightarrow{W_A/C} & \mathcal{F}/\Gamma.A.W_A(C) \\ {\scriptstyle W_C}\uparrow & & \uparrow{\scriptstyle W_{W_A(C)}} \\ \mathcal{F}/\Gamma & \xrightarrow[W_A]{} & \mathcal{F}/\Gamma.A \end{array}$$

commutes for every $A, C \in \mathcal{F}/\Gamma$. In particular, we see that $W_A(W_C(D)) = W_{W_A(C)}(W_A(D))$ for any $D \in \mathcal{F}/\Gamma$, i.e. that weakening is a self-distributive operation.

Corollary 4.36. *For any object Γ of a weakening system $\mathcal{F}$, the slice pre-weakening system on $\mathcal{F}/\Gamma$ from Definition 4.33 is a weakening system, called the **slice weakening system** on Γ.*

Example 4.37. Consider the category with term structure $\mathcal{N}$ from Example 4.20. We can equip $\mathcal{N}$ with a weakening structure as follows. Consider the functor $+k\colon \mathbb{N}/n \to \mathbb{N}/(n+k)$ that maps $(n+j, l)$ to $(n+k+j, l)$. It preserves terminal objects as in Example 4.28. Given $(n,k)\colon n+k \geq n$, define $W_{n,k}\colon \mathcal{N}/n \to \mathcal{N}/(n+k)$ as the functor $+k$ together with functions $T(n+j,l) \to T(n+k+j,l)$ defined by postcomposition



where $i_n^{n+k}\colon [n] \to [n+k]$ is the initial-segment inclusion and $i_n^{n+k} + \mathrm{id}_j$ is the function given by the universal property of the coproduct $[n] \leftarrow [n+j] \to [j]$ in **Set**.

The fact that $W_{n,k}$ is a pre-weakening homomorphism follows from the fact that initial-segment inclusions factor uniquely into inclusions whose images have codimension 1: given $(n+j,l)$ in $\mathcal{N}/n$, then

$$W_{n,k}/(n, j+l) \circ W_{n+j,l} = W_{W_{n,k}(n+j,l)} \circ W_{n,k}/(n,j)$$

since

$$(i_n^{n+k} + \mathrm{id}_{j+l}) i_{n+j}^{n+j+l} = i_{n+k+j}^{n+k+j+l}(i_n^{n+k} + \mathrm{id}_j).$$

Example 4.38. Given a clan $(\mathcal{C}, \mathcal{F})$, the induced category with term structure from Example 4.21 can be made into a weakening system if there is a choice of pullbacks in $\mathcal{F}$ which is functorial and locally functorial in the sense of Example 4.29.

Note that, since $\mathcal{F}$ is closed under pullbacks, every clan with a choice of pullbacks that gives rise to a substitution structure also has an induced weakening structure.

Example 4.39. Consider the situation of Example 4.30 above where the underlying category is a group G with term structure S.

A pre-weakening structure on G is a family of functions $W_g : G \to G$ for each $g \in G$ which preserves the identity in each coordinate (i.e. $W_e(g) = W_g(e) = g$ for any $g \in G$) and where $W_{hg} = W_g \circ W_h$ together with term structure $W_g : T(h) \to T(W_g(h))$ for any $g, h \in G$.

If each $W_g : G \to G$ is a group homomorphism, this structure is a weakening system when the following diagrams commute for every $g, h, k \in G$.

$$\begin{array}{ccc} G & \xrightarrow{W_h} & G \\ {\scriptstyle W_g}\uparrow & & \uparrow{\scriptstyle W_{W_h(g)}} \\ G & \xrightarrow[W_h]{} & G \end{array} \qquad\qquad \begin{array}{ccc} T(W_g k) & \xrightarrow{W_h} & T(W_{gh} k) \\ {\scriptstyle W_g}\uparrow & & \uparrow{\scriptstyle W_{W_h(g)}} \\ T(k) & \xrightarrow[W_h]{} & T(W_h k) \end{array}$$

Now consider the more particular example discussed in Example 4.30, where G is still an arbitrary group, but $T(g) = \mathrm{Aut}(G)$ for all $g \in G$. We can let each $W_g : G \to G$ be φ_g, the conjugation automorphism sending h to ghg^{-1}, and we can let each $W_g : T(k) \to T(k)$ be 'conjugation by conjugation' taking each automorphism $x \in T(k)$ to $\varphi_g x \varphi_g^{-1}$. Since

$\varphi_h\varphi_g = \varphi_{\varphi_h(g)}\varphi_h$, the left-hand diagram above commutes, and using that equation we find that $\varphi_h\varphi_g(-)\varphi_g^{-1}\varphi_h^{-1} = \varphi_{\varphi_h(g)}\varphi_h(-)\varphi_{\varphi_h(g)}^{-1}\varphi_h^{-1}$ so the right-hand diagram commutes.

4.2.3. *Projection systems.*

Definition 4.40. A **pre-projection system** is a pre-weakening system $\mathcal{F}$ equipped with an element $\mathsf{idtm}_A \in T(W_A(A))$ for every $A \in \mathcal{F}/\Gamma$ and $\Gamma \in \mathcal{F}$.

Definition 4.41. A **pre-projection homomorphism** $F : \mathcal{F} \to \mathcal{D}$ is a pre-weakening homomorphism for which

$$F(\mathsf{idtm}_A) = \mathsf{idtm}_{F(A)}$$

for every $A \in \mathcal{F}/\Gamma$ and $\Gamma \in \mathcal{F}$.

Definition 4.42. Let $\mathcal{F}$ be a pre-projection system and Γ an object of $\mathcal{F}$. The **slice pre-projection structure** on $\mathcal{F}/\Gamma$ is given by the slice pre-weakening structure in Definition 4.33 together with $\mathsf{idtm}^\Gamma_P := \mathsf{idtm}_P$, for every $P \in \mathcal{F}/\Gamma.A$ and $A \in \mathcal{F}/\Gamma$.

Definition 4.43. A **projection system** is a pre-projection system for which every W_A is a pre-projection homomorphism. A **projection homomorphism** is a pre-projection homomorphism between projection systems.

Corollary 4.44. *For any object Γ of a projection system $\mathcal{F}$, the slice pre-projection system on $\mathcal{F}/\Gamma$ from Definition 4.42 is a projection system, called the* ***slice projection system*** *on Γ.*

Example 4.45. Consider the weakening system on $\mathcal{N}$ from Example 4.37. We can equip it with a projection structure defining, for every (n, k) in $(\mathbb{N}, \geq)$, the element $\mathsf{idtm}_{n,k} \in T(W_{n,k}(n, k)) = \mathbf{Set}([k], [n + k])$ to be the final-segment inclusion

$$[k] \xrightarrow{i_k^{n+k}} [n + k]$$

The fact that each $W_{n,j} : \mathcal{N}/n \to \mathcal{N}/(n + j)$ is a projection homomorphism is readily verified:

$$\begin{aligned} W_{n,j}(\mathsf{idtm}_{n+m,k}) &= (i_n^{n+j} + \mathrm{id}_{m+k})i_k^{n+m+k} \\ &= i_k^{n+m+j+k} \\ &= \mathsf{idtm}_{W_{n,j}(n+m,k)}. \end{aligned}$$

Example 4.46. Given a clan $(\mathcal{C}, \mathcal{F})$, the weakening system from Example 4.38 can be upgraded to a projection system by defining the element $\mathsf{idtm}_A \in T(W_A(A))$ to be the unique section of the pullback of A along itself induced by the pair of identity arrows on $\Gamma.A$.

Note that no additional condition on the choice of pullbacks has to be imposed, besides those mentioned in Example 4.38. In particular, every clan with a choice of pullbacks that gives rise to a weakening structure has also an induced projection structure.

Example 4.47. Consider the particular example discussed in Example 4.39 where the underlying category is an arbitrary group G, each $T(g)$ is the set of automorphisms of G, and W_g is conjugation by G both on elements of G and terms (automorphisms of G).

A pre-projection system consists of an element $\mathsf{idtm}_g \in \mathrm{Aut}(G)$ for every $g \in G$. We will let idtm_g be the identity automorphism on G.

For a projection system on a group G, we need $W_g(\mathsf{idtm}_h) = \mathsf{idtm}_{Wg(h)}$ for every $g, h \in G$. In our particular example, this means $g1_Gg^{-1} = 1_G$, which holds.

4.2.4. *The definition of E-systems.* We can now give the definition of E-systems.

Definition 4.48. A **pre-E-system** $\mathbb{E}$ is a strict category $\mathcal{F}$ with term structure equipped with a chosen terminal object $[\,]$ in $\mathcal{F}$, the structure of a pre-substitution system S, the structure of a pre-weakening system W, and the structure of a pre-projection system idtm.

Definition 4.49. A **pre-E-homomorphism** from $\mathbb{E}$ to $\mathbb{D}$ is a functor $H : \mathcal{F}_{\mathbb{E}} \to \mathcal{F}_{\mathbb{D}}$ between the underlying categories with term structure such that $F([\,]_{\mathbb{E}}) = [\,]_{\mathbb{D}}$, which is a pre-substitution homomorphism, a pre-weakening homomorphism, and a pre-projection homomorphism.

Definition 4.50. For every object Γ in a pre-E-system $\mathbb{E}$, the **slice pre-E-system** $\mathbb{E}/\Gamma$ on the slice category with term structure $\mathcal{F}_{\mathbb{E}}/\Gamma$ is given by the slice structures from Definitions 4.24, 4.33 and 4.42, and the identity on Γ as terminal object.

Definition 4.51. An **E-system** is a pre-E-system $\mathbb{E}$ such that
(1) each S_x is a pre-E-homomorphism,
(2) each W_A is a pre-E-homomorphism,
(3) $S_x \circ W_A = \mathrm{id}_{\mathbb{E}/\Gamma}$ for any $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$,
(4) $S_x(\mathsf{idtm}_A) = x$ for any $x \in T(A)$ and $A \in \mathcal{F}/\Gamma$, and
(5) $S_{\mathsf{idtm}_A} \circ W_A/A = \mathrm{id}_{\mathbb{E}/\Gamma.A}$ for any $A \in \mathcal{F}/\Gamma$.

An **E-homomorphism** $H : \mathbb{E} \to \mathbb{D}$ is a pre-E-homomorphism from an E-system $\mathbb{E}$ to an E-system $\mathbb{D}$. We write **Esys** for the category of E-systems and E-homomorphisms.

Remark 4.52. The condition that each W_A is a substitution homomorphism asserts that the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.B.Q & \xrightarrow{W_A/B.Q} & \mathcal{F}/\Gamma.A.W_A(B).W_A(Q) \\ {\scriptstyle S_y}\downarrow & & \downarrow{\scriptstyle S_{W_A(y)}} \\ \mathcal{F}/\Gamma.B & \xrightarrow[W_A/B]{} & \mathcal{F}/\Gamma.A.W_A(B) \end{array}$$

of functors with term structure commutes for every $Q \in \mathcal{F}/\Gamma.B$, $B \in \mathcal{F}/\Gamma$ and each $y \in T(Q)$.

Likewise, the condition that each S_x is a weakening homomorphism asserts that the diagram

$$\begin{array}{ccc} \mathcal{F}/\Gamma.A.P & \xrightarrow{S_x/P} & \mathcal{F}/\Gamma.S_x(P) \\ {\scriptstyle W_Q}\downarrow & & \downarrow{\scriptstyle W_{S_x(Q)}} \\ \mathcal{F}/\Gamma.A.P.Q & \xrightarrow[S_x/P.Q]{} & \mathcal{F}/\Gamma.S_x(P).S_x(Q) \end{array}$$

of functors with term structure commutes for every $Q \in \mathcal{F}/\Gamma.A.P$.

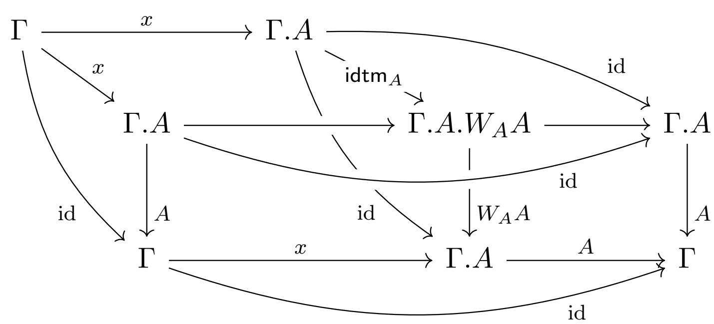
Corollary 4.53. *For any object Γ of a E-system $\mathcal{F}$, the slice pre-E-system on $\mathcal{F}/\Gamma$ from Definition 4.50 is an E-system, called the **slice E-system** on Γ.*

Example 4.54. We can finally show that the category with term structure $\mathcal{N}$ from Example 4.20 can be equipped with the structure of an E-system. It can be turned into a pre-E-system because of Examples 4.28, 4.37 and 4.45. The terminal object is $[0]$. Conditions 1 and 2 of Definition 4.51 are left to the reader. The other ones are verified as follows:

3. Given $f\colon [k] \to [n]$, it is $S_f \circ W_{n,k} = \mathrm{id}_{\mathcal{N}/n}$ since $[\mathrm{id}_n, f] i_n^{n+k} = \mathrm{id}_n$.
4. Given $f\colon [k] \to [n]$, it is $S_f(\mathsf{idtm}_{n,k}) = [\mathrm{id}_n, f] i_k^{n,k} = f$.

5. Given (n,k) in $\mathcal{N}$, it is $S_{\mathsf{idtm}_{n,k}} \circ W_{n,k}/(n,k) = \mathrm{id}_{\mathcal{N}/(n+k)}$ since $[\mathrm{id}_{n+k}, i_k^{n+k}](i_n^{n+k} + \mathrm{id}_k) = \mathrm{id}_{n+k}$.

Example 4.55. Given a clan $(\mathcal{C}, \mathcal{F})$ together with a functorial and locally functorial choice of pullbacks of arrows in $\mathcal{F}$ along arrows in $\mathcal{C}$ in the sense of Example 4.29, the induced category with term structure from Example 4.21 can be made into an E-system as follows. It is a pre-E-system because of Examples 4.29, 4.38 and 4.46. Conditions (1) and (2) are satisfied since weakening is a particular case of substitution. Condition (3) follows from $A \circ x = \mathrm{id}$ and the functoriality conditions on the choice of pullbacks. In particular, the left-hand vertical square in the commutative diagram below is a pullback, since so is the right-hand one. It follows that the upper square is a pullback too. Therefore condition (4) is also satisfied. Finally, condition (5) follows from the commutativity of the upper triangle involving idtm_A in the commutative diagram below, together, again, with the functoriality conditions on the choice of pullbacks.



Example 4.56. Consider the situation in Example 4.47 where our underlying category is an arbitrary group G, the terms of each $g \in G$ are Aut, substitution $S_x : G \to G$ is given by the automorphism x itself and substitution $S_x : T(h) \to T(S_x h)$, weakening $W_g : G \to G$, and weakening $W_g : T(h) \to T(W_g h)$ are given by conjugation.

Understood as a category, G does not have a terminal object (unless it is trivial), but we can still understand the conditions of Definition 4.51. The condition 2 in that definition means that the following diagrams must commute for $g, h, k \in G$ and $x \in T(h)$

$$\begin{array}{ccc} G & \xrightarrow{W_g} & G \\ \downarrow{\scriptstyle S_x} & & \downarrow{\scriptstyle S_{W_g(x)}} \\ G & \xrightarrow{W_g} & G \end{array} \qquad\qquad \begin{array}{ccc} T(k) & \xrightarrow{W_g} & T(W_g k) \\ \downarrow{\scriptstyle S_x} & & \downarrow{\scriptstyle S_{W_g(x)}} \\ T(S_x k) & \xrightarrow{W_g} & T(S_{W_g(x)} W_g k) \end{array}$$

Since $\varphi_g x = \varphi_g x \varphi_g^{-1} \varphi_g$, the left-hand diagram above commutes, and since $\varphi_g x(-)x^{-1}\varphi_g^{-1} = (\varphi_g x \varphi_g^{-1})\varphi_g(-)\varphi_g^{-1}(\varphi_g x \varphi_g^{-1})^{-1}$, the right-hand square above commutes.

The condition 1 in Definition 4.51 means that the following diagrams must commute for $g, h, k \in G$ and $x \in T(h)$.

$$\begin{array}{ccc} G & \xrightarrow{S_x} & G \\ \downarrow{\scriptstyle W_g} & & \downarrow{\scriptstyle W_{S_x(g)}} \\ G & \xrightarrow{S_x} & G \end{array} \qquad\qquad \begin{array}{ccc} T(k) & \xrightarrow{S_x} & T(S_x k) \\ \downarrow{\scriptstyle W_g} & & \downarrow{\scriptstyle W_{S_x(g)}} \\ T(W_g k) & \xrightarrow{S_x} & T(S_x W_g k) \end{array}$$

Since $x\varphi_g = \varphi_{x(g)}x$, the left-hand diagram commutes, and since then $x\varphi_g(-)\varphi_g^{-1}x^{-1} = \varphi_{x(g)}x(-)x^{-1}\varphi_{x(g)}^{-1}$, the right-hand diagram commutes.

Condition 3 does not hold since (on G) $S_x \circ W_g = x\varphi_g$, and in general this is not the identity.

Condition 4 does not hold since $S_x \mathsf{idtm}_g = x1_G x^{-1}$ which is not in general x.

Condition 5 does not hold since (on G) $S_{\mathsf{idtm}_g} W_g = 1_G\varphi_g = \varphi_g$ which is not the identity in general.

We introduce more convenient notation for weakening and substitution.

Definition 4.57. Let $A \in \mathcal{F}/\Gamma$. Recall that $W_A : \mathcal{F}/\Gamma \to \mathcal{F}/\Gamma.A$ acts on objects, morphisms and terms. We introduce the infix form of weakening by $A \in \mathcal{F}/\Gamma$ to be $\langle A\rangle -$. Thus, we will write

$$\begin{aligned} \langle A\rangle B &:= W_A(B) && \text{for } B \in \mathcal{F}/\Gamma \\ \langle A\rangle Q &:= W_A(Q) && \text{for } B \in \mathcal{F}/\Gamma \text{ and } Q \in \mathcal{F}/\Gamma.B \\ \langle A\rangle g &:= W_A(g) && \text{for } B \in \mathcal{F}/\Gamma,\ Q \in \mathcal{F}/\Gamma.B \text{ and } g \in T(Q) \end{aligned}$$

Definition 4.58. Let $x \in T(A)$ for a family $A \in \mathcal{F}/\Gamma$. The infix form of substitution by x is taken to be $-[x]$. Thus, we will write

$$\begin{aligned} P[x] &:= S_x(P) && \text{for } P \in \mathcal{F}/\Gamma.A \\ Q[x] &:= S_x(Q) && \text{for } P \in \mathcal{F}/\Gamma.A \text{ and } Q \in \mathcal{F}/\Gamma.A.P \\ g[x] &:= S_x(g) && \text{for } P \in \mathcal{F}/\Gamma.A,\ Q \in \mathcal{F}/\Gamma.A.P \text{ and } g \in T(Q) \end{aligned}$$

Definition 4.59. A (pre-)E-system is **stratified** if its underlying category is stratified in the sense of Definition 2.1 and the underlying functor of each W_A and S_x is stratified with respect to the stratification induced on slices.

