Stellingen

behorende bij het proefschrift

Time-varying System Identification, J-lossless Factorization and H_∞ Control

door

YU, Xiaode

1 . Time invariant is relative. Time-varying is absolute.

2 . The goal of the *standard* H_∞ control problem is to find the class of *suboptimal* controllers which attenuate the influence of disturbances to the controlled system to a (worst case) acceptable level. (Chapter 1, this thesis)

3 . The H_∞ problem is equivalent to an interpolation problem in the sense that both need to define a set of functions (operators) to fit the data set (system) and the set of functions (operators) are analytic (upper) and strictly contractive. (Chapter 7, this thesis)

4 . A state space model is becoming popular in industry because its advantage over other models is seen: its compact form, its convenience in representing MIMO systems, the guarantee of the minimality of its reduced model and the easier methodology of its controller design.

5 . Infinite horizon time-varying control can be realized only under the condition that the controlled system is known completely from time minus infinity to infinity. This happens, for example, when the system is time invariant before a certain time instant and becomes time invariant again after a certain time instant, or when the system is periodically time-varying.

6 . To identify systems is for the purpose of control. To identify people is for the purpose of making friends.

7 . Not every parent knows that to understand his child is even more important than to take good care of him.

8 . For a civilized society, rules are the first demand, democracy is the second. But a right rule itself should be democratic. For example, if a law does not benefit most of the people of the country, it is not 'democratic' and then it should be abolished.

9 . The most ingenious translation from English to Chinese I have ever seen is WWW ("World Wide Web" in English; "Wan Wei Wang (万维网)" in Chinese).

10 . Expecting a human being to be perfect is just like expecting gold to be absolutely pure (A translation of the Chinese proverb: 金无足赤，人无完人。).

Time-varying System Identification, J-lossless Factorization and H_∞ Control

Xiaode YU

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To Diguang and Junjun
To my parents

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Chapter 1

Introduction

1.1 Time-varying systems and control

Many dynamical systems exhibit time-varying behavior. For example, in electrical circuits or other kinds of networks, switching on or off a part of the network results in a time-varying process. It is also common practice in system and control engineering to treat non-linear systems as linear time-varying systems [1]. The studies by Kearnay et al. [2] [3] [4] on the performance of a human's arm joint for the purpose of obtaining a biomedical model are a good example of this kind of practice. Furthermore, periodically time-varying systems form an important class of linear systems in many mechanical and chemical applications [5]. Like in the time invariant context, the usual identification and control problems for linear time-varying systems naturally arise, and some of these are considered in this thesis.

There are many descriptions by which linear discrete time-varying systems can be represented, for example, operator descriptions, difference equations with time-varying parameters, FIR models and time-varying state equation descriptions. For multi-input multi-output systems, state equations and operator descriptions have significant advantages over others. Hence, in this thesis, we will describe all systems by operators and state equations. In particular, the input/output behavior of a linear discrete-time system can be described by a operator T which maps input sequences u,

$$u = [\; \cdots \;\; u_{-1}, \;\; \boxed{u_0}, \;\; u_1 \;\; \cdots \;]$$

to output sequences y as $y = uT$. In a *Hilbert space* context, T has a matrix representation

$$T = [T_{i,j}]_{i,j=-\infty}^{\infty} = \begin{bmatrix} \ddots & & \vdots & & \\ & T_{-1,-1} & T_{-1,0} & T_{-1,1} & \\ \cdots & T_{0,-1} & \boxed{T_{0,0}} & T_{0,1} & \cdots \\ & T_{1,-1} & T_{1,0} & T_{1,1} & \\ & & \vdots & & \ddots \end{bmatrix}$$

T is a *Toeplitz operator* if $T_{i,j} = T_{i-1,j-1}$ for all i, j, i.e. T is constant along the diagonals. Such operators correspond to time invariant systems. For time varying systems, T does not have Toeplitz structure. If $T_{i,j} = 0$ for $i > j$, T is a *causal system* and is upper triangular. Then its output y_j is determined only by the input u_i with $i \le j$. If $T_{i,j} = 0$ for $i < j$, T is an *anticausal system* and is lower triangular.

A compact description of a system can be provided by its state equations in terms of its realization. For a causal system, the equations are

$$\begin{aligned} x_{k+1} &= x_k A_k + u_k B_k \\ y_k &= x_k C_k + u_k D_k \end{aligned} \tag{1.1}$$

In this description, x is called the state and performs a memory function of the past input. $\{A_k, B_k, C_k, D_k\}$ is called a realization of the system. For a system, its realization $\{A_k, B_k, C_k, D_k\}$ has a direct connection to its operator description. Taking a stable causal system as an example, with (1.1), it is not difficult to derive that the corresponding input-output operator T is

$$T = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \\ & D_{-1} & B_{-1}C_0 & B_{-1}A_0C_1 & B_{-1}A_0A_1C_2 & \cdots \\ & & \boxed{D_0} & B_0C_1 & B_0A_1C_2 & \\ & & & D_1 & B_1C_2 & \\ & 0 & & & D_2 & \cdots \\ & & & & & \ddots \end{bmatrix}$$

The realization problem is to identify $\{A_k, B_k, C_k, D_k\}$ from the knowledge of T. Obviously, the D_k's correspond to the diagonal entries of the operator T as $D_k = T_{k,k}$. Matrices A_k, B_k and C_k can be identified from factorizations of the *generalized* (or time-varying) *Hankel operator* which is a part of the operator T. For example, the generalized Hankel operator corresponding to $k = 0$ is defined as

$$\begin{bmatrix} T_{-1,0} & T_{-1,1} & T_{-1,2} & \cdots \\ T_{-2,0} & T_{-2,1} & & \\ T_{-3,0} & & \ddots & \\ \vdots & & \ddots & \end{bmatrix} = \begin{bmatrix} B_{-1}C_0 & B_{-1}A_0C_1 & B_{-1}A_0A_1C_2 & \cdots \\ B_{-2}A_{-1}C_0 & B_{-2}A_{-1}A_0C_1 & & \\ B_{-3}A_{-2}A_{-1}C_0 & \ddots & & \\ \vdots & & & \end{bmatrix}$$

Conversely, if a realization of a system is known, the input-output operator of the system can be computed.

The state space realization problem for linear time invariant systems was discussed by many researchers, going back to the days of Kronecker. Modern insights are largely based on the pioneering work of Kalman et al [6]. It is easy to see that in time invariant case, all the Hankel operators are equal and have the familiar Hankel structure (constant along the antidiagonal). For linear time-varying systems, the realization problem was discussed by Shokoohi and Silverman [7] who called it the identification problem. In [11] and [8], an algorithm for the calculation of the realization of a linear discrete time-varying system from the input-output operator is given.

System identification

System identification is concerned with the mathematical modeling of systems from experimental input/output data. Based on the input/output data, the task of identification is twofold: (1) determine a suitable class of mathematical models such as Finite Impulse Response (FIR) models or state space models by which the system can be described, and (2) identify the best-fitting values for the parameters of the model. Often, the structural parameters of the model (eg. system orders) have to be estimated first.

For time invariant systems, there are many identification methods, both in frequency domain and time domain. Identification methods for time-varying systems are limited to the time domain because good frequency domain models are lacking.

The identification of time-varying systems can be on-line or off-line, depending on the purpose and available data. In many cases in process control, on-line identification such as adaptive identification is used. With this kind of identification method, adequate a priori information about the identified system and the variation of the system is required, and the change of the dynamics of the system has to be sufficiently small. In this thesis, we only consider off-line identification, which can be used in the identification of arbitrarily time-varying systems.

Methods for time-varying system identification were investigated by Kearney and Hunter [2] [3], both theoretically and in applications. There are two classes of approaches in time-varying system identification: (1) methods working with a single pair of input and output sequence measurements, and (2) methods working with an *ensemble* of input/output sequence pairs, each obtained from the same system or from systems with the same underlying time-varying behavior. Unlike the situation for the time invariant case, the identification of a general time-varying system from a single input/output pair is not possible without making further assumptions such as *short-term stationarity*. Less a priori knowledge about the system is required and a more accurate model can be obtained with ensemble methods but multiple copies of the same system have to be available, which is not always realistic. In [2] [3], an ensemble identification approach was used to identify the parameters of a FIR model for a biomedical application. But, a FIR model often needs many parameters. As a result, the estimated model is not expected to be accurate for finite data lengths.

The advantage of modeling systems by state space models is well known in industry and engineering. In [7], a linear time-varying system is modeled by a state equation description and the realization $\{A_k, B_k, C_k, D_k\}$ of the description is identified from the generalized Hankel matrix which is formed from the time-varying impulse response. Hence, we first have to estimate the time-varying impulse response of the system. Besides the fact that this is a two step procedure, the accurate determination of the impulse response is a difficult task.

In recent years, *subspace model identification (SMI)* methods for linear time invariant systems were developed by Moonen, de Moor, Vandenberghe and Vandewalle [10], and subsequently by Verhaegen and Dewilde [12]. The subspace-based methods have advantages in accuracy, convergence and initial conditions as compared to the FIR method and the two-step method. The state space model is estimated directly from the input/output measurements. In this thesis, we develop an ensemble method which extends the subspace model identification method to time-varying systems and periodically time-varying systems.

H_∞ control

The H_∞ *control* was first introduced by Zames [13] in 1981 and is widely regarded as a new impetus in control theory. The objective in H_∞ control is to minimize the H_∞ norm of some operator (usually an input/output operator). The H_∞ norm is the maximum amplification of any admissible (bounded) input sequence by the system. In [13], Zames considered the H_∞ control problem of minimizing the sensitivity of a system to the worst case disturbances by the design of an optimal feedback controller [13]. It was soon recognized that this H_∞ criterion could also be used in solving some robust control problems, specifically those dealing with model uncertainty. Roughly speaking, the objective of robust control is to design controllers which guarantee the closed loop stability and/or performance robustness for all systems within a given uncertainty range. For some robust control problems, the model uncertainty can be translated into a certain disturbance on the inputs of the system. This kind of robust control problems can be formulated as an H_∞ control problem.

Most research on the solution of H_∞ control problems is restricted to only a suboptimal version of the problem. Suboptimal versions aim at finding controllers which make the H_∞ norm less than some prescribed bound but not necessarily minimal. The reason for this is the mathematical difficulties that come up when trying to solve for the optimum.

Most attempts to solve H_∞ control problems focus on the so called standard H_∞ problem introduced by Doyle [14] in 1984, and treated in detail by Francis [15] in 1987. Problems such as the model-matching problem, the tracking problem and robust stabilization can all be reformulated as the standard H_∞ problem [15]. This standard set-up is shown in Figure 1.1. In this figure, P is a known plant, w is an exogenous disturbance sequence,

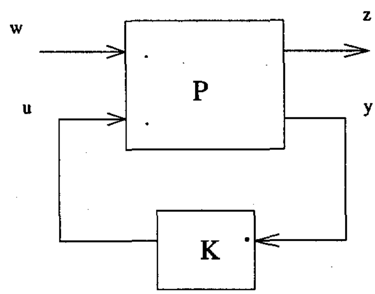

Figure 1.1: The standard H_∞ problem configuration

u is the control input sequence which is generated by a feedback controller K, y is the observed output sequence and z is the controlled error output sequence. The H_∞ control problem is to find a characterization of all admissible controllers K that stabilize the closed loop system in Figure 1.1 and that are such that the H_∞ norm of the closed loop transfer function (or operator) Φ, which is the mapping from the disturbance w to output z, is smaller than a prescribed bound γ. Hence the goal of the standard H_∞ control problem is to find suboptimal controllers which attenuate the influence of the disturbance to a certain level.

The theory and methods of H_∞ control developed in the past decade considered mostly the continuous time invariant case [15] [16] [17] [18] [19] [20] [21], and created systematic ways of controller design. Time-varying H_∞ control theory and approaches, also mostly on continuous time systems, are now gradually becoming a research topic in literature [22] [23] [24] [25]. They are largely based on the H_∞ control theory and methods for time invariant systems. Some theoretical results on the discrete time-varying system H_∞ control problem was derived by A. Halanay and V. Ionescu [26].

A recently developed approach to linear time invariant H_∞ control by Tsai and Postlethwaite [18], and Kimura [19] has been considered as a new trend in control engineering [27]. Their approach is based on two fundamental notions: the *chain scattering representation* and the *J-lossless factorization.* The chain scattering representation is an alternative way to represent a plant, and expresses the cascade structure of feedback systems. J-lossless factorization, or more precisely outor–(J_2, J_1)-lossless factorization, is a signed generalization of the outer-inner factorization to multiport systems, where J_1 and J_2 are called input and output signature operators we will discuss in Chapter 4. We have adopted this approach and put it in our framework and notation in order to develop the factorization theory and a systematic way of controller design for solving the H_∞ control problem for linear discrete time-varying systems.

1.2 Contributions

In this thesis, we consider (1) the identification problem, and (2) the H_∞ control problem for time-varying systems. We make extensive use of recently developed theory and methodology in systems and control engineering, which enable us to find solutions (to a certain extent) to these problems.

Our work is based on the following ideas:

1. Discrete time-varying system realization theory based on an efficient notation originally introduced and developed by Alpay, Dewilde and Dym [28] [29], and subsequently developed by Van der Veen [9].

2. The linear time invariant subspace model identification method, first introduced by Moonen, de Moor, Vandenberghe and Vandewalle [10] and further developed by Verhaegen and Dewilde [12].

3. Ensemble identification approach such as used by Kearney, Hunter [2] and MacNeil [3].

4. The solutions to the H_∞ problem based on chain scattering representations and J-lossless factorizations, as introduced by Tsai and Postlethwaite [18], and Kimura [19].

In the first part of the thesis, we consider the identification problem. The subspace model identification approach works directly with the input/output measurements. The main characteristic of this approach is the approximation of a subspace defined by a basis of the column or row space of a certain matrix derived from the measurements. The realization of the system is calculated from this basis and its complement. For time invariant systems, it suffices to consider only one input/output pair. By shifting the input/output sequences over several positions, each shift can be looked at as a repetition of the same experiment on different data. Hence, a single sequence of input/output data can be rearranged into an 'ensemble' for the estimation of the relevant subspaces of the system. For arbitrarily time-varying systems, however, an ensemble of input/output data can be obtained only by physically repeating the same experiment, assuming that each test exhibits the same time-varying behavior. In this thesis, we extend this ensemble subspace model identification method, and develop an algorithm to identify the state space models for arbitrarily time-varying systems, in particular periodically time-varying systems for which the repetition of the experiment is intrinsic. This algorithm belongs to the class of the MOESP (*MIMO Output – Error State Space* Model Identification) approaches which was first introduced by Verhaegen and Dewilde in [12]. This part of the work was presented at the ECC93 conference [30] and was published in [31].

The algorithm starts with an input/output ensemble and consists of several steps: 1. Determine a basis for the row space of an observability matrix from a QR factorization of an ensemble input/output observation by a singular value decomposition of part of the R factor. 2. Determine from the row space suitable values for the system matrices A and C up to similarity. 3. Determine the system matrices B and D up to similarity from the row space and the space perpendicular to the row space. 4. Determine the identified time interval. It is proved that with the algorithm, the estimated model is unbiased and consistent when the output noise is zero mean white.

The second part of the thesis deals with the H_∞ control problem for linear discrete time-varying systems. The two fundamental notions on which the method in [18] and [19] based are the chain scattering representation and the J-lossless factorization. For solving the discrete time-varying H_∞ control problem, we adopt this approach and put it in our framework and notation. Hence, lossless and J-lossless operator theory and chain scattering operator descriptions for time-varying systems forms the basis for the solution of the H_∞ control problem for time-varying systems. We first extend the factorization theory for upper operators in [9] to the general operator case, and subsequently develop a time-varying J-lossless factorization theory and algorithm.

For a general operator T, we consider the J-lossless factorization in two steps: (1) anticausal J-lossless factorization and (2) causal J-lossless factorization. In the first step, T is factorized into a bounded upper (causal) operator T_1 and an lower J-lossless (anticausal) operator Θ_a as $T = T_1\Theta_a$. In the second step, T_1 is further factorized into an invertible and outer operator T_o, and an upper J-lossless operator Θ_c as $T_1 = T_o\Theta_c$. Then $T = T_o\Theta_c\Theta_a = T_o\Theta$ with T_o invertible and outer, and $\Theta = \Theta_c\Theta_a$ J-lossless. A time-varying *Lyapunov equation* is needed to solve the first step and a set of Lyapunov-like equations are needed to solve the second step. We show that the second step is equivalent to solving a time-varying *Riccati equation* (or a Riccati recursion). The existence of the factorization is formulated in terms of the positive definiteness of the solutions to these Lyapunov and Riccati equations.

With the approach in [18] and [19], if the chain scattering representation exists and admits a J-lossless factorization, the H_∞ problem has solutions. The class of suboptimal controllers is defined in terms of a homographic transformation of the inverse of the outer operator which is obtained from this factorization and an arbitrary strictly contractive upper operator. The connection of such a controller to the system is equivalent to the homographic transformation of the J-lossless operator and the arbitrary strictly contractive upper operator. Using the strict contractiveness property of such a transformation, it follows that for any such solution the closed loop system is stable and the H_∞ norm is smaller than the prescribed bound.

1.3 Outline of the thesis

The organization of this thesis is as follows.

In Chapter 2, the relevant notation, definitions and theorems based on [9] for time-varying systems are introduced and extended.

We discuss a subspace model identification algorithm for time-varying systems in Chapter 3. We pay particular attention to periodically time-varying systems because the repetition of the experiment, which is a disadvantage of the ensemble method, is intrinsic in this case. An application of the developed algorithm to the identification of a multivariable multirate sampled data system demonstrates the usefulness of this scheme. The unbiasedness and consistency of the algorithm to zero-mean white noise on the output measurements are shown from this example.

The definition of lossless and J-lossless operators, their properties, and related theorems are considered in Chapter 4. These two kinds of operators are important in system and control engineering because of their useful and elegant properties. In this chapter, we discuss the necessary and sufficient conditions for operators to be lossless or J-lossless in terms of properties of their realizations. In our application, we are mostly interested in J-lossless operators. An important property, namely the strict contractiveness of the homographic transformation of a J-lossless operator and a strictly contractive upper operator, is proved in an operator setting. This property is the basis of our solution to the H_∞ control problem in Chapter 6.

In Chapter 5. we consider linear discrete time-varying outer–J-lossless (conjugated J-lossless–outer) factorizations in two different descriptions: an operator setting and a causal state equation setting. In the first case, we assume that a dichotomy of the time-varying system is known and given by the sum of a bounded lower operator and a bounded upper operator. The factorization is subsequently formulated in two steps: an anticausal J-lossless factorization and a causal J-lossless factorization. The second setting is a special case for which the causal form of the time-varying system exists. We extend the operator conjugation approach of [19] to the time-varying context and consider J-lossless factorizations by this method. This leads to a time-varying Riccati equation which is shown to be the same as the Riccati equation obtained with the first setting for a stable system. The explicit form of the solution of the Riccati equation and the convergence of the Riccati recursion is discussed.

The solution to the linear discrete time-varying H_∞ control problem is considered in Chapter 6. This solution is based on the chain scattering representation and outer–J-lossless factorization of Chapter 5. For a system, if its chain scattering representation exists and admits an outer–J-lossless factorization, the H_∞ control problem has solutions. The set of admissible controllers is formulated in terms of a homographic transformation of the inverse of the outer factor which is obtained from the factorization and an arbitrary strictly contractive causal system. The J-lossless factorization theory and algorithm of Chapter 5 are the main tools for this approach. In the case that the causal state equation description of a system exists, it is shown that two time-varying Riccati equations, one forward and one backward, describe the solution of the factorization of a general system. The existence of the factorization is determined by the definiteness of the solutions of these Riccati equations. The approximation of the H_∞ norm of a transfer operator is discussed for a special case. With the controllers that are thus designed, the closed loop system is stable and strictly contractive up to a known bound.

In the research of H_∞ control, there are two classes of approaches. One is the approximation method and another is the interpolation method. The method in [18] and [19] belongs to the second class. If the chain scattering representation is invertible, the H_∞ control problem is equivalent to the *interpolation problem.* The solvability of the problem is determined by the definiteness of a Pick operator which is related to the interpolation data. We discuss this topic in Chapter 7.

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Chapter 2

Time-Varying System Theory

Time-varying system and realization theory developed in [1] by Van der Veen plays an important role in this thesis. This theory was based on a notation which was originally introduced by Alpay, Dewilde in [2] and subsequently in [3] by Alpay, Dewilde and Dym. Although [1] is mainly concerned with causal stable time-varying systems, we can extend the theory to anticausal systems and, more generally, mixed causal and anticausal systems.

2.1 Non-uniform vectors, transfer operators and shift operators

Let us define an infinite vector (written as row vector) sequence u as

$$u = [u_i]_{-\infty}^{+\infty} = [\ldots u_{-1}, \boxed{u_0}, u_1 \ldots] \tag{2.1}$$

with dimensions defined by the *index sequence* $M = [M_i]_{-\infty}^{+\infty} = [\ldots M_{-1}, \boxed{M_0}, M_1 \ldots]$ with M_i a finite integer. If the M_i are not all the same, we say that u is a *non-uniform vector sequence.* A non-uniform vector sequence space $\mathcal{M}$ is defined by the set of vector sequences u with the index sequence M as the Cartesian product of the $\mathcal{M}_i$:

$$\mathcal{M} = \cdots \times \mathcal{M}_{-1} \times \boxed{\mathcal{M}_0} \times \mathcal{M}_1 \times \cdots \subset \mathbb{C}^N$$

where $\mathcal{M}_i \subset \mathbb{C}^{N_i}$. (The box indicates the entry at time point 0).

The norm of a non-uniform sequence is the standard 2-norm (vector norm) defined in terms of the usual inner product:

$$u = [u_i]_{-\infty}^{+\infty} : \qquad ||u||_2^2 = (u, u) = \sum_{-\infty}^{+\infty} ||u_i||_2^2$$

The space of non-uniform sequences in $\mathcal{M}$ with *finite 2-norm* is denoted by $\ell_2^{\mathcal{M}}$:

$$\ell_2^{\mathcal{M}} = \{u \in \mathcal{M} : ||u||_2 < \infty\}$$

Let $\mathcal{M}$ and $\mathcal{N}$ be spaces of sequences corresponding to index sequences M, N. Let us consider a system which maps the input signal vector sequence $u \in \mathcal{M}$ into an output signal vector sequence $y \in \mathcal{N}$. This mapping can be described in terms of an operator, called *transfer operator*, of the system. The mapping is described by the notation:

$$T : \mathcal{M} \rightarrow \mathcal{N}, \qquad y = uT$$

The block entry $T_{i,j}$ is an $M_i \times N_j$ matrix with i and j the position indices as same as in a normal matrix. If an operator maps signals from $\ell_2^{\mathcal{M}}$ to $\ell_2^{\mathcal{N}}$, the operator is a *bounded operator*. The set of bounded operators defines an operator space $\mathcal{X}$, that is:

$$\mathcal{X} := \{T : \ell_2^{\mathcal{M}} \longrightarrow \ell_2^{\mathcal{N}} | T \ \ is \ \ bounded\}$$

$\mathcal{X}$ is a *Hilbert space.* A Hilbert space is an inner product space which is complete, relative to the metric induced by the inner product. In $\mathcal{X}$, we define the space of the *bounded upper operators*

$$\mathcal{U}(\mathcal{M},\mathcal{N}) := \{T \in \mathcal{X}(\mathcal{M},\mathcal{N}) : \quad T_{ij} = 0 \quad (i > j)\}$$

the space of the *bounded lower operators*

$$\mathcal{L}(\mathcal{M},\mathcal{N}) := \{T \in \mathcal{X}(\mathcal{M},\mathcal{N}) : \quad T_{ij} = 0 \quad (i < j)\}$$

and the space of *bounded diagonal operators*

$$\mathcal{D} := \mathcal{U} \cap \mathcal{L}$$

Let $T \in \mathcal{X}$. The *infinite norm* of T, denoted by $||T||_\infty$ is defined as:

$$||T||_\infty := Sup\{||xT||_2, ||x||_2 = 1\}$$

For an operator $T \in \mathcal{X}$, if $TT^* = I$, then $||T||_\infty = 1$ and if $TT^* < I$, then $||T||_\infty < 1$, where T^* means the adjoint of T and $TT^* < I$ means that $I - TT^*$ is positive definite.

If $T \in \mathcal{X}$ and $\|T\|_\infty < 1$, we say T is *strictly contractive.* If T is strictly contractive and also in $\mathcal{U}$, we say T is an H_∞ *operator.*

$$H_\infty := \{T : \quad \|T\|_\infty < 1 \quad and \quad T \in \mathcal{U}\}$$

Based on the definition of bounded operator space $\mathcal{X}(\mathcal{M},\mathcal{N})$, we define the *Hilbert-Schmidt space* $\mathcal{X}_2(\mathcal{M},\mathcal{N})$ which is a subspace of $\mathcal{X}$ and also bounded in *Hilbert-Schmidt norm.* The Hilbert-Schmidt norm is defined as

$$\|A\|_{HS}^2 = \sum_{i,j} \|A_{ij}\|_2^2 \qquad (A \in \mathcal{X}(\mathcal{M},\mathcal{N}))$$

The space $\mathcal{X}_2$ is then defined by:

$$\mathcal{X}_2(\mathcal{M},\mathcal{N}) := \{A \in \mathcal{X} : \|A\|_{HS}^2 < \infty\}$$

Related spaces in $\mathcal{X}_2$ are upper, lower and diagonal Hilbert-Schmidt spaces given by:

$$\begin{aligned} \mathcal{U}_2 &:= \mathcal{X}_2 \cap \mathcal{U} \\ \mathcal{L}_2 &:= \mathcal{X}_2 \cap \mathcal{L} \\ \mathcal{D}_2 &:= \mathcal{X}_2 \cap \mathcal{D} \end{aligned}$$

Let X be a bounded operator. The *spectral radius* of X is defined as

$$r(X) := \lim_{n\to\infty} \| X^n \|^{1/n}$$

It is well known that if $r(X) < 1$, then an inverse of the operator $(I - X)$ exists and is given by the *Neumann expansion*

$$(I - X)^{-1} = I + X + X^2 + \cdots$$

We define the *shift operator* $\mathbf{Z}$ as

$$\mathbf{Z} := \begin{bmatrix} \ddots & \ddots & & 0 & \\ & 0 & I_{n_0} & & \\ & & \boxed{0} & I_{n_1} & \\ & 0 & & \ddots & \ddots \end{bmatrix} \tag{2.2}$$

where I_{n_0} is the $n_0 \times n_0$ identity matrix. Note that $\mathbf{Z}^{-1} = \mathbf{Z}^*$. Then, multiplying $\mathbf{Z}^{-1}$ on the right-hand side of a vector sequence as in (2.1)produces a one step forward shift:

$$u\mathbf{Z}^{-1} = [\ldots u_{-1}, u_0, \boxed{u_1} \ldots] \tag{2.3}$$

2.2 State space model for time-varying systems

Let us consider a time-varying *causal system* described by:

$$x_{k+1}^c = x_k^c A_k^c + u_k B_k^c \tag{2.4}$$
$$y_k^c = x_k^c C_k^c + u_k D_k^c \tag{2.5}$$

and an *anticausal system* by:

$$x_{k-1}^a = x_k^a A_k^a + u_k B_k^a \tag{2.6}$$
$$y_k^a = x_k^a C_k^a + u_k D_k^a \tag{2.7}$$

where the superscripts 'c' and 'a' stand for causal and anticausal. At time k, $x_k^c \in \mathbb{R}^{n_k^c}$ is the state of the causal system, $x_k^a \in \mathbb{R}^{n_k^a}$ the state of the anticausal system, $u_k \in \mathbb{R}^{m_k}$ the input and $y_k^c, y_k^a \in \mathbb{R}^{l_k}$ the outputs. All these are row vectors with time-varying dimensions in general. If we consider a *general time-varying system* which is the sum of the systems (2.4)–(2.5) and (2.6)–(2.7), then this system has states $x_k = [\ x_k^c \quad x_k^a\ ]$ and output $y_k = y_k^c + y_k^a$.

Definition 2.1 *For a time-varying causal system as (2.4)–(2.5), a system state space representation* $(A_k'^c, B_k'^c, C_k'^c, D_k'^c)$ *is said to be* similarly equivalent *to* $(A_k^c, B_k^c, C_k^c, D_k^c)$, *denoted as* $(A_k^c, B_k^c, C_k^c, D_k^c) \xrightarrow{T_k} (A_k'^c, B_k'^c, C_k'^c, D_k'^c)$, *if there exist a transformation* $T_k \in R^{n_k \times n_k}$ *such that* T_k *and* T_k^{-1} *are bounded for all* $k \in \mathbb{Z}$ *and* $(A_k'^c, B_k'^c, C_k'^c, D_k'^c)$ *satisfies*

$$\begin{bmatrix} A_k'^c & C_k'^c \\ B_k'^c & D_k'^c \end{bmatrix} = \begin{bmatrix} T_k & 0 \\ 0 & I_{m_k} \end{bmatrix} \begin{bmatrix} A_k^c & C_k^c \\ B_k^c & D_k^c \end{bmatrix} \begin{bmatrix} T_{k+1}^{-1} & 0 \\ 0 & I_{l_k} \end{bmatrix} \tag{2.8}$$

□

The definition of *similarity transformation* of a realization of a time-varying anticausal system is similar to time-varying causal systems.

Let us put each of the system matrices, $A_k^c, B_k^c, C_k^c, D_k^c$, for $k = \cdots, -1, 0, 1, 2, \cdots$, into infinite diagonal operators, A_c for example,

$$A_c = \begin{bmatrix} \ddots & & & \\ & A_{-1}^c & & 0 \\ & & \boxed{A_0^c} & \\ & 0 & & A_1^c \\ & & & & \ddots \end{bmatrix}$$

B_c, C_c, D_c, A_a, B_a, C_a and D_a are defined in a similar way.

For the A-operator of a causal system, A_c for example, we define $\ell_{A_c} := r(A_c\mathbf{Z})$; For the A-operator of an anticausal system, A_a for example, we define $\ell_{A_a} := r(A_a\mathbf{Z}^{-1})$.

For a causal system, it is *causally stable* if $\ell_{A_c} < 1$ and *causally unstable* if $\ell_{A_c} > 1$. For an anticausal system, it is *anticausally stable* if $\ell_{A_a} < 1$ and *anticausally unstable* if $\ell_{A_a} > 1$. From now on in this chapter, we consider the case $\ell_{A_c} < 1$ and $\ell_{A_a} < 1$ except that other cases are specified.

With the shift operation as in (2.3), the system realization (2.4)–(2.5) can be reformulated, in a global sense, into

$$X_c\mathbf{Z}^{-1} = X_cA_c + UB_c \tag{2.9}$$
$$Y_c = X_cC_c + UD_c \tag{2.10}$$

by considering the generalized inputs $U \in \mathcal{X}_2^{\mathcal{M}}$ for which the ith row, denoted by U_i, is in $\ell_2^{\mathcal{M}}$ and outputs $Y_c \in \mathcal{X}_2^{\mathcal{N}}$ for which the ith row denoted by Y_i, is in $\ell_2^{\mathcal{N}}$. U then can be explained as a stack of independent sequence $\{U_i\}$ in $\ell_2^{\mathcal{M}}$. Y_c can be explained similarly. The subscript 'c' stands for causal. With $\ell_{A_c} < 1$, $X_c = UB_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1} \in \mathcal{X}_2^{\mathcal{B}}$ with $\mathcal{B}$ the state vector sequence space. System realization (2.6)–(2.7) can be reformulated into

$$X_a\mathbf{Z} = X_aA_a + UB_a \tag{2.11}$$
$$Y_a = X_aC_a + UD_a \tag{2.12}$$

The subscript 'a' stand for anticausal.

If $\ell_{A_c} < 1$, the input-output mapping of system (2.9)–(2.10) can be expressed by an upper operator as,

$$T_u = D_c + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c \tag{2.13}$$

and we say that $T_u \in \mathcal{U}$. The identity operator has the same size as $A_c\mathbf{Z}$. Later on, if we do not mention the size of an identity operator or an identity matrix, the indexes of them are determined by the correspondent operator or matrix.

In a similar way, if $\ell_{A_a} < 1$, the input-output mapping of the system (2.11) and (2.12) can be expressed by a lower operator as,

$$T_l = D_a + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a \tag{2.14}$$

and we say that $T_l \in \mathcal{L}$.

If an upper (or lower) operator has a state realization with state-space sequence $\mathcal{B}$ where $\mathcal{B}_k$ has finite dimension, then we say that it is a *locally finite operator.*

With $\ell_{A_c} < 1$ and $\ell_{A_a} < 1$, the sum of system (2.9)–(2.10) and (2.11)–(2.12) can be expressed as a bounded operator as

$$T = D + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a \tag{2.15}$$

where $D = D_c + D_a$ and $T \in \mathcal{X}$.

Note that T can be causally unstable (or causally unbounded) if A_a is invertible and the causal form of the system exists.

If an operator $T \in \mathcal{X}$ can be split into $T = T_u + T_l$ with $T_u \in \mathcal{U}$, $T_l \in \mathcal{L}$ and both locally finite, then we say that T is a locally finite operator.

Multiplying with $\mathbf{Z}$ on the left hand side and with $\mathbf{Z}^*$ on the right hand side of a diagonal operator, A_c for example, the resulted diagonal matrix is a one step north-west shift of the original matrix. We then have:

$$\mathbf{Z}A_c\mathbf{Z}^* = \mathbf{Z}\begin{bmatrix} \ddots & & & \\ & A^c_{-1} & & \\ & & \boxed{A^c_0} & \\ & & & A^c_1 \\ & & & & \ddots \end{bmatrix}\mathbf{Z}^* = \begin{bmatrix} \ddots & & & \\ & A^c_0 & & \\ & & \boxed{A^c_1} & \\ & & & A^c_2 \\ & & & & \ddots \end{bmatrix}$$

$\mathbf{Z}A_c\mathbf{Z}^*$ is denoted as $A_c^{(-1)}$. In general, we write

$$A_c^{(k)} = \mathbf{Z}^{*k}A_c\mathbf{Z}^k$$

where $k = 0, \pm1, \pm2, ...$

With the global expression, equation (2.8) can be reformulated into

$$\begin{bmatrix} A_c' & C_c' \\ B_c' & D_c' \end{bmatrix} = \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} A_c & C_c \\ B_c & D_c \end{bmatrix}\begin{bmatrix} T^{-(-1)} & 0 \\ 0 & I \end{bmatrix}$$

with T an invertible diagonal operator.

Next, we consider the cascade connection of two bounded operators, $T_1 \in \mathcal{U}$ and $T_2 \in \mathcal{L}$. What is a state description of the resulting operator $T = T_1T_2$?

Lemma 2.1 *Let $T_1 \in \mathcal{U}$ and $T_2 \in \mathcal{L}$ be two locally finite bounded operators, where T_1 has a stable realization $\{A_1, B_1, C_1, D_1\}$ such that $T_1 = D_1 + B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}C_1$ and T_2 has a stable realization $\{A_2, B_2, C_2, D_2\}$ such that $T_2 = D_2 + B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$. Then,*

$$\begin{aligned} T = T_1T_2 = (D_1D_2 + B_1Y^{(-1)}C_2) + B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}(C_1D_2 + A_1Y^{(-1)}C_2) \\ +(D_1B_2 + B_1Y^{(-1)}A_2)\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2 \end{aligned} \tag{2.16}$$

where Y is the unique solution of the Lyapunov equation*:*

$$Y = A_1Y^{(-1)}A_2 + C_1B_2 \tag{2.17}$$

Proof:

$$\begin{aligned} T = T_1T_2 &= [D_1 + B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}C_1][D_2 + B_2\mathbf{Z}_*(I - A_2\mathbf{Z}_*)^{-1}C_2] \\ &= D_1D_2 + B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}C_1D_2 + D_1B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2 \\ &\quad +B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}C_1B_2(I - \mathbf{Z}^*A_2)^{-1}\mathbf{Z}^*C_2 \end{aligned} \tag{2.18}$$

Suppose that $(I - A_1\mathbf{Z})^{-1}C_1B_2(I - \mathbf{Z}^*A_2)^{-1}\mathbf{Z}^*$ can be separated into:

$$(I - A_1\mathbf{Z})^{-1}C_1B_2(I - \mathbf{Z}^*A_2)^{-1}\mathbf{Z}^* = (I - A_1\mathbf{Z})^{-1}X + Y(I - \mathbf{Z}^*A_2)^{-1}\mathbf{Z}^*$$

Multiplying both sides of the above equation by $(I - A_1\mathbf{Z})$ on the left and by $\mathbf{Z}(I - \mathbf{Z}^*A_2)$ on the right, we obtain:

$$C_1B_2 = X\mathbf{Z}(I - \mathbf{Z}A_2) + (I - A_1\mathbf{Z})Y = (X - A_1Y^{(-1)})\mathbf{Z} + (Y - XA_2)$$

Because C_1B_2 is diagonal, we have:

$$\begin{aligned} X &= A_1Y^{(-1)} \\ C_1B_2 &= Y - XA_2 \end{aligned}$$

Substituting the first relation above into the second we get: $C_1B_2 = Y - A_1Y^{(-1)}A_2$. This is the relation given by equation (2.17). It has a unique solution in the case both A_1 and A_2 are stable and the solution is given by:

$$Y = C_1B_2 + \sum_{i=1}^{\infty} A_1A_1^{(-1)} \cdots A_1^{(-i+1)}C_1^{(-i)}B_2^{(-i)}A_2^{(-i+1)}A_2^{(-i+2)} \cdots A_2$$

Then, T can be written as:

$$\begin{aligned} T = D_1D_2 + B_1\mathbf{Z}(I - A_2\mathbf{Z})^{-1}(C_1D_2 + A_1Y^{(-1)}C_2) + D_1B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2 \\ +B_1Y^{(-1)}(I - A_2\mathbf{Z}^*)^{-1}C_2 \end{aligned}$$

Since

$$B_1Y^{(-1)}(I - A_2\mathbf{Z}^*)^{-1}C_2 = B_1Y^{(-1)}C_2 + B_1Y^{(-1)}A_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$$

Substituting this relation into equation (2.18) we get equation (2.16). □

Conversely, if there is a locally finite operator $T \in \mathcal{X}$ which has a realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$, we can always find a factorization $T = T_1T_2$ with $T_1 \in \mathcal{U}$ (or $T_1 \in \mathcal{L}$) with a realization $\{A_1, B_1, C_c, D_c\}$ (or $\{A_2, B_2, C_a, D_a\}$ respectively) for some diagonal operators C_c and D_c (or C_a and D_a), and an operator $T_2 \in \mathcal{L}$ (or $T_2 \in \mathcal{U}$) with a realization $\{A_2, B_a, C_2, D_a\}$ (or $\{A_1, B_c, C_1, D_c\}$) for some diagonal operator B_a and D_a (or B_c and D_c) such that $T_1T_2 = T$. There choice of C_c, D_c, B_a, D_a and Y is with constraints:

$$\begin{aligned}
Y &= A_1Y^{(-1)}A_2 + C_cB_a \\
C_1 &= C_cD_a + A_1Y^{(-1)}C_2 \\
B_2 &= D_cB_a + B_1Y^{(-1)}C_2 \\
D &= D_cD_a + B_1Y^{(-1)}C_2
\end{aligned}$$

Because there are more unknowns than constraints, the choice of the realization is with certain freedom.

2.3 Reachability and observability of time-varying systems

A *Hermitian operator* A in $\mathcal{X}$ is *semipositive*, notation $A \geq 0$, if for all $u \in \ell_2^{\mathcal{M}}$,

$$(uA, u) \geq 0$$

A is *positive*, notation $A > 0$, if for all $u \in \ell_2^{\mathcal{M}}$,

$$(uA, u) > 0$$

A is *uniformly strictly positive*, notation $A \gg 0$, if there is an $\varepsilon > 0$ such that, for all $u \in \ell_2^{\mathcal{M}}$,

$$(uA, u) \geq \varepsilon(u, u)$$

If $A > 0$, A is invertible but not necessarily bounded. In the case $A \gg 0$, A is boundedly invertible.

Definition 2.2 *Let a realization of a time-varying causal system as (2.9)–(2.10) be* $\{A, B, C, D\}$ *with* $\ell_A < 1$. *Define:*

$$\mathcal{C} = \begin{bmatrix} B^{(1)} \\ B^{(2)}A^{(1)} \\ B^{(3)}A^{(2)}A^{(1)} \\ \vdots \end{bmatrix} \tag{2.19}$$

$\mathcal{C}$ *is called the* reachability *(or controllability)* operator *and* $P = \mathcal{C}^*\mathcal{C}$ *is called the* reachability Gramian. *The realization is* reachable *if* $P > 0$*; the realization is* uniformly reachable *if* $P \gg 0$. *If* $P = 0$*, the realization is called unreachable.* □

It is easy to conclude that if $B = 0$, then realization is unreachable.

Definition 2.3 *Let a realization of a time-varying causal system as (2.9)–(2.10) be* $\{A, B, C, D\}$ *with* $\ell_A < 1$. *Define:*

$$\mathcal{O} = [\, C \quad AC^{(-1)} \quad AA^{(-1)}C^{(-2)} \quad \cdots \,] \tag{2.20}$$

$\mathcal{O}$ *is called the* observability operator *and* $Q = \mathcal{O}\mathcal{O}^*$ *is called the* observability Gramian. *The realization is* observable *if* $Q > 0$*; the realization is* uniformly observable *if* $Q \gg 0$. *If* $Q = 0$*, the realization is called unobservable.* □

If $C = 0$, then the realization is unobservable.

If the realization is both reachable and observable, it is called a *minimal realization*; if the realization is both uniformly reachable and uniformly observable, it is a *uniformly minimal realization* [1].

Proposition 2.1 *[1] Let* $\{A, B, C, D\}$ *be a realization of a time-varying causal system with* $\ell_A < 1$. *The reachability Gramian* P *satisfies the Lyapunov equation*

$$A^*PA + B^*B = P^{(-1)} \tag{2.21}$$

□

Proposition 2.2 *[1] Let* $\{A, B, C, D\}$ *be a realization of a time-varying causal system with* $\ell_A < 1$. *The observability Gramian* Q *satisfies the Lyapunov equation*

$$AQ^{(-1)}A^* + CC^* = Q \tag{2.22}$$

□

In [1] and [5], the definition of reachability and observability for a time varying causal system are slightly different from Definition 2.2 and Definition 2.3. Let us consider the *causal form* of a discrete time-varying system described by the state equation:

$$XZ^{-1} = XA + UB \tag{2.23}$$

$$Y = XC + UD \tag{2.24}$$

In general, the spectral radius of the diagonal operator A is not necessarily smaller than one. Then Definition 2.2 and Definition 2.3 do not make sense any more. In this case, a finite reachability operator $\mathcal{C}_\delta$ is defined as,

$$\mathcal{C}_\delta = \begin{bmatrix} B^{(1)} \\ B^{(2)}A^{(1)} \\ \vdots \\ B^{(\delta)}A^{(\delta-1)}\dots A^{(1)} \end{bmatrix} \tag{2.25}$$

The smallest integer value of δ for which the condition $P_\delta = \mathcal{C}_\delta^*\mathcal{C}_\delta > 0$ holds is called δ_c and is indicated as the *reachability index* (similar to the linear time invariant case [6]). The finite reachability Gramian is defined as $P_{\delta_c} = \mathcal{C}_{\delta_c}^*\mathcal{C}_{\delta_c}$. The system is finitely reachable if $P_{\delta_c} > 0$; the system is uniformly finitely reachable if $P_{\delta_c} \gg 0$.

The finite observability operator Q_{δ_o} is defined similarly. Note that P_{δ_c} and Q_{δ_o} do not satisfy the Lyapunov equations (2.21) and (2.22).

With the description of (2.23) and (2.24), the definitions of *stabilizability* and *detectability* can be defined as follows:

Definition 2.4 *Let a discrete time-varying system be described by equations (2.23) and (2.24). The system is stabilizable if there exists an operator $F \in \mathcal{D}$ such that $\ell_{A+FB} < 1$.*

□

The pair $[A, B]$ is called a stabilizable pair.

Definition 2.5 *Let a discrete time-varying system be described by equations (2.23) and (2.24). The system is detectable if $[A^*, C^*]$ is a stabilizable pair.* □

The pair $[A, C]$ is called a detectable pair.

The definitions of reachability and observability etc. of a realization of a time-varying anticausal stable system are similar to that of a time-varying causal stable system but with the shift south-east instead of north-west for the reachability operator and north-west instead of south-east for the observability operator. Next, we can define the reachability and observability of a general system as Eq. (2.15) as follows. A similar definition for a general system in the time invariant case can be found in [7].

Definition 2.6 *Let a realization of a time-varying system be* $\{A_c, B_c, C_c, A_a, B_a, C_a, D\}$ *with* $\ell_{A_c} < 1$, $\ell_{A_a} < 1$ *such that the transfer operator* $T \in \mathcal{X}$ *of the system is:*

$$T = D + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a \tag{2.26}$$

Define:

$$\mathcal{C}_c = \begin{bmatrix} B_c^{(1)} \\ B_c^{(2)}A_c^{(1)} \\ B_c^{(3)}A_c^{(2)}A_c^{(1)} \\ \vdots \end{bmatrix} \tag{2.27}$$

and

$$\mathcal{C}_a = \begin{bmatrix} B_a^{(-1)} \\ B_a^{(-2)}A_a^{(-1)} \\ B_a^{(-3)}A_a^{(-2)}A_a^{(-1)} \\ \vdots \end{bmatrix} \tag{2.28}$$

$\mathcal{C}_c$ *is called the reachability operator of the causal part of the system and* $\mathcal{C}_a$ *is called the reachability operator of the anticausal part of the system. Similarly,* $P_c = \mathcal{C}_c^*\mathcal{C}_c$ *is called the reachability Gramian of the causal part and* $P_a = \mathcal{C}_a^*\mathcal{C}_a$ *is called the reachability Gramian of the anticausal part. The realization is reachable if both* $P_c > 0$ *and* $P_a > 0$*; the realization is uniformly reachable if both* $P_c \gg 0$ *and* $P_a \gg 0$. □

Definition 2.7 *Let a realization of a time-varying system be* $\{A_c, B_c, C_c, A_a, B_a, C_a, D\}$ *with* $\ell_{A_c} < 1$, $\ell_{A_a} < 1$ *such that the transfer operator* $T \in \mathcal{X}$ *of the system is:*

$$T = D + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c + B_a(I - A_a\mathbf{Z}^*)^{-1}C_a \tag{2.29}$$

Define:

$$\mathcal{O}_c = [\ C_c \quad A_cC_c^{(-1)} \quad A_cA_c^{(-1)}C_c^{(-2)} \quad \cdots\ ] \tag{2.30}$$

and

$$\mathcal{O}_a = [\ C_a \quad A_a C_a^{(1)} \quad A_a A_a^{(1)} C_a^{(2)} \quad \cdots\ ] \tag{2.31}$$

$\mathcal{O}_c$ is called the observability operator of the causal part of the system and $\mathcal{O}_a$ is called the observability operator of the anticausal part of the system. Similarly, $Q_c = \mathcal{O}_c \mathcal{O}_c^*$ is called the observability Gramian of the causal part and $Q_a = \mathcal{O}_a \mathcal{O}_a^*$ is called the observability Gramian of the anticausal part. The realization is observable if both $Q_c > 0$ and $Q_a > 0$; the realization is uniformly observable if both $Q_c \gg 0$ and $Q_a \gg 0$. □

Until now, we have only defined reachability and observability operator of a discrete time-varying system in a global sense. The system matrices of a discrete time-varying system in a global sense are in the form of infinite diagonal operators. Reachability and observability operators are then in a block infinite diagonal form. Taking the causal observability operator in (2.30) as an example, the form is as,

$$\mathcal{O} = [\ C \quad AC^{(-1)} \quad AA^{(-1)} C^{(-2)} \quad \cdots\ ]$$

$$= \begin{bmatrix} \ddots & & & \ddots & & & \ddots & & & \cdots \\ & \ddots & & & \ddots & & & \ddots & & \cdots \\ & & \ddots & & & \ddots & & & \ddots & \cdots \\ & & & \ddots & & & \ddots & & \ddots & \cdots \end{bmatrix}$$

For some time instant t, the observability matrix can also be defined locally in the form:

$$\mathcal{O}_t = \begin{bmatrix} C_t & A_t C_{t+1} & A_t A_{t+1} C_{t+2} & \cdots \end{bmatrix} \qquad t = \cdots, -1, 0, 1, \cdots \tag{2.32}$$

and for the same reason, the reachability matrix at time instant t can be defined as

$$\mathcal{C}_t = \begin{bmatrix} B_{t-1} \\ B_{t-2} A_{t-1} \\ B_{t-3} A_{t-2} A_{t-1} \\ \vdots \end{bmatrix} \qquad t = \cdots, -1, 0, 1, \cdots \tag{2.33}$$

Local reachability and observability matrices can be constructed by picking up the corresponding t-th element from every diagonal block of the reachability and observability operators and putting them in the same position corresponding to the global form.

Then, reachability and observability Gramians for time instant t are defined in a similar way as in the global form and satisfy recursions as, reachability Gramian for example, $P_{t+1} = A_t^* P_t A_t + B_t^* B_t$. P_t is the T-th entry along the diagonal of P. If the system is uniformly reachable, then every local reachability Gramian is positive definite. If the system is uniformly observable, then every local observability Gramian is positive definite[5].

2.4 State spaces of operators

In [1], the input and output state space of an upper operator are defined. Extensions to a lower operators are given in [8]. We can further extend the definitions to a general bounded operator which belongs to $\mathcal{X}$. Let us begin with some useful definitions before we look at these spaces.

Subspace: A closed linear subset in a Hilbert space $\mathcal{H}$ is called a subspace.

A sequence $\{\phi_i\}_1^\infty$ of vectors of a Hilbert spaces $\mathcal{H}$ is called a basis of this space if every vector $f \in \mathcal{H}$ can be expanded in a unique way in a series

$$f = \sum_1^\infty \alpha_i \phi_i = \lim_{n\to\infty} \sum_1^n \alpha_i \phi_i$$

which converges in the norm of $\mathcal{H}$.

Left $\mathcal{D}$-invariant subspace: A subspace $\mathcal{H}$ in $\mathcal{X}_2$ is said to be left D invariant if $\mathbf{F} \in \mathcal{H} \Rightarrow D\mathbf{F} \in \mathcal{H}$ for any diagonal $D \in \mathcal{D}$, i.e. $D\mathcal{H} \subset \mathcal{H}$.

Left shift invariant subspace: A subspace $\mathcal{H}$ in $\mathcal{X}_2$ is said to be left $\mathbf{Z}^{-1}$-shift invariant if $F \in \mathcal{H} \Rightarrow \mathbf{Z}^{-1}F \in \mathcal{H}$, $\mathbf{Z}$ is the shift operator. A subspace $\mathcal{H}$ in $\mathcal{X}_2$ is said to be left $\mathbf{Z}$-shift invariant if $F \in \mathcal{H} \Rightarrow \mathbf{Z}F \in \mathcal{H}$.

Strong basis: Let $\mathcal{H}$ be a locally finite left $\mathcal{D}$-invariant subspace in $\mathcal{X}_2$ and $\mathbf{F}$ be a basis of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{D}_2\mathbf{F}$. If the Gramian operator $\Lambda_\mathbf{F} = \mathbf{P}_0(\mathbf{F}\mathbf{F}^*) \gg 0$, where $\mathbf{P}_0$ denotes the projection onto $\mathcal{D}$, then the basis is said to be a strong basis of $\mathcal{H}$.

The concept of spaces, subspaces and a basis of a subspace can be found in standard textbooks. For the original definition of strong basis, see [1].

Let $T \in \mathcal{U}$ be a *bounded causal operator.* Define:

$$\mathcal{K}(T) = \{U \in \mathcal{L}_2\mathbf{Z}^{-1}, \mathbf{P}(UT) = 0\} \tag{2.34}$$

$$\mathcal{H}(T) = \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathcal{U}_2 T^*) \tag{2.35}$$

$$\mathcal{H}_o(T) = \mathbf{P}(\mathcal{L}_2\mathbf{Z}^{-1}T) \tag{2.36}$$

$$\mathcal{K}_o(T) = \{Y \in \mathcal{U}_2, \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(YT^*) = 0\} \tag{2.37}$$

where $\mathbf{P}$ denotes the projection onto $\mathcal{U}_2$ and $\mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}$ denotes the projection onto $\mathcal{L}_2\mathbf{Z}^{-1}$.

$\mathcal{K}$ is called the *input null space.* It is a subspace which is left $\mathcal{D}$-invariantin $\mathcal{L}_2\mathbf{Z}^{-1}$[1].

$\mathcal{H}$ is called the *input state space.* It is a subspace which is left $\mathcal{D}$-invariant in $\mathcal{L}_2\mathbf{Z}^{-1}$. The closure of $\mathcal{H}$ [1] is the complement of $\mathcal{K}$ in $\mathcal{L}_2\mathbf{Z}^{-1}$, denoted as:

$$\overline{\mathcal{H}}(T) \oplus \mathcal{K}(T) = \mathcal{L}_2\mathbf{Z}^{-1} \tag{2.38}$$

where '$\oplus$' denotes the direct sum of orthogonal spaces. Conversely, $\overline{\mathcal{H}}(T)$, the closure of $\mathcal{H}$ and the orthogonal complement of $\mathcal{K}(T)$ in $\mathcal{L}_2\mathbf{Z}^{-1}$, is likewise written as:

$$\overline{\mathcal{H}}(T) = \mathcal{L}_2\mathbf{Z}^{-1} \ominus \mathcal{K}(T)$$

$\mathcal{H}_o$ is the *output state space.* $\mathcal{K}_o$ is the *output null space.* $\mathcal{H}_o$ and $\mathcal{K}_o$ are subspaces which are left $\mathcal{D}$-invariant in $\mathcal{U}_2$, $\mathcal{K}_o$ is the complement of $\overline{\mathcal{H}_o}$ in $\mathcal{U}_2$:

$$\overline{\mathcal{H}_o}(T) \oplus \mathcal{K}_o(T) = \mathcal{U}_2 \tag{2.39}$$

The null and state spaces satisfy the following relations:

$$\begin{aligned} \mathbf{P}(\mathcal{K}T) &= 0 && (2.40)\\ \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathcal{K}_oT^*) &= 0 && (2.41)\\ \mathcal{H}_o &= \mathbf{P}(\overline{\mathcal{H}}T) && (2.42)\\ \mathcal{H} &= \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(\overline{\mathcal{H}_o}T^*) && (2.43) \end{aligned}$$

Let $T \in \mathcal{U}$ have a minimal realization $\{A, B, C, D\}$ which is locally finite with $\ell_A < 1$ such that $T = D + B\mathbf{Z}(I - A\mathbf{Z})^{-1}C$. Define:

$$\begin{aligned} \mathbf{F}^* &= B\mathbf{Z}(I - A\mathbf{Z})^{-1} && (2.44)\\ \mathbf{F}_o &= (I - A\mathbf{Z})^{-1}C && (2.45) \end{aligned}$$

In [1], the following relations for the reachability and observability Gramians are derived:

$$\begin{aligned} \mathbf{P}_0(\mathbf{F}\mathbf{F}^*) &= \mathcal{C}^*\mathcal{C} \triangleq \Lambda_F \in \mathcal{D} && (2.46)\\ \mathbf{P}_0(\mathbf{F}_o\mathbf{F}_o^*) &= \mathcal{O}\mathcal{O}^* \triangleq \Lambda_{F_o} \in \mathcal{D} && (2.47) \end{aligned}$$

$\mathbf{F}$ and $\mathbf{F}_o$ are the *strong basis representations* of the input and output state spaces, respectively. Because of the equivalent relation between $\mathbf{F}$ and $\mathcal{C}$, $\mathbf{F}$ in (2.44) is said to be the associated reachability operator. For the same reason, $\mathbf{F}_o$ in (2.45) is said to be the associated observability operator. We have the following proposition.

