BANACH FUNCTION SPACES

PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAP AAN DE TECHNISCHE HOGESCHOOL TE DELFT, KRACHTENS ARTIKEL 2 VAN HET KONINKLIJK BESLUIT VAN 16 SEPTEMBER 1927, STAATSBLAD NR. 310 EN OP GEZAG VAN DE RECTOR MAGNIFICUS DR. O. BOTTEMA, HOOGLERAAR IN DE AFDELING DER ALGEMENE WETENSCHAPPEN, VOOR EEN COMMISSIE UIT DE SENAAT TE VERDEDIGEN OP WOENSDAG 12 OCTOBER 1955, DES NAMIDDAGS TE 4 UUR, DOOR

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GEBOREN TE DELFT

TE ASSEN BIJ
VAN GORCUM & COMP. N.V. - G. A. HAK & DR. H. J. PRAKKE

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INTRODUCTION

The first chapter of this thesis is a contribution to the systematic investigation of Banach spaces the elements of which are functions. These spaces include as special cases such well-known examples as the L_p and l_p spaces $(1 \leqslant p \leqslant \infty)$ and the Orlicz spaces. Although in the older literature on Banach spaces, one may find some theorems on this subject (cf. e.g. S. Banach [1], p. 86 *)), the first serious attempts in this direction date from 1953, and were made by H. W. Ellis – I. Halperin [1] and G. G. Lorentz [3; 4]. The work of these mathematicians has had a great influence on the form and contents of Chapter 1. We consider a normed linear space X of complex functions $f(x)$ having an abstract set Δ as their domain, and we assume that these functions are measurable with respect to a totally σ-finite measure μ which is defined on a σ-ring of subsets of Δ. Furthermore, we assume that the norm $\|f\|_X$ in X satisfies, besides the usual norm conditions, some additional conditions which are immediate generalizations of well-known properties of L_p norms, and which have their origin in the theory of Lebesgue integration. These additional conditions are:

(1) The norm $\|f\|_X$ is defined for all μ-measurable complex $f(x)$ on Δ (but it may be $+\infty$ for some f), and $f \in X$ if and only if $\|f\|_X < \infty$.

(2) If $f_n(x) \geqslant 0$ $(n = 1,2, \ldots)$ and $f_n \uparrow f$ almost everywhere on Δ, then $\|f_n\|_X \uparrow \|f\|_X$.

Finally, some hypotheses of minor importance are added in order to guarantee that X contains a sufficient number of functions. As a consequence of (2) an analogue of Fatou's Lemma holds, i.e. if $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on Δ, then $\|f\|_X \leqslant \liminf \|f_n\|_X$.

In Ch. 1, section 1, we first prove the well-known fact that (1) and (2) together imply that the normed linear space X is complete, i.e. X is a Banach function space. Furthermore, we introduce the

*) Numbers in square brackets refer to the references cited at the end of the thesis.

"associate space" X' of X consisting of all μ-measurable $g(x)$ such that $\|g\|_{X'} = \sup_{\|f\|_X \leqslant 1} \int_\Delta |fg| d\mu$ is finite. It turns out that X' (with the norm $\|g\|_{X'}$) is also a Banach function space having the same properties as X. It is then a natural question to ask whether the associate space X'' of X' is again the original space X, and we prove that the answer is affirmative. The analogue of Fatou's Lemma plays an essential part in the proof of this result. In section 2 the subspace $X^\varkappa$ of all $f(x)$ having an absolutely continuous norm (roughly: if $\mu(E) \to 0$ and χ_E is the characteristic function of E, then $\|f\chi_E\| \to 0$) is introduced, and one of the theorems proved is that X is reflexive if and only if $X = X^\varkappa$ and $X' = (X')^\varkappa$. Section 3 is devoted to a discussion of several weak topologies which may be defined on X or X', and here interesting analogies and differences with the work of G. Köthe - O. Toeplitz [1] and J. Dieudonné [2] become visible. Furthermore we prove in this section that X is separable (in the norm topology) if and only if $X = X^\varkappa$ and the measure μ is separable.

In Chapter 2 we develop the theory of Orlicz spaces. These spaces are interesting not only since they give us a non-trivial example of the general theory in Ch. 1, but also since they are modulared spaces in the sense of H. Nakano, so that the characteristic features of two abstract theories are blended. In the discussion a certain non-negative, non-decreasing convex function $\Psi(x)$ plays an important part, and peculiar difficulties arise whenever this function jumps to infinity (i.e. whenever there exists a finite x_0 such that $\Psi(x) < \infty$ for $x < x_0$ and $\Psi(x) = \infty$ for $x > x_0$), so that it is a pleasant surprise that many results remain the same whether $\Psi(x)$ jumps or not. It seems that the results obtained in this thesis may be of some value for future investigations on Fourier series and integral equations.

CHAPTER I

GENERAL BANACH FUNCTION SPACES

1. Banach Function Spaces and their Associate Spaces.

In the following pages we shall assume, unless otherwise stated, that the point set Δ, the σ-ring Λ of subsets E of Δ (the "measurable" subsets of Δ) and the countably additive measure μ (defined for all $E \in \Lambda$, and satisfying $0 \leq \mu(E) \leq \infty$) form a measure space (in P.R. Halmos' terminology [1]). Moreover, we shall make the assumption that the measure μ is complete and totally σ-finite, i.e. $\mu(E) = 0$ implies $F \in \Lambda$ for any set $F \subset E$, and Δ is the union of a countable collection of sets of finite positive measure. We now choose for once and all a fixed increasing sequence $\Delta_n \in \Lambda$ ($n = 1, 2, \ldots$) of sets of finite positive measure such that Δ_n converges to Δ. For reasons of convenience, and analogous to the case of Lebesgue measure in Euclidean space, we shall call any μ-measurable set E, satisfying $E \subset \Delta_k$ for some k, a "bounded" set. Furthermore, $\int$ will always mean $\int_\Delta$.

Let P be the collection of all μ-measurable non-negative functions $f(x)$ defined on Δ, and let $\varrho(f)$ be a metric function $(0 \leq \varrho(f) \leq \infty)$ on P with the properties:

(P 1) $\varrho(f) = 0$ if and only if $f(x) = 0$ a.e. on Δ; $\varrho(f_1+f_2) \leq \varrho(f_1)+\varrho(f_2)$, and $\varrho(af) = a\varrho(f)$ for any constant $a \geqslant 0$.

(P 2) If $f_n(x) \in P$ $(n=1, 2, \ldots)$, and $f_n(x) \uparrow f(x)$ a.e. on Δ, then $\varrho(f_n) \uparrow \varrho(f)$.

(P 3) If E is any bounded set, and $\chi_E(x)$ is its characteristic function, then $\varrho(\chi_E) < \infty$.

(P 4) For every bounded set E there exists a finite constant $A_E \geqslant 0$ (depending only on the set E) such that $\int_E f\,d\mu \leqslant A_E\,\varrho(f)$ for every $f(x) \in P$.

Since for any μ-measurable complex function $f(x)$ on Δ the function $|f(x)|$ is also μ-measurable on Δ, the metric function $\|f\| = \varrho(|f|)$ is a legitimate extension for complex $f(x)$ of the above $\varrho(f)$, and this leads to the following definition:

Definition 1. *By $X = X(\Delta,\mu)$ we denote the set of all μ-measurable complex functions $f(x)$ on Δ for which $\|f\|_X = \|f\| = \varrho(|f|)$ is finite.*

As an immediate consequence of properties (P 1)–(P 4) of ϱ and Def. 1, we obtain:

Lemma 1. (a) *$\|f\|_X = 0$ is equivalent to $f(x) = 0$ a.e. on Δ.*

(b) *If $|f_1(x)| \leqslant |f_2(x)|$ a.e. on Δ, then $f_2 \in X$ implies $f_1 \in X$, and $\|f_1\|_X \leqslant \|f_2\|_X$.*

(c) *If $f \in X$, then $f(x)$ is finite a.e. on Δ.*

(d) *If $f_n \in X$ $(n = 1,2,\ldots)$, $f_n(x) \geqslant 0$, and $f_n(x) \uparrow f(x)$ a.e. on Δ, then either $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$ or $\|f_n\|_X \uparrow \infty$.*

(e) *If $f_n \in X$ $(n = 1,2,\ldots)$ and $\lim f_n(x) = f(x)$ a.e. on Δ, then $\|f\|_X \leqslant \liminf \|f_n\|_X$ ("Fatou's Lemma").*

(f) *If E is any bounded set, and $\chi_E(x)$ is its characteristic function, then $\|\chi_E\|_X < \infty$.*

(g) *For every bounded set E there exist a finite constant $A_E \geqslant 0$ (depending only on the set E) such that $\int_E |f| d\mu \leqslant A_E \|f\|_X$ for every $f \in X$.*

Proof. Only (c) and (e) need a proof. In order to prove (c), observe that, if $f \in X$, then $\|f\|_X < \infty$. Let E be the set on which $|f(x)|$ is infinite, and suppose that $\mu(E) > 0$, hence $\|\chi_E\|_X > 0$. Then it follows from (a) and (b) that $\|f\|_X \geqslant n\, \|\chi_E\|_X$ for $n = 1, 2, \ldots$, hence $\|f\|_X = \infty$, wich contradicts $f \in X$. For the proof of (e), write $h_n(x) = \inf (|f_n(x)|, |f_{n+1}(x)|, \ldots)$. Then $0 \leq h_n(x) \uparrow |f(x)|$ a.e. on Δ, hence $\|f\|_X = \lim \|h_n\|_X \leq \liminf \|f_n\|_X$ by (P_2) and (b).

The set X as introduced in Def. 1 is obviously a normed linear space with norm $\|f\|_X$, if we identify functions which are equal a.e. on Δ. The completeness of X (i.e. $\|f_m - f_n\|_X \to 0$ as $m, n \to \infty$ implies the existence of an element $f \in X$ such that $\|f - f_n\|_X \to 0$ as $n \to \infty$) will be shown now by a variation of a well-known argument which is originally due to J. von Neumann ([1], p. 111) and H. Weyl, and which was used by several other authors for similar purposes (see e.g. H. W. Ellis – I. Halperin [1], Th. 3.1, p. 579).

Theorem 1. *The space X is a Banach space.*

Proof. We have only to show that X is complete, i.e. given a sequence $f_n \in X$ such that $\|f_m - f_n\| \to 0$ as $m, n \to \infty$, we have to prove that there exists a unique $f \in X$ such that $\|f_n - f\| \to 0$ as $n \to \infty$ Since $\|f_m - f_n\| \to 0$, there exists a subsequence $g_n(x)$ such that $\Sigma_{n=1}^{\infty} \|g_{n+1} - g_n\| < \infty$. For each $x \in \Delta$ we put $g(x) = |g_1(x)| + \Sigma_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)|$, hence $\|g\| \leq \|g_1\| + \Sigma \|g_{n+1} - g_n\| < \infty$ by (P 2), which implies that the set E on which $g(x) = \infty$ is of μ-measure zero (Lemma 1, (c)). Writing now $f(x) = 0$ for $x \in E$ and $f(x) = g_1(x) + \Sigma_{n=1}^{\infty} [g_{n+1}(x) - g_n(x)]$ for $x \in \Delta - E$, we find $\|f\| \leq \|g\| < \infty$ and $\|f - g_i\| \to 0$ as $i \to \infty$, hence finally $\|f - f_n\| \leq \|f - g_i\| + \|g_i - f_n\| \to 0$ as n (and i) $\to \infty$. The uniqueness of f follows from Lemma 1, (a). Observe that in the proof no use is made of (P 3) and (P 4).

Remark. This proof shows that if $\lim \|f - f_n\| = 0$, then the sequence f_n, and also every subsequence of f_n, contains a subsequence which converges pointwise to $f(x)$ a.e. on Δ. It follows that $f_n(x)$ converges to $f(x)$ in measure on every set S of finite measure, i.e. if $\varepsilon > 0$ is arbitrarily given, and E_n is the subset of S on which $|f(x) - f_n(x)| > \varepsilon$, then $\lim \mu(E_n) = 0$. In fact, assuming this to be false for some $\varepsilon > 0$, we should have $\mu(E_{n_k}) > \delta$ for some $\delta > 0$ and some sub-sequence n_k. But (taking a subsequence of n_k if necessary) we may assume that $f_{n_k}(x)$ converges to $f(x)$ a.e. on S, so lim sup E_{n_k} has measure zero. Hence $0 = \mu$ (lim sup $E_{n_k}) \geqslant$ lim sup μ $(E_{n_k}) \geqslant \delta > 0$, which is absurd. Observe that this argument does not depend on (P 3) or (P 4); using (P 4), one may argue as follows, provided S is bounded: $\varepsilon\mu(E_n) \leq \int_S |f - f_n| d\mu \leqslant A_S \|f - f_n\| \to 0$, hence $\lim \mu(E_n) = 0$.

For any metric function $\varrho(f)$, defined on the set P of all non-negative μ-measurable functions $f(x)$, and satisfying (P 1) – (P 4), we introduce the "associate" metric function $\varrho'(f)$ on P by $\varrho'(f) = \sup \int f(x) g(x) d\mu$ for all $g(x) \in P$ such that $\varrho(g) \leqslant 1$. We shall show that ϱ' satisfies (P′1) – (P′4), analogous to (P 1) – (P 4).

(P′1) If $f(x) = 0$ a.e. on Δ, then $\varrho'(f) = 0$. Let now $\varrho'(f) = 0$, i.e. $\int fg d\mu = 0$ for every $g \in P$, such that $\varrho(g) \leqslant 1$, in particular for $g(x) = \chi_{\Delta_n} / \varrho(\chi_{\Delta_n})$. Then $f(x) = 0$ a.e. on every Δ_n, hence $f(x) = 0$ a.e. on Δ. The other conditions of (P′1) are trivially satisfied.

(P′2) Let $f_n(x) \in P$ $(n = 1, 2, \ldots)$ and $f_n(x) \uparrow f(x)$ a.e. on Δ. If $\varrho'(f)$ is finite, and $\varepsilon > 0$ is given, there exists an element $g(x) \in P$

such that $\varrho(g) \leqslant 1$ and $\int fg d\mu > \varrho'(f) - \varepsilon$. Hence, on account of $f_n(x)g(x) \uparrow f(x)g(x)$ a.e. on Δ, there exists an index $N(\varepsilon)$ such that $\int f_n g d\mu > \varrho'(f) - 2\varepsilon$ for $n > N$. It follows that $\varrho'(f_n) > \varrho'(f) - 2\varepsilon$ for $n > N(\varepsilon)$, and since $\varrho'(f_1) \leqslant \varrho'(f_2) \leqslant \ldots \leqslant \varrho'(f)$ (it is evident by definition that if $f_1, f_2 \in P$ and $f_1(x) \leq f_2(x)$ a.e. on Δ, then $\varrho'(f_1) \leqslant \varrho'(f_2)$), we have shown that $\varrho'(f_n) \uparrow \varrho'(f)$. If $\varrho'(f) = \infty$, the proof is similar.

(P'3) Let E be bounded, and χ_E its characteristic function. Then, if $\varrho(f) \leqslant 1$, we have $\int \chi_E f d\mu \leqslant A_E \varrho(f) \leqslant A_E$ by (P 4), hence $\varrho'(\chi_E) \leqslant A_E < \infty$.

(P'4) Once more, let E be bounded, and χ_E its characteristic function. We have to prove that $\int_E f d\mu \leqslant A'_E \varrho'(f)$ for all $f \in P$. If $\mu(E) = 0$ there is nothing to prove; let therefore $\mu(E) > 0$. Then $\varrho'(f) \geqslant \int [f\chi_E / \varrho(\chi_E)]\, d\mu$, hence $\int_E f d\mu \leqslant \varrho(\chi_E)\, \varrho'(f)$, so that $A'_E = \varrho(\chi_E)$ satisfies all that is required.

The preceding argument shows that if the metric functions $\varrho^{(n)}(f)$ on P are defined by $\varrho^{(0)}(f) = \varrho(f)$,
$\varrho^{(n)}(f) = \sup \int fg d\mu$ for all $g \in P$ such that $\varrho^{(n-1)}(g) \leqslant 1$ $(n = 1, 2, \ldots)$, then every $\varrho^{(n)}(f)$ satisfies (P 1) – (P 4). We extend the domain of $\varrho^{(n)}$ to complex μ-measurable functions $f(x)$ on Δ by taking $\varrho^{(n)}(|f|)$, which leads to:

Definition 2. *By* $X^{(n)}$ $(n = 0, 1, 2, \ldots)$ *we denote the set of all μ-measurable functions $f(x)$ on Δ for which $\|f\|_{X^{(n)}} = \varrho^{(n)}(|f|)$ is finite.*

Each $X^{(n)}$ $(n = 0, 1, 2, \ldots)$ is obviously a normed linear space with norm $\|f\|_{X^{(n)}}$, if we identify functions which are equal a.e. on Δ. Furthermore, since every $\varrho^{(n)}$ $(n = 0, 1, 2, \ldots)$ satisfies (P 1) – (P 4), we have (as an extension of Th. 1):

Theorem 2. *Every space* $X^{(n)}$ $(n = 0, 1, 2, \ldots)$ *is a Banach space.*

Remarks. (1) For reasons of convenience we shall denote the spaces $X^{(0)}$, $X^{(1)}$, $X^{(2)}$ and $X^{(3)}$ by X, X', X'' and X''' respectively.

(2) The space X' consists of all μ-measurable complex $f(x)$ on Δ for which

$$\|f\|_{X'} = \varrho'(|f|) = \sup \int |fg| d\mu = \sup \left|\int fg d\mu\right| < \infty,$$

where sup is over all $g \in X$ satisfying $\|g\|_X \leqslant 1$ (observe that $\|g\|_X = \|g_1\|_X$, where $g_1 = g/\operatorname{sgn} f$). An analogous statement can be made for the spaces $X^{(n)}$ $(n > 1)$.

Definition 3. *The Banach function space $X^{(n+1)}$ is called the associate space of the Banach function space $X^{(n)}$ $(n = 0, 1, 2, \ldots)$.*

In the lemmas which follow now, we have collected some simple properties of X and X'.

Lemma 2. ("*Hölder's inequality*"). *If $f(x)$ and $g(x)$ are μ-measurable on Δ, then $\int|fg|d\mu \leqslant \|f\|_X\|g\|_{X'}$. In particular, if $f\in X$, $g\in X'$, then $|\int fgd\mu| \leqslant \int|fg|d\mu \leqslant \|f\|_X\|g\|_{X'} < \infty$.*

Proof. If $\|f\|_X = 0$ or $\|f\|_X = \infty$, there is nothing to prove. Let therefore $0<\|f\|_X<\infty$. Then, by definition, $\int[|fg|/\|f\|_X]\,d\mu \leqslant \|g\|_{X'}$, hence $\int|fg|d\mu \leqslant\|f\|_X\|g\|_{X'}$.

Remark. By definition of $\|g\|_{X'}$ this inequality is sharp in the following sense: if $g\in X'$ and $\varepsilon > 0$, there exists an element $f\in X$ such that $\|f\|_X = 1$ and $0 \leqslant \|g\|_{X'} - |\int fgd\mu| < \varepsilon$.

Lemma 3. *Every $g\in X'$ defines on the space X a bounded linear functional $g^*(f) = \int fgd\mu$ with norm $\|g^*\| = \|g\|_{X'}$. The canonical mapping $g \to g^*$ of the associate space X' into the conjugate space X^* is therefore isometric.*

Proof. The linear functional $g^*(f) = \int fgd\mu$ is bounded, since by Hölder's inequality $|g^*(f)| \leqslant \|g\|_{X'}\,\|f\|_X$. This shows, moreover, that $\|g^*\| \leqslant \|g\|_{X'}$; the converse inequality follows from the definition of $\|g\|_{X'}$.

Remark. In general the canonical mapping of X' into X^* is not "onto", as the example $X = L_\infty$ shows, where $X' = L_1$ is a proper subspace of X^*.

The next lemma is an inverse of Hölder's inequality.

Lemma 4. *$f\in X'$ if and only if $f(x)g(x)$ is integrable over Δ for every $g\in X$.*

Proof. "Only if" is evident. "If" can be proved by means of the Banach-Steinhaus Theorem (A. C. Zaanen [3], p. 135), as follows: Assume that $f(x)g(x)$ is integrable over Δ for all $g\in X$, and write $f_n(x) = f(x)$ for $x\in \Delta_n, |f(x)| \leqslant n$, and $f_n(x) = 0$ elsewhere on Δ. Then $f_n \in X'$ $(n = 1, 2, ...)$ by (P 3), and $\lim f_n(x) = f(x)$ a.e. on Δ, so the bounded linear functionals $f_n{}^*(g) = \int f_ngd\mu$ $(n = 1, 2, ...)$ on X have the property that the sequence $|f_n{}^*(g)|$ is bounded for each $g\in X$ since, on account of $|f_ng| \uparrow |fg|$, we have $|f_n{}^*(g)| \leq \lim \int|f_ng|d\mu = \int|fg|d\mu$. An application of the Banach-Steinhaus Theorem now shows that $\|f_n\|_{X'} \leqslant M$ for all n, so that $\|f\|_{X'} \leqslant \liminf \|f_n\|_{X'} \leqslant M$ by "Fatou's Lemma."

A second proof, in which no use is made of the Banach-Steinhaus Theorem, is obtained by using a device due to G.G. Lorentz-D. G. Wertheim [1]. Assume that $f(x)g(x)$ is integrable over Δ for all $g \in X$, and that nevertheless $\|f\|_{X'} = \infty$. This implies the existence of a sequence $g_n(x)$ $(n = 1, 2, ...)$ such that $0 \leq g_n(x) \in X$, $\|g_n\|_X \leqslant 1$ and $\int |fg_n| d\mu > n^3$. Write $g(x) = \Sigma_{n=1}^{\infty} n^{-2} g_n(x)$. Then, by (P 2), $\|g\|_X$ is finite, hence $\int |fg| d\mu$ is finite by hypothesis. On the other hand, however, $\int |fg| d\mu > n^{-2} \int |fg_n| d\mu > n$ for $n = 1, 2, ...$ This is a contradiction.

Remark. In the same way one can prove that $f \in X^{(n)}$ for some $n = 1, 2, ...$ if and only if $f(x)g(x)$ is integrable over Δ for every $g \in X^{(n-1)}$.

Lemma 4 gives a new characterization of the Banach spaces $X^{(n)}$ for $n = 1, 2, ...$ This enables us to show that $X^{(n)}$ and $X^{(n+2)}$ for $n = 1, 2, ...$ consist of the same functions. Indeed, $X^{(n)} \subset X^{(n+2)}$ is trivial (Hölder's inequality and Lemma 4), and holds even if $n = 0$. In order to prove that $X^{(n+2)} \subset X^{(n)}$ for $n = 1, 2, ...,$ let $f \in X^{(n+2)}$, i.e. let $f(x)g(x)$ be integrable over Δ for every $g \in X^{(n+1)}$. Then, since $X^{(n-1)} \subset X^{(n+1)}$ for $n = 1, 2, ...,$ the function $f(x)g(x)$ is integrable over Δ for every $g \in X^{(n-1)}$, which implies $f \in X^{(n)}$ by Lemma 4. This argument does not give any information about the problem whether $X = X''$ or not. The only fact we have been able to prove so far in this direction is that $X \subset X''$.

The next lemma gives more precise information.

Lemma 5. *If $f(x)$ is μ-measurable on Δ, then $\|f\|_{X''} \leqslant \|f\|_X$, and $\|f\|_{X^{(n)}} = \|f\|_{X^{(n+2)}}$ for $n = 1, 2, ...,$ i.e. $X^{(n)}$ and $X^{(n+2)}$ are identical for $n = 1, 2, ...,$ so $X^{(n)}$ and $X^{(n+1)}$ are mutually associate.*

Proof. We first prove that $\|f\|_{X^{(n+2)}} \leq \|f\|_{X^{(n)}}$ for $n = 0, 1, 2, ...$ By definition, $\|f\|_{X^{(n+2)}} = \sup \int |fg| d\mu$ for all g such that $\|g\|_{X^{(n+1)}} \leqslant 1$, and this does not exceed $\|f\|_{X^{(n)}}$ by Hölder's inequality. Next, for $n = 1, 2, ...,$ we find on account of $\|g\|_{X^{(n+1)}} \leqslant \|g\|_{X^{(n-1)}}$ that $\|f\|_{X^{(n+2)}} = \sup \int |fg| d\mu$ (for $\|g\|_{X^{(n+1)}} \leqslant 1$) $\geqslant \sup \int |fg| d\mu$ (for $\|g\|_{X^{(n-1)}} \leqslant 1$) $= \|f\|_{X^{(n)}}$. Observe that this argument fails for $n = 0$.

It turns out, therefore, that among all Banach spaces $X^{(n)}$ $(n = 0, 1, 2, ...)$ which we have introduced so far there are at most three essentially different ones: X, X' and X'' (since $X^{(n)} = X'$ for $n = 1, 3, 5, ...,$ and $X^{(n)} = X''$ for $n = 2, 4, 6, ...$). The property that two spaces, like $X^{(n)}$ and $X^{(n+1)}$ for $n = 1, 2, \ldots$, are mutually associate, deserves a name of its own:

Definition 4. *The space X is called perfect whenever X and X' are mutually associate (i.e. whenever $\|f\|_X = \|f\|_{X''}$ for every μ-measurable $f(x)$ on Δ).*

Hence ,by Lemma 5:

Theorem 3. *The spaces X' and X'' are perfect.*

In 1934 G. Köthe and O. Toeplitz [1], (G. Köthe [1; 2]) gave, in connection with their investigations on sequence spaces, a definition of what they called "ein vollkommener Raum." The meaning of "perfect space" as defined here is closely related to the meaning of "vollkommener Raum", and was introduced for the first time in a paper by G. G. Lorentz-D. G. Wertheim [1], where they generalized the Köthe-Toeplitz theory to function spaces (so-called Köthe-Toeplitz spaces) with a norm topology. Another generalization to function spaces of the Köthe-Toeplitz theory has recently been given by J. Dieudonné [2]. In order to obtain a complete survey, we briefly describe here what Köthe-Toeplitz spaces are: Let C be a non-empty collection of non-negative μ-measurable functions $c(x)$ on Δ such that

(a) C is normal : if $c \in C$ and $0 \leqslant c_1(x) \leqslant c(x)$ a.e. on Δ, then $c_1 \in C$.

(b) C is convex: if $c_i \in C$ and $0 \leqslant a_i \leqslant 1$, $\Sigma_{i=1}^{n} a_i = 1$, then $\Sigma_{i=1}^{n} a_i c_i(x) \in C$.

(c) if $c_n(x) \in C$ $(n = 1, 2, \ldots)$ and $c_n(x) \uparrow c(x)$ a.e. on Δ, then $c(x) \in C$.

(d) if E is bounded, and $\chi_E(x)$ its characteristic function, then $\chi_E \in C$.

(e) for every bounded set E there exists a finite constant $A_E \geqslant 0$ such that $\int_E c(x)\, d\mu \leqslant A_E$ for every $c \in C$.

The Köthe-Toeplitz space $X = X(C)$ consists then of all μ-measurable complex functions $f(x)$ on Δ for which $\|f\|_X = \varrho(|f|) = \sup_{c \in C} \int |f(x)|\, c(x)\, d\mu < \infty$. We see at once that $\varrho(f)$, if $f \in P$, satisfies (P 1)-(P 4), and therefore every Köthe-Toeplitz space $X(C)$ is a Banach space of the type considered in the preceding pages. Moreover, it is evident that the associate space X' of any space X is a Köthe-Toeplitz space, if we take for C the set of all non-negative $c(x) \in X$ satisfying $\|c\|_X \leqslant 1$. Finally, if $X = X(C)$, the proof of

Lemma 5 shows that $\|c\|_{X'} \leqslant 1$ for all $c \in C$, hence $\|f\|_{X''} = \|f\|_X$ for all $f \in X(C)$, i.e. all Köthe-Toeplitz spaces are perfect.

The systematic investigation of Banach function spaces whose norm is derived from a metric function $\varrho(f)$ possessing the properties (P 1), (P 2) and several other properties varying from theorem to theorem, was begun by H. W. Ellis - I. Halperin [1]. Of course one finds traces of similar ideas here and there in the older literature, e.g. already in Banach's famous book ([1], p. 86). The introduction of (P 3) and (P 4) is due to A. C. Zaanen, and the resulting theoretical structure turns out to be of satisfactory elegance and generality. It remains largely a matter of future investigation to decide how much of the theory remains valid without (P 3) or (P 4), although some results in this direction are already known.