A morphism of stratified (pre-)E-systems is **stratified** if its underlying functor is stratified.

The category of stratified E-systems and stratified E-homomorphisms between them is denoted by $\mathbf{Esys_s}$.

Example 4.60. The E-system on $\mathcal{N}$ from Example 4.54 is stratified by the identity functor.

4.2.5. *Pairing and the projections.* The composition $A.P$ of $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$ behaves like a strict Σ-type. In this section we define the pairing term $\mathsf{pair}^{A,P} := \mathsf{idtm}_{A.P} \in T(W_P(W_A(A.P)))$ and the projections and prove several useful properties about them. The strictness is found, among other things, in the fact that we can prove judgmental η-equality, and that pairing is strictly associative.

In this section we make use of the infix form of the weakening and substitution operations introduced in Definitions 4.57 and 4.58.

Definition 4.61. Let $x \in T(A)$ and $u \in T(S_x(P))$ for $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. We define the **term extension of** x **and** u to be

$$x.u := \mathsf{idtm}_{A.P}[x][u] \in T(A.P).$$

It is well defined since

$$T(W_{A.P}(A.P)) \xrightarrow{S_x} T(W_{P[x]}(A.P)) \xrightarrow{S_u} T(A.P) \tag{4.1}$$

where $S_x \circ W_{A.P} = W_{P[x]}$ because S_x is a weakening homomorphism and $S_x \circ W_A = \mathrm{Id}$.

To prove anything about the term $x.u$, we need the following property.

Theorem 4.62. *Let $x \in T(A)$ and $u \in T(S_x(P))$ for $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. Then we have*

$$S_{x.u} = S_u \circ (S_x/P) \colon \mathbb{E}/\Gamma.A.P \to \mathbb{E}/\Gamma$$

Proof.

$$\begin{aligned}
S_{x.u} &= S_{S_u(S_x(\mathsf{idtm}_{A.P}))} && \text{(By 4.61)}\\
&= S_{S_u(S_x(\mathsf{idtm}_{A.P}))} \circ (S_u \circ W_{S_x(P)}) \circ (S_x \circ W_A) && \text{(By 4.51.3)}\\
&= S_u \circ S_{S_x(\mathsf{idtm}_{A.P})} \circ W_{S_x(P)} \circ S_x \circ W_A && \text{(By 4.51.1)}\\
&= S_u \circ (S_x/P) \circ S_{\mathsf{idtm}_{A.P}} \circ W_P \circ W_A && \text{(By 4.51.1)}\\
&= S_u \circ (S_x/P) \circ S_{\mathsf{idtm}_{A.P}} \circ W_{A.P} && \text{(By 4.31.2)}\\
&= S_u \circ (S_x/P). && \text{(By 4.51.5)}
\end{aligned}$$

□

Corollary 4.63. *For every $x \in T(A)$, $u \in T(S_x(P))$ and $v \in T(S_{x.u}(Q))$ we have*

$$(x.u).v = x.(u.v) \in T(A.P.Q).$$

Proof. By Theorem 4.62, we have $S_v \circ (S_{x.u}/Q) = S_v \circ (S_u/Q[x]) \circ (S_x/P.Q) = S_{u.v} \circ (S_x/P.Q)$, so associativity of term extension follows. □

Definition 4.64. Let $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. We define

$$\begin{aligned}
\mathrm{pr}_0^{A,P} &:= \langle P\rangle \mathsf{idtm}_A \in T(\langle A.P\rangle A)\\
\mathrm{pr}_1^{A,P} &:= \mathsf{idtm}_P \in T(\langle P\rangle P)
\end{aligned}$$

Lemma 4.65. *Let $F \colon \mathbb{E} \to \mathbb{D}$ be an E-homomorphism. For every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$, $x \in T(A)$ and $u \in T(S_x(P))$, it is*

$$F(x.u) = F(x).F(u), \qquad F(\mathrm{pr}_0^{A,P}) = \mathrm{pr}_0^{F(A),F(P)}, \qquad \text{and} \qquad F(\mathrm{pr}_1^{A,P}) = \mathrm{pr}_1^{F(A),F(P)}$$

Proof. We compute:

$$F(x.u) = F(\mathsf{idtm}_{A.P}[x][u]) = \mathsf{idtm}_{FA.FP}[Fx][Fu] = Fx.Fu,$$

$$F(\mathrm{pr}_0^{A,P}) = F(\langle P\rangle \mathsf{idtm}_A) = \langle FP\rangle \mathsf{idtm}_{FA} = \mathrm{pr}_0^{FA,FP},$$

$$F(\mathrm{pr}_1^{A,P}) = F(\mathsf{idtm}_P) = \mathsf{idtm}_{FP} = \mathrm{pr}_1^{FA,FP}$$

where the outer equalities hold by definition, and the inner ones since F is an E-homomorphism. □

Lemma 4.66. *For every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$, $x \in T(A)$ and $u \in T(S_x(P))$, it is*

$$\begin{aligned}
\mathrm{pr}_0^{A,P}[x.u] &= x,\\
\mathrm{pr}_1^{A,P}[x.u] &= u,\\
\mathrm{pr}_0^{A,P}.\mathrm{pr}_1^{A,P} &= \mathsf{idtm}_{A.P}.
\end{aligned}$$

Proof. To show that $\mathrm{pr}_0^{A,P}[x.u] = x$, we use that $S_{x.u} = S_u \circ S_x/P$ to show that

$$
\begin{aligned}
\mathrm{pr}_0^{A,P}[x.u] &= (\langle P\rangle \mathsf{idtm}_A)[x.u] && \text{(By 4.64)}\\
&= (\langle P\rangle \mathsf{idtm}_A)[x][u] && \text{(By Theorem 4.62)}\\
&= (\langle P[x]\rangle \mathsf{idtm}_A[x])[u] && \text{(By 4.51.1)}\\
&= \mathsf{idtm}_A[x] && \text{(By 4.51.3)}\\
&= x && \text{(By 4.51.4)}
\end{aligned}
$$

To show that $\mathrm{pr}_1^{A,P}[x.u] = u$, note that

$$\mathrm{pr}_1^{A,P}[x.u] = \mathsf{idtm}_P[x][u] = \mathsf{idtm}_{P[x]}[u] = u$$

Finally note that

$$\langle A.P\rangle A.(W_{A.P}/A)(P) = \langle A.P\rangle A.P \colon \Gamma.A.P.\langle A.P\rangle(A.P) \to \Gamma.A.P$$

and $\mathsf{idtm}_{\langle A.P\rangle A.P} = (W_{A.P}/A.P)(\mathsf{idtm}_{A.P})$. Thus $\mathrm{pr}_0^{A,P}.\mathrm{pr}_1^{A,P} = \mathsf{idtm}_{A.P}$ follows from the commutativity of the outer square in the diagram below.

$$
\begin{array}{ccc}
T(W_{\langle A.P\rangle A.P}(\langle A.P\rangle A.P)) & \xrightarrow{S_{\mathrm{pr}_0^{A,P}}} & T(W_P/P(\langle A.P\rangle A.P)) \\
\uparrow {\scriptstyle W_{A.P}/A.P} & \nearrow {\scriptstyle W_P/P} & \downarrow {\scriptstyle S_{\mathrm{pr}_1^{A,P}}} \\
T(\langle A.P\rangle A.P) & \xrightarrow{\mathrm{id}} & T(\langle A.P\rangle A.P)
\end{array}
$$

The bottom-right triangle commutes by 4.51.5. For the top-left one:

$$
\begin{aligned}
S_{\langle P\rangle \mathsf{idtm}_A} \circ W_{A.P}/A.P &= S_{\langle P\rangle \mathsf{idtm}_A} \circ W_P/(W_A(A.P)) \circ W_A/A.P && \text{(By 4.31.2)}\\
&= \left(W_P \circ S_{\mathsf{idtm}_A} \circ W_A/A\right)/P && \text{(By 4.51.2)}\\
&= W_P/P. && \text{(By 4.51.5)}
\end{aligned}
$$

□

Theorem 4.67. *For every $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$, the map*

$$
\begin{aligned}
\textstyle\coprod_{x\in T(A)} T(P[x]) &\longrightarrow T(A.P)\\
(x,u) &\longmapsto x.u
\end{aligned}
$$

is a bijection.

Proof. The inverse to the given map is defined by $w \mapsto (\mathrm{pr}_0^{A,P}[w], \mathrm{pr}_1^{A,P}[w])$. Thanks to Lemma 4.66 it is enough to show that, for every $w \in T(A.P)$, one has

$$\mathrm{pr}_0^{A,P}[w].\mathrm{pr}_1^{A,P}[w] = w.$$

Lemma 4.65 gives us that y

$$\mathrm{pr}_0^{A,P}[w].\mathrm{pr}_1^{A,P}[w] = (\mathrm{pr}_0^{A,P}.\mathrm{pr}_1^{A,P})[w].$$

Thus the claim follows from $\mathrm{pr}_0^{A,P}.\mathrm{pr}_1^{A,P} = \mathsf{idtm}_{A.P}$, which holds again by Lemma 4.66. □

One consequence of this theorem is that the set $T(\mathrm{id}_\Gamma)$ has exactly one element, see Corollary 5.17.

4.3. **Characterising B-systems as stratified E-systems.** In this section we construct an equivalence of categories between B-systems and the subcategory of **Esys** on the stratified E-systems and stratified homomorphisms. The functor from B-systems to stratified E-systems is constructed in Section 4.3.1, the one in the other direction in Section 4.3.2. That these form an equivalence is shown in Section 4.3.3.

4.3.1. *From B-systems to stratified E-systems.* Note first that we obtain a functor $\mathbf{Bfr} \to \mathbf{Cat}$ as the composition

$$\mathbf{Bfr} \xrightarrow{\mathbf{R}} \mathbf{RtTr} \xrightarrow{\mathbf{G}} \mathbf{Grph} \xrightarrow{\mathbf{F}} \mathbf{Cat}$$

where $\mathbf{G}$ and $\mathbf{F}$ are the functors from Definition 2.13 and $\mathbf{R}$ is the forgetful functor from Definition 4.6. Arrows in $\mathbf{FGR}(\mathbb{B})$ are of the form $(X, k) \colon (n + k, X) \to (n, \mathsf{ft}^k(X))$, for $X \in B_{n+k}$.

We begin by equipping $\mathbf{FGR}(\mathbb{B})$ with a term structure. The B-frame $\mathbb{B}$ already provides us with sets of terms for the edges of $\mathbf{GR}(\mathbb{B})$, namely $T(X, 1) := \partial^{-1}(X)$. In order to construct sets of terms for (X, k) for each k, which we do in Construction 4.69, we assume that $\mathbb{B}$ comes with a substitution structure in the sense of Definition 4.25. We then show in Construction 4.72 that $\mathbf{FGR}$ gives rise to a functor $\mathbf{T}$ from B-frames with substitution to strict categories with term structure. Next, in Construction 4.75 we provide $\mathbf{T}(\mathbb{B})$ with a pre-E-system structure when $\mathbb{B}$ is a B-system, and prove in Lemma 4.76 that $\mathbf{T}$ preserves and reflects weakening and projection homomorphisms. Finally, we show in Lemma 4.78 that the functor $\mathbf{T}$ lifts to a full and faithful functor from B-systems to stratified E-systems.

Problem 4.68. For every B-frame $\mathbb{B}$ with substitution structure S, to construct a term structure T on the strict category $\mathcal{F}_{\mathbb{B}} := \mathbf{FGR}(\mathbb{B})$ and to construct, for any $t \in T(X, k)$, a homomorphism of B-frames $S^k_t \colon \mathbb{B}/X \to \mathbb{B}/\mathsf{ft}^k(X)$.

Construction 4.69 (for Problem 4.68)**.** We define the term structure by induction on $n \in \mathbb{N}$. More precisely, for any $X \in B_n$ and $k \le n$ we will define a set $T(X, k)$ and, for any $t \in T(X, k)$, a homomorphism of B-frames $S^k_t \colon \mathbb{B}/X \to \mathbb{B}/\mathsf{ft}^k(X)$.

For every n and $X \in B_n$, let $T(X, 0) := \{*\}$ and $S^0_* := \mathrm{id} \colon \mathbb{B}/X \to \mathbb{B}/X$.

For every n and $X \in B_{n+1}$, let

$$T(X, 1) := \partial^{-1}(X) \subseteq \tilde{B}_{n+1} \tag{4.2}$$

and $S^1_x := S_x \colon \mathbb{B}/X \to \mathbb{B}/\mathsf{ft}(X)$ which is a homomorphism of B-frames by assumption.

Suppose now that, for every $m \le n$ and $Y \in B_m$, we have defined sets $T(Y, k)$ for $k \le m$ and, for every $t \in T(Y, k)$, a homomorphism of B-frames $S^k_t \colon \mathbb{B}/Y \to \mathbb{B}/\mathsf{ft}^k(Y)$. Let $X \in B_{n+1}$ and define, for $1 \le k \le n$,

$$T(X, k + 1) := \coprod_{t \in T(\mathsf{ft}(X), k)} T(S_t(X), 1). \tag{4.3}$$

and, for $(t, x) \in T(X, k + 1)$, a homomorphism of B-frames $S^{k+1}_{(t,x)}$ as the composite below

$$\begin{array}{ccc} \mathbb{B}/X & \xrightarrow{\quad S^{k+1}_{(t,x)} \quad} & \mathbb{B}/\mathsf{ft}^{k+1}(X) \\ & {\scriptstyle S^k_t/X} \searrow \qquad \nearrow {\scriptstyle S_x} & \\ & \mathbb{B}/S^k_t(X) & \end{array} \tag{4.4}$$

where S_x comes from the substitution structure and S^k_t from the inductive hypothesis. □

Remarks 4.70.

(1) For every B-frame $\mathbb{B}$ and $X \in B_n$, we have an isomorphism of strict categories $\mathbf{FGR}(\mathbb{B}/X) \cong \mathbf{FGR}(\mathbb{B})/(n, X)$ natural in $\mathbb{B}$ which maps (i, Y) to $(n+i, Y)$ and it is the identity on arrows. It follows that, when $\mathbb{B}$ is a B-system, we can choose the identity as the action on the term structure. Therefore this isomorphism of categories lifts to an isomorphism of categories with term structure $(\mathbf{FGR}(\mathbb{B}/X), T_{\mathbb{B}}) \cong (\mathbf{FGR}(\mathbb{B})/(n, X), T_{\mathbb{B}})$.

Once we establish an E-system structure on $\mathbf{FGR}(\mathbb{B})$, we will see that this isomorphism is in fact an isomorphism of E-systems.

(2) Let $\mathbb{A}$ and $\mathbb{B}$ be B-frames with substitution structure and $H\colon \mathbb{A} \to \mathbb{B}$ be a homomorphism of B-frames. If H preserves the substitution structure, then for every $X \in B_{n+k}$ and $t \in T(X, k)$ the square

$$\begin{array}{ccc} \mathbb{A}/X & \xrightarrow{H/X} & \mathbb{B}/H(X) \\ {\scriptstyle S^k_t}\downarrow & & \downarrow{\scriptstyle S^k_{\tilde{H}(t)}} \\ \mathbb{A}/\mathsf{ft}^k(X) & \xrightarrow{H/\mathsf{ft}^k(X)} & \mathbb{B}/\mathsf{ft}^k H(X) \end{array} \tag{4.5}$$

commutes in **Bfr**, where $\tilde{H}(t) := (\tilde{H}(t_1), \ldots, \tilde{H}(t_k))$. Indeed, by definition of S^k_t in (4.4), the square (4.5) factors vertically into k squares of the form in Definition 4.10.1, each of which commutes if H preserves the substitution structure.

(3) Let $\mathbb{A}$ and $\mathbb{B}$ be B-frames with substitution structure and $H\colon \mathbb{A} \to \mathbb{B}$ be a homomorphism of B-frames. Suppose that $\mathbb{A}$ and $\mathbb{B}$ have weakening structure and define, for every $X \in B_{n+k}$, the homomorphism of B-frames $W^k_X\colon \mathbb{B}/\mathsf{ft}^k(X) \to \mathbb{B}/X$ as the composite

$$W^k_X\colon\ \mathbb{B}/\mathsf{ft}^k(X) \xrightarrow{W_{\mathsf{ft}^{k-1}(X)}} \cdots \xrightarrow{W_{\mathsf{ft}(X)}} \mathbb{B}/\mathsf{ft}(X) \xrightarrow{W_X} \mathbb{B}/X \tag{4.6}$$

which we take to be $\mathrm{id}_{\mathbb{B}}$ if $n = k = 0$.

If H preserves weakening structure, then for every $X \in B_{n+k}$ the square

$$\begin{array}{ccc} \mathbb{A}/X & \xrightarrow{H/X} & \mathbb{B}/H(X) \\ {\scriptstyle W^k_X}\uparrow & & \uparrow{\scriptstyle W^k_{H(X)}} \\ \mathbb{A}/\mathsf{ft}^k(X) & \xrightarrow{H/\mathsf{ft}^k(X)} & \mathbb{B}/\mathsf{ft}^k H(X) \end{array} \tag{4.7}$$

commutes in **Bfr**. Indeed, by definition of W^k_X in (4.6) the square (4.7) factors vertically into k squares of the form in Definition 4.10.2, each of which commutes if H preserves the weakening structure.

Problem 4.71. To lift the functor $\mathbf{FGR}\colon \mathbf{Bfr} \to \mathbf{Cat}$ to a functor $\mathbf{T}\colon \mathbf{SubBfr} \to \mathbf{TCat}$ from the category **SubBfr** of B-frames with substitution structure and homomorphisms of B-frames that preserve the substitution structure, to the category of strict categories with term structure.

Construction 4.72 (for Problem 4.71)**.** Let $\mathbb{A}$ and $\mathbb{B}$ be B-frames with substitution structure. For every homomorphism of B-frames $H\colon \mathbb{A} \to \mathbb{B}$, the functor $\mathbf{FGR}(H)\colon \mathcal{F}_{\mathbb{A}} \to \mathcal{F}_{\mathbb{B}}$, maps an object (n, X) to $(n, H(X))$ and an arrow (X, k) to $(H(X), k)$. Since $H(*) = *$, the functor $\mathbf{FGR}(H)$ strictly preserves the (unique) terminal object.

To make $\mathbf{T}(H) := \mathbf{FGR}(H)$ into a functor with term structure note that, for every $t = (t_1, \ldots, t_k) \in T(X, k)$ and $1 \leq j \leq k$, the function $\tilde{H}$ restricts as follows

$$\begin{array}{ccc} T(\mathsf{ft}^{k-j} S_{t_{j-1}} \cdots S_{t_1}(X), 1) & \hookrightarrow & \tilde{A}_{n-k+1} \\ \big\downarrow{\scriptstyle \tilde{H}} & & \big\downarrow{\scriptstyle \tilde{H}} \\ T(\mathsf{ft}^{k-j} S_{\tilde{H}(t_{j-1})} \cdots S_{\tilde{H}(t_1)} H(Y), 1) & \hookrightarrow & \tilde{B}_{n-k+1} \end{array} \tag{4.8}$$

since H commutes with the functions ft and preserves the substitution structure in the sense of Definition 4.10. It follows that

$$\mathbf{T}(H)(t) := (\tilde{H}(t_1), \ldots, \tilde{H}(t_k)) \in T(\tilde{H}(X), k). \tag{4.9}$$

This makes $\mathbf{T}(H)\colon \mathcal{F}/(n, X) \to \mathcal{F}/(n-k, \mathsf{ft}^k(X))$ into a functor with term structure.

