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ON THE INFLUENCE OF LATERAL VIBRATIONS OF SUPPORTS FOR AN AXIALLY MOVING STRING

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In this paper the transverse oscillations in travelling strings due to arbitrary lateral vibrations of the supports will be studied. Using the method of Laplace transforms (exact) solutions will be constructed for the initial-boundary value problems which describe these transverse oscillations.

1 Introduction

About fifty years ago Sack [1], Mahalingan [2], and Archibald and Emslie [3] started to investigate the transverse oscillations in travelling strings due to the sinusoidal lateral vibrations of the supports over which the string passes. In [1], [2], and [3] it is assumed that the transverse displacement of the string can be expressed in a simple trigonometric function which is directly related to the sinusoidal lateral vibrations of the supports. In this way the resonance frequencies were found. Recently Pakdemirli and Boyaci [4] considered the transverse oscillations in travelling strings due to small lateral vibrations of the supports. A perturbation method (based on the truncation method and the method of multiple scales) has been used in [4] to approximate the transverse oscillations of the string. The applicability of this perturbation method to these moving string problems (in particular the use of the truncation method) will be discussed at the end of this paper.

The following non-dimensional equation of motion for the string problem will be considered in this paper

$$\frac{\partial^2 w}{\partial t^2} + 2v\frac{\partial^2 w}{\partial x \partial t} + (v^2-1)\frac{\partial^2 w}{\partial x^2} = 0 \quad , \quad 0 < x < 1 \quad , \quad t > 0 \; , \tag{1}$$

where $w = w(x,t)$ is the transverse displacement of the string, v is the (constant) axial transport velocity of the string, x is the spatial coordinate, and t is the time coordinate. These dimensionless quantities are in the following way related to the dimensional ones (see also for instance [1]-[7]): $w = \frac{w^*}{l}$, $x = \frac{x^*}{l}$, $t = t^*(T/\varrho A l)^{1/2}$, and $v = v^*/(T/\varrho A)^{1/2}$, in which l is the (constant) distance between the two supports, T is the (constant) tension in the string, and ϱA is the (constant) mass of the string per unit length. Furthermore, it has been assumed that the oscillations are sufficiently small such that the nonlinear terms in the equation of motion can be neglected. Also gravity effects, bending stiffness of the string, and damping effects have been neglected. In this paper the following Dirichlet boundary conditions will be considered

$$w(0,t) = 0 \quad , \quad \text{and} \quad w(1,t) = f(t) \quad , \quad t \geq 0 \; , \tag{2}$$

where $f(t)$ describes arbitrary vibrations of the support at $x = 1$. Only the vibrations of the string due to the arbitrary vibrations of the support(s) will be considered in this paper. For that reason it is assumed that the initial displacement of the string and the initial velocity of the string are both identically equal to zero, that is,

$$w(x,0) = 0 \quad , \quad \text{and} \quad w_t(x,0) = 0 \quad , \quad 0 < x < 1 \; , \tag{3}$$

respectively. In this paper the initial-boundary value problem (1)-(3) for $w(x,t)$ will be solved exactly using the Laplace transform method.
This paper is organized as follows. In section 2 of this paper the stationary belt problem will be studied (that is, problem (1)-(3) with $v = 0$). Standard techniques from the theory of partial differential equations can be used in this case (see for instance [8]). In section 3 of this paper the axially moving string problem (1)-(3) will be solved exactly using the standard method of Laplace transforms. Finally in section 4 of this paper some conclusions will be drawn and some remarks will be made.

2 The stationary belt

In this section we will study the initial-boundary value problem (1)-(3) with $v = 0$, that is,

$$\begin{aligned} &w_{tt} - w_{xx} = 0 \quad , \quad 0 < x < 1 \quad , \quad t > 0 \; , \\ &w(0,t) = 0 \quad , \quad w(1,t) = f(t) \quad , \quad t \geq 0 \; , \\ &w(x,0) = w_t(x,0) = 0 \quad , \quad 0 < x < 1 \; . \end{aligned} \tag{4}$$