We now turn to the problem whether every space X, as defined by us, is perfect or not, i.e. whether X is a Köthe-Toeplitz space or not. The answer is affirmative. The first proof was found by G. G. Lorentz [4], and the fact was communicated by letter (in answer to a question on this subject) to A. C. Zaanen in November 1954. The author independently found a proof in December 1954, which later on turned out to be quite different from Lorentz's proof. The main idea of our proof is the use of a separation theorem (N. Bourbaki [1], p. 73) for closed convex sets in locally convex linear topological vector spaces. Sticking to the same idea I. Halperin has simplified our proof, and extended the statement to his more general spaces. We reproduce here Halperin's simplified version; in section 3 of the present chapter we shall briefly outline the original version.

Theorem 4. *The space X is perfect, i.e. X is a Köthe-Toeplitz space $X = X(C)$ where C consists of all non-negative μ-measurable functions $c(x)$ on Δ for which $\|c\|_{X'} \leqslant 1$.*

Proof. We have to show that $\|f\|_{X''} = \|f\|_X$ for every μ-measurable complex $f(x)$ on Δ. For any such function we know already that $\|f\|_{X''} \leqslant \|f\|_X$, it is sufficient, therefore, to prove the inverse inequality. Write, for $n = 1, 2, \ldots$, $f_n(x) = |f(x)|$ if $|f(x)| \leqslant n$ and $x \in \Delta_n$, $f_n(x) = n$ if $|f(x)| > n$ and $x \in \Delta_n$, and $f_n(x) = 0$ if $x \in \Delta - \Delta_n$. Then $f_n(x) \uparrow |f(x)|$ a.e. on Δ and $f_n(x) \in X$ $(n = 1, 2, \ldots)$; hence, if we can prove that $\|f_n\|_X \leqslant \|f_n\|_{X''}$ for all n, the property (P 2) shows that $\|f\|_X \leqslant \|f\|_{X''}$. We choose therefore an arbitrary index N, which is kept fixed in what follows, and, denoting by S the unit sphere of X (i.e. the set of all $f \in X$ such that $\|f\|_X \leqslant 1$), we observe that in the Lebesgue space $L_1(\Delta_N,\mu)$ the set $U = S \cap L_1(\Delta_N,\mu)$ is a convex subset, closed in the norm topology of $L_1(\Delta_N,\mu)$. Indeed, the convexity of U is evident, and since every sequence $g_n \in U$ $(n = 1, 2, ...)$ which converges in the norm topology of $L_1(\Delta_N, \mu)$ to an element $g \in L_1(\Delta_N,\mu)$ contains a subsequence $g_k(x)$, $k = n_1, n_2, ...$, which converges pointwise a.e. on Δ_N to $g(x)$, we find by "Fatou's Lemma" that $\|g\|_X \leqslant \liminf \|g_k\|_X \leqslant 1$, hence $g \in U$. Without loss of generality we may suppose that $\|f_N\|_X > 0$. Then, for any constant $\varepsilon > 0$, the non-negative function $g(x) = (1 + \varepsilon) f_N(x)/\|f_N\|_X$ belongs to $L_1(\Delta_N,\mu)$, but not to U, and hence can be separated from U by a closed hyperplane in $L_1(\Delta_N,\mu)$, determined by some element $f^* \in (L_1)^*$. Every such f^* may be expressed by means of a function $h(x) \in L_\infty(\Delta_N, \mu)$, so there exists a constant $C > 0$ such that $|\int_{\Delta_N} gh d\mu| > C$ and $|\int_{\Delta_N} fh d\mu| \leqslant C$ for all $f \in U$. These inequalities remain true if we replace $h(x)$ by $k(x)$, where $k(x) = |h(x)|$ for $g(x) \neq 0$, and $k(x) = 0$ for $g(x) = 0$ (observe that $f \in U$ implies $f/\text{sgn}\, h \in U$).

Next we prove that $|\int_{\Delta_N} f\, k\, d\mu| \leqslant C$ for all $f \in S$. For this purpose, let $0 \leqslant f(x) \in S$. Then, for $n = 1, 2, ...$, the function $\min \{f(x), ng(x)\}$ is contained in $L_1(\Delta_N, \mu)$ and in S, so that $\int_{\Delta_N} fk d\mu = \sup_n \int_{\Delta_N} \min \{f(x), ng(x)\}.\, k(x) d\mu \leqslant C$. Our result so far is therefore that $\int_{\Delta_N} gk d\mu \geqslant \sup_{f \in S} |\int fk d\mu|$. Hence $\int_{\Delta_N} [(1 + \varepsilon) f_N(x)/\|f_N\|_X]\, k(x) d\mu \geqslant \|k\|_{X'}$, from which we deduce by Hölder's inequality that $\|f_N\|_X \leqslant (1 + \varepsilon) \int_{\Delta_N} [f_N(x)\; k(x)/\|k\|_{X'}]\, d\mu \leqslant (1 + \varepsilon)\, \|f_N\|_{X''}$. In this inequality $\varepsilon > 0$ is arbitrary, hence $\|f_N\|_X \leqslant \|f_N\|_{X''}$.

Remarks. (1) If $f \in X$, then $f^*(g) = \int fg d\mu$ defines on X' a bounded linear functional, and $\|f^*\| = \|f\|_X$.

(2). Hölder's inequality $\int |fg| d\mu \leqslant \|f\|_X \|g\|_{X'}$ is sharp in two respects: given $\varepsilon > 0$ and $f \in X$, there exists a function $g \in X'$ such that $\|g\|_{X'} = 1$ and $0 \leqslant \|f\|_X - |\int fg d\mu| < \varepsilon$, and given $\varepsilon > 0$ and $g \in X'$, there exists a function $f \in X$ such that $\|f\|_X = 1$ and $0 \leqslant \|g\|_{X'} - |\int fg d\mu| < \varepsilon$.

(3) If B is an arbitrary Banach space, and the unit sphere of the conjugate space B^* is denoted by S^*, then $\|f\| = \sup_{f^* \in S^*} |f^*(f)|$ for any $f \in B$. Now, if B' is a linear subspace of B^*, and $S' = S^* \cap B'$, the subspace B' is called a norm fundamental subspace of B^* whenever $\|f\| = \sup_{f^* \in S'} |f^*(f)|$ for all $f \in B$. What we

have proved therefore in Th. 4 is that X' is a norm fundamental subspace of X^*.

It is perhaps interesting to observe that (in the general Banach space situation) B' is norm fundamental if and only if S' is dense in S^* in the weak* topology of B^* (i.e. the topology generated by the elements of B). In fact, if S' is dense in S^*, and $f^* \in S^*$, $f \in B$, $\varepsilon > 0$ are given, there exists an element $g^* \in S'$ such that $|f^*(f) - g^*(f)| < \varepsilon$. Hence $|f^*(f)| < |g^*(f)| + \varepsilon \leqslant \sup_{g^* \in S'} |g^*(f)| + \varepsilon$, so $\|f\| \leqslant \sup_{f^* \in S^*} |f^*(f)| \leq \sup_{g^* \in S'} |g^*(f)| + \varepsilon$, which implies $\|f\| \leq \sup_{g^* \in S'} |g^*(f)|$. The inverse inequality is trivial, so B' is norm fundamental. Assume conversely that B' is norm fundamental, and that, nevertheless, S' is not dense in S^*. Then the closure $\overline{S'}$ (in the weak* topology) is a proper convex subset of $\overline{S^*} = S^*$, so there exists an element $f_0^* \in S^*$ not contained in $\overline{S'}$. An application of the separation theorem shows the existence of an element $f \in B$ and a constant $C \geqslant 0$ such that $|f_0^*(f)| > C$ and $|g^*(f)| \leqslant C$ for all $g^* \in S'$. Since $f_0^* \in S^*$, the first inequality implies $\|f\| > C$; since B' is norm fundamental, the second inequality implies $\|f\| \leqslant C$. This is the desired contradiction.

2. Absolute Continuity of the Norm and Reflexivity.

Let X be a Banach function space of the type considered in section 1, and let, for any set $E \subset \Delta$, $\chi_E(x)$ be the fixed notation for the characteristic function of E. We introduce the following definition (similar to a definition given by G. G. Lorentz [2; 3]):

Definition 1. *An element $f \in X$ is said to have an absolutely continuous norm whenever the following conditions are satisfied:*

(a) *If E is bounded, and E_n is a sequence of μ-measurable subsets of E such that $\mu(E_n) \to 0$ as $n \to \infty$, then $\|f\chi_{E_n}\| \to 0$ as $n \to \infty$.*

(b) $\|f\chi_{\Delta - \Delta_n}\| \to 0$ *as $n \to \infty$.*

The space X is said to have an absolutely continuous norm whenever every $f \in X$ has an absolutely continous norm.

The definition suggests that the property of possessing an absolutely continuous norm depends to a great extent on the particular choice of the sets Δ_n. This dependence, however, is only apparent:

Lemma 1. *An element $f \in X$ has an absolutely continuous norm if and only if f satisfies the following conditions:*

(a) *Given* $\varepsilon > 0$, *there exists a number* $\delta > 0$ *such that* $\mu(E) < \delta$ *implies* $\|f\chi_E\| < \varepsilon$.

(b) *If the sequence of sets* E_n $(n = 1, 2, \ldots)$ *converges to a set of measure zero, then* $\|f\chi_{E_n}\| \to 0$ *as* $n \to \infty$.

Proof. If $f \in X$ satisfies the conditions of Lemma 1, then f is evidently of absolutely continuous norm. Let now, conversely, $f \in X$ be of absolutely continuous norm, and let $\mu(E_n) \to 0$. Then, if $\varepsilon > 0$ is given, there exists an index $N(\varepsilon)$ such that $\|f\chi_{\Delta - \Delta_N}\| < \varepsilon/2$. Write now $E_n = E_n' + E_n''$, where $E_n' = E_n \cap \Delta_N$ and $E_n'' = E_n \cap (\Delta - \Delta_N)$. Then $\|f\chi_{E_n}\| \leqslant \|f\chi_{E_n'}\| + \|f\chi_{E_n''}\| \leqslant \|f\chi_{E_n'}\| + \varepsilon/2 < \varepsilon$ for n sufficiently large, since $E_n' \subset \Delta_N$ and $\mu(E_n') \to 0$.

Next, let E_n converge to a set of measure zero. Given $\varepsilon > 0$, we determine N such that $\|f\chi_{\Delta - \Delta_N}\| < \varepsilon/2$. Then $\|f\chi_{E_n}\| \leqslant \|f\chi_{E_n \cap \Delta_N}\| + \|f\chi_{E_n \cap \Delta - \Delta_N}\| \leqslant \|f\chi_{E_n \cap \Delta_N}\| + \varepsilon/2$. Hence, since the sequence $E_n \cap \Delta_N$ of subsets of Δ_N converges to a set of measure zero (so that their measures converge to zero on account of $\mu(\Delta_N) < \infty$), we obtain $\|f\chi_{E_n}\| < \varepsilon$ for n sufficiently large.

The following definition is suggested now by Def. 1:

Definition 2. *By* X^χ *we denote the set of all* $f \in X$ *which possess an absolutely continuous norm.*

Evidently the set X^χ is linear. Furthermore X^χ is normal, i.e. the relations $f \in X^\chi$, $g \in X$ and $|g(x)| \leqslant |f(x)|$ a.e. on Δ imply $g \in X^\chi$. Hence, if $f \in X^\chi$, then $f\chi_E \in X^\chi$ for every μ-measurable set E.

Theorem 1. X^χ *is a normal linear subset of* X, *closed in the norm topology of* X, *i.e.* X^χ *is a normal linear subspace of* X.

Proof. We have only to prove that X^χ is closed. Let $f_n \in X^\chi$ $(n = 1, 2, \ldots)$ and $\|f - f_n\| \to 0$ as $n \to \infty$. Then, given $\varepsilon > 0$, there exists an index $N(\varepsilon)$ such that $\|f - f_N\| < \varepsilon/2$. In order to prove now that the norm of f satisfies condition (a) of absolute continuity, let E be bounded, and $E_n \subset E$ $(n = 1, 2, \ldots)$ such that $\mu(E_n) \to 0$. Then $\|f\chi_{E_n}\| \leqslant \|(f - f_N)\, \chi_{E_n}\| + \|f_N\, \chi_{E_n}\| < \varepsilon$ for n sufficiently large. The proof for condition (b) is similar.

Remark. It is possible that the subspace $X^\chi \subset X$ contains only the null element of X as is shown for example by the case $X = L_\infty(\Delta, \mu)$ (provided the measure μ contains no atoms). For the other Lebesgue spaces $L_p(\Delta, \mu)$, $1 \leqslant p < \infty$, we have evidently $(L_p)^\chi = L_p$.

Lemma 2. *If* $f_n \in X^\chi$ $(n = 1, 2, \ldots)$, *then* f_n *converges strongly (i.e.*

in the norm topology) to an element $f \in X^{\chi}$ if and only if $f_n(x)$ converges in measure to $f(x)$ on each set S of finite measure, and the norms of f_n are uniformly absolutely continuous.

Proof. We first prove that the conditions are necessary. Since strong convergence in X implies convergence in measure on sets of finite measure (section 1, Th. 1, Remark), we have only to show that the norms of f_n are uniformly absolutely continuous. Given $\varepsilon > 0$, there exists an index $n_0(\varepsilon)$ such that $\|f - f_n\| < \varepsilon/2$ for $n \geqslant n_0(\varepsilon)$. Let now E be bounded, and $E_m \subset E$ such that $\mu(E_m) \to 0$ as $m \to \infty$. Then, since the norms of f and all f_n are absolutely continuous, there exists an index $m_0(\varepsilon)$ such that, for $m \geqslant m_0(\varepsilon)$, we have $\|f\chi_{E_m}\| < \varepsilon/2$ and $\|(f_n - f)\chi_{E_m}\| < \varepsilon/2$ for $n = 1, 2, \ldots, n_0$. Hence, if $m \geqslant m_0(\varepsilon)$ and n is arbitrary, then $\|f_n \chi_{E_m}\| \leqslant \|(f_n - f) \chi_{E_m}\| + \|f\chi_{E_m}\| < \varepsilon$. The proof for condition (b) is similar.

We now turn to the proof that the conditions are sufficient. The hypothesis that the norms of f_n are uniformly absolutely continuous implies that, given $\sigma > 0$, there exists an index $N(\sigma)$ such that $\|f_n\chi_{\Delta-\Delta_N}\| < \sigma/2$ $(n = 1, 2, \ldots)$, so $\|(f_m - f_n) \chi_{\Delta-\Delta_N}\| < \sigma$ for all m, n. If, furthermore, for any fixed $\varepsilon > 0$, we write $E_{m,n} = \{x : |f_m(x) - f_n(x)| \geqslant \varepsilon\} \cap \Delta_N$, then $\|(f_m - f_n)\chi_{\Delta_N}\| \leqslant \|(f_m - f_n)\chi_{\Delta_N - E_{m,n}}\| + \|(f_m - f_n) \chi_{E_{m,n}}\| \leqslant \varepsilon \|\chi_{\Delta_N}\| + \|(f_m - f_n) \chi_{E_{m,n}}\|$. By the convergence in measure on Δ_N and the fact that the norms of f_n are uniformly absolutely continuous, $\|(f_m - f_n) \chi_{E_{m,n}}\|$ can be made arbitrarily small for sufficiently large m, n, so that $\limsup_{m,n} \|(f_m - f_n)\chi_{\Delta_N}\| \leqslant \varepsilon \|\chi_{\Delta_N}\|$. This holds for any $\varepsilon > 0$, hence $\|(f_m - f_n)\chi_{\Delta_N}\| \to 0$ as $m, n \to \infty$. Finally, $\|f_m - f_n\| \leqslant \|(f_m - f_n)\chi_{\Delta_N}\| + \|(f_m - f_n)\chi_{\Delta-\Delta_N}\|$ implies $\limsup_{m,n} \|f_m - f_n\| < \sigma$, and since $\sigma > 0$ is arbitrary we have $\|f_m - f_n\| \to 0$ as $m, n \to \infty$. But then f_n converges strongly to some $g \in X^{\chi}$, and, from the convergence in measure of $f_n(x)$ to $g(x)$ on each set of finite measure (section 1, Th. 1, Remark), as well as to $f(x)$ (by hypothesis), we conclude that $f(x) = g(x)$ a.e. on Δ. Hence $\|f - f_n\| \to 0$ as $n \to \infty$.

Corollary. (1) *Any sequence $f_n \in X^{\chi}$ such that $|f_n(x)| \downarrow 0$ a.e. on Δ has the property that $\|f_n\| \downarrow 0$.*

(2) *$X = X^{\chi}$ if and only if any sequence $f_n \in X$ such that $|f_n(x)| \downarrow 0$ a.e. on Δ has the property that $\|f_n\| \downarrow 0$.*

Proof. We have only to prove the second part. "Only if" is evident. If, conversely, any sequence $f_n \in X$ such that $|f_n(x)| \downarrow 0$ satisfies

$\|f_n\| \downarrow 0$, and if $f \in X$ is given, we consider the sequence $f_n = f\chi_{\Delta-\Delta_n}$. Then $|f_n(x)| \downarrow 0$, hence $\|f\chi_{\Delta-\Delta_n}\| \downarrow 0$. Also, if E bounded, and E_n is any decreasing sequence of subsets of E such that $\mu(E_n) \to 0$, then similarly $\|f\chi_{E_n}\| \to 0$. It remains to prove that the same is true if E_n is not necessarily decreasing. Assuming that the statement is false, there exists a number $\varepsilon > 0$ such that $\|f\chi_{E_n}\| > \varepsilon$ for some sequence $E_n \subset E$ satisfying $\mu(E_n) \to 0$. We may assume that $\mu(E_n) < n^{-2}$. Then, if $F_n = \bigcup_{i=n}^{\infty} E_i$, the sequence F_n is decreasing, $\mu(F_n)$ is decreasing, $\mu(F_n) \to 0$ and $\|f\chi_{F_n}\| > \varepsilon$, in contradiction to what has already been proved.

Remark. If we apply this lemma to a Lebesgue space of type L_1, we find necessary and sufficient conditions in order that a sequence of integrable functions converges in mean to an integrable function (compare P. R. Halmos [1], p. 108).

We shall now consider another linear subspace of X.

Definition 3. *By X^b we denote the closure (in the norm topology of X) of the set of all bounded μ-measurable (complex) functions $f(x)$ on Δ, having the property that the set on which $f(x) \neq 0$ is bounded. The subspace $(X')^b$ of X' is defined similarly.*

Obviously, if $f \in X^b$, then the real and imaginary parts of f belong to X^b, and also $f\chi_E \in X^b$ for any μ-measurable set E. Furthermore, it is easy to see that $(X')^b$ is a norm fundamental subspace of the Banach space X^* (the conjugate space of X), i.e. if $f \in X$, then $\|f\|_X = \sup \int |fg| d\mu$ for all $g \in (X')^b$ such that $\|g\|_{X'} \leqslant 1$. In fact, if $f \in X$ and $\varepsilon > 0$ are given, there exists, since by the perfectness of X, Hölder's inequality is sharp, an element $g \in X'$ such that $\|g\|_{X'} \leqslant 1$ and $0 \leqslant \|f\|_X - \int |fg| d\mu < \varepsilon/2$. Put $g_n(x) = g(x)$ for $x \in \Delta_n$, $|g(x)| \leqslant n$, and $g_n(x) = 0$ elsewhere. Then $0 \leqslant \int |fg| d\mu - \int |fg_n| d\mu < \varepsilon/2$ for $n > n_0(\varepsilon)$. Hence, for $n > n_0(\varepsilon)$, we have $g_n \in (X')^b$, $\|g_n\|_{X'} \leqslant 1$ and $0 \leqslant \|f\|_X - \int |fg_n| d\mu < \varepsilon$, which shows that $\|f\|_X = \sup \int |fg| d\mu$ for all $g \in (X')^b$ satisfying $\|g\|_{X'} \leqslant 1$. Similarly $X^b = (X'')^b$ is a norm fundamental subspace of $(X')^*$, so certainly a norm fundamental subspace of $X = X''$. Moreover, this argument leads to a non-trivial extension of Lemma 4 in section 1, which we formulate in the following lemma:

Lemma 3. *$f \in X$ (or $f \in X'$) if and only if $f(x)g(x)$ is integrable over Δ for every $g \in (X')^b$ (or for every $g \in X^b$).*

The subspaces X^χ and X^b are closely related. A first result is embodied in:

Lemma 4. $X^\chi \subset X^b$.

Proof. Let $f \in X^\chi$, and $f_n(x) = f(x)$ for $x \in \Delta_n, |f(x)| \leqslant n$, and $f_n(x) = 0$ elsewhere. Then $f_n \in X^b$ $(n = 1, 2, ...)$, and if we write $E_n = \{x : f_n(x) \neq f(x)\}$, the sequence E_n decreases to a set of measure zero. Hence, by Lemma 1 (b), we have $\|f - f_n\| = \|f\chi_{E_n}\| \to 0$ as $n \to \infty$, which shows that $f \in X^b$.

Examples. For the Lebesgue spaces L_p $(1 \leqslant p < \infty)$ and the spaces $\Lambda(\alpha, p)$ $(1 \leqslant p < \infty)$ of G. G. Lorentz [1; 2] we have $X = X^b = X^\chi$. If $X = L_\infty(\Delta, \mu)$, where $\mu(\Delta) < \infty$ and μ contains no atoms, then $\{0\} = X^\chi \subset X^b = X$. The following example shows that $\{0\} \subset X^\chi \subset X^b \subset X$ may occur, where all inclusions are proper. Let μ be Lebesgue measure on the interval $[0, \infty)$ and $\|f\|_X = \int_0^1 |f(x)| d\mu + \|f\chi_{[1 \leqslant x < \infty]}\|_\infty$. Then X^χ consists of all $f(x)$ such that $f \in L_1(0,1)$ and $f(x) = 0$ a.e. on $[1, \infty)$, and X^b consists of all $f(x)$ such that $f \in L_1(0,1)$, $f(x)$ is bounded on $[1, \infty)$ and such that $f(x) \to 0$ as $x \to \infty$. In chapter 2 we shall discuss more examples.

Now we shall discuss the problem on what conditions we have $X^\chi = X^b$. An answer can be formulated as follows:

Theorem 2. *In order that $X^\chi = X^b$ it is necessary and sufficient that the conjugate space $(X^b)^*$ of X^b may be isometrically identified with the associate space X' of X.*

Proof. In order to prove that the condition is necessary, we have to show that every bounded linear functional $g^*(f)$ on the Banach space $X^\chi = X^b$ can uniquely be written in the form $g^*(f) = \int fg d\mu$, $g \in X'$. Assume therefore that $g^*(f)$ is such a bounded linear functional, and define the set function $F(E)$ by $F(E) = g^*(\chi_E)$ for all μ-measurable subsets $E \subset \Delta_1$. Since $|F(E)| \leqslant \|g^*\| \, \|\chi_E\|_X \to 0$ as $\mu(E) \to 0$, there exists an integrable function $g(x)$ on Δ_1 such that $g^*(\chi_E) = F(E) = \int_{\Delta_1} \chi_E g d\mu$ (Radon-Nikodym Theorem). We extend $F(E)$ and the corresponding $g(x)$ in an obvious way to all Δ_n. Hence $g^*(\chi_E) = F(E) = \int \chi_E g d\mu$ for any bounded set E, from which we immediately deduce that $g^*(f) = \int fg d\mu$ for all step functions $f(x)$ vanishing outside a bounded set. For any $f(x)$, non-negative, bounded and vanishing outside a bounded set, there exists a sequence of these step functions $f_n(x) \geqslant 0$ such that $f_n(x) \uparrow f(x)$ uniformly on Δ, hence $\|f-f_n\|_X \to 0$ as $n \to \infty$. This implies $g^*(f) = \int fg d\mu$ for every such f, and the same is then true if $f(x)$ is no longer non-negative. We next show that $g \in X'$. Let $f \in X$, and write $f_n(x) = |f(x)|/\text{sgn } g(x)$ for $x \in \Delta_n$, $|f(x)| \leqslant n$, and $f_n(x) = 0$ elsewhere. Then $\|f_n\|_X \leqslant \|f\|_X$, so $|g^*(f_n)| \leqslant \|g^*\| \, \|f_n\|_X \leqslant \|g^*\| \, \|f\|_X$. But $g^*(f_n) = \int f_n g d\mu = \int |f_n g| d\mu$, so that, since $|f_n(x) g(x)| \uparrow |f(x) g(x)|$ a.e. on Δ, we obtain $\int |fg| d\mu = \lim \int |f_n g| d\mu = \lim g^*(f_n) \leqslant \|g^*\| \, \|f\|_X < \infty$. It follows that $f(x)g(x)$ is integrable over Δ for every $f \in X$, hence $g \in X'$ by Lemma 4 in section 1. Finally, we prove that $g^*(f) = \int fg d\mu$ for any $f \in X^\chi = X^b$ First we take $f \in X^b = X^\chi$ such that $f(x)$ vanishes outside some Δ_N. For $n = 1, 2, ...,$ write $f_n(x) = f(x)$ for $|f(x)| \leqslant n$, and $f_n(x) = 0$ elsewhere. Then $\mu(E_n[x : f_n(x) \neq f(x)])$ $\to 0$ as $n \to \infty$, hence $\|f-f_n\|_X = \|f\chi_{E_n}\|_X \to 0$, and $g^*(f) = \lim g^*(f_n) = \lim \int f_n g d\mu = \int fg d\mu$ by dominated convergence. Let next $f \in X^b = X^\chi$ be arbitrary, and write $f_n(x) = f(x)$ on Δ_n, and $f_n(x) = 0$ elsewhere. Then $\|f-f_n\|_X = \|f\chi_{\Delta-\Delta_n}\|_X \to 0$ as $n \to \infty$, hence once more $g^*(f) = \int fg d\mu$. Since X^b is a norm fundamental subspace of X (see Def. 3, Remark), we have $\|g\|_{X'} = \sup |\int fg d\mu|$ (for $f \in X^b$, $\|f\|_X \leqslant 1) = \|g^*\|$, and this shows too that $g(x)$ is unique. Conversely, every $g \in X'$ obviously generates a bounded linear functional $g^*(f) = \int fg d\mu$ on X, and therefore on $X^b = X^\chi$. The final result is therefore that there exists a 1–1 isometric correspondence between $(X^b)^*$ and X'.

In the proof that the condition $(X^b)^* = X'$ is sufficient, we shall adopt a slightly more general point of view. Assuming that V is a linear subspace (i.e. a closed linear subset) of X with the property that if $f \in V$, then the real and imaginary parts of f belong to V, and also $f\chi_E \in V$ for any measurable set E, we shall prove that $V^* = X'$ implies $V \subset X^\chi$. The particular case $V = X^b$ will imply then that $X^b \subset X^\chi$, and since $X^\chi \subset X^b$ is always true by Lemma 4, we obtain the desired result $X^b = X^\chi$. Let therefore $V^* = X'$. Then, by a well-known theorem (J. Dieudonné [1], p. 128) the unit sphere S' of X' is compact in the corresponding weak* topology (i.e. the topology generated by the elements of V). If $V \subset X^\chi$ is false, there exists a function $f \in V$ which does not possess an absolutely continuous norm, and, since it follows from the hypotheses on V that any $f \in V$ is a linear combination of non-negative functions belonging to V, we may assume that $f(x) \geqslant 0$. Let first condition (a) for an absolutely continuous norm not be satisfied by this particular f. Then, for some bounded set E and some $\varepsilon > 0$, there exist subsets

$E_n \subset E$ such that $\mu(E_n) < n^{-2}$ and $\|f\chi_{E_n}\| > \varepsilon$ for $n = 1, 2, \ldots$, so, writing $F_n = \bigcup_{i=n}^{\infty} E_i$, the sequence F_n is decreasing, $\mu(F_n) \to 0$ and $\|f\chi_{F_n}\| > \varepsilon$ for all n. Consider now the sequence $A_n = \{ g \in X' : |\int f\chi_{F_n} g\, d\mu| < \varepsilon\}$ $(n = 1, 2, \ldots)$ of weak* open subsets of $X' = V^*$; these sets A_n constitute a weak* open covering of X', hence, by the compactness of S', there exists a finite number of indices $n_1, n_2, \ldots, n_k$ with the property that for any $g \in S'$ there exists an index $n_i = n_i(g)$, $1 \leqslant i \leqslant k$, such that $|\int f\chi_{F_{n_i}} g d\mu| < \varepsilon$. But then, since $f(x) \geqslant 0$ and $g \in S'$ implies $|g| \in S'$, we also have $\int |f\chi_{F_{n_j}} g| d\mu < \varepsilon$ for some $n_j = n_j(g)$, $1 \leqslant j \leqslant k$. It follows, since F_n is decreasing, that for $n \geqslant N = \max(n_1, n_2, \ldots, n_k)$ we have $\int |f\chi_{F_n} g| d\mu < \varepsilon$ for every $g \in S'$, hence $\|f\chi_{F_n}\| \leqslant \varepsilon$ for $n \geqslant N$. This, however, contradicts $\|f\chi_{F_n}\| > \varepsilon$ for all n. The proof that condition (b) for an absolutely continuous norm is satisfied by any $f \in V$ is similar.