Proposition 2.3 *[1] Let $T \in \mathcal{U}$ and $\{A, B, C, D\}$ be a bounded locally finite realization of T such that $T = D + B\mathbf{Z}(I - A\mathbf{Z})^{-1}C$. Let $\mathbf{F}$ and $\mathbf{F}_o$ defined in Eqs. (2.44) and (2.45) be strong basis representations of the input and output state spaces. Then $\mathcal{H}_o \subset \mathcal{D}_2\mathbf{F}_o$ and $\mathcal{H} \subset \mathcal{D}_2\mathbf{F}$.*

If the realization is uniformly reachable, then $\mathcal{H}_o = \mathcal{D}_2\mathbf{F}_o$.

If the realization is uniformly observable, then $\mathcal{H} = \mathcal{D}_2\mathbf{F}$. □

In a similar way, we define the null space and state space etc. for lower operators.

Let $T \in \mathcal{L}$ be a *bounded anticausal operator.* Define:

$$\mathcal{K}^a(T) = \{U = \mathcal{U}_2\mathbf{Z}, \mathbf{P}_{\mathcal{L}_2}(UT) = 0\} \tag{2.48}$$
$$\mathcal{H}^a(T) = \mathbf{P}_{\mathcal{U}_2\mathbf{Z}}(\mathcal{L}_2T^*) \tag{2.49}$$
$$\mathcal{H}^a_o(T) = \mathbf{P}_{\mathcal{L}_2}(\mathcal{U}_2\mathbf{Z}T) \tag{2.50}$$
$$\mathcal{K}^a_o(T) = \{Y \in \mathcal{L}_2, \mathbf{P}_{\mathcal{U}_2\mathbf{Z}}(YT^*) = 0\} \tag{2.51}$$

$\mathcal{K}^a$ and $\mathcal{H}^a$ are input null space and input state space of T. The superscript 'a' stands for anticausal. They are left $\mathcal{D}$-invariant spaces in $\mathcal{U}_2\mathbf{Z}$. $\mathcal{K}^a$ is the complement of $\overline{\mathcal{H}^a}$ in $\mathcal{U}_2\mathbf{Z}$:

$$\overline{\mathcal{H}^a} \oplus \mathcal{K}^a = \mathcal{U}_2\mathbf{Z}$$

$\mathcal{K}^a_o$ and $\mathcal{H}^a_o$ are output null space and output state space of T. They are left $\mathcal{D}$-invariant space in $\mathcal{L}_2$. $\mathcal{K}^a_o$ is the complement of $\overline{\mathcal{H}^a_o}$ in $\mathcal{L}_2$:

$$\overline{\mathcal{H}^a_o} \oplus \mathcal{K}^a_o = \mathcal{L}_2$$

Sometimes we title these spaces for a lower operator 'anticausal' to distinguish them from those defined for an upper operator.

The null and state spaces of a lower operator satisfy the following relations:

$$\mathbf{P}_{\mathcal{L}_2}(\mathcal{K}^aT) = 0 \tag{2.52}$$
$$\mathbf{P}_{\mathcal{U}_2\mathbf{Z}}(\mathcal{K}^a_oT^*) = 0 \tag{2.53}$$
$$\mathcal{H}^a_o = \mathbf{P}_{\mathcal{L}_2}(\overline{\mathcal{H}}^aT) \tag{2.54}$$
$$\mathcal{H}^a = \mathbf{P}_{\mathcal{U}_2\mathbf{Z}}(\overline{\mathcal{H}_o}^aT^*) \tag{2.55}$$

where $\mathbf{P}_{\mathcal{L}_2}$ denotes the projection onto $\mathcal{L}_2$ and $\mathbf{P}_{\mathcal{U}\mathbf{Z}}$ onto $\mathcal{U}\mathbf{Z}$.

Let $T \in \mathcal{L}$ have a minimal realization $\{A, B, C, D\}$ with $\ell_A < 1$ such that $T = D + B\mathbf{Z}^*(I - A\mathbf{Z}^*)^{-1}C$, Define:

$$\mathbf{F}^{a*} = B\mathbf{Z}^*(I - A\mathbf{Z}^*)^{-1} \tag{2.56}$$

$$\mathbf{F}^a_o = (I - A\mathbf{Z}^*)^{-1}C \tag{2.57}$$

We can derive that:

$$\mathbf{P}_0(\mathbf{F}^a\mathbf{F}^{a*}) = \mathcal{C}_a\mathcal{C}_a^*$$

$$\mathbf{P}_0(\mathbf{F}^a_o\mathbf{F}^{a*}_o) = \mathcal{O}_a^*\mathcal{O}_a$$

where $\mathcal{C}_a$ and $\mathcal{O}_a$ are defined as same as (2.28) and (2.31) respectively, and $\mathbf{P}_0$ denotes the projection onto the diagonal. $\mathbf{F}^a$ and $\mathbf{F}^a_o$ are strong basis representations of the input and output state spaces of T, respectively. A dual of Proposition 2.3 is as follows.

Proposition 2.4 *Let $T \in \mathcal{L}$ and $\{A, B, C, D\}$ be a bounded locally finite realization of T. Define $\mathbf{F}^a$ and $\mathbf{F}^a_o$ as in equations (2.56) and (2.57). Then $\mathcal{H}^a_o \subset \mathcal{D}_2\mathbf{F}^a_o$ and $\mathcal{H}^a \subset \mathcal{D}_2\mathbf{F}^a$.*

If the realization is uniformly reachable, then $\mathcal{H}^a_o = \mathcal{D}_2\mathbf{F}^a_o$.

If the realization is uniformly observable, then $\mathcal{H}^a = \mathcal{D}_2\mathbf{F}^a$. □

Let T be a locally finite bounded operator which can be separated into causal and an anticausal part. With the definition of state spaces and null spaces for causal operators and anticausal operators, we have the next corollary.

Corollary 2.1 *Let $T \in \mathcal{X}$ be an operator of the form:*

$$T = D + B_1(\mathbf{Z}^* - A_1)^{-1}C_1 + B_2(\mathbf{Z} - A_2)^{-1}C_2$$

where $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ is a minimal realization of T. Define $T_c = D + B_1(\mathbf{Z}^* - A_1)^{-1}C_1 \in \mathcal{U}$ and $T_a = B_2(\mathbf{Z} - A_2)^{-1}C_2 \in \mathcal{L}\mathbf{Z}^{-1}$. Then, we have

$$\begin{array}{rclrcl}
\mathcal{H}(T) & = & \mathcal{H}(T_c) & \mathcal{K}(T) & = & \mathcal{K}(T_c) \\
\mathcal{H}_o(T) & = & \mathcal{H}(T_c) & \mathcal{K}_o(T) & = & \mathcal{K}_o(T_c) \\
\mathcal{H}^a(T) & = & \mathcal{H}^a(T_a) & \mathcal{K}^a(T) & = & \mathcal{K}^a(T_a) \\
\mathcal{H}^a_o(T) & = & \mathcal{H}^a_o(T_a) & \mathcal{K}^a_o(T) & = & \mathcal{K}^a_o(T_a)
\end{array}$$

Proof: The proof is straightforward. □

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Chapter 3

A class of Subspace Model Identification Algorithms to Identify Discrete Time-Varying Systems

3.1 Introduction

Linear time-varying system identification techniques have received much attention recently [1], [2]. Although a lot of effort has been devoted to the development of identification schemes to identify linear time-invariant dynamical systems, most systems demonstrate time-varying and/or nonlinear behavior in reality. Since it is common practice in system engineering to treat non-linear systems as linear time-varying systems [3], the identification of linear time-varying systems should be an important topic in system identification.

As outlined in [2] and [4], there are two classes of approaches to identify linear time-varying systems: (1) methods working with a single time sequence of input and output quantities such as quasi-time-invariant approach, recursive identification schemes equipped with a mechanism to 'forget' the past and recursive functional series modeling estimation [5]; (2) methods working with an ensemble of input and output time sequences. The first class of time-varying system identification methods requires a priori knowledge about a system's structure and the form of its time-variation and/or the change in the dynamics of the underlying system occurs at a rate comparable to the sampling rate. If these conditions are not available, these methods are inappropriate to estimate accurate models [5] [6]. When accurate models are required in these circumstances, a possible solution can be found via the second class of methods, ensemble methods. Ensemble methods enable us to identify linear time-varying systems with changing dynamics over a time scale shorter than that of the dynamics themselves, and/or with no priori knowledge about the system's behavior or the form of its time-variation. The key consideration of ensemble methods is that an ensemble of input and output time sequences, each exhibiting the same underlying time-varying behavior, can provide enough information about an unknown time-varying system. This point was also supported by the research of B. Widrow [7] and M. Neidzwieki [5], in whose work it is recommended that multiple observations must be employed to identify a time-varying system accurately when a priori knowledge about the time-variation of the system dose not exist. A similar need was established in the analysis of identification problems for biomedical systems exhibiting fast changes in their dynamics [2] [8] [6]. In [6], it was experimentally verified that identification schemes of recursive type equipped with exponential forgetting were not able to handle such fast changing dynamics. Ensemble methods have not been used frequently for the possible reasons of the lack of algorithms and the practical difficulties associated with acquiring, storing and analyzing the large data sets.

Nevertheless, some applications of ensemble identification were reported by Kearney et al. [4] [8], MacNeil et al [2] and Yu and Verhaegen [6] in biomedical engineering where they tried to obtain more accurate models. Other possible applications of ensemble methods include the identification of non-linear systems operating along a particular trajectory, such as a robot arm executing repetitive manoeuvres, and periodically time-varying systems. With the latter class of systems, the collection of an ensemble of i/o sequences each exhibiting the same time-varying behavior is intrinsic.

In [4], [8] and [2], a numerical scheme was proposed to identify a time-varying Finite Impulse Response (FIR) model from an ensemble I/O data. A FIR model often needs many parameters especially in time-varying cases. As a result, the estimated model is not accurate for finite number of data. As in the time-invariant cases, the estimation of the impulse responses should be avoided when the underlying system is marginally stable [9] [10].

Subspace model identification (SMI) methods are viewed as a better alternative to the method mentioned above. First of all, a state space model has a more compact form than the FIR model. More important, state space models are widely used in system theory and control. In [1], a linear time-varying state space model has been identified from the so called generalized Hankel matrix in time-varying context. Despite of the fact that this is a two step procedure, the determination of the accurate impulse responses, which form the Hankel matrices, is still a difficult task. Secondly, SMI methods have the advantage that the estimated model is more accurate than the FIR model given a finite set of data. Thirdly, SMI methods do not experience the problems of convergence and sensitivity to initial estimates as iterative methods, and also do not experience problems when the data is measured on a plant with non-zero initial condition [9]. In this chapter, we formulate a class of SMI algorithms to identify arbitrary MIMO linear time-varying systems with ensemble methods. With the algorithms, the state space model can be obtained directly from the available input and output data sets. The algorithms are an extension of the recently developed subspace model identification approach MOESP (*MIMO Output – Error State Space* Model Identification) in [10] [11] [12]. The algorithms retain all the properties from the MOESP. For example, the ordinary MOESP algorithm for ensemble identification problems can provide an unbiased consistent estimate when the noise on the output is zero-mean white; the PI scheme for ensemble identification problems can give an unbiased consistent model when the noise is zero-mean coloured. We propose that they are useful in the state space model identification of a MIMO time-varying system when there is not a priori knowledge of the system's behavior and time variation, and/or the system has fast changing factors. More generally, the dimensions of the state, input and output of the system considered in the algorithms are allowed to vary. The variation of the state dimension has been observed in an application of the algorithms derived in this paper to the identification of a biomedical system [6]. The variation of the dimension of the input and output quantities occurs in the treatment of a particular class of periodic systems, multirate sampling systems. In our work, special attention is given to this class of systems.

The choice of extending the particular subspace model identification approach is twofold. First, the MOESP approach allows to address the same classes of time-invariant identification problems compared to the related SMI approaches, such as approaches in [13] and [14]. Therefore, extending the MOESP approach does in no way restrict the range of problems that can be tackled. Second, the ultimate close relationship between the MOESP approach and the related approaches in [13] and [14] would enable the interested reader to extend these other approaches without much difficulty when following the strategy outlined in this paper for the MOESP approach.

This chapter elaborates on the work presented in [15]. There it was shown that ensemble identification problems are a natural way of formulating identification problems in the non-stationary (operator-theoretic) system theory pioneered by different researchers such as Alpay, Dewilde and van der Veen [16] [17]. The theory is currently receiving a high degree of maturity, see e.g. [18].

The organization of this chapter is as follows. Section 2 summarizes some general notation used throughout the whole chapter and quickly reviews some properties of random variables in a time-varying context. Section 3 presents the ensemble identification problem when there are no errors on the input/output data and contains definitions, such as persistency of excitation of the input, relevant to solving the ensemble identification problem. In section 4, the solution to the ensemble identification problem described in section 3 is presented and followed by the solution to the ensemble identification problem when the output is disturbed by zero mean errors of white or arbitrary coloring. In Section 5 we specialize this solution to the identification of periodically time-varying systems. In this section some of the performances of the derived algorithms are presented in an illustrative example considering the identification of a multirate sampled data system. Finally, section 6 contains some concluding remarks. Finally, section 6 contains some concluding remarks. Since the identification algorithm is obtained by considering the local properties rather than the global properties of the identified systems, we discuss problems by using local notations rather than global notations in this chapter.

3.2 Notations and statistical framework of analysis

In this section, we list some frequently used notations and briefly review the statical concepts used throughout the chapter.

1. $\rho(A)$ denotes the rank of matrix A and A^T denotes the transpose of matrix A.

2. I_n denotes the identity matrix of order n.

3. The matrix inequality $A > (\geq) B$ means that $A - B$ is positive (semi positive) definite.

4. $\mathbb{R}^n$ denotes the n-dimensional vector space over the field of real numbers and $\mathbb{R}^{n\times m}$ denotes the space of $n \times m$ matrices with entries in $\mathbb{R}$. $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}^+$ denotes the set of positive integers.

5. The QR factorization: The QR factorization of a matrix $A \in \mathbb{R}^{n\times m}$ is a factorization of A into an orthonormal column matrix $Q \in \mathbb{R}^{n\times n}$ (i.e. $Q^TQ = I_n$) and an upper trapezoidal matrix $R \in \mathbb{R}^{n\times m}$ such that:

$$A = QR$$

Sometimes, when $n \geq m$, we consider a partial QR factorization in which only the first m columns of the orthonormal matrix Q and the first m rows of the matrix R are retained.

6. The vectors considered in this paper are all row vectors.

7. Matrix partitioning: We use standard Matlab [19] notation, an example is:

Example 1: Let $A \in \mathbb{R}^{m\times n}$ and let $k < n$, then a partitioning of A is represented by:

$$A = [\ A(:,1:k) \quad A(:,k+1:n)\ ]$$

where A(:,1:k) denotes the first k columns of A and A(:,k+1:n) denotes the last (n − k) columns, while the initial ':' indicates that all rows are chosen.

8. The statistical framework of analysis [22] [20]:

For a discrete stochastic process, for example $\mathbf{v}_t$, $v_{j,k}$ presents the observation in the j^{th} experiment at time instant t. The ensemble of $\mathbf{v}_t$ is a family histories $v_{j,k}$ for $j \in [1, n]$, $t \in [t_0, t_0 + T - 1]$. For a special time t, $\mathbf{v}_t$ is a random variable (RV) [22] [20] and it is assumed that the ensemble sample average and the ensemble sample covariance are asymptotically unbiased estimates of the true mean and covariance of these RVs. For example, for the mean this gives the following equality:

$$E[\mathbf{v_t}] = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} v_{j,t} \quad or \quad E[\mathbf{v_t}] = \frac{1}{n} \sum_{j=1}^{n} v_{j,t} + O_n(\varepsilon)$$

where $O_n(\varepsilon)$ is a matrix quantity of norm ε which vanishes as $n \to \infty$.

3.3 Preliminaries, problem statement and definitions

3.3.1 Model description and the ensemble identification problem

For the time being, we consider the deterministic part of the dynamic relationship between input and output quantities of the system to be identified to be represented by the class of *multi-input, multi-output (MIMO)* discrete linear time-varying state space model. Let the dimensions of the input vector u_t and output vector y_t be m_t and ℓ_t, i.e. $u_t \in \mathbb{R}^{m_t}, y_t \in \mathbb{R}^{\ell_t}$, and the dimension of the state be N_t, i.e. $x_t \in \mathbb{R}^{N_t}$. Then , the state model is given by:

$$x_{t+1} = x_t A_t + u_t B_t \tag{3.1}$$

$$y_t = x_t C_t + u_t D_t \tag{3.2}$$

The system matrices are of the following sizes:

$$\begin{array}{c} \\ N_t \\ m_t \end{array} \begin{bmatrix} N_{t+1} & \ell_t \\ A_t & C_t \\ B_t & D_t \end{bmatrix} \tag{3.3}$$

The solution to equation (3.1) is given by:

$$x_t = x_0 \Phi(t, t_0) + \sum_{i=t_0}^{t-1} u_i B_i \Phi(t, i-1) \tag{3.4}$$

where $\Phi(t, t_0)$ is defined as:

$$\Phi(t, t_0) := \begin{cases} A_{t_0}A_{t_0+1}\cdots A_{t-1} & t > t_0 \\ I & t = t_0 \\ 0 & t < t_0 \end{cases} \tag{3.5}$$

Let us repeat the Definition 2.1 of Chapter 2.

Definition 3.1 *The system state space representation* $(\alpha_t, \beta_t, \gamma_t, \delta_t)$ *is said to be* similarly equivalent *to* (A_t, B_t, C_t, D_t) *denoted as* $(A_t, B_t, C_t, D_t) \stackrel{T_t}{\rightarrow} (\alpha_t, \beta_t, \gamma_t, \delta_t)$, *if there exist a transformation* $T_t \in R^{N_t \times N_t}$ *such that* T_t *and* T_t^{-1} *are bounded for all* $t \in Z$ *and* $(\alpha_t, \beta_t, \gamma_t, \delta_t)$ *satisfies:*

$$\begin{bmatrix} \alpha_t & \gamma_t \\ \beta_t & \delta_t \end{bmatrix} = \begin{bmatrix} T_t & 0 \\ 0 & I_l \end{bmatrix} \begin{bmatrix} A_t & C_t \\ B_t & D_t \end{bmatrix} \begin{bmatrix} T_{t+1}^{-1} & 0 \\ 0 & I_m \end{bmatrix} \tag{3.6}$$

□

Let a system state space representation (A_t, B_t, C_t, D_t) be given. Consider a similarity transformation $T_t \in \mathbb{R}^{N_t \times N_t}$ to the system such that T_t and T_t^{-1} are bounded for all $t \in Z$ and $(A_t, B_t, C_t, D_t) \stackrel{T_t}{\rightarrow} (\alpha_t, \beta_t, \gamma_t, \delta_t)$. If the representation (A_t, B_t, C_t, D_t) is bounded $(A_t(.), B_t(.), C_t(.)$ is bounded) and stable ($\Phi(t, t_0) \rightarrow 0$ as $t \rightarrow \infty$), then the transformation T_t preserves the boundedness of the transformed system by equation (3.6), and stability by the equation[1]:

$$\Phi_T(t, t_0) = T_{t_0}\Phi(t, t_0)T_t^{-1} \tag{3.7}$$

where $\Phi_T(t, t_0) = \alpha_{t_0}\alpha_{t_0+1}\cdots\alpha_{t-1}$.

The deterministic ensemble identification problem:

Let the index j in $u_{j,t}$ indicate the input sample at time t of the j^{th} experiment conducted with the system (3.1) and (3.2). If $j \in [j_0, j_0 + n - 1]$ and $t \in [t_0, t_0 + T - 1]$ (where j_0 indicates the first experiment, t_0 indicates the first instant, n indicates the total number of experiments and T indicates the total measuring time a single experiment lasts), the problem is to determine a state space description:

$$\xi_{j,t+1} = \xi_{j,t}\alpha_t + u_{j,t}\beta_t \tag{3.8}$$
$$y_{j,t} = \xi_{j,t}\gamma_t + u_{j,t}\delta_t \tag{3.9}$$

such that $(\alpha_t, \beta_t, \gamma_t, \delta_t)$ is similarly equivalent to the state space representation (A_t, B_t, C_t, D_t) which is consistent with the input and output sequences contained in the chosen ensembles:

$$\begin{matrix} y_{j_0,t_0} & y_{j_0,t_0+1} & \cdots & y_{j_0,t_0+T-1} \\ y_{j_0+1,t_0} & y_{j_0+1,t_0+1} & \cdots & y_{j_0+1,t_0+T-1} \\ \vdots & \vdots & \cdots & \vdots \\ y_{j_0+n-1,t_0} & y_{j_0+n-1,t_0+1} & \cdots & y_{j_0+n-1,t_0+T-1} \end{matrix} \tag{3.10}$$

The ensemble of input sequences $u_{j,t}$ is for the same series of experiments and over the same time interval.

From the state space representation (2) and (3), it is easy to find the following relationship:

$$\begin{pmatrix} y_{j_0,t} & y_{j_0,t+1} & \cdots & y_{j_0,t+s-1} \\ y_{j_0+1,t} & y_{j_0+1,t+1} & \cdots & y_{j_0+1,t+s-1} \\ \vdots & \vdots & \cdots & \vdots \\ y_{j_0+n-1,t} & y_{j_0+n-1,t+1} & \cdots & y_{j_0+n-1,t+s-1} \end{pmatrix}$$

$$= \begin{pmatrix} x_{j_0,t} \\ x_{j_0+1,t} \\ \vdots \\ x_{j_0+n-1,t} \end{pmatrix} \begin{bmatrix} C_t & A_tC_{t+1} & \cdots & A_tA_{t+1}\cdots A_{t+s-2}C_{t+s-1} \end{bmatrix}$$

$$+ \begin{pmatrix} u_{j_0,t} & u_{j_0,t+1} & \cdots & u_{j_0,t+s-1} \\ u_{j_0+1,t} & u_{j_0+1,t+1} & \cdots & u_{j_0+1,t+s-1} \\ \vdots & \vdots & \cdots & \vdots \\ u_{j_0+n-1,t} & u_{j_0+n-1,t+1} & \cdots & u_{j_0+n-1,t+s-1} \end{pmatrix}$$

$$\begin{pmatrix} D_t & B_tC_{t+1} & \cdots & B_tA_{t+1}\cdots A_{t+s-2}C_{t+s-1} \\ & D_{t+1} & \cdots & \cdots \\ & & \ddots & \vdots \\ & & & D_{t+s-1} \end{pmatrix} \tag{3.11}$$

This equation can be denoted more compactly as:

$$Y_{t,s} = X_t\mathcal{O}_{t,s} + U_{t,s}\Delta_{t,s} \tag{3.12}$$

Following the work of [1], the matrices $Y_{t,s}$ and $U_{t,s}$ are referred to as *generalized Hankel matrices*. The index set of these generalized Hankel matrices should be $(j_0, t), s, n$. This set would allow a precise definition of that which part of the recorded output sequences in equation (3.10) and the corresponding input sequences are stored in the respective matrices. However, since the experiment index will not be relevant and we assume the number of experiments n to be fixed by the experimental circumstances, we restrict the index set of both matrices to t, s. Here t indicates the time index of the their top left entry and s determines their column width in the following way. Let

$$L_t^s = \sum_{\tau=t}^{t+s-1} l_\tau \quad \text{and} \quad M_t^s = \sum_{\tau=t}^{t+s-1} m_\tau \tag{3.13}$$

then $Y_{t,s} \in \mathbb{R}^{n \times L_t^s}$ and $U_{t,s} \in \mathbb{R}^{n \times M_t^s}$. Since $X_t \in \mathbb{R}^{n \times N_t}$, thus $\mathcal{O}_{t,s} \in \mathbb{R}^{N_t \times L_t^s}$ and $\Delta_{t,s} \in \mathbb{R}^{M_t^s \times L_t^s}$.

Based on the data representation, the solution to the deterministic ensemble identification problem will be given by subsequently treating the following *subproblems*:

1. Given the data matrices $Y_{t,s}$ and $U_{t,s}$, to determine conditions on the input sequences to retrieve the row space of the *observability matrix* $\mathcal{O}_{t,s}$ and to determine this row space.
2. To determine the matrices $[\alpha_t, \gamma_t]$ from the row space of $\mathcal{O}_{t,s}$.
3. To determine the matrices $[\beta_t, \delta_t]$.
4. To determine the minimum μ and χ such that $[\alpha_t, \beta_t, \gamma_t, \delta_t]$ can be calculated in the time interval $t \in [t_0 + \mu, t_0 + T - 1 - \chi]$.

3.3.2 Relevant lemmas and important definitions.

In this chapter, we will use the following lemma.

Lemma 3.1 *(*Sylvester's inequality *[23] p. 655) Let $M_1 \in \mathbb{R}^{m \times n}$ and $M_2 \in \mathbb{R}^{n \times p}$, then:*

$$\rho(M_1) + \rho(M_2) - n \le \rho(M_1 M_2) \le min\{\rho(M_1), \rho(M_2)\} \tag{3.14}$$

□

Let us repeat the following series of definitions which will demonstrate to be relevant in solving the ensemble identification problem.

Definition 3.2 *(*Jazwinski 1970*)[24] Let, $\delta \in \mathbb{Z}^+$,*

$$\mathcal{O}_{t,\delta} := \begin{bmatrix} C_t & A_t C_{t+1} & \cdots & A_t A_{t+1} \cdots C_{t+\delta} \end{bmatrix}$$

and observability Gramian $G_o(t, t+\delta) := \mathcal{O}_{t,s}\mathcal{O}_{t,\delta}^T$, then the pair $[A_t, C_t]$ is uniformly observable *if $\exists \delta \in \mathbb{Z}^+$, and positive constants b_1, b_2 such that :*

$$0 < b_1 I \leq G_o(t, t+\delta) \leq b_2 I, \forall t \tag{3.15}$$

□

The least integer value δ, for which the condition in (3.15) holds is denoted by δ_o. For linear time-invariant systems, δ_o refers to the *observability index (Kailath, 1980], p.356)* [23]. With the δ defined in Definition 3.2, it follows immediately via Lemma 3.1, that $\rho(\mathcal{O}_{t,\delta}) = N_t$ and this can only be true if $L_t^\delta \geq N_t$.

The dual of Definition 3.2 is Definition 3.3 below.

Definition 3.3 *(*Jazwinski 1970*)[24] Let $\mathcal{C}_{t,\delta}$ be*

$$\mathcal{C}_{t,\delta} := \begin{bmatrix} B_{t-1} \\ B_{t-2}A_{t-1} \\ \vdots \\ B_{t-\delta}A_{t-\delta+1}\cdots A_{t-2}A_{t-1} \end{bmatrix} \tag{3.16}$$

and controllability Gramian $G_c(t-\delta, t) := \mathcal{C}_{t,\delta}^T \mathcal{C}_{t,\delta}$, then the pair $[A_t, B_t]$ is uniformly controllable *if $\exists \delta \in Z^+$, and positive constant a_1, a_2 such that:*

$$0 < a_1 I \leq G_c(t-\delta, t) \leq a_2 I, \forall t \tag{3.17}$$

□

The least integer value of δ for which the condition (3.17) holds is denoted by δ_c and is indicated as the *controllability index* when the system is time-invariant [23]. When the conditions in Definition 3.3 hold, Lemma 3.1 shows that $\rho(\mathcal{C}_{t,\delta}) = N_t$ and $M_t^\delta \geq N_t$.

Definition 3.4 *A bounded realization (A_t, B_t, C_t, D_t) is said to be* uniform *if it is uniformly controllable and observable [1].*

Based on the definitions of controllability and observability, we have the following lemma.

Lemma 3.2 *Let (A_t, B_t, C_t, D_t) be a uniform realization, then its similarly equivalent realization $(\alpha_t, \beta_t, \gamma_t, \delta_t)$ is also a uniform realization.*

Proof: If $(A_t, B_t, C_t, D_t) \xrightarrow{T_t} (\alpha_t, \beta_t, \gamma_t, \delta_t)$, it is easy to see that the controllability and observability Gramian of the transformed system have the following forms:

$$G_{c_T}(t) = T_t^{-T} G_c(t) T_t^{-1} \tag{3.18}$$

$$G_{o_T}(t) = T_t G_o(t) T_t \tag{3.19}$$

These two equations imply that $G_{c_T}(t)$ and $G_{o_T}(t)$ are congruent [21] to $G_c(t)$ and $G_o(t)$ respectively. Therefore, when $G_c(t)$ and $G_o(t)$ are positive definite and bounded, so are $G_{c_T}(t)$ and $G_{o_T}(t)$. □

Definition 3.5 *A system representation (A_t, B_t, C_t, D_t) is said to be* uniformly balanced *if the following two conditions hold:*
(i) (A_t, B_t, C_t) is uniform,

(ii) $$G_o(t, t+s) = G_c(t-s, t) := \Sigma(t) = \begin{bmatrix} \sigma_1(t) & & & \\ & \sigma_2(t) & & \\ & & \ddots & \\ & & & \sigma_{N_t}(t) \end{bmatrix}$$

where $\sigma_1(t) \geq \sigma_2(t) \geq \cdots \geq \sigma_{N_t}(t) \geq 0$, $\forall t$. □

In [1], an algorithm is presented to compute a uniformly balanced realization for any given bounded uniform realization. The algorithm essentially proceeds in two steps. In the first step, a similarity transformation is performed such that the observability Gramian of the similarly equivalent realization is the identity matrix. In the second step a similarity transformation is performed such that the controllability Gramian becomes diagonal.

Our final definition, which is illustrated by an example in the appendix of this chapter, is stated next.

Definition 3.6 *The input sequences in the data matrix $U_{t,s}$ are* locally persistent excitation *for the system which has a realization $\{A_t, B_t, C_t, D_t\}$ at the time instant t if $\exists s \in \mathbb{Z}^+$ such that:*

$$\rho([U_{t,s}|X_t]) = M_t^s + N_t \tag{3.20}$$

□

3.4 A Subspace model identification solution to the ensemble identification problem

3.4.1 The deterministic ensemble identification problem

When there is no noise in the input-output data, the matrices $Y_{t,s}$ and $U_{t,s}$ are related as in equation (3.12). Based on this relationship, we address the subproblems described at the end of Subsection 3.3.1 subsequently in the following subsections.

The determination of the row space of $\mathcal{O}_{t,s}$

First, perform a *QR factorization* of the compound matrix $[U_{t,s} \mid Y_{t,s}]$, as:

$$[U_{t,s} \mid Y_{t,s}] = [Q_{1,t} \mid Q_{2,t}] \begin{matrix} M_t^s \\ n-M_t^s \end{matrix} \left(\begin{array}{c|c} \overset{M_t^s}{R_{11,t}} & \overset{L_t^s}{R_{12,t}} \\ \hline 0 & R_{22,t} \end{array} \right) \tag{3.21}$$

where $Q_{1,t}^T Q_{1,t} = I_{M_t^s}$ and $Q_{1,t}^T Q_{2,t} = 0$. Then we have the following theorem.

Theorem 3.1 *Let the system be uniformly controllable and observable. In addition let the following conditions be satisfied:*

(1) *the input sequences in the data matrix $U_{t,s}$ are locally persistent at time instant t,*

(2) $s \geq \delta_o$,

(3) $n \geq (M_t^s + N_t)$,

(4) *the QR factorization of matrix $[U_{t,s}|Y_{t,s}]$ are given and partitioned as in equation (3.21),*

then $\rho(R_{22,t}) = \rho(\mathcal{O}_{t,s}) = N_t$, and the row space of $R_{22,t}$ equals the row space of the observability matrix $\mathcal{O}_{t,s}$.

Proof: Denote the QR factorization of the matrix pair $[U_{t,s} \mid X_t]$ as:

$$[U_{t,s} \mid X_t] = [Q_{1,t} \mid Q_{x,t}] \begin{matrix} M_t^s \\ N_t \end{matrix} \left(\begin{array}{c|c} \overset{M_t^s}{R_{11,t}} & \overset{N_t}{R_{x1,t}} \\ \hline 0 & R_{x2,t} \end{array} \right) \tag{3.22}$$

where $Q_{1,t}^T Q_{x,t} = 0$. Since $n \geq (M_t^s + N_t)$ and the columns of the matrix $[Q_{1,t} \mid Q_{x,t}]$ are orthogonal, with the help of Lemma 3.1, condition (1) shows that,

$$\rho\left(\left[\begin{array}{c|c} R_{11,t} & R_{x1,t} \\ \hline 0 & R_{x2,t} \end{array}\right]\right) = \rho([U_{t,s}|X_t]) = M_t^s + N_t \tag{3.23}$$

and

$$\rho(R_{x2,t}) = N_t \tag{3.24}$$

With the QR factorization in equations (3.21) and (3.22), we can express $Y_{t,s}$ as:

$$Y_{t,s} = Q_{1,t}R_{12,t} + Q_{2,t}R_{22,t}$$

and equation (3.12) as:

$$Y_{t,s} = X_t\mathcal{O}_{t,s} + Q_{1,t}R_{11,t}\Delta_{t,s}$$

Using the expression for X_t as given in equation (3.22), the latter equation can also be denoted as:

$$Y_{t,s} = Q_{1,t}R_{x1,t}\mathcal{O}_{t,s} + Q_{x,t}R_{x2,t}\mathcal{O}_{t,s} + Q_{1,t}R_{11,t}\Delta_{t,s}$$

Hence we have:

$$Q_{1,t}R_{12,t} + Q_{2,t}R_{22,t} = Q_{1,t}R_{x1,t}\mathcal{O}_{t,s} + Q_{x,t}R_{x2,t}\mathcal{O}_{t,s} + Q_{1,t}R_{11,t}\Delta_{t,s} \tag{3.25}$$

Multiplying on the left of (3.25) with $Q_{1,t}^T$ and using the properties that $Q_{1,t}^T Q_{1,t} = I_{M_t^s}$, $Q_{1,t}^T Q_{2,t} = Q_{1,t}^T Q_{x,t} = 0$ we obtain:

$$R_{12,t} = R_{x1,t}\mathcal{O}_{t,s} + R_{11,t}\Delta_{t,s} \tag{3.26}$$

Substituting this relationship back into the right-hand side of equation (3.25) yields:

$$Q_{2,t}R_{22,t} = Q_{x,t}R_{x2,t}\mathcal{O}_{t,s} \tag{3.27}$$

Hence by equation (3.24) and the fact $\rho(\mathcal{O}_{t,s}) = N_t$, Lemma 3.1 shows that:

$$\rho(R_{22,t}) = \rho(\mathcal{O}_{t,s}) = N_t$$

Furthermore, since $Q_{2,t}^T Q_{x,t} R_{x2,t} \in \mathbb{R}^{L_t^s \times N_t}$ and $\mathcal{O}_{t,s} \in \mathbb{R}^{N_t \times L_t^s}$ and since $L_t^s \geq N_t$ by condition (2), the row space of $R_{22,t}$ equals that of $\mathcal{O}_{t,s}$. □

It should be remarked that when condition (3) has to hold for $\forall t$, the minimal number of experiments, denoted by $n_{\min}$, required is:

$$n_{\min} = \max_{t \in [t_0, t_0+n-1]} (M_t^s + N_t) \tag{3.28}$$

Therefore, we see that $n_{\min}$ depends on an upper-bound of the order of the system provided that s has been chosen such that condition (1) and (2) of Theorem 3.1 are satisfied. When s is of the same order of magnitude as the order of the system as in the time invariant case, then $n_{\min}$ turns out to be of the same order in magnitude as s. However, as we will see later on in Subsection 3.4.2, the presence of errors on the output measurements would require the number n to be very large (∞) when estimates of high accuracy (consistent) are required.

As a result of Theorem 3.1, we can find the row space of $\mathcal{O}_{t,s}$ through a *SVD* (singular value decomposition) of $R_{22,t}$. Denote a SVD of $R_{22,t}$ as

$$R_{22,t} = U_t \left(\begin{array}{c|c} S_{N_t} & 0 \\ \hline 0 & 0 \end{array} \right) \left(\begin{array}{c} V_{N_t}^T \\ \hline (V_{N_t}^{\perp})^T \end{array} \right) \tag{3.29}$$

with $U_t \in \mathbb{R}^{(n-M_t^s)\times(n-M_t^s)}$, $S_{N_t} \in \mathbb{R}^{N_t \times N_t}$, $V_{N_t} \in \mathbb{R}^{L_t^s \times N_t}$ and $V_{N_t}^{\perp} \in \mathbb{R}^{L_t^s \times (L_t^s - N_t)}$. Let $T_t \in \mathbb{R}^{N_t \times N_t}$ be a square invertible matrix, then because of Theorem 3.1 and the fact that the row space of $\mathcal{O}_{t,s}$ determines $[A_t, C_t]$ pairs up to state equivalence, we can write:

$$V_{N_t}^T = T_t^{-1} \mathcal{O}_{t,s} \tag{3.30}$$

The determination of the matrices $[\alpha_t, \gamma_t]$

From equations (3.30) and (3.6), we get:

$$V_{N_t}^T = [\ \gamma_t \quad \alpha_t \gamma_{t+1} \quad \cdots \quad \alpha_t \cdots \alpha_{t+s-2} \gamma_{t+s-1}\ ] \tag{3.31}$$

Hence, it is easy to see that

$$\gamma_t = V_{N_t}^T(:, 1 : \ell_t) \tag{3.32}$$

Lemma 3.3 *Let the conditions in Theorem 3.1 hold, with condition (2) strengthened to* $s > \delta_o$ *and condition (3) replaced by that in equation (3.28), then* α_t *can be solved from the overdetermined set of equations:*

$$\begin{aligned} V_{N_t}^T(:, \ell_t + 1 : L_t^s) &= \alpha_t[\ \gamma_{t+1} \quad \alpha_{t+1}\gamma_{t+2} \quad \cdots \quad \alpha_{t+1}\cdots\alpha_{t+s-2}\gamma_{t+s-1}\ ] \\ &= \alpha_t V_{N_{t+1}}^T(:, 1 : L_{t+1}^{s-1}) \end{aligned} \tag{3.33}$$

Proof: By Theorem 3.1, we can determine the row space of $\mathcal{O}_{t+1,s}$ via a SVD of the matrix $R_{22,t+1}$. Denote this row space by V_{N+1}^T, then there exists a non-singular matrix T_{t+1} such that:

$$V_{N_{t+1}}^T = T_{t+1}^{-1}\mathcal{O}_{t+1,s}$$

Since $s > \delta_o$, $\rho(\mathcal{O}_{t+1,s-1}) = N_{t+1}$, the matrix $V_{N_{t+1}}^T(:, 1 : L_{t+1}^{s-1})$ has full row rank and we can solve equation (3.33) for α_t. □

From the above relations we can get γ_t for time $[t_0, t_0 + T - s]$ and α_t for time $[t_0, t_0 + T - 1 - s]$.

The determination of the matrices $[\beta_t, \delta_t]$

Theorem 3.2 *Let the system be uniformly controllable and observable. In addition the following conditions be satisfied:*

(1) *the input sequences in the data matrix* $U_{t,s}$ *are locally persistent exciting at all time instants* t*, with* $t \in (t, t + s - 1)$*,*

(2) $s > \delta_o$,

(3) $n \geq \max_{t\in[t_0,t_0+n-1]}(M_t^s + N_t)$,

(4) *the QR factorization of matrix sequence* $[U_{t+1-i,s}|Y_{t+1-i,s}]$ *for* $i = 0 : 1 : s$ *be given and partitioned similar as in equation (3.21),*

(5) $\rho(V_{N_{t-i}}^{\perp}(L_{t-i}^{s-1} + 1 : L_{t-i}^s, :)) = \ell_{t-i+s-1}$ *for* $i = 0 : 1 : s - 1$*,*

and by Lemma 3.3, the matrix $\left(\begin{array}{c|c} I_{\ell_t} & 0 \\ \hline 0 & V_{N_{t+1}}^T(:, 1 : L_{t+1}^{s-1}) \end{array}\right)$ *has full rank, then,* β_t *and* δ_t *can be solved from the equation*

$$\left(\ \delta_t \mid \beta_t\ \right)\left(\begin{array}{c|c} I_{\ell_t} & 0 \\ \hline 0 & V_{N_{t+1}}^T(:, 1 : L_{t+1}^{s-1}) \end{array}\right) \times$$

$$\underbrace{L_t^s \left\{ \left[V_{N_t}^{\perp} \mid \frac{V_{N_{t-1}}^{\perp}(L_{t-1}^1+1:L_{t-1}^s,:)}{0} \mid \cdots \mid \begin{array}{c} V_{N_{t-s+1}}^{\perp}(L_{t-s+1}^{s-1}+1:L_{t-s+1}^s) \\ \hline 0 \\ \hline \vdots \\ \hline 0 \end{array} \right] \right.}=$$

$$\left[\ \phi_t(1:m_t,:) \mid \phi_{t-1}(M_{t-1}^1+1:M_{t-1}^1+m_t,:) \mid \cdots \mid \phi_{t-s+1}(M_{t-s+1}^{s-1}+1:M_{t-s+1}^{s-1}+m_t,:)\ \right] \tag{3.34}$$

where $\phi_t = R_{11,t}^{-1} R_{12,t} V_{N_t}^{\perp}$.

Proof: Under condition (2), $V_{N_{t-i}}^{\perp}$ for $i = 0, 1, 2, \cdots, s-1$ exists.

Using $V_{N_t}^T V_{N_t}^{\perp} = 0$ and equation (3.30), multipling equation (3.26) by $V_{N_t}^{\perp}$ on the right yields:

$$R_{12,t} V_{N_t}^{\perp} = R_{11,t} \Delta_{t,s} V_{N_t}^{\perp} \tag{3.35}$$

Under condition (1), $R_{11,t}$ is non-singular, then we have:

$$R_{11,t}^{-1} R_{12,t} V_{N_t}^{\perp} = \Delta_{t,s} V_{N_t}^{\perp} \tag{3.36}$$

Denote the matrix product $R_{11,t}^{-1} R_{12,t} V_{N_t}^{\perp}$ as ϕ_t, then equation (3.36) is expressed explicitly as

$$\left[\begin{array}{c} \phi_t(1:m_t,:) \\ \hline \phi_t(m_t+1:M_t^2,:) \\ \hline \vdots \\ \hline \phi_t(M_t^{s-1}+1:M_t^s,:) \end{array}\right] = \left[\begin{array}{c} \left(\ \delta_t \mid \beta_t\ \right) \left(\begin{array}{c} I_{l_t} \mid \quad 0 \\ \hline 0 \mid V_{N_{t+1}}^T(:,1:L_{t+1}^{s-1})) \end{array} \right) \\ \hline \left(\ \delta_{t+1} \mid \beta_{t+1}\ \right) \left(\begin{array}{c} 0 \mid I_{l_{t+1}} \mid \quad 0 \\ \hline 0 \mid \ 0 \ \mid V_{N_{t+2}}^T(:,1:L_{t+2}^{s-2}) \end{array} \right) \\ \hline \vdots \\ \hline \left(\ \delta_{t+s-1} \mid \beta_{t+s-1}\ \right) \left(\begin{array}{c} 0 \mid \cdots \mid I_{l_{t+s-1}} \\ \hline 0 \mid \cdots \mid \ 0 \end{array} \right) \end{array}\right] V_{N_t}^{\perp} \tag{3.37}$$

The matrices δ_t and β_t thus satisfy equation (3.34).

Under condition (5), the underbraced matrix in equation (3.34) has full rank, hence applying Lemma 3.1, we can solve δ_t and β_t from equation (3.34). □

The above analysis shows that when the conditions stipulated in Lemma 3.3 and Theorem 3.2 are satisfied over the time interval $[t_0, t_0+T-1]$, we can get δ_t and β_t for $t \in$

$[t_0 + s - 1, t_0 + T - s]$. Hence, under these conditions, the calculation of the quadruple of system matrices $\begin{bmatrix} \alpha_t & \gamma_t \\ \beta_t & \delta_t \end{bmatrix}$ is possible in the time interval $t \in [t_0 + \mu, t_0 + T - 1 - \chi]$, with minimal values for μ and χ respectively $s - 1$ and s.

We then summarize the above results into a generalization of the *ordinary* MOESP *algorithm* [10] applicable to ensemble identification problems.

The ordinary MOESP *algorithm for ensemble identification problems:*

Given:

1. a uniformly controllable and uniformly observable time-varying system,
2. an ensemble of input and output sequences such that the input sequences in the series of data matrices $U_{t,s}$ are locally persistent excitations for $[t_0, t_0 + T - s - 1]$,
3. $s > \delta_o$, with δ_o defined in paragraph following Definition 3.2.
4. $n \geq \max_{t \in [t_0, t_0+n-1]}(M_t^s + N_t)$, with n the number of experiments, M_t^s defined in equation (3.13) and N_t the order of the system at time instant t.

For $t = t_0 + s - 1 : t_0 + T - s - 1$, *do the following:*

step 1. construct the sequence of generalized Hankel matrices $U_{t+1-i,s}$, $Y_{t+1-i,s}$ for $i = 0 : 1 : s$.

Step 2: Perform a QR factorization of the sequence of compound matrices $[U_{t+1-i,s} | Y_{t+1-i,s}]$ for $i = 0 : 1 : s$, without storing Q and partition the R factor as in equation (3.21).

Step 3: perform a SVD of the sequence of matrices $R_{22,t+1-i}$ for $i = 0 : 1 : s$ as given in equation (3.29) and store the matrix sequences $V_{N_{t+1-i}}$, , $V_{N_{t+1-i}}^{\perp}$ for $i = 0 : 1 : s$.

Step 4: Solve equations (3.32), (3.33) to get γ_t, α_t and when condition (5) of Theorem 3.2 is satisfied solve equation (3.34) to determine β_t and δ_t.

For the special case the input and output dimensions are constants and denoted by m and ℓ respectively, equations (3.32), (3.33) and (3.34) specialize to:

$$\gamma_t = V_{N_t}^T(:, 1 : l) \tag{3.38}$$

$$V_{N_t}^T(:, l + 1 : sl) = \alpha_t V_{N_{t+1}}^T(:, 1 : (s - 1)l) \tag{3.39}$$

$$\left(\delta_t \mid \beta_t \right) \left(\frac{I_l \mid \quad 0}{0 \mid V^T_{N_{t+1}}(:, 1:(s-1)l)} \right) \times$$

$$\underbrace{\left[V^{\perp}_{N_t} \mid \frac{V^{\perp}_{N_{t-1}}(l+1:sl,:)}{0} \mid \cdots \mid \begin{array}{c} V^{\perp}_{N_{t-s+1}}((s-1)l+1:sl,:) \\ \hline 0 \\ \hline \vdots \\ \hline 0 \end{array} \right]} =$$

$$\left[\phi_t(1:m,:) \mid \phi_{t-1}(m+1:2m,:) \mid \cdots \mid \phi_{t-s+1}((s-1)m+1:sm,:) \right] \tag{3.40}$$

respectively. Where $L_t^s = sl$ and $M_t^s = sm$.

A property of the realization calculated by the ordinary algorithm for ensemble identification problem

If the system realization $[A_t, B_t, C_t, D_t]$ is a uniform realization, then $[\alpha_t, \gamma_t, \beta_t, \delta_t]$ is also a uniform realization (Lemma 3.2) Moreover, the resulted observability Gramian equals the identity matrix, hence the first step of the algorithm in [1] to calculate a uniformly balanced realization of a time-varying system can be skipped when such a realization is needed.

3.4.2 System with additive errors on the output

In reality, the input-output data sequences are collected by measurement. As for the time-invariant case [10], we will consider only noise on the output measurement. In this case, the system model (2-3) changes into:

$$x_{j,t+1} = x_{j,t}A_t + u_{j,t}B_t \tag{3.41}$$

$$z_{j,t} = x_{j,t}C_t + u_{j,t}D_t + v_{j,t} \tag{3.42}$$

where $z_{j,t}$ is the measured output and $v_{j,t}$ represents the noise. Let $V_{t,s}$ denote the generalized Hankel matrix constructed from the samples $v_{j,t}$ as $Y_{t,s}$ has been constructed from $y_{j,t}$, then the relevant relationship between input and output sequences, as given by (3.12) for the noise free case, changes into:

$$Z_{t,s} = X_t\mathcal{O}_{t,s} + U_{t,s}\Delta_{t,s} + V_{t,s} \tag{3.43}$$

For these types of errors, we can state a generalization of Theorem 3.1 of [12] to the present ensemble identification context.

Theorem 3.3 *Let a system be uniformly controllable and observable. In addition let the following conditions be satisfied:*

(1) *the input sequences in the series of data matrix $U_{t,s}$ are locally persistent excitations for $t \in [t_0, t_0 + T - s - 1]$,*

(2) $s > \delta_o$,

(3) *the QR factorizations of $[U_{t,s}|Z_{t,s}]$ and $[U_{t,s}|X_t]$ are given by:*

$$[U_{t,s}|Z_{t,s}] = [Q_{1,t} \mid Q_{2,t}] \begin{matrix} M_t^s \\ n-M_t^s \end{matrix} \left(\begin{array}{c|c} \overset{M_t^s}{R_{11,t}} & \overset{L_t^s}{R_{12,t}} \\ \hline 0 & R_{22,t} \end{array} \right) \tag{3.44}$$

and

$$[U_{t,s} \mid X_t] = [Q_{1,t} \mid Q_{x,t}] \begin{matrix} M_t^s \\ N_t^s \end{matrix} \left(\begin{array}{c|c} \overset{M_t^s}{R_{11,t}} & \overset{N_t}{R_{x1,t}} \\ \hline 0 & R_{x2,t} \end{array} \right) \tag{3.45}$$

respectively,

(4) *the noise $v_{j,t}$ is independent of the input and zero mean, such that the following limits hold:*

$$\lim_{n\to\infty} \frac{1}{n} V_{t1,s}^T U_{t2,r} = 0, \forall t1, t2, \quad \textit{and} \quad \forall r > 0, \forall s > 0. \tag{3.46}$$

$$\lim_{n\to\infty} \frac{1}{n} V_{t,s}^T V_{t,s} = R_v \tag{3.47}$$

$$\lim_{n\to\infty} \frac{1}{n} V_{t,s}^T X_t = 0 \tag{3.48}$$

$$\lim_{n\to\infty} \frac{1}{n} R_{x2}^T R_{x2} = P_{x2} \tag{3.49}$$

then

$$\lim_{n\to\infty} \frac{1}{n} R_{22,t}^T R_{22,t} = \mathcal{O}_{t,s}^T P_{x2} \mathcal{O}_{t,s} + R_v \tag{3.50}$$

Proof: Taking into account the treatment of the statistical quantities as outlined in Section 3.2, the proof of this theorem can be given along the lines set up when proving

Theorem 1 of [12]. From the data equation (3.43) and the QR factorization defined in item 3 of the theorem, we obtain the following relationships:

$$U_{t,s} = Q_{1,t}R_{11,t} \tag{3.51}$$

$$Z_{t,s} = Q_{1,t}R_{12,t} + Q_{2,t}R_{22} = X_t\mathcal{O}_{t,s} + U_{t,s}\Delta_{t,s} + V_{t,s} \tag{3.52}$$

$$X_t = Q_{1,t}R_{x1,t} + Q_{x,t}R_{x2,t} \tag{3.53}$$

Substituting equations (3.51) and (3.53) into equation (3.52) yields:

$$\frac{1}{\sqrt{n}}(Q_{1,t}R_{12,t} + Q_{2,t}R_{22}) = \frac{1}{\sqrt{n}}(Q_{1,t}R_{x1,t}\mathcal{O}_{t,s} + Q_{x,t}R_{x2,t}\mathcal{O}_{t,s} + Q_{1,t}R_{11,t}\Delta_{t,s} + V_{t,s}) \tag{3.54}$$

equation (3.46) can be denoted as,

$$\frac{1}{n}V_{t,s}^T U_{t,s} = \frac{1}{n}V_{t,s}^T Q_{1,t}R_{11,t} = O_n(\varepsilon)$$

From condition (1) of the theorem, it follows that there exists an $\bar{n}$ such that for $n \geq \bar{n}$, the matrix $\frac{1}{\sqrt{n}}R_{11,t}$ is invertible. Therefore, it follows from the prior equation that:

$$\frac{1}{\sqrt{n}}V_{t,s}Q_{1,t} = O_n(\varepsilon) \tag{3.55}$$

Inserting this result into equation (3.48), we obtain, since by condition (1) the matrix $\frac{1}{\sqrt{n}}R_{x2,t}$ in also invertible for $n \geq \bar{n}$, that:

$$\frac{1}{\sqrt{n}}V_{t,s}Q_{x,t} = O_n(\varepsilon) \tag{3.56}$$

Now multiply equation (3.54) on the left by $Q_{1,t}^T$ and using the orthogonality between the matrices $Q_{1,t}$, $Q_{2,t}$ and $Q_{x,t}$ and equation (3.55) yields:

$$\frac{1}{\sqrt{n}}R_{12,t} = \frac{1}{\sqrt{n}}(R_{x1,t}\mathcal{O}_{t,s} + R_{11,t}\Delta_{t,s}) + O_n(\varepsilon)$$

and hence equation (3.54) reduces to:

$$\frac{1}{\sqrt{n}}Q_{2,t}R_{22,t} = \frac{1}{\sqrt{n}}(Q_{x,t}R_{x2,t}\mathcal{O}_{t,s} + V_{t,s}) + O_n(\varepsilon)$$

Multiplying both sides of this relationship on the left by their transpose and using equation (3.56) yields:

$$\frac{1}{n}R_{22,t}^T R_{22,t} = \frac{1}{n}\mathcal{O}_{t,s}^T R_{x2,t}^T R_{x2,t}\mathcal{O}_{t,s} + \frac{1}{n}V_{t,s}^T V_{t,s} + O_n(\varepsilon)$$

Taking the limit $n \to \infty$ yields equation (3.50) □

Case 1: The additive errors are discrete zero-mean white noise.