The action of H on the sets $T(X, k)$ is clearly functorial in H. □

Remarks 4.73.

(1) The functor $\mathbf{T}\colon \mathbf{SubBfr} \to \mathbf{TCat}$ from Construction 4.72 is faithful, since the functors $\mathbf{R}\colon \mathbf{Bfr} \to \mathbf{RtTr}$, $\mathbf{G}\colon \mathbf{RtTr} \to \mathbf{Grph}$ and $\mathbf{F}\colon \mathbf{Grph} \to \mathbf{Cat}$ are faithful and the sets $T(X, 1)$ for $X \in B_n$ form a partition of $\tilde{B}_n$.

(2) For every B-frame with substitution structure $\mathbb{B}$, it follows by Propositions 2.8 and 2.14 that the underlying category of $\mathbf{T}(B)$ is stratified by the functor that maps $(X, k)\colon (n+k, X) \to (n, \mathsf{ft}^k(X))$ to $n + k \geq n$.

Problem 4.74. For every B-system $\mathbb{B}$, to construct a pre-E-system structure on the category with term structure $\mathbf{T}(\mathbb{B})$ from Construction 4.69.

Construction 4.75 (for Problem 4.74)**.** Construction 4.69 provides a homomorphism of B-frames $S^k_t\colon \mathbb{B}/X \to \mathbb{B}/\mathsf{ft}^k(X)$ for every $X \in B_{n+k}$ and $t \in T(X, k)$. The homomorphism S^k_t preserves the substitution structure since it factors, as in Remark 4.70.2, into k B-homomorphisms of the form S_{x_j}, where $x_j \in \tilde{B}_{n+k-j}$ for $j < k$. Construction 4.72 and Remark 4.70.1 then yield a functor with term structure

$$(\mathbf{T}(\mathbb{B})/(n+k, X), T_{\mathbb{B}}) \xrightarrow{S_t := \mathbf{T}(S^k_t)} (\mathbf{T}(\mathbb{B})/(n, \mathsf{ft}^k(X)), T_{\mathbb{B}}) \tag{4.10}$$

as required.

To construct the pre-weakening structure, consider the homomorphism of B-frames $W^k_X\colon \mathbb{B}/\mathsf{ft}^k(X) \to \mathbb{B}/X$ defined in Remark 4.70.3. Since $\mathbb{B}$ is a B-system, each factor of W^k_X in (4.6) is a homomorphism of B-systems and so is W^k_X. Construction 4.72 and Remark 4.70.1 provide a functor with term structure

$$\mathbf{T}(\mathbb{B})/(n, \mathsf{ft}^k(X)) \xrightarrow{W_{(X,k)} := \mathbf{T}(W^k_X)} \mathbf{T}(\mathbb{B})/(n, X). \tag{4.11}$$

It remains to construct the pre-projection structure. In fact, we will prove a little bit more. We construct by induction on $n \in \mathbb{N}$, for every $X \in B_n$ and $k \leq n$, an element

$\mathsf{idtm}_{(X,k)} \in T(W_{(X,k)}(X,k)) = T(W_X^k(X),k)$ with the property that the triangle of B-homomorphisms

$$\begin{array}{ccc} \mathbb{B}/X & \xrightarrow{\quad\mathrm{id}\quad} & \mathbb{B}/X \\ {\scriptstyle W_X^k/X}\searrow & & \nearrow{\scriptstyle S^k_{\mathsf{idtm}_{(X,k)}}} \\ & \mathbb{B}/W_X^k(X) & \end{array}$$

commutes. This additional condition is needed in the inductive construction. For every n and $X \in B_n$, let

$$\mathsf{idtm}_{(X,0)} := * \in T(W_{(X,0)}(X,0),0) = T(X,0).$$

For every n and $X \in B_{n+1}$, it is $\partial \circ \delta(X) = W_X(X) \in B_{n+2}$. Thus we can define

$$\mathsf{idtm}_{(X,1)} := \delta(X) \in T(W_{(X,1)}(X,1)) = T(W_X(X),1) \tag{4.12}$$

and $S^1_{\mathsf{idtm}_{(X,1)}} \circ W_X/X = \mathrm{id}_{\mathbb{B}/X}$ by condition 5 in Definition 4.11.

Suppose now that we have defined, for every $m \le n$, $Y \in B_m$ and $i \le m$, an element $\mathsf{idtm}_{(Y,i)} \in T(W_{(Y,i)}(Y,i))$ such that $S^i_{\mathsf{idtm}_{(Y,i)}} \circ W^i_Y/Y = \mathrm{id}_{\mathcal{T}/(m,Y)}$. Let $X \in B_{n+1}$. It follows from (4.3) that, for every $1 \le k \le n$

$$T(W_{(X,k+1)}(X,k+1)) = \coprod_{t \in T(W_X^{k+1}(\mathsf{ft}(X)),k)} T(S^k_t \circ W_X^{k+1}(X),1).$$

But $W_X^{k+1} = W_X \circ W^k_{\mathsf{ft}(X)}$, thus

$$\bar{t} := \tilde{W}_X(\mathsf{idtm}_{(\mathsf{ft}(X),k)}) \in T(W_X^{k+1}(\mathsf{ft}(X)),k)$$

and

$$\begin{aligned} S^k_{\bar{t}} \circ W_X^{k+1}/\mathsf{ft}(X) &= S^k_{\bar{t}} \circ W_X/W^k_{\mathsf{ft}(X)}(\mathsf{ft}(X)) \circ W^k_{\mathsf{ft}(X)}/\mathsf{ft}(X) \\ &= W_X \circ S^k_{\mathsf{idtm}_{(\mathsf{ft}(X),k)}} \circ W^k_{\mathsf{ft}(X)}/\mathsf{ft}(X) \\ &= W_X \end{aligned} \tag{4.13}$$

by Remark 4.70.2 and the fact that W_X preserves the substitution structure, and assumption (4.75). In particular, $T(S_{\bar{t}} \circ W_X^{k+1}(X),1) = T(W_X(X),1)$ and we can define

$$\mathsf{idtm}_{(X,k+1)} := (\bar{t}, \delta(X)). \tag{4.14}$$

It remains to check that $S^{k+1}_{\mathsf{idtm}_{(X,k+1)}} \circ W_X^{k+1}/X = \mathrm{id}_{\mathbb{B}/X}$. This is indeed the case by (4.4), (4.13) and condition 5 in Definition 4.11:

$$\begin{aligned} S^{k+1}_{\mathsf{idtm}_{(X,k+1)}} \circ W_X^{k+1}/X &= S_{\delta(X)} \circ \left(S^k_{\bar{t}} \circ W_X^{k+1}/\mathsf{ft}(X)\right)/X \\ &= S_{\delta(X)} \circ W_X/X \\ &= \mathrm{id}_{\mathbb{B}/X}. \end{aligned}$$

This completes the construction of the pre-E-system structure. □

Lemma 4.76. *Let $\mathbb{A}$ and $\mathbb{B}$ be B-systems and $H: \mathbb{A} \to \mathbb{B}$ a homomorphism of B-frames that preserves the substitution structure.*

(1) *The functor with term structure $\mathbf{T}(H): \mathcal{F}_{\mathbb{A}} \to \mathcal{F}_{\mathbb{B}}$ is a pre-substitution homomorphism.*
(2) *H preserves the weakening structure if and only if $\mathbf{T}(H)$ is a pre-weakening homomorphism.*

(3) *H preserves the structure of generic elements if and only if $\mathbf{T}(H)$ is a pre-projection homomorphism.*

Proof.

(1) By definition of the pre-substitution structure in Construction 4.75 and Remark 4.70.1, $\mathbf{T}(H)$ is a pre-substitution homomorphism if every image under $\mathbf{T}$ of any square in $\mathbf{Bfr}$ of the form (4.5) commutes. By Remark 4.70.2, such squares commute since H preserves the substitution structure.
(2) By definition of the pre-weakening structure (4.6) and Remark 4.70.1, $\mathbf{T}(H)$ is a pre-weakening homomorphism if and only if the image under $\mathbf{T}\colon \mathbf{Bsys} \to \mathbf{TCat}$ of any square in $\mathbf{Bfr}$ of the form (4.7) commutes. By Remark 4.70.3, such squares commute if H preserves the weakening structure. The converse holds since $\mathbf{T}$ is faithful by Remark 4.73.1.
(3) By (4.8) and (4.9), $\mathbf{T}(H)$ acts componentwise as $\tilde{H}$ on a term $t \in T(X,t)$. It follows that $\mathbf{T}(H)$ preserves the terms $\mathsf{idtm}_{(X,k+1)} = (\delta(\mathsf{ft}^k(X)), \dots, \delta(X))$ for $X \in B_n, k < n$ if and only if H preserves generic elements. □

Lemma 4.77. *For every B-system $\mathbb{B}$, the pre-E-system constructed in 4.75 is a stratified E-system.*

Proof. First, we need to verify conditions 1–4 in Definition 4.51, as condition 5 holds by construction.

1. It follows from Lemma 4.76 and (4.10) since S_t^k, as defined in (4.4), is a homomorphism of B-systems when $\mathbb{B}$ is a B-system.
2. As above, it follows by Lemma 4.76 and (4.11) since W_X^k, as defined in (4.6), is a homomorphism of B-systems.
3. The case $X \in B_n, * \in T(X,0)$ holds trivially. The case $X \in B_{n+1}, x \in T(X,1)$ follows from condition 3 in Definition 4.11 and functoriality of $\mathbf{T}$. The case $X \in B_{n+k+1}, (t,x) \in T(X, k+1)$, where $t \in T(\mathsf{ft}(X),k)$ and $x \in T(S_t^k(X),1)$, holds by induction and functoriality of $\mathbf{T}$ as
$$\begin{aligned} S^k_{(t,x)} \circ W_X^{k+1} &= S_x \circ S_t^k/X \circ W_X \circ W^k_{\mathsf{ft}(X)} \\ &= S_x \circ W_{S_t(X)} \circ S_t^k \circ W^k_{\mathsf{ft}(X)} \\ &= \mathrm{id}_{\mathbb{B}/\mathsf{ft}^{k+1}(X)} \end{aligned}$$
by (4.4) and (4.6), the fact that S_t is a pre-E-homomorphism, and Definition 4.11.2 and the inductive hypothesis.
4. As above, the case $k = 0$ holds trivially and the case $k = 1$ holds by condition 4 in Definition 4.11. For $X \in B_{n+1},\ k \leq n$ and $(t,x) \in T(X, k+1)$,
$$\begin{aligned} \tilde{S}^{k+1}_{(t,x)}(\mathsf{idtm}_{(X,k+1)}) &= \tilde{S}_x \circ \tilde{S}_t^k(\tilde{W}_X(\mathsf{idtm}_{(\mathsf{ft}(X),k)}), \delta(X)) \\ &= \tilde{S}_x(\tilde{W}_{S_t^k(X)} \circ \tilde{S}_t^k(\mathsf{idtm}_{(\mathsf{ft}(X),k)}), \delta(S_t^k(X))) \\ &= (\tilde{S}_x \circ \tilde{W}_{S_t^k(X)}(t), \tilde{S}_x(\delta(S_t^k(X)))) \\ &= (t,x) \end{aligned}$$
by (4.4) and (4.14), the fact that S_t is a pre-E-homomorphism, the inductive hypothesis, and conditions 3 and 4 in Definition 4.11.

Finally, the underlying category $\mathcal{F} = \mathbf{FGR}(\mathbb{B})$ is stratified by Remark 4.73.2. By definition, weakening and substitution functors preserve the $\mathbb{N}$-component of objects and arrows. It follows that $\mathbf{T}(\mathbb{B})$ is a stratified E-system. □

Lemma 4.78.

(1) *The functor* $\mathbf{T}\colon \mathbf{SubBfr} \to \mathbf{TCat}$ *described in Construction 4.72 lifts to a functor* $\mathbf{B2E}\colon \mathbf{Bsys} \to \mathbf{Esys_s}$.
(2) *The functor* $\mathbf{B2E}$ *is full and faithful.*

Proof.

1. By Lemma 4.77, it is enough to show that, for every homomorphism of B-systems $H\colon \mathbb{A} \to \mathbb{B}$, the functor with term structure $\mathbf{T}(H)\colon \mathbf{T}(\mathbb{A}) \to \mathbf{T}(\mathbb{B})$ is a stratified homomorphism of E-systems. By Lemma 4.76, $\mathbf{T}(H)$ is a homomorphism of E-systems. It is stratified since it preserves the $\mathbb{N}$-component of objects and arrows by definition.
2. The functor $\mathbf{B2E}$ is faithful by Remark 4.73.1. Let then $K\colon \mathbf{B2E}(\mathbb{A}) \to \mathbf{B2E}(\mathbb{B})$ be a stratified homomorphism of E-systems. Since K is stratified, the function on objects $K\colon \coprod_m A_m \to \coprod_n B_n$ is the identity on indices and gives rise to a family of functions $H\colon \prod_n (A_n \to B_n)$ such that, for every object (n, X) and arrow $(X, k)\colon (n+k, X) \to (n, \mathsf{ft}^k(X))$
$$K(n, X) = (n, H_n(X)) \qquad \text{and} \qquad K(X, k) = (H_{n+k}(X), k). \tag{4.15}$$
We shall show that H is a morphism of B-system such that $\mathbf{B2E}(H) = K$.
The functions H_n commute with the father functions ft since, for every $X \in A_{n+1}$, the arrow $K(X, 1)\colon (n+1, H_{n+1}(X)) \to (n, H_n(\mathsf{ft}(X)))$ in $\mathcal{F}_{\mathbb{B}}$ is necessarily of the form $(n+1, Y) \to (n, \mathsf{ft}(Y))$.
The family of sets $T(X, 1)$ indexed on $X \in A_{n+1}$ forms a partition of $\tilde{A}_{n+1}$. Therefore the functions $K_X\colon T(X, 1) \to T(H_{n+1}(X), 1)$ glue together to form a function $\tilde{H}_{n+1}\colon \tilde{A}_{n+1} \to \tilde{B}_{n+1}$ such that
$$\tilde{H}_{n+1}(x) = K_{\partial(x)}(x). \tag{4.16}$$
It follows that $\partial \circ \tilde{H} = H \circ \partial$ since $\tilde{H}_{n+1}(x) \in T(H_{n+1}(\partial(x)), 1)$. Therefore H is a homomorphism of B-frames from $\mathbb{A}$ to $\mathbb{B}$.
Let $x \in \tilde{B}_{n+1}$. Since K is a substitution homomorphism, for every $Y \in B_{n+k+1}$ such that $\mathsf{ft}^k(Y) = \partial(x)$, it is
$$\begin{aligned}(n+k, \tilde{H}_{n+k} \circ S_x(Y)) &= K \circ S_x(n+k+1, Y)\\ &= S_{K_{\partial(x)}(x)} \circ K(n+k+1, Y)\\ &= (n+k, S_{\tilde{H}_{n+1}(x)} \circ H_{n+k+1}(Y))\end{aligned}$$
and, for every $y \in \tilde{B}_{n+k+1}$ such that $\mathsf{ft}^k \circ \partial(y) = \partial(x)$, it is
$$\begin{aligned}\tilde{H}_{n+k} \circ \tilde{S}_x(y) &= K_{\partial \circ S_x(y)} \circ S_x(y)\\ &= S_{K_{\partial(x)}(x)} \circ K_{\partial(y)}(y)\\ &= \tilde{S}_{\tilde{H}_{n+1}(x)} \circ \tilde{H}_{n+k+1}(y).\end{aligned}$$
It follows that the homomorphism of B-frames H preserves the substitution structure.
We can thus apply Construction 4.72 to H and observe that $\mathbf{T}(H) = K$. Indeed $\mathbf{T}(H)$ and K have the same action on objects and arrows because of (4.15) and Construction 4.72.

To see that they also agree on the term structure, recall from Construction 4.69 that the term structure of an E-system of the form $\mathbf{B2E}(\mathbb{B})$ is given by lists of elements in the sets $\tilde{B}_n$, and then use (4.16) and (4.9). Therefore $\mathbf{B2E}(H) = \mathbf{T}(H) = K$ once we show that H is a homomorphism of B-systems.

It remains to verify that H also preserve the weakening structure and the structure of generic elements. Since K is a projection homomorphism, for every $X \in B_{n+1}$ it is

$$\mathbf{T}(H/X \circ W_X) = K/(n+1, X) \circ W_{(X,1)} = W_{K(X,1)} \circ K/(n, \mathsf{ft}(X)) = \mathbf{T}(W_{H_{n+1}(X)} \circ H/\mathsf{ft}(X)).$$

The first claim then follows from faithfulness of $\mathbf{B2E}$. Finally, H preserves generic elements

$$\tilde{H}_{n+2} \circ \delta(X) = K_{W_X(X)}(\mathsf{idtm}_{(X,1)}) = \mathsf{idtm}_{K(X,1)} = \delta \circ H_{n+1}(X)$$

since K is a projection homomorphism. □

4.3.2. *From stratified E-sytems to B-systems.* We have constructed a full and faithful functor $\mathbf{Bsys} \to \mathbf{Esys_s}$. Here we construct a functor in the opposite direction. We begin in Construction 4.82 with a functor $\mathbf{E2B}$ from stratified categories with term structures to B-frames. In Construction 4.84 we consider substitution, weakening and projection structures and prove in Lemma 4.85 that $\mathbf{E2B}$ maps homomorphisms into homomorphisms. This allows us to lift $\mathbf{E2B}$ to a functor $\mathbf{Esys_s} \to \mathbf{Bsys}$ in Construction 4.87.

Problem 4.79. Given a stratified category with term structure $(\mathcal{F}, T)$, to construct a B-frame $\mathbf{B2E}(\mathcal{F}, T)$.

Construction 4.80 (for Problem 4.79)**.** For every object X in $\mathcal{F}$, let $\overline{X}$ denote the unique indecomposable arrow with domain X given by Lemma 2.6. For every $n \in \mathbb{N}$, define sets

$$B(\mathcal{F}, T)_n := \{X \in \mathrm{Ob}(\mathcal{F}) \mid L(X) = n\} \tag{4.17}$$

$$\tilde{B}(\mathcal{F}, T)_{n+1} := \coprod_{X \in B(\mathcal{F},T)_{n+1}} T(\overline{X}) \tag{4.18}$$

and functions $\mathsf{ft}_n \colon B(\mathcal{F}, T)_{n+1} \to B(\mathcal{F}, T)_n$ and $\partial_n \colon \tilde{B}(\mathcal{F}, T)_{n+1} \to B(\mathcal{F}, T)_{n+1}$ by

$$\mathsf{ft}(X) := \mathrm{cod}(\overline{X}) \tag{4.19}$$

$$\partial(X, x) := X. \tag{4.20}$$

These definitions give rise to a B-frame $\mathbf{E2B}(\mathcal{F}, T)$. □

Problem 4.81. To construct a functor $\mathbf{E2B} \colon \mathbf{TCat_s} \to \mathbf{Bfr}$ from the category of stratified categories with term structure and stratified functors with term structure to the category of B-frames and homomorphisms.

