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Stability Margins in Adaptive Mixing Control via a Lyapunov-based Switching Criterion

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***Abstract*—This paper proposes a Lyapunov-based switching logic within the framework of adaptive mixing control (AMC), where a weighted combination of a family of candidate controllers can be inserted in the loop to regulate the output of an uncertain plant. The proposed AMC scheme employs a bank of parallel estimators, or multiple estimators, together with a switching logic that orchestrates which estimate should be evaluated by the mixer. The switching logic is driven by input/output data and uses Lyapunov-based criteria to assess the best estimate among the bank of parallel estimates. The resulting scheme guarantees convergence of the switching signal in finite time to a controller that satisfies a Lyapunov inequality implying a prescribed stability margin. The problem of convergence to the desired controller is addressed both analytically and numerically. In contrast, most classes of continuous tuning adaptive control or switching adaptive control schemes do not guarantee that after the switching stops or the adaptation is switched off the resulting closed loop linear time-invariant (LTI) system is stable, unless there is sufficient plant excitation that guarantees convergence to the desired fixed parameter controller. The proposed scheme guarantees that if the desired controller is switched on, it will never be switched off thereafter. Furthermore, simulations demonstrate that while alternative adaptation methods can converge to an LTI unstable feedback loop, the proposed scheme consistently converges to the desired controller.**

***Index Terms*—Adaptive control, mixing control, supervisory logic, linear matrix inequalities.**

I. Introduction

In the absence of any persistently exciting signals, classical adaptive control schemes, *e.g.*, model-reference or pole-placement adaptive control schemes, cannot guarantee that the estimated parameters converge to the true parameter values; therefore convergence to the desired LTI controller is not guaranteed. Consequently, in the absence of persistency of excitation, there is no guarantee that if adaptation is switched off the resulting closed loop LTI system is stable: the control scheme can possibly converge to a system whose unstable part is not excited [1]. Similarly, in adaptive schemes employing switching among a family of precalculated candidate controllers, *e.g.* [2], [3], [4], even though the boundedness of the closed-loop signals is established, there is no guarantee that the final switched-on controller is stabilizing if the switching logic is turned off. In this case it is not possible to exclude the possibility that a destabilizing controller is finally switched-on and kept in the loop because the unstable dynamics are not excited.

It is therefore important to develop adaptive switching mechanisms that can infer, from input/output data, the stability margin of a potential feedback loop and switch the corresponding candidate controller on. For this purpose switching logics employing Lyapunov-based criteria have been designed in recent years in the context of switching supervisory control [5], [6]. The objective of this work is the development of Lyapunov-based criteria for adaptive mixing control (AMC) [7], [8], where, rather than selecting a single candidate controller like in switching architectures, a weighted combination of more candidate controllers can be inserted in the loop to regulate the output of the uncertain plant. Mixing architectures, whose stability and robustness properties have been established in [7], [8], have been shown to moderate the detuning phenomenon arising in adaptive switching control due to the discrete nature of the candidate controllers versus the continuous nature of the uncertainty set [7]. The development of Lyapunov-based criteria for the selection of the control law in an adaptive mixing framework is of relevant importance for the development of adaptive mixing schemes with improved stability properties. In this paper the AMC scheme is extended to employ a bank of parallel estimators, or multiple estimators, together with a switching logic that, according to Lyapunov-based criteria, orchestrates which estimate should be evaluated by the mixer in order to determine the participation level each candidate controller. The resulting scheme guarantees that the final switched-on controller satisfies a Lyapunov inequality implying a prescribed stability margin in terms of a desired exponential decay of the norm of the states. While guaranteeing convergence to a stable LTI system is, in the absence of persistency of excitation, still an open problem, the proposed scheme guarantees that if the desired controller is switched on in feedback with the uncertain plant, it will never be switched off thereafter. Furthermore, numerical examples demonstrate that the proposed mechanism increases the chances that the final switched-on controller is the desired controller for the uncertain plant. Numerical methods based on Linear Matrix Inequalities (LMIs) for the analysis and the synthesis of the Lyapunov-based criteria are provided.

The paper is organized as follows: Section II introduces the parametrization of the uncertain plant, while Section III deals with a state-space formulation associated with the uncertain plant which is used for the implementation of the control law. Section IV revises the AMC stability results. Section V introduces the Lyapunov-based switching logic for the AMC scheme and Section VI explores LMI methods for the analysis

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and synthesis of the Lyapunov-based conditions and of the family of candidate controllers. Extension of the proposed method in the presence of disturbances is dealt with in Section VII. In Section VIII two numerical examples are used to show the effectiveness of the method.

Notation: Given the vector-valued time function $v \in \mathbb{R}^n$, v_t denotes the time truncation of the function v up to time t. The $\mathcal{L}_2$ norm of v_t is $\|v_t\|_2 := \left(\int_0^t |v(\tau)|^2 \, \mathrm{d}\tau\right)^{1/2}$, where $|v|$ is the Euclidean norm, and the $\mathcal{L}_\infty$ norm of v_t is $\|v_t\|_\infty := \sup\{|v(q)|, 0 \le q \le t\}$. We say that $v \in \mathcal{L}_2$ ($v \in \mathcal{L}_\infty$) if the $\mathcal{L}_2$ ($\mathcal{L}_\infty$) norm exists and is finite for $t \to \infty$. Finally, the notation $\lceil y \rceil$ indicates the smallest integer greater than or equal to y.

II. Problem Formulation: uncertain plant parametrization

The adaptive control problem is formulated for the class of noise-free linear time-invariant (LTI) uncertain systems. The extension of the proposed method to the case where disturbances and unmodelled dynamics are present is considered in Sect. VII. Consider the uncertain LTI SISO plant

$$y = G_0(s, \theta^*)u = \frac{\theta_b^{*T} \alpha_m(s)}{s^n + \theta_a^{*T} \alpha_{n-1}(s)} u, \qquad (1)$$

where $G_0(s, \theta^*)$ represents the transfer function of the uncertain plant; the vector $\theta^* := [\theta_b^{*T} \;\; \theta_a^{*T}]^T \in \mathbb{R}^{n+m+1}$ contains the unknown parameters of $G_0(s, \theta^*)$; the notation $\alpha_n(s)$ is used to indicate the vector containing all the powers from n to zero of the Laplace variable s, *i.e.* $\alpha_n(s) := [s^n \; s^{n-1} \cdots s \; 1]^T$.

We make the following plant assumptions, which are considered in most adaptive control designs:

P1. The degree n of the denominator of $G_0(s, \theta^*)$ is known.
P2. The plant is strictly proper, *i.e.*, $m \le n - 1$.
P3. $\theta^* \in \Omega$ for some known compact convex set $\Omega \subset \mathbb{R}^{n+m+1}$.

Remark 1: Assumptions P1-P3 are considered in most classical adaptive control designs. It must be underlined that with respect to model reference adaptive control we do not require the numerator of the plant to be Hurwitz, while with respect to adaptive pole placement control we do not require the numerator and the denominator of the plant to be coprime, *i.e.* stable zero-pole cancellations are allowed. The scheme can be extended to include tracking by using the internal model principle, where the reference signal $r \in \mathcal{L}_\infty$ is assumed to satisfy $Q_m(s) r = 0$. $Q_m(s)$ is the internal model of r, a monic polynomial of degree q with nonrepeated roots on the imaginary axis must satisfy:

P4. The numerator of the plant and Q_m are coprime.
□

The adaptive mixing law approach replaces θ^* with its estimate θ. An *on-line parameter estimator* based on the parametrization (1) of the uncertain plant is used to generate θ at each time t. In this work, a gradient law with dynamic normalization signal [9], [10] is considered:

$$\begin{aligned} \dot{\theta}(t) &= \Pr_{\Omega}\left(\Gamma \phi(t) \epsilon(t)\right), \quad \theta(0) = \theta_0, & (2)\\ \epsilon(t) &= \frac{\zeta(t) - \theta^T(t)\phi(t)}{m_s^2(t)}, & (3)\\ m_s^2(t) &= 1 + n_d(t), & (4)\\ \dot{n}_d(t) &= -\delta_0 n_d(t) + u^2(t) + y^2(t), \quad n_d(0) = 0, & (5) \end{aligned}$$

where $\theta(0) \in \Omega$, $\delta_0 \ge 0$, Pr stands for the projection operator that forces the estimated parameters to stay within the specified convex set Ω, ϵ is the normalized estimation error, $\Gamma \in \mathbb{R}^{(n+m+1)\times(n+m+1)}$ is the positive definite adaptive gain. The quantities

$$\begin{aligned} \zeta(t) &= \frac{s^n}{\Lambda_p(s)} y(t) & (6)\\ \phi(t) &= \left[\frac{\alpha_m^T(s)}{\Lambda_p(s)} u(t) \;\; - \frac{\alpha_{n-1}^T(s)}{\Lambda_p(s)} y(t)\right]^T & (7) \end{aligned}$$

are the observation and the regressor vector of the parametric model of the plant (1), and $\Lambda_p(s)$ is a Hurwitz polynomial of degree n. The adaptive law (2)-(5) guarantees [9, Sect. 4.4.1 and Table 4.2], [10, Sect. 3.3]:

E1. $\epsilon(t)$, $\epsilon(t) m_s(t)$, $\dot{\theta}(t) \in \mathcal{L}_2 \cap \mathcal{L}_\infty$.

III. State-space formulation of plant and control law

While considering an uncertain plant in the input/output form (1), a state-space formulation associated with the uncertain plant will be used for the purpose of analysis, as well as for the development of the switching logic of Sect. V. The plant (1) can be transformed into the following state-space representation

$$\begin{aligned} \dot{x}(t) &= A(\theta^*) x(t) + B u_f^{(n)}(t) & (8)\\ y(t) &= C(\theta^*) x(t). \end{aligned}$$

Here $x(t) := [y_f^{(n-1)}(t), \ldots, y_f(t), u_f^{(n-1)}(t), \ldots, u_f(t)]^T$, $y_f^{(n-1)}$ ($u_f^{(n-1)}$) denotes the $(n-1)$-th derivative of y_f (u_f), where $y_f = y/\Lambda(s)$, $u_f = u/\Lambda(s)$ and $\Lambda(s) = s^n + \lambda_{n-1} s^{n-1} + \cdots + \lambda_0$ is a Hurwitz polynomial of degree n. Besides,

$$A(\theta^*) = \left[\begin{array}{cc|c} \multicolumn{2}{c|}{-\theta_a^{*T}} & \bar{\theta}_b^{*T} \\ I_{n-1} & 0_{(n-1)\times 1} & 0_{(n-1)\times n} \\ \multicolumn{2}{c|}{0_{1\times n}} & 0_{1\times n} \\ \multicolumn{2}{c|}{0_{(n-1)\times n}} & I_{n-1} \;\; | \; 0_{(n-1)\times 1} \end{array}\right],$$

$$B = \begin{bmatrix} 0 \\ 0_{(n-1)\times 1} \\ 1 \\ 0_{(n-1)\times 1} \end{bmatrix}, \quad C(\theta^*) = \left[-\theta_a^{*T} \mid \bar{\theta}_b^{*T} - \theta_\lambda^T\right],$$

where $\theta_\lambda = [\lambda_{n-1}, \ldots, \lambda_0]^T$ and $\bar{\theta}_b^* \in \mathbb{R}^n$ denotes the vector that derives from filling θ_b^* with $n - m - 1$ zeros.