For this case, we can state the following Corollary to Theorem 3.3.

Corollary 3.1 *Let the conditions of Theorem 3.3 hold, and let the noise* $\mathbf{v}_t$ *be zero-mean white noise, then the ordinary* MOESP *algorithm to solve ensemble identification problems will determine the row space of the matrix* $\mathcal{O}_{t,s}$ *in an asymptotically unbiased way.*

Proof: In the white noise case, we know that:

$$\lim_{n\to\infty} \frac{1}{n} V_{t,s}^T V_{t,s} = \sigma_v^2 I_{L_t^s} \tag{3.57}$$

Hence, equation (3.50) in Theorem 3.3 becomes:

$$\lim_{n\to\infty} \frac{1}{n} R_{22,t}^T R_{22,t} = \mathcal{O}_{t,s}^T P_{x2} \mathcal{O}_{t,s} + \sigma_v^2 I_{L_t^s} \tag{3.58}$$

When $n \to \infty$, the invariant subspace of $\frac{1}{n} R_{22,t}^T R_{22,t}$ corresponding to the eigenvalues greater than σ_v^2 will determine the row space of $\mathcal{O}_{t,s}$ [11]. Hence the the ordinary MOESP algorithm for ensemble identification problems will determine the row space of $\mathcal{O}_{t,s}$ in an asymptotically unbiased way. □

Based on this corollary, we can show as in the time-invariant case (see [12]), that the quadruple of system matrices $\begin{bmatrix} \alpha_t & \gamma_t \\ \beta_t & \delta_t \end{bmatrix}$ can be computed asymptotically unbiasedly by the ordinary MOESP algorithm for ensemble identification problems. In Section 3.5, we illustrate this property by means of numerical simulation.

Case 2: The additive errors are discrete zero-mean noise of arbitrary coloring.

According to the description above, when the noise is not zero-mean white, then the calculation of the row space of $\mathcal{O}_{t,s}$ in the ordinary MOESP algorithm for ensemble identification problems will be biased. A possible and well-known rescue in time-invariant system identification is to introduce instrumental variables. Also for ensemble identification problems, the instrumental variable idea can be used. This is demonstrated by extending the PI-MOESP scheme of [12], in which past input data was used as instrumental variables, to the ensemble identification problem.

The key step in this generalization is presented in the next theorem, which is due to the similarity with its time-invariant counterpart in [12] stated without proof.

Theorem 3.4 *Let the conditions (1) to (4) in Theorem 3.3 hold. and let the following QR factorization be defined:*

$$\left[\begin{array}{c|c|c} U_{t,s} & Z_{t,s} & U_{t-s,s} \end{array}\right] = \left[\begin{array}{c|c|c} Q_{1,t} & Q_{2,t} & Q_{3,t} \end{array}\right] \begin{array}{c} M_t^s \\ L_t^s \\ M_t^s \end{array} \left[\begin{array}{c|c|c} \overset{M_t^s}{R_{11,t}} & \overset{L_t^s}{R_{12,t}} & \overset{M_t^s}{R_{13,t}} \\ \hline 0 & R_{22,t} & R_{23,t} \\ \hline 0 & 0 & R_{33,t} \end{array}\right] \tag{3.59}$$

we then have:

$$\lim_{n\to\infty} \frac{1}{n} R_{23,t}^T R_{22,t} = \lim_{n\to\infty} \frac{1}{n} U_{t-s,s}^T Q_{x,t} R_{x2,t} \mathcal{O}_{t,s} \tag{3.60}$$

□

When the input sequences $u_{j,t}$ are chosen such that for $n \to \infty$, the row spaces of the matrices $R_{23,t}^T R_{22,t}$ and $\mathcal{O}_{t,s}$ coincide, we can again determine this row space via a SVD of the matrix $R_{23,t}^T R_{22,t}$. Based on this result, we are able to generalize the PI scheme of [12] to ensemble identification problems. This extension, which will be summarized in the next paragraph, is then able to determine the quadruple of system matrices over the time-interval $t \in [t_0 + \mu, t_0 + T - 1 - \chi]$ for μ equal to $2s - 1$ and χ equal to s.

The PI-MOESP *scheme for ensemble identification problems:*

Given: the same information and conditions needed for the ordinary MOESP scheme for the ensemble identification problem.

Do the following:

Step 1: construct the sequence of compound generalized Hankel matrices $\left[\begin{array}{c|c|c} U_{t+1-i,s} & Z_{t+1-i,s} & U_{t+1-i-s,s} \end{array}\right]$ for $i = 0:1:s$.

Step 2: perform a QR factorization of the sequence of compound matrices $\left[\begin{array}{c|c|c} U_{t+1-i,s} & Z_{t+1-i,s} & U_{t+1-i-s} \end{array}\right]$ for $i = 0:1:s$, without storing Q and partition the R factor as in equation (3.59).

Step 3: Perform a SVD of the sequence of matrices $R_{23,t+1-i}^T R_{22,t+1-i}$ for $i = 0:1:s$ given as,

$$R_{23,t+1-i}^T R_{22,t+1-i} = U_{t+1-i} \left(\begin{array}{c|c} S_{N_{t+1-i}} & 0 \\ \hline 0 & 0 \end{array}\right) \left(\begin{array}{c} V_{N_{t+1-i}}^T \\ \hline (V_{N_{t+1-i}}^{\perp})^T \end{array}\right)$$

and store the matrix sequences $V_{N_{t+1-i}}, , V_{N_{t+1-i}}^{\perp}$ for $i = 0:1:s$.

Step 4: similar as Step 4 of the ordinary MOESP scheme for the ensemble identification problem.

3.5 Periodic discrete time system state space model identification

Linear periodic systems constitute an important class of linear systems and many mechanical and chemical plants exhibit periodic behavior[26]. Except for the intrinsic periodic systems, periodic discrete time systems also naturally arise when performing multirate (MR) (refer 5.2) or multi order (MO)[27] sampling to a linear time-invariant continuous system.

In most researches, a periodic discrete time system with m_t inputs, ℓ_t outputs and period P is first embedded into a time-invariant system with M_1^P inputs and L_1^P outputs (L_1^P and M_1^P defined as in Eq. (3.13), $L_1^P = P\ell$ and $M_1^P = Pm$ for the constant dimension case). Then all the techniques for the analysis and design of time invariant systems are applicable to periodic systems [28]. However, real time operations require that periodic systems be implemented as periodic systems. In this point of view, periodic system models become important[29].

As stated in [28], [29], [30], state space models are extremely well suited to analyze multirate sampled system. This in combination with the fact that many analysis and control design procedures are available for this class of systems makes it highly desirable of proper identification schemes which allow to identify the state space models of this class of systems. With the techniques in the previous section, we present a numerical scheme which allows us to address this task. First, we specialize the notations of the state space model (3.1)–(3.2) into present context.

Let $x_t \in \mathbb{R}^{N_t}$, $u_t \in \mathbb{R}^{m_t}$ and $y_t \in \mathbb{R}^{\ell_t}$ are state, input and output vectors in the state space description (3.1)–(3.2) and let this system be periodically time-varying with the period P. Then the system matrices A_t, B_t, C_t and D_t in (3.1)–(3.2) satisfy the following additional constraint,

$$A_{iP+t} = A_k, \quad B_{iP+k} = B_k, \quad C_{iP+k} = C_k, \quad D_{iP+k} = D_k. \tag{3.61}$$

where $1 \leq k < P$, $i \in \mathbb{Z}$.

3.5.1 Ordinary MOESP scheme for periodically time-varying systems

Let us consider $t_o = 1$ and let the following input and output sequences of a periodic system to be identified be given:

$$[u_1, u_2, \cdots u_t \cdots u_P, u_{P+1} \cdots u_{M_{tot}}]$$

$$[y_1, y_2, \cdots y_t \cdots y_P, y_{P+1} \cdots y_{M_{tot}}]$$

Then, we can rearrange the data into an ensemble of input (and output) sequences as in the ensemble identification problem, i.e.

$$\begin{bmatrix} u_1 & u_2 & \cdots & u_{M_{tot}-(n-1)P} \\ u_{P+1} & u_{P+2} & \cdots & u_{M_{tot}-nP} \\ \vdots & \vdots & \cdots & \vdots \\ u_{(n-1)P+1} & u_{(n-1)P+2} & \cdots & u_{M_{tot}} \end{bmatrix} \tag{3.62}$$

Then, it becomes possible to use the developed algorithm to identify the periodically time-varying system. However, because of the periodicity, two additional remarks need to be made.

First, we need to define the minimal length of the sequences for the identification. The minimal number of sequences given by Eq. (3.28) now is,

$$n_{\min} = \max_{t\in[1,P]}(M_t^s + N_t)$$

To obtain the last required row space $\mathcal{O}_{P,s}$, the generalized Hankel matrices $U_{P,s}$ (and $Y_{P,s}$) is needed. The $U_{P,s}$ is equal to,

$$\begin{bmatrix} u_P & u_{P+1} & \cdots & u_{P+s-1} \\ u_{2P} & u_{2P+1} & \cdots & u_{2P+s-1} \\ \vdots & \vdots & \cdots & \vdots \\ u_{nP} & u_{nP+1} & \cdots & u_{nP+s-1} \end{bmatrix} \tag{3.63}$$

As a consequence, the minimum M_{tot}

$$(M_{tot})_{min} = \max_{t\in[1,P]}(M_t^s + N_t))P + s - 1.$$

Second, we should guarantee that the calculated state space realization is also periodic. This can be done by using the appropriate row space of the extended observability matrix $\mathcal{O}_{t,s}$ at various time instances. We illustrate this only for calculation of system matrix α_t, since the periodicity of δ_t and γ_t are guaranteed by the same strategy.

The sequence of transformation matrices $\{\alpha_1, \cdots, \alpha_{P+1}\}$ are similarly equivalent with $\{A_1, A_2, \cdots, A_P, A_{P+1}\}$ as,

$$\alpha_1 = T_1^{-1} A_1 T_2, \quad \cdots, \quad \alpha_{P+1} = T_{P+1}^{-1} A_{P+1} T_{P+2} = T_{P+1}^{-1} A_1 T_{P+2} \quad \cdots \tag{3.64}$$

The periodicity of the similarly equivalent state representation gives the constraint $\alpha_{iP+k} = \alpha_k$, ect. for $1 \leq k < P$ and $i \in \mathbb{Z}$, then T_{iP+k} must be equal to T_k. In the identification, this can be done by setting the row space of the extended observability matrices $\mathcal{O}_{iP+k}$ to be equal to that of $\mathcal{O}_k$ and the correspondent complement space $V_{N_{iP+k}}^T$ to be equal to $V_{N_k}^T$ for $i \in Z$.

This analysis results in the following algorithm to identify periodically time-varying systems. We now summarize the steps of the ordinary MOESP algorithm for periodical time-varying system state space model identification:

Given:

1. a uniformly controllable and observable periodic system,
2. input and output sequences which can be rearranged into the ensemble input and output sequences as shown in Eq. (3.62) such that the series of matrices $U_{1,s}, \cdots, U_{P,s}$ are locally persistent excitation. This requirement rules out the use of periodic inputs with period equal to P,
3. $s > \delta_o$
4. $M_{tot} \geq (\max_{t \in [1,P]}(M_t^s + N_t))P + s - 1$.

For $t \in [1, P]$, do the following:

Step 1: construct the input and output matrix $U_{t,s}$ and $Y_{t,s}$.

Step 2: implement step 2 to step 3 in ordinary MOESP algorithm for ensemble identification problems to obtain the row spaces of $\mathcal{O}_{t,s}$ denoted by $V_{N_t}^T$ and their complement spaces $V_{N_t}^{\perp}$ and store these matrices.

Step 3: compute α_t by solving Eq. (3.32), and γ_t by Eq. (3.33). Note that $V_{N_{P+1}} = V_{N_1}$.

Step 4: compute β_t and δ_t by solving Eq. (3.34). Note again that $V_{N_0}^{\perp} = V_{N_P}^{\perp}$, $V_{N_{-1}}^{\perp} = V_{N_{P-1}}^{\perp}, \cdots$.

We should mention here that when the system is disturbed by a zero-mean white noise on the output, the experiment time running to infinity results in a consistent model in periodic system identification, which differs from time-varying system identification where the number of experiments running to infinity results in a consistent model.

3.5.2 A multirate sampling system example

Multirate sampling system

Multivariable multirate (MR) sampling systems occur in various areas of system implementation such as multirate controller design [30][31]. Different sampling periods for different variables are chosen in this kind of systems depending upon the characteristic of variables, frequency of measurements and other factors. Systems including multirate sampled data mechanisms with different sampling periods are called multirate sampling systems.

To demonstrate the operation of the algorithm for the identification of periodic systems, we analyze in this section a particular class of MR sampling systems where all sampling rate ratios must be rational numbers or where all sampling period must be integer multiples of a Smallest Time Period (STP). In [31], it is shown that such sampling policy applied to a linear, time-invariant system results in a periodic system description.

Let the STP be denoted by T. The period of repetition of the sampling schedule is called the Basic Time Period (BTP) which is the smallest common multiple of all sampling periods.

Model for the simulation experiment

In the present example, the linear continuous time-invariant system model is described by the following second order differential equation:

$$\frac{d^2 z}{dt^2} + C(t)\frac{dz}{dt} + K(t)z(t) = f(t)$$

where $f(t)$ is the input, $z(t)$ and $\frac{dz}{dt}$ are the outputs, and $C(t)$ and $K(t)$ are periodically time-varying scalars specified later.

Experiment data:

Smallest sampling time: T = 0.01 s,

Sampling period of input f: T,

Sampling period of output $\frac{dz}{dt}$: 2T,

Sampling period of output z: 3T,

BTP = 6T.

$C(t)$ and $K(t)$ are periodic time function with period BTP, and specified over one period as,

$$C(t) = 1 + 0.3\sin(\frac{\pi}{6}t) \qquad 0 \le t \le 6T$$

$$K(t) = 0.5 + 0.02t \qquad 0 \le t \le 6T$$

The sampling schedule is described in Figure 3.1.

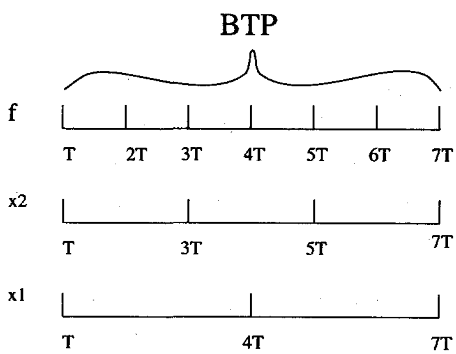

Figure 3.1: Sampling periods for input and output

This MR sampling system can be easily described and simulated as a SIMO system in a state space model with the changing dimensions of input and output at different sampling instant. Hence, the identification for such system is naturally addressed within this framework. Simulation and realization of such a system in a parametric framework appear difficult.

Let $x_1 = z$, $x_2 = \frac{dz}{dt}$, then $x = [\ x_1 \quad x_2\ ]$. The continuous time state space model is

$$\frac{dx}{dt} = x \begin{bmatrix} 0 & -K \\ 1 & -C \end{bmatrix} + f(t) \begin{bmatrix} 0 & 1 \end{bmatrix}$$

The simulation experiment is carried out on the MATLAB. By using zero-order hold method, we transform the continuous time model to the discrete time model as:

$$x_{k+1} = x_k A_k + f_k B_k$$

where A_k and B_k change periodically. Because of the multirate sampling, the C-matrices of the system are:

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad C_2 = [] \quad C_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$C_4 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C_6 = []$$

Experiment

We set up two Monte Carlo simulation tests with both SNR equal to 30 dB. Input f is a white noise with variance equal to 1. Let s equal to 10. In this example, $P = 6$, $\max_{t\in(1,P)}(M_t^{10} + N_t) = 12$, then the $min(M_{tot}) = 81$. We used two different M_{tot} to test the properties of unbiasedness and consistency of the algorithm: $M_{tot_1} = 150$ and $M_{tot_2} = 600$. 100 runs were performed for each test.

The quantities x_1 and x_2 denote the noiseless output, v_1 and v_2 denote the added output noise and they are zero-mean white. The output measurements are:

$$x_{1,m} = x_1 + v_1$$
$$x_{2,m} = x_2 + v_2$$

The identified models were compared in two aspects as discussed below.

First, we consider the transition matrix of the given discretized system over one period, that is $A_1 A_2 \cdots A_6$ and the same for the identified model, that is $\alpha_1\alpha_2\cdots\alpha_6$. Using the relationship between α_i and A_i as in Eq. (3.64) and the relation $T_1 = T_{P+1}$, we have:

$$A_1 A_2 \cdots A_6 = T_1^{-1}\alpha_1\alpha_2\cdots\alpha_6 T_1$$

Therefore, in the noise-free case, the eigenvalues of these two matrices coincide. In the noisy data case, we can only expect the eigenvalues of the matrix $\alpha_1\alpha_2\cdots\alpha_6$ to be asymptotically unbiased estimate of those of the matrix $A_1A_2\cdots A_6$. Thus, as a first measure of the performance of the developed algorithm, we depict in Figure Eig, the eigenvalues of these two matrices. In Figure 3.2, the center of the cross in the figure denotes the true eigenvalue location (for the sake of clarity, only one eigenvalue is shown, another eigenvalue is conjugate to this one). The eigenvalues of the resulted models are marked by the sign 'o'. From the figure we can see that under the condition that s remains the same, when M_{tot} increases, the eigenvalues of the obtained model unbiasedly approach the true eigenvalues when the noise on the output is zero-mean white. Thus this experiment conforms Corollary 3.1.

Second, we compare the output error between the outputs $x_{1,m}$, $x_{2,m}$ and the reconstructed outputs $\hat{x_{1,m}}$, $\hat{x_{2,m}}$ by using the estimated periodic model over a time interval of 600 input samples. The results of this comparison are presented in Figure 3.3. Here, the solid line represents the noise level on the output as indicated by $std(v_1)$ and $std(v_2)$, where 'std' denotes the standard deviation. The other types of lines show the quantities of $std(\hat{x}_1 - x_{1,m})$ and $std(\hat{x}_2 - \dot{x}_{2,m})$. When the reconstruction would be done with the original system, the latter quantities are then equal to the noise level. From this figure we can see that when M_{tot} becomes larger, these quantities approaches the noise level. When $M_{tot} = 600$, the difference between these quantities and the noise level is very small. These results also conform the consistency of the identified models.

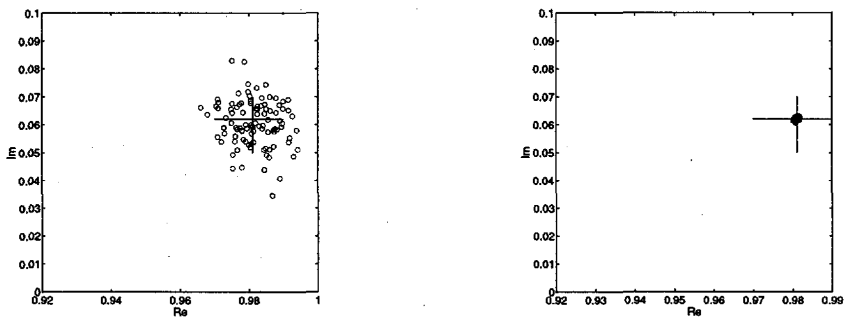

Figure 3.2: Eigenvalue distribution of the Monte Carlo tests. Left figure: $M_{tot} = 150$. Right figure: $M_{tot} = 600$.

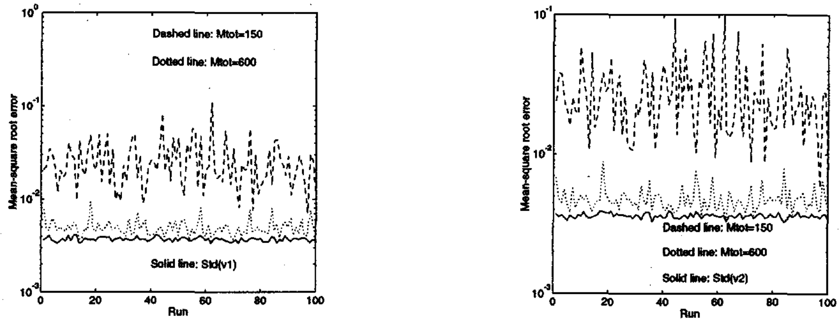

Figure 3.3: Error level comparason. Left figure: $std(v_1)$, $std(\hat{x_1} - x_{1,m})$. Right figure: $std(v_2)$, $std(\hat{x}_2 - x_{2,m})$.

3.6 Conclusions

In this chapter, algorithms for identifying a discrete, linear time-varying state space model were introduced. The algorithms require a multiple series of experiments, each time recorded the input and output sequences when the underlying system undergoes the same time-varying behavior. The algorithm were extended for periodic discrete systems. Here the repetition of the experiment is intrinsic. An application of the developed schemes to the identification of a multivariable multirate sampled data system demonstrates the usefulness of these schemes. The results of the simulation experiment showed that the algorithm allows to consistently estimate a state space model for such systems when the noise on the output is zero-mean white.

The additional usefulness of the schemes developed in this chapter was demonstrated in [6]. In this paper, a realistic identification problem, namely the identification of the human joint dynamics, was considered. For this practical application, the schemes, especially the PI scheme for the ensemble identification problem, allowed to identify low order state space models and lead to accurate reconstruction of the output. In the same paper, the results were compared with recursive output error identification schemes using exponential forgetting factors between 0.9 and 0.95 and this comparison study clearly showed that the latter approach was completely inadequate in identifying the underlying foot dynamics.

Further research on the use of the numerical schemes developed is proposed.

3.7 Appendix: An illustration of Definition 3.6.

Assume there is a uniformly controllable, linear and time-varying system, the input dimension is 1 and the input sequences are:

$$u_{j,t} = \begin{cases} 1 & j = t \\ 0 & \text{otherwise} \end{cases}$$

With $j \in [1, n]$ and $t \in [1, T]$, then the input sequence matrix $U_{1,T}$ for $T > n$ is of the type:

$$U_{1,T} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix} \tag{3.65}$$

For simplicity, suppose that the initial state is zero, the order of the system is 2 in the interval [1,T], $\delta_c = 2$ and the window length is chosen so that $n - s \geq \delta_c$. In this example, we look at s = 4. The matrix $[U_{1,4}|X_{(1,1),n}]$ reads:

$$[U_{1,4}|X_1] = \begin{bmatrix} 1 & & & & 0 & 0 \\ & 1 & & & 0 & 0 \\ & & 1 & & 0 & 0 \\ & & & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

for the zero initial condition. The rank of the matrix is 4 and the matrix is not a locally persistent excitation at time instant t=1.

Next, for the same experiment, consider the matrix $[U_{3,4}|X_3]$:

$$[U_{3,4}|X_3] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & B_1A_2 \\ 0 & 0 & 0 & 0 & B_2 \\ \hline 1 & & & & 0\,0 \\ & 1 & & & 0\,0 \\ & & 1 & & 0\,0 \\ & & & 1 & 0\,0 \\ \hline 0 & 0 & 0 & 0 & 0\,0 \\ \vdots & \vdots & \vdots & \vdots & \vdots\,\vdots \\ 0 & 0 & 0 & 0 & 0\,0 \end{array}\right]$$

Since $\delta_c = 2$, Lemma 3.3 shows that the submatrix $\left[\begin{array}{c} B_1A_2 \\ B_2 \end{array}\right]$ has full rank. Therefore the matrix $[U_{3,4}|X_3]$ has full column rank.

Since $n - s \geq \delta_c$, Lemma 3.3 shows that the last matrix in the sequence $\{[U_{t,4}|X_t]\}$ for $t > 1$ which has full column rank is the matrix

$$[U_{n-s+1,4}|X_{n-s+1}] = \left[\begin{array}{cccc|c} & & & & B_1A_2\cdots A_{n-s} \\ & & & & \vdots \\ & & & & B_{n-s}A_{n-s} \\ & & & & B_{n-s} \\ \hline 1 & & & & 0\,0 \\ & 1 & & & 0\,0 \\ & & 1 & & 0\,0 \\ & & & 1 & 0\,0 \end{array}\right]$$

In summary, when using dirac impulse as input and zero initial state condition, the matrices $[U_{t,s}|X_t]$ for $t \in [\delta_c + 1, n - s + 1]$ are full rank matrices for $n - s \geq \delta_c$. □

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Chapter 4

Lossless Operators, J-lossless Operators and Their Properties

4.1 Introduction

Lossless and J-lossless operators (functions) play an important role in system and control engineering [1] [2] [5] because of their many useful and elegant properties. For example, the energy conservation property of lossless operators (functions) is used in the orthogonal embedding problem for filter design and lossless cascade factorization [2]; the homographic transformation property of J-lossless functions is used in H_∞ control for time invariant systems [1] and for the solution of the interpolation problem [3] [4]. Lossless and J-lossless operators (functions or systems) and their properties are well known in the time invariant context [6]. In this chapter, we consider lossless and J-lossless operators and their properties in linear discrete time-varying context for the purpose of H_∞ control in this context. The properties of lossless and J-lossless operators in a time-varying context are very similar to that of their time invariant counterparts. The content of this chapter is based on [2] but contrary to the result in [2], it is not restricted to upper operators. Also we are more interested in J-lossless operators in our application. J-lossless operators are known as the chain scattering operators of the corresponding lossless operators which are known as the scattering operators. They are representations of the same physical systems. The properties of lossless operators are easier to be understood and well known. Because of the close relationship between J-lossless operators and lossless operators, a brief discussion of the latter helps us for a better understanding of the former.

4.2 Lossless operators

Definition 4.1 *An operator $\Sigma \in \mathcal{X}$ is an* isometry *if $\Sigma\Sigma^* = I$, a* co-isometry *if $\Sigma^*\Sigma = I$ and* unitary *if both $\Sigma\Sigma^* = I$ and $\Sigma^*\Sigma = I$.* □

A special case for an *isometric operator* or a *co-isometric operator* occurs when the operator is upper. We have the following definition for this kind of operators.

Definition 4.2 *An isometric operator Σ is called* lossless *if $\Sigma \in \mathcal{U}$. A co-isometric operator Σ is called* co-lossless *if $\Sigma \in \mathcal{U}$. A unitary operator Σ is called* inner *if $\Sigma \in \mathcal{U}$. In this case, Σ is both lossless and co-lossless.* □

From these two definitions, we can readily deduce that the conjugator of a lower co-isometric operator is lossless and the conjugator of a lower isometric operator is co-lossless.

Theorem 4.1 *Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ and $\ell_{A_\Sigma} < 1$. $\Sigma = D_\Sigma + B_\Sigma \mathbf{Z}(I - A_\Sigma \mathbf{Z})^{-1} C_\Sigma$. Σ is an isometry iff there exists a* Hermitian operator *$Q \in \mathcal{D}$ such that*

$$\begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix} \begin{bmatrix} Q^{(-1)} & \\ & I \end{bmatrix} \begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix}^* = \begin{bmatrix} Q & \\ & I \end{bmatrix} \tag{4.1}$$

Σ is a co-isometry iff there exists a Hermitian operator $P \in \mathcal{D}$ such that:

$$\begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix}^* \begin{bmatrix} P & \\ & I \end{bmatrix} \begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix} = \begin{bmatrix} P^{(-1)} & \\ & I \end{bmatrix} \tag{4.2}$$

Σ is unitary iff both (4.1) and (4.2) are satisfied.

If $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ is a uniform realization, then $Q \gg 0$ and $P \gg 0$; if Σ is also unitary, then $P = Q^{-1}$.

Proof: Sufficiency: *Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ and $\ell_{A_\Sigma} < 1$. Assume that the conditions given by (4.1) are satisfied. Then $\Sigma\Sigma^* = I$.*

Rewrite the conditions as

$$\begin{aligned} A_\Sigma Q^{(-1)} A_\Sigma^* + C_\Sigma C_\Sigma^* &= Q \\ A_\Sigma Q^{(-1)} B_\Sigma^* + C_\Sigma D_\Sigma^* &= 0 \\ B_\Sigma Q^{(-1)} B_\Sigma^* + D_\Sigma D_\Sigma^* &= I \end{aligned}$$

Then:

$$\Sigma\Sigma^* = [D_\Sigma + B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}C_\Sigma][D_\Sigma^* + C_\Sigma^*(\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^*]$$
$$= D_\Sigma D_\Sigma^* + D_\Sigma C_\Sigma^*(\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^* + B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}C_\Sigma D_\Sigma^*$$
$$+B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}C_\Sigma C_\Sigma^*(\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^*$$
$$= D_\Sigma D_\Sigma^* - B_\Sigma Q^{(-1)}A_\Sigma^*(\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^* - B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}A_\Sigma Q^{(-1)}B_\Sigma^*$$
$$+B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}C_\Sigma C_\Sigma^*(\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^*$$
$$= D_\Sigma D_\Sigma^* + B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}[-(\mathbf{Z}^* - A_\Sigma)Q^{(-1)}A_\Sigma^* - A_\Sigma Q^{(-1)}(\mathbf{Z} - A_\Sigma^*) + C_\Sigma C_\Sigma^*](\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^*$$
$$= D_\Sigma D_\Sigma^* + B_\Sigma(\mathbf{Z}^* - A_\Sigma)^{-1}[-\mathbf{Z}^*Q^{(-1)}A_\Sigma^* + A_\Sigma Q^{(-1)}A_\Sigma^* - A_\Sigma Q^{(-1)}\mathbf{Z} + Q](\mathbf{Z} - A_\Sigma^*)^{-1}B_\Sigma^*$$
$$= D_\Sigma D_\Sigma^* + B_\Sigma Q^{(-1)}B_\Sigma^* = I$$

$\Sigma^*\Sigma = I$ can be proved in a similar way. Then Σ is an isometry, a co-isometry or a unitary operator respectively, if condition (4.1), (4.2) or both are satisfied from Definition 4.1.

Next, we prove that the conditions given by (4.1) and (4.2) are also necessary for an isometric and a co-isometric operator, respectively. Again, we only give the proof for an isometric operator, the proof for a co-isometric operator is in a similar way.

Necessity: *Let $\Sigma \in \mathcal{U}$ be a locally finite operator with a realization $\{A_\Sigma, B_\Sigma, C_\Sigma, D_\Sigma\}$ and $\ell_{A_\Sigma} < 1$. Assume that $\Sigma\Sigma^* = I$. Then, conditions defined by (4.1) are satisfied.*

In particular we have $\mathbf{P}_0(\Sigma\Sigma^*) = \Sigma\Sigma^* = I$. Define $\mathbf{F}_o = (I - A_\Sigma\mathbf{Z})^{-1}C_\Sigma$, so that $\Sigma = D_\Sigma + B_\Sigma\mathbf{Z}\mathbf{F}_o$. Hence,

$$\begin{aligned}\mathbf{P}_0(\Sigma\Sigma^*) &= \mathbf{P}_0(D_\Sigma D_\Sigma^*) + \mathbf{P}_0(D_\Sigma\mathbf{F}_o^*\mathbf{Z}^*B_\Sigma^*) + \mathbf{P}_0(B_\Sigma\mathbf{Z}\mathbf{F}_oD_\Sigma^*) + \mathbf{P}_0(B_\Sigma\mathbf{Z}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^*B_\Sigma^*)\\ &= D_\Sigma D_\Sigma^* + 0 + 0 + B_\Sigma\mathbf{P}_0(\mathbf{Z}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^*)B_\Sigma^* \qquad (4.3)\end{aligned}$$

Let $Q = \mathbf{P}_0(\mathbf{F}_o\mathbf{F}_o^*)$, then $\Sigma\Sigma^* = I$ indicates

$$D_\Sigma D_\Sigma^* + B_\Sigma Q^{(-1)}B_\Sigma^* = I$$

and Q satisfies the recursion

$$\begin{aligned}Q = \mathbf{P}_0(\mathbf{F}_o\mathbf{F}_o^*) &= \mathbf{P}_0[(I - A_\Sigma\mathbf{Z})^{-1}C_\Sigma C_\Sigma^*(I - \mathbf{Z}^*A_\Sigma^*)^{-1}]\\ &= \mathbf{P}_0(C_\Sigma C_\Sigma^*) + \mathbf{P}_0(A_\Sigma\mathbf{Z}(I - A_\Sigma\mathbf{Z})^{-1}C_\Sigma C_\Sigma^*(I - \mathbf{Z}^*A_\Sigma^*)^{-1}\mathbf{Z}^*A_\Sigma^*)\\ &= C_\Sigma C_\Sigma^* + A_\Sigma Q^{(-1)}A_\Sigma^*\end{aligned}$$

Next, we show that $C_\Sigma D_\Sigma^* + A_\Sigma Q^{(-1)}B_\Sigma^* = 0$. Let us look at

$$\mathbf{P}_0(\mathbf{Z}^{-n}\Sigma\Sigma^*)$$
$$= \mathbf{P}_0(\mathbf{Z}^{-n}(D_\Sigma + B_\Sigma\mathbf{Z}\mathbf{F}_o)(D_\Sigma^* + \mathbf{F}_o^*\mathbf{Z}^*B_\Sigma^*))$$
$$= \mathbf{P}_0(\mathbf{Z}^{-n}D_\Sigma D_\Sigma^*) + \mathbf{P}_0(\mathbf{Z}^{-n}D_\Sigma\mathbf{F}_o^*\mathbf{Z}^*B_\Sigma^*) + \mathbf{P}_0(\mathbf{Z}^{-n}B_\Sigma\mathbf{Z}\mathbf{F}_oD_\Sigma^*)$$

$$+\mathbf{P}_0(\mathbf{Z}^{-n}B_\Sigma\mathbf{Z}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^*B_\Sigma^*) \tag{4.4}$$

If $n > 0$, the first and second terms in the expansion are equal to zero. The third term

$$\begin{aligned}
&\mathbf{P}_0(\mathbf{Z}^{-n}B_\Sigma\mathbf{Z}\mathbf{F}_oD_\Sigma^*)\\
&= \mathbf{P}_0(B_\Sigma^{(n)}\mathbf{Z}^{-(n-1)}\mathbf{F}_oD_\Sigma^*)\\
&= B_\Sigma^{(n)}\mathbf{P}_0(\mathbf{Z}^{-(n-1)}(I-A_\Sigma\mathbf{Z})^{-1})C_\Sigma D_\Sigma^*\\
&= B_\Sigma^{(n)}A_\Sigma^{\{n-1\}}C_\Sigma D_\Sigma^*
\end{aligned}$$

and the fourth term

$$\begin{aligned}
&\mathbf{P}_0(\mathbf{Z}^{-n}B_\Sigma\mathbf{Z}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^*B_\Sigma^*)\\
&= B_\Sigma^{(n)}\mathbf{P}_0(\mathbf{Z}^{-(n-1)}\mathbf{F}_o\mathbf{F}_o^*\mathbf{Z}^*)B_\Sigma^*\\
&= B_\Sigma^{(n)}A_\Sigma^{\{n-1\}}A_\Sigma Q^{(-1)}B_\Sigma^*
\end{aligned}$$

Substituting the results of the third and fourth terms into (4.4) we obtain:

$$\mathbf{P}_0(\mathbf{Z}^{-n}\Sigma\Sigma^*) = B_\Sigma^{(n)}A_\Sigma^{\{n-1\}}(C_\Sigma D_\Sigma^* + A_\Sigma Q^{(-1)}B_\Sigma^*)$$

Because $\Sigma\Sigma^*$ is diagonal, its off-diagonal elements are zero, thus $\mathbf{P}_0(Z^{-n}\Sigma\Sigma^*) = 0$ for $n \neq 0$. Then $C_\Sigma D_\Sigma^* + A_\Sigma Q^{(-1)}B_\Sigma^* = 0$.

For $n < 0$, we obtain another necessary condition which is equivalent to $C_\Sigma D_\Sigma^* + A_\Sigma Q^{(-1)}B_\Sigma^* = 0$. Thus we have proved that Σ is isometric iff the conditions given by (4.1) are satisfied.

$Q \gg 0$ and $P \gg 0$ follow immediately after the definitions of uniform reachability and uniform observability. If Σ is also unitary, both conditions of (4.1) and (4.2) should be satisfied and since Q is an invertible Hermitian operator, $\begin{bmatrix} Q & \\ & I \end{bmatrix}$ can be factorized into

$$\begin{bmatrix} Q & \\ & I \end{bmatrix} = \begin{bmatrix} R^{-1} & \\ & I \end{bmatrix}\begin{bmatrix} R^{-*} & \\ & I \end{bmatrix}$$

Define,

$$\begin{bmatrix} A & C \\ B & D \end{bmatrix} := \begin{bmatrix} R & \\ & I \end{bmatrix}\begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix}\begin{bmatrix} R^{-(-1)} & \\ & I \end{bmatrix}$$

then we have $\begin{bmatrix} A & C \\ B & D \end{bmatrix}\begin{bmatrix} A & C \\ B & D \end{bmatrix}^* = I$ and $\begin{bmatrix} A & C \\ B & D \end{bmatrix}^*\begin{bmatrix} A & C \\ B & D \end{bmatrix} = I$. In comparison with equation (4.2), it is not difficult to obtain: $P = Q^{-1}$. □

A realization matrix $\mathbf{\Sigma} \in \mathcal{D}$

$$\mathbf{\Sigma} = \begin{bmatrix} A & C \\ B & D \end{bmatrix}$$

is said to be unitary if

$$\mathbf{\Sigma\Sigma}^* = I \qquad \textit{and} \qquad \mathbf{\Sigma}^*\mathbf{\Sigma} = I$$

This kind of realization can be obtained by a similarity transformation which is based on the observability Gramian Q or reachability Gramian P. Let $\mathbf{\Sigma} = \begin{bmatrix} A_\Sigma & C_\Sigma \\ B_\Sigma & D_\Sigma \end{bmatrix}$ be a known realization of a unitary operator Σ, and let Q and P be the observability and reachability Gramians of Σ. Let $Q = (R^*R)^{-1}$ or $P = R^*R$ (This is possible since by Theorem 4.1, $Q \gg 0$ and $P = Q^{-1}$). Then R defines a state transformation which leads to a unitary realization $\{A, B, C, D\}$, where $A = RA_\Sigma R^{-(-1)}$, $B = B_\Sigma R^{-(-1)}$, $C = RC_\Sigma$ and $D = D_\Sigma$. We say that the pair $[A, C]$ is in *output normal form* because $AA^* + CC^* = I$ and $[A, B]$ is in *input normal form* because $A^*A + B^*B = I$. For an isometric operator or a co-isometric operator, we can consider a similar transformation. The difference is that for an isometric operator, we define $Q = (R^*R)^{-1}$ and only obtain an output normal pair $[A, C]$. For a co-isometric operator, we define $P = R^*R$ and only obtain an input normal pair $[A, B]$. The transformed realizations are called isometric and co-isometric, respectively.

4.3 (J_2, J_1)-lossless operators

Referring to Figure 4.1 (a), let Σ be a known operator, mapping the input $[\ a_1 \quad b_2\ ]$ to the output $[\ a_2 \quad b_1\ ]$, i.e.

$$[\ a_2 \quad b_1\ ] = [\ a_1 \quad b_2\ ]\Sigma = [\ a_1 \quad b_2\ ]\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \tag{4.5}$$

In the Figure, the variable with a dot stands for an input of the mapping and without a dot stands for an output. If Σ_{22} is invertible, we can derive the mapping from $[\ a_1 \quad b_1\ ]$

to $[\ a_2 \quad b_2\ ]$, denoted by Θ in Figure 4.1(b) from Σ as

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & -\Sigma_{12}\Sigma_{22}^{-1} \\ \Sigma_{22}^{-1}\Sigma_{21} & \Sigma_{22}^{-1} \end{bmatrix} \tag{4.6}$$

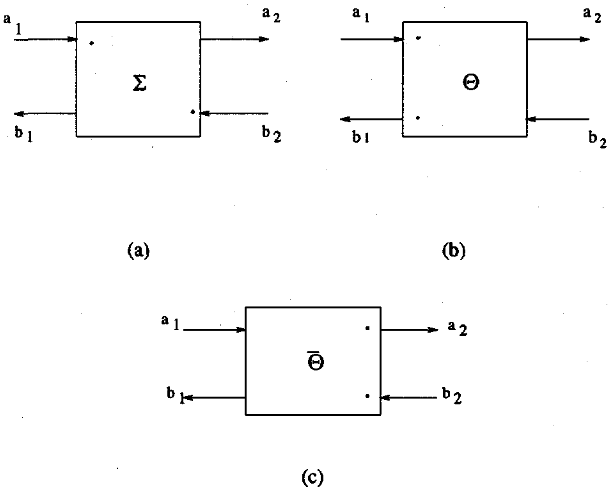

Figure 4.1: Scattering operator Σ, chain scattering operator Θ and the dual chain scattering operator $\bar{\Theta}$.

If Σ is a *scattering operator*, then Θ is known as the *chain scattering operator* of Σ.

If Σ_{11} is invertible, we can derive a mapping from $[\ a_2 \quad b_2\ ]$ to $[\ a_1 \quad b_1\ ]$, denoted by $\bar{\Theta}$, from Σ as,

$$\bar{\Theta} = \begin{bmatrix} \bar{\Theta}_{11} & \bar{\Theta}_{12} \\ \bar{\Theta}_{21} & \bar{\Theta}_{22} \end{bmatrix} = \begin{bmatrix} \Sigma_{11}^{-1} & \Sigma_{11}^{-1}\Sigma_{12} \\ -\Sigma_{21}\Sigma_{11}^{-1} & \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12} \end{bmatrix} \tag{4.7}$$

$\bar{\Theta}$ is called the *dual chain scattering operator* of Σ.

If we introduce a feedback relation $b_1 = a_1 S$ between b_1 and a_1, then the closed loop mapping from b_2 to a_2, denoted by Φ, is given by

$$\Phi = \Sigma_{21} + \Sigma_{22}(I - S\Sigma_{12})^{-1}S\Sigma_{11} \tag{4.8}$$

It is not difficult to obtain the equation

$$\Phi = \mathrm{HM}(\Theta; S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) \tag{4.9}$$

where HM stands for *HoMographic transformation* and

$$\Phi = \mathrm{DHM}(\bar{\Theta}; S) = (\bar{\Theta}_{21} - \bar{\Theta}_{22}S)(\bar{\Theta}_{12}S - \bar{\Theta}_{11})^{-1} \tag{4.10}$$

where DHM stands for the *Dual of the HoMographic transformation.* Before the discussion of J-lossless operator, let us look at port signature operators which describe the input/output relation of the variables of the system in different descriptions: the chain scattering description or the dual of the chain scattering description.

In Figure 4.1, we use a dot to indicate the variables of the *input port*, or these variables are called *input port variables.* We use arrows to indicate the flow direction of the variables, or the variables with arrows into the block are *input variables* and with arrows out of the block are *output variables.*

Let $J_1 \in \mathcal{D}$ be the *input port signature* and $J_2 \in \mathcal{D}$ the *output port signature matrices* which are defined as $J_i = \begin{bmatrix} \ddots & & & & \\ & j_{i,-1} & & & \\ & & j_{i,0} & & \\ & & & j_{i,1} & \\ & & & & \ddots \end{bmatrix}$ for $i = 1, 2$, where the entry $j_{i,k} = \begin{bmatrix} I_{(p+)_k} & \\ & -I_{(p-)_k} \end{bmatrix}$ $(k = -\infty, \cdots, +\infty)$ is determined by the dimensions of the input and output of the ports at time instant k. For a chain scattering operator, the dimension of input variables on the input port is $p+$ and the dimension of output variables on the input port is $p-$. It is reversed on the output port. For a dual chain scattering operator, the dimension of output variables on the input port is $p+$ and the dimension of input variables on the input port is $p-$. It is reversed on the *output port.*

Definition 4.3 *Let J_1 and J_2 be the input and output signature operators respectively of a known operator $\Theta \in \mathcal{X}$. Θ is a (J_2, J_1)-*isometry *if $\Theta J_2 \Theta^* = J_1$, a (J_1, J_2)-*co-isometry *if $\Theta^* J_1 \Theta = J_2$ and* J-unitary *if both $\Theta J_2 \Theta^* = J_1$ and $\Theta^* J_1 \Theta = J_2$.* □

Theorem 4.2 *Let an operator $\Sigma \in \mathcal{X}$ be isometric, co-isometric or unitary, respectively. If the corresponding chain scattering operator Θ, with J_1 and J_2 the input and output signature operators, exists, then Θ is (J_2, J_1)-isometric, (J_1, J_2)-co-isometric or J-unitary, respectively. If the corresponding dual chain scattering operator $\bar{\Theta}$, with J_1 and J_2 the input and output signature operators, exists, then $\bar{\Theta}$ is (J_2, J_1)-isometric, (J_1, J_2)-co-isometric or J-unitary, respectively.*

Proof: For the proof of the first statement, we refer to [2]. The second statement is proved in a similar way. □

If $\mathcal{H}$ is a locally finite $\mathcal{D}$-invariant subspace, then it has some strong basis representation $\mathbf{F}$ such that $\mathcal{H} = \mathcal{D}_2\mathbf{F}$. In analogy with the definition of Gramian operator $\Lambda_F = \mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$ in Section 2.4 of Chapter 2, we define the *J-Gramian operator* of this basis as the diagonal operators:

$$\Lambda_{\mathbf{F}}^J = \mathbf{P}_0(\mathbf{F}J\mathbf{F}^*) \quad \in \quad \mathcal{D}(\mathcal{B},\mathcal{B})$$

$\mathbf{F}$ is *J-orthonormal* if $\Lambda_{\mathbf{F}}^J = J_{\mathcal{B}}$, where $J_{\mathcal{B}}$ is some signature operator on $\mathcal{B}$. We call $\mathcal{H}$ *regular* if the J-Gramian operator of any strong basis is bounded invertible. Note that $\Lambda_{\mathbf{F}}^J$ bounded invertible implies $\Lambda_{\mathbf{F}} \gg 0$ but the converse is not true. From the definition of J-orthonormal basis we know that a J-orthonormal basis is a strong basis.

Let $T \in \mathcal{U}$ have a uniformly minimal realization $\{A, B, C, D\}$ with $\ell_A < 1$ and J_1 and J_2 be the input and output signature operators. Then $\mathbf{F}^* = B\mathbf{Z}(I - A\mathbf{Z})^{-1}$ and $\mathbf{F}_o = (I - A\mathbf{Z})^{-1}C$ are the strong basis of $\mathcal{H}(T)$ and $\mathcal{H}_o(T)$ respectively. If $\mathbf{P}_0(\mathbf{F}J_1\mathbf{F}^*)$ and $\mathbf{P}_0(\mathbf{F}_oJ_2\mathbf{F}_o^*)$ are invertible, we say the realization $\{A, B, C, D\}$ is regular. A *regular realization* of a bounded lower operator or a mixed operator are defined in a similar way.

The next theorem is an analogue of Theorem 4.1.

Theorem 4.3 *Let $\Theta \in \mathcal{U}$ be a locally finite operator with a realization $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ and $\ell_{A_\Theta} < 1$. Then Θ is a (J_2, J_1)-isometry iff there exists a Hermitian operator $Q \in \mathcal{D}$ such that:*

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} Q^{(-1)} & \\ & J_2 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} Q & \\ & J_1 \end{bmatrix} \tag{4.11}$$

a (J_1, J_2)-co-isometry iff there exists a Hermitian operator $P \in \mathcal{D}$ such that:

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* \begin{bmatrix} P & \\ & J_1 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} = \begin{bmatrix} P^{(-1)} & \\ & J_2 \end{bmatrix} \tag{4.12}$$

J-unitary iff both (4.11) and (4.12) are satisfied. If $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ is a regular realization of Θ, then Q and P are invertible; if Θ is also J-unitary and $J_1 = J_2$, then $P = Q^{-1}$.
□

Theorem 4.4 *Let $\Theta \in \mathcal{L}$ be a locally finite operator with a realization $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ and $\ell_{A_\Theta} < 1$ so that $\Theta = D_\Theta + B_\Theta \mathbf{Z}^*(I - A_\Theta \mathbf{Z}^*)^{-1} C_\Theta$. Θ is a (J_2, J_1)-isometry iff there exists a Hermitian operator $Q \in \mathcal{D}$ such that*

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} -Q & \\ & J_2 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} -Q^{(-1)} & \\ & J_1 \end{bmatrix} \tag{4.13}$$

a (J_1, J_2)-co-isometry iff there exists a Hermitian operator $P \in \mathcal{D}$ such that

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* \begin{bmatrix} -P^{(-1)} & \\ & J_1 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} = \begin{bmatrix} -P & \\ & J_2 \end{bmatrix} \tag{4.14}$$

and Θ is J-unitary if both (4.13) and (4.14) are satisfied. If Θ is (J_2, J_1)-isometric or (J_1, J_2)-co-isometric, and $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ is a regular realization of Θ, then Q or P are invertible; if Θ is also J-unitary and $J_1 = J_2$, then $P = Q^{-1}$. □

In the case that $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ is a regular realization of Θ, Q in equation (4.11) can be factorized as $Q = (R^* J_\mathcal{B} R)^{-1}$, P in equation (4.12) as $P = S^* J'_\mathcal{B} S$, where

$$J_\mathcal{B} = \begin{bmatrix} \ddots & & & \\ & j_{b,-1} & & & \\ & & j_{b,0} & & \\ & & & j_{b,1} & \\ & & & & \ddots \end{bmatrix}$$

$$j_{b,i} = \begin{bmatrix} I_+ & \\ & -I_- \end{bmatrix}, \text{ for } i = -\infty, \ldots, \infty.$$

$J_\mathcal{B}$ is the state signature matrix (see [2] for the detailed description) and $J'_\mathcal{B}$ has the same definition. Because Q and P are invertible diagonal operators, R and S are invertible diagonal operators. Define $A'_\Theta = R A_\Theta R^{-(-1)}$, $B'_\Theta = B_\Theta R^{-(-1)}$ and $C'_\Theta = R C_\Theta$ in equation (4.11) (or $A'_\Theta = S A_\Theta S^{-(-1)}$, $B'_\Theta = B_\Theta S^{-(-1)}$ and $C'_\Theta = S C_\Theta$). Then equation (4.11) changes into:

$$\begin{bmatrix} A'_\Theta & C'_\Theta \\ B'_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} J_\mathcal{B}^{(-1)} & \\ & J_2 \end{bmatrix} \begin{bmatrix} A'_\Theta & C'_\Theta \\ B'_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} J_\mathcal{B} & \\ & J_1 \end{bmatrix} \tag{4.15}$$

and equation (4.12) into:

$$\begin{bmatrix} A'_\Theta & C'_\Theta \\ B'_\Theta & D_\Theta \end{bmatrix}^* \begin{bmatrix} J'_\mathcal{B} & \\ & J_1 \end{bmatrix} \begin{bmatrix} A'_\Theta & C'_\Theta \\ B'_\Theta & D_\Theta \end{bmatrix} = \begin{bmatrix} J'^{(-1)}_\mathcal{B} & \\ & J_2 \end{bmatrix} \tag{4.16}$$

We say that $[A'_\Theta, C'_\Theta]$ in Eq. (4.15) is in J-output normal form because $A'_\Theta J_\mathcal{B}^{(-1)} A'^*_\Theta + C'_\Theta J_2 C'^*_\Theta = J_\mathcal{B}$ and $[A'_\Theta, B'_\Theta]$ in Eq. (4.16) in J-input normal form because $A'^*_\Theta J'_\mathcal{B} A'_\Theta + B'^*_\Theta J_1 B'_\Theta = J'^{(-1)}_\mathcal{B}$. Similar to what we have stated in the previous section about unitary operators, the realization in (4.15) is called J-isometric and the realization in (4.16) is called J-co-isometric. If both (4.15) and (4.16) are satisfied, the realization is called J-unitary.

Definition 4.4 *If an operator Σ is lossless, the corresponding chain scattering operator Θ, with J_1 and J_2 the input and output signature operator, respectively, is* (J_2, J_1)-lossless. *If an operator Σ is co-lossless, the corresponding dual chain scattering operator $\bar{\Theta}$, with J_1 and J_2 the input and output signature operator, respectively, is* conjugated (J_1,J_2)-lossless.

□

Note that a (J_2, J_1)-lossless operator Θ and conjugated (J_1, J_2)-lossless operator $\bar{\Theta}$ are not necessarily upper but a lossless operator or a co-lossless operator Σ is upper.