Corollary. *Let V be a linear subspace of X such that:*

(a) *If $f \in V$, then the real and imaginary parts of f also belong to V.*

(b) *If $f \in V$, then $f\chi_E \in V$ for any μ-measurable set $E \subset \Delta$.*

(c) *$X^b \subset V$.*

Then $V^* = X'$ if and only if $V = X^\chi = X^b$.

We do not know if this is also true if V does not satisfy (c). The following theorem is an immediate consequence:

Theorem 3. *$X^* = X'$ (isometrically) if and only if $X = X^\chi = X^b$ (i.e. if and only if the space X has an absolutely continuous norm).*

By means of this theorem we can obtain necessary and sufficient conditions in order that X is reflexive. Similar conditions have recently been obtained by I. Halperin [3] and G. G. Lorentz [4].

Theorem 4. *X is reflexive if and only if both X and its associate space X' have an absolutely continuous norm.*

Proof. The sufficiency of both X and X' having absolutely continuous norms is evident by Th. 3. The necessity can be proved as follows: As we have seen before (Lemma 3 in section 1), $X' \subset X^*$. Now, if $X' \neq X^*$ and X is reflexive (i.e. $X = X^{**}$), there exists, by the Hahn-Banach Theorem, an element $f \in X^{**} = X$ such that $\|f\|_X > 0$ and $\int fg d\mu = 0$ for all $g \in X'$, which is absurd. Hence $X' = X^*$, which implies $(X')' = X = (X^*)^* = (X')^*$. The desired result follows now by Th. 3.

Remark. It is an immediate consequence that the Lebesgue spaces $L_p(\Delta, \mu)$ are reflexive for $1 < p < \infty$ and, provided μ contains no atoms, fail to be so for $p = 1$ and $p = \infty$.

3. Linear Topologies, Separability and Reflexivity.

In the present section we shall assume that the reader is familiar with the elementary theory of locally convex linear topological vector spaces (see e.g. N. Bourbaki [1], J. Dieudonné [1, 3] and H. Nakano [2]). We briefly recall some of the most important notions.

A mapping $N: f \to N(f)$ of a linear vector space R into the real numbers such that $N(0) = 0$, $0 \leqslant N(f) < \infty$ for each $f \in R$; $N(f_1 + f_2) \leqslant N(f_1) + N(f_2)$ for each pair $f_1, f_2 \in R$; $N(af) = |a| N(f)$ for each $f \in R$ and each complex a, is called a semi-norm on R. Each set $\{N\}$ of such semi-norms on R defines a locally convex linear topology on R in the following way: The particular sets $\{f : N(f - f_0) < \varepsilon\}$ for all $N \in \{N\}$, all $f_0 \in R$ and all $\varepsilon > 0$ are the generators of the topology, i.e. the open sets are all unions of all finite intersections of these generators. We shall always assume that the set $\{N\}$ contains so many semi-norms that if $N(f) = 0$ for all $N \in \{N\}$, then $f = 0$. The resulting topology is then a Hausdorff topology.

A well-known example is the weak topology on a Banach space B, generated by a total subset Y of B^* (the statement that Y is total means that $f^*(f) = 0$ for all $f^* \in Y$ implies $f = 0$). The semi-norms $N(f)$ are defined by $N(f) = |f^*(f)|$, $f^* \in Y$. We denote this topology by $\sigma(B, Y)$. If no confusion is possible, $\sigma(B, B^*)$ is usually called "the weak topology" on B. Similarly $\sigma(B^*, B^{**})$ is the weak topology on B^*. It is stronger than $\sigma(B^*, B)$, the weak* topology on B^*. These two topologies on B^* are identical if an only if B is reflexive. It is a well-known theorem (and we have used it already in Th. 2 of section 2) that the unit sphere $\|f^*\| \leqslant 1$ of B^* is compact in the weak* topology.

The subset H of B is called $\sigma(B, Y)$ bounded if for each $f^* \in Y$ there exists a number M_{f^*} (depending on f^*) such that $|f^*(f)| \leqslant M_{f^*}$ for all $f \in H$. The sequence $f_n \in B$ $(n = 1, 2, \ldots)$ is called a $\sigma(B, Y)$ Cauchy sequence if $f^*(f_n)$ converges for each $f^* \in Y$ to a finite number, and it is called $\sigma(B, Y)$ convergent if there exists an element $f_0 \in B$ such that $\lim f^*(f_n) = f^*(f_0)$ for each $f^* \in Y$.

Lemma 1. *If Y is a norm fundamental subspace of B^*, then the subset H of B is $\sigma(B, Y)$ bounded if and only if it is bounded in norm.*

Proof. If H is bounded in norm, so $\|f\| \leqslant M$ for all $f \in H$, then $|f^*(f)| \leqslant M \|f^*\| = M_{f^*}$ for each $f^* \in Y$ and all $f \in H$. If conversely $|f^*(f)| \leqslant M_{f^*}$ for each $f^* \in Y$ and all $f \in H$, then, considering the bounded linear functionals $F(f^*) = f^*(f)$ on the Banach space Y, we have $\|F\| \leqslant M$ for all F by the Banach-Steinhaus Theorem. Furthermore, if we denote by S^* the unit sphere of B^*, $\|F\| = \sup_{f^* \in S^* \cap Y} |F(f^*)| = \sup |f^*(f)| = \|f\|$, since Y is norm fundamental. Hence $\|f\| \leqslant M$ for all $f \in H$.

Let X be a Banach function space of the type considered in sections 1, 2, and let again, for any set $E \subset \Delta$, $\chi_E(x)$ be the fixed notation for the characteristic function of E.

A semi-norm N on X will be called a normal semi-norm if it has the property that $|f_1(x)| \leqslant |f(x)|$ a.e. on Δ implies $N(f_1) \leqslant N(f)$. Hence, if N is normal, and $|f_1| = |f_2|$ a.e. on Δ, then $N(f_1) = N(f_2)$. The semi-norms $N(f) = |\int fg d\mu|$, $g \in X'$, which define the topology $\sigma(X, X')$, are not normal, for, replacing $f(x)$ by $f_1(x) = |f(x)|/\text{sgn } g(x)$, we have $|f_1(x)| = |f(x)|$, and nevertheless $N(f_1) = \int |fg| d\mu > N(f)$ in general. The semi-norm $N(f) = \int |fg| d\mu$, $g \in X'$, however, is normal.

We shall need the following lemma:

Lemma 2. *If $f_n(x) \in L_1(\Delta, \mu)$ $(n = 1, 2, \ldots)$, and the sequence of set functions $F_n(E) = \int_E f_n \, d\mu$ converges to a finite set function $F(E)$ for each μ-measurable $E \subset \Delta$, then*

(a) *the functions $F_n(E)$ are uniformly absolutely continuous, i.e. given $\varepsilon > 0$, there exist a number $\delta > 0$ and an index N such that $\int_{\Delta - \Delta_N} |f_n| \, d\mu < \varepsilon$ for all f_n, and $\int_E |f_n| \, d\mu < \varepsilon$ for all f_n if $\mu(E) < \delta$,*

(b) *there exists a function $f_0(x) \in L_1(\Delta, \mu)$ such that $F(E) = \int_E f_0 d\mu$.*

Proof. For the existence of $\delta > 0$ and $f_0(x)$ we refer to P. R. Halmos ([1], p. 170). In order to prove the existence of the index N we write $D_1 = \Delta_1$, $D_n = \Delta_n - \Delta_{n-1}$ $(n = 2, 3, \ldots)$, and $E_n = E \cap D_n$ for any μ-measurable set E. The measure ν on Δ is now defined by $\nu(E) = \Sigma_1^\infty \mu(E_n)/[2^n\{1 + \mu(D_n)\}]$ Then $\nu(\Delta) < \infty$, and $\nu(E) = 0$ if and only if $\mu(E) = 0$. Hence all $F_n(E)$ are absolutely continuous with respect to the measure ν. It follows that, given $\varepsilon > 0$, there exists a number $\delta_1 > 0$ such that $\nu(E) < \delta_1$ implies $|F_n(E)| < \varepsilon/4$ for all n. But $\nu(\Delta - \Delta_N) < \delta_1$ for N sufficiently large, so $|\int_E f_n \, d\mu| < \varepsilon/4$ for any $E \subset \Delta - \Delta_N$ and all n. Hence $\int_{\Delta - \Delta_N} |f_n| \, d\mu < \varepsilon$ for all n.

Lemma 3. *Let $L_1 = L_1(S, \mu)$ for some bounded set S, and $L_\infty = L_\infty(S, \mu)$. Then every $\sigma(L_1, L_\infty)$ Cauchy sequence is $\sigma(L_1, L_\infty)$convergent.*

Proof. Let $f_n \in L_1$ $(n = 1, 2, \ldots)$ be a $\sigma(L_1, L_\infty)$ Cauchy sequence. Then it is obviously $\sigma(L_1, L_\infty)$ bounded, hence bounded in norm by Lemma 1; so $\|f_n\| \leqslant M$. Since $\chi_E \in L_\infty$ for any μ-measurable $E \subset S$, the sequence $F_n(E) = \int_E f_n \, d\mu$ converges to a finite set function $F(E)$ on S. Hence, by Lemma 2(b), there exists a function $f_0(x) \in L_1$ such that $\lim \int_E f_n \, d\mu = \int_E f_0 \, d\mu$ for all μ-measurable $E \subset S$. It follows that $\lim \int_S f_n \, g d\mu = \int_S f_0 \, g d\mu$ for each μ-measurable step function $g(x)$ on S.

Let now $0 \leqslant g(x) \in L_\infty$. Then there exists a sequence $g_n(x)$ of non-negative step functions such that $g_n \uparrow g$ uniformly, so $\lim \|g - g_n\|_\infty = 0$. Hence

$$|\int (f_0 - f_n) g d\mu \,| \leqslant |\int f_0 (g - g_N) \, d\mu \,| + |\int (f_0 - f_n) g_N d\mu| + |\int f_n (g_N - g) \, d\mu|$$
$$\leqslant |\int (f_0 - f_n) g_N \, d\mu| + [\|f_0\| + M] \, \|g - g_N\|_\infty,$$

so that, given $\varepsilon > 0$, we may first take N so large that the second term does not exceed $\varepsilon/2$, and then n so large that the first term does not exceed $\varepsilon/2$. It follows that $\lim \int_S f_n \, g d\mu = \int_S f_0 \, g d\mu$ for such a non-negative $g(x)$, and the same is then true for any $g(x) \in L_\infty$.

Theorem 1. *Let Y be a linear subspace of X' such that $(X')^b \subset Y$, and such that $f \in Y$ implies $f\chi_E \in Y$ for any μ-measurable $E \subset \Delta$. Then every $\sigma(X, Y)$ Cauchy sequence is $\sigma(X, Y)$ convergent.*

Proof. Let $f_n \in X$ $(n = 1, 2, \ldots)$ be a $\sigma(X, Y)$ Cauchy sequence. Then it is obviously $\sigma(X, Y)$ bounded, hence bounded in norm by Lemma 1 (since Y is norm fundamental on account of $(X')^b \subset Y$); so $\|f_n\|_X \leqslant M$. Furthermore, since $f_n \in L_1(\Delta_1, \mu)$ for all n, and $L_\infty(\Delta_1, \mu) \subset (X')^b \subset Y$, the sequence f_n is a $\sigma(L_1, L_\infty)$ Cauchy sequence on Δ_1, so that by the preceding lemma there exists a function $f_0 \in L_1(\Delta_1, \mu)$ such that $\lim \int f_n g d\mu = \int f_0 g d\mu$ for each $g \in L_\infty(\Delta_1, \mu)$. Extending $f_0(x)$ in an obvious way to all Δ_k, we obtain $\lim \int f_n g d\mu = \int f_0 g d\mu$ for each $g \in L_\infty(\Delta_k, \mu)$.

We next prove that $f_0 \in X$. Let $g \in X'$ be arbitrary, and, for $m = 1, 2, \ldots$, let $g_m(x) = |g(x)|/\mathrm{sgn} f_0(x)$ for $x \in \Delta_m$, $|g(x)| \leqslant m$, and $g_m(x) = 0$ elsewhere. Then $g_m \in L_\infty(\Delta_m, \mu)$, so $\int |f_0 \, g_m| d\mu = \int f_0 g_m d\mu =$

$\lim_n \int f_n g_m d\mu \leqslant M \|g\|_{X'}$, which implies on account of $|g_m(x)| \uparrow |g(x)|$ that $\int |f_0 g| d\mu = \lim_m \int |f_0 g_m| d\mu \leqslant M \|g\|_{X'}$. Hence $f_0 g$ is integrable over Δ for each $g \in X'$, which is equivalent to $f_0 \in X$.

It remains to prove that $\lim \int f_n g d\mu = \int f_0 g d\mu$ for each $g \in Y$. Let first g vanish outside some Δ_k. Since $g\chi_E \in Y$ for any μ-measurable $E \subset \Delta_k$, the sequence $F_n(E) = \int f_n g \chi_E d\mu = \int_E f_n g d\mu$ converges to a finite set function on Δ_k. Hence, given $\varepsilon > 0$, there exists by Lemma 2(a) a number $\delta > 0$ such that $\mu(E) < \delta$ implies $|F_n(E)| < \varepsilon$ for all n. Moreover, we may take δ so small that also $|\int_E f_0 g d\mu| < \varepsilon$ if $\mu(E) < \delta$. Now split up Δ_k into two sets Δ' and Δ'' such that $\mu(\Delta') < \delta$ and $g(x)$ is bounded on Δ''. Then
$|\int (f_n - f_0)\, g d\mu| \leqslant |\int_{\Delta''} (f_n - f_0) g d\mu| + |F_n(\Delta')| + |\int_{\Delta'} f_0 g d\mu| < 3\varepsilon$
for sufficiently large n. Finally, if $g \in Y$ is arbitrary, the sequence $F_n(E) = \int f_n g \chi_E d\mu = \int_E f_n g d\mu$ converges to a finite set function on Δ. Hence, given $\varepsilon > 0$, there exists by Lemma 2(a) an index N such that $|\int_{\Delta - \Delta_N} f_n g d\mu| < \varepsilon$ for all n, and also $|\int_{\Delta - \Delta_N} f_0 g d\mu| < \varepsilon$. Then $|\int (f_n - f_0)\, g d\mu| < 3\,\varepsilon$ for sufficiently large n.

Corollary. *Every $\sigma(X, X')$ Cauchy sequence is $\sigma(X, X')$ convergent.*

The following definition is analogous to Def. 1 in section 2:

Definition 1. *The semi-norm $N(f)$ on X is called absolutely continuous whenever it has the following properties:*

(a) *If E is bounded, and E_n is a sequence of μ-measurable subsets of E such that $\mu(E_n) \to 0$ as $n \to \infty$, then $N(f\chi_{E_n}) \to 0$ as $n \to \infty$ for each $f \in X$.*

(b) *$N(f\chi_{\Delta - \Delta_n}) \to 0$ as $n \to \infty$ for each $f \in X$.*

If the set $\{N\}$ of semi-norms defines the topology T on X, and if each $N \in \{N\}$ is absolutely continuous, then the topology T is said to be absolutely continuous.

Exactly as in section 2, the dependence on the sets Δ_n is only apparent, as our next lemma shows.

Lemma 4. *The semi-norm $N(f)$ is absolutely continuous if and only if it satisfies the following conditions:*

(a) *Given $\varepsilon > 0$ and $f \in X$, there exists a number $\delta > 0$ such that $\mu(E) < \delta$ implies $N(f\chi_E) < \varepsilon$.*

(b) *If the sequence of sets E_n $(n = 1, 2, \ldots)$ converges to a set of measure zero, then $N(f\chi_{E_n}) \to 0$ as $n \to \infty$ for each $f \in X$.*

Proof. Analogous to the proof of Lemma 1 in section 2.

It is easy to give examples of absolutely continuous semi-norms. If g_i $(i = 1, \ldots, n)$ are elements of X', then $N(f) = \sup_{1 \leqslant i \leqslant n} |\int f g_i d\mu|$ is absolutely continuous; the topology $\sigma(X, X')$ is therefore absolutely continuous. A less trivial example is the following one:

Lemma 5. *If $f_n \in X$ $(n = 1, 2, \ldots)$ is a σ (X, X') Cauchy sequence, then $N(g) = \sup_n \int |f_n g| d\mu$ is an absolutely continuous normal semi-norm on X'.*

Proof. We observe first that $N(g) < \infty$ for each $g \in X'$ on account of the boundedness of the sequence $\|f_n\|_X$. The absolute continuity follows from Lemma 2(a), since the sequence of set functions $F_n(E) = \int f_n g \chi_E d\mu = \int_E f_n g d\mu$ converges to a finite set function on Δ.

In the Lemmas 6, 7 and in Theorem 2, which follow next, we assume that the topology T on X defined by the set $\{N\}$ of semi-norms, is absolutely continuous, and that all $N \in \{N\}$ are normal.

Lemma 6a. *Let $f \in X$, $f_n \in X$ $(n = 1, 2, \ldots)$, let $f_n(x)$ converge in measure to $f(x)$ on each set S of finite measure, and let finally, for each $N \in \{N\}$, the absolute continuity of $N(f_n)$ be uniform in n. Then $N(f - f_n) \to 0$ as $n \to \infty$ for each $N \in \{N\}$, i.e. f_n converges to f in the topology T.*

Proof. Analogous to the sufficiency proof of Lemma 2 in section 2 (even somewhat simpler, since $f_m - f_n$ may be replaced by $f - f_n$).

Corollary. (1) *Any sequence $f_n \in X$ such that $|f_n(x)| \downarrow 0$ a.e. on Δ has the property that $N(f_n) \downarrow 0$ for each $N \in \{N\}$.*

(2) *If $0 \leqslant f_n(x) \in X$ and $f_n(x) \uparrow f(x) \in X$, then $N(f - f_n) \to 0$ for each $N \in \{N\}$, so certainly $N(f_n) \uparrow N(f)$ for each $N \in \{N\}$.*

Our next aim is to prove that the conditions of Lemma 6a are not only sufficient but also necessary for the convergence of f_n to f in the topology T. The difficulty lies in the proof of the convergence in measure. In the corresponding proof of Lemma 2 in section 2 the success is due to the fact that $\|f - f_n\|_X \to 0$ implies the existence of a subsequence g_n of f_n such that $g_n(x)$ converges to $f(x)$ pointwise a.e. on Δ. We shall show that the convergence of f_n to f in T implies the same, but in order to do so it seems inevitable to extend each semi-norm N to a domain which may be larger than X

Let P be the set of all non-negative μ-measurable functions on Δ, and $Q = P \cap X$. Then Q is closed under addition, multiplication by non-negative constants, and the lattice operations of taking $\max(f_1, f_2)$ and $\min(f_1, f_2)$. This is evident for $\min(f_1, f_2)$, and for $\max(f_1, f_2)$ it follows by observing that $\max(f_1, f_2) \leqslant f_1 + f_2$. Let L be the class of limits of monotone increasing sequences of functions of Q. Evidently $L \supset Q$, and the class L is also closed under addition, multiplication by non-negative constants and the lattice operations. If $l(x) \in L$, and $N \in \{N\}$, we wish to define $N(l) = \lim N(f_n)$, where $f_n \in Q$ and $f_n \uparrow l$, and where $+\infty$ is admitted as a possible value of $N(l)$. In the particular case that $l \in Q$, the second part of the above Corollary shows that the new definition agrees with the old value of $N(l)$. However, we have to show first that $N(l)$ is independent of the particular sequence $f_n \uparrow l$. Let therefore f_n and g_m be increasing sequences of functions of Q, and let $\lim g_m \leqslant \lim f_n$. Then $f_n \geqslant \min(f_n, g_m)$ and $\min(f_n, g_m) \uparrow g_m$ as $n \to \infty$, hence $\lim N(f_n) \geqslant N(g_m)$, so that, letting $m \to \infty$, we obtain $\lim N(f_n) \geqslant \lim N(g_m)$. Once the uniquencess of $N(l)$ is established, the properties $N(l) \geqslant 0$, $N(l_1 + l_2) \leqslant N(l_1) + N(l_2)$ and $N(al) = a\,N(l)$ for any constant $a \geqslant 0$ are evident. Furthermore, if $l_2 \in L$ and $0 \leqslant l_1(x) \leqslant l_2(x)$, then $l_1 \in L$ and $N(l_1) \leqslant N(l_2)$. In order to prove this, let $f_n \in Q$ and $f_n \uparrow l_2$. Then $g_n = \min(f_n, l_1) \in Q$ and $g_n \uparrow l_1$, so $l_1 \in L$ and $N(l_1) = \lim N(g_n) \leqslant \lim N(f_n) = N(l_2)$. Finally, if $h(x)$ is a μ-measurable complex function on Δ such that $|h(x)| \in L$ and $N(|h|) < \infty$, we define $N(h) = N(|h|)$. Observe that for $h \in X$ this agrees with the old value (since N is a normal semi-norm).

Lemma 6b. *If $f \in X$, $f_n \in X$ $(n = 1, 2, \ldots)$, and f_n converges to f in the topology T, then $f_n(x)$ contains a subsequence which converges pointwise to $f(x)$ a.e. on Δ.*

Proof. Without loss of generality we may assume that all $f_n(x)$ are finite everywhere on Δ. We start by picking one semi-norm $N \in \{N\}$. Since $\lim N(f_m - f_n) = 0$ as $m, n \to \infty$, there exists a subsequence g_n of f_n such that $\Sigma\, N(g_{n+1} - g_n) < \infty$. Then, if $l(x) = |g_1(x)| + \Sigma_1^\infty |g_{n+1}(x) - g_n(x)|$, we have $l \in L$, $N(l) < \infty$, and, if $g(x) = g_1(x) + \Sigma_1^\infty \{g_{n+1}(x) - g_n(x)\}$ for $l(x) < \infty$ and $g(x) = 0$ for $l(x) = \infty$, we have $|g(x)| \leqslant l(x)$, hence $|g| \in L$ and $N(g) = N(|g|) < \infty$. Similarly $N(g - g_n) \to 0$ as $n \to \infty$, so, since $N(f - g_n) \to 0$ as well, $N(f - g) = 0$.

Now pick a second norm $N^* \in \{N\}$. Then $N^*(f - g_n) \to 0$, and, for a suitable subsequence $g_n^*(x)$ of $g_n(x)$, the function g^*, defined by $g^*(x) = g_1^*(x) + \Sigma_{n=1}^{\infty} \{g_{n+1}^*(x) - g_n^*(x)\}$ for $l(x) < \infty$ and $g^*(x) = 0$ for $l(x) = \infty$, satisfies $N^*(f - g^*) = 0$. But $g^* = \lim g_n^* = \lim g_n = g$ for each x at which $l(x) < \infty$, and $g^* = g = 0$ if $l(x) = \infty$. Hence $g^* = g$ for all x, so $N(f - g) = N^*(f - g) = 0$. This implies $N(f - g) = N(|f - g|) = 0$ for each $N \in \{N\}$. It remains to prove that if $k \in L$, $N(k) = 0$ for all $N \in \{N\}$, then $k = 0$ a.e. on Δ. Let $g_n \in Q$, $g_n \uparrow k$. Then $0 \leqslant N(g_n) \leqslant N(k) = 0$ for all $N \in \{N\}$, hence $N(g_n) = 0$ for all $N \in \{N\}$, so $g_n(x) = 0$ a.e. on Δ since the topology T is a Hausdorff topology. This holds for $n = 1, 2, \ldots$, so $k(x) = 0$ a.e. on Δ.

Lemma 6c. *If $f \in X$, $f_n \in X$ $(n = 1, 2, \ldots)$, and f_n converges to f in the topology T, then $f_n(x)$ converges in measure to $f(x)$ on each set of finite measure, and, for each $N \in \{N\}$, the absolute continuity of $N(f_n)$ is uniform in n.*

Proof. The convergence in measure follows from the existence of a subsequence converging pointwise to $f(x)$ a.e. on Δ (see section 1, Th. 1, Remark). The proof of the absolute continuity of $N(f_n)$, uniformly in n, is analogous to the corresponding part in the proof of Lemma 2 in section 2.

In order to prove a further important property of the topology T, we need a lemma:

Lemma 7. *Let $g^* \in X^*$. Then $g^* \in X'$ (i.e. there exists a function $g(x) \in X'$ such that $g^*(f) = \int fg d\mu$ for all $f \in X$) if and only if g^* has the property that $0 \leqslant f_n(x) \in X$ $(n = 1, 2, \ldots)$, $f_n(x) \downarrow 0$ a.e. on Δ, implies $g^*(f_n) \to 0$ as $n \to \infty$.*

Proof. If $g^* \in X'$, then it obviously has the mentioned property (dominated convergence theorem). Assume, therefore, conversely that $g^* \in X^*$ has this property. By $F(E) = g^*(\chi_E)$ we define a finitely additive set function for all μ-measurable sets $E \subset \Delta_1$. We shall prove that $F(E)$ is countably additive. If $E_k \subset \Delta_1$ $(k = 1, 2, \ldots)$ is a sequence of disjoint μ-measurable sets, and $E = \mathrm{U}_{k=1}^{\infty} E_k$, $G_n = E - \mathrm{U}_1^n E_k$, the sequence G_n is decreasing, and $\chi_{G_n} \downarrow 0$ as $n \to \infty$. Hence $F(E) - \Sigma_1^n F(E_k) = F(G_n) = g^*(\chi_{G_n}) \to 0$ by hypothesis, so $F(E)$ is countably additive. Since $\mu(E) = 0$ implies $F(E) = 0$, it is also absolutely continuous, so there exists (by the Radon-Nikodym Theorem) a function $g(x) \in L_1(\Delta, \mu)$ such that

$g^*(\chi_E) = \int \chi_E g d\mu$ for any μ-measurable $E \subset \Delta_1$. The remaining part of the proof is very similar to the corresponding part in the proof of Th. 2 in section 2.

Theorem 2. *Let the topology T on X, defined by the set $\{N\}$ of normal, absolutely continuous semi-norms, be stronger than the topology $\sigma(X, X')$ and weaker than the norm topology of X. Then X' is the dual space of X, i.e. any linear functional $G(f)$ on X, continuous in the topology T, is of the form $G(f) = \int fg d\mu$, $g \in X'$, and conversely.*

Proof. Let $G(f)$ be a linear functional on X, which is continuous in the topology T. Then, since T is weaker than the norm topology, $G(f)$ is also continuous in the norm topology, i.e. $G(f)$ is a bounded linear functional on X. Let now $0 \leqslant f_n(x) \in X$, $f_n(x) \downarrow 0$ a.e. on Δ. Then (by Lemma 6a, Coroll. (1)) f_n converges to zero in the topology T, hence $G(f_n) \to 0$ by hypothesis. This shows that the bounded linear functional G on X satisfies the conditions of Lemma 7, so $G \in X'$. Conversely, if $g(x) \in X'$ is arbitrarily chosen, then $G(f) = \int fg d\mu$ defines a mapping of X into the complex numbers which is continuous in the $\sigma(X, X')$ topology, and therefore also in the stronger topology T.

Corollary. *If H is a linear subset of X, then H is closed in the topology T if and only if it is closed in the $\sigma(X, X')$ topology.*

Proof. Since T is stronger than $\sigma(X, X')$, any set which is closed in $\sigma(X, X')$ is also closed in the topology T. Assume, conversely, that H is a linear set which is closed in the topology T. Then, by a well-known theorem, H is the intersection of all closed hyperplanes (in the topology T) which contain H. But, by Th. 2, each closed hyperplane in the topology T is determined by an equation $G(f) = \int fg d\mu = c$, $g \in X'$, c constant, and it is also known that each closed hyperplane in the topology $\sigma(X, X')$ has an equation of the same form. Hence H is the intersection of a set of closed hyperplanes in $\sigma(X, X')$, and this shows that H is closed in $\sigma(X, X')$.

Remark. The proof of Th. 2 is independent of the perfectness of X, i.e. we can prove Th. 2 (and the Lemmas 6a and 7, upon which the proof of Th. 2 is founded) without using the property that X is perfect.

We now show that the statement in Th. 2 is not an empty statement by giving an example of a topology T which actually satisfies all conditions of Th. 2. A similar topology on a Banach lattice was considered by H. Nakano [1].

