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Decoding Criteria and Zero WER Analysis for Channels with Bounded Noise and Offset

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***Abstract*—Data storage systems may not only be disturbed by noise. In some cases, the error performance can also be seriously degraded by offset mismatch. Here, channels are considered for which both the noise and offset are bounded. For such channels, Euclidean distance-based decoding, Pearson distance-based decoding, and Maximum Likelihood decoding are considered. In particular, for each of these decoders, bounds are determined on the magnitudes of the noise and offset intervals which lead to a word error rate equal to zero. Case studies with simulation results are presented confirming the findings.**

***Index Terms*—Flash memory, optical recording, maximum likelihood decoding, bounded noise, offset mismatch, zero WER**

I. Introduction

With the explosive growth of reliance on information, both for home and personal use along with business and professional needs, more data are being generated, processed, and stored. It is necessary to guarantee very high access speeds, low power consumption, and, most importantly, reliable data storage systems. In data storage systems, it is usually found that noise (which leads to unpredictable stochastic errors) is an important issue, but that also other physical factors may hamper the reliability of the stored data. For example, in Flash memories, the number of electrons of a cell decreases with time and some cells become defective over time [1]. The amount of electron leakage depends on various physical parameters, such as, the device's temperature, the magnitude of the charge, and the time elapsed between writing and reading the data. In digital optical recording, fingerprints and scratches on the surface of discs result in offset variations of the retrieved signal [2].

To address the physical-related offset issues, two approaches are usually investigated and applied in storage systems. One approach uses pilot sequences to estimate the unknown channel offset [3]. The method is often considered too expensive with respect to redundancy. Other approaches are error correcting techniques. Up to now, various coding techniques have been applied to alleviate the detection in case of channel mismatch, specifically rank modulation [4], balanced codes [5] and composition check codes [6]. These methods are often considered too expensive in terms of redundancy and complexity.

Since the retrieved data value has been offset in channels, a Euclidean distance measure will be biased or grossly inaccurate. Immink and Weber [7] showed that decoders using the Pearson distance have immunity to offset and/or gain mismatch. Use of the Pearson distance requires that the set of codewords satisfies certain special properties. Such sets are called Pearson codes. In [8], optimal Pearson codes were presented, in the sense of having the largest number of codewords and thus minimum redundancy among all q-ary Pearson codes of fixed length n. Further, in [9] a decoder was proposed based on minimizing a weighted sum of Euclidean and Pearson distances. In [10], Blackburn investigated a maximum likelihood (ML) criterion for channels with Gaussian noise and unknown gain and offset mismatch. In a subsequent study, ML decision criteria were derived for Gaussian noise channels when assuming various distributions for the offset in the absence of gain mismatch [11].

The above-summarized research results are based on the Gaussian noise model, which is the most satisfactory of reality in many cases. An increasing number of studies focus on another important class of non-Gaussian stochastic processes: bounded noise, which is motivated by the fact that the Gaussian stochastic process is an inadequate mathematical model of the physical world because it is unbounded [12], [13]. Moreover, in many relevant cases, especially in Flash memory, the impact of parameters (such as charge leakage) on the retrieved data value should not be arbitrarily large. Consequently, not taking into account the bounded nature of stochastic variations may lead to impracticable model-based inferences. In this paper, we explore decoding criteria for channels with bounded noise and bounded offset mismatch. Specifically, we consider Euclidean distance-based decoding, Pearson distance-based decoding, and Maximum Likelihood decoding. Most importantly, we investigate, for each of these decoders, under which constraints zero Word Error Rate (WER) performance can be achieved. We should stress that zero WER performance is achieved without assumptions of specific distributions for the bounded noise and offset.

The remainder of this paper is organized as follows. We first review the channel with noise and offset and the classical Euclidean and Pearson distance-based decoding criteria in Section II. In Section III, we present a ML decoding method when the noise and offset in the channel are bounded. Simulation results for specific cases are given in Section IV. In the channel with bounded noise and offset, zero WER is achievable for all detectors discussed in this paper. Conditions to achieve zero WER for these decoders are derived in Section V. We conclude the paper in Section VI.

II. Preliminaries and Channel Model

We consider transmitting a codeword $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ from a codebook $\mathcal{S} \subset \mathbb{R}^n$, where n, the length of $\mathbf{x}$, is a positive integer. In many applications, the received vector may not only be hampered by noise $\mathbf{v} = (v_1, v_2, \ldots, v_n)$, but also by gain a and/or offset b. Hence,

$$\mathbf{r} = a(\mathbf{x} + \mathbf{v}) + b\mathbf{1},$$

where $\mathbf{1} = (1, 1, \ldots, 1)$ is the real all-one vector of length n. The gain and offset values a and b may change from word to word, but are constant for all transmitted symbols within a codeword, while the noise values vary from symbol to symbol.