Construction 4.82 (for Problem 4.81)**.** The action on objects is given by Construction 4.80. Let then $F \colon (\mathcal{F}, T) \to (\mathcal{F}', T')$ be a stratified functor with term structure. We need to construct a homomorphism of B-frames $\mathbf{E2B}(F) \colon \mathbf{E2B}(\mathcal{F}, T) \to \mathbf{E2B}(\mathcal{F}', T')$. Since F is stratified, it maps $B(\mathcal{F}, T)_n$ into $B(\mathcal{F}', T')_n$. For every $X \in B(\mathcal{F}, T)_{n+1}$, the functor F

maps the indecomposable arrow $\overline{X}$ to the indecomposable arrow $\overline{F(X)}$ by Lemma 2.10. It follows first that

$$\begin{aligned} F \circ \mathsf{ft}(X) &= F \circ \mathrm{cod}(\overline{X}) \\ &= \mathrm{cod}(\overline{F(X)}) \\ &= \mathsf{ft} \circ F(X), \end{aligned}$$

and secondly that we can define, for every $n \in \mathbb{N}$, a function $\tilde{F}\colon \tilde{B}(\mathcal{F}, T)_{n+1} \to \tilde{B}(\mathcal{F}', T')_{n+1}$ such that $\partial \circ \tilde{F}(X, t) = F \circ \partial(X, t)$ by

$$\tilde{F}(X, t) := (F(X), F(t)). \tag{4.21}$$

This defines a homomorphism of B-frames $\mathbf{E2B}(F) := (F, \tilde{F})$. □

Problem 4.83. Let $(\mathcal{F}, T)$ be a stratified category with term structure and consider the B-frame $\mathbf{E2B}(\mathcal{F}, T)$ from Construction 4.80

(1) From a stratified pre-substitution structure on $(\mathcal{F}, T)$, construct a substitution structure on $\mathbf{E2B}(\mathcal{F}, T)$.
(2) From a stratified pre-weakening structure on $(\mathcal{F}, T)$, construct a weakening structure on $\mathbf{E2B}(\mathcal{F}, T)$.
(3) From a pre-projection structure on $(\mathcal{F}, T)$, construct a structure of generic elements on $\mathbf{E2B}(\mathcal{F}, T)$.

Construction 4.84 (for Problem 4.83)**.**

1. For every $(X, t) \in \tilde{B}(\mathcal{F}, T)_{n+1}$, the functor with term structure $S_t\colon (\mathcal{F}, T)/X \to (\mathcal{F}, T)/\mathsf{ft}(X)$ is stratified. Construction 4.82 then yields a homomorphism of B-frames
$$\mathbf{E2B}(\mathcal{F}, T)/X \xrightarrow{S_{(X,t)} := \mathbf{E2B}(S_t)} \mathbf{E2B}(\mathcal{F}, T)/\mathsf{ft}(X). \tag{4.22}$$
2. For every $X \in B(\mathcal{F}, T)_n$, the functor with term structure $W_{\overline{X}}\colon (\mathcal{F}, T)/\mathsf{ft}(X) \to (\mathcal{F}, T)/X$ is stratified, where $\overline{X}$ denotes the unique indecomposable arrow with domain X. Construction 4.82 then yields a homomorphism of B-frames
$$\mathbf{E2B}(\mathcal{F}, T)/\mathsf{ft}(X) \xrightarrow{W_X := \mathbf{E2B}(W_{\overline{X}})} \mathbf{E2B}(\mathcal{F}, T)/X. \tag{4.23}$$
3. For every $X \in B(\mathcal{F}, T)_{n+1}$, we can define
$$\delta(X) := (W_X(X), \mathsf{idtm}_{\overline{X}}) \in \tilde{B}(\mathcal{F}, T)_{n+2} \tag{4.24}$$
since $\overline{W_X(X)} = W_{\overline{X}}(\overline{X})$. □

Lemma 4.85. *Let $F\colon (\mathcal{F}, T) \to (\mathcal{F}', T')$ be a stratified functor with term structure.*

(1) *If $(\mathcal{F}, T)$ and $(\mathcal{F}', T')$ have stratified pre-substitution structure and F is a pre-substitution homomorphism, then $\mathbf{E2B}(F)\colon \mathbf{E2B}(\mathcal{F}, T) \to \mathbf{E2B}(\mathcal{F}', T')$ preserves the substitution structure.*
(2) *If $(\mathcal{F}, T)$ and $(\mathcal{F}', T')$ have stratified pre-weakening structure and F is a pre-weakening homomorphism, then $\mathbf{E2B}(F)\colon \mathbf{E2B}(\mathcal{F}, T) \to \mathbf{E2B}(\mathcal{F}', T')$ preserves the weakening structure.*
(3) *If $(\mathcal{F}, T)$ and $(\mathcal{F}', T')$ have stratified pre-projection structure and F is a pre-projection homomorphism, then $\mathbf{E2B}(F)\colon \mathbf{E2B}(\mathcal{F}, T) \to \mathbf{E2B}(\mathcal{F}', T')$ preserves the structure of generic elements.*

Proof.

1. We need to show that, for every $(X,t) \in \tilde{B}(\mathcal{F},T)_{n+1}$, it is $\mathbf{E2B}(F)/\mathsf{ft}(X) \circ S_{(X,t)} = S_{(F(X),F(t))} \circ \mathbf{E2B}(F)/X$. This follows from (4.22), functoriality of **E2B** and $F/\mathsf{ft}(X) \circ S_t = S_{F(t)} \circ F/X$, which holds because F is a pre-substitution homomorphism.
2. We need to show that, for every $X \in B(\mathcal{F},T)_n$, it is $\mathbf{E2B}(F)/X \circ W_X = W_{F(X)} \circ \mathbf{E2B}(F)/\mathsf{ft}(X)$. This follows from (4.23), functoriality of **E2B** and $F/X \circ W_{\overline{X}} = W_{\overline{F(X)}} \circ F/\mathsf{ft}(X)$, which holds because F is a pre-substitution homomorphism and $\overline{F(X)} = F(\overline{X})$.
3. For every $X \in B(\mathcal{F},T)_{n+1}$, it is

$$\mathbf{E2B}(F) \circ \delta(X) = (F(W_X(X)), F(\mathsf{idtm}_{\overline{X}})) = (W_{F(X)}(F(X)), \mathsf{idtm}_{\overline{F(X)}}) = \delta \circ \mathbf{E2B}(F)(X)$$

   where the first and last equality hold by (4.21) and (4.24), and the middle one because F is a pre-projection homomorphism. □

Problem 4.86. To lift the functor $\mathbf{E2B}\colon \mathbf{TCat_s} \to \mathbf{Bfr}$ to a functor $\mathbf{E2B}\colon \mathbf{Esys_s} \to \mathbf{Bsys}$.

Construction 4.87 (for Problem 4.86)**.** Let $\mathbb{E}$ be a stratified E-system. Then $\mathbf{E2B}(\mathcal{F},T)$ can be given the structure of a pre-B-system $\mathbf{E2B}(\mathbb{E})$ by Construction 4.84. To show that $\mathbf{E2B}(\mathbb{E})$ is a B-system, we need to verify conditions 1–5 of Definition 4.11.

1,2. Since, for every $A \in \mathcal{F}/\Gamma$ and $t \in T(A)$, the morphisms W_A and S_t are stratified E-homomorphism, it follows by Lemma 4.85 that the homomorphisms of B-frames constructed in (4.22) and (4.23) are homomorphisms of B-systems.

3. For $(X,t) \in \tilde{B}(\mathbb{E})_{n+1}$, it is

$$S_{(X,t)} \circ W_X = \mathbf{E2B}(S_t \circ W_{\overline{X}}) = \mathrm{id}_{\mathbf{E2B}(\mathbb{E})/\mathsf{ft}(X)}$$

   by (4.22), (4.23), functoriality of **E2B** and 4.51.3.
4. For $(X,t) \in \tilde{B}(\mathbb{E})_{n+1}$, it is

$$S_{(X,t)} \circ \delta(X) = ((S_{(X,t)} \circ W_X)(X), S_t(\mathsf{idtm}_{\overline{X}})) = (X,t)$$

   by (4.22), (4.24), condition 3 just proved and 4.51.4.
5. For every $X \in B(\mathbb{E})_{n+1}$, it is

$$S_{\delta(X)} \circ W_X/X = \mathbf{E2B}(S_{\mathsf{idtm}_{\overline{X}}} \circ W_{\overline{X}}/X) = \mathrm{id}_{\mathbf{E2B}(\mathbb{E})/X}$$

   by (4.22–4.24), functoriality of **E2B** and 4.51.5.

Finally, for every stratified E-homomorphism $F\colon \mathbb{E} \to \mathbb{D}$, the homomorphism of B-frames $\mathbf{E2B}(F)\colon \mathbf{E2B}(\mathbb{E}) \to \mathbf{E2B}(\mathbb{D})$ is a homomorphism of B-systems by Lemma 4.85. □

4.3.3. *Equivalence of B-systems and stratified E-systems.* Here we show in Theorem 4.90 that the functors **B2E** from Lemma 4.78 and **E2B** from Construction 4.87 form an equivalence of categories. We do so by showing in Construction 4.89 that $\mathbf{E2B}\colon \mathbf{Esys_s} \to \mathbf{Bsys}$ is an essential section of the full and faithful functor **B2E**.

Problem 4.88. For every stratified E-system $\mathbb{E}$, to construct an isomorphism of stratified E-systems $\mathbf{B2E}(\mathbf{E2B}(\mathbb{E})) \cong \mathbb{E}$, natural in $\mathbb{E}$.

Construction 4.89 (for Problem 4.88)**.** In this construction we decorate the structures from $\mathbf{B2E}(\mathbf{E2B}(\mathbb{E}))$ with a hat, as in $\hat{\mathcal{F}}$. Since $\mathcal{F}$ is stratified, the function mapping $(n, X) \in \coprod_n B(\mathbb{E})_n$ to X extends to an isomorphism φ between the underlying strict category $\hat{\mathcal{F}}$ of $\mathbf{B2E}(\mathbf{E2B}(\mathbb{E}))$, constructed in 4.69, and $\mathcal{F}$. In particular, it maps an arrow (X, k) to the arrow $\overline{X}^k := \overline{\mathsf{ft}^{k-1}(X)} \circ \cdots \circ \overline{X} \colon X \to \mathsf{ft}^k(X)$ in $\mathcal{F}$ as in (2.1).

In order to lift φ to an isomorphism of categories with term structure, we need to show that $\hat{T}(X, k) \cong T(\overline{X}^k)$ for every $X \in B(\mathbb{E})_n$ and $k \leq n$, where $\hat{T}(X, k)$ is the set defined in Construction 4.69. For every $X \in B_{n+1}$, by (4.2) it is

$$\hat{T}(X, 1) = \partial^{-1}(X) = \left\{(Y, y) \in \tilde{B}(\mathbb{E})_{n+1} \mid Y = X,\, y \in T(\overline{Y})\right\} \cong T(\overline{X}).$$

Suppose that $\hat{T}(Y, j) \cong T(\overline{Y}^j)$ for every $m < n, Y \in B_m$ and $j \leq m$. It follows by (4.3) that

$$\hat{T}(X, k+1) = \coprod_{t \in T(\mathsf{ft}(X), k)} T(S_t(X), 1) \cong \coprod_{t \in T(\overline{\mathsf{ft}(X)}^k)} T(\overline{S_t(X)}) \cong T(\overline{X}^{k+1})$$

where the last bijection follows from Theorem 4.67 since $\overline{X}^{k+1} = \overline{\mathsf{ft}(X)}^k \circ \overline{X}$ and $\overline{S_t(X)} = S_t(\overline{X})$. In other words, elements of $\hat{T}(X, k)$ are lists of length k of pairs $(Y, y) \in \tilde{B}(\mathbb{E})_{n+j}$ for $j = 1, \ldots, k$, where $y \in T(\overline{Y})$, and the action on terms of φ first acts componentwise dropping the first component of each pair and then applies the bijection from Theorem 4.67.

Next, we show that this choice of isos is natural in $\mathbb{E}$. Given a stratified E-homomorphism $F \colon \mathbb{E} \to \mathbb{D}$, we need to show that $\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F)) = F \circ \varphi_{\mathbb{E}}$. The functor $\mathbf{B2E}(\mathbf{E2B}(F))$ maps an arrow (X, k) to $(F(X), k)$, thus

$$\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(X, k) = \overline{F(X)}^k = F(\overline{X}^k) = F \circ \varphi_{\mathbb{E}}(X, k).$$

since F preserves indecomposable arrows by Lemma 2.10. The functor with term structure $\mathbf{B2E}(\mathbf{E2B}(F))$ maps $(X, x) \in \hat{T}(X, 1)$ to $(F(X), F(x))$ by (4.8) and (4.21), thus

$$\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(X, x) = F(x) = F \circ \varphi_{\mathbb{E}}(X, x).$$

Suppose now that, for every $m \leq n$, $Y \in B(\mathbb{E})_m$, $i \leq m$ and $(Y, t) \in \hat{T}(Y, i)$, it is $\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(Y, t) = F \circ \varphi_{\mathbb{E}}(Y, t)$. Let $X \in B(\mathbb{E})_{n+1}$ and $(t, (X, x)) \in \hat{T}(X, k+1)$, then

$$\begin{aligned}
\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(t, (X, x)) &= (\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F))(t)).F(x) \\
&= (F \circ \varphi_{\mathbb{E}}(t)).F(x) \\
&= F(\varphi_{\mathbb{E}}(t).x) \\
&= F \circ \varphi_{\mathbb{E}}(t, (X, x))
\end{aligned}$$

by definition of φ, inductive hypothesis, Lemma 4.65, and definition of φ again. Therefore we conclude that, for every E-homomorphism $F \colon \mathbb{E} \to \mathbb{D}$,

$$\varphi_{\mathbb{D}} \circ \mathbf{B2E}(\mathbf{E2B}(F)) = F \circ \varphi_{\mathbb{E}}. \tag{4.25}$$

It remains to show that each component $\varphi_{\mathbb{E}}$ is an E-homomorphism.

To show that φ is a weakening homomorphism, note that for every $X \in B(\mathbb{E})_{n+k}$, it is $\hat{W}_{(X,k)} = \mathbf{B2E}(W_X^k)$ by (4.11) and

$$\begin{aligned} W_X^k &= W_{\mathsf{ft}^{k-1}(X)} \circ \cdots W_X \\ &= \mathbf{E2B}(W_{\overline{\mathsf{ft}^{k-1}(X)}} \circ \cdots \circ W_{\overline{X}}) \\ &= \mathbf{E2B}(W_{\varphi(X,k)}) \end{aligned}$$

by, in order, (4.6); (4.23) and functoriality of **E2B**; condition 4.31.1 in the case $k = 0$ and condition 4.31.2 for $k > 0$; and definition of φ. Moreover, $W_{\varphi(X,k)}$ is an E-homomorphism, thus φ is a weakening homomorphism by (4.25).

To show that φ is a substitution homomorphism we reason by induction. The case $X \in B(\mathbb{E})_n$ and $* \in \hat{T}(X,0)$ is trivial. For every $X \in B(\mathbb{E})_{n+1}$ and $(X,x) \in \hat{T}(X,1)$, it is $\hat{S}_{(X,x)} = \mathbf{B2E}(S_{(X,x)})$ by (4.10) and

$$S_{(X,x)} = \mathbf{E2B}(S_x) = \mathbf{E2B}(S_{\varphi(X,x)})$$

by (4.22) and definition of φ. Suppose now that, for every $m \leq n$, $Y \in B(\mathbb{E})_m$, $i \leq m$ and $t \in \hat{T}(Y,i)$, it is $S_t = \mathbf{E2B}(S_{\varphi(t)})$ as homomorphisms of B-systems. Then for every $X \in B(\mathbb{E})_{n+1}$, $k \leq n$ and $(t,(X,x)) \in \hat{T}(X,k+1)$, it is $\hat{S}_{(t,(X,x))} = \mathbf{B2E}(S_{(t,(X,x))})$ by (4.10) and

$$\begin{aligned} S_{(t,(X,x))} &= S_{(X,x)} \circ S_t/X \\ &= \mathbf{E2B}(S_x \circ S_{\varphi(t)}/X) \\ &= \mathbf{E2B}(S_{\varphi(t).x}) \\ &= \mathbf{E2B}(S_{\varphi(t,(X,x))}) \end{aligned}$$

by, in order, (4.4); inductive hypothesis, (4.22) and functoriality of **E2B**; Theorem 4.62; and definition of φ. Therefore $S_t = \mathbf{B2E}(\mathbf{E2B}(S_{\varphi(t)}))$ for every $X \in B(\mathbb{E})_{n+k}$ and $t \in \hat{T}(X,k))$. We conclude that φ is a substitution homomorphism by naturality (4.25).

To show that φ is a projection homomorphism we reason by induction. The case $(X,0)$ for $X \in B(\mathbb{E})_n$ is again trivial. Let $X \in B(\mathbb{E})_{n+1}$, then

$$\hat{\mathsf{idtm}}_{(X,1)} = \delta(X) = (W_X(X), \mathsf{idtm}_{\overline{X}})$$

by (4.12) and (4.24). Therefore $\varphi(\hat{\mathsf{idtm}}_{(X,1)}) = \mathsf{idtm}_{\varphi(X,1)}$ by definition of φ. Suppose that, for every $m \leq n$, $Y \in B(\mathbb{E})_m$, $i \leq m$, it is $\varphi(\hat{\mathsf{idtm}}_{(Y,i)}) = \mathsf{idtm}_{\varphi(Y,i)}$. Let $X \in B(\mathbb{E})_{n+1}$ and $k \leq n$. Then

$$\begin{aligned} \varphi(\hat{\mathsf{idtm}}_{(X,k+1)}) &= \varphi(W_{(X,1)}(\hat{\mathsf{idtm}}_{(\mathsf{ft}(X),k)}), \delta(X)) \\ &= \Big(W_{\overline{X}}(\varphi(\hat{\mathsf{idtm}}_{(\mathsf{ft}(X),k)}))\Big).\mathsf{idtm}_{\overline{X}} \\ &= \big(W_{\overline{X}}(\mathsf{idtm}_{\varphi(\mathsf{ft}(X),k))}\big).\mathsf{idtm}_{\overline{X}} \\ &= \mathsf{idtm}_{\overline{\mathsf{ft}(X)}^k \circ \overline{X}} \\ &= \mathsf{idtm}_{\varphi(X,k+1)} \end{aligned}$$

by (4.14), definition of φ and the fact that φ is a weakening homomorphism, the inductive hypothesis, Lemma 5.19, and definition of φ again. Therefore $\varphi(\hat{\mathsf{idtm}}_{(X,k)}) = \mathsf{idtm}_{\varphi(X,k)}$ for every $X \in B(\mathbb{E})_{n+k}$. This concludes the proof that φ is an E-homomorphism. □

Finally we reach the main result of this section.