Remark 2: A state-space transformation similar to (8) also applies for any state $x(t) := [y_f^{(\bar{n}-1)}(t), \ldots, y_f(t), u_f^{(\bar{n}-1)}(t), \ldots, u_f(t)]^T$ and any Hurwitz polynomial $\Lambda(s)$ of degree $\bar{n}$, with $\bar{n} \ge n$. In this case the vectors θ_b^* and θ_a^* should be filled with $\bar{n}-m-1$ and $\bar{n}-n$ zero entries, respectively. It is useful to consider $\bar{n} \geq n$ to include control designs where the order of the controller is greater than the order of the plant. In the sequel, for simplicity, we will consider controllers of order n. Besides, without loss of generality, we will take $\Lambda(s)=\Lambda_p(s)$. □

The state and input of (8) are filtered values of the input/output pair (u,y) obtained via

$$\dot{\bar{y}}_f(t) = F\bar{y}_f(t)+Gy(t) \quad (9)$$
$$\dot{\bar{u}}_f(t) = F\bar{u}_f(t)+Gu(t), \quad (10)$$

where $\bar{y}_f(t) = [y_f^{(n-1)}(t),\ldots,y_f(t)]^T$, $\bar{u}_f(t) = [u_f^{(n-1)}(t),\ldots,u_f(t)]^T$ and

$$F=\begin{bmatrix} -\lambda_{n-1} & \ldots & -\lambda_0 \\ & I_{n-1} & 0_{(n-1)\times 1}\end{bmatrix}, \quad G=\begin{bmatrix} 1 \\ 0_{(n-1)\times 1}\end{bmatrix}. \quad (11)$$

Note that, assuming that the designer selects the initial conditions $\bar{y}_f(0)$ and $\bar{u}_f(0)$ of the filters (9)-(10), it is possible to calculate, at each time t, the state $x(t)=[\bar{y}_f^T(t)\ \bar{u}_f^T(t)]^T$, independently of the initial condition of the plant (1). It should be noted that $y_f^{(n)}(t)$ is also directly measurable at each time t via the first row of (9), *i.e.*

$$y_f^{(n)}(t) = -\lambda_{n-1}y_f^{(n-1)}(t)-\ldots-\lambda_0 y_f(t)+y(t). \quad (12)$$

The measurements of filtered inputs and outputs will be used in Sect. V to develop the switching logic among different candidate control laws. The state-space formulation (8) calls for a particular implementation of the control law, as presented in the following.

A. Controller implementation

Since the control objective is to choose the plant input u so that the plant output y is regulated to zero, we consider output-feedback control laws in the form $u(t)=-Q(s)/L(s)y(t)$, that can be written in a streamlined notation as

$$u^{(n)}+l_{n-1}u^{(n-1)}+\ldots+l_1u^{(1)}+l_0u \quad (13)$$
$$= -p_{n-1}y^{(n-1)}-\ldots-p_1y^{(1)}-p_0y, \quad (14)$$

where $u^{(n)}$ ($y^{(n)}$) is the n-th order derivative of u (y). By adopting the representation (8), the output-feedback control law (13) can be implemented in such a way to feed back the state $x(t)$. In fact, after filtering the left and the right side of (13) by the stable filter $1/\Lambda(s)$, we obtain a controller in a full-state feedback form

$$u_f^{(n)}(t) = -Kx(t), \quad (15)$$
$$K = [p_{n-1}\ \ldots\ p_1\ p_0\ l_{n-1}\ \ldots\ l_1\ l_0]. \quad (16)$$

It should be noted that (15) leads to a non-minimal transfer function representation of (13). This is due to the introduction of the filter $1/\Lambda(s)$. In particular, from (15) we obtain $u(t)=-(Q(s)\Lambda(s))/(L(s)\Lambda(s))y(t)$: thus the filtering action (9)-(10) introduces stable zero-pole cancellations. As a consequence of this representation, the internal stability of the feedback loop formed by the plant (1) and the controller (13) is equivalent to the internal stability of the feedback loop formed by the plant (1) and the controller $u(t) = -(Q(s)\Lambda(s))/(L(s)\Lambda(s))y(t)$. In the rest of the paper, the control action is supposed being implemented as shown in Algorithm 1: for the sake of implementation in a digital computer, a temporal discretization for the solutions of the ordinary differential equations is shown. In the next section we will show how to combine a family of candidate control laws in the form (15) in a multicontroller with mixing architecture.

Algorithm 1 Implementation of control action

At time t;
Given: Given the measurements of the input/output pair $(u(t),y(t))$;
1: Calculate the filtered input/output pair $(\bar{u}_f(t+\delta t),\bar{y}_f(t+\delta t))$ via (9)-(10);
2: Form the state $x(t+\delta t)=[\bar{y}_f^T(t+\delta t)\ \bar{u}_f^T(t+\delta t)]^T$ and calculate $u_f^{(n)}(t+\delta t)=-Kx(t+\delta t)$
3: Apply the input $u(t+\delta t)=u_f^{(n)}(t+\delta t)-[\lambda_{n-1},\ldots,\lambda_0]^T\bar{u}_f(t+\delta t)$.

IV. Multicontroller and mixer

The focus of the control problem is on situations where a large parameter uncertainty set Ω prevents any single linear controller, *e.g.*, designed with robust linear control techniques, from meeting the performance requirements over the whole uncertainty set Ω. In multiple model architectures, in order to cope with the large uncertainty set of the plant (1), we assume the presence of a finite family of N control laws $\{C_i(s)=P_i(s)/L_i(s)\}_{i\in\mathcal{I}}$, where $\mathcal{I} := \{1,\ldots,N\}$. When the filtering action of Sect. III is employed, the control laws can be implemented as $\{K_i\}_{i\in\mathcal{I}}$, with K_i as in (15). Given the family of N candidate controllers $\{K_i\}_{i\in\mathcal{I}}$, a multicontroller $\mathbb{C}(\beta)$ is constructed. As typically assumed in multiple model architectures, each candidate controller K_i yields a stable closed-loop system that meets the performance requirements for a compact subset Ω_i of the uncertainty set Ω. The subsets $\{\Omega_i\subset\mathbb{R}^{n+m+1}\}_{i\in\mathcal{I}}$, are a finite cover of Ω, *i.e.*, $\Omega\subset\cup_{i\in\mathcal{I}}\Omega_i$. The multicontroller is a dynamical system capable of generating the N candidate control laws, as well as a mix of candidate control laws for overlapping parameter subsets

$$u_f^{(n)}(t) = -\sum_{i=1}^{N}\beta_i(\theta)K_ix(t). \quad (17)$$

The multicontroller depends on a mixing signal $\beta = [\beta_1,\ldots,\beta_N]^T\in\mathbb{R}^N$ which determines the participation level of each of the candidate controllers. The mixer implements the mapping $\beta:\Omega\mapsto\mathcal{B}_\theta$, where $\mathcal{B}_\theta$ is the set of admissible mixing values

$$\mathcal{B}_\theta = \{\beta\in\mathbb{R}^N:\sum_{i\in\mathcal{I}}\beta_i(\theta)=1;\ \beta_i\geq 0;\ \beta_i=0 \text{ if } \theta\notin\Omega_{\rm i}\}. \quad (18)$$

The following properties of $\beta(\cdot)$ and of the multicontroller $\mathbb{C}(\beta)$ are assumed

M1. $\beta(\cdot)$ is Lipschitz in Ω.
C1. For every $\theta^*\in\Omega$, let $\beta^* := \beta(\theta^*)$; then $\mathbb{C}(\beta^*)$ internally stabilizes the plant $G_0(\theta^*)$.

Remark 3: Property M1 ensures that if θ is tuned slowly (in the $\mathcal{L}_2$ sense of property E1), then the closed-loop system varies slowly in the $\mathcal{L}_2$ sense. This property is used to establish stability of the closed loop system by using results from time-varying systems [9]. Property C1 ensures that $\mathbb{C}(\beta)$ is a certainty equivalence stabilizing controller. Note that assumption C1 is a point-wise stability requirement that needs not to be verified for every admissible mixing strategy in $\mathcal{B}_\theta$, but only for the chosen mixing strategy $\beta(\cdot)$. □

The on-line parameter estimator is now combined with the bank of linear controllers and the mixing strategy to develop an adaptive mixing control design. The stability properties of such AMC scheme are recalled [7], [8]:

Theorem 1: Let the uncertain plant be given by (1). Consider the adaptive mixing controller with the multicontroller $\mathbb{C}(\beta)$ given by (9), (10) and (17) and satisfying assumption C1. If the mixing function $\beta(\theta)$ given by (18) satisfies M1; and if θ is generated by the adaptive law (2)-(5), then the resulting adaptive mixing control scheme guarantees that all closed-loop signals are bounded, *i.e.*, u, $y \in \mathcal{L}_\infty$; furthermore $y(t) \to 0$ as $t \to \infty$.

Proof: See [7], [8]. ■

Remark 4: In the absence of sufficient plant excitation the regressor vector ϕ in the adaptive law (2) cannot be guaranteed to be persistently exciting, which implies that the estimated parameters generated by (2) may not converge to the true values [7], [8]. This in turn implies that the AMC scheme cannot be guaranteed to converge to the desired controller K_i whose index i satisfies $\theta^* \in \Omega_i$. In the following section, a Lyapunov-based switching logic is developed in order to guarantee, also in the absence of persistency of excitation, exponential decaying of the norm of the states according to a desired stability margin. □

V. LYAPUNOV-BASED SWITCHING LOGIC

In this section the AMC scheme is equipped with a Lyapunov-based switching logic. By monitoring I/O data, the switching logic inserts in the feedback loop a control law that verifies a Lyapunov inequality. Unlike adaptive switching control, that selects at each time instant a single candidate controller, adaptive mixing control allows for the weighted selection of the candidate controllers. Thus, the Lyapunov criterion to be developed will take into account the mixing policy occurring in the overlapping region of two or more subsets Ω_i.