This initial-boundary value problem (4) can be solved by using the available standard techniques as for instance described in section 6 of chapter 5 in [8]. Firstly the boundary conditions are made homogeneous by putting

$$w(x,t) = u(x,t) + x\, f(t) \; , \tag{5}$$

where $u(x,t)$ has to satisfy

$$\begin{aligned} &u_{tt} - u_{xx} = -x\, f''(t) \quad , \quad 0 < x < 1 \quad , \quad t > 0 \; , \\ &u(0,t) = u(1,t) = 0 \quad , \quad t \geq 0 \; , \\ &u(x,0) = -x\, f(0) \quad , \quad u_t(x,0) = -x\, f'(0) \quad , \quad 0 < x < 1 \; . \end{aligned} \tag{6}$$

Then, the function $u(x,t)$ is written into two parts

$$u(x,t) = u_1(x,t) + u_2(x,t) \; , \tag{7}$$

where $u_1(x,t)$ satisfies the homogeneous PDE and the inhomogeneous IVs, that is,

$$\begin{aligned} &u_{1_{tt}} - u_{1_{xx}} = 0 \quad , \quad 0 < x < 1 \quad , \quad t > 0 \; , \\ &u_1(0,t) = u_1(1,t) = 0 \quad , \quad t \geq 0 \\ &u_1(x,0) = -x\, f(0) \quad , \quad u_t(x,0) = -x\, f'(0) \quad , \quad 0 < x < 1 \; , \end{aligned} \tag{8}$$

and where $u_2(x,t)$ satisfies the inhomogeneous PDE and the homogeneous IVs, that is,

$$\begin{aligned} &u_{2_{tt}} - u_{2_{xx}} = -x\, f''(t) \quad , \quad 0 < x < 1 \quad , \quad t > 0 \; , \\ &u_2(0,t) = u_2(1,t) = 0 \quad , \quad t \geq 0 \\ &u_2(x,0) = x_{2_t}(x,0) = 0 \quad , \quad 0 < x < 1 \; . \end{aligned} \tag{9}$$

The initial-boundary value problem (8) for $u_1(x,t)$ can readily be solved, yielding

$$u_1(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos(n\pi t) + B_n \sin(n\pi t)\right) \sin(n\pi x) \ , \tag{10}$$

where

$$\begin{aligned} A_n &= -2\int_0^1 x\ f(0)\sin(n\pi x)dx &= \frac{2\cdot(-1)^n}{n\pi} f(0) \ , \quad \text{and} \\ B_n &= \frac{-2}{n\pi}\int_0^1 x\ f'(0)\sin(n\pi x)dx &= \frac{2\cdot(-1)^n}{n^2\pi^2} f'(0) \ . \end{aligned}$$

The initial-boundary value problem (9) for $u_2(x,t)$ can be solved by putting

$$u_2(x,t) = \sum_{n=1}^{\infty} u_n(t)\sin(n\pi x) \ . \tag{11}$$

Then, (11) is substituted into the initial-boundary value problem (9) to obtain

$$\begin{aligned} u_n^{''} + n^2\pi^2 u_n &= -2f''(t)\int_0^1 x\ \sin(n\pi x)dx = \frac{2\cdot(-1)^n}{n\pi} f''(t) \ , \\ u_n(0) = u_n^{'}(0) &= 0 \ . \end{aligned} \tag{12}$$

The initial value problem (12) for $u_n(t)$ can readily be solved, and so from (11) $u_2(x,t)$ follows, yielding

$$\begin{aligned} u_2(x,t) &= \sum_{n=1}^{\infty} \frac{2\cdot(-1)^n}{n^2\pi^2} \left\{ \sin(n\pi t)\int_0^t f''(\tau)\cos(n\pi\tau)d\tau - \cos(n\pi t)\int_0^t f''(\tau)\sin(n\pi\tau)d\tau \right\} \sin(n\pi x) \\ &= \sum_{n=1}^{\infty} \frac{2\cdot(-1)^n}{n^2\pi^2} \int_0^t f''(\tau)\sin(n\pi(t-\tau))d\tau\ \sin(n\pi x) \ . \end{aligned} \tag{13}$$