In the channel model under consideration in this paper, we assume that there is no gain mismatch, i.e., $a = 1$, but there is an offset $b \in \mathbb{R}$, i.e.,

$$\mathbf{r} = \mathbf{x} + \mathbf{v} + b\mathbf{1}. \tag{1}$$

The values v_i in the noise vector $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ are independently and identically distributed with probability density function ϕ, leading to a probability density function $\chi(\mathbf{v}) = \Pi_{i=1}^n \phi(v_i)$ for $\mathbf{v}$.

A well-known decoding criterion upon receipt of the vector $\mathbf{r}$ is to choose a codeword $\hat{\mathbf{x}} \in \mathcal{S}$ which minimizes the (squared) Euclidean distance between the received vector $\mathbf{r}$ and codeword $\hat{\mathbf{x}}$, i.e.,

$$\delta_E(\mathbf{r}, \hat{\mathbf{x}}) = \sum_{i=1}^n (r_i - \hat{x}_i)^2. \tag{2}$$

It is known to be optimal with regard to handling Gaussian noise.

The Pearson distance measure [7] is used in situations which require resistance towards offset and/or gain mismatch. For any vector $\mathbf{u} \in \mathbb{R}^n$, let

$$\bar{\mathbf{u}} = \frac{1}{n} \sum_{i=1}^n u_i$$

denote the average symbol value and let

$$\sigma_{\mathbf{u}} = \left(\sum_{i=1}^n (u_i - \bar{\mathbf{u}})^2 \right)^{1/2}$$

denote the unnormalized symbol standard deviation. The Pearson distance between the received vector $\mathbf{r}$ and a codeword $\hat{\mathbf{x}} \in \mathcal{S}$ is defined as

$$\delta_P(\mathbf{r}, \hat{\mathbf{x}}) = 1 - \rho_{\mathbf{r},\hat{\mathbf{x}}}, \tag{3}$$

where $\rho_{\mathbf{r},\hat{\mathbf{x}}}$ is the well-known Pearson correlation coefficient,

$$\rho_{\mathbf{r},\hat{\mathbf{x}}} = \frac{\sum_{i=1}^n (r_i - \bar{\mathbf{r}})(\hat{x}_i - \bar{\hat{\mathbf{x}}})}{\sigma_{\mathbf{r}} \sigma_{\hat{\mathbf{x}}}}.$$

A Pearson decoder chooses a codeword minimizing this distance. As shown in [7], a simpler Pearson distance-based criterion leading to the same result in the minimization process reads

$$\delta'_P(\mathbf{r}, \hat{\mathbf{x}}) = \sum_{i=1}^n \left(r_i - \hat{x}_i + \bar{\hat{\mathbf{x}}}\right)^2, \tag{4}$$

if there is no gain mismatch, as assumed in this paper.

III. Maximum Likelihood Decoding for Noise and Offset with Bounded Ranges

We assume that the noise values are restricted to a certain range. More specifically, ϕ only takes non-zero values on an interval $(-\alpha, \alpha)$, where $\alpha > 0$. Hence, $-\alpha < v_i < \alpha$ for all i. For a codeword $\hat{\mathbf{x}} = (\hat{x}_1, \ldots, \hat{x}_n) \in \mathcal{S}$, we define its noise environment as

$$U_{\hat{\mathbf{x}}} = \{\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n : \hat{x}_i - \alpha < u_i < \hat{x}_i + \alpha\}. \tag{5}$$

For the offset b we assume that it has a probability density function ζ which only takes non-zero values on an interval (γ, η). Hence, $\gamma < b < \eta$. Since the receiver can subtract $\frac{\eta+\gamma}{2}\mathbf{1}$ from $\mathbf{r}$ if the offset range is not symmetric around zero, we may assume without loss of generality that the offset is within the range $(-\beta, \beta)$, where $\beta = (\eta - \gamma)/2$, which we will do throughout the rest of this paper. We define

$$L_{\mathbf{r}} = \{\mathbf{r} - t\mathbf{1} : t \in (-\beta, \beta)\} \tag{6}$$

for a vector $\mathbf{r} \in \mathbb{R}^n$.