Definition 2. *By $|\sigma|(X, X')$ we denote the topology on X defined by means of the set of semi-norms $N_g(f) = \int |fg| d\mu$, $g \in X'$. The topology $|\sigma|(X', X)$ on X' is defined similarly.*

Each $N_g(f)$ is evidently a normal semi-norm, and if $N_g(f) = 0$ for all $g \in X'$, then $f = 0$ a.e. on Δ. Furthermore, each N_g is absolutely continuous, so $|\sigma|(X, X')$ is absolutely continuous. In order to prove that $|\sigma|(X, X')$ is stronger than $\sigma(X, X')$, it is sufficient to prove that any set $O_\sigma = \{f : |\int fg d\mu| < \varepsilon, g \in X', \varepsilon > 0\}$ contains a $|\sigma|(X, X')$ open set. The set $O_{|\sigma|} = \{f : \int |fg| d\mu < \varepsilon\}$ satisfies this condition. Similarly, since $O_{|\sigma|}$ contains the norm open set $\{f : \|f\|_X \|g\|_{X'} < \varepsilon\}$, the norm topology is stronger than $|\sigma|(X, X')$. Hence, as an immediate consequence of Th. 2:

Theorem 3. *The dual space of X in the topology $|\sigma|(X, X')$ is X'.*

Lemma 8. *If $f \in X$, $f_n \in X$ $(n = 1, 2, \ldots)$, then f_n is $|\sigma|(X, X')$ convergent to f if and only if f_n is $\sigma(X, X')$ convergent to f, and $f_n(x)$ converges in measure to $f(x)$ on each set of finite measure.*

Proof. In view of the Lemmas 6a, 6c we have only to show that the statement "f_n is $\sigma(X, X')$ convergent to f" implies the statement "for each $g \in X'$, the absolute continuity of $\int |f_n g| d\mu$ is uniform in n". This, however, is a consequence of the convergence of $F_n(E) = \int f_n g \chi_E d\mu$ (Lemma 2).

Lemma 9. *The unit sphere $S = \{f : \|f\|_X \leqslant 1\}$ is closed in the $|\sigma|(X, X')$ topology.*

Proof. If S is not closed, there exists an element $f_0 \in X$ such that $\|f_0\| > 1$ and every $|\sigma|(X, X')$ neighbourhood of f_0 contains at least one $f \in S$. Take ε such that $0 < \varepsilon < (\|f_0\| - 1)/2$, and then determine the index N such that $1 + \varepsilon < \|f_0 \chi_{\Delta_N}\| \leqslant \|f_0\|$. Next, consider for $n = 1, 2, \ldots$ the neighbourhoods $V_n = \{f : \int |f_0 - f| \chi_{\Delta_N} d\mu \leqslant n^{-1}\}$ of f_0. Each V_n contains an element $f_n \in S$. Furthermore, since the sequence $f_n \chi_{\Delta_N}$ converges in the $L_1(\Delta_N, \mu)$ norm topology to $f_0 \chi_{\Delta_N}$, it contains a subsequence $g_n \chi_{\Delta_N}$ which converges pointwise to $f_0 \chi_{\Delta_N}$ a.e. on Δ. Hence $1 + \varepsilon < \|f_0 \chi_{\Delta_N}\| \leqslant \liminf_n \|g_n \chi_{\Delta_N}\| \leqslant \lim \inf \|g_n\| \leqslant 1$ by "Fatou's Lemma", which is absurd.

Corollary. *If $f_0 \in X$, $f_n \in X$ $(n = 1, 2, \ldots)$, and f_n is $|\sigma|(X, X')$ convergent to f_0, then $\|f_0\| \leqslant \lim \inf \|f_n\|$.*

Proof. If $a = \lim\inf \|f_n\| < \|f_0\| = b$, there exists a subsequence g_n of f_n such that $a = \lim \|g_n\|$, so, if $c = (a + b)/2$, we have $\|g_n\| \leqslant c$ for sufficiently large n. The sphere $\|f\| \leqslant c$ is closed in $|\sigma|(X,X')$, all g_n (except a finite number) are contained in this sphere, but their limit f_0 is not. This is absurd.

Remarks. (1) The proof of this lemma is independent of the perfectness of X.

(2) Our first version of the proof that X is perfect is essentially based on this lemma, and the proof may be carried out as follows: If $\varepsilon > 0$ and $0 \neq f_0 \in X$ are arbitrarily chosen, then $g_0 = (1 + \varepsilon) f_0 / \|f_0\|$ is not in S, so that, since S is closed and convex, g_0 can be separated from S by a closed hyperplane which is determined by a $|\sigma|(X, X')$ continuous linear functional. Hence, by Th. 3, there exists an element $h \in X'$ such that $\int g_0 h d\mu \neq \int f h d\mu$ for all $f \in S$. Then $\int g_0 \, h d\mu \neq 0$, so $h \neq 0$; we may therefore assume that $\|h\|_{X'} = 1$. If f runs through the whole of S, the numbers $\int f h d\mu$ cover at least the open circle $|z| < 1$ in the complex plane, hence $|\int g_0 h d\mu| \geqslant 1$, i.e. $|\int f_0 h d\mu| \geqslant \|f_0\|_X (1 + \varepsilon)^{-1}$. But then, denoting by S' the unit sphere of X', we obtain $\sup_{h \in S'} \int |f_0 h| d\mu \geqslant \|f_0\|_X (1+\varepsilon)^{-1}$, so $\|f_0\|_{X''} \geqslant \|f_0\|_X (1 + \varepsilon)^{-1}$. This holds for any $\varepsilon > 0$, hence $\|f_0\|_{X''} \geqslant \|f_0\|_X$. Combined with the trivial inverse inequality, this gives the desired result $\|f_0\|_{X''} = \|f_0\|_X$.

Making use of the perfectness of X, Lemma 9 may be replaced by a stronger statement.

Lemma 10. *The unit sphere S of X is closed in the $\sigma(X, (X')^b)$ topology of X.*

Proof. In section 2 we have already found that, due to the perfectness of X, the subspace $(X')^b$ is norm fundamental, i.e. S is the set of all $f \in X$ such that $|\int f g d\mu| \leqslant 1$ for all $g \in S' \cap (X')^b$, where S' is the unit sphere of X'. For each $g \in (X')^b$, the set of all $f \in X$ satisfying $|\int f g d\mu| \leqslant 1$ is $\sigma(X, (X')^b)$ closed; since S is an intersection of such sets, S is also closed in $\sigma(X, (X')^b)$.

Corollary. *If $f_0 \in X$, $f_n \in X$ $(n = 1, 2, \ldots)$, and f_n is $\sigma(X, (X')^b)$ convergent to f_0, then $\|f_0\| \leqslant \lim\inf \|f_n\|$.*

Theorem 4. *The space X, provided with the topology $|\sigma|(X,X')$, is complete.*

Proof. Let A be a directed index set, and let $\{f_a\}$, $a \in A$, be a Cauchy directed set in X in the topology $|\sigma|(X, X')$ (i.e. to any $g \in X'$ corresponds an index $a_0 \in A$ such that $\int |(f_{a'} - f_{a''})\, g| d\mu \leqslant 1$ for all $a', a'' \succ a_0$). We have to prove the existence of a unique $f \in X$ with the property that to any $g \in X'$ corresponds an index $a_1 \in A$ such that $\int |(f_{a'} - f)\, g| d\mu \leqslant 1$ for all $a' \succ a_1$. This element f is then the limit of $\{f_a\}$. In the argument which follows we shall make use of the following facts which are easily proved: If R is a metric space (metric $d(f, g)$), complete in the ordinary sequence sense, and if $\{f_a\}$ is a Cauchy directed set in R (hence, there exist indices a_n $(n = 1, 2. ...)$ such that $d(f_{a'}, f_{a''}) \leqslant n^{-1}$ for all $a', a'' \succ a_n$), then $\{f_a\}$ contains a Cauchy subsequence $h_n = f_{a^{(n)}}$ such that $a^{(n)} \succ a_n$, $a^{(n+1)} \succ a^{(n)}$, and the unique limit f of h_n is also the unique limit of $\{f_a\}$. If R is not complete, but h_n converges nevertheless to an element $f \in R$, then f is the unique limit of $\{f_a\}$. Observing now that the given set $\{f_a\}$ in X is also a Cauchy directed set in each Lebesgue space $L_1(\Delta_n, \mu)$ $(n = 1, 2, ...)$, application of these facts easily leads to the existence of a unique μ-measurable $f(x)$ on Δ such that $\{f_a\}$ converges to $f\chi_{\Delta_n}$ in each $L_1(\Delta_n, \mu)$. We shall prove first that $f \in X$. Let g be an arbitrary element of X', and consider the topology $T_0 = T_0(g)$ on X defined by the countable set of normal semi-norms $N_0(f) = \int |fg| d\mu$, $N_n(f) = \int |f\chi_{\Delta_n}| d\mu$ $(n = 1, 2, ...)$. This topology is weaker than $|\sigma|(X, X')$, so $\{f_a\}$ is also a Cauchy directed set in T_0. Furthermore T_0 is metrizable (but not necessarily complete). Hence, choosing the subsequence h_n from $\{f_a\}$ in the indicated way, h_n is a Cauchy sequence in the topology T_0. But, since T_0 is stronger than the norm topology of $L_1(\Delta_N, \mu)$ for each N, and $\{f_a\}$ converges to $f\chi_{\Delta_N}$ in this norm topology, it follows that h_n converges to $f\chi_{\Delta_N}$ in this norm topology. We may assume therefore (passing, if necessary, to a suitable subsequence) that $h_n(x)$ converges pointwise to $f(x)$ a.e. on Δ. Now, if $E = \{x \in \Delta : |g(x)| > 0\}$, the sequence $h_n \chi_E$ is a Cauchy sequence in the space $L_1(E, \nu)$, where $d\nu = |g| d\mu$, and on account of the pointwise convergence its limit is $f\chi_E$ (observe that "almost everywhere on E" is the same for the measures μ and ν), hence $f\chi_E \in L_1(E, \nu)$, so $\int |fg| d\mu < \infty$. Since $g \in X'$ is arbitrary, this shows that $f \in X$. Finally, since $\{f_a\}$ converges to f in the topology $T_0(g)$ for each $g \in X'$, f is also the limit of $\{f_a\}$ in the topology $|\sigma|(X, X')$.

There are several notions of compactness in topology and we briefly recall their definitions. The subset H of a topological space is called semi-compact if every infinite subset of H has at least one point of accumulation which belongs to H. If the latter condition is omitted, H is called relatively semi-compact. The subset H is called sequentially compact if every sequence of points of H contains a converging subsequence whose limit belongs to H. If the latter condition is omitted, H is called relatively sequentially compact. The subset H is called compact if every open covering of H contains a finite subcovering, and H is called relatively compact if its closure is compact. Obviously, in order that H be relatively semi-compact (or relatively sequentially compact), it is sufficient that its closure be semi-compact (or sequentially compact). This, however, is not a necessary condition. It is well-known that compactness, as well as sequential compactness, implies semicompactness, and the same holds for the corresponding notions of relative compactness. In general, however, there does not exist any other implication between these three pairs of notions (for examples we refer to A. Grothendieck [1]). In a metric space the three notions of compactness are equivalent, and the same is true of the three notions of relative compactness. A. Weil [1] showed that in a complete uniform space (relative) compactness and (relative) semi-compactness are equivalent. The weak topology of a Banach space is an example of a uniform topology which in general is not complete and not metrizable, but for which relative semi-compactness implies relative compactness according to a theorem of W. F. Eberlein [1], and relative semi-compactness also implies relative sequential compactness according to a theorem of V. Šmulian [1; 2]. These results together show therefore that for the weak topology of a Banach space the three notions of relative compactness are equivalent. A. Grothendieck, in the paper cited above, proved the following two theorems (which we shall announce as lemmas), direct generalizations of Eberlein's and Šmulian's results:

Lemma α. (*A. Grothendieck* [1], *Proposition* 2, *p.* 177). *Let E be a vector space, and T_1, T_2 two locally convex linear Hausdorff topologies in E such that E is a complete space with the topology T_1. Then, if the dual space of E with T_1 and the dual space of E with T_2 are identical (i.e. if any linear functional is continuous in T_1 if and only if it is continuous in T_2), the notions of relative semi-compactness and relative compactness in the topology T_2 are equivalent.*

Remark. If E is a Banach space, we obtain Eberlein's Theorem by taking for T_1 the norm topology and for T_2 the weak topology.

Lemma β. (*A. Grothendieck* [1], *Proposition* 6, *p.* 181). *Let E be a locally convex linear Hausdorff space, containing a countable collection of neighbourhoods of the origin having the origin as intersection. Then, if E^* is its dual space, and f_n $(n = 1, 2, ...)$ a sequence in E which is relatively semi-compact in the topology $\sigma(E, E^*)$, f_n contains a subsequence which is convergent in the topology $\sigma(E, E^*)$.*

Remark. If E is a Banach space, there always exists a countable collection of neighbourhoods of the origin having the origin as intersection, and so we obtain Šmulian's Theorem.

Returning to the function space X, and basing ourselves on the Lemmas α, β, we shall consider now compactness properties in the $\sigma(X, X')$ and $|\sigma|(X, X')$ topologies.

Lemma 11. (a) *In the $\sigma(X, X')$ topology the three notions of relative compactness are equivalent.*

(b) *If T is any other locally convex linear Hausdorff topology on X such that the dual space of E provided with the topology T is X', then the three notions of relative compactness in the topology T are equivalent. This holds in particular for $T = |\sigma|(X, X')$.*

Proof. (a) Since X, provided with the topology $|\sigma|(X, X')$, is a complete space (cf. Th. 4) having X' as its dual (cf. Th. 2), it follows from Lemma α that relative semi-compactness and relative compactness in the topology $\sigma(X, X')$ are equivalent. Lemma β shows, by taking for E the space X with the topology $|\sigma|(X, X')$ and for the countable system of neighbourhoods of the origin the sets $V_{n,m} = \{f : \int |f| \chi_{\Delta_m} d\mu < n^{-1}\}$ $(m, n = 1, 2, ...)$, that relative semi-compactness implies relative sequential compactness in the topology $\sigma(X, X')$.

(b) Once more, it follows from Lemma α that relative semi-compactness and relative compactness in the topology T are equivalent. Let H be a subset of X which is relatively semi-compact in the topology T. Since T is stronger than $\sigma(X, X')$, the subset H is also relatively semi-compact in the topology $\sigma(X, X')$, and therefore relatively sequentially compact in $\sigma(X, X')$. Hence any sequence f_n taken from H contains a subsequence g_n which is $\sigma(X, X')$ converging to some $f \in X$. Now, any point of accumulation of the point set $\{g_n\}$ in the topology T is also a point of accumulation of $\{g_n\}$ in the topology $\sigma(X, X')$, and coincides therefore with f; it follows that f is the only point of accumulation, and therefore the limit of g_n in the topology T. This shows that H is relatively sequentially compact in the topology T.

Corollary. *The subset H of X is $|\sigma|(X, X')$ relatively compact if and only if it is $\sigma(X, X')$ relatively compact, and if every sequence of elements from H contains a subsequence which converges in measure to an element of X on each set of finite measure.*

Proof. By Lemma 11(b) H is $|\sigma|(X, X')$ relatively compact if and only if any sequence of elements from H contains a $|\sigma|(X, X')$ converging subsequence, and by Lemma 8 the sequence f_n is $|\sigma|(X, X')$ convergent to $f \in X$ if and only if it is $\sigma(X, X')$ convergent to f, and if $f_n(x)$ converges in measure to $f(x)$ on each set of finite measure.

Remark. A result similar to Lemma 11 was obtained for a large class of locally convex linear topological function spaces (Köthe spaces) by J. Dieudonné [2].

Theorem 5. *The subset H of X is $\sigma(X, X')$ relatively compact if and only if $N(g) = \sup_{f \in H} \int |fg| d\mu$ is an absolutely continuous normal semi-norm on X'.*

Proof. If H is $\sigma(X, X')$ relatively compact, then H is $\sigma(X, X')$ bounded, since otherwise there exist elements $g \in X'$, $f_n \in H$ ($n = 1, 2, \ldots$) such that $|\int f_n g d\mu| \geqslant n$, but this is absurd by Lemma 5 since f_n contains a $\sigma(X,X')$ converging subsequence. Hence H is bounded in norm by Lemma 1, so $N(g)$ is finite for each $g \in X'$ This shows already that $N(g)$ is a normal semi-norm on X'. Assume now that there exists a sequence E_n of bounded sets such that $\mu(E_n) \to 0$ and $N(g\chi_{E_n}) > \alpha$ for some $\alpha > 0$ and some $g \in X'$. Then $\int_{E_n} |f_n g| d\mu > \alpha$ for suitable $f_n \in H$ ($n = 1, 2, \ldots$), where (by Lemma 11) we may assume that f_n is a $\sigma(X, X')$ converging sequence. But Lemma 5 shows that $N_1(g) = \sup_n \int |f_n g| d\mu$ is absolutely continuous, so $\int_{E_n} |f_n g| d\mu > \alpha$ ($n = 1, 2, \ldots$) is absurd. It follows that $N(g)$ satisfies the first condition for absolute continuity. The proof that $N(g)$ also satisfies the second condition is similar.

Let now, conversely, $N(g)$ be an absolutely continuous semi-norm on X'. Since $N(g) < \infty$ for any $g \in X'$, the subset H is $\sigma(X, X')$

bounded, so (by Lemma 1) bounded in norm. But then, since $X = X''$ is isometrically imbedded in the conjugate space $(X')^*$ of X', the closure $\bar{H}$ of H in the weak* topology $\sigma((X')^*, X')$ is compact in this topology. In order to prove that H is $\sigma(X, X')$ relatively compact, it is sufficient, therefore, to show that $\bar{H}$ is a subset of $X = X''$. If $f^* \in \bar{H}$, there corresponds to any pair $\varepsilon > 0$, $g \in X'$ an element $f \in H$ such that $|f^*(g) - \int fg d\mu| < \varepsilon$, so $|f^*(g)| \leqslant \int |fg| d\mu + \varepsilon$, which implies $|f^*(g)| \leqslant N(g)$. Now, since $N(g)$ is normal and absolutely continuous, it has (by Corollary 1 of Lemma 6a) the property that $g_n \in X'$, $|g_n| \downarrow 0$ implies $N(g_n) \to 0$. But then (by $|f^*(g)| \leqslant N(g)$) the linear functional $f^*(g)$ has the same property so $f^* \in X'' = X$ by Lemma 7.

Theorem 6. *Let T be a locally convex linear Hausdorff topology on X which is defined by a set of normal semi-norms, and let the dual of X (with the topology T) be X'. Then the topology T is absolutely continuous.*

Proof. Let $N(f)$ be one of the semi-norms of T, $A = \{ f \in X : N(f) \leqslant 1\}$ and A^0 the polar set of A, i.e. $A^0 = \{g \in X' : |\int fg d\mu| \leqslant 1$ for all $f \in A\}$. Since A is normal we have $\sup_{f \in A} |\int fg d\mu| = \sup_{f \in A} \int |fg| d\mu$; hence we may write as well $A^0 = \{g \in X' : \int |fg| d\mu \leqslant 1$ for all $f \in A\}$, so A^0 is a normal subset of X'. We shall prove that $N(f) = \sup_{g \in A^0} \int |fg| d\mu$. It is evident that $N(f) \geqslant \sup_{g \in A^0} \int |fg| d\mu$. In order to prove the inverse inequality, let $f_0 \in X$, $N(f_0) \neq 0$. Then (by the Hahn-Banach extension theorem) there exists a linear functional f^* on X such that $f^*(f_0) = N(f_0)$ and $|f^*(f)| \leqslant N(f)$ for all $f \in X$. This shows that f^* is continuous in the topology T, hence (since the dual of X is X') there exists an element $g_0 \in X'$ such that $f^*(f) = \int fg_0 \, d\mu$ for all $f \in X$. It follows that $\sup_{f \in A} |\int fg_0 d\mu| = \sup |f^*(f)| \leqslant \sup N(f) = 1$, so $g_0 \in A^0$, and therefore $N(f_0) = f^*(f_0) \leqslant \sup_{g \in A^0} |\int f_0 g d\mu| = \sup_{g \in A^0} \int |f_0 g| d\mu$. Furthermore, since X' is the dual of X, A^0 is a $\sigma(X', X)$ compact subset of X' (the proof is similar to the proof of the theorem that the unit sphere in the conjugate space B^* of a Banach space B is $\sigma(B^*, B)$ compact), hence, by Th. 5, $N(f) = \sup_{g \in A^0} \int |fg| d\mu$ is absolutely continuous.

Remark. The hypothesis that X' is the dual of X also implies that T is stronger than $\sigma(X, X')$ and weaker than the norm topology. The present theorem is therefore an exact converse of Th. 2. If T is the norm topology in Theorems 2 and 6, we obtain Th. 3 in section 2.

In the remainder of this section we are concerned with the question under what conditions the space X is separable (in the ordinary norm topology). We obtain results which are closely related to similar results obtained recently by H. W. Ellis - I. Halperin [2] and G. G. Lorentz [4].

Theorem 7. *Let Z be a linear subspace of X such that $X^b \subset Z$, and such that $f \in Z$ implies $f\chi_E \in Z$ for any μ-measurable subset E of Δ. Then Z is separable if and only if $Z = X^b = X^\chi$, and the measure μ is separable.*

Proof. The proof that the conditions are sufficient is standard. Since μ is separable, there exists a sequence of sets F_i $(i = 1, 2, ...)$ of finite measure which is dense in measure (i.e. if E is any set of finite measure, and $\varepsilon > 0$, then $\int|\chi_E - \chi_{F_i}|d\mu < \varepsilon$ for some F_i). Then the double sequence $F_i \cap \Delta_n$ $(i, n = 1, 2, ...)$ is also dense in measure. Given $f \in X^\chi = X^b$ and $\varepsilon > 0$, we first approximate f by a function $g_1(x)$ which is bounded and vanishes outside some Δ_N: $\|f - g_1\| < \varepsilon/3$. This function g_1 may be written as $(h_1 - h_2) + i(h_3 - h_4)$, where each h_p $(p = 1, 2, 3, 4)$ is non-negative, bounded and zero outside Δ_N. Approximating these h_p uniformly from below by rational step functions, we obtain a complex rational step function $g_2(x)$, zero outside Δ_N, such that $\|g_1 - g_2\| < \varepsilon/3$. We have therefore $g_2 = \Sigma_1^k c_q \chi_{E_q}$, where all c_q are complex rational, all E_q are disjoint and contained in Δ_N. Given one of these E_q, we can make $\mu(E_q - F_i \cap \Delta_N) + \mu(F_i \cap \Delta_N - E_q)$ arbitrarily small by suitable choice of $F_i \cap \Delta_N$, hence, since $\chi_{\Delta_N} \in X^\chi$, we can choose $P_q = F_i \cap \Delta_N$ in such a way that

$$\|\chi_{E_q} - \chi_{P_q}\| = \|\chi_{\Delta_N}\, \chi_{(E_q - P_q) \cap (P_q - E_q)}\| < \varepsilon/3\, \Sigma_1^k|c_q|.$$

Taking now $g_3 = \Sigma_1^k c_q \chi_{P_q}$, we find

$$\|g_2 - g_3\| = \|\Sigma_1^k c_q (\chi_{E_q} - \chi_{P_q})\| \leqslant \Sigma_1^k|c_q|\, \|\chi_{E_q} - \chi_{P_q}\| < \varepsilon/3,$$

hence $\|f - g_3\| < \varepsilon$, where g_3 is chosen from the countable collection of finite linear combinations with complex rational coefficients of characteristic functions of a countable set system.

Now we turn to the proof that the conditions are necessary. First we shall show that if Z is separable, then $Z = X^\chi$ by an argument which is due to G. G. Lorentz - D. G. Wertheim [1]. Let the sequence f_n be dense in Z, and let $g_n \in X'$ $(n = 1, 2, ...)$ be an arbitrary sequence in the unit sphere of X'. Since $|\int f_1 g_n\, d\mu| \leqslant \|f_1\|$ by Hölder's inequality, there exists a subsequence g_{1n} of g_n

such that $\lim_{n\to\infty} \int f_1\, g_{1n}\, d\mu$ exists as a finite number. Hence, by the diagonal process and the hypothesis that f_n is dense in Z, there exists a subsequence g_{nn} of g_n such that $\lim \int f g_{nn}\, d\mu$ exists as a finite number for each $f \in Z$, so, by Th. 1 of this section, g_{nn} is $\sigma(X', Z)$ convergent to some $g \in X'$. Assume now that there exists an element $f_0 \in Z$ for which the first condition for an absolutely continuous norm is not satisfied. Then, for some Δ_k, there exists a sequence $E_n \subset \Delta_k$ such that $\mu(E_n) \to 0$ and $\|f_0\, \chi_{E_n}\| \geqslant \alpha > 0$; since X is perfect, this implies the existence of a sequence $g_n \in X'$ such that $\|g_n\| \leqslant 1$ and $\int |f_0 g_n|\, \chi_{E_n}\, d\mu \geqslant \alpha/2$. Writing $\bar{g}_n = |g_n| \chi_{E_n} / \mathrm{sgn} f_0$ we have $\|\bar{g}_n\| \leqslant 1$, $\int f_0 \bar{g}_n\, d\mu \geqslant \alpha/2$, and $\bar{g}_n$ is zero outside E_n. According to the above argument we may assume (by passing, if necessary, to a suitable subsequence) that $\bar{g}_n$ is $\sigma(X', Z)$ convergent to some $\bar{g} \in X'$; moreover, we may also assume that $\Sigma_1^\infty\ \mu(E_n) < \infty$. Consequently, if E is a subset of Δ, disjoint with $\cup_{j=p}^\infty E_j$, then $\int \bar{g} \chi_E d\mu = \lim \int \bar{g}_n \chi_E d\mu = 0$, hence $\bar{g}(x)$ vanishes outside each $\cup_{j=p}^\infty E_j$, so $\bar{g}(x) = 0$ a.e. outside $P = \lim\sup E_n$. But $\mu(P) \leqslant \Sigma_p^\infty\ \mu(E_j)$ for each p, so $\mu(P) = 0$, and this shows that $\bar{g}(x) = 0$ a.e. on Δ. Then $0 = \int f_0 \bar{g} d\mu = \lim \int f_0 \bar{g}_n d\mu \geqslant \alpha/2$, which is a contradiction. The proof that the second condition for absolute continuity is satisfied is similar. Hence $Z \subset X^\chi$, and since $X^\chi \subset X^b \subset Z$, we obtain $Z = X^\chi = X^b$.

To complete the proof we have to show that separability of Z implies separability of μ. The subset of Z formed by the characteristic functions of all μ-measurable subsets of Δ_1 is also separable; let $\chi_{E_k}^{(1)}$ be dense in this subset. Since, by section 1, $\mu(E) = \int_{\Delta_1} \chi_E\, d\mu \leqslant A_{\Delta_1} \|\chi_E\|$, there corresponds to each number $\varepsilon > 0$ a number $\delta > 0$ such that $\|\chi_E\| < \delta$, $E \subset \Delta_1$ implies $\mu(E) < \varepsilon$. Hence, if $E \subset \Delta_1$ and $\|\chi_E - \chi_{E_k^{(1)}}\| < \delta$, then $\mu(E - E_k^{(1)}) + \mu(E_k^{(1)} - E) < \varepsilon$, so the sets $E_k^{(1)}$ are dense in measure in Δ_1. Similarly we obtain sets $E_k^{(n)}$ $(k = 1, 2, \ldots)$ which are dense in measure in $\Delta_n - \Delta_{n-1}$ $(n = 1, 2, \ldots)$. It is then easy to see that the countable collection of all finite unions of sets $E_k^{(n)}$ $(k, n = 1, 2, \ldots)$ is dense in measure.

Corollary 1. *X is separable if and only if X has an absolutely continuous norm, and the measure μ is separable.*

Corollary 2. *If $X^\chi = X^b$, then X^χ is separable if and only if the measure μ is separable.*

Remark. If X is not separable, but X^b is separable (such spaces exist; see Ch. 2, section 3), and if V is a separable subspace of X such that X^b is a proper subspace of V (such a subspace may be obtained for example by taking all elements $g + \alpha f_0$, $g \in X^b$, α complex, $f_0 \in X$ fixed and not belonging to X^b), then V cannot have the property that $f \in V$ implies $f\chi_E \in V$ for each μ-measurable $E \subset \Delta$.

Theorem 8. *If the conjugate space X^* of X is separable, then X is reflexive.*

Proof. If X^* is separable, then X' (as a subset of X^*) is separable, and, according to a theorem of S. Banach ([1], p. 189) X is also separable. Hence, by Corollary 1 of Theorem 7, both X and its associate space X' have an absolutely continuous norm. But then, by Th. 4 in section 2, X is reflexive.