From (5), (7), (10), and (13) the solution $w(x,t)$ of the initial-boundary value problem (4) now follows. By introducing

$$S(x,t) = \sum_{n=1}^{\infty} \frac{2\cdot(-1)^n}{n^2\pi^2} \sin(n\pi t)\sin(n\pi x)$$

this solution $w(x,t)$ can also be written in the compact form

$$w(x,t) = f(0)S_t(x,t) + f'(0)S(x,t) + \int_0^t f''(\tau)S(x,t-\tau)d\tau + xf(t) \ . \tag{14}$$

3 The axially moving string

In this section we will study the initial-boundary value problem (1)-(3) with $v^2 \neq 1$. The method of Laplace transforms will be used to construct the (exact) solution for this problem. Let $W(x,s)$ be the Laplace transform of $w(x,t)$, that is,

$$W(x,s) = \int_0^\infty w(x,t)e^{-st}dt \ . \tag{15}$$

Then, by applying the Laplace transform to the PDE (1) and by using the IVs (3) the following equation is obtained

$$s^2W(x,s) - sw(x,0) - w_t(x,0) + 2v(sW_x(x,s) - w_x(x,0)) + (v^2-1)W_{xx}(x,s) = 0$$

$$\Rightarrow \quad W_{xx} + \frac{2vs}{v^2-1}W_x + \frac{s^2}{v^2-1}W = 0 \ . \tag{16}$$

The equation (16) for $W(x,s)$ can readily be solved, yielding

$$W(x,s) = C_1(s)\exp\left(\frac{-sx}{v+1}\right) + C_2(s)\exp\left(\frac{-sx}{v-1}\right) \ , \tag{17}$$

where $C_1(s)$ and $C_2(s)$ are still arbitrary functions, which will be determined by the BCs (2). Let $F(s)$ be the Laplace transform of $f(t)$. Then, by applying the Laplace transform to the BCs (2) it follows that

$$W(0,s) = 0 \ \ , \quad \text{and} \quad W(1,s) = F(s) \ . \tag{18}$$

From (17) and (18) it then follows that

$$C_1(s) = F(s)/\Big(\exp(\tfrac{-s}{v+1}) - \exp(\tfrac{-s}{v-1})\Big) \ , \quad \text{and} \quad C_2(s) = -C_1(s) \ ,$$

and so,

$$W(x,s) = F(s)\frac{\left(\exp\left(\frac{-sx}{v+1}\right) - \exp\left(\frac{-sx}{v-1}\right)\right)}{\left(\exp\left(\frac{-s}{v+1}\right) - \exp\left(\frac{-s}{v-1}\right)\right)} \ . \tag{19}$$

The inverse Laplace transform of $W(x,s)$ is defined to be

$$w(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} W(x,s)e^{st}ds = \sum_n \text{Res}(s_n) \tag{20}$$

for some $\gamma > 0$, and where $\text{Res}(s_n)$ is the residue of $W(x,s)e^{st}$ at $s = s_n$. Since $f(t)$ is arbitrary it follows that the poles in (20) due to $F(s)$ are unknown. For that reason (19) will be rewritten in the form

$$W(x,s) = F(s)\cdot H(x,s) \quad \text{with} \quad H(x,s) = \frac{\exp\left(\frac{-sx}{v+1}\right) - \exp\left(\frac{-sx}{v-1}\right)}{\exp\left(\frac{-s}{v+1}\right) - \exp\left(\frac{-s}{v-1}\right)} \ . \tag{21}$$

The inverse Laplace transform applied to (21) will eventually lead to a convolution integral in the (x,t)-domain. Before this inverse transform can be determined the singularities of $H(x,s)$ have to be calculated first. These singularities of $H(x,s)$ are given by

$$\begin{aligned} &\exp\left(\frac{-s}{v+1}\right)-\exp\left(\frac{-s}{v-1}\right)=0 \quad \Leftrightarrow \quad \exp\left(\frac{2s}{v^2-1}\right)=1 \quad \Rightarrow \\ &s=s_n=(v^2-1)n\pi i \end{aligned} \tag{22}$$

with $n \in \mathbb{Z}$. It should be observed that $s_0=0$ is not a pole of $H(x,s)$, and that all other poles of $H(x,s)$ are simple and are given by (22) with $n \in \mathbb{Z}\backslash\{0\}$. To obtain a solution form for $w(x,t)$ which is immediately comparable to the case $v=0$ (see formula (14) in section 2 of this paper) the expression (21) for $W(x,s)$ is rewritten into

$$W(x,s)=(s^2F(s)-s\ f(0)-f'(0))\frac{H(x,s)}{s^2}+(s\ f(0)+f'(0))\frac{H(x,s)}{s^2}\ . \tag{23}$$