Theorem 4.90. *The functors* $\mathbf{B2E}\colon \mathbf{Bsys} \to \mathbf{Esys_s}$ *from Lemma 4.78 and* $\mathbf{E2B}\colon \mathbf{Esys_s} \to \mathbf{Bsys}$ *from Construction 4.87 form an equivalence of categories.*

Proof. As the functor **B2E** is fully faithful by Lemma 4.78.2, it is enough to show that **E2B** is an essential section of **B2E**. This holds by Construction 4.89. □

5. Equivalence between B- and C-systems

In this section, we construct an equivalence between B-systems and C-systems, in several steps. We first construct an adjunction between the categories of CE-systems and of E-systems. To this end, we construct, in Section 5.1, a functor from CE-systems to E-systems, and, in Section 5.2, a functor in the other direction, from E-systems to CE-systems. In Section 5.3 we show that these functors form an adjunction that restricts to an equivalence when considering *rooted* CE-systems. Finally, in Section 5.4, we give our equivalence between B-systems and C-systems, obtained by restricting the aforementioned equivalence to *stratified* rooted CE-systems and E-systems, respectively.

5.1. From CE-sytems to E-systems.

Definition 5.1. Let $\mathbb{A}$ be a CE-system. For any $\Gamma \in \mathcal{C}$, we define the **slice CE-system** $\mathbb{A}/\Gamma$ as follows. Let $\mathcal{C}_{\mathbb{A}}(\Gamma)$ be the strict category with the same objects as $\mathcal{F}_{\mathbb{A}}/\Gamma$ and with all arrows from $I(A)$ to $I(B)$ in $\mathcal{C}_{\mathbb{A}}/\Gamma$ as arrows from A to B. The functor $I/\Gamma\colon \mathcal{F}_{\mathbb{A}}/\Gamma \to \mathcal{C}_{\mathbb{A}}/\Gamma$ factors as an identity-on-objects I_Γ followed by a full and faithful one as shown in the diagram below.

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{A}}/\Gamma & \xrightarrow{\quad I/\Gamma \quad} & \mathcal{C}_{\mathbb{A}}/\Gamma \\ & {\scriptstyle I_\Gamma}\searrow \quad \nearrow & \\ & \mathcal{C}_{\mathbb{A}}(\Gamma) & \end{array}$$

We take I_Γ to be the underlying functor of $\mathbb{A}/\Gamma$. The choice of pullback squares is induced by $\mathbb{A}$.

We shall omit the subscript $\mathbb{A}$ from $\mathcal{C}_{\mathbb{A}}(\Gamma)$ whenever the CE-system is clear from context.

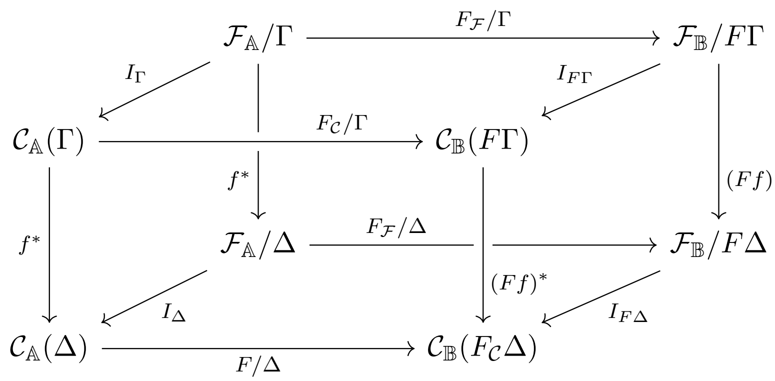
Remarks 5.2. Let $\mathbb{A}$ be a CE-system.

(1) For every object Γ, the identity id_Γ is terminal in $\mathcal{C}_{\mathbb{A}}(\Gamma)$. It follows that any slice CE-system is rooted.
(2) For every $f\colon \Delta \to \Gamma$ in $\mathcal{C}$, the functor $f^*\colon \mathcal{F}/\Gamma \to \mathcal{F}/\Delta$ lifts to a functor $f^*\colon \mathcal{C}(\Gamma) \to \mathcal{C}(\Delta)$ making the square below commute.

$$\begin{array}{ccc} \mathcal{F}/\Gamma & \xrightarrow{f^*} & \mathcal{F}/\Delta \\ {\scriptstyle I_\Gamma}\downarrow & & \downarrow{\scriptstyle I_\Delta} \\ \mathcal{C}(\Gamma) & \xrightarrow{f^*} & \mathcal{C}(\Delta) \end{array} \tag{5.1}$$

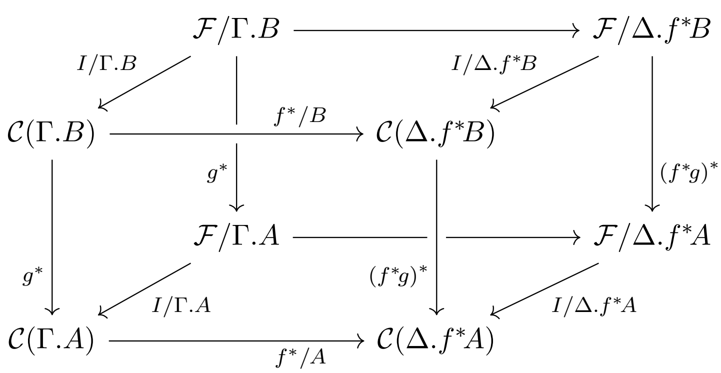
(3) For every $f\colon \Delta \to \Gamma$ the commutative square in (5.1) lifts to a CE-homomorphism $f^*\colon \mathbb{A}/\Gamma \to \mathbb{A}/\Delta$.

Lemma 5.3. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a CE-homomorphism. Then for every $f\colon \Delta \to \Gamma$ in $\mathcal{C}_{\mathbb{A}}$ the diagram below commutes.*



Proof. Commutativity of the back face follows from the fact that $(Ff)^*(FA) = F(f^*A)$ for every $A \in \mathcal{F}_{\mathbb{A}}/\Gamma$, commutativity of the front face follows from the universal property of pullbacks, and commutativity of the other faces is immediate. □

Lemma 5.4. *Let $\mathbb{A}$ be a CE-system. For every $f\colon \Delta \to \Gamma$ in $\mathcal{C}$ and every $g\colon A \to B$ in $\mathcal{C}(\Gamma)$ the diagram below commutes.*



Proof. This is Lemma 5.3 applied to f^* seen as a homomorphism of CE-systems thanks to Remark 5.2.3. □

Problem 5.5. To construct a functor $\mathbf{CE2E}\colon \mathbf{CEsys} \to \mathbf{Esys}$.

Construction 5.6 (for Problem 5.5)**.** Let $\mathbb{A}$ be a CE-system with underlying functor $I\colon \mathcal{F} \to \mathcal{C}$. The underlying category of the E-system $\mathbf{CE2E}(\mathbb{A})$ is $\mathcal{F}$. The chosen terminal object is the one in $\mathbb{A}$. To equip $\mathcal{F}$ with a term structure we define, for every $A \in \mathcal{F}/\Gamma$, the set

$$T(A) := \{x\colon \Gamma \to \Gamma.A \mid I(A) \circ x = \mathrm{id}_\Gamma\}. \tag{5.2}$$

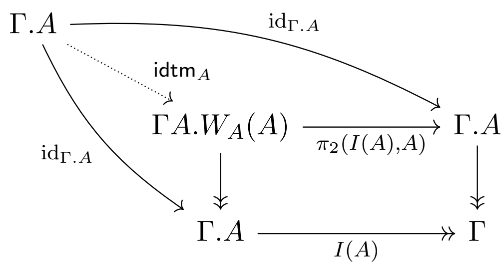
We define for any $A \in \mathcal{F}/\Gamma$, the functor

$$W_A := A^*\colon \mathcal{F}/\Gamma \to \mathcal{F}/\Gamma.A. \tag{5.3}$$

Likewise, we define for any $x \in T(A)$, the functor

$$S_x := x^*\colon \mathcal{F}/\Gamma.A \to \mathcal{F}/\Gamma. \tag{5.4}$$

These clearly extend to functors with term structure. We also define $\mathsf{idtm}_A : T(W_A(A))$ by the universal property of pullbacks as in the diagram below.



As an immediate consequence of Lemma 5.4, we get that each functor W_A and S_x is both a weakening functor and a substitution functor. It follows by the definitions that weakening and substitution preserve the terms idtm_A.

It remains to verify the remaining conditions of E-systems.

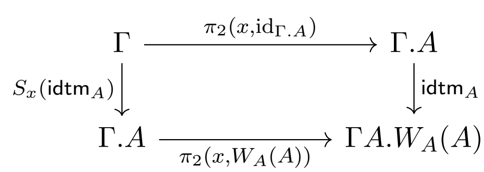
3. To show that substitution in weakened families is constant, note that

$$S_x \circ W_A = x^* \circ A^* = (A \circ x)^* = (\mathrm{id}_\Gamma)^* = \mathrm{id}_{\mathcal{C}_\mathcal{F}/\Gamma}.$$

5. The identity terms are neutral for pre-composition:

$$S_{\mathsf{idtm}_A} \circ W_A/A = S_{\mathsf{idtm}_A} \circ \pi_2(A,A)^* = (\pi_2(A,A) \circ \mathsf{idtm}_A)^* = (\mathrm{id}_{\Gamma.A})^* = \mathrm{id}_{\mathcal{C}_\mathcal{F}/\Gamma.A}.$$

4. The identity terms behave like identity functions: by the universal property, $S_x(\mathsf{idtm}_A)$ is the unique section of A such that the square



commutes. Thus, it suffices to show that this square also commutes with x in the place of $S_x(\mathsf{idtm}_A)$. Note that $\pi_2(x, \mathrm{id}_{\Gamma.A}) = x$. Since $\Gamma A.W_A(A)$ is itself a pullback, it suffices and it is straightforward to verify the equalities

$$W_A(A) \circ \pi_2(x, W_A(A)) \circ x = W_A(A) \circ \mathsf{idtm}_A \circ x$$
$$\pi_2(A,A) \circ \pi_2(x, W_A(A)) \circ x = \pi_2(A,A) \circ \mathsf{idtm}_A \circ x.$$

Let now $F : \mathbb{A} \to \mathbb{B}$ be a CE-system homomorphism. The underlying functor of $\mathbf{CE2E}(F)$ is $F_\mathcal{F} : \mathcal{F}_\mathbb{A} \to \mathcal{F}_\mathbb{B}$, which clearly preserves the choice of terminal objects, while the action on terms is given by $F_\mathcal{C}$. This functor with term structure is both a weakening and a substitution homomorphism because of Lemma 5.3. Note that commutativity of the front square in the diagram in Lemma 5.3 is needed for the equations on the action on terms. Finally, it is a projection homomorphism since it preserves identities. □

Remark 5.7. It follows immediately from the above construction that, for every CE-system $\mathbb{A}$, the E-system $\mathbf{CE2E}(\mathbb{A})$ has the property that $T(\mathrm{id}_\Gamma)$ is a singleton set for every $\Gamma \in \mathcal{F}$. As we shall see in Corollary 5.17, this is true for every E-system. In fact, it will follow from Theorem 5.34(1) that $\mathbf{CE2E}$ is essentially surjective on objects.

5.2. **From E-systems to CE-systems.** In this section we construct a functor from **Esys** to **CEsys**. We proceed in several steps: In Section 5.2.1 we define the strict category of internal morphisms of an E-system. There are two kinds of morphisms in this category: internal morphisms from A to B in context Γ, and for any internal morphism $f : A \to B$ in context Γ there are morphisms over f. There are also two kinds of composition, and in Section 5.2.2 we prove an interchange law for them. In Section 5.2.3 we complete the construction of the functor from **Esys** to **CEsys**.

5.2.1. *The strict category of internal morphisms of an E-sytem.* In this section we define for every E-system $\mathbb{E}$, and every context Γ in $\mathbb{E}$, a category $\mathcal{C}_{\mathbb{E}}(\Gamma)$. This goal is accomplished in Theorem 5.16. The empty context $[\,]$ of $\mathbb{E}$, i.e. a terminal object in $\mathcal{F}$, allows us to have a non-trivial category structure on the contexts of $\mathbb{E}$. In this case, the category structure is inherited from the category $\mathcal{C}_{\mathbb{E}} := \mathcal{C}_{\mathbb{E}}([\,])$.

Definition 5.8. For every $A, B \in \mathcal{F}/\Gamma$ we define the set

$$\mathrm{thom}(A, B) := T(\langle A\rangle B).$$

An element $f \in \mathrm{thom}(A, B)$ is called an **internal morphism in context** Γ. We sometimes write $f \in \mathrm{thom}(A, B)$ to indicate that f is an internal morphism over Γ, or we may draw a diagram of the form

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \quad \swarrow & \\ & \Gamma & \end{array}$$

or we may omit the arrows down to Γ and say instead that we have a diagram in context Γ. Note however that this is not (yet) a diagram in any category: the double-head arrows are arrows in $\mathcal{F}$, but the other ones are just elements in some $\mathrm{thom}(A, B)$.

Remark 5.9. Note that $\mathrm{thom}(\mathrm{id}_\Gamma, A) = T(A)$ for any $A \in \mathcal{F}/\Gamma$, because we have $W_{\mathrm{id}_{\Gamma.A}} = \mathrm{id}_{\mathcal{F}/\Gamma.A}$.

Note also that $\mathrm{thom}(A.P, B) = \mathrm{thom}(P, \langle A\rangle B)$ for any $P \in \mathcal{F}/\Gamma.A$ and $B \in \mathcal{F}/\Gamma$, because $W_{A.P} = W_P \circ W_A$. Once we have established a strict category of which the morphisms are given by $\mathrm{thom}(-,-)$, we therefore get that

$$A.(-) \dashv W_A.$$

The right adjoint to weakening by A, if it exists, will be the dependent product Π_A.

Definition 5.10. Let $A, B \in \mathcal{F}/\Gamma$. For any $f \in \mathrm{thom}(A, B)$ we define the **pre-composition** E-homomorphism

$$f^* := S_f \circ W_A/B : \mathbb{E}/\Gamma.B \to \mathbb{E}/\Gamma.A.$$

We shall denote the action of f^* on a family $Q \in \mathcal{F}/\Gamma.B$ as $Q \cdot f$. Similarly, for every $C \in \mathcal{F}/\Gamma$, we shall write $g \cdot f$ for the action of f^* on $g \in \mathrm{thom}(B, C) = T(W_B(C))$.

Lemma 5.11. *Let $F\colon \mathbb{E} \to \mathbb{D}$ be an E-homomorphism. Then for every $f \in \mathrm{thom}(A, B)$ in $\mathbb{E}$, the square of E-homomorphisms below commutes.*

$$\begin{array}{ccc} \mathbb{E}/\Gamma.B & \xrightarrow{F/\Gamma.B} & \mathbb{D}/F\Gamma.FB \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle (Ff)^*} \\ \mathbb{E}/\Gamma.A & \xrightarrow{F/\Gamma.A} & \mathbb{D}/F\Gamma.FA \end{array}$$

Proof. As F is both a weakening and a substitution homomorphism, it is

$$\begin{aligned}(F/\Gamma.A) \circ f^* &= (F/\Gamma.A) \circ S_f \circ (W_A/B) = S_{Ff} \circ (F/\Gamma.A.W_A(B)) \circ (W_A/B) \\ &= S_{Ff} \circ (W_{FA}/FB) \circ (F/\Gamma.B) = (Ff)^* \circ (F/\Gamma.B).\end{aligned}$$

□

Definition 5.12. Let $A, B \in \mathcal{F}/\Gamma$, $Q \in \mathcal{F}/\Gamma.A$ and $R \in \mathcal{F}/\Gamma.B$. For every $f \in \mathrm{thom}(A, B)$ we define

$$\mathrm{thom}_f(Q, R) := \mathrm{thom}(Q, R \cdot f).$$

Remarks 5.13.

(1) The terms $\mathrm{pr}_0^{A,P}$ and $\mathrm{pr}_1^{A,P}$ from Definition 4.64 are internal morphisms:

$$\mathrm{pr}_0^{A,P} \in \mathrm{thom}(A.P, A) \qquad \text{and} \qquad \mathrm{pr}_1^{A,P} \in \mathrm{thom}_{\mathrm{pr}_0^{A,P}}(\mathrm{id}_{A.P}, P).$$

(2) Note that for $g \in \mathrm{thom}(B, C)$, we have $g \cdot f \in T(S_f(W_A/B(W_B(C))))$, whereas we would like that $g \cdot f \in \mathrm{thom}(A, C)$. More generally, we can show that

$$S_f \circ (W_A/B) \circ W_B = W_A.$$

Since weakening is a weakening homomorphism, we have

$$S_f \circ (W_A/B) \circ W_B = S_f \circ W_{W_A(B)} \circ W_A.$$

By condition 3 in Definition 4.51 we get that

$$S_f \circ W_{W_A(B)} \circ W_A = W_A.$$

Remark 5.14. Note that condition 5 in Definition 4.51 asserts precisely that $(\mathsf{idtm}_A)^* = \mathrm{id}_{\mathcal{F}/\Gamma.A}$ for any $A \in \mathcal{F}/\Gamma$. In particular, it follows that $g \circ \mathsf{idtm}_A = g$ for any $g \in \mathrm{thom}(A, B)$

Lemma 5.15. *For any $f \in \mathrm{thom}(A, B)$ and $g \in \mathrm{thom}(B, C)$ we have $f^* \circ g^* = (g \cdot f)^*$.*

Proof.

$$\begin{aligned}f^* \circ g^* &= S_f \circ (W_A/B) \circ S_g \circ (W_B/C) \\ &= S_f \circ S_{W_A(g)} \circ (W_A/B.W_B(C)) \circ (W_B/C) \\ &= S_{S_f(W_A(g))} \circ (S_f/W_A(W_B(C))) \circ (W_A/B.W_B(C)) \circ W_B/C \\ &= S_{S_f(W_A(g))} \circ ((S_f \circ (W_A/B) \circ W_B)/C) \\ &= S_{S_f(W_A(g))} \circ ((S_f \circ W_{W_A(B)} \circ W_A)/C) \\ &= S_{S_f(W_A(g))} \circ W_A/C \\ &= (g \cdot f)^*.\end{aligned}$$

□

Theorem 5.16.

(1) *For every E-system $\mathbb{E}$ and every object Γ in its underlying strict category $\mathcal{F}$, objects of $\mathcal{F}/\Gamma$ and internal morphisms of $\mathbb{E}$ over Γ form a strict category $\mathcal{C}_{\mathbb{E}}(\Gamma)$.*

(2) *Every E-homomorphism $F \colon \mathbb{E} \to \mathbb{D}$ induces a functor $F_\Gamma \colon \mathcal{C}_{\mathbb{E}}(\Gamma) \to \mathcal{C}_{\mathbb{D}}(F(\Gamma))$ for every Γ in $\mathcal{F}_{\mathbb{E}}$.*

Proof.