In order to achieve ML decoding, we need to choose the codeword of maximum likelihood given the received vector. Assuming all codewords are equally likely, this is equivalent to maximizing the probability density value of the received vector $\mathbf{r}$ given the candidate codeword $\hat{\mathbf{x}}$. Denoting the probability density function of $\mathbf{v} + b\mathbf{1}$ by ψ, we find with (1) that we should thus maximize

$$\psi(\mathbf{r} - \hat{\mathbf{x}}) = \int_{-\infty}^{\infty} \chi(\mathbf{r} - \hat{\mathbf{x}} - b\mathbf{1})\zeta(b)db \tag{7}$$

over all candidate codewords $\hat{\mathbf{x}}$, where χ and ζ are the probability density functions of the noise and offset, respectively. χ and ζ can be any distribution as long as they are restricted to the indicated intervals. In Section IV, we will show simulation results assuming specific distributions.

Note from (6) and (5) that a point $\mathbf{r} - t\mathbf{1}$ of $L_{\mathbf{r}}$ is in $U_{\hat{\mathbf{x}}}$ if and only if t satisfies

$$\begin{cases} r_i - \hat{x}_i - \alpha < t < r_i - \hat{x}_i + \alpha, \forall i = 1, \ldots, n, \\ -\beta < t < \beta. \end{cases} \tag{8}$$

From this observation, we find that (7) equals

$$\begin{cases} \int_{t_1(\mathbf{r},\hat{\mathbf{x}})}^{t_0(\mathbf{r},\hat{\mathbf{x}})} \chi(\mathbf{r} - \hat{\mathbf{x}} - b\mathbf{1})\zeta(b)db & \text{if } t_0(\mathbf{r},\hat{\mathbf{x}}) > t_1(\mathbf{r},\hat{\mathbf{x}}), \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

where

$$\begin{aligned} t_0(\mathbf{r},\hat{\mathbf{x}}) &= \min\left(\{r_i - \hat{x}_i + \alpha \,|\, i = 1, ..., n\} \cup \{\beta\}\right), \\ t_1(\mathbf{r},\hat{\mathbf{x}}) &= \max\left(\{r_i - \hat{x}_i - \alpha \,|\, i = 1, ..., n\} \cup \{-\beta\}\right). \end{aligned} \tag{10}$$

IV. CASE STUDIES

In this section, we consider several noise and offset distributions. Simulated WER results are shown for the codebook

$$\mathcal{S}^* = \{(0,0,0),(1,1,0),(1,0,1),(0,1,1)\}$$

of length 3 and size 4, in combination with different decoders. This simple codebook is used to demonstrate some important WER characteristics. Codebook construction as such is beyond the scope of this paper. The interested reader is referred to [14].

A. WER for Uniform Noise and Offset

The uniform distribution is the most-commonly used for bounded random variables. Let the probability density function of a random variable which is uniformly distributed on the interval (τ_1, τ_2) be denoted by $\mathcal{U}(\tau_1, \tau_2)$. The uniform distribution $\mathcal{U}(\tau_1, \tau_2)$ has probability density function

$$\mathcal{U}(x) = \begin{cases} \frac{1}{\tau_2 - \tau_1} & \text{if } \tau_1 < x < \tau_2, \\ 0 & \text{otherwise.} \end{cases} \quad (11)$$

Hence, for the noise we assume $v_i \sim \mathcal{U}(-\alpha, \alpha)$ and for the offset $b \sim \mathcal{U}(-\beta, \beta)$.

Note that ML decoding in this case is tantamount to maximizing

$$\max\{0, t_0(\mathbf{r}, \hat{\mathbf{x}}) - t_1(\mathbf{r}, \hat{\mathbf{x}})\}, \quad (12)$$

i.e., choosing the codeword $\hat{\mathbf{x}}$ for which the part of the line segment $L_{\mathbf{r}}$ that is within $U_{\hat{\mathbf{x}}}$ is largest.

Simulated WER results for the example code $\mathcal{S}^*$ and various values of α and β are shown in Figs. 1-3 for Euclidean, Pearson, and ML decoders, respectively.

In Fig. 1, we observe that the performance of the Euclidean decoder gets worse with increasing values of α and/or β. In Fig. 2, the curves for different values of β overlap because of the Pearson decoder's intrinsic immunity to offset mismatch. Note that the performance of the Euclidean decoder is close to ML performance for $\beta = 0.15$ and that the performance of the Pearson decoder is close to ML performance for $\beta = 0.30$ in Fig. 3.

Most interestingly, for Euclidean decoding, WER approaches zero if $\alpha \leq 1/2 - \beta$, while for Pearson decoding, it happens when $\alpha < 1/4$. WER approaches zero for the ML decoding if $\alpha \leq 1/4$ or $\alpha \leq 1/2 - \beta$, i.e., $\alpha \leq \max\{1/4, 1/2 - \beta\}$. Indeed, we observe in Fig. 3 a zero WER for $\alpha \leq 0.35$ if $\beta = 0.15$, for $\alpha \leq 0.30$ if $\beta = 0.20$, and for $\alpha \leq 0.25$ if $\beta = 0.25$ or $\beta = 0.30$. We will show in Section V that, for all decoders under consideration, a WER of zero is achieved if the magnitudes of the noise and offset intervals satisfy certain conditions.