Now it should be observed that $\frac{H(x,s)}{s^2}$ has a pole of order two at $s=0$, and that $\frac{H(x,s)}{s^2}$ has poles of order one at $s=s_n=(v^2-1)n\pi i$ with $n \in \mathbb{Z}\backslash\{0\}$. The inverse Laplace transform of $\frac{H(x,s)}{s^2}$, that is, $L^{\text{inv}}\left(\frac{H(x,s)}{s^2}\right)$ now easily follows, yielding

$$\begin{aligned} &L^{\text{inv}}\left(\frac{H(x,s)}{s^2}\right) \\ &=\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{H(x,s)}{s^2}e^{st}ds=\frac{1}{1!}\lim_{s\to 0}\frac{\partial}{\partial s}\left(H(x,s)e^{st}\right)+\sum_{n\neq 0}\text{Res}(s_n) \\ &= \qquad\qquad \dots \qquad\qquad =\frac{x}{v^2-1}(v-xv+t(v^2-1)) \\ &\quad -\sum_{n=1}^{\infty}\Bigg\{\frac{\cos((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)}\Big[\cos((v-1)n\pi)(\cos((v-1)n\pi x)-\cos((v+1)n\pi x)) \\ &\qquad\qquad +\sin((v-1)n\pi)(\sin((v-1)n\pi x)-\sin((v+1)n\pi x))\Big] \\ &\qquad +\frac{\sin((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)}\Big[\sin((v-1)n\pi)(\cos((v+1)n\pi x)-\cos((v-1)n\pi x)) \\ &\qquad\qquad +\cos((v-1)n\pi)(\sin((v-1)n\pi x)-\sin((v+1)n\pi x))\Big]\Bigg\} \\ &=h_2(x,t)\ . \end{aligned} \tag{24}$$

Further it should be observed that $\frac{H(x,s)}{s}$ has only poles of order one at $s=(v^2-1)n\pi i$ with $n \in \mathbb{Z}$. The inverse Laplace transform of $\frac{H(x,s)}{s}$ now also easily follows, yielding

$$
\begin{aligned}
&L^{\text{inv}}\left(\frac{H(x,s)}{s}\right)\\
&= \frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{H(x,s)}{s}e^{st}ds = \sum_{n\in\mathbb{Z}} \text{Res}(s_n) = ... = x\\
&\quad+\sum_{n=1}^{\infty}\Bigg\{\frac{\sin((v^2-1)n\pi t)}{n\pi}\Big[\cos((v-1)n\pi)(\cos((v-1)n\pi x)-\cos((v+1)n\pi x))\\
&\qquad\qquad+\sin((v-1)n\pi)(\sin((v-1)n\pi x)-\sin((v+1)n\pi x))\Big]\\
&\qquad+\frac{\cos((v^2-1)n\pi t)}{n\pi}\Big[\sin((v-1)n\pi)(\cos((v-1)n\pi x)-\cos((v+1)n\pi x))\\
&\qquad\qquad+\cos((v-1)n\pi)(\sin((v+1)n\pi x)-\sin((v-1)n\pi x))\Big]\Bigg\} =\\
&\quad h_1(x,t) = \frac{\partial h_2}{\partial t}(x,t)\ .
\end{aligned}
\tag{25}
$$

Since the inverse Laplace transform of $s^2F(s) - s\ f(0) - f'(0)$ is $f(t)$, it then finally follows from (20), and (23)-(25) that

$$
w(x,t) = \int\limits_0^t f''(\tau)h_2(x,t-\tau)d\tau + f(0)h_1(x,t) + f'(0)h_2(x,t)\ , \tag{26}
$$

where $h_2(x,t)$ and $h_1(x,t)$ are given by (24) and (25) respectively. By putting