It is assumed that the bank of controllers has been designed to guarantee a known stability margin for every possible value of the uncertain parameter in Ω. To this end, the following assumption is introduced:

L1. There exists a family of Lyapunov functions, $V_i(x) = x^T P_i x$, $i \in \mathcal{I}$, that satisfy

$$\frac{\partial V_i}{\partial x}\dot{x}(\theta^*, \beta(\theta)) = \frac{\partial V_i}{\partial x}\left(A(\theta^*) - B\sum_{j=1}^{N}\beta_j(\theta)K_j\right)x \leq -\rho_i V_i(x), \quad \forall\theta^* \in \Omega_i, \forall\theta \in \Omega_i, \forall x, \tag{19}$$

where P_i, $i \in \mathcal{I}$, are positive definite symmetric matrices and ρ_i, $i \in \mathcal{I}$, are positive constants.

Note that inequality (19) is a simultaneous stabilizability condition that can be checked using LMI-based tools [11, Sect. 2.2]: the LMI-based numerical procedure for determining the positive definite symmetric matrices P_i and the positive constants ρ_i will be explained in Section VI.

Remark 5: Lyapunov stability arguments can be used to establish that (19) guarantees exponential stability in the subset Ω_i (*cf.* [12, Thm. 4.10]), *i.e.*, (19) guarantees the existence of two positive constants α_1 and α_2 such that:

$$|x(t)| \leq \alpha_1 e^{-\alpha_2 t}\,|x(0)|\,, \tag{20}$$

where $\alpha_2 = \rho_i/2$ is the stability margin, $\alpha_1 = (\lambda_{max}(P_i)/\lambda_{min}(P_i))^{1/2}$, and $\lambda_{max}(P_i)$, $\lambda_{min}(P_i)$ are the maximum and the minimum eigenvalue of P_i, respectively. □

Inequality (19) is used in the proposed adaptive mixing approach to guarantee a desired stability margin in terms of an exponential decay rate of the Lyapunov function. Using similar ideas as in [5], define as $\dot{x}^i(t)$ the derivative that the state x would have if the controller K_i is placed in the loop at time t, that is

$$\begin{aligned}\dot{x}^i(t) &= A(\theta^*)x(t) - BK_i x(t) \\ &= \dot{x}(t) - BK_i x(t) - Bu_f^{(n)}(t).\end{aligned} \tag{21}$$

The idea behind the development of the switching logic is to verify which are the controller indexes i that satisfy the inequality

$$\frac{\partial V_i}{\partial x}\dot{x}^i(t) = 2x^T(t)P_i\dot{x}^i(t) \leq -\rho_i x^T(t)P_i x(t), \tag{22}$$

and choose the corresponding estimated parameter vector θ_i to be evaluated by the mixer. Thanks to the particular transformation in (8), the components of $\dot{x}$ are directly measurable or can be calculated. In fact, $\dot{x}(t) := [y_f^{(n)}(t), \ldots, y_f^{(1)}(t), u_f^{(n)}(t), \ldots, u_f^{(1)}(t)]^T$. The components, $y_f^{(n-1)}(t), \ldots, y_f^{(1)}(t)$ and $u_f^{(n-1)}(t), \ldots, u_f^{(1)}(t)$ are directly available from the state $x(t)$. The quantity $u_f^{(n)}(t)$ is available from the applied control law, while $y_f^{(n)}$ can be calculated from (12).

A. Multiple estimators and hysteresis switching logic

A parallel estimation architecture is used, in an effort to improve initial learning performance. We use as many estimators as the N candidate controllers: each estimator differs in its initial condition and it is designed to project its estimate on a subset Ω_i:

$$\dot{\theta}_i(t) = \Pr_{\Omega_i}(\Gamma\phi(t)\epsilon_i(t)) \tag{23}$$

$$\epsilon_i(t) = \frac{\zeta(t) - \theta_i^T(t)\phi(t)}{m_s^2(t)}, \tag{24}$$

where $\theta_i(0) \in \Omega_i$ and m_s^2 is as in (4)-(5). A hysteresis switching mechanism is introduced to choose which is the best estimate among the N estimated parameters vectors $\theta_1, \cdots, \theta_N$. Based on the Lyapunov conditions (19), for each parameter estimator we consider the performance signal $\mathcal{J}_i$:

$$\Lambda_i(t) = \max\left\{2x^T(t)P_i\dot{x}^i(t) + \rho_i x^T(t)P_i x(t), 0\right\} \tag{25}$$

$$\mathcal{J}_i(t) = \max_{0\leq q\leq t}\{\Lambda_i(q)\}, \tag{26}$$

where $\dot{x}^i$ is as in (21). A supervisory logic compares the N performance signals $\{\mathcal{J}_i(t)\}_{i\in\mathcal{I}}$, and selects, at each time t, the estimate $\theta_{\sigma(t)} := \theta_\sigma(t)$ of index σ via the following *hysteresis switching logic* [13]:

$$\sigma(t) = \left\{ \text{the least } j: \; j = \arg\min_{i\in\mathcal{I}} \left\{\mathcal{J}_i(t) - h\,\delta_{i,\sigma^-(t)}\right\} \right\}, \quad \sigma^-(0) \in \mathcal{I}, \tag{27}$$

where $\delta_{i,j}$ is the Kronecker's index ($\delta_{i,j} = 1$ if $i = j$, $\delta_{i,j} = 0$ otherwise), $\sigma^-(t)$ is the limit of $\sigma(\tau)$ from below as $\tau \to t$ and h, a (typically small) positive real number, is the *hysteresis constant*. When more than one index minimizes $\left\{\mathcal{J}_i(t) - h\,\delta_{i,\sigma^-(t)}\right\}$, the least index is selected [13]: alternatively, any other minimizing index can be chosen without altering the properties of the switching logic.

The following Hysteresis Switching Logic (HSL) lemma establishes the behavior of the switching system arising from (27).

HSL Lemma [13] *Let $\mathscr{S}$ denote the class of all possible switching sequences $\sigma(\cdot)$. Consider the following assumptions:*

A1. For each $\sigma(\cdot) \in \mathscr{S}$ and $i \in \mathcal{I}$, $\mathcal{J}_i(t)$ admits a limit as $t \to \infty$, or the limit goes to infinity;

A2. For each $\sigma(\cdot) \in \mathscr{S}$, there exist integers $\mu \in \mathcal{I}$ such that $\mathcal{J}_\mu(\cdot)$ is bounded.

Let σ be the switching sequence resulting from (27). Then, if A1 and A2 hold, there is a finite time $t^* \in \mathbb{R}_+$, after which no more switching occurs. Moreover, $\mathcal{J}_{\sigma(t^*)}(\cdot)$ is bounded.

B. Stability of Lyapunov-based Adaptive Mixing Control

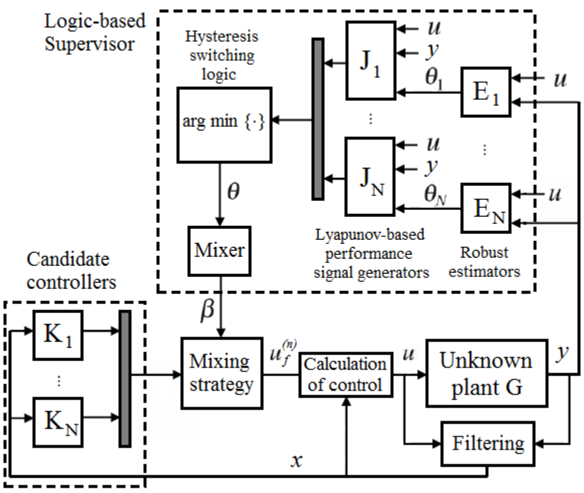


Fig. 1. Lyap-AMC architecture

The stability properties of the resulting Lyapunov-based AMC (Lyap-AMC), whose architecture is shown in Figure 1, are established by the following Theorem:

Theorem 2: Let the uncertain plant be given by (1). Consider the bank of parallel estimators (23)-(24),(4)-(5). Consider the adaptive mixing controller with the multicontroller $\mathbb{C}(\beta)$ given by (9),(10) and (17) and satisfying assumption L1. Let $\Lambda(s)$ be analytic in $\Re[s] < -\bar{\rho}/2$, where $\bar{\rho} = \min_i \{\rho_i, i \in \mathcal{I}\}$. If the multicontroller is driven by the mixing strategy $\beta(\theta_{\sigma(\cdot)})$, with $\beta(\cdot)$ given in (18) and the index $\sigma(\cdot)$ selected according to the switching logic (27), (25) and (26), then the following hold:

1) (*Final switching time*) There is a final switching time t^* for the index σ and $\mathcal{J}_{\sigma(t^*)} < h$.
2) (*Transient performance before final switching time*) Before the final switching time $\mathcal{J}_{\sigma(t)} < h$, which guarantees that each controller is switched-on at most once, there are at most N switches before the final switching time and
$$|y(t)| \le c_1 e^{-\frac{c_2}{2}(t)}|x(0)| + N\underline{h}, \quad \forall\, 0 \le t \le t^*, \tag{28}$$
where $c_1 = c_\lambda \prod_{i=1}^N \left(\lambda_{max}(P_i)/\lambda_{min}(P_i)\right)^{1/2}$, c_λ depends on the coefficients of the filter $\Lambda(s)$, $c_2 = \min_i \rho_i$, and $\underline{h} = \mathcal{O}(h)$. Furthermore, whenever a (desired) controller whose index i satisfies $\theta^* \in \Omega_i$ is inserted in the loop, it will never be switched-off thereafter.
3) (*Steady-state performance after final switching time*) The final switched-on controller, namely $\mathbb{C}(\beta(\theta_{\sigma(t^*)}))$ guarantees
$$|y(t-t^*)| \le \alpha_1 e^{-\frac{\rho_{\sigma(t^*)}}{2}(t-t^*)}|x(t^*)| + \underline{h}, \quad \forall\, t \ge t^*, \tag{29}$$
where $\alpha_1 = c_\lambda \left(\lambda_{max}(P_{\sigma(t^*)})/\lambda_{min}(P_{\sigma(t^*)})\right)^{1/2}$, $\underline{h} = \mathcal{O}(h)$. In addition, whenever $\theta^* \in \Omega_{\sigma(t^*)}$, $\lim_{t\to\infty} y(t) = 0$.