B. WER for Uniform Noise and Various Offset Distributions

In this subsection, we consider again uniform noise, but various options for the offset distribution. In particular, $v_i \sim \mathcal{U}(-0.3, 0.3)$, while the offset is (i) uniform, i.e., $b \sim \mathcal{U}(-\beta, \beta)$, as in the previous subsection, (ii) triangular, i.e., $b \sim \mathcal{T}(-\beta, 0, \beta)$, as specified next, or (iii) Gaussian with mean zero and variance σ^2, i.e., $b \sim \mathcal{N}(0, \sigma^2)$. The last option is

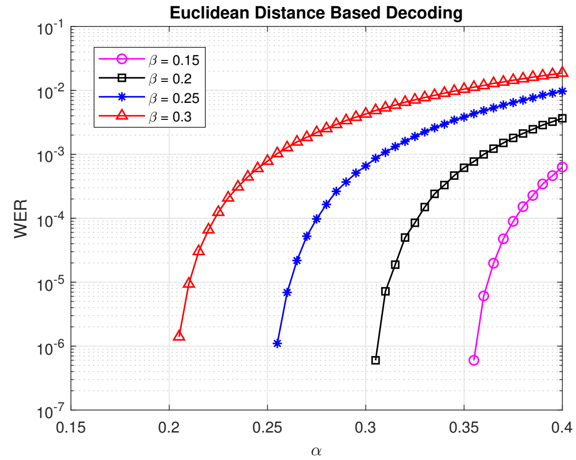

Fig. 1. Simulated WER of Euclidean distance-based decoding for codebook $\mathcal{S}^*$ on channels with uniform noise $v_i \sim \mathcal{U}(-\alpha, \alpha)$ and uniform offset $b \sim \mathcal{U}(-\beta, \beta)$.

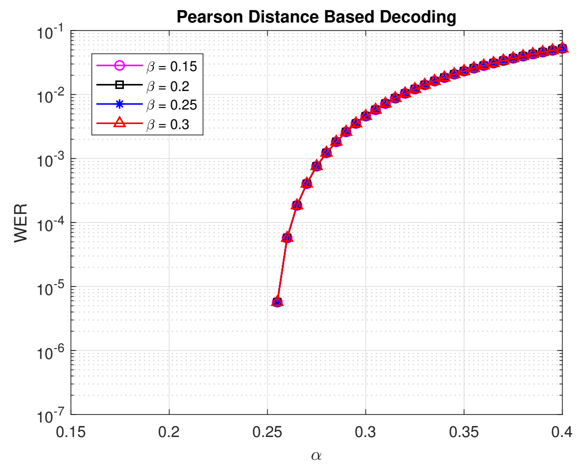

Fig. 2. Simulated WER of Pearson distance-based decoding for codebook $\mathcal{S}^*$ on channels with uniform noise $v_i \sim \mathcal{U}(-\alpha, \alpha)$ and uniform offset $b \sim \mathcal{U}(-\beta, \beta)$.

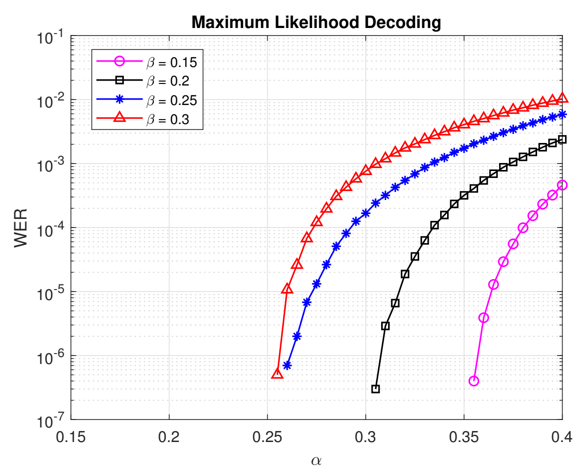

Fig. 3. Simulated WER of ML decoding for codebook $\mathcal{S}^*$ on channels with uniform noise $v_i \sim \mathcal{U}(-\alpha, \alpha)$ and uniform offset $b \sim \mathcal{U}(-\beta, \beta)$.