Remark. For Banach lattices this theorem was proved by T.T. Ogasawara [1, 2] and for function spaces by G. G. Lorentz [4]. The theorem, however, is not true for general Banach spaces as shown by R. C. James [1], who gave an example of a non-reflexive Banach space B for which even B^{**} is separable.

Addendum: Unfortunately, the proof of Lemma 6b in section 3 is false. The proof can be saved if $\{N\}$ contains a countable subset $\{N_n\}$ such that $N_n(f) = 0$ $(n = 1, 2, \ldots)$ implies $f = 0$ a.e. Lemma 6c holds therefore with the same restriction. Lemma 8 remains true since the restrictive condition is satisfied with $N_n(f) = \int |f\chi_{\Delta n}| \, d\mu$.

CHAPTER II

ORLICZ SPACES

1. Young's Inequality and the Young Classes $\mathbf{P}_\Phi$

Let $v = \varphi(u)$, $u \geqslant 0$, be a non-decreasing real function of u such that $\varphi(0) = 0$. We assume that $\varphi(u)$ is left continuous (hence $\varphi(u) = \varphi(u-)$ for $u > 0$), and does not vanish identically (Example: $\varphi(0) = 0$, $\varphi(u) = 1$ for all $u > 0$). By $u = \psi(v)$ we denote the left continuous inverse (if $\varphi(u)$ is discontinuous at $u = a$, then $\psi(v) = a$ for $\varphi(a-) < v \leqslant \varphi(a+)$, and if $\varphi(u) = c$ for $a < u \leqslant b$, but $\varphi(u) < c$ for $u < a$, then $\psi(c) = a$). Furthermore $\psi(0) = 0$, and, if $\lim_{u\to\infty} \varphi(u) = l$ is finite, then $\psi(v) = \infty$ for $v > l$ (in the example above $\psi(v) = 0$ for $0 \leqslant v \leqslant 1$ and $\varphi(v) = \infty$ for $v < 1$).

Definition 1. *If the non-decreasing functions $v = \varphi(u)$ and $u = \psi(v)$, mutually inverse, satisfy the above conditions, then the functions $\Phi(u)$ and $\Psi(v)$, defined for $u \geqslant 0$, $v \geqslant 0$ by the Lebesgue integrals*

$$\Phi(u) = \int_0^u \varphi(t)\, dt, \quad \Psi(v) = \int_0^v \psi(t)\, dt,$$

are called complementary Young functions.

The functions Φ and Ψ are obviously absolutely continuous and convex functions, $\Phi(u)$ for $0 \leqslant u < \infty$, and $\Psi(v)$ in the interval where it is finite. The reader should keep well in mind in all which follows the possibility that $\Psi(v)$ may be $+\infty$ for all $v > l$, where l is finite.

We recall the important inequality due to W. H. Young (1912 [1]) (for the proof and further details cf. A. C. Zaanen [2], [3]):

Theorem 1 (*Young's inequality*). *If $\Phi(u)$ and $\Psi(v)$ are complementary Young functions, then*

$$uv \leqslant \Phi(u) + \Psi(v)$$

for arbitrary $u \geqslant 0$, $v \geqslant 0$, and equality occurs if and only if one at least of the relations $v = \varphi(u)$, $u = \psi(v)$ is satisfied.

Corollary. *We have*

$$\Phi(u) = max_{v \geqslant 0}\,(uv - \Psi(v)),\ \Psi(v) = sup_{u \geqslant 0}\,(uv - \Phi(u)),$$

where sup may be replaced by max if $\psi(v) < \infty$.

Remarks. (1) If $\Phi(0) = 0$, $\Phi(u) \geqslant 0$ for $u \geqslant 0$ and $\Phi(u)$ is convex, then $\Phi(u)$ is a Young function.

(2) The condition $\lim_{u\to\infty} \Phi(u)/u = \infty$ is equivalent to $\lim_{u\to\infty} \varphi(u) = \infty$, and this is equivalent to $\psi(v) < \infty$ for $0 \leqslant v < \infty$.

Let $\Phi(u)$ and $\Psi(v)$ be complementary. Furthermore, let μ be the same totally σ-finite measure on the set Δ as in Chapter 1. Then, if $f(x)$ is a μ-measurable function, real or complex, on Δ, the functions $\varphi|f(x)|$, $\psi|f(x)|$, $\Phi|f(x)|$ and $\Psi|f(x)|$ are evidently also μ-measurable on Δ.

Definition 2. *By the Young class* $P_\Phi = P_\Phi(\Delta, \mu)$ *we shall mean the set of all complex functions* $f(x)$, *μ-measurable on* Δ, *for which* $M_\Phi(f) = \int \Phi|f(x)|d\mu < \infty$. *The Young class* $P_\Psi = P_\Psi(\Delta, \mu)$ *is defined similarly, i.e.* P_Ψ *consists of all μ-measurable complex* $f(x)$ *such that* $M_\Psi(f) = \int \Psi|f|d\mu < \infty$.

If $\Phi(u) = u^p/p$ $(1 \leqslant p < \infty)$, then P_Φ consists of the same functions as the Lebesgue space L_p. For $p > 1$ we have $\Psi(v) = v^q/q$, where $1/p + 1/q = 1$, so that the complementary class P_Ψ consists of the same functions as L_q. In the case $p = 1$ it is readily seen that $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$ and $\Psi(v) = \infty$ for $v > 1$, so P_Ψ consists of all μ-measurable $f(x)$ satisfying $|f(x)| \leqslant 1$ almost everywhere on Δ. Hence, in this case, P_Ψ is a proper non-linear subset of L_∞.

We shall try to find necessary and sufficient conditions for P_Φ and P_Ψ to be linear. With this purpose in mind we first discuss the following problem: Given two Young functions Ψ and Ψ_1, we wish to state a necessary and sufficient condition in order that $P_\Psi \subset P_{\Psi_1}$ (i.e. $M_\Psi(f) < \infty$ implies $M_{\Psi_1}(f) < \infty$). The discussion which follows extends two theorems by W. Orlicz and Z.W. Birnbaum [1].

Lemma 1. *Assume that* $\mu(\Delta) < \infty$, *and that* μ *is not purely atomic, i.e.* Δ *contains a subset* E_0 *of positive measure which is free of atoms. Then, if* $\Psi(v) = 0$ *for* $0 \leqslant v \leqslant l < \infty$ *and* $\Psi(v) = \infty$ *for* $v > l$, *we have* $P_\Psi \subset P_{\Psi_1}$ *if and only if* $\Psi_1(l) < \infty$. *In all other cases* $P_\Psi \subset P_{\Psi_1}$ *if and only if there exist two constants* $a > 0$, $b > 0$ *such that* $\Psi(a) < \infty$ *and* $\Psi_1(v) \leqslant b\,\Psi(v)$ *for* $v \geqslant a$.

Proof. In the exceptional case the statement is evident, since P_Ψ consists of all μ-measurable $f(x)$ such that $|f(x)| \leqslant l$ a.e. on Δ. In all other cases there exists a value v_0 such that $0 < \Psi(v_0) < \infty$. In these cases the sufficiency of the condition is evident on account of $\Psi(a) < \infty$ and $\mu(\Delta) < \infty$. We prove the necessity. If the condition is not satisfied, there exists an increasing sequence v_n ($n = 1, 2, ...$) such that $0 < \Psi(v_n) < \infty$, $\Psi_1(v_n) > 2^n \Psi(v_n)$. Let E_n be a sequence of disjoint μ-measurable subsets of E_0 such that $\mu(E_n) = \mu(E_0) \Psi(v_1)/[2^n \Psi(v_n)]$. This is possible, since E_0 is free of atoms and $\Sigma\mu(E_n) \leqslant \mu(E_0)$. Then, if $f(x) = v_n$ on E_n and vanishes elsewhere, we have

$$\int \Psi|f|d\mu = \Sigma_1^\infty \Psi(v_n) \mu(E_n) = \Sigma_1^\infty \mu(E_0) \Psi(v_1)/2^n < \infty,$$

$$\int \Psi_1|f|d\mu = \Sigma_1^\infty \Psi_1(v_n) \mu(E_n) \geqslant \Sigma_1^\infty 2^n \Psi(v_n) \mu(E_n) = \Sigma_{n=1}^\infty \mu(E_0) \Psi(v_1) = \infty,$$

in contradiction to $P_\Psi \subset P_{\Psi_1}$.

Lemma 2. *Let $\mu(\Delta) < \infty$, where Δ is the union of a countably infinite number of atoms p_n of measure b_n (arranged such that $b_{n+1} \leqslant b_n$) and a set of measure zero. Let moreover lim inf $b_{n+1}/b_n > 0$. Then $P_\Psi \subset P_{\Psi_1}$ if and only if the same condition as in the preceding lemma is satisfied.*

Proof. As above, the exceptional case and the sufficiency of the condition in all other cases are evident. Assume therefore that we are not in the exceptional case, and that $P_\Psi \subset P_{\Psi_1}$. Since lim inf $b_{n+1}/b_n > 0$, there exist a constant $k > 1$ and an index N such that for any $c < b_N$ the interval $c/k \leqslant t \leqslant c$ contains at least one b_n ($n > N$). Determining the numbers v_n as in the preceding lemma, we can take $c = \mu(\Delta)/[2^n \Psi(v_n)]$ for n sufficiently large, hence

$$\mu(\Delta)/[2^n k \Psi(v_n)] \leqslant b_{i_n} \leqslant \mu(\Delta)/[2^n \Psi(v_n)] \text{ for } n \geqslant n_0.$$

It may happen that different n give the same b_{i_n}, but in any case we find an infinity of different b_{i_n} for $n = n_0, n_0 + 1, ...$ For a moment disregarding this small complication, the choice $f(x) = v_n$ on the atom p_{i_n} leads to

$$\int \Psi|f|d\mu = \Sigma_{n_0}^\infty \Psi(v_n) b_{i_n} \leqslant \Sigma_{n_0}^\infty \mu(\Delta)/2^n < \infty,$$

$$\int \Psi_1|f|d\mu = \Sigma_{n_0}^\infty \Psi_1(v_n) b_{i_n} \geqslant \Sigma_{n_0}^\infty 2^n \Psi(v_n) b_{i_n} \geqslant \Sigma_{n=n_0}^\infty \mu(\Delta)/k = \infty,$$

in contradiction to $P_\Psi \subset P_{\Psi_1}$. If the mentioned complication occurs, summation over a suitable subsequence gives the same result.

Remarks. (1) It remains an open question whether Lemma 2 in the non-exceptional case remains true in the absence of an additional hypothesis such as $\liminf b_{n+1}/b_n > 0$.

(2) If Δ is the union of a finite number of atoms of finite positive measure, then $P_\Psi \subset P_{\Psi_1}$ if and only if $\Psi_1(v) < \infty$ for any v for which $\Psi(v) < \infty$.

Lemma 3. *If $\mu(\Delta) = \infty$, and Δ contains a subset E_0 of infinite measure which is free of atoms, then $P_\Psi \subset P_{\Psi_1}$ if and only if there exists a constant $b > 0$ such that $\Psi_1(v) \leqslant b\,\Psi(v)$ for all $v \geqslant 0$.*

Proof. The sufficiency of the condition is evident. We prove the necessity. Let therefore $P_\Psi \subset P_{\Psi_1}$. If $\Psi(v) = 0$ for $0 \leqslant v \leqslant l < \infty$ and $\Psi(v) = \infty$ for $v > l$, then $f(x) = l \in P_\Psi \subset P_{\Psi_1}$, hence $\Psi_1(l) = 0$ so the desired condition is trivially satisfied. Assuming therefore that we are not in this exceptional case, Lemma 1 shows already the existence of $a > 0$ and $b_1 > 0$ such that $\Psi(a) < \infty$ and $\Psi_1(v) \leqslant b_1\,\Psi(v)$ for $v \geqslant a$. Furthermore it is evident that $\Psi(v) = 0$ implies $\Psi_1(v) = 0$. Let $c = \max\,\{v | \Psi(v) = 0\}$. If the desired condition is not satisfied, there exists a sequence $w_\nu \downarrow c$ such that $\Psi_1(w_\nu) \geqslant \nu\,\Psi(w_\nu)$, and a subsequence $v_n = w_{\nu_n}$ such that $\Psi(v_n) \leqslant n^{-2}$. We next determine integers $\lambda_n \geqslant 1$ such that $n^{-2} \leqslant \lambda_n \Psi(v_n) \leqslant 2\,n^{-2}$ and disjoint subsets E_i $(i = 1, 2, \ldots)$ of E_0 such that $\mu(E_i) = 1$. Then, if $f(x) = v_n$ on $\bigcup_{k=1}^{\lambda_n} E_{\lambda_1 + \cdots + \lambda_{n-1} + k}$, and $f(x) = 0$ elsewhere, we have

$$\int \Psi |f| d\mu = \Sigma\, \lambda_n\, \Psi(v_n) \leqslant \Sigma\, 2n^{-2} < \infty,$$

$$\int \Psi_1 |f| d\mu = \Sigma\, \lambda_n\, \Psi_1(v_n) \geqslant \Sigma\, n\, \lambda_n\, \Psi(v_n) \geqslant \Sigma n^{-1} = \infty$$

contradicting $P_\Psi \subset P_{\Psi_1}$.

Lemma 4. *If $\mu(\Delta) = \infty$, and Δ is the union of a countably infinite number of atoms p_n of measure b_n such that $0 < \liminf b_n \leqslant \limsup b_n < \infty$, then $P_\Psi \subset P_{\Psi_1}$ if and only if $\Psi_1(v) < \infty$ where $\Psi(v) < \infty$, and there exist two constants $c > 0$, $d > 0$ such that $0 < \Psi(c) \leqslant \infty$ and $\Psi_1(v) \leqslant d\,\Psi(v)$ for $0 \leqslant v \leqslant c$.*

Proof. Let the condition be satisfied, and assume that $f \in P_\Psi$, hence $\Sigma\, b_n\, \Psi\, |f(p_n)| < \infty$. Then f is bounded; so $\max |f(p_n)| = a < \infty$. Hence $\Psi(a) < \infty$, so that also $\Psi_1(a) < \infty$ by hypothesis. But then, by our further hypothesis, there exists a constant A (depending on a) such that $\Psi_1(v) \leqslant A\,\Psi(v)$ for $0 \leqslant v \leqslant a$, hence $f \in P_{\Psi_1}$

The proof that the condition is necessary is very similar to the corresponding part of the proof of the preceding lemma.

Corollary. *$\Sigma\ \Psi|x_n| < \infty$ implies $\Sigma\ \Psi_1|x_n| < \infty$ for any complex sequence x_n if and only if $\Psi_1(v) < \infty$ where $\Psi(v) < \infty$, and $c > 0$, $d > 0$ exist such that $0 < \Psi(c) \leqslant \infty$, $\Psi_1(v) \leqslant d\ \Psi(v)$ for $0 \leqslant v \leqslant c$.*

Remarks. (1) By combining Lemma 4 and Lemma 1 or Lemma 2, several other cases may now immediately be discussed.

(2) If $1 \leqslant q < p < \infty$, and μ is Lebesgue measure on the interval $0 \leqslant x \leqslant 1$, then Lemma 1 shows that $L_p \subset L_q$ since $v^q \leqslant v^p$ for $v \geqslant 1$. If however Δ is the interval $0 \leqslant x < \infty$, then Lemma 3 shows that neither of the classes L_p and L_q is included in the other one. Lemma 4 shows that the sequence spaces l_p and l_q satisfy $l_q \subset l_p$, since $v^p \leqslant v^q$ for $0 \leqslant v \leqslant 1$.

We return to the question of the linearity of P_Ψ.

Theorem 2. *The class P_Ψ is linear if and only if $P_{\Psi(v)} \subset P_{\Psi(2v)}$, and in this case $P_{\Psi(v)} = P_{\Psi(2v)}$.*

Proof. If $P_{\Psi(v)}$ is linear, then $\int\Psi|f|d\mu < \infty$ implies $\int\Psi|2f|d\mu < \infty$, hence $P_{\Psi(v)} \subset P_{\Psi(2v)}$. Conversely, if $P_{\Psi(v)} \subset P_{\Psi(2v)}$ and $f \in P_\Psi$, then $2^k f \in P_\Psi$ for any integer $k \geqslant 1$, hence $af \in P_\Psi$ for any complex constant a. Furthermore, if $f_1, f_2 \in P_{\Psi(v)} \subset P_{\Psi(2v)}$, then

$$\int\Psi|f_1 + f_2|d\mu \leqslant \int\Psi[\tfrac{1}{2}(2|f_1| + 2|f_2|)]d\mu \leqslant \tfrac{1}{2}\int\Psi|2f_1|d\mu + \tfrac{1}{2}\int\Psi|2f_2|d\mu < \infty$$

by the convexity of Ψ. Hence $f_1 + f_2 \in P_\Psi$. It follows that P_Ψ is linear.

Since $P_{\Psi(2v)} \subset P_{\Psi(v)}$ is always true on account of $\Psi(v) \leqslant \Psi(2v)$, the relations $P_{\Psi(v)} \subset P_{\Psi(2v)}$ and $P_{\Psi(v)} = P_{\Psi(2v)}$ are equivalent.

Remark. The linearity condition $P_{\Psi(v)} = P_{\Psi(2v)}$ may be replaced of course by $P_{\Psi(v)} = P_{\Psi(kv)}$, where k is constant and $k > 1$.

It will be evident now that by choosing $\Psi_1(v) = \Psi(2v)$ in the Lemmas 1–4, we get linearity conditions for P_Ψ. For reasons of convenience we introduce some abbreviations.

Definition 3. *The Young function $\Psi(v)$ is said to have the property δ_2 if $\Psi(v) > 0$ for all $v > 0$, and there exist two constants $a > 0$, $m > 0$ such that $\Psi(2v) \leqslant m\Psi(v)$ for $0 \leqslant v \leqslant a$, and $\Psi(v)$ is said to have the property Δ_2 if there exist two constants $b > 0$, $M > 0$ such that $\Psi(b) < \infty$ and $\Psi(2v) \leqslant M\Psi(v)$ for all $v \geqslant b$ (hence, in this case, $\Psi(v)$ is finite for all v). If $\Psi(v)$ has both properties, so if there* *exists a constant $M > 0$ such that $\Psi(2v) \leqslant M\,\Psi(v)$ for all $v \geqslant 0$, then $\Psi(v)$ is said to have the property (δ_2, Δ_2).*

Property δ_2 is equivalent to the apparently weaker property that there exist constants $a > 0$, $m > 0$ such that $\Psi(a) > 0$ and $\Psi(2v) \leqslant m\,\Psi(v)$ for $0 \leqslant v \leqslant a$. Combining Theorem 2 and the Lemmas 1–4, we obtain

Theorem 3. *If $\mu(\Delta) < \infty$, and Δ contains a subset of positive measure which is free of atoms, then P_Ψ is linear if and only if $\Psi(v)$ satisfies Δ_2. If $\mu(\Delta) = \infty$, and Δ contains a subset of infinite measure which is free of atoms, then P_Ψ is linear if and only if $\Psi(v)$ satisfies (δ_2, Δ_2). If $\mu(\Delta) = \infty$, and Δ is the union of atoms p_n of measure b_n such that $0 < \liminf b_n \leqslant \limsup b_n < \infty$, then P_Ψ is linear if and only if $\Psi(v)$ is finite for all $v \geqslant 0$ and $\Psi(v)$ satisfies δ_2.*

Remarks. (1) If $\Psi(v)$ has the property Δ_2, so $\Psi(v)$ finite for all $v \geqslant 0$ and $\Psi(2v) \leqslant M\Psi(v)$ *for* $v \geqslant v_0$, there exist constants $p \geqslant 1$, $N > 0$ such that $\Psi(v) \leqslant Nv^p$ for $v \geqslant v_0$. In order to prove this, we write $M = 2^p$ (hence $p \geqslant 1$), so $\Psi(2v)/(2v)^p \leqslant \Psi(v)/v^p$ for $v \geqslant v_0$. If N is the maximal value of $\Psi(v)/v^p$ in $v \leqslant v_0 \leqslant 2v_0$, it follows easily that $\Psi(v) \leqslant Nv^p$ for $v \geqslant v_0$. The converse statement is not true however as the following example shows: Let $v_n = n!/2$ $(n = 2, 3, \ldots)$, and let $\Psi(v)$ be such that $\Psi(v) < v^2$, $\Psi(v_n) = (n-1)!\,v_n$ and $\Psi(2v_n) - \Psi(v_n) = n!v_n$. Then $\Psi(2v_n)/\Psi(v_n) = n + 1$.

(2) If $\lim_{n\to\infty} \varphi(u) = l < \infty$, then $\Phi(u) \leqslant lu$ for all $u \geqslant 0$, hence the Lebesgue space L_1 satisfies $L_1 \subset P_\Phi$. Furthermore, if $\varepsilon > 0$ is such that $\varepsilon < l/2$, there exists a number u_0 such that $\varphi(u_0) > l - \varepsilon$, and a number u_1 such that $1 - u_0/u > (l-2\varepsilon)/(l-\varepsilon)$ for $u \geqslant u_1$. Hence, if $u \geqslant u_1$,

$$\Phi(u) \geqslant (l-\varepsilon)\,(u-u_0) = (l-\varepsilon)\,u\,(1-u_0/u) > u\,(l-2\varepsilon).$$

This shows that for $\mu(\Delta) < \infty$ the classes P_Φ and L_1 consist of the same functions.

(3) If $\Phi(u)$ and $\Psi(v)$ are complementary, one may ask if it is always true that one at least of these functions has the property Δ_2. The answer is negative, as shown by the following example: Take a sequence $1 < M_1 < M_2 < \ldots$ such that $\lim M_n = \infty$. The graph of $\varphi(u)$ consists of straight line segments connecting the points $(0, 0)$, (u_1, v_1), (u_2, v_2), ..., where u_1, v_1 are arbitrarily positive; $u_2 = 2u_1$, $v_2 = M_1 v_1$; $u_3 = M_2 u_2$, $v_3 = 2v_2$; $u_4 = 2u_3$, $v_4 = M_3 u_3$, and so on. Then $\Phi(u_2) > \frac{1}{2} M_1 \Phi(u_1)$, $\Psi(v_3) > \frac{1}{2} M_2\,\Psi(v_2)$, and so on.

2. Definition and some Properties of Orlicz Spaces.

We consider $M_\Psi(f) = \int \Psi|f(x)|d\mu$. By the properties of the Young function Ψ we have:

(a) $M_\Psi(0) = 0$.
$M_\Psi(kf) = 0$ for all $k \geqslant 0$ is equivalent to $f = 0$ a.e. on Δ.
$M_\Psi(kf) \leqslant 1$ for all $k \geqslant 0$ is equivalent to $f = 0$ a.e. on Δ.
If $\Psi(v) > 0$ for all $v > 0$, then $M_\Psi(f) = 0$ is equivalent to $f = 0$ a.e. on Δ.

(b) $M_\Psi(e^{i\varphi}f) = M_\Psi(f)$ for real φ.
If $a \geqslant 0$, $b \geqslant 0$, $a + b = 1$, then $M_\Psi(af + bg) \leqslant aM_\Psi(f) + b\,M_\Psi(g)$.

(c) $M_\Psi(kf)$ is a convex left continuous function of k for $k > 0$, and if $M_\Psi(af) < \infty$ for some $a > 0$, then $M_\Psi(kf)$ is a finite convex continuous function of k for $0 \leqslant k \leqslant a$.

By means of $M_\Psi(f)$ we now define the following metric function (Minkowski functional), at first only for non-negative functions:

Definition 1. *To every non-negative μ-measurable function $f(x)$ on Δ we assign the non-negative number $\varrho(f) = inf\, k^{-1}$ for all $k \geqslant 0$ such that $M_\Psi(kf) \leqslant 1$.*

It follows at once that $0 \leqslant \varrho(f) \leqslant \infty$, and $\varrho(f)$ is finite if and only if there exists a constant $k > 0$ such that $M_\Psi(kf)$ is finite.

As an example we consider the case that $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$, $\Psi(v) = \infty$ for $v > 1$. Then $M_\Psi(kf) \leqslant 1$ if and only if $kf(x) \leqslant 1$ a.e., so $k^{-1} \geqslant$ ess sup $f(x)$. Hence $\varrho(f) =$ ess sup $f(x)$, the L_∞ norm of $f(x)$.

We shall prove that the metric function $\varrho(f)$ posesses the properties (P 1) - (P 4) of Ch. 1, section 1.

(P 1) $\varrho(f) = 0$ if and only if $M_\Psi(kf) \leqslant 1$ for all $k \geqslant 0$, and this is equivalent to $f(x) = 0$ a.e. on Δ. If $\alpha > 0$, then $\varrho(\alpha f) = \inf k^{-1}$ for all $k \geqslant 0$ such that $M_\Psi(k\alpha f) \leqslant 1$, and this is equivalent to $\varrho(\alpha f) = \alpha \inf (\alpha k)^{-1}$ for all $k \geqslant 0$ such that $M_\Psi(\alpha k f) \leqslant 1$. Hence $\varrho(\alpha f) = \alpha\, \varrho(f)$. Finally we shall prove that $\varrho(f_1 + f_2) \leqslant \varrho(f_1) + \varrho(f_2)$. If $\varrho(f_1) + \varrho(f_2) = \infty$, there is nothing to prove. Assume therefore that $\varrho(f_1) + \varrho(f_2) = \gamma < \infty$. Let $\varrho(f_1) = a\gamma$ and $\varrho(f_2) = b\gamma$, $a + b = 1$. Then $M_\Psi[(f_1 + f_2)/\gamma] = M_\Psi[a(f_1/a\gamma) + b(f_2/b\gamma)] \leqslant a\,M_\Psi(f_1/a\gamma) + b\,M_\Psi(f_2/b\gamma) = a\,M_\Psi\,[f_1/\varrho(f_1)] + b\,M_\Psi\,[f_2/\varrho(f_2)] \leqslant a + b = 1$, hence $\varrho(f_1 + f_2) \leqslant \gamma = \varrho(f_1) + \varrho(f_2)$.

(P 2) First we observe that $f_1 \leqslant f_2$ implies $\varrho(f_1) \leqslant \varrho(f_2)$. If $\varrho(f_2) = \infty$ this is evident; if $\varrho(f_2) < \infty$, then $M_\Psi[f_1/\varrho(f_2)] \leqslant M_\Psi[f_2/\varrho(f_2)] \leqslant 1$, hence $\varrho(f_1) \leqslant \varrho(f_2)$. Let now $f_n \uparrow f$ a.e. on Δ, and let $\varrho(f_n) = a_n \uparrow a$. If $a = 0$, then $a_n = 0$ $(n = 1, 2, \ldots)$, hence $f_n(x) = 0$ a.e. for all n, so $f(x) = 0$ a.e., and $\varrho(f) = 0$. Since $\varrho(f)$ and all $\varrho(f_n)$ are zero, $\varrho(f_n) \uparrow \varrho(f)$ is satisfied. If $a = \infty$, so $\varrho(f_n) \uparrow \infty$, the relation $\varrho(f_n) \uparrow \varrho(f)$ is trivial. Let therefore $0 < a < \infty$. Then, since $a \geqslant a_n$ and $M_\Psi(f_n/a_n) \leqslant 1$, we find $M_\Psi(f_n/a) \leqslant 1$, hence $M_\Psi(f/a) \leqslant 1$ by Fatou's Lemma, and this implies $\varrho(f) \leqslant a$. It follows that $\varrho(f) \leqslant a = \lim \varrho(f_n) \leqslant \varrho(f)$, so $\varrho(f_n) \uparrow \varrho(f)$.

(P 3) Let $\chi_E(x)$ be the characteristic function of the bounded subset E of Δ. Then $\varrho(\chi_E) = \inf k^{-1}$ for all $k \geqslant 0$ such that $\int_E \Psi(k)\, d\mu \leqslant 1$; hence, if the range of $\Psi(v)$ covers the whole interval $0 \leqslant v < \infty$, we find $\varrho(\chi_E) = [\overset{-1}{\Psi}\{\mu(E)\}^{-1}]^{-1}$. If however there exists a number $l > 0$ such that $0 \leqslant \Psi(l) < \infty$, and $\Psi(v) = \infty$ for all $v > l$, then $\varrho(\chi_E) = l^{-1}$ if $\mu(E)\ \Psi(l) \leqslant 1$, and $l^{-1} < \varrho(\chi_E) = [\overset{-1}{\Psi}\{\mu(E)\}^{-1}]^{-1}$ if $\mu(E)\ \Psi(l) > 1$. Observe that the proof not only shows that $\varrho(\chi_E) < \infty$ for any bounded E, but even for any E of finite measure.