Proof: See the Appendix. ■

Remark 6: In supervisory adaptive control schemes with multiple models, the performance signals $\mathcal{J}_i$ are typically based on some norm of the estimation error, *e.g.*, $\mathcal{J}_i(t) = \int_0^t |\epsilon_i m_s(\tau)|^2 \, d\tau$ [14], [15]. Such performance signals have been used also in the context of AMC [16]. The resulting AMC scheme guarantees the same stability properties as the single-estimator AMC presented in Theorem 1. Similar stability properties are also achieved by classical adaptive control schemes, namely model-reference and pole-placement adaptive control as well as by many supervisory adaptive control schemes with multiple models. The developed performance signals (25)-(26), based on a Lyapunov criterion, guarantee additional stability properties, *i.e.*, a desired stability margin. □

Remark 7: For purely switching adaptive architectures, selecting a single candidate control law without mixing the candidate controllers inside the overlapping subsets, condition L1 reduces to

$$\begin{aligned} \frac{\partial V_i}{\partial x}\dot{x}(\theta^*) &= \frac{\partial V_i}{\partial x}\left(A(\theta^*) - BK_i\right)x(t) \\ &\le -\rho_i V_i(x), \quad \forall \theta^* \in \Omega_i, \forall x, \end{aligned} \tag{30}$$

which is similar to the conditions that can be found e.g. in [5] (cf. eq.(3)) and in [6] (cf. eq.(3)). The results of Theorem 2 apply to an adaptive switching control architecture satisfying (30) and equipped with the Lyapunov-based switching logic (25), (26) and (27) in a straightforward manner. The switching control architecture can be seen as the limiting case of a mixing architecture, with the overlapping regions of the subsets Ω_i shrinking to zero. As a consequence, condition L1 includes (30) as a limiting case and it is no more restrictive than other Lyapunov-based approaches available in literature. □

Remark 8: The focus of the results in Theorem 2 is on guaranteeing stability of the adaptive system under the condition that θ^* is unknown but constant. It is possible to verify that the presented results are still valid if $\theta^*(t) \in \Omega_i \ \forall t$, *i.e.*, if the parameter vector θ^* is time-varying but it never leaves its initial uncertainty subset Ω_i. In the case of a time-varying parameter vector $\theta^*(t)$ going from one uncertainty subset to another, the $\max$ operator in (26) must be modified. In fact, such an operator makes $\mathcal{J}_i$ monotonically nondecreasing, thus not allowing to insert in the loop a candidate controller which performed unsatisfactorily in the past, before all the remaining candidate controllers perform at least as badly. Modifications of (25)-(26) in order to deal with time-varying parameters include the use of finite windows or fading memories, *e.g.*

$$\Lambda_i(t) = \max\left\{2x^T(t)P_i\dot{\hat{x}}^i(t) + \rho_i x^T(t)P_i x(t), 0\right\} \quad (31)$$

$$\mathcal{J}_i(t) = \max_{0\le q\le t}\left\{e^{-r(t-q)}\Lambda_i(q)\right\}, \quad (32)$$

where $r > 0$ is the discount factor. The stability analysis in the time-varying case is not straightforward and will be the subject of future studies. Designing controllers for plants with time varying parameters even when the parameters of the plant are known is not straightforward and requires a completely different approach than in the LTI case as documented in [17]. □

VI. Linear Matrix Inequalities for analysis and synthesis of candidate controllers

Linear Matrix Inequalities are a powerful tool that allow the construction of quadratic Lyapunov functions for stability and performance analysis of linear systems [11]. In this section LMI-based numerical methods are developed to solve the following two problems, associated to the Lyapunov-based condition (19):

1) Given a family of candidate controllers $\{K_i\}_{i\in\mathcal{I}}$, find the symmetric matrices P_i and the scalars ρ_i, $i \in \mathcal{I}$, such that condition (19) is verified (Analysis problem).
2) Find the symmetric matrices P_i and the scalars ρ_i, $i \in \mathcal{I}$, as well as the state-feedback gains K_i, $i \in \mathcal{I}$, such that condition (19) is verified (Synthesis problem).

A. Analysis problem

Inequality (19) can be written as

$$x^T\left[A^T(\theta^*)P_i + P_iA(\theta^*) + \rho_iP_i - \sum_{j=1}^N \beta_j(\theta)K_j^TB^TP_i^T \right.$$
$$\left. -P_iB\sum_{j=1}^N \beta_j(\theta)K_j\right]x \le 0, \quad \forall\theta^*\in\Omega_i, \forall\theta\in\Omega_i, \forall x,$$

which implies

$$A^T(\theta^*)P_i + P_iA(\theta^*) + \rho_iP_i - \sum_{j=1}^N \beta_j(\theta)K_j^TB^TP_i^T$$
$$-P_iB\sum_{i=1}^N \beta_j(\theta)K_j \preceq 0, \quad \forall\theta^*\in\Omega_i, \forall\theta\in\Omega_i. \quad (33)$$

Condition (33) is a parameter dependent LMI that should be satisfied over the whole subset Ω_i. Since a parameter-dependent LMI is equivalent to a set of infinitely many LMIs, it is difficult to solve in general. In literature we can distinguish two main approaches aiming at the solution of parameter-dependent LMIs: if the uncertainty subset Ω_i belongs to a polytope with vertices V^k, $k = 1,\dots,n+m+1$, the first approach uses convexity properties to formulate the LMI only at the vertices [18], [19]:

$$A^T(\theta^{*[k]})P_i + P_iA(\theta^{*[k]}) + \rho_iP_i - K_j^TB^TP_i^T$$
$$-P_iBK_j \preceq 0, \quad \forall\theta^{*[k]}\in V^k, \forall j\in\mathcal{I}_i, \quad (34)$$

where $\mathcal{I}_i$ is the subset of $\mathcal{I}$ indicating the mixing signals that can be active in the subset Ω_i. Note that in (34) we took into account the fact that $\beta(\theta) \in \mathbb{R}^N$ belongs to a polytope with vertices $e_1,\dots,e_N$, where e_j is the orthogonal basis with zero entries and a 1-entry in the j-th position. The number of LMIs in (34) is $(n+m+1)*\dim(\mathcal{I}_i)$. A second approach to the solution of (33) consists of gridding the parameter set by taking M sample points $\left\{\theta^{[1]},\dots,\theta^{[M]}\right\} \in \Omega_i$, and formulating the LMIs at the grid points. Gridding methods based on deterministic or randomized sampling have been developed for several LMI problems [20], [21], [22]. If Ω_i is not a polytope, the gridding approach may reduce the level of conservatism: however, one drawback is the fact that there is no guarantee that the LMIs are satisfied between the grid points. A practical approach is selecting two sets of grid points, the first set to be used for the solution of the LMIs and the second one, possibly denser than the first set, to be used for validating the solution. An appropriate selection of the hysteresis constant will help to address the quantization error introduced by the grid: in fact, solving (33) by using a gridding approach implies that there exists always an index j such that

$$2x^T(t)P_j\dot{\hat{x}}^j(t) + \rho_jx^T(t)P_jx(t) \le \kappa, \quad (35)$$

where $\kappa \ge 0$ takes into account the quantization error introduced by the grid, which decreases by increasing the number of grid points. By choosing the hysteresis constant $h \ge \kappa$ we can avoid spurious switching due to the quantization error. With the gridding approach the number of LMIs to be solved for every subset Ω_i is $M * M$. Such a formulation, although tractable in many practical applications, might suffer from dimensionality problem due to the fact that the number of grid points increases exponentially with the dimension of the parameter. At the current state-of-the-art, Semi-Definite Programming (SDP) solvers based on interior-point methods [23], [24], can efficiently handle medium-scale problems with $< 20,000$ optimization variables; SDP solvers based on augmented-Lagrangian methods [25], [26] scale to larger problems. Algorithm 2 proposes an algorithm to solve the problem of maximizing the decaying rate ρ_i.

B. Synthesis problem

If the problem is to find both the Lyapunov matrices P_i and the state-feedback gains K_i, $i \in \mathcal{I}$, then inequality (33) is not linear in the unknown terms. Besides, because of the mixing architecture, the state-feedback gains are related to

Algorithm 2 Analysis problem

Given: Given a family of candidate controllers $\{K_i\}_{i\in\mathcal{I}}$;

1: Select M sample points $\{\theta^{[1]},\ldots,\theta^{[M]}\}\in\Omega_i$, and a positive constant $\Delta_{\rho_i}>0$. Set $c=1$ and $\rho_i^c=0$;

2: Solve the feasibility problem FB c-th:

$$\begin{aligned}&\text{find } P_i\\&\text{s.t.}\\&A^T(\theta^{[l]})P_i+P_iA(\theta^{[l]})+\rho_i^cP_i-\sum_{j=1}^N\beta_j(\theta^{[r]})K_j^TB^TP_i^T\\&\quad -P_iB\sum_{j=1}^N\beta_j(\theta^{[r]})K_j\preceq 0,\quad l=1,\ldots,M,\ r=1,\ldots,M\\&P_i\succ 0\end{aligned}$$

3: If FB c-th is feasible, set $c=c+1$, $\rho_i^c=\rho_i^{c-1}+\Delta_{\rho_i}$, then go to Step 2. Else $\rho_i=\rho_i^c-\Delta_{\rho_i}$. Stop.

Repeat Steps 1-3 for every subsets Ω_i.