Proposition 4.1 *Let $\Theta \in \mathcal{U}$ have a uniform realization $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ and $\ell_{A_\Theta} < 1$ such that $\Theta = D_\Theta + B_\Theta \mathbf{Z}(I - A_\Theta \mathbf{Z})^{-1} C_\Theta$, and be a (J_2, J_1)-isometric operator with Σ as the corresponding isometric operator. If there exists a $Q \gg 0$ such that equation (4.11) is satisfied, then Σ is lossless and Θ is (J_2, J_1)-lossless.*

Proof: We adopt the proof of [2] (P. 166), where it is proved that Θ is J-lossless iff $J_\mathcal{B} = I$ in (4.15). This implies that all the states of the corresponding Σ are causal with the spectral radius of the corresponding A-operator of Σ is smaller than 1 and then Σ is lossless. The condition for $J_\mathcal{B} = I$ in the factorization $Q = (R^* J_\mathcal{B} R)^{-1}$ is equal to the condition $Q \gg 0$. Then Proposition 4.1 is proved. □

The dual of Proposition 4.1 is stated as follows:

Proposition 4.2 *Let an operator $\Theta \in \mathcal{L}$ have a uniformly minimal realization $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ with $\ell_{A_\Theta} < 1$ and $\Theta = D_\Theta + B_\Theta(\mathbf{Z} - A_\Theta)^{-1} C_\Theta$. If Θ is a (J_2, J_1)-isometric operator and there exists a $Q \gg 0$ such that equation (4.13) is satisfied, then the corresponding isometric operator Σ is lossless and Θ is (J_2, J_1)-lossless.* □

Remark 4.1 : *Similar propositions as Proposition 4.1 and 4.2 can be stated for a J-co-isometric operator.*

Theorem 4.5 *Let $\Theta \in \mathcal{X}$ be a locally finite operator and $\{A_1, B_1, C_1, A_2, B_2, C_2, D_\Theta\}$ be a regular realization with $\ell_{A_1} < 1$ and $\ell_{A_2} < 1$ such that $\Theta = D_\Theta + B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}C_1 + B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$. Θ is (J_2, J_1)-isometric if there exists a Hermitian operator $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \in \mathcal{D}$ such that:*

$$\left[\begin{array}{cc|c} A_1 & & C_1 \\ & I & C_2 \\ \hline B_1 & & D_\Theta \end{array}\right] \left[\begin{array}{cc|c} Q_{11}^{(-1)} & Q_{12}^{(-1)} & \\ Q_{21}^{(-1)} & Q_{22}^{(-1)} & \\ \hline & & J_2 \end{array}\right] \left[\begin{array}{cc|c} A_1 & & C_1 \\ & I & C_2 \\ \hline B_1 & & D_\Theta \end{array}\right]^*$$

$$= \left[\begin{array}{cc|c} I & & \\ & A_2 & \\ \hline & B_2 & I \end{array}\right] \left[\begin{array}{cc|c} Q_{11} & Q_{12} & \\ Q_{21} & Q_{22} & \\ \hline & & J_1 \end{array}\right] \left[\begin{array}{cc|c} I & & \\ & A_2 & \\ \hline & B_2 & I \end{array}\right]^* \tag{4.17}$$

Proof: (4.17) specifies the following conditions:

1. $A_1 Q_{11}^{(-1)} A_1^* + C_1 J_2 C_1^* = Q_{11}$
2. $A_1 Q_{12}^{(-1)} + C_1 J_2 C_2^* = Q_{12} A_2^*$
3. $Q_{22}^{(-1)} = A_2 Q_{22} A_2^* - C_2 J_2 C_2^*$
4. $B_1 Q_{11}^{(-1)} A_1^* - B_2 Q_{21} + D_\Theta J_2 C_1^* = 0;$
5. $B_1 Q_{12}^{(-1)} + D_\Theta J_2 C_2^* - B_2 Q_{22} A_2^* = 0$
6. $B_1 Q_{11}^{(-1)} B_1^* + D_\Theta J_2 D_\Theta^* - B_2 Q_{22} B_2^* = J_1$

then,

$$\begin{aligned}
&\Theta J_2 \Theta^* \\
&= [D_\Theta + B_1(\mathbf{Z}^* - A_1)^{-1}C_1 + B_2(\mathbf{Z} - A_2)^{-1}C_2] J_2 [D_\Theta^* + C_1^*(\mathbf{Z} - A_1^*)^{-1}B_1^* + C_2^*(\mathbf{Z}^* - A_2^*)^{-1}B_2^*] \\
&= D_\Theta J_2 D_\Theta^* + D_\Theta J_2 C_1^*(\mathbf{Z} - A_1^*)^{-1}B_1^* + D_\Theta J_2 C_2^*(\mathbf{Z}^* - A_2^*)^{-1}B_2^* + B_1(\mathbf{Z}^* - A_1)^{-1}C_1 J_2 D_\Theta^* \\
&+B_1(\mathbf{Z}^* - A_1)^{-1}C_1 J_2 C_1^*(\mathbf{Z} - A_1^*)^{-1}B_1^* + B_1(\mathbf{Z}^* - A_1)^{-1}C_1 J_2 C_2^*(\mathbf{Z}^* - A_2^*)^{-1}B_2^* \\
&+B_2(\mathbf{Z} - A_2)^{-1}C_2 J_2 D_\Theta^* + B_2(\mathbf{Z} - A_2)^{-1}C_2 J_2 C_1^*(\mathbf{Z} - A_1^*)^{-1}B_1^* \\
&\qquad +B_2(\mathbf{Z} - A_2)^{-1}C_2 J_2 C_2^*(\mathbf{Z}^* - A_2^*)^{-1}B_2^* \\
&= D_\Theta J_2 D_\Theta^* + (-B_1 Q_{11}^{(-1)} A_1^* + B_2 Q_{21})(\mathbf{Z} - A_1^*)^{-1}B_1^* + (-B_1 Q_{12}^{(-1)} + B_2 Q_{22} A_2^*)(\mathbf{Z}^* - A_2^*)^{-1}B_2^* \\
&+B_1(\mathbf{Z}^* - A_1)^{-1}(-A_1 Q_{11}^{(-1)} B_1^* + Q_{12} B_2^*) + B_1(\mathbf{Z}^* - A_1)^{-1}(Q_{11} - A_1 Q_{11}^{(-1)} A_1^*)(\mathbf{Z} - A_1^*)^{-1}B_1^* \\
&+B_1(\mathbf{Z}^* - A_1)^{-1}(-A_1 Q_{12}^{(-1)} + Q_{12} A_2^*)(\mathbf{Z}^* - A_2^*)^{-1}B_2^* + B_2(\mathbf{Z} - A_2)^{-1}(-Q_{21}^{(-1)} B_1^* + A_2 Q_{22} B_2^*) \\
&+B_2(\mathbf{Z} - A_2)^{-1}(-Q_{21}^{(-1)} A_1^* + A_2 Q_{21})(\mathbf{Z} - A_1^*)^{-1}B_1^* \\
&\qquad +B_2(\mathbf{Z} - A_2)^{-1}(A_2 Q_{22} A_2^* - Q_{22}^{(-1)})(\mathbf{Z}^* - A_2^*)^{-1}B_2^*
\end{aligned}$$

The terms with B_1 on the left and B_1^* on the right are:

$$\begin{aligned}
&B_1(-Q_{11}^{(-1)} A_1^*(\mathbf{Z} - A_1^*)^{-1} - (\mathbf{Z}^* - A_1)^{-1}A_1 Q_{11}^{(-1)} + (\mathbf{Z}^* - A_1)^{-1}(Q_{11} - A_1 Q_{11}^{(-1)} A_1^*)(\mathbf{Z} - A_1^*)^{-1})B_1^* \\
&= B_1(\mathbf{Z}^* - A_1)^{-1}(-(\mathbf{Z}^* - A_1)Q_{11}^{(-1)} A_1^* - A_1 Q_{11}^{(-1)}(\mathbf{Z} - A_1^*) + Q_{11} - A_1 Q_{11}^{(-1)} A_1^*)(\mathbf{Z} - A_1^*)^{-1}B_1^* \\
&= B_1 Q_{11}^{(-1)} B_1^*
\end{aligned}$$

The terms with B_1 on the left and B_2^* on the right are:

$$
\begin{aligned}
&-B_1 Q_{12}^{(-1)}(\mathbf{Z}^* - A_2^*)^{-1}B_2^* + B_1(\mathbf{Z}^* - A_1)^{-1}Q_{12}B_2^* \\
&\qquad + B_1(\mathbf{Z}^* - A_1)^{-1}(-A_1 Q_{12}^{(-1)} + Q_{12}A_2^*)(\mathbf{Z}^* - A_2^*)^{-1}B_2^* \\
&= 0
\end{aligned}
$$

In a similar way, we can derive that the terms with B_2 on the left and B_1^* on the right are equal to zero, and the terms with B_2 on the left and B_2^* on the right are equal to $-B_2 Q_{22} B_2^*$. It shows that

$$
\Theta J_2 \Theta^* = B_1 Q_{11}^{(-1)} B_1^* - B_2 Q_{22} B_2^* + D_\Theta J_2 D_\Theta^* = J_1
$$

and thus that Θ is (J_2, J_1)-isometric. □

Conversely, if $\Theta \in \mathcal{X}$ is J-isometric with a regular realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$, then there exists a Hermitian operator $Q \in \mathcal{D}$ such that the conditions given by (4.17) are satisfied. The proof of this part is given in Proposition 4.3.

It is well known that the cascade connection of J-lossless operators results in a J-lossless operator and conversely, a J-lossless operator can be expressed as a cascade connection of several J-lossless operators, if the factorization is possible. In particular, the cascade connection of an upper J-lossless operator and a lower J-lossless operator results in a J-lossless operator which is general not upper or lower any more.

Lemma 4.1 *Let $\Theta_1 \in \mathcal{X}$ and $\Theta_2 \in \mathcal{X}$ be two known operators where Θ_2 is J_2-unitary. Then $\Theta = \Theta_1\Theta_2$ is (J_2, J_1)-isometric iff Θ_1 is (J_2, J_1)-isometric.*

Proof: The proof is straightforward. □

Let $T \in \mathcal{X}$ have a uniformly minimal realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$. Suppose that T has a factorization $T = T_1 T_2$ with T_1 and T_2 such that one is upper and another is lower, and one's A matrix is A_1 and another one's is A_2. Then the realizations of T_1 and T_2 are also uniformly minimal. The consequence of this is that with the calculation of factorization, we do not increase the order of the upper part and the lower part of the system.

The next proposition shows that the conditions in Theorem 4.5 are also *necessary* for a (J_2, J_1)-isometric operator in $\mathcal{X}$.

Proposition 4.3 *Let $\Theta \in \mathcal{X}$ have a regular realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ with $\ell_{A_1} < 1$, $\ell_{A_2} < 1$ and port signature matrices (J_1, J_2). If Θ is (J_2, J_1)-isometric, then there exists a Hermitian operator $Q \in \mathcal{D}$ such that conditions given by (4.17) are satisfied.*

Proof: As we have discussed in Section 2.2, Θ can be factorized as $\Theta = \Theta_1\Theta_2$ with $\Theta_1 = D_c + B_1(\mathbf{Z}^* - A_1)^{-1}C_c$ for some $D_c \in \mathcal{D}$ and $C_c \in \mathcal{D}$, and $\Theta_2 = D_a + B_a(\mathbf{Z} - A_2)^{-1}C_2$ for some $D_a \in \mathcal{D}$ and $B_a \in \mathcal{D}$. Since the realization is regular, there exists a Hermitian operator $Q_{22} \in \mathcal{D}$ which is invertible such that Condition 3 in Theorem 4.5 is satisfied. Then we can find some D_a and B_a such that Θ_2 is J_2-unitary such that

$$\begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix} \begin{bmatrix} Q_{22} & \\ & -J_2 \end{bmatrix} \begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix}^* = \begin{bmatrix} Q_{22}^{(-1)} & \\ & -J_2 \end{bmatrix} \tag{4.18}$$

is satisfied. Assume that with such a Θ_2, the factorization $\Theta = \Theta_1\Theta_2$ with Θ_1 having the assumed realization exists. With Lemma 2.1 in Chapter 2, we have:

$$\begin{aligned} \Theta_1\Theta_2 &= (D_cD_a + B_1Y^{(-1)}C_2) + B_1(\mathbf{Z}^* - A_1)^{-1}(C_cD_a + A_1Y^{(-1)}C_2) \\ &\quad + (D_cB_a + B_1Y^{(-1)}A_2)(\mathbf{Z} - A_2)^{-1}C_2 \end{aligned}$$

where $Y \in \mathcal{D}$ is the solution of the Lyapunov equation:

$$C_cB_a = Y - A_1Y^{(-1)}A_2 \tag{4.19}$$

This yields for the realization of Θ

$$C_1 = C_cD_a + A_1Y^{(-1)}C_2 \tag{4.20}$$

$$B_2 = D_cB_a + B_1Y^{(-1)}A_2 \tag{4.21}$$

$$D = D_cD_a + B_1Y^{(-1)}C_2 \tag{4.22}$$

With these identities in addition to (4.18) which gives the constraints of Θ_2 to be J-unitary and (4.19) which comes from the assumed factorization $\Theta = \Theta_1\Theta_2$, we can then conclude condition 2 and 5 in Theorem 4.5 as follows.

Multiplying (4.19) by $Q_{22}A_2^*$ on the right results in:

$$C_cB_aQ_{22}A_2^* = YQ_{22}A_2^* - A_1Y^{(-1)}A_2Q_{22}A_2^* \tag{4.23}$$

and multiplying (4.20) by $J_2C_2^*$ on the right and substituting the relation in (4.18) results in:

$$\begin{aligned} C_1J_2C_2^* &= C_cD_aJ_2C_2^* + A_1Y^{(-1)}C_2J_2C_2^* \\ &= C_cB_aQ_{22}A_2^* + A_1Y^{(-1)}A_2Q_{22}A_2^* - A_1Y^{(-1)}Q_{22}^{(-1)} \end{aligned} \tag{4.24}$$

The sum of (4.23) and (4.24) gives:

$$C_1J_2C_2^* = YQ_{22}A_2^* - A_1Y^{(-1)}Q_{22}^{(-1)}$$

Define $YQ_{22} = Q_{12} \in \mathcal{D}$, then

$$C_1 J_2 C_2^* = Q_{12} A_2^* - A_1 Q_{12}^{(-1)} \tag{4.25}$$

Multiplying (4.21) by $Q_{22}A_2^*$ on the right we obtain,

$$B_2 Q_{22} A_2^* = D_c B_a Q_{22} A_2^* + B_1 Y^{(-1)} A_2 Q_{22} A_2^*$$

By substituting (4.18) into the above expression and using $YQ_{22} = Q_{12}$, we obtain

$$B_2 Q_{22} A_2^* = D J_2 C_2^* + B_1 Q_{12}^{(-1)} \tag{4.26}$$

Then we know that if $\Theta = \Theta_1\Theta_2$ with Θ_1 and Θ_2 having the assumed realization and Θ_2 J_2-unitary, there is a $Q_{12} \in \mathcal{D}$ such that Condition 2 and 5 in Theorem 4.5 are satisfied.

From Corollary 4.1 we know that with $\Theta = \Theta_1\Theta_2$ and Θ_2 J-unitary, Θ is (J_2, J_1)-isometric iff Θ_1 is (J_2, J_1)-isometric. From Theorem 4.3, Θ_1 is (J_2, J_1)-isometric iff there is a Hermitian operator $M \in \mathcal{D}$ such that:

$$\begin{bmatrix} A_1 & C_c \\ B_1 & D_c \end{bmatrix} \begin{bmatrix} M^{(-1)} & \\ & J_2 \end{bmatrix} \begin{bmatrix} A_1 & C_c \\ B_1 & D_c \end{bmatrix}^* = \begin{bmatrix} M & \\ & J_1 \end{bmatrix} \tag{4.27}$$

is satisfied. The (1,1) block identity of (4.27) is

$$A_1 M^{(-1)} A_1^* + C_c J_2 C_c^* = M \tag{4.28}$$

Next we show that if this condition is satisfied, then there is a Hermitian operator $Q_{11} = M + Q_{12} Q_{22}^{-1} Q_{12}^*$ which satisfies condition 1 in Theorem 4.5:

$$A_1 Q_{11}^{(-1)} A_1^* + C_1 J_2 C_1^* = Q_{11} \tag{4.29}$$

With (4.18), equation (4.19) and (4.20), the following identity holds

$$C_c = C_1 J_2 D_a^* J_2 - Y Q_{22} B_a^* J_2$$

Note that $YQ_{22} = Q_{12}$. Substituting C_c in the above expression into relation (4.28) results in

$$A_1 M^{(-1)} A_1^* + (C_1 J_2 D_a^* - Q_{12} B_a^*) J_2 (C_1 J_2 D_a^* - Q_{12} B_a^*)^* = M \tag{4.30}$$

Since Θ_2 is J-unitary, the following conditions are satisfied:

$$\begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix}^* \begin{bmatrix} Q_{22}^{-(-1)} & \\ & -J_2 \end{bmatrix} \begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix} = \begin{bmatrix} Q_{22}^{-1} & \\ & -J_2 \end{bmatrix}$$

By using this relation we can derive the following equation from (4.30)

$$A_1 M^{(-1)} A_1^* + C_1 J_2 C_1^* + (C_1 J_2 C_2^* - Q_{12} A_2^*) Q_{22}^{-(-1)} (C_2 J_2 C_1^* - A_2 Q_{12}^*) = M + Q_{12} Q_{22}^{-1} Q_{12}^*$$

With equation (4.25) and $Q_{11} = M + Q_{12} Q_{22}^{-1} Q_{12}^*$ which is a Hermitian in $\mathcal{D}$, we obtain (4.29) which is condition 1 in Theorem 5.1.

Next we show that Condition 4 and 6 in Theorem 4.5 are also satisfied.

With the expression in (4.20) and (4.22) we have:

$$DJ_2C_1^* = (D_c D_a + B_1 Y^{(-1)} C_2) J_2 (D_a^* C_c^* + C_2^* Y^{(-1)*} A_1^*)$$

Substituting the relation in (4.18) into the above equation results in

$$\begin{aligned} DJ_2C_1^* &= (B_1 Y^{(-1)} A_2 + D_c B_a) Q_{22} (A_2^* Y^{(-1)*} A_1^* + B_a^* C_c^*) - B_1 Y^{(-1)} Q_{22}^{(-1)} Y^{(-1)*} A_1^* + D_c J_2 C_c^* \\ &= B_2 Q_{22} Y^* - B_1 Y^{(-1)} Q_{22}^{(-1)} Y^{(-1)*} A_1^* + D_c J_2 C_c^* \end{aligned}$$

Substituting the relation $D_c J_2 C_c^* = -B_1 M^{(-1)} A_1^*$ into the above equation and with the definition of Q_{12} and Q_{11} we obtain Condition 4 in Theorem 5.1

$$DJ_2C_1^* = B_2 Q_{12}^* - B_1 Q_{11}^{(-1)} A_1^* \tag{4.31}$$

Condition 6

$$DJ_2D^* = J_1 + B_2 Q_{22} B_2^* - B_1 Q_{11}^{(-1)} B_1^* \tag{4.32}$$

is obtained in a similar way. Then the proposition is proved. □

Corollary 4.1 *Let an operator $\Theta \in \mathcal{X}$ be (J_2, J_1)-isometric and $\{A_1, B_1, C_1, A_2, B_2, C_2, D_\Theta\}$ be a uniformly minimal realization of Θ with $\ell_{A_1} < 1$, $\ell_{A_2} < 1$ such that $\Theta = D_\Theta + B_1(\mathbf{Z}^* - A_1)^{-1} C_1 + B_2 \mathbf{Z}^* (I - A_2 \mathbf{Z}^*)^{-1} C_2$. Θ is (J_2, J_1)-lossless iff Q in (4.17) is uniformly positive definite.*

Proof: The proof follows the proof of Proposition 4.3. Assume that Θ can be factorized as $\Theta = \Theta_1\Theta_2$ with Θ_2 J-unitary and Θ_1 (J_2, J_1)-isometric, and Θ_1 and Θ_2 have the realizations as in Proposition 4.3. Θ is (J_2, J_1)-lossless iff Θ_1 is (J_2, J_1)-lossless and Θ_2 is J_2-lossless. This requires that in (4.18), $Q_{22} \gg 0$ and in (4.27), $M = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^* \gg 0$. These two conditions result in:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \gg 0$$

with $Q_{21} = Q_{12}^*$. □.

4.4 Homographic transformation property of J-lossless operators

The next theorem reveals an important property of J-lossless operators. We call this homographic property of J-lossless operators because it comes from the homographic transformation of J-lossless operators and any upper strictly contractive operator.

Theorem 4.6 *Let an operator $\Theta \in \mathcal{X}$ be (J_2, J_1)-lossless and have a partitioning as $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ and let an operator $S \in \mathcal{U}$ be strictly contractive ($\|S\|_\infty < 1$). Let*

$$\Phi = HM(\Theta; S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) \tag{4.33}$$

Then Φ is upper and $\|\Phi\|_\infty < 1$ (or Φ is an H_∞ operator).

Proof: First, we show the invertibility of $(S\Theta_{12} + \Theta_{22})$.

Since Θ is (J_2, J_1)-lossless, the corresponding $\Sigma \in \mathcal{U}$ is lossless and has a partitioning $\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}$ with Σ_{22} invertible. Under these conditions, $||\Sigma_{12}||_\infty < 1$ and $(I - S\Sigma_{12})$ is invertible. With the relation $\Sigma_{12} = -\Theta_{12}\Theta_{22}^{-1}$, we have $(I + S\Theta_{12}\Theta_{22}^{-1})$ invertible and then $(\Theta_{22} + S\Theta_{12})$ invertible.

Now we show that under the given conditions, Φ is upper.

As we discussed in Section 4.3, Φ can be expressed with Σ and S as,

$$\Phi = \Sigma_{21} + \Sigma_{22}(I - S\Sigma_{12})^{-1}S\Sigma_{11}$$

The expansion of $(I - S\Sigma_{12})^{-1}$ is

$$(I - S\Sigma_{12})^{-1} = I + S\Sigma_{12} + (S\Sigma_{12})^2 + \cdots$$

so that

$$\Phi = \Sigma_{21} + \Sigma_{22}S\Sigma_{11} + \Sigma_{22}S\Sigma_{12}S\Sigma_{11} + \cdots$$

Under the given conditions, the Neumann series converges to an upper matrix, i.e. Φ is upper. Next, we prove that $\|\Phi\|_\infty < 1$.

Rewrite equation (4.33) as,

$$(S\Theta_{12} + \Theta_{22}) \begin{bmatrix} \Phi & I \end{bmatrix} = \begin{bmatrix} S & I \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \tag{4.34}$$

Because Θ is J-lossless, we have that $\Theta J \Theta^* = J$. Denote $\varphi = (S\Theta_{12} + \Theta_{22})$. Multiplying both sides of equation (4.34) on the right side first with the J operator and then multiplying each side with the conjugate transpose of themselves, we obtain,

$$\varphi(\Phi\Phi^* - I)\varphi^* = SS^* - I$$

From the condition $\|S\|_\infty < 1$, we then have that $\|\Phi\|_\infty < 1$. □

A dual theorem of the above theorem is:

Theorem 4.7 *Let $\Theta \in \mathcal{X}$ be a given operator such that Θ^* is (J_1, J_2)-lossless and partitioned as $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ and let an operator $S \in \mathcal{U}$ be strictly contractive ($\|S\|_\infty < 1$). Let:*

$$\Phi = DHM(\Theta; S) = (\Theta_{21} - \Theta_{22}S)(\Theta_{12}S - \Theta_{11})^{-1}$$

Then, Φ is upper and $\|\Phi\|_\infty < 1$. □

4.5 State space properties of J-unitary operators

As stated in [2], a bounded, block upper J-unitary operators Θ has input and output state spaces $\mathcal{H}(\Theta)$ and $\mathcal{H}_o(\Theta)$ that are closed, regular subspaces [2]. It has properties discussed below.

Proposition 4.4 *[2] Let $\Theta \in \mathcal{U}$ be a J-unitary operator. Then,*

The input null space is: $\mathcal{K}(\Theta) = \mathcal{L}_2\mathbf{Z}^{-1}\Theta^*J$;

The input state space is: $\overline{\mathcal{H}}(\Theta) = \mathcal{L}_2\mathbf{Z}^{-1} \ominus \mathcal{L}_2\mathbf{Z}^{-1}\Theta^*J$;

The output null space is: $\mathcal{K}_o(\Theta) = \mathcal{U}_2\Theta J$;

The output state space is: $\overline{\mathcal{H}}_o(\Theta) = \mathcal{U}_2 \ominus \mathcal{U}_2\Theta J$. □

Proposition 4.5 *[2] Let $\Theta \in \mathcal{U}$ be J-unitary. $\mathcal{H}(\Theta)$ and $\mathcal{H}_o(\Theta)$ are closed subspaces, then:*

$$\begin{aligned} \mathcal{H}_o(\Theta) &= \mathcal{H}(\Theta)J\Theta \\ \mathcal{H}(\Theta) &= \mathcal{H}_o(\Theta)J\Theta^* \end{aligned}$$

□

See [2] for the proof of Proposition 4.4 and 4.5. The next proposition is an extension of the above propositions.

Proposition 4.6 *Let $\Theta \in \mathcal{U}$ and be a (J_2, J_1)-isometry, i.e. $\Theta J_2 \Theta^* = J_1$. Then:*

$$\mathcal{K}_o = \mathcal{U}_2\Theta J_2 \oplus Ker(.\Theta^* |_{\mathcal{U}_2})$$

$$Ker(.\Theta^* |_{\mathcal{U}_2}) = 0 \qquad \Rightarrow \qquad \Theta \text{ is J-unitary.}$$

Proof: Let $\Theta J_2 \Theta^* = J_1$. Now we show that $\mathcal{U}_2\Theta J_2$ is closed.

Denote $\mathcal{S} = \mathcal{U}_2\Theta J_2$ and then $\mathcal{S}\Theta^* J_1 = \mathcal{U}_2$. Since $\mathcal{U}_2$ is closed, $\mathcal{S}\Theta^* J_1$ is closed. This is possible if $\mathcal{S}$ is closed so that $\mathcal{U}_2\Theta J_2$ is closed.

Since $\mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}((\mathcal{U}_2\Theta J_2)\Theta^*) = 0$, then $\mathcal{U}_2\Theta J_2 \subset \mathcal{K}_o(\Theta)$.

$$\begin{aligned}\mathcal{K}_o(\Theta) \ominus \mathcal{U}_2\Theta J_2 &= \{X \in \mathcal{U}_2, \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(X\Theta^*) = 0 \wedge \mathbf{P}(X\Theta^*) = 0\} \\ &\quad (X \perp \mathcal{U}_2\Theta J_2 \wedge \mathbf{P}((\mathcal{U}_2\Theta J_2)\Theta^*) = \mathcal{U}_2) \\ &= \{X \in \mathcal{U}_2, X\Theta^* = 0\} \\ &= Ker(.\Theta^*|_{\mathcal{U}_2})\end{aligned}$$

□

This proposition together with (2.39) allows us to conclude that

$$\mathcal{U}_2\Theta J_2 \oplus ker(.\Theta^*|_{\mathcal{U}_2}) \oplus \overline{\mathcal{H}}_o(\Theta) = \mathcal{U}_2 \tag{4.35}$$

With the definition of state space of a lower operator in Chapter 2, the state space properties of a lower J-unitary operator are stated as the dual of Proposition 4.4 and 4.5:

Proposition 4.7 *Let $\Theta \in \mathcal{L}$ be a J-unitary operator. Then,*

The input null space is: $\mathcal{K}^a(\Theta) = \mathcal{U}_2\mathbf{Z}\Theta^*J$;

The input state space is: $\overline{\mathcal{H}^a}(\Theta) = \mathcal{U}_2\mathbf{Z} \ominus \mathcal{U}_2\mathbf{Z}\Theta^*J$;

The output null space is: $\mathcal{K}_o^a(\Theta) = \mathcal{L}_2\Theta J$;

The output state space is: $\overline{\mathcal{H}_o^a}(\Theta) = \mathcal{L}_2 \ominus \mathcal{L}_2\Theta J$. □

Proposition 4.8 *[2] Let $\Theta \in \mathcal{L}$ be J-unitary. $\mathcal{H}^a(\Theta)$ and $\mathcal{H}_o^a(\Theta)$ are closed subspaces, and:*

$$\begin{aligned}\mathcal{H}_o^a(\Theta) &= \mathcal{H}^a(\Theta)J\Theta \\ \mathcal{H}^a(\Theta) &= \mathcal{H}_o^a(\Theta)J\Theta^*\end{aligned}$$

The proof of Proposition 4.7 and 4.8 are just analogs of Proposition 4.4 and 4.5. □

Definition 4.5 *An operator $T_o \in \mathcal{U}$ is said to be* outer *if*

$$\overline{\mathcal{U}_2 T_o} = \mathcal{U}_2$$

□

If an outer operator is invertible, its inverse is again upper.

4.6 Conclusions

In this chapter, we have considered J-lossless operators in space $\mathcal{X}$ and have shown the realization conditions of such an operator. One important property of J-lossless operators, the so-called homographic transformation property, has been proved in the operator setting. This property can be used in solving the H_∞ control problems in time-varying systems, which we will deal with in Chapter 6.

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Chapter 5

J-lossless Factorization

5.1 Introduction

In this chapter, we consider the factorization

$$G = T_o\Theta \quad \in \quad \mathcal{X}$$

with T_o invertible and outer, and Θ (J_2, J_1)-lossless in the discrete time-varying context. This kind of factorization is called an outer–(J_2, J_1)-lossless factorization [1]. Here, we consider the case where the dimension sequence of the output of G is pointwise greater than or equal to the dimension sequence of the input. T_o is invertible. Θ has the same size as G. Under this dimension condition, Θ can be (J_2, J_1)-lossless.

The fact that this kind of factorization is closely related to J-spectral factorizations is well known and can be observed from [2], [3]. Spectral factorization problems can be solved by solving Riccati equations [2] [3]. Riccati equations usually have more than one solution. The most interesting solution in control engineering, the one connected to the stability of a closed loop function, is the maximal positive solution. In time invariant case, this solution can be obtained by an analysis of the eigenvalues and invariant spaces of the associated Hamiltonian matrix [4] [5]. The monotonicity of the maximal solution of the Algebraic Riccati Equation (ARE) and the convergence properties of the Difference Riccati Equation(DRE) have been discussed for linear time invariant case by many researchers [6] [7] [8] [9].

In [10], inner-outer (outer-inner) and spectral factorizations of discrete time-varying systems are studied. In this case, the Riccati equation is replaced by a recursion with time-varying coefficients that can also have time-varying dimensions. The solution of the equation requires an initial point of the recursion. Exact initial points can be computed in specific situations; in other cases, the recursion converges to the solution starting from an incorrect initial point which is different from zero, an identity matrix for example. On the other hand, the eigenvalue analysis to classify stable and unstable systems in the time variant case is no longer applicable and less is known about the structure of the solution, although some results are presented on the derivation of the equation, the convergence of the solution starting from an approximate initial point and the explicit solution based on the original system realization in [10]. We treat the extension to outer–J-lossless factorizations in discrete time-varying systems.

We consider the factorization from two different descriptions. In the first setting, we assume that a dichotomy form of a time-varying system is known and given by an operator in $\mathcal{X}$. We will need the help of a subspace concept discussed in [10] with an extension of a J factor. Here we consider the factorization in two steps by solving a Lyapunov equation and a set of Lyapunov-type equations. We also find that the second step of the factorization is equivalent to solving a recursive Riccati equation. The conditions for the existence of the factorization are discussed. The convergence of the recursion is discussed. The second algorithm works on a state equation description. We assume that the dichotomy exists for the system. We extend the conjugation concept in [1] to the time-varying context and introduce the corresponding factorization algorithm. The solution based on the J-conjugation method is also connected to solutions of Riccati equations.

5.2 Factorization based on operator description

Assume an operator $G \in \mathcal{X}$ is specified by the representation,

$$G = D + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a \tag{5.1}$$

with $\ell_{A_c} < 1$, $\ell_{A_a} < 1$, the dimension of the output of G is pointwise greater than or equal to the dimension of the input and with (J_1, J_2) the port signature operators. Suppose that G admits a factorization:

$$G = G_1\Theta_a \tag{5.2}$$

where the operator $\Theta_a \in \mathcal{L}$ is anticausal and J_2-lossless (the subscript 'a' stands for anticausal), and G_1 is causal. Furthermore, suppose that G_1 admits a factorization as,

$$G_1 = T_o\Theta_c \tag{5.3}$$

where $\Theta_c \in \mathcal{U}$ (the subscript 'c' stands for causal) is (J_2, J_1)-lossless and $T_o \in \mathcal{U}$ is outer. Define

$$\Theta = \Theta_c \Theta_a \tag{5.4}$$

then, G has an outer–(J_2, J_1)-lossless factorization $G = T_o\Theta$.

With this strategy, we consider the outer–(J_2, J_1)-lossless factorization of G in two steps, first we take out the anticausal J_2-lossless part and then the causal (J_2, J_1)-lossless part.

5.2.1 Anticausal J_2-lossless factorization

Let $G \in \mathcal{X}$ be a given chain scattering operator specified by (5.1) with $\ell_{A_a} < 1$ and $\ell_{A_c} < 1$, with port signature matrices (J_1, J_2), and with (A_a, C_a) uniformly observable. Let us consider the factorization in equation (5.2).

Proposition 5.1 *Let $G \in \mathcal{X}$ be a given operator with port signature matrices (J_1, J_2), specified by (5.1) with $\ell_{A_a} < 1$, $\ell_{A_c} < 1$ and (A_a, C_a) uniformly observable. Let $\mathbf{F}_o^a = (I - A_a\mathbf{Z}^*)^{-1}C_a$. Define a J_2-unitary operator $\Theta_a \in \mathcal{L}$ with its anticausal output state space $\mathcal{H}_o^a(\Theta_a) = \mathcal{D}_2\mathbf{F}_o^a$. Assume that there is a Hermitian invertible operator $Q \in \mathcal{D}$ such that*

$$A_a Q A_a^* - C_a J_2 C_a^* = Q^{(-1)} \tag{5.5}$$

is satisfied. Under this condition, we embed $[A_a, C_a]$ with a pair $[B_{\Theta_a}, D_{\Theta_a}]$ such that:

$$\begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix} \begin{bmatrix} Q & \\ & -J_2 \end{bmatrix} \begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix}^* = \begin{bmatrix} Q^{(-1)} & \\ & -J_2 \end{bmatrix} \tag{5.6}$$

and

$$\begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix}^* \begin{bmatrix} P^{(-1)} & \\ & -J_2 \end{bmatrix} \begin{bmatrix} A_a & C_a \\ B_{\Theta_a} & D_{\Theta_a} \end{bmatrix} = \begin{bmatrix} P & \\ & -J_2 \end{bmatrix} \tag{5.7}$$

are satisfied. Define a J_2-unitary operator $\Theta_a = D_{\Theta_a} + B_{\Theta_a}\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a \in \mathcal{L}$ and let $G_1 = GJ_2\Theta_a^*J_2$. Then, G_1 is upper and has a realization

$$G_1 = D_g + B_g\mathbf{Z}(I - A_g\mathbf{Z})^{-1}C_g \tag{5.8}$$

where A_g, B_g, C_g and D_g are equal to,

$$A_g = \begin{bmatrix} A_c & C_cJ_2C_a^* \\ & A_a^* \end{bmatrix} \tag{5.9}$$

$$B_g = [\ B_c \quad DJ_2C_a^* - B_aQA_a^*\ ] \tag{5.10}$$

$$C_g = \begin{bmatrix} C_cJ_2D_{\Theta_a}^*J_2 \\ B_{\Theta_a}^*J_2 \end{bmatrix} \tag{5.11}$$

$$D_g = DJ_2D_{\Theta_a}^*J_2 - B_aQB_{\Theta_a}^*J_2 \tag{5.12}$$

If $Q \gg 0$, then Θ_a is J_2-lossless and G has a factorization $G = G_1\Theta_a$ with G_1 upper and Θ_a lower and J_2-lossless.

Proof: Rewrite equation (5.5) as

$$\begin{bmatrix} A_a & C_a \end{bmatrix} \begin{bmatrix} Q & \\ & -J_2 \end{bmatrix} \begin{bmatrix} A_a & C_a \end{bmatrix}^* = Q^{(-1)}$$

For Q invertible, we embed $\begin{bmatrix} A_a & C_a \end{bmatrix}$ with $[\ B_{\Theta_a} \quad D_{\Theta_a}\ ]$ such that (5.6) and (5.7) are satisfied. In this case, $P = Q^{-1}$ and the realization $\{A_a, B_{\Theta_a}, C_a, D_{\Theta_a}\}$ is regular. We construct $\Theta_a = D_{\Theta_a} + B_{\Theta_a}\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a$. With Theorem 4.4 we know that Θ_a is J_2-unitary. Let $G_1 = GJ_2\Theta_a^*J_2$, then,

$$\begin{aligned} G_1 =& GJ_2\Theta_a^*J_2 \\ =& [D + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a]J_2[D_{\Theta_a}^* + C_a^*(I - \mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*]J_2 \\ =& DJ_2D_{\Theta_a}J_2 + [D + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c]J_2C_a^*(I - \mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*J_2 \\ & + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_cJ_2D_{\Theta_a}^*J_2 + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_aJ_2D_{\Theta_a}^*J_2 \\ & + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_aJ_2C_a^*(I - \mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*J_2 \end{aligned}$$

The first three terms are obviously upper. What we need to show is that the last two terms are upper as well. Let us rewrite this part as:

$$\begin{aligned} & B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_aJ_2D_{\Theta_a}^*J_2 + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_aJ_2C_a^*(I - \mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*J_2 \\ &= B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}A_aQB_{\Theta_a}^*J_2 + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}(A_aQA_a^* - Q^{(-1)})(I - \mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*J_2 \\ &= B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}[A_aQ\mathbf{Z}^*(I - \mathbf{Z}A_a^*) + A_aQA_a^* - Q^{(-1)}](I - \mathbf{Z}A_a^*)^{-1}\mathbf{Z}B_{\Theta_a}^*J_2 \\ &= -B_aQ(I - A_a^*\mathbf{Z})^{-1}B_{\Theta_a}^*J_2 \\ &= -B_aQB_{\Theta_a}J_2 - B_aQA_a^*\mathbf{Z}(I - A_a^*\mathbf{Z})^{-1}B_{\Theta_a}^*J_2 \end{aligned}$$

We see that this part is also upper. Then G_1 is upper. By combining the first three terms with the last two terms of G_1, it is not difficult to derive that:

$$G_1 = DJ_2D^*_{\Theta_a}J_2 - B_aQB_{\Theta_a}J_2 + [\ B_c \quad DJ_2C_a^* - B_aQA_a^*\ ](\mathbf{Z}^* - \begin{bmatrix} A_c & C_cJ_2C_a^* \\ & A_a^* \end{bmatrix})^{-1} \begin{bmatrix} C_cJ_2D^*_{\Theta_a}J_2 \\ B^*_{\Theta_a}J_2 \end{bmatrix}$$

Then G_1 has the realization $\{A_g, B_g, C_g, D_g\}$ of (5.9), (5.10), (5.11) and (5.12). Since $\Theta_a^* J_2 \Theta_a = J_2$, G admits a factorization $G = G_1\Theta_a$. If $Q \gg 0$, Θ_a is anticausally J_2-lossless as factorization (5.2) requires. □

After the anticausal J-lossless factorization, we obtain an anticausal J_2-lossless factor and an upper operator. In the next subsection, we consider the *causal J-lossless factorization* of an upper operator in (5.3).

5.2.2 Causal (J_2, J_1)-lossless factorization

Theorem 5.1 *Let $T \in \mathcal{U}$ with port signature matrices (J_1, J_2). Suppose that there exists a $\Theta \in \mathcal{U}$ which is (J_2, J_1)-isometric with its realization regular, such that $\overline{\mathcal{U}_2 T J_2} = \mathcal{U}_2 \Theta J_2$. Then T has a factorization*

$$T = T_o\Theta$$

with $T_o \in \mathcal{U}$ outer.

Proof: Define $T_o = TJ_2\Theta^* J_1$. Then

$$\overline{\mathcal{U}_2 T_o} = \overline{\mathcal{U}_2 T J_2 \Theta^* J_1} = \overline{\overline{\mathcal{U}_2 T J_2}\Theta^* J_1} = \overline{\mathcal{U}_2 \Theta J_2 \Theta^* J_1} = \mathcal{U}_2$$

so that T_o is outer. Next we show that $T = T_o\Theta$.

If Θ is J-unitary, $\Theta^{-1} = J_2\Theta^* J_1$, then it is always true that if $T_o = TJ_2\Theta^* J_1$, then $T = T_o\Theta$. In the case that Θ is only (J_2, J_1)-isometric but with its realization regular, there always exists an $\Omega \in \mathcal{U}$ which is the J-complement of Θ such that:

$$\begin{bmatrix} \Omega \\ \Theta \end{bmatrix} J_2 [\ \Omega^* \quad \Theta^*\ ] = \begin{bmatrix} J_c & \\ & J_1 \end{bmatrix} \tag{5.13}$$

and

$$\begin{bmatrix} \Omega^* & \Theta^* \end{bmatrix} \begin{bmatrix} J_c & \\ & J_1 \end{bmatrix} \begin{bmatrix} \Omega \\ \Theta \end{bmatrix} = J_2 \tag{5.14}$$

where J_c is called the complement port signature matrix of J_1. Then we have: $\Omega^* J_c \Omega + \Theta^* J_1 \Theta = J_2$ or $J_2 \Omega^* J_c \Omega = I - J_2 \Theta^* J_1 \Theta$ and $\Theta J_2 \Omega^* = 0$. On the other hand, because $\mathcal{U}_2 T J_2 \Omega^* \subset \mathcal{U}_2 \Theta J_2 \Omega^* = 0$, $T J_2 \Omega^* = 0$. Hence $T = T J_2 \Theta^* J_1 \Theta = T_o \Theta$ iff there is an Ω such that equations (5.13) and (5.14) are satisfied. Since in the case that the realzation of Θ is regular, there always exists such an Ω. Then for a (J_2, J_1)-isometric operator Θ such that $\overline{\mathcal{U}_2 T J_2} = \mathcal{U}_2 \Theta J_2$, T has a factorization $T = T_o \Theta$ with T_o outer. □

In Section 4.3, we have defined the input and output signature matrices J_1 and J_2 for a chain scattering operator. In general, their entries are time-varying and the relation between J_1 and J_2 can not be given by a simple expression. But in some special cases, J_1 and J_2 are explicitly related. Let us consider the relation of J_1 and J_2 in a special case which is related to the problem we deal with.

Let a chain scattering operator $T \in \mathcal{U}$. The factorization we are interested in is $T = T_o \Theta$ with T_o outer, Θ (J_2, J_1)-lossless and upper. Let Θ be partitioned as $\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ with Θ_{22} invertible. Because Θ is upper, Θ_{22} is upper. On the other hand, the corresponding scattering operator, Σ, is lossless. Thus Θ_{22}^{-1} must be upper as well. Let $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ be a realization of Θ. Suppose D_Θ is partitioned as $\begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$ following the partitioning of Θ. Since both Θ_{22} and Θ_{22}^{-1} are upper, D_{22} is invertible. Because $D_{22} \in \mathcal{D}$, the invertibility of D_{22} implies that every entry of D_{22} is invertible and then square. The row and column dimensions of D_{22}, which are corresponding to the dimensions of the negative part of J_1 and J_2, are thus equal to each other. This equality of in addition with the condition that the dimension of the output is pointwise greater than or equal to the dimension of the input implies that $j_{2,i} = \begin{bmatrix} I & \\ & j_{1,i} \end{bmatrix}$ for $i = -\infty ... + \infty$. In the global notation, we denote the relation $j_{2,i} = \begin{bmatrix} I & \\ & j_{1,i} \end{bmatrix}$ as

$$J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix} \tag{5.15}$$

For the rest of the chapter we assume that the relation of equation (5.15) exists. In this case, J_c in the proof of Theorem 5.1 equals the identity operator.

Let $\Theta \in \mathcal{U}$ be a (J_2, J_1)-isometric operator. Then $\mathcal{K}_o(\Theta) = \mathcal{K}_o'(\Theta) \oplus \mathcal{K}_o''(\Theta)$, where $\mathcal{K}_o'(\Theta) = \mathcal{U}_2\Theta J_2$ and $\mathcal{K}_o'' = ker(.\Theta^*|_{\mathcal{U}_2}) = \{\chi \in \mathcal{U}_2, \chi\Theta^* = 0\}$, and $\mathcal{K}_o(\Theta) \oplus \overline{\mathcal{H}}_o(\Theta) = \mathcal{U}_2$. Let $T \in \mathcal{U}$ be an operator with port signature matrices (J_1, J_2). If we find a Θ such that $\mathcal{K}_o'(\Theta) = \overline{\mathcal{U}_2 T J_2}$, then $\overline{\mathcal{U}_2 T J_2} = \mathcal{U}_2\Theta J_2$. We then have the following proposition.

Proposition 5.2 *Let $T \in \mathcal{U}$ be an operator with port signature matrices (J_1, J_2). Let Θ be a (J_2, J_1)-isometric operator such that $\mathcal{K}_o'(\Theta) = \overline{\mathcal{U}_2 T J_2}$. Then, $\overline{\mathcal{H}_o}(\Theta) J_2 T^* \subset \overline{\mathcal{H}}(T)$.*

Proof: Since $\mathcal{K}_o'(\Theta) = \overline{\mathcal{U}_2 T J_2} = \mathcal{U}_2\Theta J_2$,

$$\mathcal{U}_2 \ominus \overline{\mathcal{U}_2 T J_2} = \mathcal{U}_2 \ominus \mathcal{U}_2\Theta J_2 = \overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}_o'' \tag{5.16}$$

where $\mathcal{K}_o''(\Theta) = ker(.\Theta^*|_{\mathcal{U}_2})$ and hence, $\overline{\mathcal{U}_2 T J_2} \perp \overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}_o''$.

For any $\chi \in [\overline{\mathcal{H}_o}(\Theta) \oplus \mathcal{K}_o'']J_2$, $\mathbf{P}_o(\mathcal{U}_2 T \chi^*) = 0$.

So that $\chi T^* \in \mathcal{L}_2 \mathbf{Z}^{-1}$. From this result and (4.35) we have:

$$(\overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}_o'')J_2 = \{\chi \in \mathcal{U}_2, \chi T^* \in \mathcal{L}_2 \mathbf{Z}^{-1}\} \tag{5.17}$$

From the definition of $\mathcal{H}(T)$ we have:

$$\chi T^*|_{\chi \in [\overline{\mathcal{H}}_o(\Theta) \oplus \mathcal{K}_o'']J_2} \in \mathcal{H}(T) \subset \overline{\mathcal{H}}(T) \tag{5.18}$$

So that in particular, $\overline{\mathcal{H}_o}(\Theta) J_2 T^* \subset \overline{\mathcal{H}}(T)$ □

Let $T \in \mathcal{U}$ be an operator with port signature matrices (J_1, J_2). Define a (J_2, J_1)-isometric operator Θ such that $\mathcal{K}_o'(\Theta) = \overline{\mathcal{U}_2 T J_2}$. Let $\mathbf{E}_o$ be a J-orthonormal basis representation of $\overline{\mathcal{H}}_o(\Theta)$: $\overline{\mathcal{H}}_o(\Theta) = \mathcal{D}_2 \mathbf{E}_o$ and let $\mathbf{F}$ be a basis representation of $\overline{\mathcal{H}}(T)$. Because $\overline{\mathcal{H}}_o(\Theta) J_2 T^* \subset \overline{\mathcal{H}}(T)$, we must have $\mathbf{E}_o J_2 T^* = X\mathbf{F}$ for some bounded diagonal operator X which plays an instrumental role in the derivation of a state realization of Θ.

Suppose that $\mathbf{E}_o J_2$ has a component in $\mathcal{K}_o''$ so that $D\mathbf{E}_o J_2 \in \mathcal{K}_o''$ for some $D \in \mathcal{D}_2$. Since $\mathcal{K}_o'' = ker(.\Theta^*|_{\mathcal{U}_2}) = ker(.T^*|_{\mathcal{U}_2})$ ($T^* = \Theta^* T_o^*$ and $\ker(.T_o^*) = 0$), we have $D\mathbf{E}_o J_2 T^*|_{D\mathbf{E}_o J_2 \in \mathcal{K}_o''} = DX\mathbf{F}|_{D\mathbf{E}_o J_2 \in \mathcal{K}_o''} = 0$ so that $D \in ker(.X)$. Hence $\overline{\mathcal{H}}_o(\Theta) = \mathcal{D}_2 \mathbf{E}_o$ can be described as the largest subspace $\mathcal{D}_2 \mathbf{E}_o$ (and then $\overline{\mathcal{H}}_o(\Theta) J_2 = \mathcal{D}_2 \mathbf{E}_o J_2$ is also the largest subspace) for which: $\mathbf{E}_o J_2 T^* = X\mathbf{F}$ with $ker(.X) = 0$. The two conditions $\mathbf{E}_o J_2 T^* = X\mathbf{F}$ and $ker(.X) = 0$ in addition with the property of J-lossless operators define a realization of a J-lossless Θ such that $\overline{\mathcal{U}_2 T J_2} = \mathcal{U}_2\Theta J_2$. If such a Θ exists, then, according to Theorem 5.1, the factorization $T = T_o\Theta$, where T_o is outer and Θ (J_2, J_1)-lossless, exists.

Proposition 5.3 *Let $T \in \mathcal{U}$ be a locally finite transfer operator with port signature matrices (J_1, J_2) such that $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$ and a uniformly reachable realization $\{A, B, C, D\}$ such that $\ell_A < 1$ and $(TJ_2T^*)^{-1}$ exists. T has a factorization $T = T_o\Theta$, where T_o is invertible and outer, and $\Theta \in \mathcal{U}$ is (J_2, J_1)-lossless iff there is a pair $\{A_\Theta, C_\Theta\}$ which corresponds to a J-orthonormal basis representation of $\overline{\mathcal{H}_o}(\Theta)$, the output state space of Θ, with $\ell_{A_\Theta} < 1$, and a diagonal operator X such that the following conditions are satisfied,*

(i) $A_\Theta X^{(-1)}A^* + C_\Theta J_2 C^* = X$

(ii) $A_\Theta X^{(-1)}B^* + C_\Theta J_2 D^* = 0$

(iii) $A_\Theta A_\Theta^* + C_\Theta J_2 C_\Theta^* = I$

(iv) $Ker(.X) = 0$

If such an X exists, it is unique up to a left diagonal unitary factor, i.e, X^*X is unique.

Proof: The given is a locally finite transfer operator $T \in \mathcal{U}$ with a uniformly reachable realization $\{A, B, C, D\}$, where $\ell_A < 1$. Let $\mathbf{F} = (I - \mathbf{Z}^*A^*)^{-1}\mathbf{Z}^*B^*$ and $\mathbf{F}_o = (I - A\mathbf{Z})^{-1}C$. Suppose that a pair $\{A_\Theta, C_\Theta\}$ and a diagonal operator X fulfilling $(i) - (iii)$ exist and let $\mathbf{E}_oJ_2 = (I - A_\Theta\mathbf{Z})^{-1}C_\Theta J_2$, we have the following equations:

$$\mathbf{E}_oJ_2 = C_\Theta J_2 + A_\Theta\mathbf{Z}\mathbf{E}_oJ_2 \tag{5.19}$$

$$\mathbf{Z}\mathbf{F} = B^* + A^*\mathbf{F} \tag{5.20}$$

$$T^* = D^* + C^*\mathbf{F} \tag{5.21}$$

As analyzed before, T has a factorization $T = T_o\Theta$ with T_o outer and Θ (J_2, J_1)-lossless, iff the conditions that $\mathbf{E}_oJ_2T^* = X\mathbf{F}$ with $Ker(.X) = 0$ and Θ (J_2, J_1)-lossless are satisfied. Now we show that conditions $(i) - (iv)$ are equivalent to these conditions.

Uniform reachability implies that $\mathcal{H}(T) = \mathcal{D}_2\mathbf{F}$. According to Proposition 5.2, we need to find a (J_2, J_1)-lossless operator Θ such that $\overline{\mathcal{H}_o}(\Theta)J_2T^* \subset \overline{\mathcal{H}}(T)$. That is $\mathbf{E}_oJ_2T^* = X\mathbf{F}$ for some bounded $X \in \mathcal{D}$. Because $\mathbf{F} \in \mathcal{L}_2\mathbf{Z}^{-1}$, $P_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{E}_oJ_2T^*) = X\mathbf{F}$. Next, we show that X is given by a solution of the equation in condition (i). With $P_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{E}_oJ_2T^*) = X\mathbf{F}$ and equation (5.20), $P_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{Z}X\mathbf{F}) = X^{(-1)}P_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{Z}\mathbf{F}) = X^{(-1)}A^*\mathbf{F}$. On the other hand:

$$\begin{aligned} A_\Theta P_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{Z}\mathbf{E}_oJ_2T^*) &= P_{\mathcal{L}\mathbf{Z}^{-1}}([A_\Theta\mathbf{Z}\mathbf{E}_oJ_2]T^*) \\ &= P_{\mathcal{L}_2\mathbf{Z}^{-1}}([\mathbf{E}_oJ_2 - C_\Theta J_2]T^*) \\ &= P_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{E}_oJ_2T^*) - P_{\mathcal{L}\mathbf{Z}^{-1}}(C_\Theta J_2T^*) \\ &= X\mathbf{F} - C_\Theta J_2C^*\mathbf{F} \end{aligned}$$

Because $P_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{Z}\mathbf{E}_o J_2 T^*) = P_{\mathcal{L}\mathbf{Z}^{-1}}(\mathbf{Z}X\mathbf{F})$, then:

$$A_\Theta X^{(-1)} A^* \mathbf{F} = X\mathbf{F} - C_\Theta J_2 C^* \mathbf{F}$$

Since the realization is uniformly reachable, we have:

$$A_\Theta X^{(-1)} A^* + C_\Theta J_2 C^* = X$$

Condition (ii) is derived with the condition $\mathbf{E}_o J_2 T^* = X\mathbf{F} \in \mathcal{L}_2\mathbf{Z}^{-1}$ as follows:

$$\begin{aligned}
\mathbf{P}_0(\mathbf{E}_o J_2 T^*) &= \mathbf{P}_0([C_\Theta + A_\Theta \mathbf{Z}\mathbf{E}_o] J_2 T^*) \\
&= C_\Theta J_2 D^* + A_\Theta \mathbf{P}_0(\mathbf{Z}\mathbf{E}_o J_2 T^*) \\
&= C_\Theta J_2 D^* + A_\Theta \mathbf{P}_0(\mathbf{Z}X\mathbf{F}) \\
&= C_\Theta J_2 D^* + A_\Theta X^{(-1)} B^* = 0
\end{aligned}$$

Condition (iii) is given by the fact that $\mathbf{E}_o J_2$ is a J-orthonormal basis representation of the output state space of a J-lossless operator and condition (iv) has been derived before.

Conversely, if conditions $(i) - (iv)$ are satisfied, then conditions for the existence of the outer–J-lossless factorization $T = T_o\Theta$ are satisfied. We show this by substituting conditions $(i) - (ii)$ into $\mathbf{E}_o J_2 T^*$ to verify that the condition $\mathbf{E}_o J_2 T^* = X\mathbf{F}$ is satisfied and conditions $(iii) - (iv)$ are the same in both directions.

The uniqueness of X is seen from the following analysis.