1. For $A, B \in \mathcal{F}/\Gamma$, the set of arrows from A to B is $\mathrm{thom}(A, B)$. The fact that composition is associative is a direct corollary of Lemma 5.15. The axiom $(\mathsf{idtm}_A)^* = \mathrm{id}_{\Gamma.A}$ implies that the identity morphisms satisfy the right identity law. It remains to show that $f \cdot \mathsf{idtm}_B = f$, which is a simple calculation:

$$f \cdot \mathsf{idtm}_B = S_f \circ W_A(\mathsf{idtm}_B) = S_f(\mathsf{idtm}_{W_A B}) = f.$$

2. The action of F_Γ on arrows is given by the term structure of F. Functoriality of F_Γ follows from Lemma 5.11 and the fact that F is a projection homomorphism. $\square$

Now that we have a category structure, we can state and prove the following consequence of Theorem 4.67.

Corollary 5.17.

(1) *Let $A \in \mathcal{F}/\Gamma$ and $Q \in \mathcal{F}/\Gamma.B$, then for every $f \in \mathrm{thom}(A, B)$ there is a bijection*

$$\varphi \colon T(Q \cdot f) \xrightarrow{\sim} \left\{ h \in \mathrm{thom}(A, B.Q) \mid \mathrm{pr}_0^{B,Q} \cdot h = f \right\}.$$

given by $\varphi(t) = f.t$.

(2) *For every object Γ, $T(\mathrm{id}_\Gamma) = \{\mathsf{idtm}_{\mathrm{id}_\Gamma}\}$.*

Proof.

1. Theorem 4.67 yields the following bijection:

$$\begin{aligned} \mathrm{thom}(A, B.Q) &= T(\langle A\rangle(B.Q)) \\ &= T(\langle A\rangle B.\langle A\rangle Q) \\ &\cong \coprod_{f \in T(\langle A\rangle B)} T(\langle A\rangle Q[f]) \\ &= \coprod_{f \in \mathrm{thom}(A,B)} T(Q \cdot f). \end{aligned}$$

Also, we find $\mathrm{pr}_0^{\langle A\rangle B, \langle A\rangle Q}[h] = \langle A\rangle \mathrm{pr}_0^{B,Q}[h] = \mathrm{pr}_0^{B,Q} \cdot h$.

2. The above bijection becomes in this case

$$T(\mathrm{id}_{\Gamma.A}) \cong \{h \in \mathrm{thom}(A, A) \mid \mathrm{pr}_0^{A,\mathrm{id}_{\Gamma.A}} \cdot h = \mathsf{idtm}_A\} = \{\mathsf{idtm}_A\}$$

where the second equality follows from $\mathrm{pr}_0^{A,\mathrm{id}_{\Gamma.A}} = \mathsf{idtm}_A$. Since $\mathrm{id}_\Gamma = W_{\mathrm{id}_\Gamma}(\mathrm{id}_\Gamma)$, the only element in $T(\mathrm{id}_\Gamma)$ is $\mathsf{idtm}_{\mathrm{id}_\Gamma}$. $\square$

Theorem 5.18. *Let $A \in \mathcal{F}/\Gamma$ and $P \in \mathcal{F}/\Gamma.A$. Precomposition with $\mathrm{pr}_0^{A,P}$ is weakening by P, i.e.*

$$\mathbb{E}/\Gamma.A \xrightarrow{\left(\mathrm{pr}_0^{A,P}\right)^* = W_P} \mathbb{E}/\Gamma.A.P$$

Proof.

$$\begin{aligned} \left(\mathrm{pr}_0^{A,P}\right)^* &= S_{\mathrm{pr}_0^{A,P}} \circ W_{A.P}/A \\ &= S_{\langle P\rangle \mathsf{idtm}_A} \circ W_P/W_A(A) \circ W_A/A \\ &= W_P \circ S_{\mathsf{idtm}_A} \circ W_A/A \\ &= W_P \end{aligned}$$

$\square$

We conclude this section with a description of the projections and the pairing operation of an E-system in the image of the functor **CE2E** from Construction 5.6 in terms of the underlying CE-system structure.

Lemma 5.19. *Let $\mathbb{A}$ be a CE-system and consider the E-system $\mathbb{E} := \mathbf{CE2E}(\mathbb{A})$. For every object Γ, every $A \in \mathcal{F}/\Gamma$, $P \in \mathcal{F}/\Gamma.A$, it is*

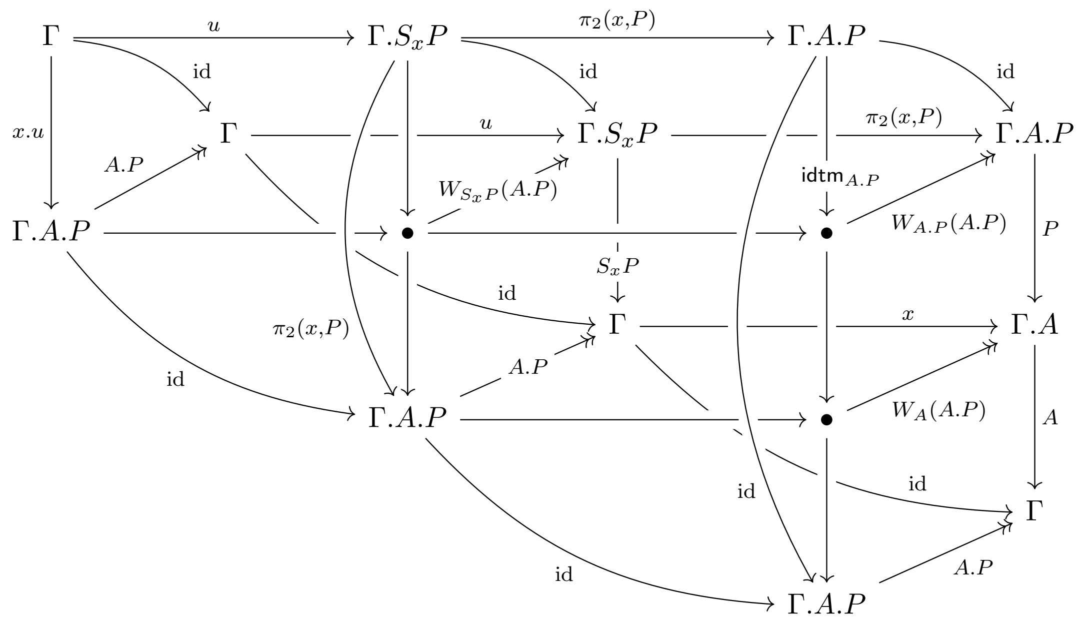
$$\mathrm{pr}_0^{A,P} = \langle \mathrm{id}_{\Gamma.A.P}, P\rangle \in \mathcal{C}_{\mathbb{A}}(\Gamma.A.P, \Gamma.A.P.\langle A.P\rangle A)$$

$$\mathrm{pr}_1^{A,P} = \langle \mathrm{id}_{\Gamma.A.P}, \mathrm{id}_{\Gamma.A.P}\rangle \in \mathcal{C}_{\mathbb{A}}(\Gamma.A.P, \Gamma.A.P.\langle P\rangle P)$$

and, for every $x \in T(A)$ and $u \in T(S_x P)$, it is

$$x.u = \pi_2\,(x, P) \circ u \in \mathcal{C}_{\mathbb{A}}(\Gamma, \Gamma.A.P).$$

Proof. The first two claims follow immediately from Definition 4.64 and the definitions in Construction 5.6. The third claim follows from commutativity of the front-left face in the diagram below.



This diagram commutes by definition, in the sense that every square not involving the top row is a chosen pullback in $\mathbb{A}$, and the remaining part commutes by definition of $\mathsf{idtm}_{A.P}$ and $x.u$ in Construction 5.6 and Definition 4.61, respectively. In this diagram we drop occurrences of the functor I and freely use notation from the E-system $\mathbf{CE2E}(\mathbb{A})$ to increase readability. □

5.2.2. *The interchange laws.* We are now in the position to define vertical and horizontal composition, and prove properties of them. In particular, we conclude the section showing in Theorem 5.28 that every pair $f \in \mathrm{thom}(A, B)$ and $F \in \mathrm{thom}_f(P, Q)$ induces a morphism, *i.e.* a commuting square, from $\mathrm{pr}_0^{A,P}$ to $\mathrm{pr}_0^{B,Q}$.

Definition 5.20. Let $f \in \mathrm{thom}(A, B)$ and $F \in \mathrm{thom}_f(P, Q)$. Then we define

$$f \ltimes F := (\langle P\rangle f).F \in \mathrm{thom}(A.P, B.Q).$$

This is well defined: we have $\langle P\rangle(Q \cdot f) = Q \cdot (\langle P\rangle f) = S_{\langle P\rangle f} \circ W_{A.P}/B(Q)$ since W_P is a substitution homomorphism, therefore $f \ltimes F \in T((\langle A.P\rangle B).W_{A.P}/B(Q)) = T(\langle A.P\rangle(B.Q))$ by Definition 4.61 and functoriality of $W_{A.P}$.

Whenever we say that we have a diagram of the form

$$\begin{array}{ccc} R & \xrightarrow{f_2} & S \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ P & \xrightarrow{f_1} & Q \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ A & \xrightarrow{f_0} & B \end{array}$$

we mean that we have $f_0 \in \mathrm{thom}(A, B)$, $f_1 \in \mathrm{thom}_{f_0}(P, Q)$ and $f_2 \in \mathrm{thom}_{f_0 \ltimes f_1}(R, S)$.

Lemma 5.21. *Let $H : \mathbb{E} \to \mathbb{D}$ be an E-homomorphism. For every $f \in \mathrm{thom}(A, B)$ and $F \in \mathrm{thom}_f(P, Q)$ it is*

$$H(f \ltimes F) = H(f) \ltimes H(F).$$

Proof. $H(f \ltimes F) = H(\langle Q\rangle f.F) = \langle HQ\rangle Hf.HF = H(f) \ltimes H(F)$. □

Lemma 5.22. *Vertical composition is associative.*

Proof. Consider the diagram

$$\begin{array}{ccc} R & \xrightarrow{f_2} & S \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ P & \xrightarrow{f_1} & Q \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ A & \xrightarrow{f_0} & B \end{array}$$

in context Γ. Because weakening distributes over term extension, and term extension is associative, we have

$$\begin{aligned} (f_0 \ltimes f_1) \ltimes f_2 &= \langle R\rangle(\langle P\rangle f_0.f_1).f_2 \\ &= (\langle R\rangle\langle P\rangle f_0.\langle R\rangle f_1).f_2 \\ &= \langle P.R\rangle f_0.(\langle R\rangle f_1.f_2) && \text{(By Corollary 4.63)} \\ &= f_0 \ltimes (f_1 \ltimes f_2). \end{aligned}$$

□

Definition 5.23. Let $f \in \mathrm{thom}(A, B)$ and $F \in \mathrm{thom}_f(P, Q)$. Then we define the E-homomorphism

$$F^\bullet := F^* \circ (f^*/Q) : \mathbb{E}/\Gamma.B.Q \to \mathbb{E}/\Gamma.A.P.$$

The infix notation of $F^\bullet$ is taken to be $- \bullet F$.

Lemma 5.24. *Let $f \in \mathrm{thom}(A, B)$ and $F \in \mathrm{thom}_f(P, Q)$. Then we have the equality*

$$F^\bullet = (f \ltimes F)^*.$$

Proof.

$$\begin{aligned} F^* \circ (f^*/Q) &= S_F \circ W_P \circ S_f/(W_A(Q)) \circ W_A/B.Q \\ &= S_F \circ S_{W_P(f)}/W_P(W_A(Q)) \circ W_P/W_A(B.Q) \circ W_A/B.Q \\ &= S_F \circ S_{W_P(f)}/W_P(W_A(Q)) \circ W_{A.P}/B.Q \\ &= S_{W_P(f).F} \circ W_{A.P}/B.Q && \text{(By Theorem 4.62)} \\ &= (f \ltimes F)^*. \end{aligned}$$

□

In the next theorem we prove the interchange law of horizontal and vertical composition. Its proof uses the following fact.

Lemma 5.25. *Let $f \in \mathrm{thom}(A, B)$ be an internal morphism in context Γ. Then one has*

$$f^* \circ W_B = W_A.$$

Proof. The proof is a simple calculation:

$$f^* \circ W_B = S_f \circ W_A/B \circ W_B = S_f \circ W_{W_A(B)} \circ W_A = W_A. \qquad \square$$

Theorem 5.26. *Consider the diagram*

$$\begin{array}{ccccc} P & \xrightarrow{F} & Q & \xrightarrow{G} & R \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

in context Γ. Then the equality

$$(g \ltimes G) \cdot (f \ltimes F) = (g \cdot f) \ltimes (G \bullet F)$$

of morphisms from $A.P$ to $C.R$ in context Γ holds.

Proof. By Lemma 5.24, we have

$$\begin{aligned} (g \ltimes G) \cdot (f \ltimes F) &= F^* \circ (f^*/Q)(((\langle Q\rangle g).G) && \text{(By Lemma 5.24)} \\ &= F^*(((\langle Q \cdot f\rangle g \cdot f).(f^*/Q(G))) \\ &= (F^*(\langle Q \cdot f\rangle g \cdot f)).(F^* \circ f^*/Q(G)) \\ &= (F^*(\langle Q \cdot f\rangle g \cdot f)).(G \bullet F) \\ &= (\langle P\rangle g \cdot f).(G \bullet F) && \text{(By Lemma 5.25)} \\ &= (g \cdot f) \ltimes (G \bullet F). && \square \end{aligned}$$

Theorem 5.27. *Consider the diagram*

$$\begin{array}{ccccc} P & \xrightarrow{F} & Q & \xrightarrow{G} & R \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

in context Γ. Then $F^\bullet \circ G^\bullet = (G \bullet F)^\bullet$. In other words the composition $- \bullet -$ is associative.

Proof.

$$\begin{aligned} F^\bullet \circ G^\bullet &= (f \ltimes F)^* \circ (g \ltimes G)^* && \text{(By Lemma 5.24)} \\ &= (g \ltimes G \cdot f \ltimes F)^* && \text{(By Lemma 5.15)} \\ &= ((g \cdot f) \ltimes (G \bullet F))^* && \text{(By Theorem 5.26)} \\ &= (G \bullet F)^\bullet. && \text{(By Lemma 5.24)} \end{aligned}$$

$\square$

Theorem 5.28. *Let $f \in \mathrm{thom}(A, B)$ and $F \in \mathrm{thom}_f(P, Q)$. Then $f \ltimes F$ is the unique morphism from $A.P$ to $B.Q$ with the property that both the diagram*

$$\begin{array}{ccc} A.P & \xrightarrow{f \ltimes F} & B.Q \\ {\scriptstyle \mathrm{pr}_0^{A,P}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_0^{B,Q}} \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes and $\mathrm{pr}_1^{B,Q} \cdot (f \ltimes F) = F$.

Proof. We first note that

$$\begin{aligned} \mathrm{pr}_0^{B,Q} \cdot (f \ltimes F) &= F^* \circ (f^*/Q)(\langle Q \rangle \mathsf{idtm}_B) && \text{(By Lemma 5.24)} \\ &= F^* \circ \langle Q \cdot f \rangle (\mathsf{idtm}_B \cdot f) \\ &= F^* \circ \langle Q \cdot f \rangle f \\ &= \langle P \rangle f && \text{(By Lemma 5.25)} \\ &= f \cdot \mathrm{pr}_0^{A,P}. && \text{(By Theorem 5.18)} \end{aligned}$$

Also, we have

$$\begin{aligned} \mathrm{pr}_1^{B,Q} \cdot (f \ltimes F) &= F^* \circ (f^*/Q)(\mathsf{idtm}_Q) && \text{(By Lemma 5.24)} \\ &= \mathsf{idtm}_{Q \cdot f} \cdot F \\ &= F. \end{aligned}$$

Thus, we conclude that $f \ltimes F$ has indeed the stated property. For the uniqueness, let $G : A.P \to B.Q$ be a morphism such that $\mathrm{pr}_0^{B,Q} \cdot G = f \cdot \mathrm{pr}_0^{A,P}$ and $\mathrm{pr}_1^{B,Q} \cdot G = F$. Then it follows that

$$G = \left(f \cdot \mathrm{pr}_0^{A,P}\right).F = \langle P \rangle f.F = f \ltimes F. \qquad \square$$

5.2.3. *The functor from E-systems to CE-systems.* Let $\mathbf{Esys}_*$ be the category of pointed E-systems: objects are pairs $(\mathbb{E}, \Gamma)$ of an E-systems $\mathbb{E}$ and an object Γ in its underlying strict category, and arrows are E-homomorphisms that preserve the distinguished object. There is an evident forgetful functor $\mathbf{Esys}_* \to \mathbf{Esys}$ together with an embedding $\mathbf{E2E}_* \colon \mathbf{Esys} \hookrightarrow \mathbf{Esys}_*$ which picks out the terminal object of an E-system.

Problem 5.29. To construct a functor $\mathbf{E_*2CE} \colon \mathbf{Esys}_* \to \mathbf{CEsys}$.

Construction 5.30 (for Problem 5.29)**.** Let $(\mathbb{E}, \Gamma)$ be a pointed E-system and consider the category of terms $\mathcal{C}_{\mathbb{E}}(\Gamma)$ from Theorem 5.16. Define a functor $I_{\mathbb{E}}^{\Gamma} \colon \mathcal{F}/\Gamma \to \mathcal{C}_{\mathbb{E}}(\Gamma)$ as follows. It is the identity on objects and maps an arrow $Q \colon A.Q \to A$ in $\mathcal{F}/\Gamma$ to $\mathrm{pr}_0^{A,Q} \in \mathrm{thom}(A.Q, A)$. For functoriality, we compute $\mathrm{pr}_0^{A,\mathrm{id}_{\Gamma.A}} = \langle \mathrm{id}_{\Gamma.A} \rangle \mathsf{idtm}_A = \mathsf{idtm}_A$ and

$$\begin{aligned} \mathrm{pr}_0^{A,Q.R} &= \langle Q.R \rangle \mathsf{idtm}_A = \langle R \rangle (\langle Q \rangle \mathsf{idtm}_A) \\ &= \langle R \rangle \mathrm{pr}_0^{A,Q} = \left(\mathrm{pr}_0^{Q,R}\right)^* \left(\mathrm{pr}_0^{A,Q}\right) \\ &= \mathrm{pr}_0^{A,Q} \cdot \mathrm{pr}_0^{Q,R}. \end{aligned}$$

Next, we show that $\mathcal{C}_{\mathbb{E}}(\Gamma)$ admits a functorial choice of pullbacks of arrows in the image of $I_{\mathbb{E}}$. Given $f \in \mathrm{thom}(A,B)$ and R in $\mathcal{F}/\Gamma.B$, there is $R \cdot f$ in $\mathcal{F}/\Gamma.A$. We define

$$\pi_2(f,R) := f \ltimes \mathsf{idtm}_{R\cdot f} \colon A.R\cdot f \to B.R. \tag{5.5}$$

Then the following diagram in $\mathcal{C}_{\mathbb{E}}(\Gamma)$

$$\begin{array}{ccc} A.(R\cdot f) & \xrightarrow{\pi_2(f,R)} & B.R \\ {\scriptstyle \mathrm{pr}_0^{A,R\cdot f}}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_0^{B,R}} \\ A & \xrightarrow{\quad f \quad} & B \end{array} \tag{5.6}$$

commutes. The functoriality conditions follow immediately from the interchange laws proven in Section 5.2.2. To show that (5.6) is a pullback square, consider a morphism $g : X \to A$ in $\mathcal{C}_{\mathbb{E}}(\Gamma)$ and use the isomorphisms

$$\begin{aligned} \{h \in \mathrm{thom}(X, B.Q) \mid \mathrm{pr}_0^{B,Q} \cdot h = f \cdot g\} &\cong T(Q\cdot(f\cdot g)) = T((Q\cdot f)\cdot g) \\ &\cong \{u \in \mathrm{thom}(X, A.(Q\cdot f)) \mid \mathrm{pr}_0^{A,Q\cdot f} \cdot u = g\} \end{aligned}$$

given by Corollary 5.17 and Lemma 5.15.