Return: Return: P_i and ρ_i, $i\in\mathcal{I}$.

each other by nonlinear constraints, which make the problem more difficult than a classical state-feedback synthesis problem based on LMIs. In other words, while in the analysis problem we can solve each optimization problem separately for each Ω_i, the same thing is not true in the synthesis problem, since the solution of the problem for a certain subset $\Omega_{\bar{i}}$ affects the solution of the other subsets, because of the mixing architecture. Next, an iterative LMI-based procedure for solving inequality (19) is proposed. For compactness of notation (33) is rearranged as

$$\begin{aligned}A^T(\theta^*)P_i+P_iA(\theta^*)+\rho_iP_i-\bar{\beta}(\theta)^T\mathbf{K}^TB^TP_i\\-P_iB\mathbf{K}\bar{\beta}(\theta)\preceq 0,\quad\forall\theta^*\in\Omega_i,\forall\theta\in\Omega_i,\end{aligned}\tag{36}$$

where

$$\bar{\beta}(\theta)=\begin{bmatrix}\beta_1I_m\\\vdots\\\beta_NI_m\end{bmatrix},\quad \mathbf{K}=[K_1\ \cdots\ K_N].\tag{37}$$

Inequality (36) has the same form as static output feedback stabilization: several numerical methods have been proposed in literature for the solution of this problem [27], [28], [29], [30], [31]. Next, the iterative method proposed in [29] is elaborated for the adaptive mixing control architecture in order to solve the synthesis problem. Inequality (33) is satisfied if and only if there exist matrices $P_i\succ 0$, $i\in\mathcal{I}$ and $\mathbf{K}$ satisfying the following matrix inequality, $\forall\theta^*\in\Omega_i$, $\forall\theta\in\Omega_i$:

$$\begin{aligned}A^T(\theta^*)P_i+P_iA(\theta^*)+\rho_iP_i-P_iBB^TP_i+\\(B^TP_i-\mathbf{K}\bar{\beta}(\theta))^T(B^TP_i-\mathbf{K}\bar{\beta}(\theta))\preceq 0.\end{aligned}\tag{38}$$

Inequality (38) is a quadratic matrix inequality. In order to solve it we introduce additional design variables X_i that satisfy

$$X_iBB^TP_i+P_iBB^TX_i-X_iBB^TX_i\preceq P_iBB^TP_i.\tag{39}$$

Inequality (39) holds since $(X_i-P_i)^TBB^T(X_i-P_i)\succeq 0$. The equality holds if and only if $X_iB=P_iB$. By combining inequalities (38) and (39) we obtain a sufficient condition to solve inequality (36):

$$\begin{aligned}A^T(\theta^*)P_i+P_iA(\theta^*)+\rho_iP_i+\Psi_i+\\(B^TP_i-\mathbf{K}\bar{\beta}(\theta))^T(B^TP_i-\mathbf{K}\bar{\beta}(\theta))\preceq 0,\end{aligned}$$

with $\Psi_i=-X_iBB^TP_i-P_iBB^TX_i+X_iBB^TX_i$. The last inequality can be solved by applying a Schur complement and the iterative algorithm presented in Algorithm 3. The LMI conditions should be satisfied over the whole subsets Ω_i, $i\in\mathcal{I}$. If we grid each subset Ω_i by taking M sample points $\{\theta^{[1i]},\ldots,\theta^{[Mi]}\}\in\Omega_i$, $i\in\mathcal{I}$, the total number of LMIs to be solved in Algorithm 3 is $M*M*N$. An interesting problem is finding the greatest ρ_i which make the LMI problem feasible, in order to maximize the decaying rate of the regulation error. In Algorithm 3 an algorithm is proposed to solve such a problem. Similar comments as the analysis problem regarding the computational tractability of the gridding approach also apply to the synthesis problem.

Algorithm 3 Synthesis problem

Given: Given $M*N$ sample points $\{\theta^{[1i]},\ldots,\theta^{[Mi]}\}\in\Omega_i$, $i\in\mathcal{I}$;

1: Select $Q_i\succ 0$, $i\in\mathcal{I}$, and solve for P_i, $i\in\mathcal{I}$, for some points $\theta^{[ki]}$ inside Ω_i, the algebraic Riccati equations:

$$A^T(\theta^{[ki]})P_i+P_iA(\theta^{[ki]})-P_iBB^TP_i+Q_i=0,\qquad i\in\mathcal{I}\tag{40}$$

Set $c=1$ and $\bar{\rho}^0=0$, $X_i^1=P_i$, $i\in\mathcal{I}$;

2: Solve for P_i, $\mathbf{K}$ and ρ^c

$$\begin{aligned}&\bar{\rho}^c=\min\rho^c\\&\text{s.t.}\\&\begin{bmatrix}A^T(\theta^{[li]})P_i+P_iA(\theta^{[li]})+\Psi_i^c & (B^TP_i-\mathbf{K}\bar{\beta}(\theta^{[ri]}))^T\\ B^TP_i-\mathbf{K}\bar{\beta}(\theta^{[ri]}) & -I\end{bmatrix}\preceq 0\\&\Psi_i^c=-X_i^cBB^tP_i-P_iBB^tX_i^c+X_i^cBB^tX_i^c+\rho^cP_i\\&P_i\succ 0\qquad l=1,\ldots,M,\ r=1,\ldots,M,\ i\in\mathcal{I}\end{aligned}$$

3: If $\bar{\rho}^c<0$ and $|\bar{\rho}^c-\bar{\rho}^{c-1}|<\kappa_1$, with κ_1 a prescribed tolerance, $\mathbf{K}$ are the desired controller gains. **Return:** Return: P_i, $\rho_i=\bar{\rho}^c$, $i\in\mathcal{I}$, and $\mathbf{K}$. Stop.

4: Solve for P_i, $\mathbf{K}$

$$\begin{aligned}&[\bar{P}_1,\ldots,\bar{P}_N]=\arg\min[\text{tr}(P_1)+\ldots+\text{tr}(P_N)]\\&\text{s.t.}\\&\begin{bmatrix}A^T(\theta^{[li]})P_i+P_iA(\theta^{[li]})+\Psi_i^c & (B^TP_i-\mathbf{K}\bar{\beta}(\theta^{[ri]}))^T\\ B^TP_i-\mathbf{K}\bar{\beta}(\theta^{[ri]}) & -I\end{bmatrix}\preceq 0\\&\Psi_i^c=-X_i^cBB^tP_i-P_iBB^tX_i^c+X_i^cBB^tX_i^c+\bar{\rho}^cP_i\\&P_i\succ 0\qquad l=1,\ldots,M,\ r=1,\ldots,M,\ i\in\mathcal{I}\end{aligned}$$

5: If $\|B^TX_i^c-B^T\bar{P}_i\|<\kappa_2$, $i\in\mathcal{I}$, with κ_2 a prescribed tolerance, the synthesis problem may not be solvable, Stop. Else set $c=c+1$ and $X_i^c=\bar{P}_i$, then go to Step 2.

Even if, as typical in iterative schemes, finding the global optimum is not guaranteed, the optimization problem of Step 2 is a generalized eigenvalue minimization problem, which guarantees the progressive reduction of ρ_c. The initial choice of Q_i might affect the number of iterations required and the final solution of the algorithm.

VII. Extension to $\mathcal{H}_\infty$ control designs

Despite the achieved convergence and stability properties, the proposed approach is not immune to the 'model-mismatch' instability problems that have been addressed in unfalsified methods [32], and that are inherent in model-based adaptive design approaches. In case of mismatch between the plant model and the real plant, due to nonlinearities, time-delays and/or unstructured uncertainties, the stability of the adaptive scheme might be lost. In order to address such phenomena, the proposed method is extended to $\mathcal{H}_\infty$-based control designs, thus handling in an effective way bounded disturbances and/or unmodelled dynamics [11], [33]. Such an extension goes through a modification of both the multicontroller design criterion L1 and the performance signals (25). In this section such modifications are presented and discussed. When the uncertain plant (1) is affected by bounded input and/or output disturbances, using similar transformations as in Sect. II, we can transform the uncertain plant into

$$\dot{x}(t) = A(\theta^*)x(t) + Bu(t) + L(\theta^*)d(t), \quad (41)$$

where d is a bounded disturbance, *i.e.*, $|d(t)| \leq d_0$, $\forall t \in \mathbb{R}_+$.

Condition L1 must be modified in order to account for the disturbance term. The following assumption is supposed to hold:

L2. The multi-controller $u_f^{(n)} = -\sum_{j=1}^N \beta_j(\theta)K_j x$ has been designed to guarantee the existence of a family of Lyapunov functions, $V_i(x) = x^T P_i x$, $i \in \mathcal{I}$, that satisfy

$$\dot{V}_i(x) = \frac{\partial V_i}{\partial x}\dot{x}(\theta^*, \beta(\theta)) \leq -\rho_i V_i(x) - y_f^T y_f + \gamma_i d^T d, \quad \forall \theta^*, \theta \in \Omega_i, \forall x, (42)$$

where ρ_i and γ_i, $i \in \mathcal{I}$, are known positive constants.

Remark 9: The condition (42) guarantees, inside the uncertainty subset Ω_i that

$$\int_0^\infty \left[y_f^T(t)y_f(t) - \gamma_i d^T(t)d(t)\right] \mathrm{d}t \leq 0 \Rightarrow \frac{\|y_f\|_2^2}{\|d\|_2^2} \leq \gamma_i, \quad (43)$$

thus achieving finite $\mathcal{L}_2$ gain [11]. Similarly to condition (19), also (42) can be transformed into a set LMI, to be used both for analysis and synthesis problems, in order to find the Lyapunov functions P_i, the feedback gains K_i and the constants ρ_i and γ_i. An optimal criterion for solving the LMI could be the one of maximizing ρ_i, while minimizing γ_i, so as to maximize the decaying rate of the regulation error and minimize the effect of the disturbance term. □

The performance signals (25) must be modified as well, to take into account the additional term. A natural choice for the performance signal is:

$$\Lambda_i(t) = \max\left\{2x^T(t)P_i\dot{x}^i(t) + \rho_i x^T(t)P_i x(t) + y_f^T(t)y_f(t), 0\right\} \quad (44)$$

$$\mathcal{J}_i(t) = \max_{0\leq q\leq t}\left\{\left(\frac{\Lambda_i(q)}{\gamma_i}\right)^{1/2}\right\}. \quad (45)$$

Note that, even in the presence of disturbances, $\dot{x}^i(t)$ can be calculated from Eq. (21).