(P 4) Let $\varrho(f) < \infty$, and let E be a bounded subset of Δ. Then, if $k = 1/\varrho(f)$, we have $\int \Psi(kf)\, d\mu \leqslant 1$, hence, by Jensen's inequality,

$$(1) \qquad \Psi\{\tfrac{1}{\mu(E)} \textstyle\int_E kf\, d\mu\} \leqslant \tfrac{1}{\mu(E)} \textstyle\int \Psi(kf)\, d\mu \leqslant \tfrac{1}{\mu(E)}.$$

If the range of $\Psi(v)$ covers the whole interval $0 \leqslant v < \infty$, this implies $\{\mu(E)\}^{-1} \int_E kf d\mu \leqslant \overset{-1}{\Psi}\{\mu(E)\}^{-1}$, hence

$$\int_E f d\mu \leqslant \overset{-1}{\Psi}\{1/\mu(E)\}\ \mu(E)\ 1/k = \overset{-1}{\Psi}\{1/\mu(E)\}\ \mu(E)\, \varrho(f).$$

In this case, therefore, we may choose $A_E = \overset{-1}{\Psi}\{\mu(E)\}^{-1}\ \mu(E)$. If however there exists a number $l > 0$ such that $0 \leqslant \Psi(l) < \infty$, and $\Psi(v) = \infty$ for all $v > l$, we distinguish between $\mu(E)\Psi(l) > 1$ and $\mu(E)\Psi(l) \leqslant 1$. In the first case $1/\mu(E) < \Psi(l)$, so once more $\int_E f d\mu \leqslant \overset{-1}{\Psi}\{\mu(E)\}^{-1} \mu(E)\ \varrho(f)$. In the second case, since the left side of (1) is finite, we find $\{\mu(E)\}^{-1} \int_E kf d\mu \leqslant l$, hence $\int_E f d\mu \leqslant l\mu(E)\, \varrho(f)$.

Once the properties (P 1) - (P 4) for $\varrho(f)$ are verified, we give, in accordance with Ch. 1, section 1, the following definition:

Definition 2. *The Orlicz space $L_{M\Psi} = L_{M\Psi}(\Delta,\mu)$ is the set of all complex functions $f(x)$, μ-measurable on Δ, and satisfying $\|f\|_{M\Psi} = \varrho(|f|) < \infty$.*

Identifying elements $f_1, f_2 \in L_{M\Psi}$ if and only if $\mu(E|f_1 \neq f_2) = 0$, the space $L_{M\Psi}$ is obviously a normed linear space with norm $\|f\|_{M\Psi}$. By Ch. 1, section 1, Th. 1 we have

Theorem 1. *The Orlicz space $L_{M\Psi}$ is a Banach space.*

If $\Psi(v) = v^p/p$, $1 \leqslant p < \infty$, then $L_{M\Psi} = L_p$ and $\|f\|_{M\Psi} = (1/p)^{1/p} \|f\|_p$. If $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$ and $\Psi(v) = \infty$ for $v > 1$, then $L_{M\Psi} = L_\infty$ and $\|f\|_{M\Psi} = \|f\|_\infty$.

Lemma 1. (a) *$f \in L_{M\Psi}$ if and only if $M_\Psi(kf) < \infty$ for some constant $k > 0$. Hence $P_\Psi \subset L_{M\Psi}$.*

(b) *$\|f\|_{M\Psi} \leqslant 1$ if and only if $M_\Psi(f) \leqslant 1$. More precisely, $\|f\|_{M\Psi} \leqslant 1$ implies $M_\Psi(f) \leqslant \|f\|_{M\Psi}$, and $\|f\|_{M\Psi} > 1$ implies $M_\Psi(f) \geqslant \|f\|_{M\Psi}$.*

(c) *$M_\Psi(f) = 1$ implies $\|f\|_{M\Psi} = 1$; if $M_\Psi(k_0 f) < \infty$ for some constant $k_0 > 1$, then $\|f\|_{M\Psi} = 1$ implies $M_\Psi(f) = 1$.*

(d) *$\lim_{n\to\infty} \|f_n - f\|_{M\Psi} = 0$ if and only if $\lim_{n\to\infty} M_\Psi\{k(f_n - f)\} = 0$ for all constants $k \geqslant 0$, and in this case $M_\Psi(kf) \leqslant \liminf M_\Psi(kf_n)$ for every $k \geqslant 0$. If moreover $M_\Psi(k_0 f_n) < \infty$ for some k_0 and all n, then $M_\Psi(kf) = \lim M_\Psi(kf_n)$ for all k such that $0 \leqslant k < k_0$.*

Proof. (a) Follows immediately from the definition of the norm $\|f\|_{M\Psi}$.

(b) Let first $k^{-1} = \|f\| \leqslant 1$. Then $k M_\Psi(f) \leqslant M_\Psi(kf) = M_\Psi(f/\|f\|) \leqslant 1$, hence $M_\Psi(f) \leqslant k^{-1} = \|f\|$. Let next $\|f\| > 1$. Then, for any constant k such that $1 < k < \|f\|$, we have $M_\Psi(k^{-1}f) > 1$ by the definition of $\|f\|$; hence $k^{-1} M_\Psi(f) \geqslant M_\Psi(k^{-1}f) > 1$, so $M_\Psi(f) > k$. Since this holds for every $k < \|f\|$, we obtain $M_\Psi(f) \geqslant \|f\|$.

(c) It is evident by (b) that $M_\Psi(f) = 1$ implies $\|f\| = 1$. Let now $M_\Psi(k_0 f) < \infty$ for some $k_0 > 1$, and $\|f\| = 1$. Then, if $1 < k < k_0$, we have $\|kf\| = k > 1$, hence $M_\Psi(kf) > 1$ by (b), so $M_\Psi(f) \geqslant 1$ by continuity. But also $M_\Psi(f) \leqslant 1$ since $\|f\| = 1$. Hence $M_\Psi(f) = 1$.

(d) $\lim \|f_n - f\| = 0$ implies $\lim \|k(f_n - f)\| = 0$ for any constant $k \geqslant 0$, hence $\lim M_\Psi\{k(f_n - f)\} = 0$ by (b). Conversely, if $\lim M_\Psi\{k(f_n - f)\} = 0$ for every $k \geqslant 0$, and $\varepsilon > 0$ is given, then $M_\Psi\{\varepsilon^{-1}(f_n - f)\} < 1$ for $n \geqslant n_0(\varepsilon)$, so $\|\varepsilon^{-1}(f_n - f)\| \leqslant 1$ by (b). Hence $\|f_n - f\| \leqslant \varepsilon$ for $n \geqslant n_0$. Assuming now that $\lim \|f_n - f\| = 0$ is satisfied, we choose a subsequence g_n of f_n such that $\lim M_\Psi(kg_n) = \liminf M_\Psi(kf_n)$. Since $\lim \|g_n - f\| = 0$, the sequence g_n contains a subsequence g_{n_k}, converging pointwise to f a.e. on Δ. So, by Fatou's lemma, $M_\Psi(kf) \leqslant \lim M_\Psi(kg_{n_k}) = \liminf M_\Psi(kf_n)$. Let now, moreover, $M_\Psi(k_0 f_n) < \infty$ for some k_0 and all n, and let $0 \leqslant k < k_0$. Then

$$M_\Psi(kf) = M_\Psi\{kf_n + k(f - f_n)\} =$$
$$M_\Psi\{kk_0^{-1}.\, k_0 f_n + (1 - kk_0^{-1}).\, k(1 - kk_0^{-1})^{-1}(f - f_n)\}$$
$$\leqslant kk_0^{-1} M_\Psi(k_0 f_n) + (1 - kk_0^{-1})\, M_\Psi[k(1 - kk_0^{-1})^{-1}(f - f_n)],$$

so $M_\Psi(kf) < \infty$. Choose ε such that $0 < \varepsilon < 1$ and k_1 such that $k < k_1 < k_0$ (hence $M_\Psi(k_1 f) < \infty$). Then

$$M_\Psi(kf_n) = M_\Psi\{kf + k(f_n - f)\} =$$
$$M_\Psi\{k(1-\varepsilon)\, f + k\varepsilon f + k\,(f_n - f)\} =$$
$$M_\Psi\{(1-\varepsilon)\, kf + \varepsilon kk_1^{-1}.\, k_1 f + \varepsilon(1 - kk_1^{-1}).\, k\varepsilon^{-1}(1 - kk_1^{-1})^{-1}(f_n - f)\}$$
$$\leqslant (1-\varepsilon)\, M_\Psi(kf) + \varepsilon\, kk_1^{-1} M_\Psi(k_1 f) + \varepsilon(1 - kk_1^{-1})\, M_\Psi[k\varepsilon^{-1}(1 - kk_1^{-1})^{-1}$$
$$(f_n - f)] \leqslant M_\Psi(kf) + \varepsilon\, M_\Psi(k_1 f) + \varepsilon\, M_\Psi[k\varepsilon^{-1}(1 - kk_1^{-1})^{-1}(f_n - f)].$$

By choosing n sufficiently large the last term can be made smaller than ε; hence $\limsup M_\Psi(kf_n) \leqslant M_\Psi(kf)$, from which the desired result follows. That $\lim \|f_n - f\| = 0$ does not always imply $M_\Psi(kf) = \lim M_\Psi(kf_n)$ is shown by the following example: Let $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$, and $\Psi(v) = \infty$ for $v > 1$, so $\|f\|_{M\Psi} = \|f\|_\infty$. Then, if $f_n(x) = 1 + n^{-1}$ for all x $(n = 1, 2, \ldots)$ and $f(x) = 1$ for all x, we have $\lim \|f_n - f\| = 0$, but $M_\Psi(f_n) = \infty$ $(n = 1, 2, \ldots)$ and $M_\Psi(f) = 0$.

Lemma 2. *If $f \in L_{M\Psi}$, and E_n $(n = 1, 2, \ldots)$ is a finite or countable collection of disjoint subsets of Δ of finite positive measure, then $\|f\|$ is not increased if on each E_n we replace $f(x)$ by its average on E_n, i.e. by $\int_{E_n} f d\mu / \mu(E_n)$.*

Proof. Writing $\bar{f}$ for the function obtained by replacing f by its average on all E_n, we find by Jensen's inequality

$\int \Psi(|\bar{f}|/\|f\|)\, d\mu = \Sigma_{n=1}^\infty \int_{E_n} \Psi(|\bar{f}|/\|f\|)\, d\mu + \int_{\Delta - \cup E_n} \Psi(|f|/\|f\|)\, d\mu$
$\leqslant \Sigma_{n=1}^\infty \int_{E_n} \Psi\,[1/\mu(E_n) \int_{E_n} |f|/\|f\|\, d\mu]\, d\mu + \int_{\Delta - \cup E_n} \Psi(|f|/\|f\|)\, d\mu \leqslant$
$\Sigma_{n=1}^\infty \int_{E_n} \Psi(|f|/\|f\|)\, d\mu + \int_{\Delta - \cup E_n} \Psi(|f|/\|f\|)\, d\mu = M_\Psi(f/\|f\|) \leqslant 1$,
hence $\|\bar{f}\| \leqslant \|f\|$ by the definition of the norm.

This lemma shows that the norm $\|f\|_{M\Psi}$ is "levelling" in the terminology of H. W. Ellis – I. Halperin [1].

In Ch. 1, section 1 we have assigned to each Banach function space X its associate Banach function space X'. Choosing $X = L_{M\Psi}$, we now introduce the associate space $(L_{M\Psi})'$.

Definition 3. *Let Φ and Ψ be complementary Young functions. Then the Orlicz space $L_\Phi = L_\Phi(\Delta, \mu)$ is defined to be the associate space $(L_{M\Psi})'$, i.e. L_Φ consists of all complex functions $f(x)$, μ-measurable* on Δ, *for which* $\|f\|_\Phi = \sup \int |fg|\,d\mu < \infty$, *where* sup *is for all* $g(x)$ *satisfying* $\|g\|_{M\Psi} \leqslant 1$.

The reader will ask why $(L_{M\Psi})'$ is denoted by L_Φ, since the above definition involves only the function Ψ, and not the complementary function Φ. The notation L_Φ is justified by Theorem 2 below. The reader is also warned not to confuse the new notations L_Φ, $\|f\|_\Phi$ and the old notations $L_{M\Phi}$, $\|f\|_{M\Phi}$. The Orlicz space $L_{M\Phi}$ consists, according to Def. 2, of all $f(x)$ such that $\|f\|_{M\Phi} < \infty$, where $\|f\|_{M\Phi}$, according to Def. 1, is equal to inf k^{-1} for all $k \geqslant 0$ such that $M_\Phi(kf) = \int \Phi|kf|\,d\mu \leqslant 1$. What we shall prove in Theorem 2, and this is the justification for the notation L_Φ, is that L_Φ and $L_{M\Phi}$ contain the same functions f, although $\|f\|_\Phi$ and $\|f\|_{M\Phi}$ are not necessarily equal. Before doing so, we shall make some additional observations. In the first place, one may replace the definition of $\|f\|_\Phi$ in Def. 3 by $\|f\|_\Phi = \sup \int |fg|\,d\mu$ for all g satisfying $\int \Psi|g|\,d\mu \leqslant 1$, since $M_\Psi(g) \leqslant 1$ is equivalent to $\|g\|_{M\Psi} \leqslant 1$ by Lemma 1 (b). Furthermore, we have Hölder's inquality

$$\int |fg|\,d\mu \leqslant \|f\|_\Phi \|g\|_{M\Psi},$$

which is sharp in the sense explained in Remark (2) at the end of Ch. 1, section 1. Finally, by Lemma 4 of Ch. 1, section 1, $f \in L_\Phi$ if and only if $\int fg\,d\mu$ exists as a finite number for every $g \in L_{M\Psi}$.

Theorem 2. *The spaces* L_Φ *and* $L_{M\Phi}$ *consist of the same functions, i.e.* $f \in L_\Phi$ *if and only if* $M_\Phi(kf) < \infty$ *for some constant* $k > 0$. *More precisely,* $M_\Phi(f/\|f\|_\Phi) \leqslant 1$ *for every* $f \in L_\Phi$ *which does not vanish identically.*

Proof. If f is such that $M_\Phi(kf) < \infty$ for some constant $k > 0$, and $M_\Psi(g) \leqslant 1$, then $\int k|fg|\,d\mu \leqslant M_\Phi(kf) + 1$ by Young's inequality, hence $\|f\|_\Phi \leqslant k^{-1}\{M_\Phi(kf) + 1\} < \infty$. In order to prove the converse, we shall show that $M_\Phi(f/\|f\|_\Phi) \leqslant 1$ for every $f \in L_\Phi$ which does not vanish identically. If f satisfies these conditions, then $\int |fg|\,d\mu \leqslant \|f\|_\Phi$ if $M_\Psi(g) \leqslant 1$, and $\int |fg|\,d\mu \leqslant \|f\|_\Phi M_\Psi(g)$ if $M_\Psi(g) > 1$ by Hölder's inequality and Lemma 1(b). Hence $\int |fg|\,d\mu \leqslant \|f\|_\Phi M'_\Psi(g)$, where $M'_\Psi(g) = \max\,(M_\Psi(g), 1)$. If first f is bounded and vanishing outside the set Δ_n, then $M_\Phi(f/\|f\|_\Phi) < \infty$ and $M_\Psi[\varphi(|f|/\|f\|_\Phi)] < \infty$; hence, since Young's inequality becomes an equality for $g = \varphi(|f|/\|f\|_\Phi)$,

$$M'_\Psi(g) \geqslant \int [|f|/\|f\|_\Phi]\,g\,d\mu = M_\Phi(f/\|f\|_\Phi) + M_\Psi(g).$$

If $M'_\Psi(g) = M_\Psi(g)$, then $M_\Phi(f/\|f\|_\Phi) = 0 \leqslant 1$; if $M'_\Psi(g) = 1$, then $M_\Phi(f/\|f\|_\Phi) + M_\Psi(g) \leqslant 1$, hence $M_\Phi(f/\|f\|_\Phi) \leqslant 1$. Let now

$f \in L_\Phi$ be arbitrary. Defining $f_n(x)$ $(n = 1, 2, ...)$ by $f_n(x) = 0$ outside Δ_n, $f_n(x) = f(x)$ on Δ_n $(|f| \leqslant n)$ and $f_n(x) = n$ on Δ_n $(|f| > n)$, we have $|f_n| \uparrow |f|$, $M_\Phi(f_n/\|f_n\|_\Phi) \leqslant 1$ and $\|f_n\|_\Phi \leqslant \|f\|_\Phi$. Hence $M_\Phi(f_n/\|f\|_\Phi) \leqslant 1$, so $M_\Phi(f/\|f\|_\Phi) \leqslant 1$ by Fatou's lemma.

Theorem 3. *The norms $\|f\|_{M\Phi}$ and $\|f\|_\Phi$ are equivalent, i.e. the norm topologies generated by these norms are identical. More precisely,*

$$\|f\|_{M\Phi} \leqslant \|f\|_\Phi \leqslant 2\,\|f\|_{M\Phi}$$

for any μ-measurable $f(x)$.

Proof. The preceding theorem shows that $\|f\|_{M\Phi} = \infty$ if and only if $\|f\|_\Phi = \infty$. Let therefore $f \in L_\Phi$. If $\|f\|_\Phi = 0$, then $f = 0$ a.e., so $\|f\|_{M\Phi} = 0$, and conversely. Hence, we may assume that $\|f\|_\Phi \neq 0$ and $\|f\|_{M\Phi} \neq 0$. The inequality $M_\Phi(f/\|f\|_\Phi) \leqslant 1$ implies $\|f\|_{M\Phi} \leqslant \|f\|_\Phi$. For $k = \{\|f\|_{M\Phi}\}^{-1}$ we have $M_\Phi(kf) \leqslant 1$, hence, by Young's inequality, $\|f\|_\Phi \leqslant k^{-1}\{M_\Phi(kf) + 1\} \leqslant 2\,\|f\|_{M\Phi}$.

Now that we have obtained the three Orlicz spaces $L_{M\Psi}$, $L_{M\Phi}$ and L_Φ, it will not be difficult for the reader to guess how the discussion is brought to a close. The fourth Orlicz space L_Ψ is introduced as the associate space $(L_{M\Phi})'$ of $L_{M\Phi}$, and $\|f\|_\Psi = \sup \int |fg|\,d\mu$ for all g satisfying $\|g\|_{M\Phi} \leqslant 1$. Analogous to what we did in Theorem 2, we wish to prove that L_Ψ and $L_{M\Psi}$ consist of the same functions, and that $M_\Psi(f/\|f\|_\Psi) \leqslant 1$ for any $f \in L_\Psi$ which does not vanish identically. The proof requires some care, since it may happen that $\Psi(v) = \infty$ for finite v (if $\Psi(v) < \infty$ for all finite v, Theorem 2 gives the desired result). Assume therefore that $\lim_{u\to\infty} \varphi(u) = l < \infty$, so $\Psi(v) = \infty$ for $v > l$. We prove first that in this case $f \in L_\Psi$ implies $|f|/\|f\|_\Psi \leqslant l$ a.e. on Δ. In fact, assuming that $|f(x)| > l\|f\|_\Psi$ on a set E of finite positive measure, we might put $g(x) = [l\mu(E)]^{-1}$ on E and $g(x) = 0$ elsewhere, and then, in virtue of $\Phi(u) \leqslant lu$ for all u, we should have $M_\Phi(g) = \mu(E)\Phi[\{l\mu(E)\}^{-1}] \leqslant 1$ and $\int |fg|\,d\mu = \int_E |f|\,d\mu/[l\mu(E)] > \|f\|_\Psi$ simultaneously, which is absurd. Hence, supposing first that $f(x)$ vanishes outside some Δ_n, and choosing the arbitrary but fixed number δ such that $0 < \delta < 1$, we see that both $\Psi[\delta|f|/\|f\|_\Psi]$ and $\Phi|g|$, where $g = \psi\,[\delta|f|/\|f\|_\Psi]$, are bounded a.e. on Δ_n, and therefore integrable over Δ. Writing $M'_\Phi(g) = \max\,(M_\Phi(g), 1)$, we have now

$$\delta M'_\Phi(g) \geqslant \int |\delta f/\|f\|_\Psi\, g|\,d\mu = M_\Psi\,(\delta f/\|f\|_\Psi) + M_\Phi(g),$$

from which we derive, since $M'_\Phi(g) = 1$ on account of $M_\Phi(g) \leqslant$

$\delta M'_\Phi(g) < M'_\Phi(g)$, that $M_\Psi(\delta f/\|f\|_\Psi) \leqslant \delta < 1$. Making $\delta \uparrow 1$, we obtain the desired result. The extension to the case that $f(x) \neq 0$ on a set which is not bounded is now evident. Analogous to what we proved in Theorem 3, it follows finally that

$$\|f\|_{M\Psi} \leqslant \|f\|_\Psi \leqslant 2\,\|f\|_{M\Psi}.$$

Example: If $\Phi(u) = u^p/p$, $1 < p < \infty$, then $\Psi(v) = v^q/q$, where $p^{-1} + q^{-1} = 1$. We have already found that in this case $\|f\|_{M\Phi} = (1/p)^{1/p}\,\|f\|_p$ and $\|f\|_{M\Psi} = (1/q)^{1/q}\,\|f\|_q$, and an easy computation gives $\|f\|_\Phi = q^{1/q}\|f\|_p$ and $\|f\|_\Psi = p^{1/p}\,\|f\|_q$. If $p = q = 2$, then $\|f\|_{M\Phi} = \|f\|_{M\Psi} = \frac{1}{2}\sqrt{2}\,\|f\|_2$ and $\|f\|_\Phi = \|f\|_\Psi = \sqrt{2}\,\|f\|_2$. If $\Phi(u) = u$, then $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$ and $\Psi(v) = \infty$ for $v > 1$, so in this case $\|f\|_{M\Phi} = \|f\|_\Phi = \|f\|_1$ and $\|f\|_{M\Psi} = \|f\|_\Psi = \|f\|_\infty$. This example shows therefore that the inequalities in Theorem 3 for $\|f\|_\Phi$ and $\|f\|_{M\Phi}$ (and also the corresponding inequalities for $\|f\|_\Psi$ and $\|f\|_{M\Psi}$) are sharp.

Remarks. (1) In 1932 W. Orlicz [1] defined for the first time the spaces L_Φ and L_Ψ (L^*_Φ and L^*_Ψ in Orlicz's notation), assuming that Φ and Ψ are complementary Young functions possessing some additional properties ($\varphi(u)$ and $\psi(v)$ continuous, strictly increasing and tending to infinity). Some years earlier, however, W. H. Young [2], making an extensive use of his important inequality, had found a number of properties of the Young classes P_Φ and P_Ψ in his investigation of what he called supersummability (summability of $\Phi|f|$ or $\Psi|f|$). The spaces L_Φ and L_Ψ, as defined by Orlicz, did not include spaces of the L_1 or L_∞ types, owing to the restrictions imposed upon $\varphi(u)$ and $\psi(v)$, and this may be felt as a weakness, since in many respects the spaces L_1 and L_∞ are, so to speak, the cornerstones on which the whole structure of L_p spaces rests. This defect was remedied by A. C. Zaanen [2] in 1949, who extended Young's inequality to the case where $\Psi(v)$ may jump to infinity, and defined L_Φ, L_Ψ in such a way that all L_p ($1 \leqslant p \leqslant \infty$) were included. However, there still remained another defect: the spaces L_Φ and L_Ψ are not associate in the sense of Ch. 1, section 1, and this makes the relation between $\|f\|_\Phi$ and $\|g\|_\Psi$ not sufficiently clear (Hölder's inequality in the form $\int |fg|\,d\mu \leqslant \|f\|_\Phi\,\|g\|_\Psi$ is true but not sharp). This is the reason why we have introduced here the spaces $L_{M\Phi}$ and $L_{M\Psi}$, so that finally one has to deal with the intertwined associate pairs $L_{M\Phi}$, L_Ψ and L_Φ, $L_{M\Psi}$. It is to the point here to observe that H. Nakano [1, 2] has recently published a comprehensive abstract theory of what he calls modulared spaces, i.e. linear spaces on which a non-negative functional, a modular, is defined which has some but not all properties of a norm. In our case $M_\Phi(f)$ and $M_\Psi(f)$ are modulars in the sense of Nakano.

(2) By Ch. 1, section 1, Th. 4 the Orlicz spaces $L_{M\Phi}$ and $L_{M\Psi}$ are perfect, and by Ch. 1, section 1, Th. 3 the same is true of the associate spaces L_Ψ and L_Φ. Hence

$$\|f\|_{M\Phi} = \sup \int |fg| d\mu \text{ for all } g \text{ satisfying } \|g\|_\Psi \leqslant 1,$$
$$\|f\|_{M\Psi} = \sup \int |fg| d\mu \text{ for all } g \text{ satisfying } \|g\|_\Phi \leqslant 1.$$

(3) If Φ and Ψ are complementary, and E is a subset of Δ of finite measure (characteristic function χ_E), then $\mu(E) = \|\chi_E\|_\Phi \|\chi_E\|_{M\Psi}$ (compare H. W. Ellis - I. Halperin [1], p. 580). If $\mu(E) = 0$, this is evidently true; if $0 < \mu(E) < \infty$, then $\|\chi_E\|_\Phi = \sup \int_E |f| d\mu$ for all f such that $\|f\|_{M\Psi} \leqslant 1$. Replacing $|f|$ on E by its average $\bar{f} = [\mu(E)]^{-1} \int_E |f| d\mu$, we have $\int_E |\bar{f}| d\mu = \int_E |f| d\mu$ and $\|\bar{f}\|_{M\Psi} \leqslant \|f\|_{M\Psi}$ by Lemma 2, hence $\|\chi_E\|_\Phi = \sup \int_E k d\mu$ for all constants $k \geqslant 0$ such that $\|k\chi_E\|_{M\Psi} \leqslant 1$. It follows that $\|\chi_E\|_\Phi = \mu(E)\ [\|\chi_E\|_{M\Psi}]^{-1}$. This result throws a new light on the constant A_E in part (P 4) of the proof (preceding Definition 2) that $\varrho(|f|) = \|f\|_{M\Psi}$ satisfies the conditions (P 1) - (P 4) of Ch. 1, section 1. In the notations of that proof, we have found in (P 3) that $\|\chi_E\|_{M\Psi} = [\overset{-1}{\Psi}\{\mu(E)\}^{-1}]^{-1}$ or $\|\chi_E\|_{M\Psi} = l^{-1}$ for any set E of finite positive measure, hence by our present result $\|\chi_E\|_\Phi = \mu(E)\ \overset{-1}{\Psi}\{\mu(E)\}^{-1}$ or $\|\chi_E\|_\Phi = l\mu(E)$. The value $A_E = \mu(E)\ \overset{-1}{\Psi}\{\mu(E)\}^{-1}$ or $A_E = l\mu(E)$ in (P 4) is therefore sharp, since $\int_E |f| d\mu \leqslant \|\chi_E\|_\Phi\ \|f\|_{M\Psi}$ is sharp.

Lemma 3. *Let Φ and Ψ be complementary. If $f \in L_\Phi$ (or $f \in L_\Psi$), and E_n $(n = 1, 2, \ldots)$ is a finite or countable collection of disjoint subsets of Δ of finite positive measure, then $\|f\|_\Phi$ (or $\|f\|_\Psi$) is not increased if on each E_n we replace $f(x)$ by its average on E_n, i.e. by $\int_{E_n} f d\mu / \mu(E_n)$.*

Proof. Writing $\bar{f}$ for the function obtained by replacing f by its average on all E_n, and assuming that $\|g\|_{M\Psi} \leqslant 1$, we find

$$\int |\bar{f}g| d\mu = \Sigma_{n=1}^{\infty} \int_{E_n} 1/\mu(E_n) |\int_{E_n} f d\mu| . |g|\, d\mu + \int_{\Delta - \cup E_n} |fg| d\mu \leqslant$$
$$\Sigma_{n=1}^{\infty} \int_{E_n} |f|\ [1/\mu(E_n) \int_{E_n} |g| d\mu]\ d\mu + \int_{\Delta - \cup E_n} |fg| d\mu = \int |f\bar{g}| d\mu,$$

where $\bar{g}$ is obtained by replacing $|g|$ by its average, so $\|\bar{g}\|_{M\Psi} \leqslant 1$ by Lemma 2. Hence $\|\bar{f}\|_\Phi = \sup \int |\bar{f}g| d\mu \leqslant \sup \int |f\bar{g}| d\mu \leqslant \|f\|_\Phi$.

In section 1 of the present chapter we have obtained (Lemmas 1–4) some necessary and sufficient conditions in order that $P_\Psi \subset P_{\Psi_1}$.