With the same strategy given by Theorem 3.28 in [10] we can prove that $.H_T^* = \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(.T^*)|_{\mathcal{U}_2} = \mathbf{P}_o(.\mathbf{F}_o^*)\mathbf{F}$. Hence $\mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}(\mathbf{E}_o J_2 T^*) = \mathbf{P}_o(\mathbf{E}_o J_2 \mathbf{F}_o^*)\mathbf{F}$. Since T is uniformly reachable, $X = \mathbf{P}_o(\mathbf{E}_o J_2 \mathbf{F}_o^*)$. X^*X is obtained as:

$$\begin{aligned}
X^*X &= \mathbf{P}_o(\mathbf{F}_o J_2 \mathbf{E}_o^*)\mathbf{P}_o(\mathbf{E}_o J_2 \mathbf{F}_o^*) \\
&= \mathbf{P}_o(\mathbf{P}_o(\mathbf{F}_o J_2 \mathbf{E}_o^*)\mathbf{E}_o J_2 \mathbf{F}_o^*) \\
&= \mathbf{P}_o(\mathbf{P}_{\mathcal{H}}^{J_2}(\mathbf{F}_o) J_2 \mathbf{F}_o^*) \qquad (\mathbf{P}_{\mathcal{H}}^{J_2}(.) = \mathbf{P}_o(.J_2\mathbf{E}_o^*)\mathbf{E}_o)
\end{aligned}$$

Then we see that X^*X is unique. □

Then we have $X^*X = \mathbf{P}_o(\mathbf{P}_{\mathcal{H}}^{J_2}(\mathbf{F}_o) J_2 \mathbf{F}_o^*)$. Since $X^*X \geq 0$, then we need $\mathbf{P}_o(\mathbf{P}_{\mathcal{H}}^{J_2}(\mathbf{F}_o) J_2 \mathbf{F}_o^*) \geq 0$. This gives another explanation of the difference of this factorization from the outer–inner factorization of T: when the latter exists while the former exists only under the condition that $\mathbf{P}_o(\mathbf{P}_{\mathcal{H}}^{J_2}(\mathbf{F}_o) J_2 \mathbf{F}_o^*)$ is semipositive definite. To obtain a unique X, we can choose X_k at every step to be in an upper triangular form with all its diagonal entries positive.

If we have found such a X that conditions $(i)-(iv)$ are satisfied, then we have the pair $\{A_\Theta, C_\Theta\}$ which corresponds to a realization of a (J_2, J_1)-lossless operator Θ. Embedding $\{A_\Theta, C_\Theta\}$ with $\{B_\Theta, D_\Theta\}$ such that,

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} I & \\ & J_2 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$$

then, $\Theta = D_\Theta + B_\Theta \mathbf{Z}(I - A_\Theta \mathbf{Z})^{-1} C_\Theta$ and $\Theta J_2 \Theta^* = J_1$. With $T = T_o \Theta$, the outer operator T_o is derived as follows,

$$\begin{aligned} T_o &= T J_2 \Theta^* J_1 \\ &= D J_2 D_\Theta^* J_1 + B\mathbf{Z}(I - A\mathbf{Z})^{-1} C J_2 D_\Theta^* J_1 + T J_2 C_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 \end{aligned} \quad (5.22)$$

The third term of the above equation is

$$\begin{aligned} & T J_2 C_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 \\ &= D J_2 C_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 + B(\mathbf{Z}^* - A)^{-1} C J_2 C_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 \\ &= -B X^{(-1)*} A_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 + B(\mathbf{Z}^* - A)^{-1} C J_2 C_\Theta^* (I - \mathbf{Z}^* A_\Theta^*)^{-1} \mathbf{Z}^* B_\Theta^* J_1 \\ &= B(I - \mathbf{Z}A)^{-1} X^{(-1)*} B_\Theta^* J_1 \\ &= B X^{(-1)*} B_\Theta^* J_1 + B\mathbf{Z}(I - A\mathbf{Z})^{-1} A X^{(-1)*} B_\Theta^* J_1 \end{aligned}$$

By substituting the above result back into (5.22), we obtain the realization of T_o,

$$T_o \sim \left[\begin{array}{c|c} A & C J_2 D_\Theta^* J_1 + A X^{(-1)*} B_\Theta^* J_1 \\ \hline B & D J_2 D_\Theta^* J_1 + B X^{(-1)*} B_\Theta^* J_1 \end{array} \right] \quad (5.23)$$

The invertibility of T_o follows from condition of the invertibility of $T J_2 T^*$.

The next lemma is dealing with the J-unitary *embedding problem* which is part of the factorization procedure.

Lemma 5.1 *Let α be an $(m \times n)$ matrix and γ be an $(m \times l)$ matrix. Let j_1 and j_2 be signature matrices such that* $j_2 = \begin{bmatrix} I & \\ & j_1 \end{bmatrix}$ *and*

$$\alpha\alpha^* + \gamma j_2 \gamma^* = I_m$$

Define $j = \begin{bmatrix} I_n & \\ & j_2 \end{bmatrix} = \begin{bmatrix} I_m & \\ & j_1 \end{bmatrix}$. *Then, there exists matrices β and δ such that*

$$\theta = \begin{array}{c} m \\ n+l-m \end{array} \begin{bmatrix} \overset{n}{\alpha} & \overset{l}{\gamma} \\ \beta & \delta \end{bmatrix}$$ *is a j-unitary matrix in the sense*

$$\theta j \theta^* = j \qquad \theta^* j \theta = j$$

Proof: With the given conditions, the first block row of θ is already (j, I)-isometric,

$$\begin{bmatrix} \alpha & \gamma \end{bmatrix} \begin{bmatrix} I_n & \\ & j_2 \end{bmatrix} \begin{bmatrix} \alpha^* \\ \gamma^* \end{bmatrix} = I_m \tag{5.24}$$

We now show that this block row can be completed to a j-unitary matrix. From (5.24) we know that the m rows of $\begin{bmatrix} \alpha & \gamma \end{bmatrix}$ are linearly independent. Choose a matrix $\begin{bmatrix} b & d \end{bmatrix}$ with $(n + l - m)$ independent rows such that

$$\begin{bmatrix} \alpha & \gamma \end{bmatrix} \begin{bmatrix} I_n & \\ & j_2 \end{bmatrix} \begin{bmatrix} b^* \\ d^* \end{bmatrix} = 0 \qquad \Longleftrightarrow \qquad \begin{bmatrix} b & d \end{bmatrix} = \begin{bmatrix} \alpha & \gamma j_2 \end{bmatrix}^{\perp}$$

We claim that the square matrix $\begin{bmatrix} \alpha & \gamma \\ b & d \end{bmatrix}$ is invertible. Suppose that for some $[\ x_1 \quad x_2\ ]$

$$[\ x_1 \quad x_2\ ] \begin{bmatrix} \alpha & \gamma \\ b & d \end{bmatrix} = [\ 0 \quad 0\ ]$$

then

$$[\ x_1 \quad x_2\ ] \begin{bmatrix} \alpha & \gamma \\ b & d \end{bmatrix} \begin{bmatrix} I_n & \\ & j_2 \end{bmatrix} \begin{bmatrix} \alpha^* \\ \gamma^* \end{bmatrix} = [\ x_1 \quad 0\ ] = [\ 0 \quad 0\ ]$$

Hence $x_1 = 0$ and $x_2[\ b \quad d\ ] = 0$. But the rows of $[\ b \quad d\ ]$ are linearly independent, so that $x_2 = 0$. Hence

$$\begin{bmatrix} \alpha & \gamma \\ b & d \end{bmatrix} \begin{bmatrix} I_n & \\ & j_2 \end{bmatrix} \begin{bmatrix} \alpha^* & b^* \\ \gamma^* & d^* \end{bmatrix} = \begin{bmatrix} I_m & \\ & N \end{bmatrix}$$

where N is an $(n + l - m) \times (n + l - m)$ invertible matrix. By the usual *inertia* argument [10], the signature of N is equal to j_1 and then N has a factorization $N = R j_1 R^*$. Thus putting

$$[\ \beta \quad \delta\ ] = R^{-1}[\ b \quad d\ ], \qquad \theta = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}$$

ensures that θ is j-unitary as required. □

This lemma is a special case of the more general embedding problem of Lemma 5.16 in [10].

To compute the outer–J-lossless factorization of an operator $T \in \mathcal{U}$ such that $T = T_o\Theta$ with T_o invertible and outer, and Θ J-lossless, we need to compute the instrumental operator X and the realization of Θ first and then the realization of the outer operator can be computed with equation (5.23). Summarizing the above results, we obtain the following algorithm, where the computation is carried out locally and backward recursively.

Outer–J-lossless factorization algorithm:

In: $\{T_k\}$ (a reachable realization of a upper operator T)

X_0 (initial condition of X)

$j_{1,k}, j_{2,k}$ (input and output signature operator)

Out: $\{\Theta_k\}, \{(T_o)_k\}$ (realization of J-lossless and outer factors)

For $k = -1, \cdots$

(1). $\begin{bmatrix} A'^*_{\Theta_k} \\ C'^*_{\Theta_k} \end{bmatrix} = \begin{bmatrix} X_{k+1}B_k^* \\ j_{2,k}D_k^* \end{bmatrix}^{\perp}$

(2). $M_k = A'_{\Theta_k}A'^*_{\Theta_k} + C'_{\Theta_k}j_{2,k}C'^*_{\Theta_k}$.

Under the condition that $M_k > 0$, factorize $M_k = r_k r_k^*$ and define $A''_{\Theta_k} = r_k^{-1}A'_{\Theta_k}$ and $C''_{\Theta_k} = r_k^{-1}C'_{\Theta_k}$.

(3). $X'_k = [\ A''_{\Theta_k} \quad C''_{\Theta_k}\ ] \begin{bmatrix} X_{k+1}A_k^* \\ j_{2,k}C_k^* \end{bmatrix}$

(4). $\begin{bmatrix} X_k \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{1,k} \\ Q_{2,k} \end{bmatrix} X'_k$ (QR factorization of X'_k)

(5). $[\ A_{\Theta_k} \quad C_{\Theta_k}\ ] = Q_{1,k}[\ A''_{\Theta_k} \quad C''_{\Theta_k}\ ]$

(6). $[\ B_{\Theta_k} \quad D_{\Theta_k}\ ] = [\ A''_{\Theta_k} \quad C''_{\Theta_k}j_{2,k}\ ]^{\perp}$

(7) $(T_o)_k = \begin{bmatrix} A_k & C_k j_{2,k} D^*_{\Theta_k} j_{1,k} + A_k X^*_{k+1} B^*_{\Theta_k} j_{1,k} \\ C_k & D_k j_{2,k} D^*_{\Theta_k} j_{1,k} + B_k X^*_{k+1} B^*_{\Theta_k} j_{1,k} \end{bmatrix}$

In the algorithm, we have used the condition $TJ_2\Omega^* = 0$.

With Lemma 5.1, we consider Step (6) in 4 steps as follows:

(i) Find $[\ B'_{\Theta_k} \quad D'_{\Theta_k}\ ]$ such that $[\ B'_{\Theta_k} \quad D'_{\Theta_k}\ ] = [\ A''_{\Theta_k} \quad C''_{\Theta_k}j_{2,k}\ ]^{\perp}$;

(ii) Calculate $N = B'_{\Theta_k}B'^*_{\Theta_k} + D'_{\Theta_k}j_{2,k}D'^*_{\Theta_k}$;

(iii) Factorize $N = Rj_{1,k}R^*$;

(iv) Calculate $[\ B_{\Theta_k} \quad D_{\Theta_k}\ ] = R^{-1}[\ B'_{\Theta_k} \quad D'_{\Theta_k}\ ]$.

Remark: As stated before, Θ is J-lossless iff $M_k > 0$. Then the second step in the above computation can be carried out. Otherwise condition (iii) in Proposition 5.3 will not be satisfied.

In the algorithm, one problem remains is the initialization of X. For a finite operator, $X_0 = [.]$ because the dimension of the states after time instant 0 is zero. For a system which is time invariant after time instant 0, the initial condition is determined by the solution of the time invariant system. X_0 now has to satisfy:

$$\begin{aligned}
&X_0 = A_{\Theta_0} X_0 A_0^* + C_{\Theta_0} j_{2,0} C_0^* \\
&A_{\Theta_0} X_0 B_0^* + C_{\Theta_0} j_{2,0} D_0^* = 0 \\
&A_{\Theta_0} A_{\Theta_0}^* + C_{\Theta_0} j_{2,0} C_{\Theta_0}^* = I
\end{aligned}$$

For a periodic system, the initial condition is determined by the solution of the equivalent time invariant system within one period. The time invariant system solution can be obtained from an analysis of the eigen space of a corresponding Riccati equation which we will discuss later or be approximated with the recursive algorithm. The initial condition for the recursion can be an identify matrix with the conrrespondent dimension.

With the outer–J-lossless algorithm, we implement the causal outer–(J_2, J_1)-lossless factorization of G_1.

5.2.3 The realization of the outer factor of a stable system and recursive Riccati equation

In the time invariant case, it is well know that the outer factor T_o, if it exists, of T in the factorization $T = T_o\Theta$ can be expressed in terms of the original system matrices $\{A, B, C, D\}$ of T and an intermediate quantity which is the solution of a Riccati equation with $\{A, B, C, D\}$ as parameters [2]. The Riccati equation can be obtained by performing a *J-spectral factorization* $TJ_2T^* = T_oJ_1T_o^*$ [2] and an outer–J-lossless factorization similar to the time invariant case with the J-lossless conjugation method [1]. The algorithm to compute the realization T_o in equation (5.23) contains the intermediate quantities C_Θ and D_Θ besides X. Now we show that in the time-varying case, the realization T_o in equation (5.23) can be written in terms of the solution of a Riccati recursion and the realization of the system, also we show how the corresponding Riccati equation can be derived in terms of the algorithm discussed previously.

Theorem 5.2 . *Let $T \in \mathcal{U}$ be a locally finite transfer operator with a uniform realization $\{A, B, C, D\}$ and assume $\ell_A < 1$. If the outer–J-lossless factorization $T = T_o\Theta$, with T_o outer and Θ J-lossless, exists, then a realization of T_o is given by:*

$$T_o \sim \left[\begin{array}{c|c} A & CJ_2D^* + AY^{(-1)}B^* \\ \hline B & DJ_2D^* + BY^{(-1)}B^* \end{array}\right] \left[\begin{array}{cc} I & \\ & R^*J_1 \end{array}\right] \tag{5.25}$$

where $Y = X^*X \geq 0$ with X defined in Proposition 5.3. R is diagonal with its entries square and satisfies:

$$J_1 = R(BY^{(-1)}B^* + DJ_2D^*)R^* \tag{5.26}$$

If $BY^{(-1)}B^* + DJ_2D^*$ is invertible, then Y is the solution to the recursive Riccati equation:

$$\begin{aligned} Y &= AY^{(-1)}A^* + CJ_2C^* \\ &\quad -(AY^{(-1)}B^* + CJ_2D^*)(BY^{(-1)}B^* + DJ_2D^*)^{-1}(AY^{(-1)}B^* + CJ_2D^*)^* \end{aligned} \tag{5.27}$$

Proof: We give the proof in terms of the computation steps for the outer–J-lossless factorization given by Proposition 5.3. With step (*), we mean the computation step for the computation of the factorization of the algorithm in the previous subsection with a global view.

Assume that the factorization $T = T_o\Theta$, with T_o outer and Θ (J_2, J_1)-lossless, exists, M in step (2) then is uniformly positive definite and r in step (2) is invertible. T_o then is given by equation (5.23) so that B_Θ and D_Θ are given, according to step (6), (2) and (1), by,

$$\begin{aligned} \left[\begin{array}{cc} B_\Theta & D_\Theta \end{array}\right] &= [\begin{array}{cc} A''_\Theta & C''_\Theta J_2 \end{array}]^\perp \\ &= \left[\begin{array}{cc} r^{-1}A'_\Theta & r^{-1}C'_\Theta J_2 \end{array}\right]^\perp \\ &= \left[\begin{array}{cc} A'_\Theta & C'_\Theta J_2 \end{array}\right]^\perp \\ &= R\left[\begin{array}{cc} BX^{*(-1)} & D \end{array}\right] \end{aligned} \tag{5.28}$$

with some diagonal operator R. Because $B_\Theta B_\Theta^* + D_\Theta J_2 D_\Theta^* = J_1$, from equation (5.28) we have:

$$J_1 = R(BY^{(-1)}B^* + DJ_2D^*)R^* \tag{5.29}$$

Since $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$ and the dimension of $(BY^{(-1)}B^* + DJ_2D^*)$ equals the dimension of J_1, R is a diagonal operator with its entries square. If $(BY^{(-1)}B^* + DJ_2D^*)$ is invertible, R is invertible, then $(R^*J_1R)^{-1} = (BY^{(-1)}B^* + DJ_2D^*)$. Substitute the relation given by equation (5.28) and $Y = X^*X$ into (5.23), we obtain the expression (5.25).

According to step (4), $X^*X = X'^*X'$. Then from step (3), equation (5.28) and step (6), we obtain,

$$\begin{aligned} Y = \;& X^*X = X'^*X' \\ = \;& \begin{bmatrix} AX^{*(-1)} & CJ_2 \end{bmatrix} \begin{bmatrix} A''^*_\Theta \\ C''^*_\Theta \end{bmatrix} \begin{bmatrix} A''_\Theta & C''_\Theta \end{bmatrix} \begin{bmatrix} X^{(-1)}A^* \\ J_2C^* \end{bmatrix} \\ = \;& \begin{bmatrix} AX^{(-1)} & CJ_2 \end{bmatrix} \left(\begin{bmatrix} I & \\ & J_2 \end{bmatrix} - \begin{bmatrix} X^{*(-1)}B^* \\ D^* \end{bmatrix} R^*J_1R \begin{bmatrix} BX^{*(-1)} & D \end{bmatrix} \right) \begin{bmatrix} X^{(-1)}A^* \\ J_2C^* \end{bmatrix} \end{aligned}$$

Substituting R^*J_1R into the above expression yields the Riccati equation as:

$$\begin{aligned} Y \;=\;& AY^{(-1)}A^* + CJ_2C^* \\ & -(AY^{(-1)}B^* + CJ_2D^*)(BY^{(-1)}B^* + DJ_2D^*)^{-1}(AY^{(-1)}B^* + CJ_2D^*)^* \end{aligned}$$

□

This equation has many solutions for Y. Since $X \in \mathcal{D}(\mathcal{B}_\Theta, \mathcal{B})$ has $\mathcal{B}_\Theta$ of maximal possible dimensions such that $ker(.X) = 0$, the solution Y of the Riccati equation must be semipositive and of maximal rank to yield an outer factor T_o.

From the above discussion we know that a J-spectral factorization such that $TJ_2T^* = T_oJ_1T_o^*$ for a discrete time-varying stable system T can be computed by solving a Riccati equation. The result is a resemblance to the problem in time invariant systems [2], the only difference is that the Riccati equation becomes a recursion if we consider the solution in time-varying systems locally. The recursion is obtained by taking the k-th entry of each diagonal in equation (5.27), that is,

$$\begin{aligned} Y_k \;=\;& A_kY_{k+1}A_k^* + C_kj_2C_k^* \\ & -(A_kY_{k+1}B_k^* + C_kj_2D_k^*)(B_{k+1}Y_kB_k^* + D_kj_2D_k^*)^{-1}(A_kY_{k+1}B_k^* + C_kj_2D_k^*)^* \end{aligned}$$

Initial conditions for the recursion can be obtained in special cases. For this part, we refer to [1].

Note that if DJ_2D^* is invertible ($D_kj_{2,k}D_k^*$ is invertible) and the initial condition for the recursion is zero, then $Y = 0$ is always a solution. In general, this is of course not the solution we are interested in (except for T is outer). The solution we are interested in is the maximal stablizing solution which is semipositive definite. From Proposition 5.3 we know that this solution is unique, if it exists. If we do not know the exact initial condition, we should put the initial condition different from zero, an identity for example, in order to obtain the maximal stablizing solution. In section 5.5, we will see that if the initial condition is different from zero, whether it is correct or not, the influence of the initial condition is disappearing as the recursion goes back to minus infinite. The solution then converges to the unique solution.

The Riccati equation can also be obtained in another way: by using J-lossless conjugation method to solve the outer–J-lossless factorization problem for a stable system.

5.2.4 Conjugated J-lossless–outer factorization

In this subsection, we consider the conjugated J-lossless–outer factorization which is a dual factorization of the outer–J-lossless factorization.

When we consider an outer–J-lossless factorization of an operator $T \in \mathcal{X}$, we have discussed the dimension condition for this factorization. The dimension of the output of T should be greater than or equal to the dimension of the input. In the conjugated J-lossless–outer factorization, the dimension condition is reversed.

Let $T \in \mathcal{X}$ with port signature matrices (J_1, J_2). Now we consider the factorization

$$T = \Theta T_o$$

with T_o invertible and outer, and Θ conjugated (J_1, J_2)-lossless. All the propositions and corollaries used in this kind of factorization are duals of the propositions and corollaries in the case of a J-lossless factorization. That is, the output state (or null) space is replaced by the input state (or null) space in the propositions and corollaries, reachability changes into observability etc. Here we give several important dual propositions, corollaries and results without proof.

The dual proposition of Proposition 5.1 is:

Proposition 5.4 *Let $T \in \mathcal{X}$ be a given operator with port signature matrices (J_1, J_2), specified by (5.1) with $\ell_{A_a} < 1$, $\ell_{A_c} < 1$ and (A_a, B_a) uniformly observable. Let $\mathbf{F}^{a*} = B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}$. Define a J-unitary $\Theta_a \in \mathcal{L}$ with its anticausal input state space $\mathcal{H}^a(\Theta_a) = \mathcal{D}_2\mathbf{F}^a$. If there is a Hermitian operator $P \in \mathcal{D}$ which is invertible such that*

$$A_a^* P^{(-1)} A_a - B_a^* J_1 B_a = P$$

is satisfied. Under this condition, we embed $[A_a, B_a]$ *with* $[C_{\Theta_a}, D_{\Theta_a}]$ *such that*

$$\begin{bmatrix} A_a & C_{\Theta_a} \\ B_a & D_{\Theta_a} \end{bmatrix}^* \begin{bmatrix} P^{(-1)} & \\ & -J_1 \end{bmatrix} \begin{bmatrix} A_a & C_{\Theta_a} \\ B_a & D_{\Theta_a} \end{bmatrix} = \begin{bmatrix} P & \\ & -J_1 \end{bmatrix} \tag{5.30}$$

and

$$\begin{bmatrix} A_a & C_{\Theta_a} \\ B_a & D_{\Theta_a} \end{bmatrix} \begin{bmatrix} P^{-1} & \\ & -J_1 \end{bmatrix} \begin{bmatrix} A_a & C_{\Theta_a} \\ B_a & D_{\Theta_a} \end{bmatrix}^* = \begin{bmatrix} P^{-(-1)} & \\ & -J_1 \end{bmatrix} \tag{5.31}$$

are satisfied. Define $\Theta_a = D_{\Theta_a} + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_{\Theta_a} \in \mathcal{L}$. *Let* $T_1 = J_1\Theta_a^* J_1 T$. *Then,* T_1 *is upper and has a realization*

$$T_1 = D_t + B_t\mathbf{Z}(I - A_t\mathbf{Z})^{-1}C_t$$

where A_t, B_t, C_t *and* D_t *are equal to,*

$$A_t = \begin{bmatrix} A_c & \\ B_a^* J_1 B_c & A_a^* \end{bmatrix}$$

$$B_t = \begin{bmatrix} J_1 D_{\Theta_a}^* J_1 B_c \\ J_1 C_{\Theta_a}^* \end{bmatrix}$$

$$C_t = \begin{bmatrix} C_c B_a^* J_1 D - A_a^* P^{(-1)} C_a \end{bmatrix}$$

$$D_t = J_1 D_{\Theta_a}^* J_1 D - J_1 C_{\Theta_a}^* P^{(-1)} C_a$$

If $P \gg 0$, Θ_a *is conjugated* J_1*-lossless. Then* T *has a factorization* $T = \Theta_a T_1$ *with* T_1 *upper and* Θ_a *lower and conjugated* J_1*-lossless.*

Before we give the dual proposition of Proposition 5.3, let us discuss the dimension condition between J_1 and J_2 in the special case where we consider the conjugated J-lossless–outer factorization.

Let a dual chain scattering operator $T \in \mathcal{U}$. The factorization we are interested in in this case is $T = \Theta T_o$ with T_o invertible and outer, and Θ upper and conjugated J-lossless. Let Θ be partitioned as $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ with Θ_{11} invertible. The analysis is similar to the case of the outer–J-lossless factorization of a chain scattering operator in subsection 5.2.2. The relation we obtain is $J_1 = \begin{bmatrix} J_2 & \\ & -I \end{bmatrix}$.

The dual proposition of Proposition 5.3 is

Proposition 5.5 *Let $T \in \mathcal{U}$ be a locally finite transfer operator with port signature matrices (J_1, J_2), where $J_1 = \begin{bmatrix} J_2 & \\ & -I \end{bmatrix}$ and a uniformly observable realization $\{A, B, C, D\}$ such that $\ell_A < 1$ and $(T^* J_1 T)^{-1}$ exists. Then, T has a factorization $T = \Theta T_o$, where T_o is invertible and outer, $\Theta \in \mathcal{U}$ is conjugated (J_1, J_2)-lossless, iff there is a pair $\{A_\Theta, B_\Theta)$ which corresponds to a J-orthonormal basis representation of $\overline{\mathcal{H}}(\Theta)$, the input state space of Θ, with $\ell_{A_\Theta} < 1$ and a diagonal operator X such that the following conditions are satisfied,*

(i) $A_\Theta^* X A + B_\Theta^* J_1 B = X^{(-1)}$

(ii) $A_\Theta^* X C + B_\Theta^* J_1 D = 0$

(iii) $A_\Theta^* A_\Theta + B_\Theta^* J_2 B_\Theta = I$

(iv) $Ker(.X) = 0$

If such a X exists, it is unique up to a left diagonal unitary factor ($X^* X$ is unique).

□

Embedding $\{A_\Theta, B_\Theta\}$ with $\{C_\Theta, D_\Theta\}$ such that,

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* \begin{bmatrix} I & \\ & J_1 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} = \begin{bmatrix} I & \\ & J_2 \end{bmatrix}$$

Then, Θ has a realization $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ and the realization of the outer T_o is

$$\mathbf{T}_o \sim \left[\begin{array}{c|c} A & C \\ \hline J_2 C_\Theta^* X A + J_2 D_\Theta J_1 B & J_2 C_\Theta^* X C + J_2 D_\Theta^* J_1 D \end{array} \right]$$

Define $Y = X^* X$, under the condition that $(C^* Y C + D^* J_1 D)$ is invertible, the realization of T_o can also be written as:

$$\mathbf{T}_o \sim \begin{bmatrix} I & \\ & J_2 R^* \end{bmatrix} \left[\begin{array}{c|c} A & C \\ \hline C^* Y A + D^* J_1 B & C^* Y C + D^* J_1 D \end{array} \right]$$

with $J_2 = R^*(C^* Y C + D^* J_1 D) R$ and Y is the solution of the Riccati equation:

$$Y^{(-1)} = A^* Y A + B^* J_1 B - (A^* Y C + B^* J_1 D) R^* J_2 R (A^* Y C + B^* J_1 D)^* \qquad (5.32)$$

5.3 J-lossless conjugation and J-lossless factorization

5.3.1 The general form of discrete time-varying Riccati equation

Consider the discrete time-varying (forward) algebraic Riccati equation of the form [11]:

$$X = KX^{(-1)}K^* - KX^{(-1)}W_1^*(W_2 + W_1X^{(-1)}W_1^*)^{-1}W_1X^{(-1)}K^* + H \tag{5.33}$$

where $H, X \in \mathcal{D}(\mathcal{B},\mathcal{B})$, $K \in \mathcal{D}(\mathcal{B},\mathcal{B}^{(-1)})$, $W_1 \in \mathcal{D}(\mathcal{M},\mathcal{B}^{(-1)})$, $W_2 \in \mathcal{D}(\mathcal{M},\mathcal{M})$ and $W_2 = W_2^*$, $H = H^*$. We assume:

(1) (K, W_1) is a stabilizable pair (Definition 2.4);

(2) (C, K) is a detectable pair (Definition 2.5, where $CC^* = H$);

(3) W_2 is invertible.

We define: $W := W_1^*W_2^{-1}W_1$. Then equation (5.33) can be rewritten as:

$$X = KX^{(-1)}K^* - KX^{(-1)}W(I + X^{(-1)}W)^{-1}X^{(-1)}K^* + H \tag{5.34}$$

Symplectic operator

Define an operator:

$$\Gamma = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \tag{5.35}$$

If an operator $\mathcal{F}$ is invertible and satisfies:

$$\Gamma^{-1}\mathcal{F}^*\Gamma = \mathcal{F}^{-1}$$

then, $\mathcal{F}$ is called *symplectic.*

Assume that K in (5.34) is invertible. Let us define a symplectic operator $\mathcal{F}$ based on the Riccati equation (5.34) as:

$$\mathcal{F} = \begin{bmatrix} K & -H \\ & I \end{bmatrix} \begin{bmatrix} I & \\ W & K^* \end{bmatrix}^{-1} = \begin{bmatrix} K + HK^{-*}W & -HK^{-*} \\ -K^{-*}W & K^{-*} \end{bmatrix} \tag{5.36}$$

We can use this to compute the desired solution of the Riccati equation (5.34).

The Equivalent relation between Riccati equation and the Symplectic matrix

Theorem 5.3 *[11] Assume that there exists a stabilizing (or antistabilizing) solution of the Riccati equation (5.34) such that:*

$$K^{\times} = K - KX^{(-1)}(I + WX^{(-1)})^{-1}W \tag{5.37}$$

fulfills $\ell_{K^\times} < 1$ *(or* $\ell_{(K^\times)^{-1}} < 1$*). Then such a solution* X *can be written as* $X = X_1^{-1}X_2$ *for any nonsingular* X_1*, and* X_1 *and* X_2 *satisfy:*

$$\begin{bmatrix} X_1 & X_2 \end{bmatrix} \mathcal{F} = S \begin{bmatrix} X_1^{(-1)} & X_2^{(-1)} \end{bmatrix} \tag{5.38}$$

where $\mathcal{F}$ *is defined in (5.36) and* $S = X_1 K^\times X_1^{-(-1)}$.

Proof: For the proof, see [11]. □

From this theorem we know that solving equation (5.38) is equivalent to solving equation (5.34). Hence, symplectic operator (5.36) corresponds to the Riccati equation (5.34). The solution X can also be denoted by $X = Ric(\mathcal{F})$.

5.3.2 J-lossless conjugation

J-lossless conjugation is regarded as a special class of conjugation and gives a natural state space interpretation of the classical interpolation theory [12]. This class of conjugation has been introduced by Ball and Helton [13] [14] and used by Kimura for the outer–J-lossless factorization and H_∞ control [1]. Although [1] only considers the problem for continuous time invariant systems, it is possible to extend the technique to discrete time-varying systems.

A very important factor in J-lossless conjugation is the J-lossless factor we have discussed in the previous chapter. In this section, we discuss such a factor in causal state equation description rather than in input-output operator description. Although causal state equation description has limitations in describing some discrete-time systems such as anticausally delayed systems and descriptor systems, it can still be used to represent a large set of systems. With this description, the spectral radius that is defined for the A-operator of a system is not necessary smaller than 1, i.e., the system is not necessarily causally stable. The operator description of these systems require a *'dichotomy'*: a splitting of spaces into a part that determines the upper part and a part that determines the lower part, i.e., the causally stable part and the anticausally stable part, respectively.

The anticausal stable part corresponds to the antistable part for the causal state equation description. The computation of dichotomy is not a trivial task in the time-varying case.

We define the degree of a system T as the irreducible state dimension of T, denoted as $deg(T)$. As discussed in Chapter 2, it is defined by an index sequence. The degree of the stable part of a system is corresponding to the causally stable part and the degree of the antistable part is corresponding to the causally antistable part or lower part of its dichotomy.

With the J-lossless conjugation method, we consider the outer–J-lossless factorization of causal state equation description setting with some extra assumptions. The same assumptions are used for the description of a J-lossless system.

Theorem 5.4 *Let a discrete time-varying system* Θ *have a uniform realization* $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ *with port signature matrices* (J_1, J_2) *and the partitions:* $B_\Theta = \begin{bmatrix} B_{\Theta 1} \\ B_{\Theta 2} \end{bmatrix}$, $C_\Theta = \begin{bmatrix} C_{\Theta 1} & C_{\Theta 2} \end{bmatrix}$, $D_\Theta = \begin{bmatrix} D_{\Theta 11} & D_{\Theta 12} \\ D_{\Theta 21} & D_{\Theta 22} \end{bmatrix}$ *where* $D_{\Theta 22}$ *is invertible such that:*

$$XZ^{-1} = XA_\Theta + \begin{bmatrix} U_1 & Y_2 \end{bmatrix} \begin{bmatrix} B_{\Theta 1} \\ B_{\Theta 2} \end{bmatrix}$$

$$\begin{bmatrix} Y_1 & U_2 \end{bmatrix} = X \begin{bmatrix} C_{\Theta 1} & C_{\Theta 2} \end{bmatrix} + \begin{bmatrix} U_1 & Y_2 \end{bmatrix} \begin{bmatrix} D_{\Theta 11} & D_{\Theta 12} \\ D_{\Theta 21} & D_{\Theta 22} \end{bmatrix}$$

where $[\, U_1 \;\; Y_2 \,]$, $[\, Y_1 \;\; U_2 \,]$ *and* X *are inputs, outputs and states time sequences respectively.* Θ *is* (J_2, J_1)*-lossless iff there exists a Hermitian diagonal operator* $Q \gg 0$ *such that:*

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} Q^{(-1)} & \\ & J_2 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} Q & \\ & J_1 \end{bmatrix}$$

or

$$\begin{bmatrix} A_\Theta & C_{\Theta 1} & C_{\Theta 2} \\ B_{\Theta 1} & D_{\Theta 11} & D_{\Theta 12} \\ B_{\Theta 2} & D_{\Theta 21} & D_{\Theta 22} \end{bmatrix} \begin{bmatrix} Q^{(-1)} & & \\ & I & \\ & & -I \end{bmatrix} \begin{bmatrix} A_\Theta & C_{\Theta 1} & C_{\Theta 2} \\ B_{\Theta 1} & D_{\Theta 11} & D_{\Theta 12} \\ B_{\Theta 2} & D_{\Theta 21} & D_{\Theta 22} \end{bmatrix}^* = \begin{bmatrix} Q & & \\ & I & \\ & & -I \end{bmatrix} \quad (5.39)$$

The dimensions of the identity matrices (and thus J_1 and J_2) in equation (5.39) correspond to the dimension of the input and output vectors $[\ U_1 \ \ Y_2\ ]$ and $[\ Y_1 \ \ U_2\ ]$, we do not express the dimension explicitly for simplification.

Proof: Suppose that the corresponding scattering system, say Σ, of Θ is lossless and assume that Σ has a uniform realization $\{A, B, C, D\}$ with the partitions: $B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$, $C = [\ C_1 \ \ C_2\ ]$, $D = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$. Because $D_{\Theta 22}$ is invertible, D_{22} is invertible. The state equation of Σ then is:

$$XZ^{-1} = XA + [\ U_1 \ \ U_2\ ] \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$[\ Y_1 \ \ Y_2\ ] = X[\ C_1 \ \ C_2\ ] + [\ U_1 \ \ U_2\ ] \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

According to Theorem 4.1 of Chapter 4, Σ is lossless iff $\ell_A < 1$ and there is a Hermitian diagonal operator $Q \gg 0$ such that,

$$\begin{bmatrix} A & C_1 & C_2 \\ B_1 & D_{11} & D_{12} \\ B_2 & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} Q^{(-1)} & & \\ & I & \\ & & I \end{bmatrix} \begin{bmatrix} A & C_1 & C_2 \\ B_1 & D_{11} & D_{12} \\ B_2 & D_{21} & D_{22} \end{bmatrix}^* = \begin{bmatrix} Q & & \\ & I & \\ & & I \end{bmatrix} \tag{5.40}$$

Q is the observability Gramian of Σ. (5.40) specifies,

$$AQ^{(-1)}A^* + C_1C_1^* + C_2C_2^* = Q \tag{5.41}$$
$$AQ^{(-1)}B_1^* + C_1D_{11}^* + C_2D_{12}^* = 0 \tag{5.42}$$
$$AQ^{(-1)}B_2^* + C_1D_{21}^* + C_2D_{22}^* = 0 \tag{5.43}$$
$$B_1Q^{(-1)}B_1^* + D_{11}D_{11}^* + D_{12}D_{12}^* = I \tag{5.44}$$
$$B_1Q^{(-1)}B_2^* + D_{11}D_{21}^* + D_{12}D_{22}^* = 0 \tag{5.45}$$
$$B_2Q^{(-1)}B_2^* + D_{21}D_{21}^* + D_{22}D_{22}^* = I \tag{5.46}$$

Because D_{22} is invertible, from equation (5.43), (5.45) and (5.46) we have,

$$\begin{aligned} C_2 &= -AQ^{(-1)}B_2^*D_{22}^{-*} - C_1D_{21}^*D_{22}^{-*} \\ D_{12} &= -B_1Q^{(-1)}B_2^*D_{22}^{-*} - D_{11}D_{21}^*D_{22}^{-*} \\ D_{22}^{-1}B_2Q^{(-1)}B_2^*D_{22}^{-*} &= -I - D_{22}^{-1}D_{21}D_{21}^*D_{22}^{-*} + D_{22}^{-1}D_{22}^{-*} \end{aligned}$$

It is not difficult to derive the relation between the corresponding realizations of Σ and Θ as,

$$\begin{bmatrix} A_\Theta & C_{\Theta 1} & C_{\Theta 2} \\ B_{\Theta 1} & D_{\Theta 11} & D_{\Theta 12} \\ B_{\Theta 2} & D_{\Theta 21} & D_{\Theta 22} \end{bmatrix} = \begin{bmatrix} A - C_2 D_{22}^{-1} B_2 & C_1 - C_2 D_{22}^{-1} D_{21} & -C_2 D_{22}^{-1} \\ B_1 - D_{12} D_{22}^{-1} B_2 & D_{11} - D_{12} D_{22}^{-1} D_{21} & -D_{12} D_{22}^{-1} \\ D_{22}^{-1} B_2 & D_{22}^{-1} D_{21} & D_{22}^{-1} \end{bmatrix}$$

without state transformation. With the above relations, now we show that if conditions given by (5.40) are satisfied, then conditions given by (5.39) are satisfied. First we look at

$$\begin{aligned}
&A_\Theta Q^{(-1)} A_\Theta^* + C_{\Theta 1} C_{\Theta 1}^* - C_{\Theta 2} C_{\Theta 2}^* \\
&= (A - C_2 D_{22}^{-1} B_2) Q^{(-1)} (A^* - B_2^* D_{22}^{-*} C_2^*) + (C_1 - C_2 D_{22}^{-1} D_{21})(C_1^* - D_{21}^* D_{22}^{-*} C_2^*) - C_2 D_{22}^{-1} D_{22}^{-*} C_2^* \\
&= AQ^{(-1)} A^* - AQ^{(-1)} B_2^* D_{22}^{-*} C_2^* - C_2 D_{22}^{-1} B_2 Q^{(-1)} A^* + C_2 D_{22}^{-1} B_2 Q^{(-1)} B_2^* D_{22}^{-*} C_2^* \\
&\qquad + C_1 C_1^* - C_1 D_{21}^* D_{22}^{-*} C_2^* - C_2 D_{22}^{-1} D_{21} C_1^* + C_2 D_{22}^{-1} D_{21} D_{21}^* D_{22}^{-*} C_2^* - C_2 D_{22}^{-1} D_{22}^{-*} C_2^* \\
&= AQ^{(-1)} A^* + AQ^{(-1)} B_2^* D_{22}^{-*} D_{22}^{-1} B_2 Q^{(-1)} A^* + AQ^{(-1)} B_2^* D_{22}^{-*} D_{22}^{-1} D_{21} C_1^* \\
&\qquad + AQ^{(-1)} B_2^* D_{22}^{-*} D_{22}^{-1} B_2 Q^{(-1)} A^* + C_1 D_{21}^* D_{22}^{-*} D_{22}^{-1} B_2 Q^{(-1)} A^* - C_2 C_2^* \\
&\qquad - C_2 D_{22}^{-1} D_{21} D_{21}^* D_{22}^{-*} C_2^* + C_2 D_{22}^{-1} D_{22}^{-*} C_2^* + C_1 C_1^* + C_1 D_{21}^* D_{22}^{-*} D_{22}^{-1} B_2 Q^{(-1)} A^* \\
&\qquad + C_1 D_{21}^* D_{22}^{-*} D_{22}^{-1} D_{21} C_1^* A Q^{(-1)} B_2^* D_{22}^{-*} D_{22}^{-1} D_{21} C_1^* + C_1 D_{21}^* D_{22}^{-*} D_{22}^{-1} D_{21} C_1^* \\
&\qquad + C_2 D_{22}^{-1} D_{21} D_{21}^* D_{22}^{-*} C_2^* - C_2 D_{22}^{-1} D_{22}^{-*} C_2^* \\
&= AQ^{(-1)} A^* + C_1 C_1^* + 2(AQ^{(-1)} B_2^* D_{22}^{-*} + C_1 D_{21}^* D_{22}^{-*})(AQ^{(-1)} B_2^* D_{22}^{-*} + C_1 D_{21}^* D_{22}^{-*})^* - C_2 C_2^* \\
&= AQ^{(-1)} A^* + C_1 C_1^* + C_2 C_2^* = Q
\end{aligned}$$

In a similar way, we can show that

$$\begin{aligned}
A_\Theta Q^{(-1)} B_{\Theta 1}^* + C_{\Theta 1} D_{\Theta 11}^* - C_{\Theta 2}^* D_{\Theta 12}^* &= 0 \\
A_\Theta Q^{(-1)} B_{\Theta 2}^* + C_{\Theta 1} D_{\Theta 21}^* - C_{\Theta 2}^* D_{\Theta 22}^* &= 0 \\
B_{\Theta 1} Q^{(-1)} B_{\Theta 1}^* + D_{\Theta 11} D_{\Theta 11}^* - D_{\Theta 12} D_{\Theta 12}^* &= I \\
B_{\Theta 1} Q^{(-1)} B_{\Theta 2}^* + D_{\Theta 11} D_{\Theta 21}^* - D_{\Theta 12} D_{\Theta 22}^* &= 0 \\
B_{\Theta 2} Q^{(-1)} B_{\Theta 2}^* + D_{\Theta 21} D_{\Theta 21}^* - D_{\Theta 22} D_{\Theta 22}^* &= -I
\end{aligned}$$

are satisfied with conditions given by (5.40). With this analysis we know that for a time-varying system Σ, it is lossless iff there is a Hermitian diagonal operator $Q \gg 0$ such that the realization of Σ satisfies condition (5.40), and then its corresponding chain scattering system Θ which is J-lossless, the realization $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ of Θ satisfies condition (5.39) with the same Q.

Conversely, it follows straightforwardly that if a chain scattering system realization $\{A_\Theta, B_\Theta, C_\Theta, D_\Theta\}$ satisfies condition (5.39), the realization $\{A, B, C, D\}$ of the corresponding scattering system satisfies condition (5.40). □

We see that Q in equation (5.39) is the observability Gramian of Σ and that it is positive definite.

Now, let us look at the definition of J-lossless conjugation.

Definition 5.1 *[1] A J-lossless system Θ is said to be a stabilizing (anti-stabilizing)* J-lossless conjugator *of a system T, where T has a uniform realization $\{A, B, C, D\}$, iff:*

(1) ΘT is stable (antistable);

(2) $deg(\Theta)$ is equal to the degree of the antistable (stable) part of T (pointwise).

We call ΘT the J-lossless conjugation of T.

With the above theorem and definition, we have the next theorem.

Theorem 5.5 *Let a system T have a uniformly minimal realization $\{A, B, C, D\}$. Assume that A is invertible and that there exists a dichotomy of A. A stabilizing (or anti-stabilizing) J-lossless conjugator Θ of T exists if the Riccati equation:*

$$X = A(I + X^{(-1)}B^*JB)^{-1}X^{(-1)}A^* \tag{5.47}$$

has a solution $X \geq 0$ such that:

$$\hat{A} = A(I + X^{(-1)}B^*JB)^{-1} \tag{5.48}$$

is stable (or antistable). The corresponding symplectic matrix of (5.47) is:

$$\mathcal{F} = \begin{bmatrix} A & \\ -A^{-*}B^*JB & A^{-*} \end{bmatrix} \tag{5.49}$$

with $K = A$, $H = 0$ and $W = B^*JB$ corresponding to K, H and W in the Riccati equation (5.34).

In addition, a realization of a J-lossless conjugator Θ is

$$\Theta \sim \left[\begin{array}{c|c} A^{-*} & -A^{-*}B^*J \\ \hline D_\Theta B X^{(-1)} & D_\Theta \end{array}\right] \tag{5.50}$$

and a realization of the conjugated system is

$$\Theta T \sim \left[\begin{array}{c|c} \hat{A} & C - XA^{-*}B^*JD \\ \hline D_\Theta B & D_\Theta D \end{array}\right] \tag{5.51}$$

Proof: We prove the stabilizing conjugation, the antistabilizing conjugation can be proved in a similar way.

Assume that (A, B) is uniformly reachable [10] [15] and that there exists a stabilizing J-lossless conjugator Θ with a uniformly minimal realization:

$$\Theta \sim \left[\begin{array}{c|c} A_\Theta & C_\Theta \\ \hline B_\Theta & D_\Theta \end{array}\right] \tag{5.52}$$

and Θ is stable.

Then, a realization of the series connection of Θ and T is:

$$\Theta T \sim \left[\begin{array}{cc|c} A_\Theta & C_\Theta B & C_\Theta D \\ & A & C \\ \hline B_\Theta & D_\Theta B & D_\Theta D \end{array}\right] \tag{5.53}$$

Assume that A has a dichotomy, then the A-matrix of the above equation can be decomposed, with a similarity transformation $\begin{bmatrix} M & U \\ N & V \end{bmatrix}$, as:

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta B \\ & A \end{bmatrix} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} M^{(-1)} & U^{(-1)} \\ N^{(-1)} & V^{(-1)} \end{bmatrix} \tag{5.54}$$

with A_1 antistable ($\ell_{A_1^{-1}} < 1$) and A_2 stable ($\ell_{A_2} < 1$). A_Θ is stable. Since Θ is stable and ΘT is stable, the J-lossless conjugation cancels the antistable part of T, the antistable part of equation (5.54) must be unreachable. Therefore, there must be a matrix B_1 such that:

$$[\ B_\Theta \quad D_\Theta B\ ] = B_1[\ N^{(-1)} \quad V^{(-1)}\ ] \tag{5.55}$$

We can prove that V is invertible because (A, B) is reachable (see Appendix 1 of this chapter for the proof). Let $S = V^{-1}N$, then from equation (5.54) and equation (5.55) we have:

$$B_\Theta = D_\Theta B S^{(-1)} \tag{5.56}$$

and

$$V^{-1} A_2 V^{(-1)} = A + S C_\Theta B \tag{5.57}$$

Since A_2 is stable, $A + SC_\Theta B$ is stable. From equation (5.54) we have:

$$NA_\Theta = A_2 N^{(-1)} \tag{5.58}$$

Then,

$$SA_\Theta = V^{-1} A_2 N^{(-1)} = (A + SC_\Theta B) S^{(-1)} \tag{5.59}$$

From equation (5.39) we have:

$$A_\Theta Q^{(-1)} A_\Theta^* + C_\Theta J C_\Theta^* = Q \tag{5.60}$$

and

$$A_\Theta Q^{(-1)} B_\Theta^* + C_\Theta J D_\Theta^* = 0 \tag{5.61}$$

multiplying by S on the left hand side and by S^* on the right hand side of (5.60) and from (5.61) and (5.56) we have $C_\Theta = -A_\Theta Q^{(-1)} S^{(-1)*} B^* J$. Denoting $X = SQS^*$, we obtain the following Riccati equation:

$$X = A(I + X^{(-1)} B^* J B)^{-1} X^{(-1)} A^* \tag{5.62}$$

On the other hand,

$$A + SC_\Theta B = A - SA_\Theta Q^{(-1)} S^{(-1)*} B^* J B = A - (A + SC_\Theta B) X^{(-1)} B^* J B \tag{5.63}$$

we have that $(A+SC_\Theta B)(I+X^{(-1)}B^*JB) = A$. Under the assumption that A is invertible, $(I + X^{(-1)} B^* J B)$ is invertible. Then,

$$A + SC_\Theta B = A(I + X^{(-1)} B^* J B)^{-1} = \hat{A} \tag{5.64}$$

Because $A + SC_\Theta B$ is stable, $\hat{A}$ is stable. We have proved the first part of the theorem.

For the second part, we start with:

$$\Theta = D_\Theta + B_\Theta (\mathbf{Z}^* - A_\Theta)^{-1} C_\Theta = D_\Theta + D_\Theta B S^{(-1)} (\mathbf{Z}^* - A_\Theta)^{-1} (-A_\Theta Q^{(-1)} S^{(-1)*} B^* J)$$

$$= D_\Theta + D_\Theta B (\mathbf{Z}^* - \hat{A})^{-1} (-X A^{-*} B^* J) \sim \left[\begin{array}{c|c} \hat{A} & -XA^{-*}B^*J \\ \hline D_\Theta B & D_\Theta \end{array}\right] \tag{5.65}$$

Here we have used the relationships $S^{(-1)}(\mathbf{Z}^* - A_\Theta)^{-1} = (\mathbf{Z}^* - \hat{A})^{-1}S$ and $XA^{-*} = AX^{(-1)}(I + B^*JBX^{(-1)})^{-1}$. Furthermore,

$$XA^{-*} = A(I + X^{(-1)}B^*JB)^{-1}X^{(-1)} = \hat{A}X^{(-1)} \tag{5.66}$$

From (5.66) we can derive that Θ has a realization (see Appendix 2):

$$\Theta \sim \left[\begin{array}{c|c} A^{-*} & -A^{-*}B^*J \\ \hline D_\Theta BX^{(-1)} & D_\Theta \end{array}\right] \tag{5.67}$$

ΘT can be derived straightforwardly as:

$$\Theta T \sim \left[\begin{array}{c|c} \hat{A} & C - XA^{-*}B^*JD \\ \hline D_\Theta B & D_\Theta D \end{array}\right] \tag{5.68}$$

with

$$B_\Theta Q^{(-1)}B_\Theta^* + D_\Theta J D_\Theta^* = J \tag{5.69}$$

and $B_\Theta = D_\Theta BS^{(-1)}$, D_Θ should satisfies:

$$D_\Theta BX^{(-1)}B^*D_\Theta^* + D_\Theta JD_\Theta^* = J \tag{5.70}$$

□

Next, we use the J-lossless conjugation operation for obtaining the J-lossless factorization.

5.3.3 J-lossless factorization of a stable system

We only discuss the J-lossless factorization of a stable system in this subsection. The theorem is as follows.

Theorem 5.6 *Let T be a stable system with port signature matrices (J_1, J_2), where $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$, and with a uniformly minimal realization $\{A, B, C, D\}$ with D right invertible. T has a outer–(J_2, J_1)-lossless factorization $T = T_o\Theta$ if the Riccati equation :*

$$\begin{aligned} Y &= AY^{(-1)}A^* + CJ_2C^* \\ &\quad -(CJ_2D^* + AY^{(-1)}B^*)(DJ_2D^* + BY^{(-1)}B^*)^{-1}(CJ_2D^* + AY^{(-1)}B^*)^* \end{aligned} \tag{5.71}$$

has a solution $Y \geq 0$ *such that*

$$\hat{A} = A + FB \tag{5.72}$$

is stable with $F = -(CJ_2D^* + AY^{(-1)}B^*)(DJ_2D^* + BY^{(-1)}B^*)^{-1}$.

Let L *be a factor of* $LJ_1L^* = DJ_2D^* + BY^{(-1)}B^*$. *In that case,*

$$\Theta \sim \left[\begin{array}{c|c} A+FB & C+FD \\ \hline L^{-1}B & L^{-1}D \end{array}\right] \tag{5.73}$$

and

$$T_o \sim \left[\begin{array}{c|c} A & -FL \\ \hline B & L \end{array}\right] \tag{5.74}$$

is invertible and outer.