The stability properties of the resulting Lyapunov-based AMC are:

Theorem 3: Let the uncertain plant (1) be affected by bounded input and/or output disturbances. Consider the bank of parallel estimators (23)-(24),(4)-(5). Consider the adaptive mixing controller with the multicontroller $\mathbb{C}(\beta)$ given by (9),(10) and (17) and satisfying assumption L2; if the multicontroller is driven by the mixing strategy $\beta(\theta_{\sigma(\cdot)})$, with $\beta(\cdot)$ given in (18) and the index $\sigma(\cdot)$ selected according to the switching logic (27), (44) and (45), then the following hold:

1) (*Final switching time*) There is a final switching time t^* for the index σ and $\mathcal{J}_{\sigma(t^*)} < h + d_0$, where d_0 is the bound for d defined after (41).
2) (*Transient performance before final switching time*) Before the final switching time $\mathcal{J}_{\sigma(t)} < h + d_0$, which guarantees that there are at most $N\lceil d_0/h\rceil$ switches before the final switching time. Furthermore, if $h = d_0$, there are at most N switches before the final switching time, and whenever the (desired) controller whose index i satisfies $\theta^* \in \Omega_i$ is inserted in the loop, it will never be switched-off thereafter.
3) (*Steady-state performance after final switching time*) The final switched-on controller, namely $\mathbb{C}(\beta(\theta_{\sigma(t^*)}))$ guarantees

$$|y(t-t^*)| \leq \alpha_1 e^{-\frac{\rho_{\sigma(t^*)}}{2}(t-t^*)}|y(t^*)| + \underline{h}, \quad \forall\, t \geq t^*, \quad (46)$$

where $\alpha_1 = c_\lambda\left(\lambda_{max}(P_{\sigma(t^*)})/\lambda_{min}(P_{\sigma(t^*)})\right)^{1/2}$, and $\underline{h} = \mathcal{O}(\gamma_{\sigma(t^*)}(h+d_0)^2)$.
4) Finally

$$\int_{t^*}^\infty \left[y_f^T(\tau)y_f(\tau) - \gamma_{\sigma(t^*)}d_0^T d_0\right] \mathrm{d}\tau \leq \gamma_{\sigma(t^*)}\mathcal{O}(h^2). \quad (47)$$

Proof: See the Appendix. ■

Similarly to traditional adaptive control schemes with and without switching, also Theorems 2 and 3 do not guarantee that when adaptation is switched off the resulting LTI system is stable. The reason that in theory the closed loop system may settle at an unstable equilibrium point for some appropriate initial conditions and remain at that point till it is externally disturbed. Consider the example $\dot{x} = \mathcal{A}x$ with $x(t_0) = 0$; then $x(t) = 0$ for all $t \geq t_0$, $\mathcal{J}_i = 0$, $\forall i$, and the final switching time is $t^* = 0$, no matter what $\mathcal{A}$ is. Proving analytically such a scenario is difficult if at all possible. Since analysis cannot exclude it, simulation-based evaluations can be used to investigate whether it is possible for the adaptive scheme to settle to an unstable controller when the adaptive part is switched off. In the following section we use a numerical example to evaluate this scenario.

VIII. Numerical Examples

1) Example 1: A simple numerical example is presented to show the effectiveness of the proposed Lyapunov-based AMC scheme. Consider the first-order uncertain system

$$y(t) = G_0(s)u(t) = \frac{1}{s-\theta_1^*}u(t), \quad (48)$$

where θ_1^* is an uncertain parameter belonging to the interval

$$\Omega = \{\theta_1 :\ 1 \leq \theta_1 \leq 2.5\}. \quad (49)$$

The system (48) can be written in the streamlined notation

$$\dot{y}(t) = \theta_1^* y(t) + u(t), \quad y(0) = y_0, \quad (50)$$

where $y(0)$ is the initial condition of the plant. Despite the fact that for this first-order uncertain system the state $y(t)$ is completely measurable, we adopt the transformation (8) and the controller implementation (15) for tutorial purposes. Taking $\Lambda(s) = s + 1$, the system (48) is associated with

$$\begin{bmatrix} \dot{y}_f(t) \\ \dot{u}_f(t) \end{bmatrix} = \begin{bmatrix} \theta_1^* & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_f(t) \\ u_f(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \dot{u}_f(t), \quad (51)$$

where

$$\dot{y}_f(t) = -y_f(t) + y(t), \quad y_f(0) = y_{f_0}, \quad (52)$$
$$\dot{u}_f(t) = -u_f(t) + u(t), \quad u_f(0) = u_{f_0}, \quad (53)$$

and $y_f(0)$, $u_f(0)$ are the initial states of the filter $1/\Lambda(s)$, which are assumed to be known to the designer. It is also assumed that the state of the filter can be completely measured. Define the state $x(t) = [y_f(t) \;\; u_f(t)]^T$. Note that $x(0)$ is different than the initial condition $y(0)$ of the plant: in particular, we have $\dot{y}_f(0) = -y_{f_0} + y_0$ and $\dot{u}_f(0) = -u_{f_0} + u_0$.

The uncertainty set Ω is divided into three subsets and three output-feedback candidate controllers in the form

$$u(t) = -C_i(s)y(t) = -\frac{k_{i1}}{s + k_{i2}} y(t), \quad (54)$$

are designed for the nominal values $\theta_1 = 1,\ 1.8,\ 2.4061$. The controllers have been designed to place the closed loop eigenvalues of the three nominal feedback control loops to the roots of the polynomial $s^2 + 2s + 1.5$. The candidate controller implementation (15) is adopted, *i.e.*

$$\dot{u}_f(t) = -K_i x(t) = -[k_{i1} \;\; k_{i2}] \begin{bmatrix} y_f(t) \\ u_f(t) \end{bmatrix}, \quad (55)$$

The actual input to be applied is thus $u(t) = \dot{u}_f(t) + u_f(t)$. The candidate controllers, as well as the three subsets Ω_i, are reported in Table I. The subsets Ω_i have been found by taking into account the stability intervals of each candidate controller. The last column of Table I reports the nominal values of the uncertain parameter vector for which the candidate controller K_i is marginally stable, that is, if the controller K_1 is placed in feedback with the plant corresponding to $\theta_1^* = 1.8$ (belonging to Ω_2), the resulting feedback-loop is marginally stable. Analogously, if the controller K_2 is placed in feedback with the plant corresponding to $\theta_1^* = 2.4061$ (belonging to Ω_3), the resulting feedback-loop is marginally stable. For each subset Ω_i, an adaptive law estimating the unknown parameter θ_1^* is developed as in (2)-(5).

K_i	Ω_i	Marg. Stab. for θ_1^*
$[4.5 \;\; 2.5]$	$\theta_1 \in [1,\ 1.5]$	1.8
$[7.94 \;\; 3.3]$	$\theta_1 \in [1.4,\ 2.1]$	2.4061
$[11.3982 \;\; 3.9061]$	$\theta_1 \in [2.0,\ 2.5]$	

TABLE I
CONTROLLER COEFFICIENTS

Given the parameter subsets Ω_i, the mixer can be constructed on the basis of any Lipschitz function $\varphi(x)$, that is greater than zero on a compact set and zero elsewhere. For this simple example, a function that satisfies this requirement is the smooth bump function

$$\varphi(x) = \begin{cases} e^{-\frac{1}{1-x^2}} & \text{if } |x| < 1 \\ 0 & \text{otherwise} \end{cases} \quad (56)$$

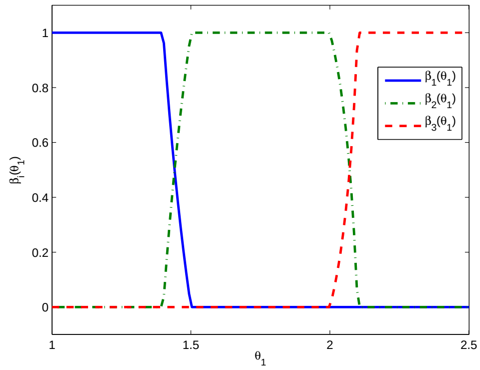


Fig. 2. Mixing strategy: $\beta_1(\theta_1)$ (solid), $\beta_2(\theta_1)$ (dash-dotted), $\beta_3(\theta_1)$ (dashed)

Other functions can be used, *e.g.*, the trapezoidal function or sinusoidal functions. Consider the pre-normalized weights, $\tilde{\beta}_i(\theta_1) = \varphi\left((2\theta_1 - U_i - L_i) / (U_i - L_i)\right)$, $i = 1, 2, 3$, where U_i, L_i are the upper and lower bounds, respectively, of the subset $\Omega_i = \{\theta_1 : L_i \leq \theta_1 \leq U_i\}$. The mixing signal $\beta(\theta_1)$ is generated by normalizing $\tilde{\beta} = [\tilde{\beta}_1 \ldots \tilde{\beta}_3]'$, *i.e.*, $i = 1, 2, 3$

$$\beta_i(\theta_1) = \frac{\tilde{\beta}_i(\theta_1)}{\sum_{j=1}^{3} \tilde{\beta}_j(\theta_1)} \quad (57)$$

The mixing function derived from the described procedure, using the bump function (56), is shown in Figure 2.

The resulting multicontroller constructed using gain interpolation (17) has been verified to satisfy assumption C1. The Lyapunov-based AMC (Lyap-AMC) is compared both with the standard AMC scheme employing one single estimator and with a switching scheme, namely Unfalsified Adaptive Switching Control (UASC) scheme [34], [3]. The designed variables used for the Lyap-AMC scheme are: $\Gamma = 5$, $\delta_0 = 0.2$, $h = 0.001$. Using the LMI (33), condition L1 is verified with

$$\begin{array}{ll} P_1 = \begin{bmatrix} 0.871 & 0.329 \\ 0.329 & 0.161 \end{bmatrix}, & P_2 = \begin{bmatrix} 0.899 & 0.296 \\ 0.296 & 0.129 \end{bmatrix} \\ P_3 = \begin{bmatrix} 0.916 & 0.947 \\ 0.947 & 0.219 \end{bmatrix}, & \begin{array}{l} \rho_1 = 0.6 \quad \rho_2 = 0.13 \\ \rho_3 = 1.2 \end{array} \end{array} \quad (58)$$

The hysteresis constant used for the UASC scheme is 0.001.

The three schemes are simulated for two values of the uncertain parameter $\theta_1^* = 1.8,\ 2.4061$, with plant initial conditions $y_0 = \mathcal{U}[\pm 0.1]$. The notation $\mathcal{U}[\pm a]$ stands for a random uniform distribution on the interval $[-a, a]$. The initial state of the filter $1/\Lambda(s)$ is also selected randomly, in particular $[y_{f_0}\ u_{f_0}]^T = [\mathcal{U}[\pm 0.1]\ \mathcal{U}[\pm 0.1]]^T$. For each experiment we run 100 Monte-Carlo simulations: the results of the simulations for the three adaptive schemes are shown in Table II. Each simulation has a time-length of $25s$. Every time the schemes are initialized with the candidate controller giving a marginally stable feedback loop. The stability of the frozen feedback control system after $25s$ is recorded. The last three columns of Table II show how many times (out of 100) the final switched-on controller was the marginally stable one.