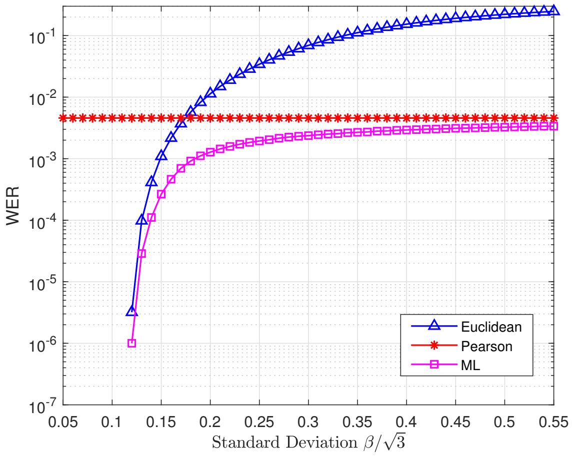

Fig. 4. Simulated WER for codebook $\mathcal{S}^*$ on channels with uniform noise $v_i \sim \mathcal{U}(-0.3, 0.3)$ and uniform offset $b \sim \mathcal{U}(-\beta, \beta)$.

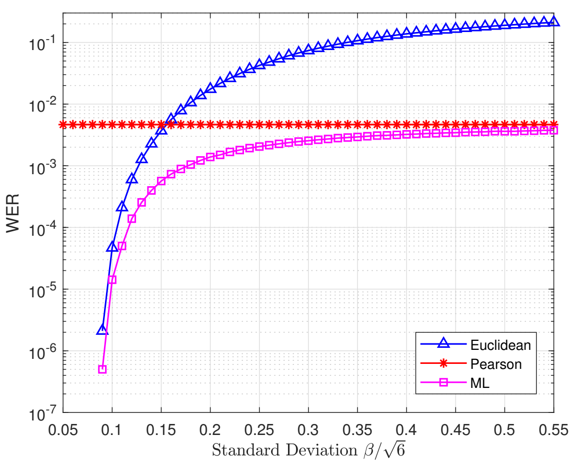

Fig. 5. Simulated WER for codebook $\mathcal{S}^*$ on channels with uniform noise $v_i \sim \mathcal{U}(-0.3, 0.3)$ and triangular offset $b \sim \mathcal{T}(-\beta, 0, \beta)$.

included since it is the most important representative of unbounded distributions. The triangular distribution $\mathcal{T}(-\beta, 0, \beta)$ has probability density function

$$\mathcal{T}(x) = \begin{cases} \frac{1}{\beta}(1 - \frac{1}{\beta}|x|) & \text{if } -\beta < x < \beta, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

In Figs. 4-6 we present WER results for the example code $\mathcal{S}^*$ for the three offset options under consideration. For comparison purposes, the WER is presented as a function of the standard deviation of the offset in each case.

In general, note that the WER of Pearson decoding has the same constant value for all cases, since it does not depend on the offset. It is close to ML performance in case of large standard deviations. The performance of Euclidean decoding is close to ML performance for small standard deviations. For medium standard deviations, ML decoding clearly outperforms both Euclidean and Pearson decoding in all three cases.

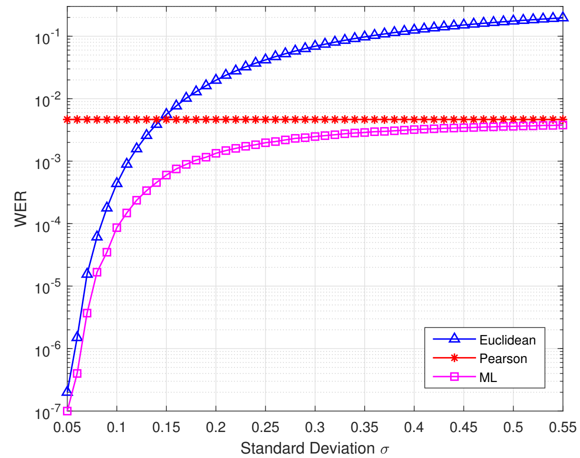

Fig. 6. Simulated WER for codebook $\mathcal{S}^*$ on channels with uniform noise $v_i \sim \mathcal{U}(-0.3, 0.3)$ and Gaussian offset $b \sim \mathcal{N}(0, \sigma^2)$.

We also observe in Fig. 4 that the WERs of Euclidean and ML decoders approach zero if the standard deviation $\beta/\sqrt{3}$ of the uniform offset distribution is at most 0.12, and in Fig. 5 that the WER approaches zero if the standard deviation $\beta/\sqrt{6}$ of the triangular offset distribution is at most 0.08. On the other hand, we see in Fig. 6 that for Gaussian offset zero WER can only be achieved by extremely small noise, as expected, due to the unbounded nature of the Gaussian distribution. In the next section we will analyse the zero WER constraints for different detectors.

V. Zero Error Analysis

In this section, we will show that, for all decoders under consideration, a WER of zero is achieved if the magnitudes of the noise and offset intervals satisfy certain conditions.