Equation (5.71) corresponds to the symplectic matrix:

$$\mathcal{F} = \begin{bmatrix} A & -CJ_2C^*A^{-*} \\ & A^{-*} \end{bmatrix} \begin{bmatrix} CJ_2(D^* - C^*A^{-*}B^*) \\ A^{-*}B^* \end{bmatrix} \begin{bmatrix} DJ_2(D^* - C^*A^{-*}B^*) \end{bmatrix}^{-1} \begin{bmatrix} B & -DJ_2C^*A^{-*} \end{bmatrix}$$

Proof: Because of the dimension condition $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$, the dimension of the outputs is larger than the dimension of the inputs. Since D is right invertible, we can define an augmentation of T as:

$$\begin{bmatrix} T \\ T' \end{bmatrix} \sim \left[\begin{array}{c|c} A & C \\ \hline B & D \\ B' & D' \end{array}\right] = \left[\begin{array}{c|c} A & C \\ \hline \hat{B} & \hat{D} \end{array}\right] \tag{5.75}$$

such that both $\hat{D}$ and $A - C\hat{D}^{-1}\hat{B}$ are invertible. Then:

$$\begin{bmatrix} T \\ T' \end{bmatrix}^{-1} = \begin{bmatrix} T^+ & T^\perp \end{bmatrix} \sim \left[\begin{array}{c|c} A - C\hat{D}^{-1}\hat{B} & -C\hat{D}^{-1} \\ \hline \hat{D}^{-1}\hat{B} & \hat{D}^{-1} \end{array}\right] \tag{5.76}$$

Note that $TT^+ = I$, $TT^\perp = 0$ and then $\Theta T^\perp = 0$ and $T_\circ = (\Theta T^+)^{-1}$. From Theorem 5.5 we know that if the Riccati equation:

$$(A - C\hat{D}^{-1}\hat{B})(I + Y^{(-1)}\hat{B}^*\hat{D}^{-*}J_2\hat{D}^{-1}\hat{B})^{-1}Y^{(-1)}(A - C\hat{D}^{-1}\hat{B})^* = Y \tag{5.77}$$

has a solution $Y \geq 0$ such that:

$$\hat{A} = (A - C\hat{D}^{-1}\hat{B})(I + Y^{(-1)}\hat{B}^*\hat{D}^{-*}J_2\hat{D}^{-1}\hat{B})^{-1} \tag{5.78}$$

is stable, then:

$$\Theta \sim \left[\begin{array}{c|c} \hat{A} & -Y(A - C\hat{D}^{-1}\hat{B})^{-*}\hat{B}^*\hat{D}^{-*}J_2 \\ \hline D_\Theta\hat{D}^{-1}\hat{B} & D_\Theta \end{array}\right] \tag{5.79}$$

$$\Theta\left[\begin{array}{cc} T^+ & T^\perp \end{array}\right] \sim \left[\begin{array}{c|c} \hat{A} & -(C + Y(A - C\hat{D}^{-1}\hat{B})^{-*}\hat{B}^*\hat{D}^{-*}J_2)\hat{D}^{-1} \\ \hline D_\Theta\hat{D}^{-1}\hat{B} & D_\Theta\hat{D}^{-1} \end{array}\right] \tag{5.80}$$

where D_Θ satisfies:

$$D_\Theta\hat{D}^{-1}(\hat{D}J_2\hat{D}^* + \hat{B}Y^{(-1)}\hat{B}^*)\hat{D}^{-*}D_\Theta^* = J_1 \tag{5.81}$$

Equation (5.77) can be rewritten as:

$$\begin{aligned} Y &= (A - C\hat{D}^{-1}\hat{B})Y^{(-1)}(A - C\hat{D}^{-1}\hat{B})^* \\ &-(A - C\hat{D}^{-1}\hat{B})Y^{(-1)}\hat{B}^*(\hat{D}J_2\hat{D}^*)^{-1}\hat{B}(I + Y^{(-1)}\hat{B}^*\hat{D}^{-*}J_2\hat{D}^{-1}\hat{B})^{-1}Y^{(-1)}(A - C\hat{D}^{-1}\hat{B})^* \end{aligned}$$

Define:

$$\begin{aligned} \Omega &= -FD = C + Y(A - C\hat{D}^{-1}\hat{B})^{-*}\hat{B}^*\hat{D}^{-*}J_2 \\ &= C + (A - C\hat{D}^{-1}\hat{B})Y^{(-1)}(I + \hat{B}^*(\hat{D}J_2\hat{D}^*)^{-1}\hat{B}Y^{(-1)})^{-1}\hat{B}\hat{D}^{-*}J_2 \end{aligned}$$

Then:

$$\begin{aligned} \hat{A} &= A - \Omega\hat{D}^{-1}\hat{B} \\ \Omega\hat{D}^{-1} &= (CJ_2\hat{D}^* + AY^{(-1)}\hat{B}^*)(\hat{D}J_2\hat{D}^* + \hat{B}Y^{(-1)}\hat{B}^*)^{-1} \\ \hat{A}Y^{(-1)}\hat{B}^* &= (\Omega - C)J_2\hat{D}^* \end{aligned}$$

Then, the Riccati equation can be written as:

$$\begin{aligned}
&\hat{A}Y^{(-1)}(A - C\hat{D}^{-1}\hat{B})^* \\
&= \hat{A}Y^{(-1)}\hat{A}^* + (\Omega - C)J_2(\Omega - C)^* \\
&= AY^{(-1)}A^* - \Omega\hat{D}^{-1}(\hat{D}J_2\hat{D}^* + \hat{B}Y^{(-1)}\hat{B}^*)\hat{D}^{-*}\Omega^* + CJ_2C^* \\
&= AY^{(-1)}A^* - (CJ_2\hat{D}^* + AY^{(-1)}\hat{B}^*)(\hat{D}J_2\hat{D}^* + \hat{B}Y^{(-1)}\hat{B}^*)^{-1}(CJ_2\hat{D}^* + AY^{(-1)}\hat{B}^*)^* + CJ_2C^*
\end{aligned}$$

Because $\Omega = -FD$, we have:

$$\Omega\hat{D}^{-1} = -FD[\; D^+ \quad D^- \;] = -F[\; I \quad 0 \;]$$

and:

$$(CJ_2\hat{D}^* + AY^{(-1)}\hat{B}^*) = -[\; F \quad 0 \;](\hat{D}J_2\hat{D}^* + \hat{B}Y^{(-1)}\hat{B}^*) \tag{5.82}$$

Then, we find:

$$F = -(CJ_2D^* + AY^{(-1)}B^*)(DJ_2D^* + BY^{(-1)}B^*)^{-1} \tag{5.83}$$

It is not difficult to obtain the relationship: $D_\Theta = L^{-1}D$ from the definition: $LJ_1L^* = DJ_2D^* + BY^{(-1)}B^*$ and $\Theta T^\perp$. Then, $D_\Theta\hat{D}^{-1}\hat{B} = L^{-1}B$.

Furthermore, the Riccati equation becomes:

$$Y = AY^{(-1)}A^* - F(DJ_2D^* + BY^{(-1)}B^*)F^* + CJ_2C^* \tag{5.84}$$

and

$$\hat{A} = A + FB = A - (CJ_2D^* + AY^{(-1)}B^*)(DJ_2D^* + BY^{(-1)}B^*)^{-1}B \tag{5.85}$$

is stable.

With the above analysis we prove equation (5.73). Equation (5.74) is proved next.

Equation (5.80) becomes:

$$\Theta\left[\; T^+ \quad T^- \;\right] \sim \left[\begin{array}{c|c} A + FB & FD\hat{D}^{-1} \\ \hline L^{-1}B & L^{-1}D\hat{D}^{-1} \end{array}\right] = \left[\begin{array}{c|c} A + FB & [\; F \quad 0 \;] \\ \hline L^{-1}B & [\; L^{-1} \quad 0 \;] \end{array}\right] \tag{5.86}$$

Then we obtain:

$$\Theta T^{+} \sim \left[\begin{array}{c|c} A+FB & F \\ \hline L^{-1}B & L^{-1} \end{array}\right] \quad (5.87)$$

A realization of T_o then is derived as:

$$T_o = (\Theta T^{+})^{-1} \sim \left[\begin{array}{c|c} A & -FL \\ \hline B & L \end{array}\right] \quad (5.88)$$

□

This theorem gives also a Riccati equation for the J-lossless factorization of a stable system. The equation is the same as we have derived from the operator description. Then we see that these two methods are equivalent to each other.

If we consider the conjugated J-lossless factorization of T, i.e. $T = \Theta T_o$, the dual theorem of the Theorem 5.6 is as follows.

Theorem 5.7 *Let T be a given time-varying system which is stable with the port signature matrices (J_1, J_2), where $J_1 = \begin{bmatrix} J_2 & \\ & -I \end{bmatrix}$ and has a uniform realization $\{A, B, C, D\}$ with D right invertible. Then T has a conjugated (J_1, J_2)-lossless factorization $T = \Theta T_o$ if the Riccati equation:*

$$\begin{aligned} Y^{(-1)} &= A^{*}YA^{*} + B^{*}J_1B \\ &\quad -(B^{*}J_1D + A^{*}YC)(D^{*}J_1D + C^{*}YC)^{-1}(B^{*}J_1D + A^{*}YC)^{*} \end{aligned} \quad (5.89)$$

has a solution $Y \geq 0$ such that:

$$\hat{A} = A + CF \quad (5.90)$$

is stable and where $F = -(D^{*}J_1D + C_{*}YC)^{-1}(B^{*}J_1D + A^{*}YC)^{*}$. Then:

$$\Theta \sim \left[\begin{array}{c|c} A+CF & CL^{-1} \\ \hline B+DF & DL^{-1} \end{array}\right] \quad (5.91)$$

and

$$T_o \sim \left[\begin{array}{c|c} A & C \\ \hline -LB & L \end{array}\right] \quad (5.92)$$

is outer and where $L^{*}J_2L = D^{*}J_1D + C^{*}YC$.

5.4 Explicit form of the solution of Riccati equation

The solution of Riccati equation (5.27) can be given in an explicit form from the original parameters with the discrete time-varying system technique.

Let $T \in \mathcal{U}$ which maps $U \in \mathcal{U}_2$ to $Y \in \mathcal{U}_2$ such that $Y = UT$ and let T have a state realization $\{A, B, C, D\}$ with $\ell_A < 1$. Define $T_{[k]} = \mathbf{P}_0(\mathbf{Z}^{-k}T)$, the k-th diagonal above the main diagonal of T. U, Y and T can be expressed as

$$\begin{aligned} U &= U_{[0]} + \mathbf{Z}U_{[1]} + \mathbf{Z}^2U_{[2]} + \cdots \\ Y &= Y_{[0]} + \mathbf{Z}Y_{[1]} + \mathbf{Z}^2Y_{[2]} + \cdots \\ T &= T_{[0]} + \mathbf{Z}T_{[1]} + \mathbf{Z}^2T_{[2]} + \cdots \end{aligned}$$

Define the diagonal expansions of them as

$$\begin{aligned} \tilde{U} &= [\ U_{[0]} \quad U_{[1]} \quad U_{[2]} \quad \cdots\ ] \\ \tilde{Y} &= [\ Y_{[0]} \quad Y_{[1]} \quad Y_{[2]} \quad \cdots\ ] \end{aligned}$$

and

$$\tilde{T} = \begin{bmatrix} T_{[0]} & T_{[1]}^{(-1)} & T_{[2]}^{(-2)} & \cdots \\ & T_{[0]}^{(-1)} & T_{[1]}^{(-2)} & \cdots \\ & & \ddots & \end{bmatrix}$$

respectively. Then, $\tilde{Y} = \tilde{U}\tilde{T}$.

$\tilde{T}$ can be further expressed as

$$\tilde{T} = \begin{bmatrix} D & B\mathcal{O}^{(-1)} \\ & \tilde{T}^{(-1)} \end{bmatrix} \tag{5.93}$$

where $\mathcal{O}^{(-1)}$ is the observability matrix defined in chapter 2.

Theorem 5.8 *Let $T \in \mathcal{U}$ be a locally finite transfer operator with state realization $\{A, B, C, D\}$ such that $\ell_A < 1$ and $(TJ_2T^*)^{-1}$ exists. Let $M \in \mathcal{D}$:*

$$M = \mathcal{O}[\tilde{J} - \tilde{J}\tilde{T}^*(\tilde{T}\tilde{J}\tilde{T}^*)^{-1}\tilde{T}\tilde{J}]\mathcal{O}^*$$

where $\tilde{T}$ *is as (5.93),* $\tilde{J} = \begin{bmatrix} J_2 & & & \\ & J_2^{(-1)} & & \\ & & J_2^{(-2)} & \\ & & & \ddots \end{bmatrix} = \begin{bmatrix} J_2 & \\ & \tilde{J}^{(-1)} \end{bmatrix}$ *and*

$$\mathcal{O} = [\, C \quad AC^{(-1)} \quad AA^{(-1)}C^{(-2)} \quad \cdots \,] = [\, C \quad A\mathcal{O}^{(-1)} \,]$$

Then $DJ_2D^* + BM^{(-1)}B^*$ *is invertible and* M *satisfies the Riccati equation*

$$\begin{aligned} M &= AM^{(-1)}A^* + CJ_2C^* \\ &\quad -(AM^{(-1)}B^* + CJ_2D^*)(BM^{(-1)}B^* + DJ_2D^*)^{-1}(AM^{(-1)}B^* + CJ_2D^*)^* \end{aligned} \quad (5.94)$$

Proof: The invertibility of TJ_2T^* implies the invertibility of $\tilde{T}\tilde{J}\tilde{T}^*$. Let $M = \mathcal{O}[\tilde{J} - \tilde{J}\tilde{T}^*(\tilde{T}\tilde{J}\tilde{T}^*)^{-1}\tilde{T}\tilde{J}]\mathcal{O}^*$. With the relation in equation (5.93), we have,

$$(\tilde{T}\tilde{J}\tilde{T}^*)^{-1} = \begin{bmatrix} DJ_2D^* + B\mathcal{O}^{(-1)}\tilde{J}^{(-1)}\mathcal{O}^{(-1)*}B^* & B\mathcal{O}^{(-1)}\tilde{J}^{(-1)}\tilde{T}^{(-1)*} \\ \tilde{T}^{(-1)}\tilde{J}^{(-1)}\mathcal{O}^{(-1)*}B^* & \tilde{T}^{(-1)}\tilde{J}^{(-1)}\tilde{T}^{(-1)*} \end{bmatrix}^{-1}$$

Because $\tilde{T}\tilde{J}\tilde{T}^*$ is invertible, with *Schur's inversion formula* [10] we know that the term:

$$\begin{aligned} \Phi^2 &= DJ_2D^* + B\mathcal{O}^{(-1)}\tilde{J}^{(-1)}\mathcal{O}^{(-1)*}B^* \\ &\quad -B\mathcal{O}^{(-1)}\tilde{J}^{(-1)}\tilde{T}^{(-1)*}(\tilde{T}^{(-1)}\tilde{J}^{(-1)}\tilde{T}^{(-1)*})^{-1}\tilde{T}^{(-1)}\tilde{J}^{(-1)}\mathcal{O}^{(-1)*}B^* \\ &= DJ_2D^* + BM^{(-1)}B^* \end{aligned}$$

is invertible. Applying Schur's inversion formula to $(\tilde{T}\tilde{J}\tilde{T}^*)^{-1}$gives,

$$(\tilde{T}\tilde{J}\tilde{T}^*)^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & (\tilde{T}^{(-1)}\tilde{J}^{(-1)}\tilde{T}^{(-1)*})^{-1} \end{bmatrix} + \Xi\Phi^{-2}\Xi^*$$

where Ξ is a matrix defined by:

$$\Xi = \begin{bmatrix} I \\ -(\tilde{T}^{(-1)}\tilde{J}^{(-1)}\tilde{T}^{(-1)*})^{-1}\tilde{T}^{(-1)}\tilde{J}^{(-1)}\mathcal{O}^{(-1)*}B^* \end{bmatrix}$$

With the definition of M, we can derive:

$$\begin{aligned} M &= \mathcal{O}[\tilde{J} - \tilde{J}\tilde{T}^*(\tilde{T}\tilde{J}\tilde{T}^*)^{-1}\tilde{T}\tilde{J}]\mathcal{O}^* \\ &= [\, C \quad A\mathcal{O}^{(-1)} \,](\tilde{J} - \tilde{J}\begin{bmatrix} D^* & \\ \mathcal{O}^{(-1)*}B^* & \tilde{T}^{(-1)*} \end{bmatrix}(\tilde{T}\tilde{J}\tilde{T}^*)^{-1}\begin{bmatrix} D & B\mathcal{O}^{(-1)} \\ & \tilde{T}^{(-1)} \end{bmatrix}\tilde{J})\begin{bmatrix} C^* \\ \mathcal{O}^{(-1)*}A^* \end{bmatrix} \\ &= AM^{(-1)}A^* + CJ_2C^* \\ &\quad -(AM^{(-1)}B^* + CJ_2D^*)(BM^{(-1)}B^* + DJ_2D^*)^{-1}(AM^{(-1)}B^* + CJ_2D^*)^* \end{aligned}$$

This is the Ricatti equation (5.94). □

5.5 Convergence of the Riccati recursion

As we know, the Riccati recursion can be obtained by taking the k-th entries from every diagonal operator in the Riccati equation (5.27). The following question can be posed now: the recursion starts with a wrong initial condition, does the recursion then converge to the correct solution? The next proposition gives the answer.

Let:

$$\mathcal{O}_i = [\ C_i \quad A_iC_{i+1} \quad A_iA_{i+1}C_{i+2} \quad \cdots\ ]$$

$$T_i = \begin{bmatrix} D_i & B_iC_{i+1} & B_iA_{i+1}C_{i+2} & \cdots \\ & D_{i+1} & B_{i+1}C_{i+2} & \cdots \\ & & D_{i+2} & \cdots \\ & & & \ddots \end{bmatrix}$$

and

$$J_{2,i} = \begin{bmatrix} j_{2,i} & & & \\ & j_{2,i+1} & & \\ & & j_{2,i+2} & \\ & & & \ddots \end{bmatrix}$$

then, $M_i = \mathcal{O}_i[J_{2,i} - J_{2,i}T_i^*(T_iJ_{2,i}T_i^*)^{-1}T_iJ_{2,i}]\mathcal{O}_i^*$.

Consider a system T' which is related to T as: $T'_{i,j} = 0$ for $j > 0$ and $T'_{i,j} = T_{i,j}$ for $j \leq 0$. The sequence M_i corresponds to T and at each point i in time given by $M_i = \mathcal{O}_i[J_{2,i} - J_{2,i}T_i^*(T_iJ_{2,i}T_i^*)^{-1}T_iJ_{2,i}]\mathcal{O}_i^*$ for $i \leq 0$, we then can partition T_i, $\mathcal{O}_i$ and $J_{2,i}$ as,

$$T_i = \begin{bmatrix} T_i' & H_0^r \\ 0 & T_0 \end{bmatrix} \tag{5.95}$$

where T_i' is an $(i \times i)$ block matrix,

$$\mathcal{O}_i = [\ \mathcal{O}_i' \quad A^{[i,-1]}\mathcal{O}_0\ ] \tag{5.96}$$

where $\mathcal{O}_i'$ is equal to the first i block columns of $\mathcal{O}_i$, $A^{[i,-1]} = A_iA_{i+1}\cdots A_{-1}$ and $H_0^r = \mathcal{C}_0^r\mathcal{O}_0$ related to the Hankel matrix H_0 [10], where $\mathcal{C}_0^r = \begin{bmatrix} \vdots \\ B_{-3}A_{-2}A_{-1} \\ B_{-2}A_{-1} \\ B_{-1} \end{bmatrix}$ and $\mathcal{O}_0 = \begin{bmatrix} C_0 & A_0C_1 & A_0A_1C_2 & \cdots \end{bmatrix}$, and

$$J_{2,i} = \begin{bmatrix} J_{2,i}' & \\ & J_{2,0} \end{bmatrix} \tag{5.97}$$

with $J_{2,i}' = \begin{bmatrix} j_{2,i} & & & \\ & j_{2,i+1} & & \\ & & \ddots & \\ & & & j_{2,-1} \end{bmatrix}$.

In terms of these quantities, M_i', the solution of Riccati equation for T', is given at $i \le 0$ by

$$M_i' = \mathcal{O}_i'[J_{2,i}' - J_{2,i}'T_i'^*(T_i'J_{2,i}'T_i'^*)^{-1}T_i'J_{2,i}']\mathcal{O}_i'^*$$

Proposition 5.6 *Let $\{A,B,C,D\}$ be a strictly stable realization ($\ell_A < 1$) of a locally finite transfer operator $T \in \mathcal{U}$ and $M_i = \mathcal{O}_i[J_{2,i} - J_{2,i}T_i^*(T_iJ_{2,i}T_i^*)^{-1}T_iJ_{2,i}]\mathcal{O}_i^*$ be the exact solution of the Riccati equation (5.94) corresponding to T. Let T' be a operator which is related to T: $T_{i,j}' = 0$ for $j > 0$ and $T_{i,j}' = T_{i,j}$ for $j \le 0$ and $M_i' = \mathcal{O}_i'[J_{2,i}' - J_{2,i}'T_i'^*(T_i'J_{2,i}'T_i'^*)^{-1}T_i'J_{2,i}']\mathcal{O}_i'^*$ be the exact solution of the Riccati equation for T'. Then $M_i' \to M_i$ for $i \to -\infty$.*

Proof: Under the given condition, we can partition T_i as in (5.95). With this partition and Schur's inversion Lemma, we have:

$$\begin{aligned}(T_iJ_{2,i}T_i^*)^{-1} &= \begin{bmatrix} 0 & 0 \\ 0 & (T_0J_{2,0}T_0^*)^{-1} \end{bmatrix} \\ &+ \begin{bmatrix} I \\ -(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}H_0^{r*} \end{bmatrix} \phi^{-2} \begin{bmatrix} I - H_o^rJ_{2,0}T_o^*(T_0J_{2,0}T_0^*)^{-1} \end{bmatrix}\end{aligned}$$

for $i \le 0$. Where

$$\begin{aligned}\phi^2 &= T_i'J_{2,i}'T_i'^* + H_0^rJ_{2,0}H_0^{r*} - H_0^rJ_{2,0}T_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}H_0^{r*} \\ &= T_i'J_{2,i}'T_i'^* + \mathcal{C}_0^r\mathcal{O}_0[J_{2,0} - J_{2,0}T_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}]\mathcal{O}_0^*\mathcal{C}_0^{r*}\end{aligned}$$

Denote $M_0' = \mathcal{O}_0[J_{2,0} - J_{2,0}T_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}]\mathcal{O}_0^*$, then,

$$\phi^2 = T_i'J_{2,i}'T_i'^* + C_0^r M_0' C_0^{r*}$$

M_0' is the solution of M at $i = 0$. Inserting the expression for $\mathcal{O}_i$ in (5.96) yields:

$$\begin{aligned}
&M_i = M_i' + A^{[i,-1]}\mathcal{O}_0[J_{2,0} - J_{2,0}T_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}]\mathcal{O}_0^*A^{[i,-1]*} + \\
&A^{[i,-1]}\mathcal{O}_0[J_{2,0} - J_{2,0}T_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}]H_0^{r*}\phi^{-2}H_o^r[J_{2,0} - J_{2,0}T_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}]\mathcal{O}_0^*A^{[i,-1]*} \\
&-A^{[i,-1]}\mathcal{O}_0[J_{2,0} - J_{2,0}T_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}]H_0^{r*}\phi^{-2}T_i'J_{2,i}'\mathcal{O}_i'^* \\
&-\mathcal{O}_i'J_{2,i}'T_i'^*\phi^{-2}H_o^r[J_{2,0} - JT_0^*(T_0J_{2,0}T_0^*)^{-1}T_0J_{2,0}]\mathcal{O}_0^*A^{[i,-1]*} \\
&+\mathcal{O}_i'J_{2,i}'T_i'^*(T_i'J_{2,i}'T_i'^*)^{-1}C_0^r[I + M_0'C_0^{r*}(T_i'J_{2,i}'T_i'^*)^{-1}C_0^r]M_0'C_0^{r*}(T_i'J_{2,i}'T_i'^*)^{-1}T_i'J_{2,i}'\mathcal{O}_i'^*
\end{aligned}$$

Each of the four terms in the middle has a factor $A^{[i,-1]}$. An examination of the term $\mathcal{O}_i'J_{2,i}'T_i'^*(T_i'J_{2,i}'T_i'^*)^{-1}C_0^r$ in more details reveals that it consists of a summation of i terms, each of them has a factor $A^{[i,k-1]}$ and $A^{[k,i-k]}$ for $0 > k > i$. The stability condition $\ell_A < 1$ implies that the products of either $A^{[i,k-1]}$ or $A^{[k,i-k]}$ goes to zero as $i \to -\infty$. Since other factors are bounded, the last five terms go to zero as $i \to -\infty$ and then this equation gives $M_i' \to M_i$ as $i \to -\infty$. □

From this proposition we see that the influence of the initial condition, whether it is correct or not correct, under some additional conditions, is disappearing and the solution of the Riccati recursion converges to M_i' as $i \to -\infty$.

5.6 Solution of the Riccati equation via a J-RQ factorization

Let $T \in \mathcal{U}$ be a locally finite operator and have a realization $\{A, B, C, D\}$ with $\ell_A < 1$. As we discussed above, the J-lossless factorization of T, if it exists, such that $T = T_o\Theta$ with T_o outer and Θ J-lossless can be computed by solving Riccati equation (5.27). It is well known that the J-lossless factorization of T exists if the stabilizing solution to the algebraic Riccati equation is positive semidefinite.

However, the solutions of Riccati equations can be computed more efficiently by using a square-root algorithm, which is a kind of "*RQ*" ("*QR*") factorization, we call it a J-*RQ* (J-*QR*) factorization. In such algorithms, the squared root of Y, say X, is computed.

Let $T \in \mathcal{U}$ with a stable realization $\{A, B, C, D\}$ and port signature matrices (J_1, J_2). Its local realizations, denoted by $\{A_k, B_k, C_k, D_k\}$'s, are finite dimensional. The J-lossless factorization, if it exists, $T = T_o\Theta$ with $\Theta J_2 \Theta^* = J_1$ and T_o outer can be computed by the following factorization. Suppose that at step k, we know the matrix X_{k+1}. Let us consider this factorization:

$$\begin{bmatrix} A_k X_{k+1}^* & C_k \\ B_k X_{k+1}^* & D_k \end{bmatrix} = \underbrace{\begin{bmatrix} X_k^* & C_{o_k} \\ & D_{o_k} \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} A_{\Theta_k} & C_{\Theta_k} \\ B_{\Theta_k} & D_{\Theta_k} \end{bmatrix}}_{Q} \tag{5.98}$$

where $\{A_{\Theta_k}, B_{\Theta_k}, C_{\Theta_k}, D_{\Theta_k}\}$ is a realization of a J-lossless operator and satisfies the condition

$$\begin{bmatrix} A_{\Theta_k} & C_{\Theta_k} \\ B_{\Theta_k} & D_{\Theta_k} \end{bmatrix} \begin{bmatrix} I & \\ & J_{2_k} \end{bmatrix} \begin{bmatrix} A_{\Theta_k} & C_{\Theta_k} \\ B_{\Theta_k} & D_{\Theta_k} \end{bmatrix}^* = \begin{bmatrix} I & \\ & J_{1_k} \end{bmatrix} \tag{5.99}$$

X_k^* has full column rank. Then, a realization of the outer factor is $T_{ok} = \{A_k, B_k, C_{o_k}, D_{o_k}\}$ and a realization of the J-lossless factor is $\Theta_k = \{A_{\Theta_k}, B_{\Theta_k}, C_{\Theta_k}, D_{\Theta_k}\}$.

We rewrite equation (5.98) and (5.99) into global forms as

$$\begin{bmatrix} AX^{(-1)*} & C \\ BX^{(-1)*} & D \end{bmatrix} = \begin{bmatrix} X^* & C_o \\ & D_o \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \tag{5.100}$$

and

$$\begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix} \begin{bmatrix} I & \\ & J_2 \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta \\ B_\Theta & D_\Theta \end{bmatrix}^* = \begin{bmatrix} I & \\ & J_1 \end{bmatrix} \tag{5.101}$$

Now we show that $Y = X^*X$ is the solution of equation (5.27).

Multiplying both side of equation (5.100) with $\begin{bmatrix} I & \\ & J_2 \end{bmatrix}$ and the conjugation of themselves on the right and from the relation (5.101) we have

$$\begin{bmatrix} AX^{(-1)*} & C \\ BX^{(-1)*} & D \end{bmatrix} \begin{bmatrix} I & \\ & J_2 \end{bmatrix} \begin{bmatrix} AX^{(-1)*} & C \\ BX^{(-1)*} & D \end{bmatrix}^* = \begin{bmatrix} X^* & C_o \\ & D_o \end{bmatrix} \begin{bmatrix} I & \\ & J_1 \end{bmatrix} \begin{bmatrix} X & \\ C_o^* & D_o^* \end{bmatrix}$$

This expression gives three equations:

$$\begin{aligned} AX^{(-1)*}X^{(-1)}A^* + CJ_2C^* &= X^*X + C_oJ_1C_o^* \\ AX^{(-1)*}X^{(-1)}B^* + CJ_2D^* &= C_oJ_1D_o^* \\ BX^{(-1)*}X^{(-1)}B^* + DJ_2D^* &= D_oJ_1D_o^* \end{aligned}$$

and from these equations and with $Y = X^*X$ which is obvious semi-positive definite, we obtain the time-varying Riccati equation (5.27). Thus $Y = X^*X$ is the solution of equation (5.27).

Furthermore, it is not difficult to verify the relation $TJ_2T^* = T_oJ_1T_o^*$, then $T = T_o\Theta$ and T_o is outer. That Θ is (J_2, J_1)-lossless is a consequence of direct relation of (5.101).

Next, we will show that if we use the outer–J-lossless factorization algorithm in subsection 5.2.2 to do the factorization, the J-lossless factor satisfies the relation given by (5.98). First, we can add the complement part of Q in (5.98), say $[\ B'_{\Theta_k} \quad D'_{\Theta_k}\ ]$ to form a J-unitary matrix $\boldsymbol{\Theta}$ as:

$$\boldsymbol{\Theta} = \begin{bmatrix} B'_{\Theta_k} & D'_{\Theta_k} \\ A_{\Theta_k} & C_{\Theta_k} \\ B_{\Theta_k} & D_{\Theta_k} \end{bmatrix}$$

$\boldsymbol{\Theta}$ satisfies that $\boldsymbol{\Theta}\begin{bmatrix} I & \\ & J_{2_k} \end{bmatrix}\boldsymbol{\Theta}^* = \begin{bmatrix} I & & \\ & I & \\ & & J_{1_k} \end{bmatrix}$ and $\boldsymbol{\Theta}^*\begin{bmatrix} I & & \\ & I & \\ & & J_{1_k} \end{bmatrix}\boldsymbol{\Theta} = \begin{bmatrix} I & \\ & J_{2_k} \end{bmatrix}$ with $\begin{bmatrix} I & \\ & J_{1_k} \end{bmatrix} = J_{2_k}$. For the equality, the R factor in (5.98) becomes:

$$R' = \begin{bmatrix} 0 & X_k^* & C_{o_k} \\ 0 & 0 & D_{o_k} \end{bmatrix}$$

such that:

$$\begin{bmatrix} A_kX_{k+1}^* & C_k \\ B_kX_{k+1}^* & D_k \end{bmatrix} = R'\boldsymbol{\Theta}$$

Since $\boldsymbol{\Theta}$ is J-unitary, the above relation can be rewritten as:

$$\begin{bmatrix} A_kX_{k+1}^* & C_k \\ B_kX_{k+1}^* & D_k \end{bmatrix}\begin{bmatrix} I & \\ & J_{2_k} \end{bmatrix}\begin{bmatrix} B'^*_{\Theta_k} & A^*_{\Theta_k} & B^*_{\Theta_k} \\ D'^*_{\Theta_k} & C^*_{\Theta_k} & D^*_{\Theta_k} \end{bmatrix}\begin{bmatrix} I & \\ & J_{2_k} \end{bmatrix} = \begin{bmatrix} 0 & X_k^* & C_{o_k} \\ 0 & 0 & D_{o_k} \end{bmatrix}$$

With the relations in the outer–J-lossless factorization algorithm, the left hand side of the above equation can be written as:

$$\left[\begin{array}{c|c} [\ A_kX_{k+1}^* \quad C_kJ_{2_k}\ ]\begin{bmatrix} A''^*_{\Theta_k} \\ C''^*_{\Theta_k} \end{bmatrix}[\ Q_{2_k}^* \quad Q_{1_k}^*\ ] & [\ A_kX_{k+1}^* \quad C_kJ_{2_k}\ ]\begin{bmatrix} B^*_{\Theta_k} \\ D^*_{\Theta_k} \end{bmatrix} \\ \hline [\ B_kX_{k+1}^* \quad D_kJ_{2_k}\ ]\begin{bmatrix} A''^*_{\Theta_k} \\ C''^*_{\Theta_k} \end{bmatrix}[\ Q_{2_k}^* \quad Q_{1_k}^*\ ] & [\ B_kX_{k+1}^* \quad D_kJ_{2_k}\ ]\begin{bmatrix} B^*_{\Theta_k} \\ D^*_{\Theta_k} \end{bmatrix} \end{array}\right]\begin{bmatrix} I & \\ & J_{1_k} \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 0 & X_k^* & C_{o_k} \\ \hline 0 & 0 & D_{o_k} \end{array}\right]$$

with A''_{Θ_k}, C''_{Θ_k}, Q_{1_k} and Q_{2_k} the same as in the algorithm description. The results of $(1,1)$ and $(2,1)$ block matrices are directly from the algorithm, we can also show that C_{o_k} and D_{o_k} are the same as we defined in equation (5.98) and also as in equation (5.23). Then we know if the outer–J-lossless factorization exists, the factorization given by (5.98) exists. Now the problem left is the possibility to implement the factorization directly with (5.98) and to realize it.

If we consider the J-RQ factorization with R upper triangular and Q J-orthonormal, we can compute the factorization with the following strategy.

Computational aspects: In the ordinary RQ factorization, an easy way to construct the orthonormal space of Q is by using the *elementary orthonormal operator* $EO = \begin{bmatrix} cos\varphi & -sin\varphi \\ cos\varphi & sin\varphi \end{bmatrix}$. To obtain J-orthonormal Q, we need to use another elementary operator, J-orthonormal operator, besides EO. An *elementary J-orthonormal operator* is defined by

$$EJ = \frac{1}{cos\varphi}\begin{bmatrix} 1 & sin\varphi \\ sin\varphi & 1 \end{bmatrix}$$

It is not difficult to check the J-orthonormal property of the operator.

In the computing process, which elementary operator should be used in a step is determined by the column of the elements considered in the step and the corresponding column signs in matrix $\begin{bmatrix} I & \\ & J \end{bmatrix}$. If the elements considered in the step have the same column sign, the elementary operator is EO; if the elements have different column sign, the elementary operator is EJ. For example, we consider the J-orthonormal RQ factorization of matrix $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \end{bmatrix} = RQ$ and assume that $\begin{bmatrix} I & \\ & J_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. When we zero element $(3,3)$, we consider element A_{33} and A_{34}. The corresponding column signs are 1 and -1. Then EJ is used in the step. If we zero element $(2,2)$, A_{22} and A_{23} are considered, the corresponding column signs are 1 and 1, then EO is used.

What we should point out is that at this stage, we do not have a nice algorithm to guarantee the existence of the factorization of this form even when we are sure that the corresponding Riccati equation has a semipositive solution and that the outer–J-lossless factorization exists. The reason may be that for the factorization of (5.98), we do not really need the R factor to be upper triangular. With the strategy we consider above, R is factorized to be upper triangular. During the computation, it happens that when EJ operator should be used, the absolute value of the entries we try to zero is greater than the absolute value of the corresponding diagonal element, this makes it impossible to construct a EJ operator to zero the entries we want to zero. But in many cases, the factorization exists. This gives us a possible easier way to consider the outer–J-lossless factorization for these special cases.

To conclude this chapter, we give two numerical examples of outer–J-lossless factorizations to the stable time invariant systems by using J-RQ factorization algorithm. A realization of this system is

$$T = \left[\begin{array}{c|ccc} 0.8 & 3 & 4 & 0.4 \\ \hline 2 & 4 & 3 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

Assume that the initial R: $R_0 = 1$. Then,

$$\begin{bmatrix} AR_0 & C \\ BR_0 & D \end{bmatrix} = \begin{bmatrix} 0.8 & 3 & 4 & 0.4 \\ 2 & 4 & 3 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

Assume $J_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and then

$$\begin{bmatrix} I & \\ & J_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Compute J-orthonormal RQ factorization recursively, the results is as follows:

$$R = -2.1333$$

$$T_o = \left[\begin{array}{c|cc} 0.8 & 7.7197 & -6.0326 \\ \hline 2 & 8.9928 & -6.4550 \\ 1 & 0 & 1.5650 \end{array}\right]$$

$$\Theta = \left[\begin{array}{c|ccc} -0.6028 & 0.0561 & -0.8150 & 0.1757 \\ \hline -1.4529 & 0.9035 & 0.7923 & 1.5984 \\ -1.3632 & 0.6390 & 0.6390 & 1.9170 \end{array}\right]$$

Then T_o has a pole equal to 0.8 and a zero -0.6028.

Another example:

$$T = \left[\begin{array}{cc|cc} 0.7 & 0 & 1 & 0.1 \\ 0 & 0.8 & 0 & 0.2 \\ \hline 0.7271 & -3.4225 & -0.6364 & 0 \\ 0.2747 & -12.2919 & 0.3532 & -1.9050 \end{array}\right]$$

Initial $R_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The resulted $R = \begin{bmatrix} 0 & -1.4072 \\ 0 & -0.0420 \end{bmatrix}$ or $R = \begin{bmatrix} -1.4072 \\ -0.0420 \end{bmatrix}$.

$$T_o = \left[\begin{array}{cc|cc} 0.7 & 0 & -0.1721 & 0.2227 \\ 0 & 0.8 & 0.0064 & 0.2017 \\ \hline 0.7271 & -3.4225 & -1.1006 & -0.1815 \\ 0.2747 & -12.2919 & 0 & -1.8675 \end{array}\right]$$

$$\Theta = \left[\begin{array}{c|cc} 0.5899 & 0.8151 & -0.1109 \\ \hline -0.8105 & 0.6095 & -0.1682 \\ 0.0695 & -0.1891 & 1.0201 \end{array}\right]$$

T has two zeros: 1.6952 and -0.5071.

T_o has two poles: 0.7 and 0.8; two zeros: 0.5899 and -0.5071.

5.7 Appendix

5.7.1 Appendix 1: The proof of the invertibility of matrix V in Theorem 5.5

Repeat equations (5.54) and (5.55) as follows:

$$\begin{bmatrix} M & U \\ N & V \end{bmatrix} \begin{bmatrix} A_\Theta & C_\Theta B \\ & A \end{bmatrix} = \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix} \begin{bmatrix} M^{(-1)} & U^{(-1)} \\ N^{(-1)} & V^{(-1)} \end{bmatrix} \tag{5.102}$$

$$[\ B_\Theta \quad D_\Theta B\ ] = B_1[\ N^{(-1)} \quad V^{(-1)}\ ] \tag{5.103}$$

We partition the transformation operator to ensure that $V \in \mathcal{D}$ is square. Assume that there is a column vector $\xi \neq 0$ such that

$$V\xi = 0$$

We can also derive that

$$V^{(-1)}\mathbf{Z}\xi = 0$$

Multiplying equation (5.103) on the right with $\mathbf{Z}\xi$ and then from the second column of it, we can obtain that:

$$B\mathbf{Z}\xi = 0$$

Multiplying the (2,1) entry of equation (5.102) on the right with $\mathbf{Z}\xi$, we get:

$$NC_\Theta B\mathbf{Z}\xi + VA\mathbf{Z}\xi = A_2V^{(-1)}\mathbf{Z}\xi$$

Since $V^{(-1)}\mathbf{Z}\xi = 0$ and $B\mathbf{Z}\xi = 0$, we have

$$VA\mathbf{Z}\xi = 0$$

We then know that ξ is in a $A\mathbf{Z}$ invariant space. Define the space as:

$$\Xi = \{\xi, V\xi = 0\}$$

then,

$$A\mathbf{Z}\Xi \subset \Xi$$

With $B\mathbf{Z}\xi = 0$ and $A\mathbf{Z}\Xi \subset \Xi$, we obtain:

$$\begin{array}{lllll}
B\mathbf{Z}\Xi = 0 & \Rightarrow & \mathbf{Z}B^{(1)}\Xi = 0 & \Rightarrow & B^{(1)}\Xi = 0 \\
B\mathbf{Z}A\mathbf{Z}\Xi = 0 & \Rightarrow & \mathbf{Z}B^{(2)}A^{(1)}\Xi = 0 & \Rightarrow & B^{(2)}A^{(1)}\Xi = 0 \\
 & & \vdots & & \\
B\mathbf{Z}(A\mathbf{Z})^{\delta-1}\Xi = 0 & \Rightarrow & \mathbf{Z}B^{(\delta)}A^{(\delta-1)}\cdots A^{(1)}\Xi = 0 & \Rightarrow & B^{(\delta)}A^{(\delta-1)}\cdots A^{(1)}\Xi = 0
\end{array}$$

with δ defined in chapter 2. This means that

$$\mathcal{C}_\delta \Xi = 0$$

$\mathcal{C}_\delta$ is the observability matrix. Because $\{A, B\}$ is a reachable pair, $\mathcal{C}_\delta \Xi = 0$ iff $\Xi = 0$. This is contradict the assumption that $\xi \neq 0$. Therefore $V\xi = 0$ iff $\xi = 0$ and V is invertible. □

5.7.2 Appendix 2: The proof of (5.67)

From (5.65), Θ has a realization:

$$\Theta = \left[\begin{array}{c|c} \hat{A} & -XA^{-*}B^*J \\ \hline D_\Theta B & D_\Theta \end{array}\right]$$

The state equation of the system can be:

$$\chi \mathbf{Z}^{-1} = \chi \hat{A} + U D_\Theta B$$

and the output equation is:

$$Y = -\chi X A^{-*} B^* J + U D_\Theta$$

Multiplying $X^{(-1)}$ on the right of the state equation, we have:

$$\chi \mathbf{Z}^{-1} X^{(-1)} = \chi \hat{A} X^{(-1)} + U D_\Theta B X^{(-1)}$$

With the relation of (5.66), the above equation can be rewritten into:

$$\chi X \mathbf{Z}^{-1} = \chi X A^{-*} + U D_\Theta B X^{(-1)}$$

Denote $\chi' = \chi X$, we obtain the realization of Θ as (5.67). □

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Chapter 6

A Solution to the H_∞ Control Problem in Discrete Time-Varying Systems

6.1 Introduction

The H_∞ control problem was introduced by Zames [1] in 1981 and H_∞ control becomes a popular control method explored in the control engineering literature in the past decade [2] [3]. Only a few researchers have worked on this topic in the context of time-varying systems, e.g. [7] [8] [9] [10]. Most solutions aim at time invariant systems [2] [3] [4] [5] [6] [11] [12].

In recent years, a unified framework of the H_∞ control theory based on two fundamental notions, the chain scattering representation and outer–J-lossless (J-lossless–outer) factorization, was developed by Tsai and Postlethwaite [13], and Kimura [14] with different approaches. Both of their researches were on continuous time invariant systems. The former works with coprime factorization approach and the latter works with conjugation method. The key step of this framework, no matter what approach is based, is obtaining the outer–J-lossless (J-lossless–outer) factorization of a chain scattering representation (a dual chain scattering representation) of a known plant. The H_∞ control problem for a general system is reduced to a H_∞ control problem for a J-lossless system. Then the H_∞ control problem is clarified in this way.

Lossless and J-lossless operator theory has been developed for discrete time-varying systems by Dewilde and Van der Veen et al.[15] [16] [17] [18]. We extended this theory to more general systems in Chapter 4 and developed outer–J-lossless (J-lossless–outer) factorization algorithms in chapter 5. With all these results, we can extend the method in [13] and [14] to the discrete time-varying context. This is the main topic of this chapter.

Let us consider the standard set-up as in [14] shown in Figure 6.1. In the figure, $P \in \mathcal{X}$ is the input-output operator of a known plant, $w = [w_k]_{-\infty}^{\infty}$, where $w_k \in \mathbb{R}^r$, is the exogenous disturbance sequence, $u = [u_k]_{-\infty}^{\infty}$, where $u_k \in \mathbb{R}^p$, is the control input sequence, $y = [y_k]_{-\infty}^{\infty}$, where $y_k \in \mathbb{R}^q$, is the observed output sequence and $z = [z_k]_{-\infty}^{\infty}$, where $z_k \in \mathbb{R}^m$, is the controlled error output sequence. In the figure, the variables with a dot indicate inputs of the mapping and the variables without a dot indicate outputs. The mapping from the inputs $[u\ w]$ to the output $[z\ y]$ is given by the operator P as

$$[\, z \quad y \,] = [\, u \quad w \,] P = [\, u \quad w \,] \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \tag{6.1}$$

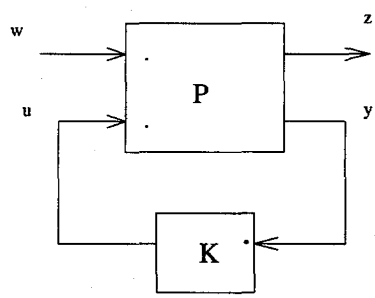

Figure 6.1: The standard system configuration

Let us introduce a feedback control operator K to the system such that

$$u = yK \tag{6.2}$$

Then, the H_∞ control problem is described as follows.

H_∞ **control problem**: For a known plant P and a given number γ, find a characterization for all admissible controllers K such that

(i) The closed loop system in Figure 6.1 is causally stable (upper);

(ii) The H_∞ norm of the closed loop operator Φ, which is the mapping from the disturbance w to output z, is smaller than γ.

Assume that $(I - KP_{12})^{-1}$ exists, then the closed-loop transfer operator Φ is given by

$$\Phi = P_{21} + P_{22}(I - KP_{12})^{-1} K P_{11} \tag{6.3}$$

We assume that $\gamma = 1$ from now on as this can always be obtained by scaling [4] the system realization with γ. Then the H^∞ control problem now is to find a class of K which satisfies (i) in the H_∞ control problem and achieves

$$||\Phi||_\infty < 1 \tag{6.4}$$

Operator P gives the mapping from $[u\ w]$ to $[z\ y]$. This system can be described in another way under certain conditions.

Assuming that P_{22} in equation (6.1) is invertible, then operator G which maps $[u\ y]$ to $[z\ w]$ is as

$$[\ z \quad w\ ] = [\ u \quad y\ ]G = [\ u \quad y\ ]\begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \tag{6.5}$$

It is not difficult to derive G from P as

$$G = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} P_{11} - P_{12}P_{22}^{-1}P_{21} & -P_{12}P_{22}^{-1} \\ P_{22}^{-1}P_{21} & P_{22}^{-1} \end{bmatrix} \tag{6.6}$$

We call G a chain scattering operator.

We redraw Figure 6.1 as Figure 6.2 by using G, the mapping from port $[u\ y]$ to port $[z\ w]$. With G, the transfer operator Φ from w to z can be described as

$$\Phi = \mathrm{HM}(G; K) = (KG_{12} + G_{22})^{-1}(KG_{11} + G_{21}) \tag{6.7}$$

where HM stands for the HoMographic transformation. The advantage of the chain scattering representation is that the cascade connection of systems, say $G_1, G_2, ...$, is the product of G_1, $G_2, ...$, ie. $G = G_1G_2 ...$. This is a very useful characteristic of the solution given by the method in [13] and [14].

In the case that P_{11} is invertible, we can define the mapping from $[z\ w]$ to $[u\ y]$ as,

$$[\ u \quad y\ ] = [\ z \quad w\ ]\bar{G} = [\ z \quad w\ ]\begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{21} & \bar{G}_{22} \end{bmatrix} \tag{6.8}$$

and

$$\begin{bmatrix} \bar{G}_{11} & \bar{G}_{12} \\ \bar{G}_{21} & \bar{G}_{22} \end{bmatrix} = \begin{bmatrix} P_{11}^{-1} & P_{11}^{-1}P_{12} \\ -P_{12}P_{11}^{-1} & P_{22} - P_{21}P_{11}^{-1}P_{12} \end{bmatrix} \tag{6.9}$$

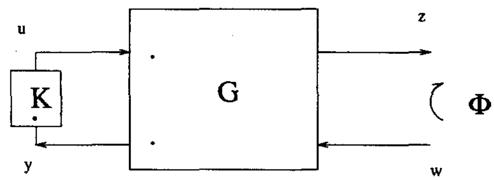

Figure 6.2: Chain scattering repesentation configuration

We call $\bar{G}$ the dual of the chain scattering representation. With $\bar{G}$, the transfer operator Φ can be described as,

$$\Phi = \mathrm{DHM}(\bar{G}; K) = (\bar{G}_{21} - \bar{G}_{22}K)(\bar{G}_{12}K - \bar{G}_{11})^{-1} \tag{6.10}$$

where DHM stands for the Dual of the HoMographic transformation.

The method in [13] and [14] is based on the chain scattering representation G and the dual of the chain scattering representation $\bar{G}$. It is standard to assume the following dimension conditions of a plant [14],

$$r \geq q \qquad\qquad m \geq p$$

in H_∞ control [14]. In this thesis, we only consider the cases that $q = r$, $m \geq p$ or $r \geq q$, $m = p$. In especially, we pay our attention to the first case, the second case can be considered dually.

Suppose in the first case that G has a factorization

$$G = T_o\Theta \tag{6.11}$$

with Θ J-lossless and T_o invertible and outer. This is equivalent to the cascade connection of two operators T_o and Θ. as in Figure 6.3. If we connect an operator, the inverse of the outer operator T_o, to the left hand side of G as in Figure 6.3, then the cascade system is equivalent to Θ. Let us repeat a very important theorem about the property of a J-lossless operator in chapter 4 for this solution:

Theorem 6.1 *Let an operator $\Theta \in \mathcal{X}$ be J-lossless and partitioned as $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ and let operator $S \in \mathcal{U}$ be strictly contractive ($\|S\|_\infty < 1$). Let*

$$\Phi = HM(\Theta; S) = (S\Theta_{12} + \Theta_{22})^{-1}(S\Theta_{11} + \Theta_{21}) \tag{6.12}$$

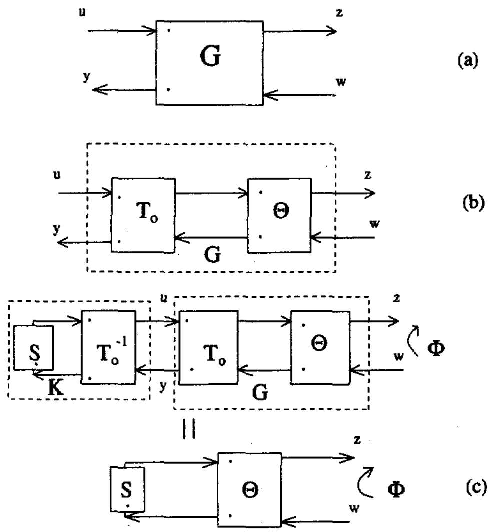

Figure 6.3: (a) and (b). Block diagrams of outer–J-lossless factorization of chain scattering representation G. (c). The block diagram of a solution of H_∞ controller and the equivalent closed-loop transfer function Φ.

Then Φ *is upper and* $\|\Phi\|_\infty < 1$. □

This theorem gives an easy way to solve the H^∞ control problem to a known plant P. If the chain scattering representation G of P exists and has a factorization as in equation (6.11), all K such that $||\mathrm{HM}(G;K)||_\infty = ||\mathrm{HM}(\Theta;S)||_\infty < 1$ are given by

$$K = HM(T_o^{-1}; S) \tag{6.13}$$

for all S which is strictly contractive.

Thus, the H^∞ control problem for a general system is reduced to H^∞ control problem for a J-lossless system.

In [14], Kimura discussed outer–J-lossless (J-lossless–outer) factorizations for continuous time invariant systems and used J-lossless conjugation to compute the outer–J-lossless (J-lossless–outer) factorization.

The method consists of finding a J-lossless transfer function Θ which contains all the unstable poles and zeros of the chain scattering transfer function and a left term T_o such that $G = T_o\Theta$, where T_o is outer. Two Riccati equations have to be solved to do the factorization. In the time-varying case, the concept of poles and zeros does not exist any more. If we consider the problem in operator description, a general operator consists of causal part and anticausal part. We note that the unstable poles in time invariant cases corresponds to the anticausal part (or antistable part in a causal state equation description) of operators and the unstable zeros to the anticausal part (or antistable part) of the 'inverse' of operators. What we need to find is a J-lossless operator which contains these two parts. If the realization of the system is given by state space description and when the existence of the dichotomy of the system is assumed, that means the system can be split into causal and anticausal parts (or stable and antistable parts), then what we need to find is a J-lossless factor Θ which contains the anticausal part (or antistable part) of G and the anticausal part (or antistable part) of the 'inverse' of G, and the left factor T_o such that $G = T_o\Theta$, where T_o is outer.

6.2 Operator description based algorithm

Let a chain scattering operator $G \in \mathcal{X}$ with the port signature matrix (J_1, J_2) such that $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$ and a uniform realization $\{A_c, B_c, C_c, A_a, B_a, C_a, D\}$, where $\ell_{A_c} < 1$ and $\ell_{A_a} < 1$, and specified by

$$G = D + B_c\mathbf{Z}(I - A_c\mathbf{Z})^{-1}C_c + B_a\mathbf{Z}^*(I - A_a\mathbf{Z}^*)^{-1}C_a \tag{6.14}$$

In the previous chapter, we have discussed the J-lossless factorization of such a G. Assume the outer–J-lossless factorization $G = T_o\Theta$, with Θ J-lossless and T_o outer, exists. A class of controller K, which results in a class of closed-loop transfer operators in the H_∞ space, are the homographic transformation of T_o^{-1} and any strictly contractive operator S as

$$K = \mathrm{HM}(T_o^{-1}; S) \tag{6.15}$$

and the transfer operator Φ then is:

$$\Phi = \mathrm{HM}(\Theta; S) \tag{6.16}$$

Because Θ is J-lossless, Φ is strictly contractive or belongs to the H_∞ group.

If the dual operator, say $\bar{G}$, of the chain scattering operator exists, we consider the conjugated J-lossless-outer operator factorization as discussed in Subsection 5.2.4 as $\bar{G} = \Theta T_o$ such that Θ^* is J-lossless and T_o is outer. Then the controller $K = \mathrm{DHM}(T_o^{-1}; S)$ with S any contractive operator and $\Phi = \mathrm{DHM}(\Theta; S)$.

6.3 Outer–(J_2 J_1)-lossless factorization — J-lossless conjugation method

Let G be a chain scattering representation of a known plant P with port signature matrices (J_1, J_2) such that $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$ and have a uniform realization $\{A, B, C, D\}$ such that:

$$XZ^{-1} = XA + UB$$

$$Y = XC + UD$$

where U, Y and X are inputs, outputs and states time sequences respectively. $\{A, B, C, D\}$ are big diagonal system matrices.

A realization of the *conjugated system* of G, denoted by G^*, is $\{A^{-*}, -C^*A^{-*}, A^{-*}B^*, D^* - C^*A^{-*}B^*\}$.

We consider the following factorization first:

$$G = G_1\Theta_+ \tag{6.17}$$

where Θ_+ is a antistable J-lossless operator and G_1 is stable. G_1 is further factorized into:

$$G_1 = T_o\Theta_- \tag{6.18}$$

where Θ_- is a stable J-lossless operator and T_o is outer. Then G has a outer–J-lossless factorization:

$$G = T_o\Theta_-\Theta_+ = T_o\Theta \tag{6.19}$$

where $\Theta = \Theta_-\Theta_+$.

In previous chapter, we have discussed J-lossless conjugation. We now use J-lossless conjugation operation to consider the factorization.

Let the antistable J-lossless conjugation of J_2G^*, if it exists, be denoted by:

$$\Theta_+J_2G^* = J_2G_1^* \tag{6.20}$$

where $J_2G_1^*$ is antistable. Then,

$$G_1 = GJ_2\Theta_+^* J_2 = G\Theta_+^{-1} \tag{6.21}$$

is stable. According to Theorem 5 in chapter 5, when Riccati equation (5.71) corresponding to G_1 has a semi-positive solution, the stable operator G_1 can be further factorized into:

$$G_1 = T_o\Theta_- \tag{6.22}$$

Then, we can find outer–(J_2 J_1)-lossless factorization of G as:

$$G = T_o\Theta_-\Theta_+ \tag{6.23}$$

with T_o outer and $\Theta_-\Theta_+$ (J_2 J_1)-lossless.