Therefore, we have constructed a CE-system $\mathbf{E_*2CE}(\mathbb{E},\Gamma)$ on $I_{\mathbb{E}}^{\Gamma} \colon \mathcal{F}/\Gamma \to \mathcal{C}_{\mathbb{E}}(\Gamma)$.

Let now $(\mathbb{D},\Delta)$ be a pointed E-system and let $F \colon \mathbb{E} \to \mathbb{D}$ be an E-homomorphism such that $F\Gamma = \Delta$. In particular, for every $A, B \in \mathcal{F}_{\mathbb{E}}/\Gamma$ there is a function $F \colon T(\langle A\rangle B) \to T(\langle FA\rangle FB)$. These functions give the action on arrows of a functor $F_\Gamma \colon \mathcal{C}_{\mathbb{E}}(\Gamma) \to \mathcal{C}_{\mathbb{D}}(F\Gamma)$ whose action on objects is given by $F/\Gamma \colon \mathcal{F}_{\mathbb{E}}/\Gamma \to \mathcal{F}_{\mathbb{D}}/F\Gamma$. Functoriality of F_Γ follows from the fact that F is a projection homomorphism and Lemma 5.11. Using Lemma 4.65, we see that $F_\Gamma \circ I_{\mathbb{E}}^{\Gamma} = I_{\mathbb{D}}^{F\Gamma} \circ (F/\Gamma)$. Finally, it follows from Lemma 5.11 and Lemma 5.21 that F_Γ preserves the choice of pullback squares.

We have described the action of $\mathbf{E_*2CE}$ on objects and arrows. Its functoriality is straightforward. □

We obtain a functor $\mathbf{E2CE} \colon \mathbf{Esys} \to \mathbf{CEsys}$ defining $\mathbf{E2CE} := \mathbf{E_*2CE} \circ \mathbf{E2E_*}$.

Remark 5.31. The CE-system $\mathbf{E2CE}(\mathbb{E}) = \mathbf{E_*2CE}(\mathbb{E},[\,])$ is on the functor $I_{\mathbb{E}}^{[\,]} \colon \mathcal{F}/[\,] \to \mathcal{C}_{\mathbb{E}}([\,])$. It is also possible to have a CE-system with category of families given by $\mathcal{F}$ itself. Consider the commutative square below, where the top functor ! maps an object Γ to the unique arrow $!_\Gamma \colon \Gamma \to [\,]$.

$$\begin{array}{ccc} \mathcal{F}_{\mathbb{E}} & \xrightarrow{\;!\;} & \mathcal{F}_{\mathbb{E}}/[\,] \\ {\scriptstyle I}\downarrow & & \downarrow{\scriptstyle I_{\mathbb{E}}^{[\,]}} \\ \mathcal{C} & \longrightarrow & \mathcal{C}_{\mathbb{E}}([\,]) \end{array}$$

The left and bottom functors are obtained as the factorisation of the composite of the top and right functors into an identity-on-objects functor I followed by a fully faithful one. The above Construction 5.30 can be easily adapted to obtain a CE-system on the functor I.

Alternatively, one could rephrase the results in Section 5.2.1 leading to Theorem 5.16, as happening over the terminal object of the E-system. In this case, the version of the results over a generic object Γ can be recovered using the slice E-system $\mathbb{E}/\Gamma$ from Corollary 4.53.

Keeping the category of families fixed in the process might seem to be an advantage of this construction. However, as we discuss in Remark 5.38, it does not seem to have any actual useful consequence on the adjunction of which $\mathbf{E2CE}$ is the left adjoint. For this reason, and because it is not an instance of the general construction from Theorem 5.16, we prefer to use the one given in Construction 5.30.

Remark 5.32. For every E-system $\mathbb{E}$ and every Γ, the CE-system $\mathbf{E_*2CE}(\mathbb{E},\Gamma)$ is rooted. The canonical terminal object id_Γ of $\mathcal{F}_\mathbb{E}/\Gamma$ is terminal in $\mathcal{C}_\mathbb{E}(\Gamma)$ by Corollary 5.17 since for every $A \in \mathcal{F}_\mathbb{E}/\Gamma$

$$\mathrm{thom}(A, \mathrm{id}_\Gamma) = T(W_A(\mathrm{id}_\Gamma)) = T(\mathrm{id}_{\Gamma.A}).$$

Next we give the choice of pullbacks in a CE-system in the image of **E2CE** in terms of the underlying E-system structure.

Lemma 5.33. *For $\mathbb{E}$ an E-sytem and Γ an object in $\mathbb{E}$, consider the CE-system $\mathbb{A} :=$ $\mathbf{E_*2CE}(\mathbb{E},\Gamma)$. For every $A \in \mathcal{F}/\Gamma$ and $P, Q \in \mathcal{F}/\Gamma.A$ it is*

$$\left(\mathrm{pr}_0^{A,P}\right)^* Q = \langle P\rangle Q \in \mathcal{F}/\Gamma.A.P$$

and

$$\pi_2(\mathrm{pr}_0^{A,P}, Q) = \mathrm{pr}_1^{P,\langle P\rangle Q} \in \mathrm{thom}(P.\langle P\rangle Q, Q).$$

Proof. The first equality follows from Theorem 5.18. For the second one:

$$\begin{aligned}
\pi_2(\mathrm{pr}_0^{\mathrm{id}_\Gamma,A}, B) &= \mathrm{pr}_0^{\mathrm{id}_\Gamma,A} \ltimes \mathsf{idtm}_{\langle A\rangle B} \\
&= \left(W_{\langle A\rangle B}\langle A\rangle \mathsf{idtm}_{\mathrm{id}_\Gamma}\right).\mathsf{idtm}_{\langle A\rangle B} \\
&= (W_A\langle B\rangle \mathsf{idtm}_{\mathrm{id}_\Gamma}).(W_A \mathsf{idtm}_B) \\
&= \langle A\rangle(\langle B\rangle \mathsf{idtm}_{\mathrm{id}_\Gamma}.\mathsf{idtm}_B) \\
&= \langle A\rangle\left(\mathrm{pr}_0^{\mathrm{id}_\Gamma,B}.\mathrm{pr}_1^{\mathrm{id}_\Gamma,B}\right) \\
&= \langle A\rangle \mathsf{idtm}_B = \mathsf{idtm}_{\langle A\rangle B} \\
&= \mathrm{pr}_1^{A,\langle A\rangle B}.
\end{aligned}$$

□

5.3. **Equivalence between E-systems and CE-systems.** In this section, we show that the functors constructed in Sections 5.1 and 5.2 form an adjunction that, when suitably restricted, yields an equivalence of categories between rooted CE-systems and E-systems.

Specifically, we prove the following results:

Theorem 5.34.

(1) *The functor* **E2CE** *is left adjoint to the functor* **CE2E**.

$$\mathbf{CEsys} \underset{\mathbf{CE2E}}{\overset{\mathbf{E2CE}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Esys}$$

(2) *The unit of the adjunction is invertible. In particular, the left adjoint* **E2CE** *is full and faithful and the right adjoint* **CE2E** *is essentially surjective on objects.*
(3) *The counit component at a CE-system $\mathbb{A}$ is invertible if and only if $\mathbb{A}$ is rooted.*

Corollary 5.35. *The adjoint functors* **E2CE** *and* **CE2E** *induce an (adjoint) equivalence between the category* **Esys** *of E-systems and the category* **rCEsys** *of rooted CE-systems.*

Proof. The equivalence follows from Theorem 5.34, the observation in Remark 5.32 that, for every E-system $\mathbb{E}$, the CE-system $\mathbf{E2CE}(\mathbb{E})$ is rooted, and the fact that **rCEsys** is a full subcategory of **CEsys** by Remark 3.14. □

To prove Theorem 5.34 we construct unit and counit and prove the triangular identities. In this proof we denote as

$$\mathcal{F}/1 \underset{!}{\overset{\mathrm{d}}{\rightleftarrows}} \mathcal{F} \tag{5.7}$$

the canonical isomorphism of strict categories, for any strict category $\mathcal{F}$ with a terminal object 1. We may still leave this isomorphism implicit when doing so creates no confusion.

Problem 5.36. To construct, for each E-system $\mathbb{E}$, an invertible E-homomorphism $\eta_{\mathbb{E}} \colon \mathbb{E} \to$ **CE2E** $\circ$ **E2CE**$(\mathbb{E})$, naturally in $\mathbb{E}$.

Construction 5.37 (for Problem 5.36)**.** Let $\mathbb{E}$ be an E-system and denote its terminal object by $[\,]$. In this proof we shall decorate with a hat the constituents of the E-system structure of $\hat{\mathbb{E}} :=$ **CE2E** $\circ$ **E2CE**$(\mathbb{E})$. The underlying strict category of $\hat{\mathbb{E}}$ is $\mathcal{F}_{\mathbb{E}}/[\,]$ and, for every $X \in (\mathcal{F}_{\mathbb{E}}/[\,])/!_\Gamma$, we have

$$\hat{T}(X) = \left\{ h \in \mathrm{thom}(!_\Gamma, !_{\Gamma.X}) \mid \mathrm{pr}_0^{!_\Gamma, !_{\Gamma.X}} \cdot h = \mathsf{idtm}_{!_\Gamma} \right\}.$$

We define $\eta_{\mathbb{E}}$ as the functor $! \colon \mathcal{F}_{\mathbb{E}} \to \mathcal{F}_{\mathbb{E}}/[\,]$ in (5.7) with term structure given by the bijections

$$T(A) \xrightarrow{\varphi} \hat{T}(!(A)) \tag{5.8}$$

from Corollary 5.17, that is, for $A \in \mathcal{F}_{\mathbb{E}}/\Gamma$ and $t \in T(A)$, it is $\eta_{\mathbb{E}}(t) := \mathsf{idtm}_{!_\Gamma}.t$. Therefore we have an invertible functor with term structure.

To conclude that this defines an invertible E-homomorphism, we compute for $A \in \mathcal{F}_{\mathbb{E}}/\Gamma$

$$\begin{aligned} \hat{W}_{!(A)} \circ (!/\Gamma) &= \left(\mathrm{pr}_0^{!_\Gamma, A}\right)^* \circ (!/\Gamma) = W_A \circ (!/\Gamma) \\ &= (!/\Gamma.A) \circ W_A, \end{aligned}$$

and for $t \in T(A)$,

$$\begin{aligned} \hat{S}_{!(t)} \circ (!/\Gamma.A)) &= (\mathsf{idtm}_{!_\Gamma}.t)^* \\ &= S_{\mathsf{idtm}_{!_\Gamma}.t} \circ (W_{!_\Gamma}/!_{\Gamma.A}) \\ &= S_t \circ (S_{\mathsf{idtm}_{!_\Gamma}} \circ (W_{!_\Gamma}/!_\Gamma))/A \\ &= (!/\Gamma) \circ S_t, \end{aligned}$$

and finally

$$\begin{aligned} \varphi(\mathsf{idtm}_A) &= \mathsf{idtm}_{!_\Gamma}.\mathsf{idtm}_A \\ &= \mathsf{idtm}_{!_{\Gamma.A}} \\ &= \hat{\mathsf{idtm}}_{!(A)}. \end{aligned}$$

Finally, naturality in $\mathbb{E}$ requires that any E-homomorphism $F \colon \mathbb{E} \to \mathbb{D}$ commutes with η as functors with term structures. This follows from Lemmas 5.11 and 4.65. □

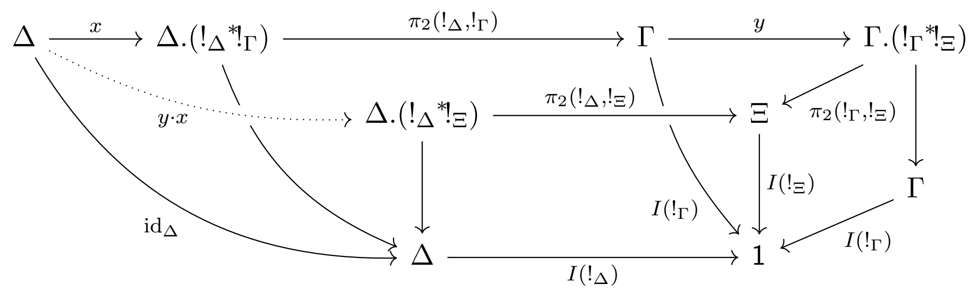
Remark 5.38. With the alternative construction for **E2CE** described in Remark 5.31, the underlying category of **CE2E** $\circ$ **E2CE**$(\mathbb{E})$ is $\mathcal{F}$ itself. In this case we could replace the isomorphism from (5.7) with an identity. However, the unit $\eta_{\mathbb{E}}$ would not become an identity, as the term structures would still be different (though isomorphic).

Problem 5.39. To construct, for each CE-system $\mathbb{A}$, a CE-homomorphism $\varepsilon_{\mathbb{A}} \colon$ **E2CE** $\circ$ **CE2E**$(\mathbb{A}) \to \mathbb{A}$, naturally in $\mathbb{A}$.

Construction 5.40 (for Problem 5.39)**.** Let $\mathbb{A}$ be a CE-system and let $\mathbb{E} := \mathbf{CE2E}(\mathbb{A})$ be the associated E-system. The underlying functor of the CE-system $\hat{\mathbb{A}} := \mathbf{E2CE} \circ \mathbf{CE2E}(\mathbb{A})$ is $I_{\mathbb{E}} \colon \mathcal{F}/[\,] \to \mathcal{C}_{\mathbb{E}}$ defined in Construction 5.30. As before, we decorate with a hat the constituents of the CE-system structure of $\hat{\mathbb{A}}$. For Γ, Δ in $\mathcal{F}$, recall that $\mathrm{thom}(!_\Delta, !_\Gamma) = \{\Delta \xrightarrow{x} \Delta.(!_\Delta{}^*!_\Gamma) \mid I(!_\Delta{}^*!_\Gamma) \circ x = \mathrm{id}_\Delta\}$ and let

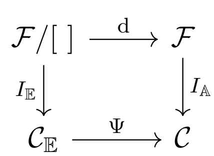
$$\mathrm{thom}(!_\Delta, !_\Gamma) \xrightarrow{\psi} \mathcal{C}(\Delta, \Gamma) \tag{5.9}$$

be the function that maps x to the arrow $\pi_2\,(I(!_\Delta), !_\Gamma) \circ x$ of $\mathcal{C}$. The functions ψ give rise to a functor $\Psi \colon \mathcal{C}_{\mathbb{E}} \to \mathcal{C}$ as follows. It maps identities to identities since the identity on Γ in $\mathcal{C}_{\mathbb{E}}$ is the only $h \in \mathrm{thom}(!_\Gamma, !_\Gamma)$ such that $\pi_2\,(I(!_\Gamma, !_\Gamma) \circ h = \mathrm{id}_\Gamma$. To see that it preserves composites, consider the commutative diagram below which defines the composite $y \cdot x$ of $x \in \mathrm{thom}(!_\Delta, !_\Gamma)$ and $y \in \mathrm{thom}(!_\Gamma, !_\Xi)$ in $\mathcal{C}_{\mathbb{E}}$.



Functoriality of Ψ amounts to the commutativity of the upper face.

To conclude that (d, Ψ) is a CE-homomorphism it remains to show that it preserves chosen pullbacks, since the square below commutes by definition of Ψ.



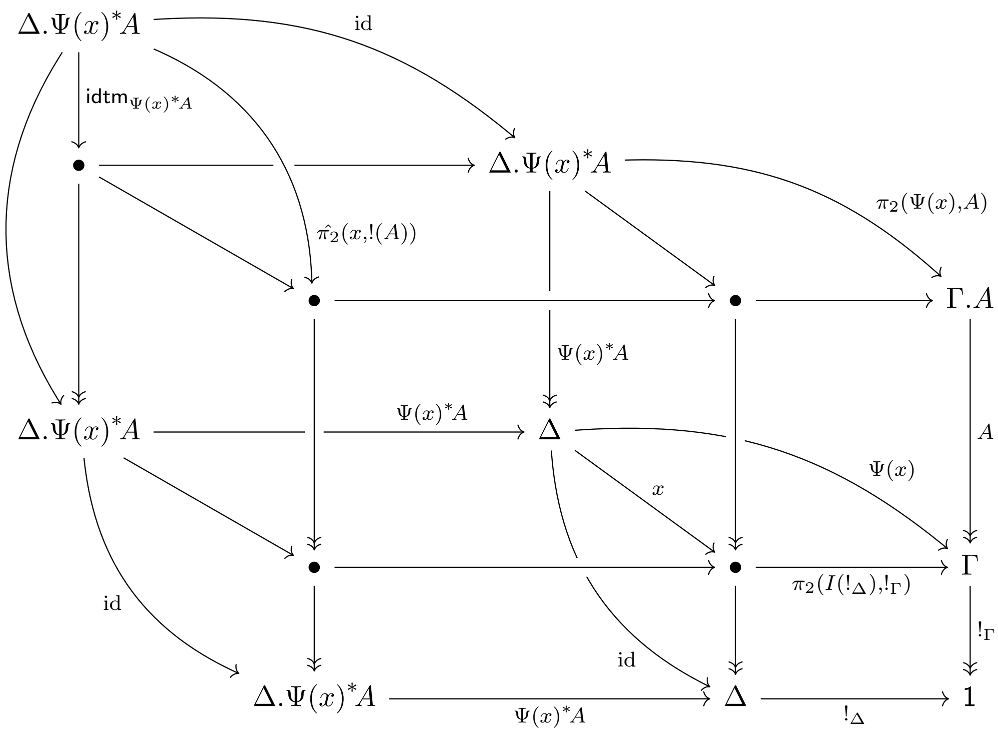
Let then $x \in \mathrm{thom}(!_\Delta, !_\Gamma)$ and $A \in \mathcal{F}/\Gamma$. It is

$$\begin{aligned} x^{\hat{*}}(!(A)) &= S_x \circ (W_{!_\Delta}/!_\Gamma) \circ !(A) \\ &= x^* \circ (I(!_\Delta)^*/!_\Gamma) \circ !(A) \\ &= !\left((\pi_2\,(I(!_\Delta), !_\Gamma) \circ x)^* A\right) \\ &= !\left(\Psi(x)^* A\right) \end{aligned}$$

whereas

$$\Psi(\hat{\pi_2}\,(x, !(A))) = \pi_2(I(!_{\Delta.(\Psi(x)^*A)}), !_{\Gamma.A}) \circ \hat{\pi_2}\,(x, !(A)) = \pi_2\,(\Psi(x), A)$$

holds by commutativity of the upper face in



This diagram commutes because all the squares not involving the top-left object are chosen pullback squares in $\mathbb{A}$, two of the remaining triangles commute by definition of idtm, and the third one involving $\hat{\pi_2}(x, !(A))$ commutes by (5.5) and Lemma 5.19.