θ_1^*	Initial controller	UASC	AMC	Lyap-AMC
1.8	K_1	9 of 100	11 of 100	0 of 100
2.4061	K_2	15 of 100	18 of 100	0 of 100

TABLE II
SIMULATION RESULTS

While, for some initial conditions, both the AMC and UASC scheme may keep for a long time the marginally stable controller, the Lyap-AMC scheme switches off the marginally stable controller and the switching signal rapidly converges to the appropriate desired controller. The reason why the standard AMC keeps in the loop the initially marginally stable controller, can be seen from Figure 3(a): for some initial conditions, the parameter estimate of the single estimator takes longer time to converge close to $\theta_1^* = 1.8$, so that the initial marginally stable controller is kept in the loop for a longer time (Figure 3(b)). The reason for the good behavior of Lyap-AMC can be explained by comparing the UASC performance signals of Figure 4(a), with the Lyap-AMC performance signals of Figure 4(b) for one experiment among the 100 experiments performed with $\theta_1^* = 1.8$. Here $\mathcal{J}_2$, the dotted performance signal, is the one corresponding to the most appropriate controller. While the UASC algorithm might take some time before discriminating with sufficient accuracy the performance signals (eventually keeping in the loop the initial marginally stable controller), the Lyap-AMC rapidly detects the controller satisfying the Lyapunov inequality (implying desired stability margin) and discards the others.

2) Example 2: This second numerical example presents the effect of disturbances on the proposed architecture. The example consists of a mass coupled with the wall via a spring and a damping, as depicted in Figure 5. The objective is to keep the mass at a constant position in spite of disturbances acting on the wall. Despite its apparent simplicity, such dynamics lie behind many practical problems like active suspension systems in cars, vibration reduction in platforms, mechanical structures and other smart flexible structures. Assuming without loss of generality a unitary mass, the equation of motion can be described by

$$\ddot{y}(t) = -k^*(y(t) - d(t)) - c\dot{y}(t) + u(t) \tag{59}$$

with initial conditions $\dot{y}(0)$, $y(0)$, where y is the position of the mass, u is the force control input, d is the disturbance, k is the spring stiffness and c is the damping coefficient. In this example we take $c = 0.2$ and assume that the spring stiffness k^* is uncertain and belongs to the uncertainty set

$$\Omega = \{k : \ 0.08 \le k \le 1.0\} . \tag{60}$$

The uncertainty set is divided into 7 subsets and for each subset an $\mathcal{H}_\infty$ controller has been designed according to the mixed-sensitivity criterion $\min_{d \in \mathcal{L}_2} \frac{\|y\|_2 + \|u\|_2}{\|d\|_2}$. The controllers have the output-feedback form

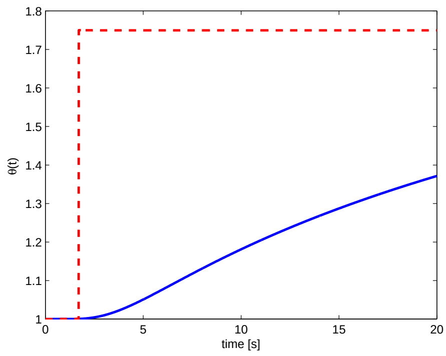
$$u = -\frac{r_1 s + r_0}{s^2 + s_1 s + s_0} y. \tag{61}$$

Both the uncertainty subsets Ω_i and the coefficients of the controllers $K_i = [r_{1_i} \ r_{0_i} \ s_{1_i} \ s_{0_i}]$ are indicated in Table III. Similarly to the first example, the candidate controller implementation described in Sect. III is adopted with $\Lambda(s) = s^2 + \sqrt{2}s + 1$,

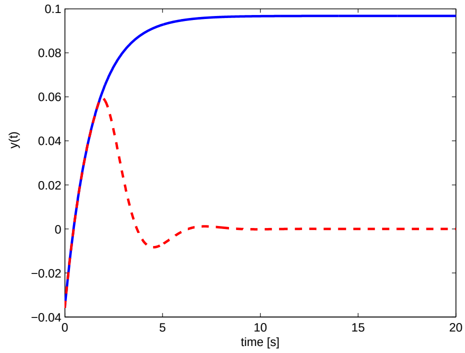
$$\begin{bmatrix} \ddot{y}_f(t) \\ \dot{y}_f(t) \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{y}_f(t) \\ y_f(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} y(t), \tag{62}$$

$$\begin{bmatrix} \ddot{u}_f(t) \\ \dot{u}_f(t) \end{bmatrix} = \begin{bmatrix} -\sqrt{2} & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \dot{u}_f(t) \\ u_f(t) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \tag{63}$$

and $\ddot{u}_f = -K_i x(t)$, with $x(t) = [\dot{y}_f(t) \ y_f(t) \ \dot{u}_f(t) \ u_f(t)]$. Due to the disturbance rejection nature of the control problem, a set of LMIs arising from condition L2 has been solved in order to find the family of Lyapunov function, decaying and attenuation rates. For lack of space, only the last two quantities are reported in Table IV.



(a) Estimate $\theta_1(t)$: standard AMC (solid), Lyap-AMC (dashed)

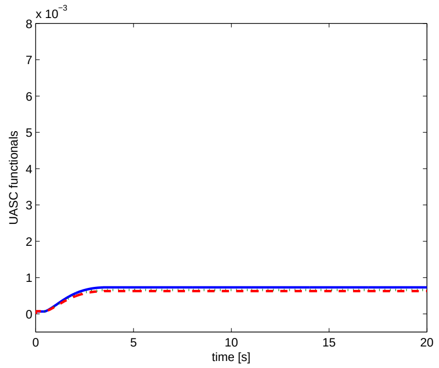


(b) Regulation of $y(t)$: standard AMC (solid), Lyap-AMC (dashed)

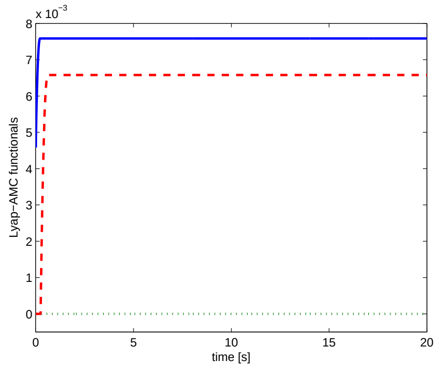
Fig. 3. $\theta_1^* = 1.8$: Estimate and regulation tasks for standard AMC and Lyap-AMC

K_i	Ω_i
[9604.8 3633.8 51.9 1278.1]	$k \in [0.08, \ 0.12]$
[1894.3 734.0 29.7 366.3]	$k \in [0.11, \ 0.18]$
[23627.9 4434.8 116.9 2363.5]	$k \in [0.16, \ 0.29]$
[6241.0 938.7 43.9 841.9]	$k \in [0.25, \ 0.4]$
[5041.7 574.1 43.8 930.2]	$k \in [0.35, \ 0.57]$
[3900.7 444.7 47.7 1129.2]	$k \in [0.5, \ 0.75]$
[3944.9 375.2 54.0 1455.0]	$k \in [0.65, \ 1]$

TABLE III
CONTROLLER COEFFICIENTS



(a) UASC func.: $\mathcal{J}_1$ (solid), $\mathcal{J}_2$ (dotted), $\mathcal{J}_3$ (dashed)



(b) Lyap-AMC func.: $\mathcal{J}_1$ (solid), $\mathcal{J}_2$ (dotted), $\mathcal{J}_3$ (dashed)

Fig. 4. $\theta_1^* = 1.8$: Performance signals for UASC and Lyap-AMC

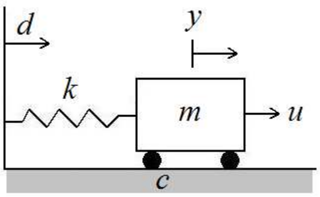
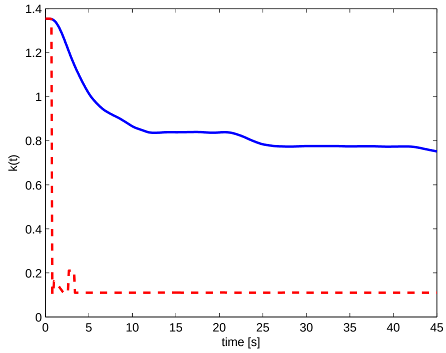


Fig. 5. Mass-spring system

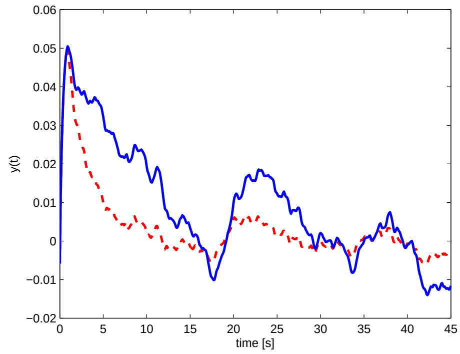
	# 1	# 2	# 3	# 4	# 5	# 6	# 7
ρ_i	0.08	0.10	0.12	0.11	0.18	0.17	0.20
γ_i	0.05	0.08	0.12	0.20	0.32	0.46	0.63

TABLE IV
DECAYING AND ATTENUATION RATES

Figure 6 shows, for $k^* = 0.2$ and for the initial condition $\dot{y}(0) = 0.05$, $y(0) = -0.007$, the output estimate and regulation task of the standard AMC as compared with the proposed Lyap-AMC. Similar results can be found for different initial conditions. The switching logic leads to a faster adaptation of $k(t)$, thus resulting in superior transient and attenuation performance. A random uniform disturbance $d(t)$ between -0.1 and 0.1 has been chosen ($d_0 = 0.1$). In Figure 7 the performance signals of the proposed Lyap-AMC are plotted: in a dashed line, the signal $\mathcal{J}_3$ associated with the desired controller is shown. Theorem 3 guarantees such signal to be below $h+d_0$. Despite the fact that, for the chosen disturbance, the bound $h+d_0$ results conservative, since all the performance signals are below 0.1, Figure 7 shows that, after a short transient, the performance signal $\mathcal{J}_3$ is the smallest, so that the desired controller is finally selected.