A. Euclidean Distance-Based Decoding

The Euclidean decoder can achieve zero WER for channels with bounded noise and offset when $\alpha+\beta$ is sufficiently small, as shown in the following result.

Theorem 1. *If the noise and offset are restricted to the intervals $(-\alpha, \alpha)$ and $(-\beta, \beta)$, respectively, with*

$$\alpha + \beta \leq \min_{\mathbf{s},\mathbf{c} \in \mathcal{S}, \mathbf{s} \neq \mathbf{c}} \left(\frac{\sum_{i=1}^{n} (s_i - c_i)^2}{2 \sum_{i=1}^{n} |s_i - c_i|} \right), \tag{14}$$

then the Euclidean decoder achieves a WER equal to zero.

Proof. Assume that $\mathbf{x} \in \mathcal{S}$ is sent and $\mathbf{r} = \mathbf{x} + \mathbf{v} + b\mathbf{1}$ is received. Then, for all codewords $\hat{\mathbf{x}} \neq \mathbf{x}$, it holds that

$$
\begin{aligned}
&\delta_E(\mathbf{r},\hat{\mathbf{x}}) - \delta_E(\mathbf{r},\mathbf{x}) \\
&= \sum_{i=1}^{n}(r_i-\hat{x}_i)^2 - \sum_{i=1}^{n}(r_i-x_i)^2 \\
&= \sum_{i=1}^{n}(r_i-x_i-\hat{x}_i+x_i)^2 - \sum_{i=1}^{n}(r_i-x_i)^2 \\
&= \sum_{i=1}^{n}(\hat{x}_i-x_i)^2 - 2\sum_{i=1}^{n}(\hat{x}_i-x_i)(r_i-x_i) \\
&= \sum_{i=1}^{n}(\hat{x}_i-x_i)^2 - 2\sum_{i=1}^{n}(\hat{x}_i-x_i)(v_i+b) \\
&\geq 2(\alpha+\beta)\sum_{i=1}^{n}|\hat{x}_i-x_i| - 2\sum_{i=1}^{n}|\hat{x}_i-x_i|\,|v_i+b| \\
&= 2\sum_{i=1}^{n}|\hat{x}_i-x_i|\,(\alpha+\beta-|v_i+b|) \\
&> 0,
\end{aligned}
$$

where the fourth equality follows from $r_i = x_i + v_i + b$, the first inequality follows from (14) and the last inequality from the fact that $|v_i + b| \leq |v_i| + |b| < \alpha + \beta$ for all i. Hence, if decoding is based on minimizing (2), the transmitted codeword is always chosen as the decoding result, leading to a WER equal to zero. □

B. Pearson Distance-Based Decoding

Since Pearson distance based decoding features its immunity to offset mismatch, zero WER performance only requires a limited value of α, which is shown in the next theorem.

Theorem 2. *If the noise and offset are restricted to the intervals $(-\alpha,\alpha)$ and $(-\beta,\beta)$, respectively, with*

$$
\alpha < \min_{\mathbf{s},\mathbf{c}\in\mathcal{S},\mathbf{s}\neq\mathbf{c}} \left(\frac{\sum_{i=1}^{n}(s_i-\bar{\mathbf{s}}-c_i+\bar{\mathbf{c}})^2}{\frac{n-1}{n}4\sum_{i=1}^{n}|s_i-\bar{\mathbf{s}}-c_i+\bar{\mathbf{c}}|} \right), \tag{15}
$$

then the Pearson decoder achieves a WER equal to zero.

Proof. Assume that $\mathbf{x} \in S$ is sent, and $\mathbf{r} = \mathbf{x} + \mathbf{v} + b\mathbf{1}$ is received. Then, for all codewords $\hat{\mathbf{x}} \neq \mathbf{x}$, it holds that