The component $\varepsilon_{\mathbb{A}} \colon \mathbf{E2CE} \circ \mathbf{CE2E}(\mathbb{A}) \to \mathbb{A}$ of the counit at $\mathbb{A}$ is defined to be the pair (d, Ψ). To see that this choice is natural in $\mathbb{A}$ it is enough to show that the square of functors

$$\begin{array}{ccc} \mathcal{C}_{\mathbf{CE2E}(\mathbb{A})} & \xrightarrow{\Psi_{\mathbb{A}}} & \mathcal{C}_{\mathbb{A}} \\ \downarrow & & \downarrow F_{\mathcal{C}} \\ \mathcal{C}_{\mathbf{CE2E}(\mathbb{B})} & \xrightarrow{\Psi_{\mathbb{B}}} & \mathcal{C}_{\mathbb{B}} \end{array}$$

commutes for every CE-homomorphism $F \colon \mathbb{A} \to \mathbb{B}$. Note that the action of the left-hand functor coincide with that of F. Commutativity of the square thus follows from

$$F(\pi_2\,(x, A)) = \pi_2\,(Fx, FA)$$

which holds by definition of CE-homomorphism. □

Next we prove the second claim in Theorem 5.34.

Lemma 5.41. *For every CE-system $\mathbb{A}$, the CE-homomorphism $\varepsilon_{\mathbb{A}}$ from Problem 5.39 is invertible if and only if $\mathbb{A}$ is rooted.*

Proof. Note first that each function ψ in (5.9) induces a bijection

$$\mathrm{thom}(!_\Delta, !_\Gamma) \xrightarrow{\ \sim\ } \{f \in \mathcal{C}(\Delta, \Gamma) \mid I(!_\Gamma) \circ f = I(!_\Delta)\} \qquad (5.10)$$

with inverse given by the universal property of the canonical pullback square below.

$$\begin{array}{ccc} \Delta.!_\Delta{}^*!_\Gamma & \longrightarrow & \Gamma \\ \big\downarrow & & \big\downarrow{\scriptstyle I(!_\Gamma)} \\ \Delta & \xrightarrow{I(!_\Delta)} & 1 \end{array}$$

As soon as 1 is terminal in $\mathcal{C}$, the right-hand set in (5.10) coincides with $\mathcal{C}(\Delta,\Gamma)$. Conversely, if the counit components are invertible it follows from (5.10) that $\mathcal{C}(\Delta,1)=\{!_\Delta\}$. □

Proof of Theorem 5.34. 1. To complete the proof we show that, for an E-system $\mathbb{E}$ and a CE-system $\mathbb{A}$

$$\mathbf{CE2E}(\varepsilon_{\mathbb{A}})\circ\eta_{\mathbf{CE2E}(\mathbb{A})}=\mathrm{Id}_{\mathbf{CE2E}(\mathbb{A})}\qquad\text{and}\qquad\varepsilon_{\mathbf{E2CE}(\mathbb{E})}\circ\mathbf{E2CE}(\eta_{\mathbb{E}})=\mathrm{Id}_{\mathbf{E2CE}(\mathbb{E})}.$$

It is clear that these equations hold between functors on families by the isomorphism in (5.7). It remains to show that they hold also between the term structures in the left-hand one, and between functors on substitutions in the right-hand one.

For a CE-system $\mathbb{A}$, a family $A\in\mathcal{F}/\Gamma$ and $y\in T(A)=\{x\colon\Gamma\to\Gamma.A\mid I(A)\circ x=\mathrm{id}_\Gamma\}$, Lemma 5.33 yields $\eta_{\mathbf{CE2E}(\mathbb{A})}(y)=\pi_2(\mathsf{idtm}_{!_\Gamma},\pi_2(!_\Gamma,!_\Gamma)^*A)\circ y$. It follows that

$$\begin{aligned}\mathbf{CE2E}(\varepsilon_{\mathbb{A}})\circ\eta_{\mathbf{CE2E}(\mathbb{A})}(y)&=\pi_2(!_\Gamma,!_{\Gamma.A})\circ\pi_2(\mathsf{idtm}_{!_\Gamma},\pi_2(!_\Gamma,!_\Gamma)^*A)\circ y\\&=\pi_2(\pi_2(!_\Gamma,!_\Gamma),A)\circ\pi_2(\mathsf{idtm}_{!_\Gamma},\pi_2(!_\Gamma,!_\Gamma)^*A)\circ y\\&=\pi_2(\pi_2(!_\Gamma,!_\Gamma)\circ\mathsf{idtm}_{!_\Gamma},A)\circ y\\&=y.\end{aligned}$$

For an E-system $\mathbb{E}$, objects Δ and Γ and $f\in\mathrm{thom}(!_\Delta,!_\Gamma)$, Lemmas 5.33 and 4.66 yield

$$\varepsilon_{\mathbf{E2CE}(\mathbb{E})}\circ\mathbf{E2CE}(\eta_{\mathbb{E}})(f)=\mathrm{pr}_1^{!_\Delta,\langle !_\Delta\rangle !_\Gamma}[\mathsf{idtm}_{!_\Gamma}.f]=f.$$

This concludes the proof of the adjunction.

3. This is Lemma 5.41. □

5.4. **Equivalence between B-systems and C-systems.** Here we describe the main contribution of our work: the construction of an equivalence of categories between the category of C-systems of Section 3 and the category of B-systems of Section 4.

Lemma 5.42. *The functor* $\mathbf{CE2E}\colon\mathbf{CEsys}\to\mathbf{Esys}$ *from Construction 5.6 restricts to a functor* $\mathbf{CE2E}\colon\mathbf{rCEsys_s}\to\mathbf{Esys_s}$ *between stratified systems.*

Proof. To see that the E-system $\mathbf{CE2E}(\mathbb{A})$ is stratified whenever the rooted CE-system $\mathbb{A}$ is stratified, note first that the underlying category $\mathcal{F}$ is stratified by assumption. Weakening and substitution homomorphisms are stratified since the pullback functor that defines them in Construction 5.6.(5.3,5.4) is stratified.

For a stratified CE-homomorphism F, the underlying functor of the E-homomorphism $\mathbf{CE2E}(F)$ is the component $F_{\mathcal{F}}$ of F on families, which is stratified by assumption. □

Lemma 5.43. *The functor* $\mathbf{E2CE}\colon\mathbf{Esys}\to\mathbf{rCEsys}$ *restricts to a functor* $\mathbf{E2CE}\colon\mathbf{Esys_s}\to\mathbf{rCEsys_s}$ *between stratified systems.*

Proof. Let $\mathbb{E}$ be a stratified E-system. In particular, the underlying category $\mathcal{F}$ is stratified. Since weakening and substitution homomorphisms are also stratified by assumption, so is the precomposition homomorphisms from Definition 5.10. It follows that the CE-system $\mathbf{E2CE}(\mathbb{E})$ is stratified.

For a stratified E-homomorphism F, the component on families of the CE-homomorphism $\mathbf{E2CE}(F)$ is the underlying functor of F, which is stratified by assumption. □

Lemma 5.44.

(1) *For every stratified E-system $\mathbb{E}$, the unit component $\eta_{\mathbb{E}}$ of Construction 5.37 is a stratified E-homomorphism.*
(2) *For every stratified CE-system $\mathbb{A}$, the counit component $\varepsilon_{\mathbb{A}}$ of Construction 5.40 is a stratified CE-homomorphism.*
(3) *The adjunction $\mathbf{E2CE} \dashv \mathbf{CE2E}$ from Theorem 5.34.1 restricts to an adjunction*

$$\mathbf{CEsys_s} \underset{\mathbf{CE2E}}{\overset{\mathbf{E2CE}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathbf{Esys_s}.$$

between subcategories of stratified structures.

Proof.

(1) The underlying functor of the unit component $\eta_{\mathbb{E}}$ is the functor $!\colon \mathcal{F} \to \mathcal{F}/[\,]$ from (5.7). This functor is stratified since $L([\,]) = 0$.
(2) The underlying functor of the counit component $\varepsilon_{\mathbb{A}}$ on families is the inverse $\mathrm{d}\colon \mathcal{F}/1 \to \mathcal{F}$ of $!\colon \mathcal{F} \to \mathcal{F}/1$, and it is stratified for the same reason.
(3) This is a consequence of Lemmas 5.42 and 5.43 and Items 1 and 2 just proved. □

Define a functor $\mathbf{C2B}\colon \mathbf{Csys} \to \mathbf{Bsys}$ as the composite

$$\mathbf{Csys} \xrightarrow{\mathbf{CE}} \mathbf{rCEsys_s} \xrightarrow{\mathbf{CE2E}} \mathbf{Esys_s} \xrightarrow{\mathbf{E2B}} \mathbf{Bsys} \tag{5.11}$$

where the functors are, in order, $\mathbf{CE}$ from Construction 3.25 $\mathbf{CE2E}$ from Construction 5.6 and $\mathbf{E2B}$ from Construction 4.87. Similarly, we obtain a functor $\mathbf{B2C}\colon \mathbf{Bsys} \to \mathbf{Csys}$ in the other direction as the composite

$$\mathbf{Bsys} \xrightarrow{\mathbf{B2E}} \mathbf{Esys_s} \xrightarrow{\mathbf{E2CE}} \mathbf{rCEsys_s} \xrightarrow{\mathbf{C}} \mathbf{Csys} \tag{5.12}$$

where the functors are, in order, $\mathbf{B2E}$ from Lemma 4.78, $\mathbf{E2CE}$ from Construction 5.30 and $\mathbf{C}$ from Definition 3.29.

Theorem 5.45. *The pair of functors* $\mathbf{C2B}$ *and* $\mathbf{B2C}$ *establish an equivalence between the category of C-systems and the category of B-systems.*

Proof. The functors defining $\mathbf{C2B}$ in (5.11) and $\mathbf{B2C}$ in (5.12) are essentially inverse to each other by Theorems 3.31 and 4.90 and Corollary 5.35. The claim follows since equivalences compose. □

6. Conclusion

We have constructed an equivalence between the category of C-systems and the category of B-systems, each equipped with a suitable notion of morphism. The equivalence does not rely on classical reasoning principles such as the axiom of choice or excluded middle. This equivalence constitutes a crucial piece in Voevodsky's research program on the formulation and solution of an initiality conjecture.

Some questions that remain open:

- Voevodsky has studied different type constructions on C-systems, in particular, dependent function types [Voe16a, Voe17b] and identity types [Voe23b]. The equivalence constructed in the present paper should be extended to type and term constructors on C-systems and B-systems.
- Via Generalized Algebraic Theories, B-systems and C-systems relate to Garner's algebras for a monad on type-and-term systems [Gar15], in the form of an equivalence of categories. It would be very useful to have an explicit description of the maps back and forth, without passing through GATs.
- E-systems and CE-systems should be related to other unstratified categorical structures for the interpretation of type theory, such as categories with families [Dyb96].
- Voevodsky envisioned a formalization, in a computer proof assistant, of his theory of type theories; some work by Voevodsky towards this goal is available online.[4] A formalization of the equivalence between B- and C-systems is still missing.
- As remarked in the introduction, B-systems seem more suitable than other semantics to accommodate for modifications of the syntax (either restricting to substructural rules, or extending it with type constructors and operators). Carrying these modifications over along the equivalence could yield corresponding formulations for C-systems and more traditional semantics.

References

[AENR23] Benedikt Ahrens, Jacopo Emmenegger, Paige Randall North, and Egbert Rijke. B-systems and C-systems are equivalent. *The Journal of Symbolic Logic*, page 1–9, 2023. doi:10.1017/jsl.2023.41.

[ALV18] Benedikt Ahrens, Peter LeFanu Lumsdaine, and Vladimir Voevodsky. Categorical structures for type theory in univalent foundations. *Logical Methods in Computer Science*, 14(3), 2018. doi:10.23638/LMCS-14(3:18)2018.

[Awo18] Steve Awodey. Natural models of homotopy type theory. *Mathematical Structures in Computer Science*, 28(2):241–286, 2018. doi:10.1017/S0960129516000268.

[Bla91] Javier Blanco. Relating categorical approaches to type dependency, 1991. Master thesis, Radboud University.

[Car78] John W. Cartmell. *Generalised Algebraic Theories and Contextual Categories*. PhD thesis, Oxford University, 1978.

[Car86] John Cartmell. Generalised algebraic theories and contextual categories. *Annals of Pure and Applied Logic*, 32:209–243, 1986. doi:10.1016/0168-0072(86)90053-9.

[Car14] John Cartmell. Meta-theory of generalised algebraic theories, 2014. URL: https://www.researchgate.net/publication/325763545_Meta-Theory_of_Generalised_Algebraic_Theories.

[Car18] John Cartmell. Generalised algebraic axiomatisations of contextual categories, 2018. URL: https://www.researchgate.net/publication/325763538_Generalised_Algebraic_Axiomatisations_of_Contextual_Categories.

[4] https://github.com/UniMath/lBsystems, https://github.com/UniMath/lCsystems

[dBBLM] Menno de Boer, Guillaume Brunerie, Peter LeFanu Lumsdaine, and Anders Mörtberg. Initiality for Martin-Löf type theory. Talk given at HoTTEST seminar, https://www.uwo.ca/math/faculty/kapulkin/seminars/hottest.html.

[Dyb96] Peter Dybjer. Internal type theory. In *Types for proofs and programs*, volume 1158 of *Lecture Notes in Computer Science*, pages 120–134, 1996. doi:10.1007/3-540-61780-9_66.

[FPT99] Marcelo P. Fiore, Gordon D. Plotkin, and Daniele Turi. Abstract syntax and variable binding. In *14th Annual IEEE Symposium on Logic in Computer Science, Trento, Italy, July 2-5, 1999*, pages 193–202. IEEE Computer Society, 1999. doi:10.1109/LICS.1999.782615.

[FV20] Marcelo Fiore and Vladimir Voevodsky. Lawvere theories and C-systems. *Proceedings of the American Mathematical Society*, 148:2297–2315, February, 2020. doi:10.1090/proc/14660.

[Gar15] Richard Garner. Combinatorial structure of type dependency. *Journal of Pure and Applied Algebra*, 219(6):1885–1914, 2015. doi:10.1016/j.jpaa.2014.07.015.

[Hof97] Martin Hofmann. Syntax and semantics of dependent types. In *Semantics and logics of computation (Cambridge, 1995)*, volume 14 of *Publ. Newton Inst.*, pages 79–130. Cambridge Univ. Press, Cambridge, 1997. doi:10.1017/CBO9780511526619.004.

[Joy17] André Joyal. Notes on Clans and Tribes. arXiv:1710.10238, 2017.

[KL21] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. The simplicial model of univalent foundations (after Voevodsky). *Journal of the European Mathematical Society*, 23(6):2071–2126, 2021. doi:10.4171/JEMS/1050.

[LA24] Thomas Lamiaux and Benedikt Ahrens. An introduction to different approaches to initial semantics, 2024. arXiv:2401.09366.

[Mac98] Saunders Mac Lane. *Categories for the working mathematician*. Springer, New York, NY, 2nd edition, 1998.

[Nor19] Paige Randall North. Identity types and weak factorization systems in Cauchy complete categories. *Mathematical Structures in Computer Science*, 29(9):1411–1427, 2019. doi:10.1017/S0960129519000033.

[Pit01] Andrew M. Pitts. Categorical logic. In *Handbook of Logic in Computer Science: Volume 5: Logic and Algebraic Methods*, page 39–123. Oxford University Press, Inc., USA, 2001.

[Shu15] Michael Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, 25(5):1203–1277, 2015. doi:10.1017/S0960129514000565.

[Str91] Thomas Streicher. *Semantics of Type Theory*. Birkhäuser, 1991. doi:10.1007/978-1-4612-0433-6.

[Tay99] Paul Taylor. *Practical Foundations of Mathematics*, volume 59 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1999.

[Uem21] Taichi Uemura. *Abstract and concrete type theories*. PhD thesis, Institute for Logic, Language and Computation, Amsterdam, 2021. URL: https://dare.uva.nl/search?identifier=41ff0b60-64d4-4003-8182-c244a9afab3b.

[Voe] Vladimir Voevodsky. Dependent type theories. Lectures in the Max Planck Institute in Bonn. February 10-14, 2016. URL: https://www.math.ias.edu/vladimir/lectures.

[Voe14] Vladimir Voevodsky. B-systems. Preprint available as arXiv:1410.5389. Latest version available at https://www.math.ias.edu/Voevodsky/files/files-annotated/Dropbox/Unfinished_papers/Type_systems/Notes_on_Type_Systems/Bsystems/B_systems_current.pdf, 2014.

[Voe15] Vladimir Voevodsky. A C-system defined by a universe category. *Theory and Applications of Categories*, 30:Paper No. 37, 1181–1215, 2015. URL: http://www.tac.mta.ca/tac/volumes/30/37/30-37abs.html.

[Voe16a] Vladimir Voevodsky. Products of families of types and (π, λ)-structures on C-systems. *Theory and Applications of Categories*, 31:Paper No. 36, 1044–1094, 2016. URL: http://www.tac.mta.ca/tac/volumes/31/36/31-36abs.html.

[Voe16b] Vladimir Voevodsky. Subsystems and regular quotients of C-systems. In *A panorama of mathematics: pure and applied*, volume 658 of *Contemporary Mathematics*, pages 127–137. Amer. Math. Soc., Providence, RI, 2016. doi:10.1090/conm/658/13124.

[Voe16c] Vladimir Voevodsky. Mathematical theory of type theories and the initiality conjecture, April, 2016. Research proposal to the Templeton Foundation for 2016-2019, project description. URL: https://www.math.ias.edu/Voevodsky/other/Voevodsky%20Templeton%20proposal.pdf.

[Voe17a] Vladimir Voevodsky. C-systems defined by universe categories: presheaves. *Theory and Applications of Categories*, 32:Paper No. 3, 53–112, 2017. URL: `http://www.tac.mta.ca/tac/volumes/32/3/32-03abs.html`.

[Voe17b] Vladimir Voevodsky. The (π, λ)-structures on the C-systems defined by universe categories. *Theory and Applications of Categories*, 32:Paper No. 4, 113–121, 2017. URL: `http://www.tac.mta.ca/tac/volumes/32/4/32-04abs.html`.

[Voe23a] Vladimir Voevodsky. C-system of a module over a Jf-relative monad. *Journal of Pure and Applied Algebra*, 227(6), 2023. `doi:10.1016/j.jpaa.2022.107283`.

[Voe23b] Vladimir Voevodsky. Martin-Löf identity types in the C-systems defined by a universe category. *Publications mathématiques de l'IHÉS*, 138:1–67, 2023. `doi:10.1007/s10240-023-00138-2`.