(a) Estimate $k(t)$: standard AMC (solid), Lyap-AMC (dashed)



(b) Regulation of $y(t)$: standard AMC (solid), Lyap-AMC (dashed)

Fig. 6. $k^* = 0.2$: Estimate and regulation tasks for standard AMC and Lyap-AMC

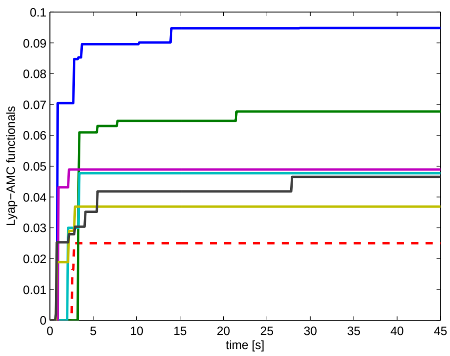


Fig. 7. $k^* = 0.2$: Performance signals for Lyap-AMC: in a dashed line the signal $\mathcal{J}_3$ associated with the desired controller is shown

IX. CONCLUSIONS

We developed an adaptive scheme based on mixing and multiple estimators that guarantees that the closed loop system converges in finite time to a controller that satisfies a Lyapunov inequality implying a prescribed stability margin. Simulation results demonstrate that, in contrast to most popular adaptive control techniques and structures, the proposed approach converges to a stable closed loop system even if adaptation is switched off at steady state.

APPENDIX: PROOF OF THEOREM 2

1) As a first step of the proof it is shown that assumptions A1 and A2 of the hysteresis switching lemma hold. Consider the switched system given by the parallel estimates $\theta_1, \ldots, \theta_N$ and the switching logic (27) selecting the index $\sigma(\cdot)$ of $\theta_{\sigma(\cdot)}$. In addition, consider the performance signals (25)-(26). It can be shown that the max operator in (26) guarantees the existence of a limit of $\mathcal{J}_i$, $i \in \mathcal{I}$ for every possible switching sequence $\sigma(\cdot) \in \mathcal{S}$ (the max operator makes $\mathcal{J}_i$ monotonically non decreasing, so that the limit exists, in case the performance signal $\mathcal{J}_i$ is bounded, or that the limit goes to infinity, in case the performance signal $\mathcal{J}_i$ grows unbounded). Now we must show that there exists at least one index $\mu \in \mathcal{I}$ such that $\mathcal{J}_\mu$ is bounded for every possible switching sequence $\sigma(\cdot) \in \mathcal{S}$. Note that, inequality (22) is satisfied for some index $j \in \mathcal{I}$, then the following inequality holds for $t \geq 0$:

$$2x^T(t)P_j\dot{x}^j(t) + \rho_j x^T(t)P_j x(t) \leq 0 \tag{64}$$

The design criterion L1 guarantees that there exists always at least one index $\mu \in \mathcal{I}$ such that (64) holds and thus $\mathcal{J}_\mu$ in (26) is bounded for every possible switching sequence. So the HSL holds, there is a finite switching time t^* and $\mathcal{J}_{\sigma(t^*)}(\cdot)$ is bounded. Because of the fact that there exists always at least one index $\mu \in \mathcal{I}$ such that (64) holds, then $\mathcal{J}_{\sigma(t^*)}(\cdot)$ cannot be greater than the hysteresis constant h.

2) The hysteresis switching logic (27) together with the fact that there exists always at least one index $\mu \in \mathcal{I}$ such that (64) holds guarantees $\mathcal{J}_{\sigma(t)}(t) < h$, $0 \leq t \leq t^*$. In fact, if $\theta^* \in \Omega_i$ then $\mathcal{J}_i(t) = 0$ $\forall t \geq 0$. So, if the index i is switched-on, it will never be switched-off thereafter. If another index j is switched-on, such an index will stay in the loop till $\lim_{\tau \to t} \mathcal{J}_j(t) = h$. As soon as the last equality is verified another index k satisfying $\mathcal{J}_k(t) < h$ will be switched-on. The new index can be $k = i$ or $k \neq i$; in any case, $\mathcal{J}_{\sigma(t)}(t) < h$, $\forall t \geq 0$. We also notice that whenever a controller is switched on twice in feedback to the plant, its performance signal grows at least by h. We conclude that every candidate controller is switched on at most once and that there are at most N switches before the final switching time. Call $t_1, t_2, \ldots, t^*$ the instants at which a switching occurs: then, between two switching instants we have

$$\max_{t_i \leq t \leq t_{i+1}} \Big\{ 2x^T(t)P_{\sigma(t_i)}\dot{x}^{\sigma(t_i)}(t) + \rho_{\sigma(t_i)} x^T(t)P_{\sigma(t_i)}x(t) - h \Big\} < 0 \tag{65}$$

which implies

$$|x(t - t_i)| \leq \kappa_{\sigma(t_i)} e^{-\frac{\rho_{\sigma(t_i)}}{2}(t-t_i)} |x(t_i)| + \underline{h}, \quad t_i \leq t \leq_{i+1},$$

where $\kappa_{\sigma(t_i)} = \big(\lambda_{max}(P_{\sigma(t_i)})/\lambda_{min}(P_{\sigma(t_i)})\big)^{1/2}$ and $\underline{h} = \mathcal{O}(h)$. By observing that $u(t) = C_1^T\dot{x}(t) + d_{11}x(t)$, $y(t) = C_2^T\dot{x}(t) + d_{21}x(t)$, with C_1, C_2, d_{11}, d_{21} depending of the coefficients of $\Lambda(s)$, (28) follows.

3) After the final switching time t^*, $\mathcal{J}_{\sigma(t^*)}(\cdot) < h$ which is equivalent to

$$\max_{t \geq t^*} \Big\{ 2x^T(t)P_{\sigma(t^*)}\dot{x}^{\sigma(t^*)}(t) + \rho_{\sigma(t^*)} x^T(t)P_{\sigma(t^*)}x(t) - h \Big\} < 0 \tag{66}$$

which implies (29). In order to establish the convergence of y in the case that $\theta^* \in \Omega_{\sigma(t^*)}$ we proceed as follows: after the final switching time t^* we can can show that the following two equations hold [9, Thm. 7.4.1]

$$\begin{aligned} Q_{\sigma(t^*)}y_\Lambda + L_{\sigma(t^*)}u_\Lambda &= 0, \\ \hat{R}_p y_\Lambda - \hat{Z}_p u_\Lambda &= \epsilon_{\sigma(t^*)}m_s^2 \end{aligned} \tag{67}$$

where $u_\Lambda = \frac{1}{\Lambda_p}u$, $y_\Lambda = \frac{1}{\Lambda_p}y$, $Q_{\sigma(t^*)}$ and $L_{\sigma(t^*)}$ are the numerator and the denominator of the final switched-on controller and $\hat{R}_p = s^n + \theta_a^T\alpha_{n-1}(s)$, $\hat{Z}_p = \theta_b^T\alpha_m(s)$, where $\theta = [\theta_a^T \; \theta_b^T]^T$ are the estimated parameters. Then, Eq. (67) can be rearranged as

$$\dot{x}(t) = A(t)x(t) + b_1(t)\epsilon_{\sigma(t^*)}m_s^2(t) \tag{68}$$

$$u(t) = C_1^T\dot{x}(t) + d_{11}x(t) \tag{69}$$

$$y(t) = C_2^T\dot{x}(t) + d_{21}x(t) \tag{70}$$

where $A(t)$ is a time-varying matrix whose determinant, for each frozen time t, is equal to

$$\det(sI - A(t)) = \hat{R}_p\hat{L} + \hat{Q}\hat{Z}_p = A^*(s,t) \tag{71}$$

where $A^*(z,k)$ is the characteristic polynomial of the closed-loop formed by the estimated plant and the controller. Thanks to C1, $A^*(z,k)$ is Hurwitz at each frozen time k, so that $A(k)$ has stable eigenvalues at each frozen time k. Using a similar procedure as in [9, Sect. 7.7.1], we can establish that

$$\|(y)_t\|_{2\delta} \leq c\left\|(\epsilon_{\sigma(t^*)}m_s^2)_k\right\|_{2\delta} + c \tag{72}$$

for $0 < \delta < 2\lambda_0$, where $\lambda_0 < 0$ is the exponential convergence rate of the homogeneous part of (68). Defining the fictitious signal $m_f^2(k) \triangleq 1 + \phi^T(k)\phi(k) + \|(y)_t\|_{2\delta}^2 + \|(u)_t\|_{2\delta}^2$ and applying the Bellman-Gronwall Lemma [9, Lemma 3.3.7] to

$$m_f^2 \leq c + c\left\|(\epsilon_{\sigma(t^*)}m_s m_f)_t\right\|_{2\delta}^2 \tag{73}$$

we establish $m_f \in l_\infty$. Using the boundedness of m_f, we can establish the boundedness of all closed-loop signals, *i.e.*, ϕ, u, $y \in \mathcal{L}_\infty$. Finally, applying [9, Lemma 3.3.3] to (68), we observe that if the input $\epsilon_{\sigma(t^*)}m_s^2 \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, which is guaranteed by the adaptive law, then y will be in $\mathcal{L}_2$ and in addition $y \to 0$ as $t \to \infty$.

APPENDIX: PROOF OF THEOREM 3

The proof proceeds by following similar steps as in the proof of Theorem 2. It is first shown that assumptions A1 and A2 of the hysteresis switching lemma hold: this happens because L2 guarantees that there exists an index $j \in \mathcal{I}$ such that

$$\left(\frac{2x^T(t)P_j\dot{x}^j(t)+\rho_j x^T(t)P_j x(t)+y_f^T(t)y_f(t)}{\gamma_i}\right)^{1/2} \le d_0$$

and thus there exists at least one index $\mu \in \mathcal{I}$ such that $\mathcal{J}_\mu$ in (45) is bounded for every possible switching sequence $\sigma(\cdot) \in \mathcal{S}$. The hysteresis switching logic (27) guarantees that whenever a controller i is switched on twice in feedback to the plant, its performance signal $\mathcal{J}_i$ has grown at least by h. This implies that there are at most $N\lceil d_0/h \rceil$ switches before the final switching time. Consequently, if $h = d_0$, there are at most N switches before the final switching time. In addition, if $\theta^* \in \Omega_i$ then $\mathcal{J}_i(t) < d_0 \ \forall t \ge 0$. So, if $h = d_0$ and the index i is switched-on, it will never be switched-off thereafter. Finally, after the final switching time, the following condition, deriving from $\mathcal{J}_{\sigma(t^*)}(\cdot) < h$

$$\max_{t\ge t^*}\left\{\left(2x^T(t)P_{\sigma(t^*)}\dot{x}^{\sigma(t^*)}(t)+\rho_{\sigma(t^*)}x^T(t)P_{\sigma(t^*)}x(t) + y_f^T(t)y_f(t)\right)^{1/2}\gamma_{\sigma(t^*)}^{-1/2}-d_0-h\right\}<0 \quad (74)$$

guarantees (47).

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