$$
\begin{aligned}
&\delta'_P(\mathbf{r},\hat{\mathbf{x}}) - \delta'_P(\mathbf{r},\mathbf{x}) \\
&= \sum_{i=1}^{n}(r_i-\hat{x}_i+\bar{\hat{\mathbf{x}}})^2 - \sum_{i=1}^{n}(r_i-x_i+\bar{\mathbf{x}})^2 \\
&= \sum_{i=1}^{n}(r_i-\hat{x}_i+\bar{\hat{\mathbf{x}}}-\bar{\mathbf{r}})^2 - \sum_{i=1}^{n}(r_i-x_i+\bar{\mathbf{x}}-\bar{\mathbf{r}})^2 \\
&= \sum_{i=1}^{n}(x_i-\bar{\mathbf{x}}-\hat{x}_i+\bar{\hat{\mathbf{x}}})^2 \\
&\qquad + \sum_{i=1}^{n}2(x_i-\bar{\mathbf{x}}-\hat{x}_i+\bar{\hat{\mathbf{x}}})(r_i-x_i+\bar{\mathbf{x}}-\bar{\mathbf{r}}) \\
&= \sum_{i=1}^{n}(x_i-\bar{\mathbf{x}}-\hat{x}_i+\bar{\hat{\mathbf{x}}})^2 + \sum_{i=1}^{n}2(x_i-\bar{\mathbf{x}}-\hat{x}_i+\bar{\hat{\mathbf{x}}})(v_i-\bar{\mathbf{v}}) \\
&> \tfrac{n-1}{n}4\alpha\sum_{i=1}^{n}\left|x_i-\bar{\mathbf{x}}-\hat{x}_i+\bar{\hat{\mathbf{x}}}\right| \\
&\qquad - \sum_{i=1}^{n}2\left|x_i-\bar{\mathbf{x}}-\hat{x}_i+\bar{\hat{\mathbf{x}}}\right|\,|v_i-\bar{\mathbf{v}}| \\
&= \sum_{i=1}^{n}\left|x_i-\bar{\mathbf{x}}-\hat{x}_i+\bar{\hat{\mathbf{x}}}\right|\left(\tfrac{n-1}{n}4\alpha-2\,|v_i-\bar{\mathbf{v}}|\right) \\
&\geq 0,
\end{aligned} \tag{16}
$$

where the fourth equality follows by substituting $r_i = x_i + v_i + b$ and $\bar{\mathbf{r}} = \bar{\mathbf{x}} + \bar{\mathbf{v}} + b$, the first inequality from (15), and the last inequality from the fact that $|v_i - \bar{\mathbf{v}}| < \frac{n-1}{n}2\alpha$ for all i. Hence, if decoding is based on minimizing (4), the transmitted codeword is always chosen as the decoding result, leading to a WER equal to zero. □

C. Maximum Likelihood Decoding

Finally, we show that zero WER for ML decoding is achieved if α or $\alpha + \beta$ is sufficiently small.

Theorem 3. *If the noise and offset are restricted to the intervals $(-\alpha,\alpha)$ and $(-\beta,\beta)$, respectively, with*

$$
\alpha \leq \min_{\mathbf{s},\mathbf{c}\in\mathcal{S},\mathbf{s}\neq\mathbf{c}} \left(\frac{\max_{1\leq i,j\leq n}\{(s_i-c_i)-(s_j-c_j)\}}{4} \right) \tag{17}
$$

or

$$
\alpha + \beta \leq \min_{\mathbf{s},\mathbf{c}\in\mathcal{S},\mathbf{s}\neq\mathbf{c}} \left(\frac{\max_{i=1,\ldots,n}(|s_i-c_i|)}{2} \right) \tag{18}
$$

then the ML decoder achieves a WER equal to zero.

Proof. Assume that $\mathbf{x} \in \mathcal{S}$ is sent and $\mathbf{r} = \mathbf{x} + \mathbf{v} + b\mathbf{1}$ is received. We will show that if (17) or (18) holds, then $\psi(\mathbf{r} - \hat{\mathbf{x}}) = 0$ for all codewords $\hat{\mathbf{x}} \neq \mathbf{x}$. First of all, note that

$$
\begin{aligned}
&t_0(\mathbf{r},\hat{\mathbf{x}}) - t_1(\mathbf{r},\hat{\mathbf{x}}) \\
&= \min\left(\{r_i-\hat{x}_i+\alpha\,|i=1,\ldots,n\}\cup\{\beta\}\right) \\
&\qquad - \max\left(\{r_i-\hat{x}_i-\alpha\,|i=1,\ldots,n\}\cup\{-\beta\}\right) \\
&= \min\left(\{r_i-\hat{x}_i+\alpha\,|i=1,\ldots,n\}\cup\{\beta\}\right) \\
&\qquad + \min\left(\{-(r_i-\hat{x}_i)+\alpha\,|i=1,\ldots,n\}\cup\{\beta\}\right) \\
&= \min(\{2\beta\}\cup\{\min_{i=1,\ldots,n}\{-|r_i-\hat{x}_i|\}+\alpha+\beta\} \\
&\qquad \cup\{\min_{1\leq i,j\leq n}\{(r_i-\hat{x}_i)-(r_j-\hat{x}_j)\}+2\alpha\}).
\end{aligned} \tag{19}
$$

Next, we will show that if (17) or (18) holds, this expression is negative whenever $\hat{\mathbf{x}} \neq \mathbf{x}$.