In the next theorems we shall discuss a similar problem: we wish to find necessary and sufficient conditions in order that $L_\Psi \subset L_{\Psi_1}$ (i.e. in order that $\|f\|_\Psi < \infty$ implies $\|f\|_{\Psi_1} < \infty$).

Theorem 4. *If Ψ and Ψ_1 are Young functions, then $L_\Psi \subset L_{\Psi_1}$ if and only if there exists a constant $C > 0$ such that $\|f\|_{\Psi_1} \leqslant C \|f\|_\Psi$ for every $f \in L_\Psi$.*

Proof. If such a $C > 0$ exists, then evidently $L_\Psi \subset L_{\Psi_1}$. Let conversely $L_\Psi \subset L_{\Psi_1}$, and let the sequence $f_n \in L_\Psi$ $(n = 1, 2, \ldots)$ be such that $\lim \|f_n - f\|_\Psi = 0$, $\lim \|f_n - g\|_{\Psi_1} = 0$. Then f_n contains a subsequence g_n which converges pointwise to f, and, since $\lim \|g_n - g\|_{\Psi_1} = 0$, g_n contains a subsequence which converges pointwise to g. Hence $f = g$. This shows that the identity mapping of L_Ψ into L_{Ψ_1} is closed, and therefore bounded by the closed graph theorem. A second proof, which does not use the closed graph theorem, is as follows: If Φ and Φ_1 are complementary to Ψ and Ψ_1 respectively, and $L_\Psi \subset L_{\Psi_1}$, then $L_\Phi \supset L_{\Phi_1}$, since $g \in L_{\Phi_1}$ implies $\int |fg| d\mu < \infty$ for every $f \in L_{\Psi_1}$, hence for every $f \in L_\Psi$, so $g \in L_\Phi$. Assuming now that $\|f\|_{\Psi_1} \leqslant C \|f\|_\Psi$ is violated for every $C > 0$ by some $f \in L_\Psi$, there exists a sequence f_n such that $\|f_n\|_\Psi = 1$, $\|f_n\|_{\Psi_1} \geqslant n$. The linear functionals $\int f_n g d\mu$ on L_{Φ_1} $(\subset L_\Phi)$ are uniformly bounded in n for every $g \in L_{\Phi_1}$ since $|\int f_n g d\mu| \leqslant \|f_n\|_\Psi \|g\|_{M\Phi} = \|g\|_{M\Phi}$ for all n; hence, by the Banach-Steinhaus Theorem, $\|f_n\|_{\Psi_1} \leqslant M$ for some constant M, contradicting $\|f_n\|_{\Psi_1} \geqslant n$.

Next we ask if it is possible to find a condition for the functions Ψ and Ψ_1 themselves which is equivalent to $L_\Psi \subset L_{\Psi_1}$. The reader is reminded that in the Lemmas 1 – 4 of section 1 we have derived that $P_\Psi \subset P_{\Psi_1}$ for the Young classes P_Ψ and P_{Ψ_1} if and only if there exists a constant $M > 0$ such that $\Psi_1(v) \leqslant M\Psi(v)$ for large v, small v or all v, depending upon the further particulars of the situation. It is certainly not true that $L_\Psi \subset L_{\Psi_1}$ is equivalent to the same condition $\Psi_1(v) \leqslant M\Psi(v)$, since, choosing $\Psi_1(v) = \Psi(2v)$, we have always $L_\Psi \subset L_{\Psi_1}$, but not always $\Psi(2v) \leqslant M\,\Psi(v)$. Theorem 5 below, however, will show that insertion of an additional positive constant saves the situation. Before stating and proving this theorem we collect some remarks about L_∞ in a lemma. For reasons of convenience we shall say that Ψ jumps whenever there is a number v_0 $(0 < v_0 < \infty)$ such that $\Psi(v) = \infty$ for $v > v_0$.

Lemma 4. (a) *If Ψ jumps, then always $L_\Psi \subset L_\infty$.*

(b) *If $\mu(\Delta) < \infty$, and Ψ, Ψ_1 are Young functions such that Ψ*

jumps, then $L_\Psi = L_\infty \subset L_{\Psi_1}$ *(the sign of equality means that* L_Ψ *and* L_∞ *contain the same functions, and not necessarily that* $\|f\|_\Psi = \|f\|_\infty$*).*

(c) *If* $\mu(\Delta) = \infty$, *then* $L_\infty \subset L_{\Psi_1}$ *if and only if there exists a number* $v_0 > 0$ *such that* $\Psi_1(v) = 0$ *for* $v \leqslant v_0$.

(d) *If* $\mu(\Delta) = \infty$, *and* Δ *is the union of a countable number of atoms* p_n *of measure* b_n *such that* $0 < \lim\inf b_n \leqslant \lim\sup b_n < \infty$, *then* $L_\Psi \subset L_\infty$ *for any* Ψ.

(e) *Under the same hypotheses on* Δ *and* μ *as in (d),* $L_\infty = L_{\Psi_1}$ *if and only if* $\Psi_1(v) = 0$ *for* $v \leqslant v_0$, *where* $v_0 > 0$.

Proof. (a) Has been proved already in the proof of $M_\Psi(f/\|f\|_\Psi) \leqslant 1$.

(b) It is sufficient to prove that $L_\infty \subset L_{\Psi_1}$ (since, once this is proved, the choice $\Psi_1 = \Psi$ gives $L_\infty \subset L_\Psi \subset L_\infty$ by (a)). If $f \in L_\infty$, then $|f(x)| \leqslant M$ a.e., and $\Psi_1(kM) < \infty$ for some $k > 0$. Hence $M_{\Psi_1}(kf) < \infty$, so $f \in L_{\Psi_1}$.

(c) If $L_\infty \subset L_{\Psi_1}$, then $f(x) = 1 \in L_{\Psi_1}$. Hence $g(x) = k \in P_{\Psi_1}$ for some $k > 0$, so $\Psi_1(k) = 0$. Conversely, if $\Psi_1(v) = 0$ for $v \leqslant v_0$, where $v_0 > 0$, then $f(x) = v_0 \in P_{\Psi_1}$, so $f \in L_\infty$ implies $f \in L_{\Psi_1}$.

(d) If $f \in L_\Psi$, then $kf \in P_\Psi$ for some $k > 0$, hence $\Sigma b_n \Psi(kf_n) < \infty$. It follows that $kf(x)$ is bounded, so $f \in L_\infty$.

(e) On account of (c) it is sufficient to prove that $L_\infty \subset L_{\Psi_1}$ implies $L_\infty = L_{\Psi_1}$. This follows from (d).

Theorem 5. (a) *Let* $\mu(\Delta) < \infty$, *and let* Δ *contain a subset of positive measure which is free of atoms. Then, if* Ψ *jumps,* $L_\Psi = L_\infty \subset L_{\Psi_1}$ *is true for any* Ψ_1. *If* Ψ *does not jump, then* $L_\Psi \subset L_{\Psi_1}$ *if and only if there exist constants* $a > 0$, $b > 0$, $v_0 > 0$ *such that* $\Psi_1(av) \leqslant b\Psi(v)$ *for* $v \geqslant v_0$.

(b) *Let* $\mu(\Delta) = \infty$, *and let* Δ *contain a subset of infinite measure which is free of atoms. Then* $L_\Psi \subset L_{\Psi_1}$ *if and only if there exist constants* $a > 0$, $b > 0$ *such that* $\Psi_1(av) \leqslant b\,\Psi(v)$ *for all* $v \geqslant 0$.

(c) *Let* $\mu(\Delta) = \infty$, *and let* Δ *be the union of atoms* p_n *of measure* b_n *such that* $0 < \lim\inf b_n \leqslant \lim\sup b_n < \infty$. *Then* $L_\Psi \subset L_{\Psi_1}$ *if and only if there exist constants* $a > 0$, $b > 0$, $v_0 > 0$ *such that* $\Psi(v_0) > 0$ *and* $\Psi_1(av) \leqslant b\Psi(v)$ *for* $v \leqslant v_0$.

Proof. (a) The statement for a function Ψ which jumps has already been proved in Lemma 4(b). Assume therefore that Ψ does not jump. It is evident that $\Psi_1(av) \leqslant b\,\Psi(v)$ for $v \geqslant v_0$ implies $L_\Psi \subset L_{\Psi_1}$. In order to prove the converse, we restate (in slightly different words) what has been proved in section 1, Lemma 1: There exists a constant $A > 0$, not depending upon Ψ or Ψ_1, but only on the set Δ (in fact, one may choose $A = \mu(\Delta)$), with the property that if any f satisfying $\int \Psi|f| d\mu \leqslant A$ also satisfies $\int \Psi_1|f| d\mu < \infty$, then there exist constants $b > 0$, $v_0 > 0$ such that $\Psi_1(v) \leqslant b\,\Psi(v)$ for $v \geqslant v_0$. This follows from the fact that otherwise there is an increasing sequence v_n such that $\Psi_1(v_n) > 2^n \Psi(v_n)$ and $\Psi(v_n) > 1$ for all n ($\Psi(v_n) > 1$ since Ψ does not jump). Choosing now the disjoint sets E_n such that $\mu(E_n) = \mu(E_0)/[2^n \Psi(v_n)]$, we find as in section 1, Lemma 1 a function f satisfying $\int \Psi|f| d\mu \leqslant \mu(\Delta)$, $\int \Psi_1|f| d\mu = \infty$. Furthermore, before starting on the principal part of the proof, we observe that $\int \Psi|f| d\mu \leqslant A$ implies $\|f\|_\Psi \leqslant A'$, where $A' = \max(2, 2A)$. In fact, if $A \leqslant 1$, then $\|f\|_{M\Psi} \leqslant 1$ by Lemma 1(b), hence $\|f\|_\Psi \leqslant 2$, and if $A > 1$, then $\int \Psi(|f|/A)\, d\mu \leqslant A^{-1} \int \Psi|f| d\mu \leqslant 1$, hence $\|f\|_{M\Psi} \leqslant A$, so $\|f\|_\Psi \leqslant 2A$.

Let now $L_\Psi \subset L_{\Psi_1}$, so $\|f\|_{\Psi_1} \leqslant C\|f\|_\Psi$ for every $f \in L_\Psi$ by Theorem 4. We define $\Psi_2(v)$ by $\Psi_2(v) = \Psi_1(v/CA')$. For any f satisfying $\int \Psi|f| d\mu \leqslant A$ we have evidently $f \in L_\Psi \subset L_{\Psi_1}$, hence $M_{\Psi_1}(f/\|f\|_{\Psi_1}) \leqslant 1$, so that certainly $M_{\Psi_1}(f/C\|f\|_\Psi) \leqslant 1$. But then also $M_{\Psi_1}(f/CA') \leqslant 1$, so $\int \Psi_2|f| d\mu \leqslant 1$. Applying now to Ψ and Ψ_2 what has been observed above, we may conclude that there exist constants $b > 0$, $v_0 > 0$ such that $\Psi_2(v) \leqslant b\,\Psi(v)$ for $v \geqslant v_0$. Hence, recalling the definition of $\Psi_2(v)$, and writing $a = (CA')^{-1}$, we finally obtain $\Psi_1(av) \leqslant b\Psi(v)$ for $v \geqslant v_0$.

(b) It is evident that $\Psi_1(av) \leqslant b\Psi(v)$ for all $v \geqslant 0$ implies $L_\Psi \subset L_{\Psi_1}$. Let conversely $L_\Psi \subset L_{\Psi_1}$, and suppose first that Ψ is such that $\Psi(v) = 0$ for $0 \leqslant v \leqslant l$, $\Psi(v) = \infty$ for $v > l$, hence $L_\Psi = L_\infty$. Then, by Lemma 4 (c), there is a number $v_0 > 0$ such that $\Psi_1(v) = 0$ for $v \leqslant v_0$. Hence $\Psi_1(av) \leqslant \Psi(v)$ for some $a > 0$ and all $v \geqslant 0$. Assume now that Ψ is not of this exceptional type. Then, using now section 1, Lemma 3, and arguing similarly as in (a) above, we obtain $\Psi_1(a_1 v) \leqslant b_1\,\Psi(v)$ for $v \geqslant v_1$ and $\Psi_1(a_2 v) \leqslant b_2\,\Psi(v)$ for $v \leqslant v_2$, where $\Psi(v_2) > 0$. Hence if $a = \min(a_1, a_2)$ and $b = \max(b_1, b_2, \max \Psi_1(av)/\Psi(v)$ for $v_2 \leqslant v \leqslant v_1)$, we have $\Psi_1(av) \leqslant b\Psi(v)$ for all $v \geqslant 0$.

(c) We first prove that $\Psi_1(av) \leqslant b\,\Psi(v)$ for $v \leqslant v_0$, where $\Psi(v_0) > 0$, implies $L_\Psi \subset L_{\Psi_1}$. If $f \in L_\Psi$, then $kf \in P_\Psi$ for some $k > 0$, hence $\Sigma b_n \Psi(kf_n) < \infty$. Since $\Psi(kf_n) \to 0$ as $n \to \infty$, we have $\Psi(kf_n) < \Psi(v_0)$ for $n \geqslant N$, hence $\Sigma_{N+1}^\infty b_n\, \Psi_1(akf_n) < \infty$. It may happen that $\Sigma_1^N b_n\, \Psi_1(akf_n) = \infty$ if Ψ_1 jumps; there exists, however, a positive number $a' < a$ such that $\Sigma_1^N b_n\, \Psi_1(a'kf_n) < \infty$. It follows that $a'kf \in P_{\Psi_1}$, so $f \in L_{\Psi_1}$. Let conversely $L_\Psi \subset L_{\Psi_1}$, and suppose first that Ψ is such that $\Psi(v) = 0$ for $0 \leqslant v \leqslant l$, $\Psi(v) = \infty$ for $v > l$, hence $L_\Psi = L_\infty$. Then, by Lemma 4(c), there is a number $v_0 > 0$ such that $\Psi_1(v) = 0$ for $v \leqslant v_0$. Hence $\Psi_1(av) \leqslant \Psi(v)$ for some $a > 0$ and all $v \geqslant 0$. If Ψ is not of this exceptional type, an appeal to section 1, Lemma 4 leads to $\Psi_1(av) \leqslant b\,\Psi(v)$ for $v \leqslant v_0$, where $\Psi(v_0) > 0$.

Remarks. (1) A direct proof, similar to the proofs of the Lemmas 1–4 in section 1, is possible. No use is made then of Theorem 4.

(2) A theorem of similar nature, but of a more restricted type (Δ is a compact subset of R_n, μ is Lebesgue measure), has been stated recently by M. A. Krasnoselskiĭ and Y. B. Rutickiĭ [1] without proof.

3. Reflexivity and Separability.

Let Φ be an arbitrary Young function, and L_Φ the corresponding Orlicz space. The main purpose of the present section is to investigate what conditions are necessary and sufficient in order that L_Φ be reflexive or separable. As we have seen in section 2, the Orlicz spaces L_Φ and $L_{M\Phi}$ consist of the same elements, and their norms are equivalent; hence L_Φ is reflexive or separable if and only if $L_{M\Phi}$ is so. We need not distinguish, therefore, between these spaces, and for reasons of convenience we shall formulate all theorems for L_Φ. In Ch. 1, section 2 the subspaces X^χ and X^b of X were introduced, and a first step towards solving the proposed problems will be the investigation of these subspaces in the case that $X = L_\Phi$ (obviously L_Φ^χ and L_Φ^b contain the same elements as $L_{M\Phi}^\chi$ and $L_{M\Phi}^b$ respectively). For this purpose it is convenient to introduce the following definition (compare H. Nakano [1, 2]):

Definition 1. *The element $f \in L_\Phi$ is said to be a finite element if $M_\Phi(kf) < \infty$ for every constant $k \geqslant 0$. The class of all finite elements of L_Φ is denoted by L_Φ^f.*

A similar definition may be given for the complementary Orlicz space L_Ψ. As observed before, it may happen that the Young function Ψ jumps, and this happens if and only if L_Ψ^f contains only the null function. In order to prove this statement, let first $\Psi(v) = \infty$ for $v > l$, and assume that there exists an element $f \in L_\Psi^f$ such that $f \neq 0$ on a set of positive measure. Then there exists a number $\varepsilon > 0$ such that $|f(x)| > \varepsilon$ on a set E of finite positive measure, hence $g(x) = \varepsilon\chi_E \in L_\Psi^f$. But $M_\Psi(kg) = \mu(E)\Psi(k\varepsilon)$, so $M_\Psi(kg) = \infty$ for all $k > l/\varepsilon$, which leads to a contradiction. If, conversely, L_Ψ^f consists only of the null function and E is a set of finite positive measure, then $M_\Psi(k\chi_E) = \infty$ for some $k > 0$, hence $\mu(E)\Psi(k) = \infty$, and this shows that $\Psi(v) = \infty$ for sufficiently large v. On account of the property that $L_\Psi^f = \{0\}$, the case that Ψ jumps shows exceptional features, and for that reason we shall discuss it separately. It hardly needs mentioning that the theory for a function Ψ which does not jump (i.e. if v is finite, then $\Psi(v)$ is finite) is not different from the theory for the function Φ.

Lemma 1. *The class L_Φ^f is a normal subspace of L_Φ.*

Proof. It is evident that L_Φ^f is linear and normal. In order to prove that L_Φ^f is closed, let $f_n \in L_\Phi^f$ and $\|f_n - f\| \to 0$ as $n \to \infty$. Then, by Lemma 1(d) in section 2, $M_\Phi\{k(f-f_n)\} \to 0$ as $n \to \infty$ for every $k \geqslant 0$. Hence, given $k > 0$, there exists an index $n_0(k)$ such that $M_\Phi\{2k(f-f_{n_0})\}$ is finite, which implies by the convexity properties of M_Φ that $2\,M_\Phi(kf) \leqslant M_\Phi\{2k(f-f_{n_0})\} + M_\Phi(2kf_{n_0}) < \infty$.

Theorem 1. $L_\Phi^b = L_\Phi^f = L_\Phi^\chi$.

Proof. We shall prove that $L_\Phi^b \subset L_\Phi^f \subset L_\Phi^\chi$. On account of the general inclusion property $L_\Phi^b \subset L_\Phi^f$ (cf. Ch. 1, section 2, Lemma 4), the desired result will follow then. In order to show that $L_\Phi^b \subset L_\Phi^f$, it is sufficient to prove that any μ-measurable bounded $f(x)$, vanishing outside a bounded set E, belongs to L_Φ^f. This, however, is evident, since $M_\Phi(kf) \leqslant \Phi(kM)\,\mu(E) < \infty$ for any $k \geqslant 0$, where $M = \sup|f(x)|$. Next, assuming that $f \in L_\Phi^f$, we shall prove that the norm of f is absolutely continuous. If E is bounded, and E_n is any decreasing sequence of subsets of E such that $\mu(E_n) \to 0$, then $g_n = |f\chi_{E_n}| \downarrow 0$ a.e. Hence, since $\Phi(kg_n) \leqslant \Phi(kf)$, we have $M_\Phi(kg_n) \to 0$ for any $k > 0$ by dominated convergence, so $\|f\chi_{E_n}\| = \|g_n\| \to 0$ by Lemma 1(d) in section 2. It remains to prove that the same is true if E_n is not necessarily decreasing. Assuming it to be false, there exists a number $\varepsilon > 0$ such that $\|f\chi_{E_n}\| > \varepsilon$ for some sequence $E_n \subset E$ satisfying $\mu(E_n) \to 0$. We may assume that $\mu(E_n) < n^{-2}$. Then, if $F_n = \cup_{i=n}^\infty E_i$, the sequence F_n is decreasing, $\mu(F_n) \to 0$ and $\|f\chi_{F_n}\| > \varepsilon$, in contradiction to what has already been proved. The proof of $\|f\chi_{\Delta-\Delta_n}\| \to 0$ is similar.

Remark. This theorem shows that L_Φ^b does not depend on the sequence $\Delta_n \subset \Delta$, but that $L_\Phi^\chi = L_\Phi^b$ is the closure (in the norm topology of L_Φ) of the set of all essentially bounded μ-measurable functions which vanish outside some set of finite measure. Moreover, if $\mu(\Delta) < \infty$, then the closure (in the norm topology of L_Φ) of $L_\infty(\Delta, \mu)$ (considered as a subset of L_Φ) is equal to $L_\Phi^\chi = L_\Phi^b$.

Theorem 2. *Let Φ and Ψ be complementary Young functions. Then*

$$(L_\Phi^\chi)^* = L_{M\Psi} \text{ and } (L_{M\Phi}^\chi)^* = L_\Psi \text{ (isometrically)},$$

and if Ψ does not jump, then

$$(L_\Psi^\chi)^* = L_{M\Phi} \text{ and } (L_{M\Psi}^\chi)^* = L_\Phi \text{ (isometrically)}.$$

Proof. Follows from the general result that $X^\chi = X^b$ implies $(X^\chi)^* = X'$ (cf. Ch. 1, section 2, Th. 2).

To complete our discussion of the subspaces L_Ψ^b and L_Ψ^χ, we finally have to investigate the case that Ψ jumps. We shall distinguish between the following cases:

(A 1) $0 < \mu(\Delta) < \infty$; Δ is free of atoms, and $\Delta_n = \Delta$ for all n.

(A 2) $0 < \mu(\Delta) < \infty$; μ is purely atomic such that Δ is the union of a countably infinite number of atoms p_n of measure b_n (arranged such that $0 < b_{n+1} \leqslant b_n$) and a set of measure zero; moreover, $\liminf b_{n+1}/b_n > 0$; $\Delta_n = \Delta$ for all n.

(A 3) $0 < \mu(\Delta) < \infty$; Δ contains atoms, but μ is not purely atomic (i.e. Δ contains a subset of finite positive measure which is free of atoms); $\Delta_n = \Delta$ for all n.

(B 1) $\mu(\Delta) = \infty$, and Δ is free of atoms.

(B 2) $\mu(\Delta) = \infty$, and Δ is the union of a set of measure zero and a countably infinite number of atoms p_n of measure b_n such that $0 < \liminf b_n \leqslant \limsup b_n < \infty$.

(B 3) $\mu(\Delta) = \infty$; Δ contains atoms, but also a set of infinite measure which is free of atoms.

In the next two lemmas we shall formulate some properties of the subspaces L_Ψ^χ and L_Ψ^b of L_Ψ in the case that Ψ jumps.

Lemma 2. *Let Ψ jump. Then, in the cases (A 1), (A 2) (even in the absence of the hypothesis that $\liminf b_{n+1}/b_n > 0$) and (A 3) the space $L_\Psi(\Delta, \mu)$ consists of the same functions as $L_\infty(\Delta, \mu)$, and $L_\Psi^b = L_\Psi$. In the case (A 1) the subspace L_Ψ^χ contains only the null function, and in the cases (A 2) and (A 3) we have $\{0\} \neq L_\Psi^\chi \neq L_\Psi$.*

Proof. We have already proved in Lemma 4(b) of section 2 that $L_\Psi(\Delta, \mu) = L_\infty(\Delta, \mu)$ if $\mu(\Delta) < \infty$ and Ψ jumps; hence, since $\Delta_n = \Delta$ for all n, it follows that $L_\Psi^b = L_\Psi$. Since $L_\Psi(\Delta, \mu) = L_\infty$

(Δ, μ), there exist (by Th. 4 in section 2) positive constants C_1 and C_2 such that $C_1\|f\|_\infty \leqslant \|f\|_\Psi \leqslant C_2 \|f\|_\infty$ for all $f \in L_\Psi$, hence $L_\Psi^\chi = L_\infty^\chi$. In the case (A 1) it is evident therefore that $L_\Psi^\chi = L_\infty^\chi = \{0\}$. In the case (A 2) L_Ψ may be considered as a space of sequences of complex numbers $f = (f_1, f_2, \ldots)$ and we have $f \in L_\Psi^\chi$ if and only if $\lim f_n = 0$. In order to prove this, we need only observe that f has an absolutely continuous norm if and only if to each $\varepsilon > 0$ corresponds and index $N(\varepsilon)$ such that $\|f_N\|_\Psi < \varepsilon$, where $f_N = (0, 0, \ldots, 0, f_{N+1}, f_{N+2}, \ldots)$, and, on account of $L_\Psi^\chi = L_\infty^\chi$, this is equivalent to $\lim f_n = 0$. The case (A 3) is combination of (A 1) and (A 2), and in this case L_Ψ^χ consists of functions f which differ from zero only on the atoms of Δ, so $f = (f_1, f_2, \ldots)$, and have the property indicated in the case (A 2) that $\lim f_n = 0$.

Lemma 3. *Let Ψ jump. Then, in the cases (B 1), (B 2) and (B 3) we have $L_\Psi(\Delta, \mu) \subset L_\infty(\Delta, \mu)$, and $L_\Psi(\Delta, \mu) = L_\infty(\Delta, \mu)$ if and only if there exists a number $v_0 > 0$ such that $\Psi(v) = 0$ for $0 \leqslant v \leqslant v_0$; furthermore $f \in L_\Psi^b$ if and only if $\lim \|f\chi_{\Delta-\Delta_n}\| = 0$. In the case (B 1) the subspace L_Ψ^χ contains only the null function, in the case (B 2) we have $L_\Psi^\chi = L_\Psi^b$, and in the case (B 3) we have $\{0\} \neq L_\Psi^\chi \neq L_\Psi^b$.*

Proof. The statements that $L_\Psi \subset L_\infty$, and that $L_\Psi = L_\infty$ if and only if there exists a number $v_0 > 0$ such that $\Psi(v) = 0$ for $0 \leqslant v \leqslant v_0$ are merely restatements of what has been proved in Lemma 4(a)(c) of section 2. Next, let $f \in L_\Psi$ and $\lim \|f\chi_{\Delta-\Delta_n}\| = 0$; we shall show that $f \in L_\Psi^b$. Since $L_\Psi \subset L_\infty$, f is essentially bounded. Hence, if $f_n(x) = f(x)$ for $x \in \Delta_n$ and $f_n(x) = 0$ elsewhere, then $f_n \in L_\Psi^b$ $(n = 1, 2, \ldots)$, so $\|f - f_n\| = \|f\chi_{\Delta-\Delta_n}\| \to 0$ as $n \to \infty$. It follows that $f \in L_\Psi^b$. If, conversely, $f \in L_\Psi^b$, then to each $\varepsilon > 0$ corresponds a bounded function g vanishing outside some bounded set such that $\|f-g\| < \varepsilon$, hence $\|f\chi_{\Delta-\Delta_n}\| \leqslant \|f-g\| < \varepsilon$ for sufficiently large n.

Since $L_\Psi \subset L_\infty$, it follows (by Th. 4 in section 2) that $\|f\|_\infty \leqslant C\|f\|_\Psi$ for some constant $C > 0$ and all $f \in L_\Psi$, hence $L_\Psi^\chi \subset L_\infty^\chi$. In the case (B 1) we know already that $L_\infty^\chi = \{0\}$, so $L_\Psi^\chi = \{0\}$. In the case (B 2), since $\liminf b_n > 0$, we have $L_\Psi^b \subset L_\Psi^\chi$, and so $L_\Psi^b = L_\Psi^\chi$ on account of the general inclusion property $X^\chi \subset X^b$ (Observe incidentally that in the case (B 2) we have $X^b = X^\chi$ for any space X of the kind discussed in Ch. 1). In the case (B 3) we have $\{0\} \neq L_\Psi^\chi \neq L_\Psi^b$, and L_Ψ^χ consists of those $f \in L_\Psi^b$ which differ from zero only on the atoms of Δ.

Corollary. *Let Φ and Ψ be complementary Young functions such that $\Psi(v) = \infty$ for sufficiently large v; let furthermore Δ be the union of a countably infinite number of atoms p_n of measure b_n such that $0 < \liminf b_n \leqslant \limsup b_n < \infty$. Then $(L_\Psi^\chi)^* = L_{M\Phi}$ and $(L_{M\Psi}^\chi)^* = L_\Phi$ (isometrically).*

Proof. Since in this case $L_\Psi^\chi = L_\Psi^b$, the result follows from the more general result that $X^\chi = X^b$ implies $(X^\chi)^* = X'$ (cf. Ch. 1, section 2, Th. 2).

Example: Let $\Phi(u) = u$ for all $u \geqslant 0$, hence $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$ and $\Psi(v) = \infty$ for $v > 1$. Let furthermore Δ be the union of a countably infinite number of atoms of measure one. Then $L_\Psi = L_{M\Psi} = l_\infty$ is the space of all bounded sequences $x = (x_1, x_2, \ldots)$ with norm $\|x\| = \sup |x_n|$, and L_Ψ^χ is the subspace c_0 of all null sequences. The corollary shows that the conjugate space of the Banach space c_0 is isometric with the space l_1 of all absolutely convergent sequences (compare S. Banach [1], p. 65).