$$
\begin{aligned}
&h(x,t) =\\
&-\sum_{n=1}^{\infty}\Bigg\{\frac{\cos((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)}\Big[\cos((v-1)n\pi)(\cos((v-1)n\pi x)-\cos((v+1)n\pi x))\\
&\qquad\qquad+\sin((v-1)n\pi)(\sin((v-1)n\pi x)-\sin((v+1)n\pi x))\Big]\\
&\qquad+\frac{\sin((v^2-1)n\pi t)}{n^2\pi^2(v^2-1)}\Big[\sin((v-1)n\pi)(\cos((v+1)n\pi x)-\cos((v-1)n\pi x))\\
&\qquad\qquad+\cos((v-1)n\pi)(\sin((v-1)n\pi x)-\sin((v+1)n\pi x))\Big]\Bigg\}
\end{aligned}
\tag{27}
$$

the solution of the initial-boundary value problem (1)-(3) can be rewritten in

$$w(x,t) = \int_0^t f''(\tau)h(x,t-\tau)d\tau + f(0)\frac{\partial h}{\partial t}(x,t) + f'(0)h(x,t) + \frac{vx(1-x)}{v^2-1}f'(t) + x\,f(t)\ . \tag{28}$$

For $v = 0$ it immediately follows from (27) and (28) that the respresentation (14) for $w(x,t)$ is again obtained. It can also be observed from (27) and (28) that resonances (that is, unbounded behaviour of $w(x,t)$ in time) can occur when for (at least) one $n \in \mathbb{Z}^+$

$$\begin{aligned} &\int_0^t f''(\tau)\cos((v^2-1)n\pi(t-\tau))d\tau \quad , \quad \text{and/or} \\ &\int_0^t f''(\tau)\sin((v^2-1)n\pi(t-\tau))d\tau \end{aligned} \tag{29}$$

are unbounded in time. For instance for $f(t) = \cos(\omega t)$ with ω equal to (or close to) $(v^2-1)n\pi$ it follows from (29) that resonance occurs (see also [1], [2], or [3] for a similar result). Also for a more general periodic function $f(t)$ with period $\frac{2}{m(v^2-1)}$ (for some fixed $m \in \mathbb{Z}^+$) it easily follows from (29) that resonances will occur.

4 Conclusions and remarks

In this paper the transverse oscillations in travelling strings due to arbitrary lateral vibrations of the support(s) have been studied. It has been shown how the initial-boundary value problem (1)-(3) which describes these oscillations can be solved using the method of Laplace transforms. The (exact) solution has been constructed explicitly, and resonance conditions have been derived.
In this paper a homogeneous boundary condition at $x = 0$ and an inhomogeneous one at $x = 1$ have been considered. When both boundary conditions are inhomogeneous the method of Laplace transforms can still be applied. In fact the method of Laplace transforms can be applied to the following initial-boundary value problem

$$\begin{aligned} &w_{tt} + 2vw_{xt} + (v^2-1)w_{xx} = g(x,t) \quad , \quad 0 < x < 1 \quad , \quad t > 0\ , \\ &w(0,t) = h(t) \quad , \quad w(1,t) = f(t) \quad , \quad t \geq 0\ , \\ &w(x,0) = w_0(x) \quad , \quad w_t(x,0) = w_1(x) \quad , \quad 0 < x < 1\ . \end{aligned} \tag{30}$$

The calculations, however, become lengthy but remain completely similar to the calculations as presented in section 3 of this paper. Also when the Dirichlet boundary conditions in (30) are replaced by for instance Neumann or Robin boundary conditions similar calculations can be performed as presented in this paper.
Recently in [4] the transverse oscillations in travelling strings due to small lateral vibrations of the supports have been considered. A perturbation method which is based on the truncation method has been used in [4] to approximate the transverse oscillations of the string. In fact in [4] the approximation is truncated to a single mode of vibration. The authors of [4] and their co-authors (see the list of references in [4]) have applied this truncation method to a lot of problems which are related to axially moving strings or beams. As can be seen from (27) and (28), however, the exact solution for the initial-boundary value problem (1)-(3) will always consist of infinitely many modes of vibration. So, truncating the approximation to a

single mode of vibration (as has been done in [4]) will in general not give accurate approximations on long time-scales. A similar conclusion on the applicability of the truncation method to problems for axially moving strings or beams with a time-varying velocity has been drawn in [9] and [10].

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