If (17) holds, then

$$
\begin{aligned}
&\min_{1\leq i,j\leq n}\{(r_i-\hat{x}_i)-(r_j-\hat{x}_j)\}+2\alpha \\
&= \min_{1\leq i,j\leq n}\{(r_i-\hat{x}_i)-(r_j-\hat{x}_j)\}-2\alpha+4\alpha \\
&< \min_{1\leq i,j\leq n}\{(r_i-\hat{x}_i)-(r_j-\hat{x}_j)-(v_i-v_j)\}+4\alpha \\
&= \min_{1\leq i,j\leq n}\{[(r_i-\hat{x}_i)-(r_j-\hat{x}_j)] \\
&\qquad -[(r_i-x_i-b)-(r_j-x_j-b)]\}+4\alpha \\
&= \min_{1\leq i,j\leq n}\{(x_i-\hat{x}_i)-(x_j-\hat{x}_j)\}+4\alpha \\
&= -\max_{1\leq i,j\leq n}\{(\hat{x}_i-x_i)-(\hat{x}_j-x_j)\}+4\alpha \\
&\leq 0.
\end{aligned} \tag{20}
$$

where the first inequality follows from the fact that $v_i - v_j \leq |v_i| + |v_j| < 2\alpha$ and the second inequality from (17).

If (18) holds, then

$$
\begin{aligned}
&\min_{i=1,\dots,n}\{-|r_i-\hat{x}_i|\}+\alpha+\beta \\
&=\min_{i=1,\dots,n}\{-|r_i-\hat{x}_i|\}-\alpha-\beta+2(\alpha+\beta) \\
&<\min_{i=1,\dots,n}\{-|r_i-\hat{x}_i|-|v_i+b|\}+2(\alpha+\beta) \\
&=\min_{i=1,\dots,n}\{-|r_i-\hat{x}_i|-|r_i-x_i|\}+2(\alpha+\beta) \qquad (21)\\
&\le\min_{i=1,\dots,n}\{-|x_i-\hat{x}_i|\}+2(\alpha+\beta) \\
&=-\max_{i=1,\dots,n}\{|x_i-\hat{x}_i|\}+2(\alpha+\beta) \\
&\le 0,
\end{aligned}
$$

where the first inequality follows from the fact that $|v_i+b| \le |v_i|+|b| < \alpha+\beta$ and the last inequality from (18).

Combining (19), (20), and (21) with (7) and (9), we find that indeed $\psi(\mathbf{r}-\hat{\mathbf{x}})=0$ for all codewords $\hat{\mathbf{x}} \neq \mathbf{x}$, while the probability density value of the received vector $\mathbf{r}$ given the transmitted codeword $\mathbf{x}$ is larger than 0, i.e., $\psi(\mathbf{r}-\mathbf{x})>0$. This implies that if decoding is based on maximizing (7), the transmitted codeword is always chosen as the decoding result, leading to a WER equal to zero. □

For the codebook $\mathcal{S}^*$, the bound on $\alpha+\beta$ for a Euclidean decoder in (14) is $1/2$, the bound on α for a Pearson decoder in (15) is $3/16$, and the bounds on α and $\alpha+\beta$ for a ML decoder in (17) and (18) are $1/4$ and $1/2$, respectively.

Considering Figs. 1-3, results from Theorems 1-3 are confirmed. The zero WER of Pearson decoding is indeed achieved if $\alpha < 3/16$. However, the shown results suggest that this may not be the best upper bound for the code under consideration. In addition, for $\alpha = 0.3$ and the example code $\mathcal{S}^*$, Theorems 1 and 3 give that, for both Euclidean and ML decoding, the WER is equal to zero if the offset is restricted to the interval $(-\beta, \beta)$ with $\beta \le 0.5-0.3 = 0.2$. This confirms the results from Figs. 4-5: for uniform offset, the zero WER is achieved if standard deviation $0.2/\sqrt{3} \approx 0.12$; for triangular offset, the zero WER is achieved if standard deviation $0.2/\sqrt{6} \approx 0.08$.

VI. Discussion and Conclusion

We have investigated Euclidean, Pearson, and ML decoders for channels which suffer from bounded noise and offset mismatch. In particular, it has been shown that the WER for such decoders is equal to zero if the noise and offset ranges satisfy certain conditions. The findings have been confirmed by simulation results.

Further investigations about how codebooks can be generated satisfying Theorems 1-3 given α and β will be of interest. Suppose α, β are fixed, it would be very interesting to fully explore the capacity of this channel and what rates can be achieved for the three decoding schemes satisfying zero WER conditions. Also, another interesting option for future research is to include the possibility of gain mismatch as well by considering various distributions for this phenomenon.

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