Theorem 6.2 *Let a chain scattering representation G of a time-varying system have a uniform state space realization $\{A, B, C, D\}$ with the port signature matrices (J_1, J_2) such that $J_2 = \begin{bmatrix} I & \\ & J_1 \end{bmatrix}$. Assume that A is invertible and there exists a dichotomy of the system. Then G has an outer–(J_2 J_1)-lossless factorization if both Riccati equations:*

$$X^{(-1)} = A^*(I - XCJ_2C^*)^{-1}XA \tag{6.24}$$

and

$$\begin{aligned} Y &= AY^{(-1)}A^* + CJ_2C^* \\ &\quad - (CJ_2D^* + AY^{(-1)}B^*)(DJ_2D^* + BY^{(-1)}B^*)^{-1}(CJ_2D^* + AY^{(-1)}B^*)^* \end{aligned} \tag{6.25}$$

have semi-positive definite solutions such that:

$$YX < I \tag{6.26}$$

In this case,

$$T_o \sim \left[\begin{array}{c|c} \hat{A} & -\hat{F}L \\ \hline \hat{B} & L \end{array}\right] \tag{6.27}$$

with:

$\hat{A} = A + CJ_2C^*A^{-*}X^{(-1)}$,

$\hat{B} = B + DJ_2C^*A^{-*}X^{(-1)}$,

$$\begin{aligned}\hat{F} &= -(\hat{C}J_2\hat{D}^* + \hat{A}(I - Y^{(-1)}X^{(-1)})^{-1}Y^{(-1)})\hat{B}^*) \\ &\quad (\hat{D}J_2\hat{D}^* + \hat{B}(I - Y^{(-1)}X^{(-1)})^{-1}Y^{(-1)})\hat{B}^*)^{-1}\end{aligned} \tag{6.28}$$

and $\hat{C} = CD_\Theta^{-1}$, $\hat{D} = DD_\Theta^{-1}$.

Where D_Θ *and* L *are determined by:*

$$D_\Theta^{-1}J_2D_\Theta^{-*} = J_2C^*A^{-*}X^{(-1)}A^{-1}CJ_2 + J_2$$

and

$$LJ_1L^* = \hat{D}J_2\hat{D}^* + \hat{B}(I - Y^{(-1)}X^{(-1)})^{-1}Y^{(-1)})\hat{B}^*$$

respectively and

$$\begin{aligned}\Theta &= \Theta_-\Theta_+ \\ &\sim \left[\begin{array}{cc|c} \hat{A} + \hat{F}\hat{B} & -(\hat{C} + \hat{F}\hat{D})D_\Theta J_2C^*A^{-*}X^{(-1)} & (\hat{C} + \hat{F}\hat{D})D_\Theta \\ & A & C \\ \hline L^{-1}\hat{B} & -L^{-1}\hat{D}D_\Theta J_2C^*A^{-*}X^{(-1)} & L^{-1}\hat{D}D_\Theta \end{array}\right]\end{aligned} \tag{6.29}$$

Proof: Let us first consider the antistabilizing J-lossless conjugator Θ_+ of JG^* which has a realization:

$$JG^* \sim \left[\begin{array}{c|c} A^{-*} & A^{-*}B^* \\ \hline -JC^*A^{-*} & J(D^* - C^*A^{-*}B^*) \end{array}\right] \tag{6.30}$$

From Theorem 5.5, the Riccati equation can be written as:

$$X = A^{-*}(I + X^{(-1)}A^{-1}CJC^*A^{-*})^{-1}X^{(-1)}A^{-1} \tag{6.31}$$

or

$$X^{(-1)} = A^*(I - XCJC^*)^{-1}XA \tag{6.32}$$

When there is a unique solution $X \geq 0$ such that:

$$\bar{A} = A + CJC^*A^{-*}X^{(-1)} \tag{6.33}$$

is stable and then, the realization of the antistablizing J-lossless conjugator Θ_+ is:

$$\Theta_+ \sim \left[\begin{array}{c|c} A & C \\ \hline -D_\Theta JC^*A^{-*}X^{(-1)} & D_\Theta \end{array}\right] \tag{6.34}$$

Θ_+ is antistable and invertible. Then:

$$G_1 = G\Theta_+^{-1} \sim \left[\begin{array}{c|c} A + CJC^*A^{-*}X^{(-1)} & -CD_\Theta^{-1} \\ \hline -(B + DJC^*A^{-*}X^{(-1)}) & DD_\Theta^{-1} \end{array}\right] = \left[\begin{array}{c|c} \hat{A} & \hat{C} \\ \hline \hat{B} & \hat{D} \end{array}\right] \tag{6.35}$$

is stable. D_Θ^{-1} can be solved from the equation:

$$JC^*A^{-*}X^{(-1)}A^{-1}CJ + J = D_\Theta^{-1}JD_\Theta^{-*} \tag{6.36}$$

From Theorem 5.6 we know that G_1 has an outer (J_2, J_1)-lossless factorization if the Riccati equation:

$$\begin{aligned} Z &= \hat{A}Z^{(-1)}\hat{A}^* + \hat{C}J_2\hat{C}^* \\ &\quad -(\hat{C}J_2\hat{D}^* + \hat{A}Z^{(-1)}\hat{B}^*)(\hat{D}J_2\hat{D}^* + \hat{B}Z^{(-1)}\hat{B}^*)^{-1}(\hat{C}J_2\hat{D}^* + \hat{A}Z^{(-1)}\hat{B}^*)^* \end{aligned} \tag{6.37}$$

has a semipositive definite solution. Next we prove that $Z = (I - YX)^{-1}Y$, where Y is the semipositive definite solution of the Riccati equation (6.25).

Denote $\mathcal{F}$,

$$\begin{aligned} \mathcal{F} &= \begin{bmatrix} A & -CJ_2C^*A^{-*} \\ & A^{-*} \end{bmatrix} \\ &\quad - \begin{bmatrix} CJ_2(D^* - C^*A^{-*}B^*) \\ A^{-*}B^* \end{bmatrix} \begin{bmatrix} DJ_2(D^* - C^*A^{-*}B^*) \end{bmatrix}^{-1} \begin{bmatrix} B & -DJ_2C^*A^{-*} \end{bmatrix} \end{aligned}$$

as the corresponding symplectic matrix of (6.25) and $\mathcal{F}_o$,

$$\begin{aligned} \mathcal{F}_o &= \begin{bmatrix} \hat{A} & -\hat{C}J_2\hat{C}^*\hat{A}^{-*} \\ & \hat{A}^{-*} \end{bmatrix} \\ &\quad - \begin{bmatrix} \hat{C}J_2(\hat{D}^* - \hat{C}^*\hat{A}^{-*}\hat{B}^*) \\ \hat{A}^{-*}\hat{B}^* \end{bmatrix} \begin{bmatrix} \hat{D}J_2(\hat{D}^* - \hat{C}^*\hat{A}^{-*}\hat{B}^*) \end{bmatrix}^{-1} \begin{bmatrix} \hat{B} & -\hat{D}J_2\hat{C}^*\hat{A}^{-*} \end{bmatrix} \end{aligned}$$

as the corresponding symplectic matrix of Riccati equation (6.37). It is not difficult to prove that:

$$\mathcal{F}_o = \begin{bmatrix} I & \\ X & I \end{bmatrix} \mathcal{F} \begin{bmatrix} I & \\ -X^{(-1)} & I \end{bmatrix} \tag{6.38}$$

X is the semipositive definite solution of the Riccati equation (6.31). Because

$$\begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \mathcal{F}_o = S_o \begin{bmatrix} Z_1^{(-1)} & Z_2^{(-1)} \end{bmatrix} \tag{6.39}$$

then,

$$\begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} I & \\ X & I \end{bmatrix} \mathcal{F} \begin{bmatrix} I & \\ -X^{(-1)} & I \end{bmatrix} = S_o \begin{bmatrix} Z_1^{(-1)} & Z_2^{(-1)} \end{bmatrix} \tag{6.40}$$

such that $Z = Z_2 Z_1^{-1}$. From this expression we know that:

$$\begin{bmatrix} Z_1 & Z_2 \end{bmatrix} \begin{bmatrix} I & \\ X & I \end{bmatrix} = \begin{bmatrix} Y_1 & Y_2 \end{bmatrix} \tag{6.41}$$

$Y = Y_2 Y_1^{-1}$ is the semipositive definite solution of Eq. (6.25). Then $Z = (I - YX)^{-1}Y$ can be easily derived from equation (6.41). Because we need that $Z \geq 0$, then we must have that $YX < I$.

From Theorem 5.6 we know that:

$$T_o \sim \left[\begin{array}{c|c} \hat{A} & \hat{F}L \\ \hline -\hat{B} & L \end{array} \right]$$

and

$$\Theta_- \sim \left[\begin{array}{c|c} \hat{A} + \hat{F}\hat{B} & \hat{C} + \hat{F}\hat{D} \\ \hline L^{-1}\hat{B} & L^{-1}\hat{D} \end{array} \right]$$

with $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$ defined in Eq. (6.35), and $\hat{F}$:

$$\hat{F} = -(\hat{C}J_2\hat{D}^* + \hat{A}Z^{(-1)}\hat{B}^*)(\hat{D}J_2\hat{D}^* + \hat{B}Z^{(-1)}\hat{B}^*)^{-1}$$

This is equation (6.28) with $Z = (I - YX)^{-1}Y$.

Then, finally the expression of Θ in equation (6.29) comes from the cascade connection of Θ_- and Θ_+. □

When T_o and Θ are determined, the controller K and the transfer operator Φ are determined in the same way discussed in the previous section.

In the case the dual representation of the chain scattering representation exists, the conjugated J-lossless-outer factorization is considered. All the computations are the duals of the computation in Theorem 6.2 as we considered in operator description case in Chapter 5.

6.4 About the Riccati recursion

In Chapter 5, we have shown the convergence of the Riccati recursion based on a stable realization $\{A, B, C, D\}$ ($\ell_A < 1$). In this chapter, the Riccati equations we need to solve are not limited in the stable systems any more.

In the early 60's, Kalman and Bucy investigated the algebraic Riccati equation in the time invariant case under the assumption of controllability and observability to derive the uniqueness of the symmetric positive semidefinite solution, the stability of the closed loop systems and the asymptotic convergence properties [19]. They showed that their main results also hold true in time-varying case. Later on and until now, many researches have been working on the solutions of Riccati equations by weaking the assumptions to stabilizability and detectability [20] [21] [23] [24], and even more in the study of the nonstabilizable case [22]. All of these work are on time invariant case.

One of the special classes of time-varying system we are interested in periodically time-varying systems. In this case, the theory of the periodic Riccati equation appears almost as complete as its time invariant comrade [25]. The Riccati recursion discussed in our problem is different from the Riccati recursion discussed in the pioneering works such that the former is with a J factor. This makes that there may not exist a positive semidefinite solution of the algebraic Riccati equation. (For the factorization, we can multiplying part of the system matrices C, D (or B, D) with a γ factor to make sure that there exists a positive semidefinite solution). But if the solution exists, with all the pioneering results [20] [21], [24] we declare that for a periodically time-varying system, if it is detectable (or stabilizable), the Riccati recursion will converge to the stabilizing solution, which is a hermitain and semipositive definite.

Another special case is that one step time-varying system like switching networks. Then the convergence of the stablizing solution of Riccati equation of these systems is as same as in time invariant case with the initial condition brings from the solution of the previous system (or the following system in the backward recursion).

In a recent paper, De Nicolao studied Riccati difference equation (or Riccati recursion) of the arbitrarily time-varying case [25]. He discussed the conditions for the existence of the maximal and stabilizing solutions under the assumption of uniform detectability (or stabilizability).

In time-varying case, the Riccati recursion arise naturally from the time-varying behavior of the system. Under the assumption of uniform detectability (or stabilizability), if the initial condition of the recursion is correct, for example, the initial condition comes from the stabilizing solution of the algebraic Riccati equation of a time invariant system which works in a certain time period before or after the time-varying period, the recursion is the only natural way to continue the computation of the solution and then what we obtain is what we want. If the initial condition is not correct, we should use the two dimensional recursive algorithm in [25] to obtain the stabilizing and symmetric positive semidefinite solution.

For the initial condition in some special cases, we refer to [18].

6.5 Numerical examples of time-varying control systems

It is well understood that the sensitivity minimization problem is in fact a kind of H_∞ control problem. In this section, we show two numerical examples of the sensitivity minimization problem by using time-varying controller which is designed with the algorithm we discussed before. An instantaneous controller, which is the solution designed by considering every time step individually, is used as comparison in both examples.

6.5.1 An RL circuit

The first example is an RL circuit as shown in Figure 6.4. We only consider the small-signal model, ie. deviations from the working point of the circuit.

In the figure, i_u is a voltage controlled current source driven by the controller K. Resistance R_1 represents the influence of the internal resistance of the current source and a constant load in the system. Resistance R_3 is a large constant load. Current i_w, which is randomly changing, represents a random load. The R_2–L branch is an inductive load.

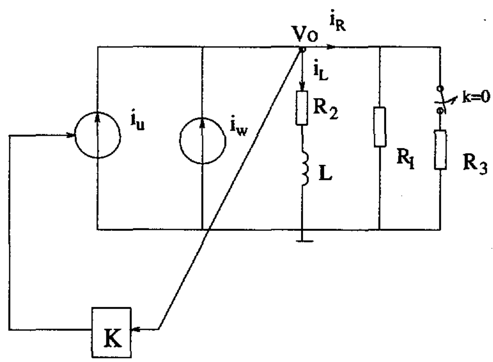

Figure 6.4: An example of a switch circuit

Because this is an inductive circuit, the output voltage v_o is sensitive to the current disturbance i_w. Suppose that at time $k = 0$, the switch is opened and then the system can be viewed as a time-varying system. We want to reduce the sensitivity of the voltage v_o and the controlled input i_u [1] to the disturbance i_w. In terms of the notation of Section 6.1, the disturbance w is i_w; the output z is v_o and i_u. Since v_o is sensitive to i_w, we take v_o as the measurement y and i_u as the controlled input u.

The state equation of the system is:

$$L\frac{di_L}{dt} = -(R + R_2)i_L + Ri_u + Ri_w$$

and the output equation is:

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} v_o \\ i_u \\ \hline v_o \end{bmatrix} = \begin{bmatrix} -R \\ 0 \\ -R \end{bmatrix} i_L + \begin{bmatrix} R & R \\ 1 & 0 \\ R & R \end{bmatrix} \begin{bmatrix} i_u \\ i_w \end{bmatrix}$$

where $R = \frac{R_1 R_3}{R_1+R_3}$ when the switch is closed and $R = R_1$ when the switch is open. We use Matlab to do the simulation. The parameters of the system are: $R_1 = 50$; $R_2 = 0.1$; $R_3 = 1.02$ and $L = 0.1$.

The system is discretized by zero order hold method with the sampling time 0.01. [2] The realization of the chain scattering representation is calculated from the discrete time

[1] It is a common practice in H_∞ control to consider the controlled input as an output [26] in the case we want to reduce the sensitivity of the controlled input to the disturbance.

[2] The sampling time should be smaller than the smallest time constant in order to obtain a valid discrete time model. In this example, the smallest time constant of this circuit is approximate to 0.1.

system. The realizations of the chain scattering representation of System 1 (before $k = 0$) and System 2 (after $k = 0$) are denoted by $\{A_1, B_1, C_1, D_1\}$ and $\{A_2, B_2, C_2, D_2\}$ respectively.

$$A_1 = 0.9905 \qquad B_1 = \begin{bmatrix} 0 \\ 0.0947 \end{bmatrix} \qquad C_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A_2 = 0.9980 \qquad B_2 = \begin{bmatrix} 0 \\ 0.0198 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 0.02 \end{bmatrix}$$

Both systems are stable so that we only need to consider the outer–J-lossless factorization of a stable system. Since the dimensions of w, u, z and y are 1, 1, 2 and 1 respectively, $J_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ and $J_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

In this simple example, we use a recursive method to calculate the outer–J-lossless factorization of the time invariant System 1 and 2. The algorithm is given in Subsection 5.2.2.. This algorithm is for time-varying systems. For time invariant systems, the recursion is the same but the realization does not change. First, let us consider the central controller which is designed by taking $S = 0$ in (6.13) for System 1 and 2. The scaling coefficient $\gamma = 1$ (the minimal γ of this system). We call the corresponding controller for these two time invariant systems Controller $K_{(1)}$ and Controller $K_{(2)}$. Similarly, the instrumental matrices of System 1 and System 2 are called $X_{(1)}$ and $X_{(2)}$ respectively. The results of the computation for the time invariant system 1 and 2 are given in the following table.

		H_∞ norm of the sensitivity function	
System	X	Open loop system	Closed loop system
1	2.25028	1.05	0.72418
2	50.00111	99.2388	0.99995

At $k = 0$, system changes from System 1 to System 2. A time-varying controller is designed by considering this change. With the algorithm given in Subsection 5.2.2, we compute the factorization. The algorithm is a backward recursion. Then, the initial value of X is $X_0 = X_{(2)}$. For $k = -1, -2, \cdots$, solution X_k is time-varying but is converging to $X_{(1)}$. The realization of System 1 is used in the recursion. The relative error between $X_{(1)}$ and X_k is smaller than 1 percent. 47 recursions takes place to reach this level. The time varying controller is connected to System 1 47 steps before the change takes place. Before the time-varying control, Controller $K_{(1)}$ is connected to the system; when and after the change takes place, Controller $K_{(2)}$ is connected to the system.

Figure 6.5 shows the realization of the time-varying controller.

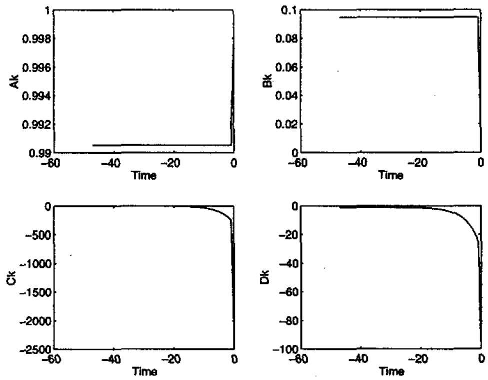

Figure 6.5: Realization of the time-varying controller

An instantaneous (IT) controller ($K_k = K_{(1)}, k < 0$; $K_k = K_{(2)}, K \geq 0$), is used as a comparison. The realizations are taken from the realization of time-varying controller at $k = -47$ for $K_{(1)}$ and $k = 0$ for $K_{(2)}$.

The H_∞ norm of the sensitivity operator is approximated with the method given in the appendix of this chapter. The results are given in the following table.

H_∞ norm of the sensitivity function		
Open loop system	Time-varying control	Instantaneous control
99.2388	0.999985	3.497388

Typical outputs of v_o and i_u of the closed loop system are given in Figure 6.6. The noise input is zero mean white with covariance equal to 1. The change takes place at $k = 0$.

From the result in the table we can see that the sensitivity of the closed loop system is largely reduced with the controller designed by the method in [13] and [14]. The time-varying controller gives a better result than the instantaneous controller with the tested data. This is because with the IT controller, v_o is much larger before $k = 0$ than it was after $k = 0$ and then i_R is relatively large before $k = 0$ with a relatively smaller R. When system is changed at $k = 0$, R becomes much larger than it is before $k = 0$. On the other hand, at $k = 0$, v_o keeps a relatively larger value, this causes a big current change in R branch and also R_2-L branch. The current change then causes an output voltage shot at T_0. The TV controller is designed by considering the influence of the second system before the change. The current in R branch is decreasing before the change takes place (i_R changes in the same way as v_o). This is the result that i_u is increasing against i_w

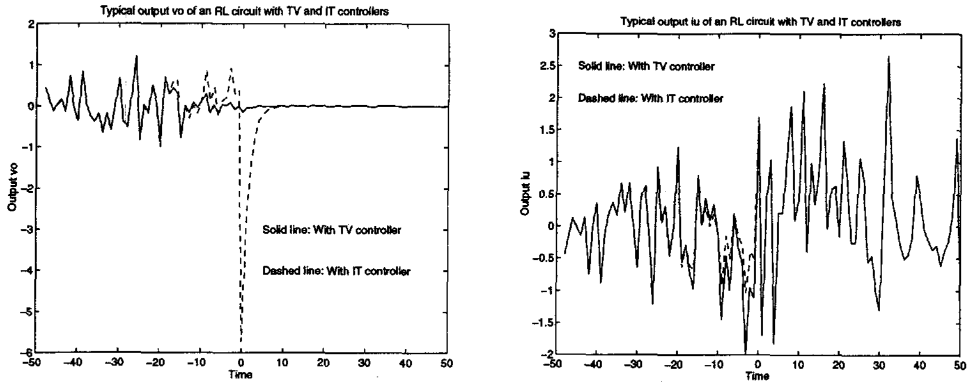

Figure 6.6: Output comparison of a TV controller and IT controller

during the transient time period. Then there is no shot in output voltage at instant T_0 when a TV controller is used.

We change the scaling coefficient γ from 1 to 2 with step length 0.1, and take $S = -0.9, 0, 0.9$. The H_∞ norm of the the sensitivity function of the closed loop systems are shown in Figure 6.7.

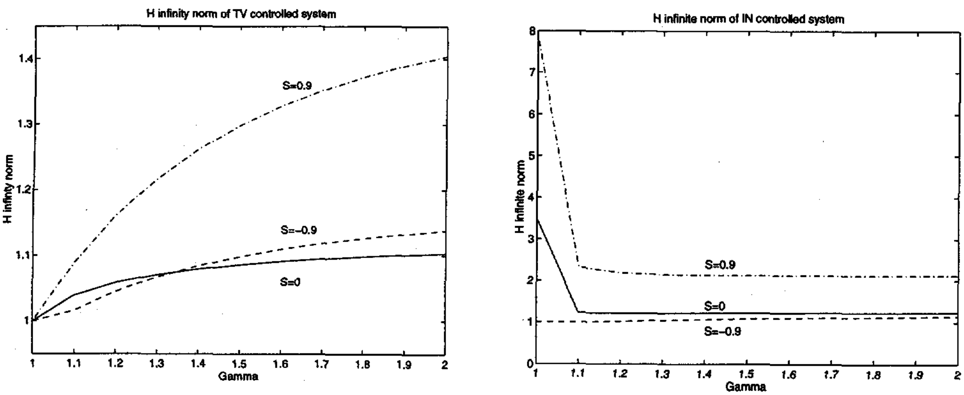

Figure 6.7: H infinity norm comparison of time-varying method and instantaneous method

The result shows that with the time-varying control method, we can always achieve the H_∞ norm of the sensitivity function of the closed loop system smaller than γ if S is strictly contractive; with the instantaneous method, this kind of result can be achieved only in some cases and in many cases, the H_∞ norm of the sensitivity function of the closed loop system is larger than γ. The minimal H_∞ norm is obtained with the time-varying controller. An observation of the outputs shows that the voltage shot never happens when a time-varying controller is used but it happens in many cases with the instantaneous method.

We can also see from Figure 6.7 that within the data we tested, the best S for time-varying control is $S = 0$ (central controller) and the best S for the instantaneous controller is $S = -0.9$. Figure 6.8 shows that with $S = 0$, the minimal H_∞ norm of the sensitivity function of System 1 is obtained; with $S = -0.9$, this norm is much larger for any γ than it is with $S = 0$. The H_∞ norm of the closed loop system of System 2 does not change much for different S and Γ, and is about 0.9999. A comprehensive review of the whole system shows that with the time-varying control method, we can obtain a significantly better controller than with the instantaneous control method.

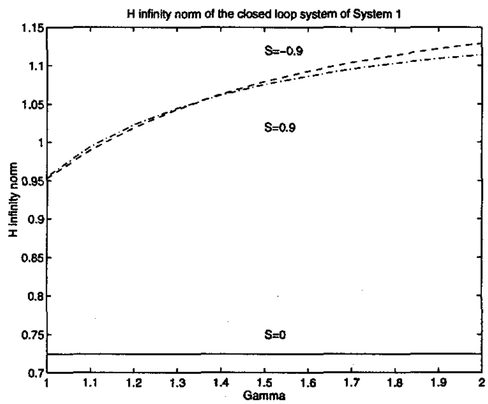

Figure 6.8: H infinity norm of the sensitivity function of the closed loop system of System 1 with different S

6.5.2 A robot arm along a given trajectory

A robot arm is a non-linear system. It can be described by a linear time-varying model along a given trajectory if we consider the deviation of the output which is caused by a certain kind of disturbance. Here is a simple example of such a system (Figure 6.9).

The system is a single beam to which a mass is applies. Let m be the mass, T be the applied torque, φ be the angle between the arm and the vertical axis, $f = mg$ be the force gravity and l be the length of the arm. Assume that the mass of the rod can be ignored compared to m. φ changes according to $\sin(\omega t)$ with time t under a reference torque. The

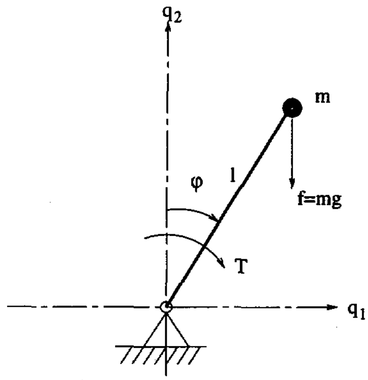

Figure 6.9: A simple robot

dynamic equation of the system is

$$ml^2\ddot{\varphi} = fl\sin\varphi + T - d\dot{\varphi} \tag{6.42}$$

where d is the damping coefficient.

Because $\varphi = \sin(\omega t)$, the first and second differentiation of φ are

$$\dot{\varphi} = \omega\cos(\omega t) \qquad and \qquad \ddot{\varphi} = -\omega^2\sin(\omega t)$$

respectively. From equation (6.42), we have

$$\ddot{\varphi} = \frac{f}{ml}\sin\varphi + \frac{1}{ml^2}T - \frac{d}{ml^2}\dot{\varphi}$$

The variation $\Delta\ddot{\varphi}$ of $\ddot{\varphi}$ under variation of T

$$\Delta\ddot{\varphi} = \frac{f}{ml}\cos\varphi\Delta\varphi + \frac{1}{ml^2}\Delta T - \frac{d}{ml^2}\Delta\dot{\varphi} \tag{6.43}$$

The position q of m is determined by φ and l as

$$q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} l\sin\varphi \\ l\cos\varphi \end{bmatrix}$$

The first and second differentiation of q are

$$\dot{q} = \begin{bmatrix} l\cos\varphi\dot{\varphi} \\ -l\sin\varphi\dot{\varphi} \end{bmatrix} \qquad and \qquad \ddot{q} = \begin{bmatrix} -l\sin\varphi\dot{\varphi}^2 + l\cos\varphi\ddot{\varphi} \\ -l\cos\varphi\dot{\varphi}^2 - l\sin\varphi\ddot{\varphi} \end{bmatrix}$$

The variation of $\ddot{q}$ is

$$\Delta\ddot{q} = \begin{bmatrix} -l\cos\varphi\dot{\varphi}^2 - l\sin\varphi\ddot{\varphi} & -2l\sin\varphi\dot{\varphi} \\ l\sin\varphi\dot{\varphi}^2 - l\cos\varphi\ddot{\varphi} & -2l\cos\varphi\ddot{\varphi} \end{bmatrix} \begin{bmatrix} \Delta\varphi \\ \Delta\dot{\varphi} \end{bmatrix} + \begin{bmatrix} l\cos\varphi \\ -l\sin\varphi \end{bmatrix} \Delta\ddot{\varphi} \tag{6.44}$$

Consider a disturbance in the torque, denoted by T_w, which is caused by wind and other external factors and a controlled input T_u which is also a torque. Then the variation of the reference torque T is $\Delta T = T_u + T_w$. equation (6.43) becomes

$$\Delta\ddot{\varphi} = \frac{f}{ml}\cos\varphi\Delta\varphi + \frac{1}{ml^2}(T_u + T_w) - \frac{d}{ml^2}\Delta\dot{\varphi} \tag{6.45}$$

We find the state equation of equation (6.45)

$$\begin{bmatrix} \Delta\dot{\varphi} & \Delta\ddot{\varphi} \end{bmatrix} = \begin{bmatrix} \Delta\varphi & \Delta\dot{\varphi} \end{bmatrix} \begin{bmatrix} 0 & \frac{f}{ml}\cos\varphi \\ 1 & -\frac{d}{ml^2} \end{bmatrix} + \begin{bmatrix} T_u & T_w \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{ml^2} \\ 0 & \frac{1}{ml^2} \end{bmatrix} \tag{6.46}$$

with state $x = \begin{bmatrix} \Delta\varphi & \Delta\dot{\varphi} \end{bmatrix}$ (For reason of consistency, we use row vectors). As we know the trajectory, the nominal φ is given as a function of time. System (6.46) is linear. Suppose that T is such that $\varphi = \sin(\omega t)$, then the system is a linear periodic system.

The output is the variation of the angle: $\Delta\varphi$ and T_u. The measurement variable is the acceleration variation $\Delta\ddot{q}_1 = (-l\cos\varphi\dot{\varphi}^2 - l\sin\varphi\ddot{\varphi} + l\cos\varphi\frac{f}{ml}\cos\varphi)\Delta\varphi - (2l\sin\varphi\dot{\varphi} + \frac{d}{ml}\cos\varphi)\Delta\dot{\varphi} + \frac{\cos\varphi}{ml}(T_u + T_w)$. This expression is obtained by substituting $\ddot{\varphi}$ into the first row of equation (6.44).

The system variables are:

Disturbance: $w = T_w$;

Controlled input: $u = T_u$;

Measurement: $y = \Delta\ddot{q}_1$;

Controlled output: $z = [\ \Delta\varphi \quad T_u\ ]$.

State: $x = [\ \Delta\varphi \quad \Delta\dot{\varphi}\ ]$.

Then the continuous time state matrices of the plant: $[\ u \quad w\ ] \rightarrow [\ z \quad y\ ]$ are:

$$A = \begin{bmatrix} 0 & \frac{f}{ml}\cos\varphi \\ 1 & -\frac{d}{ml^2} \end{bmatrix} \qquad B = \begin{bmatrix} 0 & \frac{1}{ml^2} \\ 0 & \frac{1}{ml^2} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & -l\cos\varphi\dot{\varphi}^2 - l\sin\varphi\ddot{\varphi} + \frac{f}{m}\cos^2\varphi \\ 0 & 0 & -(2l\sin\varphi\dot{\varphi} + \frac{d}{ml}\cos\varphi) \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 1 & \frac{1}{ml}\cos\varphi \\ 0 & 0 & \frac{1}{ml}\cos\varphi \end{bmatrix}$$

We use **Matlab** to do the simulation. The parameters used in the simulation are:

$$m = 1 \qquad l = 1 \qquad d = 1, \qquad \omega = 8$$

Period: $P = 2\pi/\omega$

We use the bilinear approximation in the discretization. Sampling time: $T_s = P/10$ (10 samples in one period).

The plant is unstable and sensitive to any noise. The chain scattering representation G is stable. Because the dimension of the output is larger than the dimension of the controlled input (in this example, the controlled input is part of the output) and the chain scattering representation is neither outer nor J-lossless, we need to compute the outer–J-lossless factorization of G for the controller design. With the dimension of the port, the port signature matrices J_1 and J_2 have diagonal blocks entries equal to: $j_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $j_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ respectively.

We do the following steps:

1. Discretize the model of the plant at every sampling time in one period by the bilinear approximation. Change this representation into the chain scattering representation $\{A_g, B_g, C_g, D_g\}$ of the corresponding discrete time model of the periodic system.

2. Scale the chain scattering representation by γ. In this example, we choose $\gamma > 1$. This is done by multiplying C_g and D_g by $J(\gamma)$, where $J(\gamma) = \begin{bmatrix} I & \\ & \gamma I \end{bmatrix}$. It is well known that the choice of the minimal γ depends on the system.

3. Solve the Riccati equation:

$$\begin{aligned} Y &= A_g Y^{(-1)} A_g^* + C_g J_2 C_g^* \\ &\quad -(C_g J_2 D_g^* + A_g Y^{(-1)} B_g^*)(D_g J_2 D_g^* + B_g Y^{(-1)} B_g^*)^{-1}(C_g J_2 D_g^* + A_g Y^{(-1)} B_g^*)^* \end{aligned}$$

The initial Y_0 is determined by the solution of the Riccati equation of the equivalent time invariant system within one period (Write out the recursion of one period, then the conclusion can be found).

4. With the solution, compute the realization of the outer factor T_o and the (J_2, J_1)-lossless operator Θ.

The central controller then is designed as $\mathrm{HM}(T_o^{-1}; 0)$.

The H_∞ norm of the sensitivity operator is estimated by considering one period as a time invariant system. Instantaneous controller is designed as a comparison. The results are given in the table.

γ	H_∞ norm of the sensitivity function	
	Time-varying control	Instantaneous control
1.000001	1.00000091	1.000001068
1.01	1.009146	1.01061
1.1	1.0924	1.0986
4	1.7206	1.8182

From the results we can see that within the tested γ, the H_∞ norm of the sensitivity function of the closed loop system is always smaller than γ if the system is controlled by a time-varying controller; if it is controlled by a instantaneous controller, the H_∞ norm of the sensitivity function is larger than γ when $\gamma < 1.1$. Compared with instantaneous control, time-varying control method always gives a better result in this example.

6.5.3 Remarks

From these two examples we have the following remarks:

1. In the first example, the recursive algorithm for outer–J-lossless factorization in Section 5.2.2 is used for time invariant system solutions. In the second example, Riccati recursion is used for the solution of the algebraic Riccati equation. As we mentioned in previous section that under the assumption of the detectability (or stabilizability), it is proved in literature that the recursion leads to the maximal stabilizing solution, if the solution exists. The convergent speed is depended on the realization of the system. In some cases, the square-root recursion method is more efficient than directly working on Riccati recursion. Take these two examples as example, the Riccati recursion converges very fast and it is comparable to the square-root algorithm for the RL circuit example. On the other hand, the square-root algorithm converges even faster than directly working on the

Riccati recursion for the second example. Nevertheless, the most efficient and accurate method to obtain the maximal stabilizing solution of the algebraic Riccati equation is possibly analysis of the eigenvalues and invariant subspaces of an associated Hamitonian matrix of the Riccati equation. A overview and many references about this method can be found in [24].

2. The choice of the measurement is very important for the sensitivity minimalization problem. In the RL circuit example, we chose the voltage v_o as the measurement because it is sensitive to the disturbance current in the inductive circuit. In a capacitive circuit, the current of the capacitor branch is sensitive to a disturbance voltage, so that a current which involves the capacitor current can be chosen as the measurement. If the measurement is not sensitive to the noise, then feedback can not obtain useful information about the disturbance and the control becomes impossible.

6.5.4 A digital network

A digital network is shown in figure 6.10. In this figure, the "Delay" indicates a one step

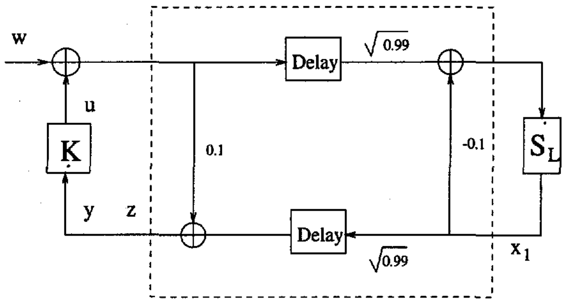

Figure 6.10: A digital network

delay, S_L is a time-varying system and K is a controller which we are going to design. The output of K is u which is the controlled input, w is an external disturbance, y is the measurement which is the input of the controller K, and z is the controlled output. Here, $y = z$. The H_∞ control problem in this network is to attenuate the influence of w to z through a feedback controller K.

Suppose that S_L is a time-varying scalar:

$$\begin{aligned} S_{L_k} &= 1 \qquad && k < 0 \\ S_{L_k} &= 0.2 && 0 \leq k \leq 10 \\ S_{L_k} &= 1 && k > 10 \end{aligned}$$

where k is the time index. For ease of discussion, we refer "System 1" when $S_{L_k} = 1$ and "System 2" when $S_{L_k} = 0.2$.

The dynamic equation of the open loop system (without K) is:

$$\begin{bmatrix} x_{1_{k+1}} & x_{2_{k+1}} \end{bmatrix} \begin{bmatrix} 1 + 0.1S_{L_{k+1}} & \\ & 1 \end{bmatrix} = \begin{bmatrix} x_{1_k} & x_{2_k} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} u_k & w_k \end{bmatrix} \begin{bmatrix} \sqrt{0.99}S_{L_{k+1}} & 0 \\ \sqrt{0.99}S_{L_{k+1}} & 0 \end{bmatrix}$$

and the output equation is:

$$\begin{bmatrix} z_k & y_k \end{bmatrix} = \begin{bmatrix} x_{1_k} & x_{2_k} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{0.99} & \sqrt{0.99} \end{bmatrix} + \begin{bmatrix} u_k & w_k \end{bmatrix} \begin{bmatrix} 0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}$$

The system is causally stable for any value of S_L. The chain scattering representation of System 1 is denoted by G_1 and of System 2 is denoted by G_2. Realizations of G_1 and G_2 are:

$$G_1: \quad A_1 = \begin{bmatrix} 0 & 1 \\ -9 & 0 \end{bmatrix} \qquad B_1 = \begin{bmatrix} 0 & 0 \\ 9.0453 & 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 0 \\ 0 & -9.9499 \end{bmatrix} \qquad D_1 = \begin{bmatrix} 0 & -1 \\ 1 & 10 \end{bmatrix}$$

$$G_2: \quad A_2 = \begin{bmatrix} 0 & 1 \\ -1.9412 & 0 \end{bmatrix} \qquad B_2 = \begin{bmatrix} 0 & 0 \\ 1.9510 & 0 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 0 \\ 0 & -9.9499 \end{bmatrix} \qquad D_2 = \begin{bmatrix} 0 & -1 \\ 1 & 10 \end{bmatrix}$$

It is not difficult to establish the minimality of the realizations. Note that both G_1 and G_2 are antistable, so that the time-varying system is antistable. However the representations can be transformed into anticausal stable forms. We denote such anticausal realizations of System G_1 and System G_2 as $\{A_{a_1}, B_{a_1}, C_{a_1}, D_{a_1}\}$ and $\{A_{a_2}, B_{a_2}, C_{a_2}, D_{a_2}\}$ respectively. The relation between the realizations of the anticausal form and the causal form are:

$$A_{a_i} = A_i^{-1}, \quad B_{a_i} = -B_i A_i^{-1}, \quad C_{a_i} = A_i^{-1} C_i, \quad D_{a_i} = D_i - B_i A_i^{-1} C_i$$

for $i = 1, 2$.

With this system, the outer–J-lossless factorization can be carried out in two steps:

1. Anticausal J-lossless factorization of the chain scattering represetation, denoted by G, of the time-varying system to obtain an upper operator and an anticausal J-lossless operator.

2. Causal J-lossless factorization of the upper operator to obtain an outer operator and a causal J-lossless operator.

The suboptimal controller K then is determined by the homographic transformation of the inverse of the outer operator and any strictly contractive upper operator.

Since the dimensions of z and u are equal, the port signature matrices J_1 and J_2 of this system are equal and given by

$$J_1 = J_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

For the first step (anticausal J-lossless factorization of G), we have to solve Lyapunov equation (5.5)

$$A_a Q A_a^* - C_a J_2 C_a^* = Q^{(-1)}$$

where A_a etc. are diagonal operators which are formed with the realization of System 1 and System 2 according to the time-varying schedule of the system. The initial condition of Q is the solution of the time invariant Lyapunov equation of System 1. At $k = 0$, the system switchs to System 2, and at $k = 11$, it changes to System 1 again, so that Q_k converges to this linear time invariant solution again. The solution Q is seen to satisfy $Q \gg 0$. Under this condition, embedding (A_a, C_a) with $(C_{\Theta_a}, D_{\Theta_a})$ such that (5.6) and (5.7) is possible. With (5.9), (5.10), (5.11) and (5.12), the realization of the upper operator is computed.

The second step is carried out with the algorithm in Subsection 5.2.2.

In this example, we use a different scaling coefficient γ in each time period. $\gamma_1 = 0.9$ for System 1 (it is approximately equal to the minimal γ of System 1). For comparison, for γ_2 of System 2 are used: $\gamma_2 = 0.2$ (it is approximately the minimal γ of System 2) to $\gamma_2 = 1$. The free parameter S in the homographic transformation (6.13) is set to $S = 0$. The H_∞ norm of the open loop system is 1. The H_∞ norm of the closed loop system is shown in figure 6.11. The H_∞ norm using an instantaneous control method is shown for comparison. The H_∞ of the closed loop system with a time-varying controller is 0.8911 independent of γ_2 within the tested range which is indeed smaller than the maximum of γ_1 and γ_2. With the instantaneous controller, the H_∞ norm of the closed loop system is larger than 0.9 when $\gamma_2 > 0.3$. It is larger than 1 when $\gamma_2 > 0.7$ and in this case, the closed loop system is worse than the open loop system. With the above results we can conclude that the H_∞ norm of the closed loop system with a TV controller is always smaller than the maximum of the scaling coefficients in accordance to theory. With an IT controller, the result can be worse. Testing with other S turns out to the same conclusion.

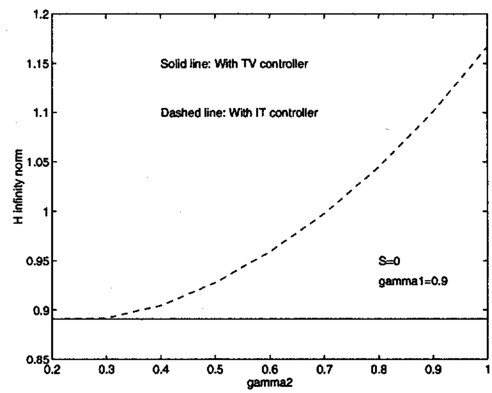

Figure 6.11: H infinity norm of the closed loop system

6.6 Conclusions

The H_∞ control problem of discrete time-varying systems is considered in this chapter in two cases:

1. when the chain scattering operator, say G, of the system exists;

2. when the dual operator of the chain scattering operator, say $\bar{G}$ exists.

With the method in [13] and [14], the H_∞ control problem is solvable if an outer-J-lossless factorization $G = T_o\Theta$ with T_o invertible and outer, and Θ lossless (or a conjugated J-lossless-outer factorization $\bar{G} = \Theta T_o$ with T_o invertible and outer, and Θ^* J-lossless) exists. The existence of the factorization depends on the definiteness of the solutions of a Lyapunov equation and a Riccati equation. If the factorization exists, the controller is designed as the homographic transformation of the inverse of the outer factor T_o and any strictly contractive operator S and the closed loop transfer operator is the homographic transformation of the J-lossless operator Θ and S (in the second case, the controller is designed as the dual of the homographic transformation of the inverse of the outer factor T_o and any strictly contractive operator S and the closed loop transfer operator is the dual of the homographic transformation of the conjugated J-lossless operator Θ and S). The H_∞ norm of the closed transfer operator is smaller than 1.

In the case that the chain scattering representation of the system can be represented by a causal state equation, J-conjugation method can be used in the factorization. Two Riccati equations need be solved with this method. The existence of the solution for the H_∞ control problem is determined by the definiteness of solutions of the two Riccati equations.

6.7 Appendix: Approximation of H_∞ norm

With the definition of the H_∞ norm of an operator, we now show that the H_∞ norm of an operator is the largest singular value of this operator.

Let T be a bounded operator. With the definition of $||T||_\infty$, we have:

$$||T||_\infty^2 = Sup\{||xT||^2, ||x|| = 1\} = Sup_{||x||=1} xTT^*x^*$$

$$= Sup_{||x||=1} xv \begin{bmatrix} \ddots & & & & \\ & \lambda_{-1}^2 & & & \\ & & \lambda_0^2 & & \\ & & & \lambda_1^2 & \\ & & & & \ddots \end{bmatrix} v^*x^*$$

$$= Sup_{||z||=||xv||=1} z \begin{bmatrix} \ddots & & & & \\ & \lambda_{-1}^2 & & & \\ & & \lambda_0^2 & & \\ & & & \lambda_1^2 & \\ & & & & \ddots \end{bmatrix} z^* = Sup_{||z||=1} \sum_{i=-\infty}^{\infty} \lambda_i^2 z_i^2 = \lambda_{max}^2$$

In the time invariant case, the H_∞ norm of a system can be calculated in frequency domain with its transfer faction: it is the maximum over all frequences of its largest singular value.

If $T \in \mathcal{U}$ has a strictly stable realization, then it has a band structure: the off diagonal entries are becoming smaller and finally approach zero. In the time invariant case for example, H_∞ norm of a system operator can also be approximated from a finite operator which is a part of the infinite operator along the main diagonal. As the finite operator becomes bigger, the maximal singular value approaches the maximum singular value of the system operator. If the finite operator is big enough, we can obtain a good approximation of the H_∞ norm of the system operator.

If the system is only time-varying in a short time period, this approximation is also valid if the finite matrix is big enough to include all the time-varying part and time invariant part. We use this method to approximate the H_∞ norm of an operator in some special case.

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Chapter 7

Interpolation and H_∞ Control

7.1 Introduction

In the previous chapter, we have discussed the method in [1] [2] of H_∞ control in discrete time-varying systems. The essential part of this method is the J-lossless factorization of a chain scattering operator of a system. This is equivalent to computing a J-spectral factorization [3]. If we can find a J-lossless operator Θ for a known chain scattering operator $G \in \mathcal{X}$ of a system P, such that both $GJ_2\Theta^*J_1$ and $(GJ_2\Theta^*J_1)^{-1}$ are upper, then the H_∞ problem for the system is solvable [2]. The set of suboptimal controller is given by the homographic transformation of the inverse of the outer factor which is obtained from the factorization and any strictly contractive outer operators. The closed loop transfer operator is given by the homographic transformation of Θ and the strictly contractive operators. The H_∞ norm of the closed loop transfer operator then is smaller than 1. This method can be interpreted by considering a given set of data (realization of G) for which we must find a J-lossless operator Θ such that the function resulting from the homographic transformation is analytic (upper) and strictly contractive. This interpretation shows that the H_∞ problem is equivalent to an interpolation problem in the sense that both need to define a set of functions (operators) to fit the data set (system) and the set of functions (operators) are analytic (upper) and strictly contractive.

In this chapter, we will look at the equivalence of these two problems in the case the system is invertible. We drop the subscripts to simplify the notation.

7.2 The equivalence of H_∞ control problem (G invertible) and interpolation problem

Before we discuss the equivalence, we recall the relation between a J-lossless operator, say Θ, and its corresponding lossless operator, say Σ.

Let a known J-lossless operator $\Theta \in \mathcal{X}$ be invertible and be partitioned as

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix} \tag{7.1}$$

Then the corresponding lossless operator Σ, which is upper and lossless, can be expressed as,

$$\Sigma = \begin{bmatrix} \Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21} & -\Theta_{12}\Theta_{22}^{-1} \\ \Theta_{22}^{-1}\Theta_{21} & \Theta_{22}^{-1} \end{bmatrix} \tag{7.2}$$

It is easy to obtain from the unitarity of Θ that $\Theta_{11} - \Theta_{12}\Theta_{22}^{-1}\Theta_{21} = \Theta_{11}^{-*}$ and $\Theta_{22}^{-1}\Theta_{21} = \Theta_{12}^{*}\Theta_{11}^{-*}$. Then we have,

$$\Sigma = \begin{bmatrix} \Theta_{11}^{-*} & -\Theta_{12}\Theta_{22}^{-1} \\ \Theta_{12}^{*}\Theta_{11}^{-*} & \Theta_{22}^{-1} \end{bmatrix} \tag{7.3}$$

Because Σ is upper, we have that Θ_{11}^{-*}, Θ_{22}^{-1}, etc. are upper.

Let $G \in \mathcal{X}$ be a known invertible operator. G and G^{-1} have uniformly minimal realizations which result in the following expressions of G and G^{-1} as,

$$G = D + B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}\begin{bmatrix} C_{21} & C_{22} \end{bmatrix} + B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}C_1 \tag{7.4}$$

$$G^{-1} = D_i + \begin{bmatrix} B_{21}' \\ B_{22}' \end{bmatrix}\mathbf{Z}^*(I - A_2'\mathbf{Z}^*)^{-1}C_2' + B_1'\mathbf{Z}(I - A_1'\mathbf{Z})^{-1}C_1' \tag{7.5}$$

where $\ell_{A_1} < 1$, $\ell_{A_1'} < 1$, $\ell_{A_2} < 1$ and $\ell_{A_2'} < 1$. Recall that all the system matrices are diagonal. For convenience, we denote the upper part of G as $D + B_1\mathbf{Z}(I - A_1\mathbf{Z})^{-1}C_1 = \begin{bmatrix} R_{11} & R_{12} \end{bmatrix}$ and of G^{-1} as $D_i + B_1'\mathbf{Z}(I - A_1'\mathbf{Z})^{-1}C_1' = \begin{bmatrix} R_{21} \\ R_{22} \end{bmatrix}$.

Now we look at necessary conditions for the J-lossless factorization of G.

Assume that there exists a J-lossless factorization of G such that $G = T_o\Theta$ with T_o invertible and outer, and Θ J-lossless. Because Θ is J-lossless and invertible, we have,

$$T_o = G\Theta^{-1} = GJ\Theta^* J$$

Define $T_o = \begin{bmatrix} \Gamma & \Delta \end{bmatrix}$ with the partition corresponding to the partition of Θ in (7.1). Γ can be written as,

$$\Gamma = B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}(C_{21}\Theta_{11}^* - C_{22}\Theta_{12}^*) + R_{11}\Theta_{11}^* - R_{12}\Theta_{12}^*$$

Since T_o is upper, Γ is upper. Multiplying by Θ_{11}^{-*} on the right of the above equation, we obtain:

$$\Gamma\Theta_{11}^{-*} = B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}(C_{21} - C_{22}\Theta_{12}^*\Theta_{11}^{-*}) + R_{11} - R_{12}\Theta_{12}^*\Theta_{11}^{-*} \tag{7.6}$$

Because both Γ and Θ_{11}^{-*} are upper, $\Gamma\Theta_{11}^{-*}$ is upper. On the other hand, we know that $\Theta_{12}^*\Theta_{11}^{-*}$ is upper and the last two terms on the right hand side of (7.6) are upper. We conclude that $B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}(C_{21} - C_{22}\Theta_{12}^*\Theta_{11}^{-*})$ is upper. This gives the first necessary condition for the existence of the factorization $G = T_o\Theta$. Because $\{A_2, B_2\}$ is a reachable pair, we can redefine the condition as $(\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}\Theta_{12}^*\Theta_{11}^{-*}) \in \mathcal{U}$. Define $S = \Theta_{12}^*\Theta_{11}^{-*}$, then,

$$(\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}\Theta_{12}^*\Theta_{11}^{-*}) = (\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}S) \quad \in \quad \mathcal{U}$$

Since Σ is lossless, $||S||_\infty < 1$ according to Theorem 4.6 in Chapter 4.

The second necessary condition is from T_o^{-1}. Because G is invertible, we have

$$T_o^{-1} = \Theta G^{-1}$$

Define $T_o^{-1} = \begin{bmatrix} \Gamma' \\ \Delta' \end{bmatrix}$ with the corresponding partition of Θ. Since T_o^{-1} is upper, Γ' and Δ' are upper. Δ' can be written as,

$$\Delta' = (\Theta_{21}B_{21}' + \Theta_{22}B_{22}')\mathbf{Z}^*(I - A_2'\mathbf{Z}^*)^{-1}C_2' + \Theta_{21}R_{21} + \Theta_{22}R_{22}$$

Multiplying on the left of the above expression by Θ_{22}^{-1}, we obtain:

$$\Theta_{22}^{-1}\Delta' = (\Theta_{22}^{-1}\Theta_{21}B_{21}' + B_{22}')\mathbf{Z}^*(I - A_2'\mathbf{Z}^*)^{-1}C_2' + \Theta_{22}^{-1}\Theta_{21}R_{21} + R_{22} \tag{7.7}$$

Because both Δ' and Θ_{22}^{-1} are upper, $\Theta_{22}^{-1}\Delta'$ is upper. We also know that $\Theta_{22}^{-1}\Theta_{21}$ is upper and the last two terms on the right hand side of (7.7) are upper. We conclude that $(\Theta_{22}^{-1}\Theta_{21}B'_{21} + B'_{22})\mathbf{Z}^*(I - A'_2\mathbf{Z}^*)^{-1}C'_2$ must be upper. This gives the second necessary condition. Because $\{A'_2, C'_2\}$ is an observable pair, we can redefine the condition as $(\Theta_{22}^{-1}\Theta_{21}B'_{21} + B'_{22})\mathbf{Z}^*(I - A'_2\mathbf{Z}^*)^{-1} \in \mathcal{U}$. From (7.1) we know that $\Theta_{22}^{-1}\Theta_{21} = \Theta_{12}^*\Theta_{11}^{-*} = S$, then,

$$(\Theta_{22}^{-1}\Theta_{21}B'_{21} + B'_{22})(\mathbf{Z} - A'_2)^{-1} = (SB'_{21} + B'_{22})(\mathbf{Z} - A'_2)^{-1} \quad \in \mathcal{U}$$

Now we can briefly formulate the necessary solvability conditions of the H_∞ problem in the case that G is invertible.

Let be given an invertible chain scattering representation G of a system given by (7.4) and its inverse given by (7.5). The H_∞ problem is solvable if there exists a J-lossless factorization such that $G = T_o\Theta$ with Θ J-lossless and T_o invertible and outer. The J-lossless factorization exists only if there is a $S \in \mathcal{U}$, $||S||_\infty < 1$ and

1. $(\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}S) \in \mathcal{U}$;
2. $(SB'_{21} + B'_{22})(\mathbf{Z} - A'_2)^{-1} \in \mathcal{U}$.

With the partitioning of Θ in (7.1), S is defined as $S = \Theta_{22}^{-1}\Theta_{21} = \Theta_{12}^*\Theta_{11}^{-*}$. Then the problem becomes that we look for a J-lossless operator Θ such that with the partitioning in (7.1) and the definition of S, condition 1 and 2 should be satisfied.

Let us look at the interpolation problem. The *double sided interpolation problem* is stated as follows:

Let be given two sets of three diagonal operators $\{V, \xi, \eta\}$ and $\{W, \zeta, \iota\}$ with $\ell_V < 1$ and $\ell_W < 1$. Does a set of strictly contractive operators $S \in \mathcal{U}$ exist such that,

1. $(\mathbf{Z} - V)^{-1}(\xi S - \eta) \in \mathcal{U}$;
2. $(S\zeta - \iota)(\mathbf{Z} - W)^{-1} \in \mathcal{U}$.

If such solutions exist, give the solutions.

From the above discussion we can see that the H_∞ problem is similar to the double sided interpolation problem. This gives a suggestion that if the interpolation problem based on the data set given by the system matrices, i.e. $\{A_2, A'_2, C_{21}, C_{22}, B'_{21}, B'_{22}\}$ (corresponding to
$\{V, W, -\xi, -\eta, \zeta, -\iota\}$), is solvable, then the H_∞ control problem is solvable. The conditions of solvability of the H_∞ control problem are similar to the conditions of the solvability of the interpolation problem.

Lemma 7.1 *Let $G \in \mathcal{X}$ be a known invertible chain scattering operator of a system, and let the output state space of the anticausal part of G be $\mathcal{H}_o^a(G) = \mathcal{D}_2(I - A_2\mathbf{Z}^*)^{-1}C_2$ and the input state space of the anticausal part of G^{-1} be $\mathcal{H}^a(G^{-1}) = \mathcal{D}_2(I - \mathbf{Z}A_2'^*)^{-1}\mathbf{Z}B_2'^*$, and both $\ell_{A_2} < 1$ and $\ell_{A_2'} < 1$. If G has a factorization such that $G = T_o\Theta$ with T_o invertible and outer, and Θ J-unitary, then,*

$$\mathcal{H}_o^a(\Theta) = \mathcal{H}_o^a(G) \tag{7.8}$$

$$\mathcal{H}^a(\Theta^{-1}) = \mathcal{H}^a(G^{-1}) \tag{7.9}$$

A uniformly minimal realization of Θ which satisfies condition (7.8) and condition (7.9) is
$\{A_2'^*, B_1, B_2'^*J, A_2, B_2, C_2, D_\Theta\}$ for some diagonal operator B_1, B_2 and D_Θ such that $\Theta = D_\Theta + B_1(\mathbf{Z}^* - A_2'^*)^{-1}B_2'^*J + B_2(\mathbf{Z} - A_2)^{-1}C_2$.

Proof: Assume that there exists a factorization $G = T_o\Theta$ with T_o invertible and outer. Since G is invertible, we have,

$$\Theta = T_o^{-1}G \qquad \textit{and} \qquad \Theta^{-1} = G^{-1}T_o$$

With $G \in \mathcal{X}$ and $\mathcal{H}_o^a(G) = \mathcal{D}_2(I - A_2\mathbf{Z}^*)^{-1}C_2$, G can be described as $G = G_{\mathcal{U}} + B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$ with $G_{\mathcal{U}}$ the part of G in $\mathcal{U}$ and B_2 the part of B matrix of G in $\mathbf{Z}^*\mathcal{L}$. With $\mathcal{H}^a(G^{-1}) = \mathcal{D}_2(I - \mathbf{Z}A_2'^*)^{-1}\mathbf{Z}B_2'^*$, G^{-1} can be described as $G^{-1} = G_{\mathcal{U}}' + B_2'\mathbf{Z}^*(I - A_2'\mathbf{Z}^*)^{-1}C_2'$ with $G_{\mathcal{U}}'$ the part of G^{-1} in $\mathcal{U}$ and C_2' the part of C matrix of G^{-1} in $\mathbf{Z}^*\mathcal{L}$.

Because T_o is invertible and outer, $T_o \in \mathcal{U}$ and $T_o^{-1} \in \mathcal{U}$.

$$\Theta = T_o^{-1}G = T_o^{-1}G_{\mathcal{U}} + T_o^{-1}B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$$

The term $T_o^{-1}G_{\mathcal{U}}$ is in $\mathcal{U}$ and does not effect the space of $\mathcal{H}_o^a(G)$. The term $T_o^{-1}B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$, as the result as a cascade connection of a upper operator and a lower operator, has the same anticausal output space as $B_2\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$ which is equal to $\mathcal{H}_o^a(G)$, then we have

$$\mathcal{H}_o^a(\Theta) = \mathcal{H}_o^a(G) = \mathcal{D}_2(I - A_2\mathbf{Z}^*)^{-1}C_2 \tag{7.10}$$

and in a similar way we can show that

$$\mathcal{H}^a(\Theta^{-1}) = \mathcal{H}^a(G^{-1}) = \mathcal{D}_2(I - \mathbf{Z}A_2'^*)^{-1}\mathbf{Z}B_2'^* \tag{7.11}$$

It is easy to check that such a $\Theta = D_\Theta + B_1(\mathbf{Z}^* - A_2'^*)^{-1}B_2'^*J + B_2(\mathbf{Z} - A_2)^{-1}C_2$ satisfies (7.10) and (7.11). □

Note that for this Θ, $\mathcal{H}_o(\Theta) = \mathcal{D}_2(I - A_2'^*\mathbf{Z})^{-1}B_2'^*J = \mathbf{Z}^*\mathcal{H}^a(\Theta^{-1})J$.

The next corollary is the converse of Lemma 7.1.

Corollary 7.1 *Let $T_1 \in \mathcal{X}$ and $T_2 \in \mathcal{X}$ be invertible operators. If T_1 and T_2 have the same anticausal output state space, their inverses have the same anticausal input state space and there is an operator T_o such that*

$$T_1 = T_oT_2 \tag{7.12}$$

then T_o is invertible and outer.

Proof: The proof of the invertibility of T_o is straightforward. What we need to show is that if $\mathcal{H}_o^a(T_1) = \mathcal{H}_o^a(T_2)$, $\mathcal{H}^a(T_1^{-1}) = \mathcal{H}^a(T_2^{-1})$ and $T_1 = T_oT_2$, then T_o is invertible and outer or both T_o and T_o^{-1} are upper. The proof is given in the appendix.

□

Lemma 7.1 and Corollary 7.1 are equivalent to the proposition 4.2 in [5] for time invariant systems and gives us a suggestion that if we have an invertible operator G, J-lossless factorization $G = T_o\Theta$ exists iff we can find a J-lossless operator Θ such that Θ has the same anticausal output state space of G and Θ^{-1} has the same anticausal input state space of G^{-1}.

Proposition 7.1 *[4] Let Θ be an invertible J-lossless operator which has a uniformly minimal realization $\{A_2'^*, B_1, B_2'^*J, A_2, B_2, C_2, D\}$ with $\ell_{A_2'} < 1$ and $\ell_{A_2} < 1$ such that*

$$\Theta = D + B_1(\mathbf{Z}^* - A_2'^*)^{-1}B_2'^*J + B_2(\mathbf{Z} - A_2)^{-1}C_2 \tag{7.13}$$

where $B_2' = \begin{bmatrix} B_{21}' \\ B_{22}' \end{bmatrix}$ and $C_2 = \begin{bmatrix} C_{21} & C_{22} \end{bmatrix}$. Let $S = HM(\Theta; 0) = \Theta_{22}^{-1}\Theta_{21} = \Theta_{12}^*\Theta_{11}^{-*}$. Then,

1. $(\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}S) \in \mathcal{U}$;

2. $(SB_{21}' + B_{22}')(\mathbf{Z} - A_2')^{-1} \in \mathcal{U}$.

Proof: Partition $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ such that Θ_{11} and Θ_{22} are square with appropriate dimensions. Denote $D + B_1(\mathbf{Z}^* - A_2'^*)^{-1}B_2'^*J = \begin{bmatrix} R_{11} & R_{12} \end{bmatrix}$ in (7.13) and multiplying on the right of it with $\Theta^{-1} = J\Theta^*J$ we find,

$$I = \left\{ \begin{bmatrix} R_{11} & R_{12} \end{bmatrix} + B_2(\mathbf{Z} - A_2)^{-1} \begin{bmatrix} C_{21} & C_{22} \end{bmatrix} \right\} J \begin{bmatrix} \Theta_{11}^* & \Theta_{21}^* \\ \Theta_{12}^{-1} & \Theta_{22}^* \end{bmatrix} J$$

Because Θ is invertible, Θ_{11} is invertible. The first block column multiplied by Θ_{11}^{-*} on the right gives

$$\begin{bmatrix} \Theta_{11}^{-*} \\ 0 \end{bmatrix} = B_2(\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}S) + R_{11} - R_{12}S$$

Θ is J-lossless, Θ_{11}^{-*} is upper, hence $B_2(\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}S) \in \mathcal{U}$. Because $\{A_2, B_2\}$ is a reachable pair, we have,

$$(\mathbf{Z} - A_2)^{-1}(C_{21} - C_{22}S) \in \mathcal{U}$$

$\Theta^{-1} = J\Theta^*J = JD^*J + B_2'(\mathbf{Z} - A_2')^{-1}B_1^*J + JC_2^*(Z^* - A_2^*)^{-1}B_2^*J$. A similar result follows from the premultiplying of Θ^{-1} with Θ, taking the second block row and premultiplying with Θ_{22}^{-1}. □

According to this proposition we know that a J-lossless operator Θ which has the form (7.13) satisfies all the necessary conditions for the J-lossless factorization. Conversely, if we know that there is a strictly contractive operator $S \in \mathcal{U}$ such that Condition 1 and 2 in Proposition 7.1 are satisfied, we can construct a Θ which is J-lossless and has the same anticausal output state space as that of G and the inverse of Θ has the same anticausal input state space as that of the inverse of G. Then with Corollary 7.1 we know that with such a Θ, the factorization $G = T_o\Theta$ exists such that T_o is invertible and outer. Hence the four necessary conditions are also sufficient conditions for the J-lossless factorizations. Then we see that in the case that G is invertible, H_∞ control problem and the double sided interpolation problem are equivalent.

In Chapter 5, we have considered the algorithm for an outer-J-lossless factorization for a given chain scattering operator G which is described by (5.1). The existence of such a factorization is determined by the condition defined in Proposition 5.1 concerning with anticausal J-lossless factorization and the conditions defined in Proposition 5.3 concerning with the causal J-lossless factorization. The condition in Proposition 5.1 is that the anticausal J-lossless operator exists iff there is a uniformly positive definite solution for the Lyapunov equation:

$$A_a Q_{22} A_a^* - C_a J C_a^* = Q_{22}^{(-1)}$$

The conditions in Proposition 5.3 for the causal J-lossless factorization are defined by the intermediate results from the step of the anticausal J-lossless factorization. Later on we have shown that the conditions given by Proposition 5.3 are equivalent to the condition given by the solution of the Riccati equation (5.27). The causal J-lossless factorization exists iff the solution of the Riccati equation is semi-positive definite. In this chapter, we have discussed that in the case the inverse of G exists, the solvability of the H_∞ problem is equivalent to the solvability of a double sided interpolation problem. Then the conditions for the solvability of these two problems are the same. Next we will show that the solvability of the interpolation problem is determined by the definiteness of a *Pick operator* which is specified by the interpolation data (or the realization of G in the H_∞ control case).

7.3 Solution of double sided interpolation problem

We repeat the double sided interpolation problem as: let be given interpolation data (A_1, C_1, A_2, C_2), where (A_1, C_1) is uniformly observable, (A_2, C_2) is uniformly reachable and $\ell_{A_1} < 1$, $\ell_{A_2} < 1$. Construct (if possible) a upper and strictly contractive operator S such that:

(1) $(Z - A_2)^{-1} C_2 \begin{bmatrix} I \\ -S \end{bmatrix} \in \mathcal{U}$;

(2) $\begin{bmatrix} S & -I \end{bmatrix} C_1^* (Z - A_1^*)^{-1} \in \mathcal{U}$.

From Proposition 7.1 we know that if we can construct a J-lossless operator Θ which has a uniformly minimal realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ for some diagonal operators B_1, B_2 and D and

$$\Theta = D + B_1(\mathbf{Z}^* - A_1)^{-1} C_1 + B_2(\mathbf{Z} - A_2)^{-1} C_2 \tag{7.14}$$

and partition $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ with Θ_{11} and Θ_{22} invertible, then $S = HM(\Theta; 0) = \Theta_{22}^{-1}\Theta_{21}$ which is upper and strictly contractive satisfies condition (1) and (2). We say that such a S is an interpolant for the double sided interpolation problem.

Then when can we construct such a Θ?

Assume that $\Theta = \Theta_c\Theta_a$ with $\Theta_c \in \mathcal{U}$, $\Theta_a \in \mathcal{L}$ and both J-lossless. With the cascade connection role given by Lemma 2.1 in Chapter 2 we know that Θ_a has a uniformly minimal realization $\{A_2, B_a, C_2, D_a\}$ for some B_a and D_a and

$$\Theta_a = D_a + B_a(\mathbf{Z} - A_2)^{-1}C_2 \tag{7.15}$$

Θ_a is J-lossless iff there exists a uniformly positive definite Hermitian operator Q_{22} such that the Lyapunov equation is satisfied:

$$\begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix} \begin{bmatrix} Q_{22} & \\ & -J \end{bmatrix} \begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix}^* = \begin{bmatrix} Q_{22}^{(-1)} & \\ & -J \end{bmatrix}$$

Next problem is that what is the uniformly minimal realization of Θ_c such that $\Theta = \Theta_c\Theta_a$ and under what condition Θ_c is J-lossless?

Suppose that Θ is as (7.14) and Θ_a is as (7.15) and the realizations of both are uniform, then,

$$\Theta_c = \Theta J \Theta_a^* J$$

and Θ_c has a realization:

$$A_c = \begin{bmatrix} A_1 & C_1JC_2^* \\ & A_2^* \end{bmatrix} \qquad B_c = \begin{bmatrix} B_1 & DJC_2^* - B_2Q_{22}A_2^* \end{bmatrix}$$
$$C_c = \begin{bmatrix} C_1JD_a^*J \\ B_a^*J \end{bmatrix} \qquad D_c = DJD_a^*J - B_2Q_{22}B_a^*J$$

Is $\{A_c, B_c, C_c, D_c\}$ a uniformly minimal realization of Θ_c? Since we need that Θ is J-lossless, from Theorem 4.5 and Proposition 4.3 we know that this is possible if there is a $Q_{12} \in \mathcal{D}$ such that:

$$C_1J_2C_2^* - Q_{12}A_2^* + A_1Q_{12}^{(-1)} = 0$$
$$B_1Q_{12}^{(-1)} + DJC_2^* - B_2Q_{22}A_2^* = 0$$

are satisfied. Defining a similarity transformation $T = \begin{bmatrix} I & -Q_{12} \\ & I \end{bmatrix}$ and with the above conditions we have:

$$TA_cT^{-(-1)} = \begin{bmatrix} A_1 & \\ & A_2^* \end{bmatrix} \qquad B_cT^{-(-1)} = \begin{bmatrix} B_1 & 0 \end{bmatrix}$$

$$TC_c = \begin{bmatrix} C_1 J D_a^* J - Q_{12} B_a^* J \\ B_a^* J \end{bmatrix}$$

We then derive a uniformly minimal realization of Θ_c as $\{A_1, B_1, C_1 J D_a^* J - Q_{12} B_a^* J, D_c\}$. Θ_c is J-lossless iff there exists a Hermintian operator $M \gg 0$ such that:

$$\begin{bmatrix} A_1 & C_1 J D_a^* J - Q_{12} B_a^* J \\ B_1 & D_c \end{bmatrix} \begin{bmatrix} M^{(-1)} & \\ & J \end{bmatrix} \begin{bmatrix} A_1 & C_1 J D_a^* J - Q_{12} B_a^* J \\ B_1 & D_c \end{bmatrix}^* = \begin{bmatrix} M & \\ & J \end{bmatrix} \quad (7.16)$$

Cascade such a Θ_c with Θ_a, we have a $\Theta = \Theta_c \Theta_a$ which has a uniformly minimal realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ and is J-lossless.

Proposition 7.2 *Let be known a set of interpolation data* (A_1, C_1, A_2, C_2), *where* (A_1, C_1) *is uniformly observable,* (A_2, C_2) *is uniformly reachable and* $\ell_{A_1} < 1$, $\ell_{A_2} < 1$, *and a diagonal operator* Q_{12} *which satisfies:*

$$C_1 J C_2^* = Q_{12} A_2^* - A_1 Q_{12}^{(-1)} \quad (7.17)$$

Let Q_{11} *and* Q_{22} *be the solutions of the following Lyapunov equations:*

$$Q_{11} = A_1 Q_{11}^{(-1)} A_1^* + C_1 J C_1^* \quad (7.18)$$

$$Q_{22}^{(-1)} = A_2 Q_{22} A_2^* - C_2 J C_2^* \quad (7.19)$$

The double sided interpolation problem has solutions iff the Hermitian operator

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \quad (7.20)$$

is uniformly positive definite. In this case, a realization of a J-lossless operator which solves the interpolation problem is $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ *for some diagonal operators* B_1, B_2 *and* D. *The realization satisfies:*

$$\begin{bmatrix} A_1 & & C_1 \\ & I & C_2 \\ B_1 & & D_\Theta \end{bmatrix} \begin{bmatrix} Q_{11}^{(-1)} & Q_{12}^{(-1)} & \\ Q_{21}^{(-1)} & Q_{22}^{(-1)} & \\ & & J \end{bmatrix} \begin{bmatrix} A_1 & & C_1 \\ & I & C_2 \\ B_1 & & D_\Theta \end{bmatrix}^*$$

$$= \begin{bmatrix} I & & \\ & A_2 & \\ & B_2 & I \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & \\ Q_{21} & Q_{22} & \\ & & J \end{bmatrix} \begin{bmatrix} I & & \\ & A_2 & \\ & B_2 & I \end{bmatrix}^* \tag{7.21}$$

Furthermore, Θ can be partitioned as $\begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$. The solutions S then can be expressed as:

$$S = (S_L\Theta_{12} + \Theta_{22})^{-1}(S_L\Theta_{11} + \Theta_{21}) \tag{7.22}$$

where S_L is an arbitrary operator which is upper and $||S_L||_\infty < 1$.

Proof: Step 1. If such a Q exists, then there is a solution.

Assume that for the given data (A_1, C_1, A_2, C_2), there is a diagonal Hermitian operator $Q \gg 0$ which is partitioned as in equation (7.20) and satisfies the conditions given by (7.17), (7.18) and (7.19). The condition $Q \gg 0$ indicates that $Q_{11} \gg 0$, $Q_{22} \gg 0$ and $Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^* \gg 0$. With conditions (7.19) and $Q_{22} \gg 0$, we can find some diagonal operators B_a and D_a such that the condition:

$$\begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix} \begin{bmatrix} Q_{22} & \\ & -J \end{bmatrix} \begin{bmatrix} A_2 & C_2 \\ B_a & D_a \end{bmatrix}^* = \begin{bmatrix} Q_{22}^{(-1)} & \\ & -J \end{bmatrix} \tag{7.23}$$

is satisfied. With $\{A_2, B_a, C_2, D_a\}$, we can define an anticausal J-lossless operator Θ_a as:

$$\Theta_a = D_a + B_a\mathbf{Z}^*(I - A_2\mathbf{Z}^*)^{-1}C_2$$

Define a Hermitian operator $M = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^*$ which is uniformly positive definite, then $Q_{11} = M + Q_{12}Q_{22}^{-1}Q_{12}^*$. With condition (7.18), we have:

$$M + Q_{12}Q_{22}^{-1}Q_{12}^* = A_1(M^{(-1)} + Q_{12}^{(-1)}Q_{22}^{-(-1)}Q_{12}^{(-1)*})A_1^* + C_1JC_1^*$$

With condition (7.17), the above expression can be written as

$$M + Q_{12}Q_{22}^{-1}Q_{12}^* = A_1M^{(-1)}A_1^* + (C_1JC_2^* - Q_{12}A_2^*)Q_{22}^{-(-1)}(C_1JC_2^* - Q_{12}A_2^*)^* + C_1JC_1^*$$

and with the relation in (7.23), we can further formulate the above equation as

$$M = A_1M^{(-1)}A_1^* + (C_1JD_a^*J - Q_{12}B_a^*J)J(C_1JD_a^*J - Q_{12}B_a^*J)^*$$

Define $C_c = C_1 J D_a^* J - Q_{12} B_a^* J$. With this condition and the condition $M \gg 0$, we can find some diagonal operators B_1 and D_c such that the condition:

$$\begin{bmatrix} A_1 & C_c \\ B_1 & D_c \end{bmatrix} \begin{bmatrix} M^{(-1)} & \\ & J \end{bmatrix} \begin{bmatrix} A_1 & C_c \\ B_1 & D_c \end{bmatrix}^* = \begin{bmatrix} M & \\ & J \end{bmatrix}$$

We then can define a causal J-lossless operator Θ_c as

$$\Theta_c = D_c + B_1 \mathbf{Z}(I - A_1 \mathbf{Z})^{-1} C_c$$

With $\Theta_c \in \mathcal{U}$ and a $\Theta_a \in \mathcal{L}$ and both J-lossless, the cascade connection of them $\Theta = \Theta_c \Theta_a$ is J-lossless and with Proposition 4.3, the realization of Θ is $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ with $B_2 = D_c B_a + B_1 Y^{(-1)} A_2$ and $D = D_c D_a + B_1 Y^{(-1)} C_2$, where $Y \in \mathcal{D}$ is the solution of the Lyapunov equation $C_c B_a = Y - A_1 Y^{(-1)} A_2$. This J-lossless operator produces, from Proposition 7.1, a strictly contractive $S = \mathrm{HM}(\Theta; 0) = \Theta_{22}^{-1} \Theta_{12}^*$ which solves the interpolation problem.

Conversely, if there is a J-lossless operator Θ which has a realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ with some diagonal operators B_1, B_2 and D which solve the interpolation problem, then there is a $Q \gg 0$ which satisfies the conditions given by (7.17), (7.18) and (7.19).

Step 2. The realization of Θ satisfies (7.21).

Since Θ_c is causally J-lossless, (7.16) is satisfied. With Proposition 4.3 in Chapter 4 we also have:

$$D J_2 C_1^* = B_2 Q_{12}^* - B_1 Q_{11}^{(-1)} A_1^*$$

and

$$D J D^* = J + B_2 Q_{22} B_2^* - B_1 Q_{11}^{(-1)} B_1^*$$

Put all the conditions into one expression we obtain (7.21).

Step 3. The solutions has the form $\mathrm{HM}(\Theta; S_L)$ with S_L any strictly contractive operator in $\mathcal{U}$.

If an operator S is an interpolant, it should satisfies the four interpolation conditions:

1. $S \in \mathcal{U}$;

2. $||S||_\infty < 1$;

3. $(\mathbf{Z} - A_2)^{-1} C_2 \begin{bmatrix} I \\ -S \end{bmatrix} \in \mathcal{U}$;

4. $\begin{bmatrix} S & -I \end{bmatrix} C_1^*(\mathbf{Z} - A_1^*)^{-1} \in \mathcal{U}$.

Let S_L be a strictly contractive upper operator and $\Theta = D + B_1(\mathbf{Z}^* - A_1)^{-1}C_1 + B_2(\mathbf{Z} - A_2)^{-1}C_2$ be a J-lossless operator as we defined before. Then $(S_L\Theta_{12} + \Theta_{22})$ is invertible. Let $S = \mathrm{HM}(\Theta; S_L)$, that is

$$S = \mathrm{HM}(\Theta; S_L) = (S_L\Theta_{12} + \Theta_{22})^{-1}(S_L\Theta_{11} + \Theta_{21}) \tag{7.24}$$

With Theorem 4.6 in Chapter 4 we know that since Θ is J-lossless, $S \in \mathcal{U}$ and $||S||_\infty < 1$ in (7.24) for any S_L such that $S_L \in \mathcal{U}$ and $||S_L||_\infty < 1$. Then we see that $S = \mathrm{HM}(\Theta; S_L)$ satisfies the first two interpolation conditions.

By using the relation of J-unitarity, S can also be expressed as,

$$S = (\Theta_{12}^* + \Theta_{22}^* S_L)(\Theta_{21}^* S_L + \Theta_{11}^*)^{-1} \tag{7.25}$$

Define $\phi = \Theta_{21}^* S_L + \Theta_{11}^*$ and $\phi^{-1} = (\Theta_{21}^* S_L + \Theta_{11}^*)^{-1} = (\Theta_{11}^{-*}\Theta_{21}^* S_L + I)^{-1}\Theta_{11}^{-*}$. Since Θ is J-lossless, both Θ_{11}^{-*} and $-\Theta_{11}^{-*}\Theta_{21}^*$ are upper and strictly contractive. Thus. ϕ^{-1} is upper. (7.25) can be expressed as

$$\begin{bmatrix} I \\ S \end{bmatrix} = \begin{bmatrix} \Theta_{11}^* & \Theta_{21}^* \\ \Theta_{12}^* & \Theta_{22}^* \end{bmatrix} \begin{bmatrix} I \\ S_L \end{bmatrix} \phi^{-1} \tag{7.26}$$

Multiplying on the left of (7.26) by ΘJ results:

$$(D + B_1(\mathbf{Z}^* - A_1)^{-1}C_1 + B_2(\mathbf{Z} - A_2)^{-1}C_2) J \begin{bmatrix} I \\ S \end{bmatrix} = J \begin{bmatrix} I \\ S_L \end{bmatrix} \phi^{-1}$$

The right hand side of this expression is upper. Since $(D + B_1(\mathbf{Z}^* - A_1)^{-1}C_1) \begin{bmatrix} I \\ -S \end{bmatrix}$ is also upper, $B_2(\mathbf{Z} - A_2)^{-1}C_2 \begin{bmatrix} I \\ -S \end{bmatrix}$ must be upper. Because the realization is uniformly minimal, condition 3 is satisfied.

Define $\phi' = S_L\Theta_{12} + \Theta_{22}$. ϕ'^{-1} is also upper. Rewrite the relation (7.24) in the form:

$$\begin{bmatrix} S & I \end{bmatrix} = \phi'^{-1} \begin{bmatrix} S_L & I \end{bmatrix} \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$$

Multiplying by $J\Theta^*$ on the right of this expression and following the same procedure as for Condition 3, we can prove that condition 4 is also satisfied with such a S. So the set of $S = \mathrm{HM}(\Theta; S_L)$ is solutions of the interpolation problem. □

If we take $S_L = 0$, then we obtain the solution $S = \Theta_{12}^*\Theta_{11}^{-*} = \Theta_{22}^{-1}\Theta_{21}$, which is sometimes called the central solution of the interpolation problem. From this proposition we can see that the solutions of interpolation problem are the same as what we have found before for the H_∞ control problem.

7.4 Conclusions

Let G be a chain scattering representation of a time-varying plant P. The H_∞ control problem in the case that G is invertible is equivalent to the interpolation problem. The solvability of the problem is determined by the definiteness of a Pick operator which is specified by the interpolation data (or system realization). The solution of the problem is given by a homographic transformation of a J-lossless operator which is constructed under the condition the Pick operator is uniformly positive definite, and a strictly contractive upper operator. For the H_∞ control problem, we can compute the J-lossless factorization with the algorithm represented in chapter 5 and 6 if the factorization exists. With the result of the factorization we can design controllers which solve the H_∞ control problem.

For a given set of interpolation data (A_1, C_1, A_2, C_2), where (A_1, C_1) is uniformly observable and (A_2, C_2) is uniformly reachable, and a diagonal operator Q_{12} which satisfies:

$$C_1 J C_2^* = Q_{12} A_2^* - A_1 Q_{12}^{(-1)}$$

Let Q_{11} and Q_{22} be the solutions of Lyapunov equations $Q_{11} = A_1 Q_{11}^{(-1)} A_1^* + C_1 J C_1^*$ and $Q_{22}^{(-1)} = A_2 Q_{22} A_2^* - C_2 J C_2^*$ respectively. Under the condition:

$$\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \gg 0$$

we can calculate a uniformly minimal realization of a J-lossless operator $\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{21} & \Theta_{22} \end{bmatrix}$ who's homographic transformation $\mathrm{HM}(\Theta; S_L)$ with S_L a strictly contractive and upper operator solves the interpolation problem of the given data. The calculation is as follows:

Step 1: Compute the uniformly minimal realization $\{A_2, B_a, C_2, D_a\}$ of a anticausal J-lossless operator Θ_a such that (7.23) is satisfied.

Step 2. Compute the uniformly minimal realization $\{A_1, B_1, C_1 J D_a^* J - Q_{12} B_a^* J, D_c\}$ of a causal J-lossless operator Θ_c such that (7.16) is satisfied.

Step 3. With Lemma 2.1 in Chapter 2, cascade Θ_c with Θ_a, we can determine the uniformly minimal realization of Θ.

The computation is carried out locally and recursively. As we analyzed before, since $\ell_{A_1} < 1$ and $\ell_{A_2} < 1$, the recursive calculation of Q_{11} and Q_{22} converges to the true solution despite of the wrong initial point.

7.5 Appendix: The proof of Corollary 7.1

The proof requires some preliminaries and lemmas.

Let $T \in \mathcal{X}$ be an invertible locally finite operator with a strong basis of its anticausal output state space $\mathbf{F}_o^a(T) = (I - A_a\mathbf{Z}^*)^{-1}C_a$. Using (A_a, C_a), define an inner operator $U \in \mathcal{U}$ such that $\mathcal{H}_o^a(U^*) = \mathcal{H}_o^a(T)$. then $TU \in \mathcal{U}$ (see Proposition 5.1). Let $U^*T^{-1} \in \mathcal{X}$ with a strong basis of its anticausal input state space $\mathbf{F}^{a*}(U^*T^{-1}) = B_a'(I - A_a'\mathbf{Z}^*)^{-1}$. Similarly, using (A_a', B_a'), define an inner operator $V \in \mathcal{U}$ such that $\mathcal{H}^a(V^*) = \mathcal{H}^a(U^*T^{-1})$, then $VU^*T^{-1} \in \mathcal{U}$. Define $\Delta = VU^*T^{-1} \in \mathcal{U}$. Then T has a factorization $T = \Delta^{-1}VU^*$.

We first show that this is an inner-outer factorization. Indeed, since $TU\mathcal{U}_2 \subset \mathcal{U}_2$, we have $U\mathcal{U}_2 \subset T^{-1}\mathcal{U}_2$. Hence,

$$T^{-1}\mathcal{U}_2 \ominus U\mathcal{U}_2 = UV^*\Delta\mathcal{U}_2 \ominus U\mathcal{U}_2 = U(V^*\Delta\mathcal{U}_2 \ominus \mathcal{U}_2) \tag{7.27}$$

It follows that $\mathcal{U}_2 \subset V^*\Delta\mathcal{U}_2$. This implies that the closure of $\Delta\mathcal{U}_2$ contains $\mathcal{U}_2$ so that Δ is outer.

Equation (7.27) can be rewritten as

$$\mathcal{L}_2T^{-*} \ominus \mathcal{L}_2U^* = (\mathcal{L}_2\Delta^*V \ominus \mathcal{L}_2)U^*$$

Define $\mathcal{K}'(T) = \mathcal{L}_2U^* \subset \mathcal{L}_2$ and $\mathcal{H}''(T) = \mathcal{L}_2\Delta^*V \ominus \mathcal{L}_2 = \mathcal{L}_2V \ominus \mathcal{L}_2$. Then $\mathcal{H}''(T) \subset \mathcal{U}_2\mathbf{Z}$ and

$$\mathcal{L}_2T^{-*} = \mathcal{H}''(T)U^* \oplus \mathcal{K}'(T) \tag{7.28}$$

Lemma 7.2 *Let $T_1 \in \mathcal{X}$, $T_2 \in \mathcal{X}$, T_1 and T_2 invertible. Suppose that $T_1 = T_oT_2$, $\mathcal{H}''(T_1) = \mathcal{H}''(T_2)$ and $\mathcal{K}'(T_1) = \mathcal{K}'(T_2)$, then $T_o \in \mathcal{U}$.*

Proof: Since $\mathcal{K}'(T_1) = \mathcal{K}'(T_2)$, with the definition of $\mathcal{K}'$ we know that there is an inner operator U such that $T_1 U \subset \mathcal{U}$ and $T_2 U \subset \mathcal{U}$.

Let $u \in \mathcal{L}_2$ and $y = uT_2^{-*}$. With (7.28), we have that $y \in \mathcal{H}''(T_2)U^* \oplus \mathcal{K}'(T_2)$.

Since also $\mathcal{H}''(T_1) = \mathcal{H}''(T_2)$,

$$\mathcal{L}_2 T_1^{-*} = \mathcal{H}''(T_1)U^* \oplus \mathcal{K}'(T_1) = \mathcal{H}''U^*(T_2) \oplus \mathcal{K}'(T_2)$$

and $y \in \mathcal{H}''(T_1)U^* \oplus \mathcal{K}'(T_1)$. With (7.28)we have that $yT_1^* \in \mathcal{L}_2$. Hence $uT_o^* = yT_1^* \in \mathcal{L}_2$ and it follows that $T_o^* \in \mathcal{L}$, or $T_o \in \mathcal{U}$. □

We now derive the dual of this result.

Let $T \in \mathcal{X}$ be an invertible locally finite operator with a strong basis of the anticausal input state space of its inverse $\mathbf{F}^{a*}(T^{-1}) = B_a''\mathbf{Z}^*(I - A_a'\mathbf{Z}^*)^{-1}$. Using (A_a', B_a''), define an inner operator $V_1 \in \mathcal{U}$ such that $\mathcal{H}^a(V_1^*) = \mathcal{H}^a(T^{-1})$. Then $V_1 T^{-1} \in \mathcal{U}$. Let $TV_1^* \in \mathcal{X}$ with a strong basis of its anticausal output state space $\mathbf{F}_o^a(TV_1^*) = (I - A_a\mathbf{Z}^*)^{-1}C_a'$. Using (A_a, C_a'), define an inner operator $U_1 \in \mathcal{U}$ such that $TV_1^*U_1 = \Delta_1 \in \mathcal{U}$. Then $T^{-1} = V_1^*U_1\Delta_1^{-1}$. Define $\mathcal{K}''(T) = \mathcal{U}_2 V_1 \in \mathcal{U}_2$ and $\mathcal{H}'(T) = \mathcal{U}_2 U_1^* \ominus \mathcal{U}_2 \in \mathcal{L}_2\mathbf{Z}^{-1}$. A similar analysis as above for Lemma 7.2 shows in that Δ_1 is outer and we obtain:

$$\mathcal{U}_2 T = \mathcal{H}'(T)V_1 \oplus \mathcal{K}''(T) \tag{7.29}$$

Dually, we have the next lemma.

Lemma 7.3 *Let $T_1 \in \mathcal{X}$, $T_2 \in \mathcal{X}$ and both invertible. Suppose $T_1 = T_o T_2$ so that $T_1^{-1} = T_2^{-1}T_o^{-1}$. Suppose that $\mathcal{H}'(T_1) = \mathcal{H}'(T_2)$ and $\mathcal{K}''(T_1) = \mathcal{K}''(T_2)$. Then $T_o^{-1} \in \mathcal{U}$.*

Proof: The proof is similar to Lemma 7.2. □

With Lemma 7.2 and 7.3, proof of Corollary 7.1 is as follows.

Proof of Corollary 7.1: In accordance with Lemma 7.2, let $T_1 = T_o T_2$, if $\mathcal{H}''(T_1) = \mathcal{H}''(T_2)$ and $\mathcal{K}'(T_1) = \mathcal{K}'(T_2)$, so that $T_o \in \mathcal{U}$.

Suppose that T_1 has a minimal realization $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ and suppose that T_1^{-1} has a minimal realization $\{A_1', B_1', C_1', A_2', B_2', C_2', D'\}$. First, we prove the corollary for a special T_2, denoted by W, which is unitary and has a realization $\{A_2'^*, B_{u_1}, B_2'^*, A_2, B_{u_2}, C_2, D_u\}$ for some diagonal operators B_{u_1}, B_{u_2} and D_u, and is such that $T_1 = T_o'W$ for some operator T_o'. We have to show that under the conditions $\mathcal{H}_o^a(T_1) =$

$\mathcal{H}^a_o(W)$ and $\mathcal{H}^a(T_1^{-1}) = \mathcal{H}^a(W^{-1})$, we have $\mathcal{H}''(T_1) = \mathcal{H}''(W)$ and $\mathcal{K}'(T_1) = \mathcal{K}'(W)$, so that T_o' is upper.

Indeed, since $\mathcal{H}^a_o(T_1) = \mathcal{H}^a_o(W)$, we can choose the inner operator U such that $\mathcal{H}^a_o(U^*) = \mathcal{H}^a_o(T_1) = \mathcal{H}^a_o(W)$, $T_1U \subset \mathcal{U}$ and $WU \subset \mathcal{U}$. Then, $\mathcal{K}'(T_1) = \mathcal{K}'(W)$. With the definition of $\mathcal{H}^a$, we have

$$\mathcal{H}^a(T_1^{-1}) = \mathbf{P}(\mathcal{L}_2 T_1^{-*}) = \mathbf{P}[\mathcal{H}''(T_1)U^* \oplus \mathcal{K}'(T_1)] = \mathbf{P}[\mathcal{H}''(T_1)U^*]$$
$$\mathcal{H}^a(W^{-1}) = \mathbf{P}(\mathcal{L}_2 W^{-*}) = \mathbf{P}[\mathcal{H}''(W)U^* \oplus \mathcal{K}'(W)] = \mathbf{P}[\mathcal{H}''(W)U^*]$$

Since $\mathcal{H}^a(T_1^{-1}) = \mathcal{H}^a(W^{-1})$ then $\mathbf{P}[\mathcal{H}''(T_1)U^*] = \mathbf{P}[\mathcal{H}''(W)U^*]$. Because both $\mathcal{H}''(T_1) \in \mathcal{U}_2\mathbf{Z}$ and $\mathcal{H}''(W) \in \mathcal{U}_2\mathbf{Z}$, thus $\mathcal{H}''(T_1) = \mathcal{H}''(W)$. Then with Lemma 7.2, we have $T_o' \in \mathcal{U}$.

Similarly, with $T_1^{-1} = W^* T_o'^{-1}$, under the conditions $\mathcal{H}^a_o(T_1) = \mathcal{H}^a_o(W)$, $\mathcal{H}^a(T_1^{-1}) = \mathcal{H}^a(W^{-1})$ and according to Lemma 7.3, we can show that $T_o'^{-1} \in \mathcal{U}$. Thus T_o' is outer.

Finally, we translate this result for W to a result for T_2. There is an operator T_o'' such that $T_o''W = T_2$. Under the conditions $\mathcal{H}^a(T_2^{-1}) = \mathcal{H}^a(T_1^{-1}) = \mathcal{H}^a(W^{-1})$ and $\mathcal{H}^a_o(T_2) = \mathcal{H}^a_o(T_1) = \mathcal{H}^a_o(W)$, we can show that T_o'' is outer with the same strategy. With $T_1 = T_oT_2$ we have $T_o = T_o'T_o''^{-1}$. $\overline{T_o\mathcal{U}_2} = \overline{T_o'T_o''^{-1}\mathcal{U}_2} = \mathcal{U}_2$. Thus T_o is outer.

The corollary is proved.

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Chapter 8

Conclusions

In this thesis, we have introduced a subspace model identification approach to identify discrete time-varying systems. Special attention has been paid to the periodic discrete time system identification in this part (Chapter 3). Secondly, by using and extending the time-varying system theory in [1] [2] and [3], the factorization theory in [3] for operators is not restricted to the upper operator case but extended to the more general bounded operator case. The resulting Outer–J-lossless factorization algorithms for time-varying systems in combination with the chain scattering representation method produce a solution to H_∞ control problem in discrete time-varying systems. From this research, the following conclusions can be drawn.

- Discrete time-varying system state space models can be identified with a MOESP identification approach under the condition that an ensemble of input and output measurements is available. In the algorithm that was developed, we have only considered the measurement error at the output of the system. If this output noise is zero-mean white, the estimated model is unbiased and consistent. A modified approach, named the PI scheme, takes the past measurement as instrumental variable and can be used to estimate the state space model unbiasedly even if the output noise is colored. The algorithm is particularly suited to identify periodic systems because the repetition of the experiments of such a system is intrinsic. Another feature of the algorithm is that it allows the dimensions of the inputs, outputs and states to vary as well. The varying state dimensions are identified from the data. This is very useful in the identification of multi-rate sampled systems.

- With the notation of anticausal systems defined by lower operators, several factorization theorems in time-varying systems can be extended to general bounded operators of mixed causality. Although we only considered a special factorization, namely the outer–J-lossless factorization, other types of factorizations, such as the inner-outer factorization, can also be obtained with the same strategy. For an outer–J-lossless factorization of a mixed upper/lower operator, one time-varying Lyapunov equation has to be solved in order to obtain a factorization into a stable operator and an anticausal J-lossless operator, and then three time-varying Lyapunov-type equations should be solved to factor the state operator into an outer factor and a causal J-lossless operator. The second step is equivalent to solving a time-varying Riccati equation. The existence of the factorization depends on the definiteness of the solution of the first Lyapunov equation and the solution of the Riccati equation.

- If a time-varying system can be expressed as a causal system, the J-lossless conjugation method can be used for the outer–J-lossless (J-lossless–outer) factorization of this system. This results in one forward and one backward time-varying Riccati recursion. The existence of the factorization depends on the definiteness of the solutions of these Riccati equations.

- The outer–J-lossless (J-lossless–outer) factorization can be used to solve the H_∞ control problem or sensitivity minimization problem in time-varying systems. The solution to a system exists if the outer–J-lossless (J-lossless–outer) factorization of the chain scattering representation of the system exists. In that case, the set of admissible controllers is given by the homographic transformation of the inverse of the outer factor and any strictly contractive upper operator. With these controllers, the H_∞ norm of the closed loop operator is smaller than a prescribed bound. Numerical simulations have shown the usefulness of this approach and the factorization algorithm developed in this thesis for solving the H_∞ control problems of time-varying systems.

- If the chain scattering operator is invertible, the H_∞ problem is similar to the interpolation problem in the sense that both need to define a set of functions (operators) to fit a data set (system) under the condition that the functions (operators) are analytic (upper) and contractive. The condition for the solvability of interpolation problem is given by the definiteness of the Pick matrix (operator) which is determined by the original data set.

The following is proposed for future research:

1. For the H_∞ control problem, we only considered the case $q = r$, $m \geq p$ and $r \geq q$, $m = p$ where q, r, m and p are the dimensions of the measurement, disturbance, output and controlled input respectively. This corresponds to the so-called two block problem in H_∞ control. The more general case where $r \geq q$, $m \geq p$ is left for further study because the structure of the Riccati equations in discrete time case are much more complicated than in the continuous time case. This extension still requires significant efforts.

2. How to compute Q_{12}, the off diagonal term of the Pick operator $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix}$ for the interpolation problem in Chapter 7, remains a difficult task. Recall that to solve the two sided interpolation problem for a given data set (A_1, C_1, A_2, C_2), three Lyapunov equations have to be solved. In particular, we have to find Q_{12} in equation

$$C_1 J C_2^* = Q_{12} A_2^* - A_1 Q_{12}^{(-1)}$$

The difficult lies in the fact that A_1 and A_2 are not necessarily invertible. It is not clear how to solve for Q_{12} recursively.

3. A J-RQ factorization is supposed to be an efficient way for computing the outer–J-lossless factorization. We need an efficient and effective algorithm for this.

4. Besides H_∞ control, there are maybe other applications of outer–J-lossless factorizations.

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Glossary of notation

1. $\mathbb{R}$ denotes the real numbers. $\mathbb{R}^n$ is a n-dimensional vector over $\mathbb{R}$ and $\mathbb{R}^{n\times m}$ indicates a matrix over $\mathbb{R}$ with n rows and m columns.

2. $\mathbb{Z}$ denotes the set of integers and $\mathbb{Z}^+$ the set of positive integers.

3. $\mathbb{C}^n$ denotes the complex vector space with dimension n.

4. $\mathcal{M} = \mathbb{C}^M$ where $M = [\ \cdots\ M_{-1},\ M_0,\ M_1\ \cdots\ ]$ with $M_i \in \mathbb{Z}^+$: Space of (non-uniform) sequences with i-th entry in $\mathbb{C}^{M_i}$.

5. $M = \sharp\mathcal{M}$: Dimension sequence of $\mathcal{M}$.

6. $\ell_2^{\mathcal{M}}$: Space of bounded vector sequences with dimensions defined by the dimension sequence $M = \sharp\mathcal{M}$; $\ell_2^{\mathcal{M}} = \{u \in \mathcal{M} : ||u||_2 < \infty\}$.

7. $\mathcal{X}(\mathcal{M},\mathcal{N})$: Space of bounded operators $\ell_2^{\mathcal{M}} \to \ell_2^{\mathcal{N}}$.

8. $\mathcal{U},\mathcal{L},\mathcal{D}$: Upper/lower/diagonal bounded operators in $\mathcal{X}$.

9. $\mathcal{X}_2,\mathcal{U}_2,\mathcal{L}_2,\mathcal{D}_2$: (Hilbert) spaces of operators in $\mathcal{X},\mathcal{U},\mathcal{L},\mathcal{D}$ with bounded *Hilbrt – Schmidt*-norm. The *HS*-norm of an operator $A \in \mathcal{X}(\mathcal{M},\mathcal{N})$ is defined by: $||A||_{HS}^2 = \sum_{i,j} ||A_{i,j}||_2^2$.

10. A^*: The adjoint of operator A.

11. $\mathbf{Z}$: Bilateral causal shift operator. $\mathbf{Z}^{-1} = \mathbf{Z}^*$: The inverse of $\mathbf{Z}$.

12. $\mathcal{U}\mathbf{Z},\mathcal{L}\mathbf{Z}^{-1}$: Strictly upper/lower bounded operators in $\mathcal{X}$. Similar notations are used for $\mathcal{U}_2,\mathcal{L}_2$ spaces as $\mathcal{U}_2\mathbf{Z},\mathcal{L}_2\mathbf{Z}^{-1}$.

13. I_m: Identity matrix with dimension m. The subscript m is often omitted.

14. $T^{(-k)} = \mathbf{Z}^k T \mathbf{Z}^{-k}$: Diagonal shift of $T \in \mathcal{X}$ over k positions into north-west direction (see section 2.1).

15. $A^{\{k\}} = A^{(k)}A^{(k-1)}\cdots A^{(1)}$.

16. $A^{[k]} = AA^{(1)} \cdots A^{(k-1)}$.

17. $\mathbf{P}, \mathbf{P}_0, \mathbf{P}_{\mathcal{L}_2}, \mathbf{P}_{\mathcal{U}_2\mathbf{Z}}, \mathbf{P}_{\mathcal{L}_2\mathbf{Z}^{-1}}$: projection onto $\mathcal{U}_2, \mathcal{D}_2, \mathcal{L}_2, \mathcal{U}_2\mathbf{Z}, \mathcal{L}_2\mathbf{Z}^{-1}$.

18. $T_{[k]} = \mathbf{P}_0(\mathbf{Z}^{-k}T)$: The k-th diagonal above the main diagonal of T.

19. $\{A, B\} = \mathbf{P}_0(AB^*)$: Diagonal inner product.

20. $A \gg 0$: A is uniformly strictly positive definite. ($A \in \mathcal{X}(\mathcal{M}, \mathcal{M})$ is a bounded Hermitian operator. $A \gg 0$ if $\exists \varepsilon > 0$, for all $U \in \mathcal{X}_2$, $\{UA, U\} \geq \varepsilon\{U, U\}$.)

21. $\mathcal{C}, \mathcal{O}$ or $\mathcal{C}_c, \mathcal{O}_c$: Reachability, observability operators of causal systems.

22. $\mathcal{C}_a, \mathcal{O}_a$: Reachability, observability operators of anticausal systems.

23. $\mathcal{H}(T), \mathcal{H}_o(T), \mathcal{K}(T), \mathcal{K}_o(T)$: Input state space, output state space, input null space, output null space of an operater $T \in \mathcal{U}$.

24. $\mathcal{H}^a(T), \mathcal{H}^a_o(T), \mathcal{K}^a(T), \mathcal{K}^a_o(T)$: Input state space, output state space, input null space, output null space of an operator $T \in \mathcal{L}$.

25. $\mathbf{F}, \mathbf{F}_o, \mathbf{F}^a, \mathbf{F}^a_o$: Strong basis representation of $\mathcal{H}, \mathcal{H}_0, \mathcal{H}^a, \mathcal{H}^a_0$.

26. $\Lambda_F = \mathbf{P}_0(\mathbf{F}\mathbf{F}^*)$: The Gramian operator associated to a basis representation $\mathbf{F}$.

27. $\left[\begin{array}{c|c} A & C \\ \hline B & D \end{array}\right]$ and $\{A, B, C, D\}$: State realization of a causal system. The State equation of the system is:

$$\begin{aligned} X\mathbf{Z}^{-1} &= XA + UB \\ Y &= XC + UD \end{aligned}$$

where A, B, C and D are diagonal operators, and U, Y and X are input, output and state sequences respectively.

We also use $\{A, B, C, D\}$ to represent a realization of a bounded upper operator or a bounded lower operator and $\{A_1, B_1, C_1, A_2, B_2, C_2, D\}$ a realization of a bounded operator. We always declare what the representation is before we use it.

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Summary

This thesis deals with linear discrete time-varying system state space model identification and H_∞ control problems. We use two kinds of descriptions for discrete time-varying systems: bounded operator descriptions and state equation representations. These two settings are well connected to each other by a notation which was first introduced by Alpay, Dewilde and Dym at the beginning of this decade and the realization theory developed by Van der Veen in recent years for discrete time-varying systems.

An algorithm for state space model identification of discrete time-varying systems is developed. It is based on two approaches: (1) ensemble identification approach and (2) a subspace model approach. The algorithm works directly on an ensemble of repeated experimental input/output measurements and it is an extension of a specific subspace model identification approach, the MOESP approach, for time invariant systems. The algorithm retains all the properties from the MOESP and it can provide an unbiased consistent state space model when the noise on the output is zero-mean white or zero-mean coloured. Particular attention has been paid to periodic systems where the repetition of the experiments is intrinsic.

The H_∞ control problem for discrete time-varying systems has been considered with a so-called standard set-up in this thesis. The solution to the problem is an extension of the recently developed solution introduced by Tsai, Postlethwaite and Kimura to the continuous time invariant H_∞ control problem. This solution is based on two fundamental notions: the chain scattering representation and the J-lossless factorization. The H_∞ control problem is solvable if a J-lossless factorization of the chain scattering representation of the system exists. In that case, the controllers for the closed loop system can be designed in a systematic way as a homographic transformation of the inverse of the outer factor which comes from the factorization and any strictly contractive stable functions (strictly contractive upper operators). These controllers guarantee the H_∞ norm of the closed loop transfer function (operator) smaller than a prescribed bound. Hence, J-lossless factorization theory for time-varying systems is the basis for our solution. We extend the factorization theory for operator description from upper operators to the general operator case and have developed a time-varying J-lossless factorization theory and algorithm. It

turns out that for a general operator, we need to solve one Lyapunov equation and a set of Lyapunov type equations for the factorization. We have shown that the latter set of equations is equivalent to a Riccati equation. The existence of the factorization is determined by the definiteness of the solutions to these equations. In the case that the chain scattering representation of a system has a causal realization, the J-lossless conjugation method which is used by Kimura in J-lossless factorization for time invariant systems can be extended to the time-varying setting. Two Riccati equations are needed to solve for the factorization in this case and the existence of the factorization depends on the definiteness to the solutions of these two Riccati equations.

The H_∞ control problem has a description in terms of a certain constrained interpolation problem. In the case that the chain scattering representation is invertible, we have shown the equivalence of the H_∞ control problem and the double sided interpolation problem. The solvability of the problems is determined by the positive definiteness of a Pick operator which is specified by the interpolation data (or system realization).

Samenvatting

Tijdsvariërende Systeemidentificatie, J-verliesvrije Factorisatie en $\mathbf{H}_\infty$-regeling

Deze thesis behandelt twee onderwerpen: lineaire discrete tijdsvariërende systeemidentificatie, en het H_∞ control probleem voor zulke systemen. De modelbeschrijvingen die worden gebruikt zijn normbegrensde operatoren en toestandsmodellen. Deze beschrijvingen zijn nauw verbonden via een notatie die begin negentiger jaren geïntroduceerd is door Dewilde, Dym en Alpay, in samenhang met een discrete tijdsvariërende realisatietheorie ontwikkeld door Van der Veen.

Het voorgestelde identificatiealgoritme voor LTV systemen combineert twee aanpakken, de eerste gebaseerd op ensemble identificatie, en de tweede op een subspace modelleringstechniek. Het algoritme werkt direct op een ensemble van meetdata verkregen door herhaalde experimenten op het tijdsvariërende systeem, en het is een uitbreiding van de zogenaamde MOESP methode voor identificatie van tijdsinvariante systemen. Het algoritme behoudt de eigenschappen van MOESP en geeft een ongebiasde consistente schatting van het toestandsmodel als de ruis op de uitgang wit is, of gekleurd met gemiddelde waarde nul. Speciale aandacht wordt gegeven aan de klasse van periodieke systemen, waarvoor de herhaling van het experiment intrinsiek is.

Het H_∞-probleem voor discrete tijdsvariërende systemen dat onderzocht wordt in deze thesis is het zogenaamde standaard-H_∞ probleem. De oplossingsmethode die gebruikt wordt is een uitbreiding van de recente techniek van Tsai, Postlethwaite en Kimura, die ontwikkeld werd voor continue tijdsinvariante systemen. De oplossing is gebaseerd op twee fundamentele begrippen: de chain-scattering representatie en de J-verliesvrije factorisatie. Het H_∞ probleem is oplosbaar als de J-verliesvrije factorisatie van de chain-scattering representatie van het systeem bestaat. In dat geval kunnen de regelaars voor het gesloten-lus systeem op een systematische manier ontworpen worden middels de homografische transformatie van de inverse van de outer factor, geparametriseerd door een willekeurige stabiele en causale operator. Deze regelaars garanderen dat de H_∞-norm van

het gesloten-lus systeem kleiner is dan een vooraf opgelegde waarde. Het blijkt dus dat J-verliesvrije factorisaties de basis vormen voor onze oplossingstechniek, zodat hieraan bijzondere aandacht gegeven wordt. Deze factorisatietheorie is uitgebreid van causale (bovendriehoeks) operatoren naar operatoren met gemengde causaliteit, en we ontwikkelen een algoritme voor de factorisatie. Het blijkt dat we één Lyapunov vergelijking en een stelsel van twee Lyapunov-achtige vergelijkingen moeten oplossen. Het stelsel is equivalent aan een Riccativergelijking. Het bestaan van de factorisatie is bepaald door het bestaan van positief-definiete oplossingen van al deze vergelijkingen. Een andere factorisatietechniek is de J-verliesvrije conjugatiemethode van Kimura. In het geval dat de chain-scattering representatie van het systeem causaal is, kan deze methode uitgebreid worden naar het discrete tijdsvarierende domein. Het bestaan van de factorisatie hangt dan af van het positief-definiet zijn van de oplossingen van twee Riccativergelijkingen.

Het H_∞-probleem heeft een beschrijving in termen van een bepaald interpolatieprobleem met randcondities. Als de chain-scattering representatie inverteerbaar is, tonen we aan dat deze problemen equivalent zijn en dezelfde condities voor oplosbaarheid hebben. De oplossing bestaat als de zogenaamde Pick-operator die de interpolatiecondities beschrijft positief-definiet is.