It is interesting to observe that if Ψ jumps (i.e. if there exists a number $l > 0$ such that $\Psi(v) < \infty$ for $v < l$ and $\Psi(v) = \infty$ for $v > l$), the inclusion $L_\Psi^b \subset L_\Psi$ is always proper in the cases (B 1) and (B 3). In fact, if E is a set of finite measure such that $\mu(E - \Delta_n) > 0$ for all $n = 1, 2, \ldots$ (such sets do not exist in the case (B 2)), then $\chi_E \in L_\Psi$, but $\lim \|\chi_E \chi_{\Delta - \Delta_n}\|_{M\Psi} = \lim \|\chi_{E - \Delta_n}\|_{M\Psi} = l^{-1} > 0$, as shown by the computations in section 2 where we discussed the property (P 3). Lemma 3 shows therefore that χ_E does not belong to L_Ψ^b. This example illustrates that in general the subspace L_Ψ^b heavily depends on the choice of the sequence Δ_n ($n = 1, 2, \ldots$). We shall prove below that in the case (B 2) it may happen that $L_\Psi^b = L_\Psi$ even if Ψ jumps.

For any Young function Φ (and also for any Young function Ψ which does not jump) we have proved in Th. 1 that $L_\Phi^b = L_\Phi^f = L_\Phi^\chi \subset L_\Phi$; hence, the Orlicz space L_Φ has an absolutely continuous norm if and only if all elements of L_Φ are finite elements. This, however, is equivalent to the linearity of the Young class P_Φ. On account of what we have proved about linearity of P_Φ in Th. 3 of section 1, we obtain therefore the following theorem:

Theorem 3. *Let Φ be a Young function. Then in the cases (A 1), (A 2) and (A 3), the space L_Φ has an absolutely continuous norm if and only if Φ has the property Δ_2. In the cases (B 1) and (B 3) the* *space L_Φ has an absolutely continuous norm if and only if Φ satisfies (δ_2, Δ_2), and in the case (B 2) the space L_Φ has an absolutely continuous norm if and only if Φ satisfies δ_2.*

If Ψ jumps, then it is evident that in the cases (A 1), (A 2), (A 3), (B 1) and (B 3) the space L_Ψ never has an absolutely continuous norm, since in all these cases $L_\Psi^\chi \neq L_\Psi$. In the case (B 2), however, Th. 3 remains true even though Ψ jumps, as shown by the following theorem:

Theorem 4. *Let Ψ jump. Then, in the case (B 2) the space L_Ψ has an absolutely continuous norm if and only if Ψ has the property δ_2.*

Proof. Let $\Psi(v) < \infty$ for $v < l$ and $\Psi(v) = \infty$ for $v > l$. We replace Ψ by a Young function Ψ_1 such that Ψ_1 does not jump and $\Psi_1 = \Psi$ for small values of v ($\Psi_1(v)$ may be defined e.g. as follows: if ε is a constant such that $0 < \varepsilon < l$, we put $\Psi_1(v) = \Psi(v)$ for $0 \leqslant v \leqslant l-\varepsilon$, and for $v > l-\varepsilon$ we define $\Psi_1(v)$ as a suitable linear function). Then, by Th. 5(c) in section 2, L_Ψ and L_{Ψ_1} consist of the same functions, and so (by Th. 4 in section 2) their norms are equivalent. Hence $L_\Psi^\chi = L_{\Psi_1}^\chi$, which implies that $L_\Psi^\chi = L_\Psi$ if and only if $L_{\Psi_1}^\chi = L_{\Psi_1}$. But then it follows from Th. 3 that L_Ψ has an absolutely continuous norm if and only if Ψ has the property δ_2.

Remark. We have already stated above that in the case (B 2) it may happen that $L_\Psi^b = L_\Psi$ even though Ψ jumps. Since $L_\Psi^b = L_\Psi^\chi$ in this case, Th. 4 gives a complete answer: If Ψ jumps, and we find ourselves in the case (B 2), then $L_\Psi^b = L_\Psi$ if and only if Ψ has the property δ_2.

We add the following example: If $\Phi(u) = e^u-1$, then the complementary function Ψ satisfies $\Psi(v) = 0$ for $0 \leqslant v \leqslant 1$ and $\Psi(v) = v(\log v - 1) + 1$ for $v > 1$. The function Φ has the property δ_2, but not the property Δ_2, and the function Ψ has the property Δ_2, but not the property δ_2. Hence, in the cases (A 1), (A 2) and (A 3), we have $L_\Psi^\chi = L_\Psi$, but $L_\Phi^\chi \neq L_\Phi$. In the cases (B 1) and (B 3) we have $L_\Phi^\chi \neq L_\Phi$, $L_\Psi^\chi \neq L_\Psi$, and finally, in the case (B 2), $L_\Phi^b = L_\Phi$ but $L_\Psi^\chi \neq L_\Psi$. If, in the case (B 2), all atoms have measure one, then $L_\Phi = P_\Phi$ consists of all absolutely convergent sequences, so L_Ψ consists of all bounded sequences. If $\Phi(u) = e^u - u - 1$, then $\Psi(v) = (v+1)\log(v+1) - v$, so Φ has the property δ_2, but not the property Δ_2, and Ψ has the property (δ_2, Δ_2).

In section 2 of Ch. 1 we have found in Th. 4 that the space X

is reflexive if and only if both X and X' have an absolutely continuous norm. If we apply this result to Orlicz spaces, and we base ourselves on Th. 3 and Th. 4 of the present section, we obtain the following theorem:

Theorem 5. *Let Φ and Ψ be complementary Young functions. Then in the cases (A 1), (A 2) and (A 3), the space L_Φ is reflexive if and only if both Φ and Ψ have the property Δ_2. In the cases (B 1) and (B 3) the space L_Φ is reflexive if and only if both Φ and Ψ have the property (δ_2, Δ_2). In the case (B 2) the space L_Φ is reflexive if and only if both Φ and Ψ have the property δ_2 (where it may happen in this case that Ψ jumps). Evidently, L_Φ is reflexive if and only if L_Ψ is reflexive.*

Remarks. (1) In the cases (A 1), (A 2), (A 3), (B 1) and (B 3) it is obvious that if L_Φ is reflexive, then Ψ does not jump, so that in these cases we can state that if Ψ jumps, then L_Φ is not reflexive.

(2) Let Φ and Ψ be complementary Young functions. Then Ψ has the property (δ_2, Δ_2) (i.e. $\Psi(2v) \leqslant M\,\Psi(v)$ for some fixed $M \geqslant 2$ and all $v \geqslant 0$) if and only if there exists a constant $k > 1$ such that $\Phi(ku) \geqslant 2k\Phi(u)$ for all $u \geqslant 0$. In order to prove this, let first $\Psi(2v) \leqslant M\Psi(v)$ for all $v \geqslant 0$. We may assume that $M > 2$. Then, for $k = M/2$, we have $\Phi(ku) = \max\,\{2kuv - \Psi(2v)\} \geqslant 2k$ $\max\,\{uv - \Psi(v)\} = 2k\Phi(u)$ for all $u \geqslant 0$. Conversely, if $\Phi(ku) \geqslant 2k\Phi(u)$ for all $u \geqslant 0$, then $\Psi(2v) = \sup\,\{2kuv - \Phi(ku)\} \leqslant 2k$ $\sup\,\{uv - \Phi(u)\} = 2k\Psi(v)$ for all $v \geqslant 0$. Observe that it follows from this proof that if $\Phi(ku) \geqslant 2k\Phi(u)$ is satisfied for some $k = k_0 > 1$, then it is also satisfied for all $k > k_0$. Similarly we may prove that Ψ has the property Δ_2 or δ_2 if and only if there exists a constant $k > 1$ such that $\Phi(ku) \geqslant 2k\,\Phi(u)$ for large u or small u respectively. This argument shows that it is possible to replace the conditions for reflexivity of L_Φ in Th. 5, which are in terms of both Φ and Ψ, by other conditions in terms of Φ alone. The theorem may then be restated as follows:

In the cases (A 1), (A 2) and (A 3) the space L_Φ is reflexive if and only if there exist constants $k > 1$, $M \geqslant 2k$ such that $2k \leqslant \Phi(ku)/\Phi(u) \leqslant M$ for sufficiently large u. In the cases (B 1) and (B 3) the space L_Φ is reflexive if and only if the condition is satisfied for all $u \geqslant 0$, and in the case (B 2) the space L_Φ is reflexive if and only if the condition is satisfied for sufficiently small u. In the last case it may happen that Ψ jumps, but in all other cases the condition excludes this.

We finally derive some conditions for separability of L_Φ and L_Φ^χ. In the first corollary of Th. 7 in Ch. 1, section 3 we have proved that X is separable if and only if X has an absolutely continuous norm and μ is separable. Combining this result with the Theorems 3 and 4 of the present section, we obtain:

Theorem 6. *Let Φ be an arbitrary Young function. In the cases* $(A\ 1)$ *and* $(A\ 3)$ *the space L_Φ is separable if and only if Φ has the property Δ_2 and μ is separable; in the case* $(A\ 2)$ *the space L_Φ is separable if and only if Φ has the property Δ_2. In the cases* $(B\ 1)$ *and* $(B\ 3)$ *the space L_Φ is separable if and only if Φ has the property* (δ_2, Δ_2) *and μ is separable. If Ψ is a Young function which perhaps jumps, and we are in the case* $(B\ 2)$, *then L_Ψ is separable if and only if Ψ has property δ_2.*

Remark. For the particular case that Δ is the interval $[0,1]$ and μ is Lebesgue measure (so the conditions of (A 1) are satisfied), Th. 6 was proved by W. Orlicz [2] by a completely different method. It was also recently obtained by G. G. Lorentz [4] by a method which is somewhat similar to ours. Furthermore M. A. Krasnoselskiĭ ([1], p. 69) has recently announced that he and I. Sobolev have obtained necessary and sufficient conditions in order that an Orlicz space be separable.

In the second corollary of Th. 7 in Ch. 1, section 3 we have found that if $X^\chi = X^b$, then X^χ is separable if and only if μ is separable. Hence:

Theorem 7. *In the cases* $(A\ 1)$, $(A\ 3)$, $(B\ 1)$ *and* $(B\ 3)$ *the subspace L_Φ^χ of L_Φ is separable if and only if μ is separable, and in the case* $(A\ 2)$ *the subspace L_Φ^χ is always separable. If Ψ is a Young function which perhaps jumps, and we are in the case* $(B\ 2)$, *then L_Ψ^χ is always separable.*

We finally observe that if Ψ jumps, and we find ourselves in one of the cases (A 1), (A 2), (A 3), (B 1) or (B 3), then the space L_Ψ is not separable even if μ is separable, since in these cases Ψ cannot satisfy Δ_2 or (δ_2, Δ_2).

As an example we conclude this section with the following extension of a result by J. Schauder [1]:

Theorem 8. *If μ is Lebesgue measure on the interval* $[0, 1]$ *and Φ is an arbitrary Young function, then Haar's orthogonal system* $\{\varphi_n(x)\}$ $(n = 1, 2, \ldots)$ *is a basis in the subspace* $L_\Phi^\chi([0,1], \mu)$.

Proof. We have to prove that if $f \in L_\Phi^\chi$, then $\|f - \Sigma_{i=1}^n c_i \varphi_i\|_{M\Phi} \to 0$ as $n \to \infty$, where the numbers c_i $(i = 1, 2, ...)$ are the Fourier coefficients $c_i = \int_0^1 f(x)\varphi_i(x)\,dx$ of $f(x)$. We write $s_n(f; x) = \Sigma_{i=1}^n c_i \varphi_i(x)$. Now, as A. Haar [1] has proved, $s_n(f;x)$ is a step function; more precisely, the interval $[0, 1]$ may be decomposed into $m(n)$ subintervals $[a_0, a_1], [a_1, a_2], ..., [a_{m-1}, a_m]$ such that if $a_p < x < a_{p+1}$ $(0 \leqslant p \leqslant m-1)$, then $s_n(f;x) = \int_{a_p}^{a_{p+1}} f(x)\,dx/(a_{p+1} - a_p)$. Hence, if k is any positive constant, we have by Jensen's inequality $M_\Phi(ks_n) = \Sigma_{p=0}^{m-1} \int_{a_p}^{a_{p+1}} \Phi(ks_n)\,dx = \Sigma_{p=0}^{m-1}(a_{p+1}-a_p)\, \Phi\left[\int_{a_p}^{a_{p+1}} kf(x)\,dx/(a_{p+1}-a_p)\right] \leqslant \Sigma_{p=0}^{m-1}\int_{a_p}^{a_{p+1}} \Phi(kf)\,dx = M_\Phi(kf)$, so $\|s_n\|_{M\Phi} \leqslant \|f\|_{M\Phi}$ for all n. A. Haar has also proved that if $f(x)$ is essentially bounded, then $s_n(f;x)$ converges pointwise to $f(x)$ a.e. on $[0, 1]$, and $\|s_n\|_\infty \leqslant \|f\|_\infty = M < \infty$. It follows that for such functions $\Phi(k|f-s_n|) \leqslant \Phi(2kM)$ for each constant $k > 0$, so $M_\Phi\{k(f-s_n)\} \to 0$ by the dominated convergence theorem. Hence $\|f-s_n\|_{M\Phi} \to 0$ as $n \to \infty$, which proves our statement for bounded functions.

Let now $f \in L_\Phi^\chi$ be arbitrary, and write $f_n(x) = f(x)$ for $|f(x)| \leqslant n$ $(n = 1, 2, ...)$, and $f_n(x) = 0$ elsewhere. Then $\lim M_\Phi\{k(f-f_n)\} = 0$ for each constant $k > 0$. This shows that, given $k > 0$ and $\varepsilon > 0$, we may write $f = f_1 + f_2$, where f_1 is bounded and $M_\Phi(kf_2) < \varepsilon$. Hence, by the convexity properties of M_Φ,

$$\begin{aligned}2\, M_\Phi\{k(f-s_n)/4\} &\leqslant M_\Phi\,[k\{f_1 - s_n(f_1;x)\}/2] + M_\Phi[k\{f_2-s_n(f_2;x)\}/2] \\ &\leqslant M_\Phi\,[k\{f_1-s_n(f_1;x)\}/2] + \tfrac{1}{2}\, M_\Phi(kf_2) + \tfrac{1}{2}\, M_\Phi\{ks_n(f_2;x)\} \leqslant \\ &\qquad M_\Phi[k\{f_1-s_n(f_1;x)\}/2] + \varepsilon,\end{aligned}$$

so $\lim M_\Phi\{k(f-s_n)/4\} = 0$ for any $k > 0$. It follows that $\lim \|f-s_n\|_{M\Phi} = 0$.

Remark. The above proof is similar to Schauder's proof for the Lebesgue spaces L_p, $p \geqslant 1$. Recently H. W. Ellis and I. Halperin [2] have obtained very general results on function spaces in connection with Haar's orthogonal system.

4. Uniformly Convex Orlicz Spaces.

A Banach space (elements $f, g, ...$) is called uniformly convex if to each number ε $(0 < \varepsilon \leqslant 2)$ corresponds a number $\delta(\varepsilon) > 0$ such that the conditions $\|f\| = \|g\| = 1$, $\|f-g\| \geqslant \varepsilon$ imply $\|(f+g)/2\|$

$\leqslant 1 - \delta(\varepsilon)$. This definition is due to J. A. Clarkson [1], and he proved that the Lebesgue spaces L_p and l_p $(1 < p < \infty)$ are uniformly convex. His proof was rather laborious, and has been simplified by R. P. Boas, Jr [1], but it was finally E. J. Mc Shane [1] who gave an extremely simple proof of this result. I am indebted to A. C. Zaanen for pointing out to me that Mc Shane's method can be used to obtain a sufficient condition in terms of the function $\Phi(u)$ in order that the Orlicz space $L_{M\Phi}$ be uniformly convex. It is also possible to find by the same method a sufficient condition, in terms of the metric function $\varrho(|f|)$, for the space X of Chapter 1 to be uniformly convex (cf. I. Halperin [2]).

Definition 1. *The Young function $\Phi(u)$ is said to be strictly convex if to each number a $(0 < a < 1)$ corresponds a number $\delta(a)$ $(0 < \delta(a) < 1)$ such that*

$$\Phi\{(u+bu)/2\} \leqslant \{1 - \delta(a)\}\{\Phi(u)+\Phi(bu)\}/2$$

for all $u \geqslant 0$ and all b satisfying $0 \leqslant b \leqslant a$.

As one can see easily, $\Phi(u) = u^p/p$, $p > 1$, is strictly convex in this sense. Furthermore, if $\Phi(u)$ is strictly convex and $0 < k < 1$, there exists a number $m(k)$ such that $0 < m(k) < 1$ and $\Phi(ku) \leqslant km(k)\Phi(u)$ for all $u \geqslant 0$. Hence, if $\Psi(v)$ is the complementary function of $\Phi(u)$, we have (by the Corollary of Th. 1 in section 1).

$$\Psi\{m^{-1}(k)\, v\} = \sup\,\{um^{-1}(k)\, v - \Phi(u)\} \leqslant$$
$$\sup\,\{ku\, k^{-1}\, m^{-1}(k)\, v - k^{-1}\, m^{-1}(k)\, \Phi\,(ku)\} = k^{-1}\, m^{-1}(k)\, \Psi(v)$$

for all $v \geqslant 0$, and this shows that $\Psi(v)$ has the property (δ_2, Δ_2).

Definition 2. *The functional $M_\Phi(f) = \int\Phi|f|d\mu$ is called uniformly convex if to each number $\varepsilon > 0$ corresponds a number $q(\varepsilon) > 0$ such that the conditions $M_\Phi(f) = M_\Phi(g) = 1$, $M_\Phi(f - g) \geqslant \varepsilon$ imply $M_\Phi\{(f+g)/2\} \leqslant 1 - q(\varepsilon)$.*

We wish to prove that if Φ is strictly convex and satisfies the condition (δ_2, Δ_2), then $M_\Phi(f)$ is uniformly convex. For this purpose we first prove the following lemma:

Lemma 1. *If Φ is strictly convex, and $\varepsilon > 0$, there exists a number $p(\varepsilon) > 0$ such that*

$$\Phi\left|\frac{x+y}{2}\right| \leqslant \{1 - p(\varepsilon)\}\,\frac{\Phi|x|+\Phi|y|}{2}$$

for all complex x, y satisfying $|x - y| \geqslant \varepsilon$ max $(|x|, |y|)$.

Proof. Without loss of generality we may assume that $\varepsilon \leqslant 1$, $x > 0$ and $|y| \leqslant x$, hence $|x-y| \geqslant \varepsilon\, x$. Now let Γ_1 be the closed circle in the complex plane with centre 0 and radius x, and Γ_2 the open circle with centre x and radius εx. Then y may vary in the closed set $\Gamma_1 - \Gamma_2$. On the circumference of Γ_2 we take the point a such that $\arg(a - x) = 3\pi/4$, and Γ_3 is the open circle with centre 0 and radius $|\alpha|$. Obviously there exists a number $\gamma(\varepsilon) < 1$ such that $|x + y| \leqslant \gamma(\varepsilon)\,(x + |y|)$ for all y in $(\Gamma_1 - \Gamma_2) - \Gamma_3$, hence

$$\Phi\left|\frac{x+y}{2}\right| \leqslant \gamma(\varepsilon)\,\Phi\left(\frac{x+|y|}{2}\right) \leqslant \gamma(\varepsilon)\,\frac{\Phi(x)+\Phi|y|}{2}$$

for such y. If y belongs to $\Gamma_1 - \Gamma_2$ and Γ_3 simultaneously, there exists a number $\beta(\varepsilon) < 1$ such that $|y| \leqslant \beta x$ (in fact, $\beta = |\alpha|/x$). Hence since Φ is strictly convex,

$$\Phi\left|\frac{x+y}{2}\right| \leqslant \Phi\left(\frac{x+|y|}{2}\right) \leqslant \{1-\delta(\beta)\}\,\frac{\Phi(x)+\Phi|y|}{2}$$

for such y. Choosing $1 - p(\varepsilon) = \max\{\gamma(\varepsilon), 1 - \delta(\beta)\}$, we obtain the desired result.

Theorem 1. *If Φ is strictly convex and satisfies the condition (δ_2, Δ_2), then $M_\Phi(f)$ is uniformly convex.*

Proof. Let $M_\Phi(f) = M_\Phi(g) = 1$, $M_\Phi(f-g) \geqslant \varepsilon > 0$. Without loss of generality we may assume that $\varepsilon \leqslant 1$. Let $a = \varepsilon/4$ and $E = \{x \in \Delta : |f-g| \geqslant a \max(|f|, |g|)\}$. If $x \in E$, then $\Phi|2^{-1}(f+g)| \leqslant \{1-p(a)\}\,2^{-1}(\Phi|f| + \Phi|g|)$ by Lemma 1, hence

$$1 - M_\Phi\left(\frac{f+g}{2}\right) = \frac{M_\Phi(f) + M_\Phi(g)}{2} - M_\Phi\left(\frac{f+g}{2}\right) \geqslant$$

$$\frac{M_\Phi(f\chi_E) + M_\Phi(g\chi_E)}{2} - M_\Phi\left(\frac{f+g}{2}\,\chi_E\right) \geqslant p(a)\,\frac{M_\Phi(f\chi_E) + M_\Phi(g\chi_E)}{2}.$$

If $x \in \Delta - E$, then $|f-g| \leqslant a(|f| + |g|)$, hence $\Phi|f-g| \leqslant \Phi\{2a\,(|f| + |g|)/2\} \leqslant 2a\,\Phi\{(|f| + |g|)/2\} \leqslant a\,(\Phi|f| + \Phi|g|)$, so $M_\Phi\{(f-g)\chi_{\Delta-E}\} \leqslant 2a = \varepsilon/2$. But $M_\Phi(f-g) \geqslant \varepsilon$ by hypothesis, so $M_\Phi\{(f-g)\chi_E\} \geqslant \varepsilon/2$. Since $M_\Phi\{(f-g)\,\chi_E\} = M_\Phi\{(2f-2g)\,\chi_E/2\} \leqslant 2^{-1}\{M_\Phi(2f\chi_E) + M_\Phi(2g\chi_E)\}$, this implies $M_\Phi(2f\chi_E) + M_\Phi(2g\chi_E) \geqslant \varepsilon$. But, on account of the property (δ_2, Δ_2), we have $\Phi(2u) \leqslant M\Phi(u)$ for some fixed $M > 0$ and all $u \geqslant 0$, hence $M_\Phi(f\chi_E) + M_\Phi(g\chi_E) \geqslant \varepsilon/M$. It follows that $1 - M_\Phi\{(f+g)/2\} \geqslant p(\varepsilon/4)\,\varepsilon/2M = q(\varepsilon)$, which is the desired result.

In order to prove that under the same conditions the Banach space $L_{M\Phi}$ is uniformly convex, we first prove two simple lemmas.

Lemma 2. *If Φ has the property (δ_2, Δ_2) and if $\varepsilon > 0$, there exists a number $\varepsilon_1(\varepsilon)$ such that $\|f\|_{M\Phi} \geqslant \varepsilon$ implies $M_\Phi(f) > \varepsilon_1$.*

Proof. We have to show that $M_\Phi(f_n) \to 0$ as $n \to \infty$ implies $\|f_n\|_{M\Phi} \to 0$. Since Φ has the property (δ_2, Δ_2), $M_\Phi(f_n) \to 0$ implies $M_\Phi(kf_n) \to 0$ for each constant $k > 0$, and this is equivalent to $\|f_n\|_{M\Phi} \to 0$ by Lemma 1(d) in section 2.

Remark. In this lemma we may replace $\|f\|_{M\Phi}$ by $\|f\|_\Phi$.

Lemma 3. *If Φ has the property (δ_2, Δ_2) and if $\varepsilon > 0$, there exists a number $\eta(\varepsilon) > 0$ such that $M_\Phi(f) \leqslant 1 - \varepsilon$ implies $\|f\|_{M\Phi} \leqslant 1 - \eta$.*

Proof. If the statement is false there exists a sequence f_n $(n = 1, 2, \ldots)$ such that $M_\Phi(f_n) \leqslant 1 - \varepsilon$ and $\|f_n\|_{M\Phi} \uparrow 1$. Then, for $a_n = \|f_n\|_{M\Phi}^{-1}$, we have $a_n \downarrow 1$ and $\|a_n f_n\|_{M\Phi} = 1$, hence, since $M_\Phi(kf_n) < \infty$ for each constant $k > 0$ on account of the property (δ_2, Δ_2), we have also $M_\Phi(a_n f_n) = 1$ by Lemma 1(c) in section 2. But then $1 = M_\Phi(a_n f_n) = M_\Phi\{(a_n - 1) 2 f_n + (2 - a_n) f_n\} \leqslant (a_n - 1) M_\Phi(2f_n) + (2 - a_n) M_\Phi(f_n) \leqslant (a_n - 1) M\, M_\Phi(f_n) + (2 - a_n) M_\Phi(f_n) \leqslant \{1 + (a_n - 1)(M - 1)\}(1 - \varepsilon)$, which contradicts $\varepsilon > 0$ for sufficiently large n.

Theorem 2. *If Φ is strictly convex and satisfies the condition (δ_2, Δ_2), then the Orlicz space $L_{M\Phi}$ is uniformly convex.*

Proof. Let $\varepsilon > 0$, and $\|f\|_{M\Phi} = \|g\|_{M\Phi} = 1$, $\|f - g\|_{M\Phi} \geqslant \varepsilon$. Then, by Lemma 1(c) in section 2, $M_\Phi(f) = M_\Phi(g) = 1$, and by Lemma 2 above $M_\Phi(f - g) \geqslant \varepsilon_1(\varepsilon)$. Hence $M_\Phi\{(f + g)/2\} \leqslant 1 - q(\varepsilon_1)$ by Theorem 1, so $\|(f + g)/2\|_{M\Phi} \leqslant 1 - \eta(q)$ by Lemma 3. Writing $\delta(\varepsilon) = \eta(q(\varepsilon_1(\varepsilon)))$, we obtain the desired result.

Remarks. (1) According to a result of D. Milman [1], every uniformly convex Banach space is reflexive (for a short proof we refer to H. Nakano [2]). Hence, if Φ is strictly convex and satisfies (δ_2, Δ_2) then $L_{M\Phi}$ is reflexive. This result, however, is already incorporated in Th. 5 of section 3 since, as we have shown, strict convexity of Φ implies the property (δ_2, Δ_2) for the complementary Young function Ψ.

(2) It is possible to prove that under the same hypotheses on Φ not only $L_{M\Phi}$, but also L_Φ is uniformly convex (for the method of proof we refer to H. Nakano [2], §§ 87, 88).

5. The Conjugate Space of an Orlicz Space.

Since the associate space $L_{M\Psi}$ of L_Φ and the conjugate space L_Φ^* of L_Φ are identical if and only if $L_\Phi^\chi = L_\Phi$, we may state that in general $L_{M\Psi}$ is a proper subspace of L_{Φ}^{*}. Denoting by $\|f^*\|_{M\Psi}$ the norm of an element $f^* \in L_{\Phi}^{*}$, and by $\|f^*\|_{\Psi}$ the norm of an element $f^* \in L_{M\Phi}^{*}$, these norms in L_{Φ}^{*} and $L_{M\Phi}^{*}$ are therefore significant extensions of $\|g\|_{M\Psi}$ and $\|g\|_{\Psi}$ in the spaces $L_{M\Psi}$ and L_{Ψ} respectively. Evidently $\|f\|_{M\Phi} \leqslant \|f\|_{\Phi} \leqslant 2\,\|f\|_{M\Phi}$ implies $\|f^*\|_{M\Psi} \leqslant \|f^*\|_{\Psi} \leqslant 2\,\|f^*\|_{M\Psi}$ for any $f^* \in L_{\Phi}^{*}$. The norms $\|g^*\|_{M\Phi}$ and $\|g^*\|_{\Phi}$ are introduced similarly. However, we may go further. For any $f^* \in L_{\Phi}^{*}$ we define $M_{\Psi}(f^*) = \sup\,\{|f^*(f)| - M_{\Phi}(f)\}$ for all $f \in L_{\Phi}$, and for any $g^* \in L_{\Psi}^{*}$ we define $M_{\Phi}(g^*) = \sup\,\{|g^*(g)| - M_{\Psi}(g)\}$ for all $g \in L_{\Psi}$. It may be proved now that if in particular $f^* = g \in L_{\Psi}$, then this new $M_{\Psi}(f^*)$ is the old $M_{\Psi}(g)$, and similarly, if $g^* = f \in L_{\Phi}$, then the new $M_{\Phi}(g^*)$ is the old $M_{\Phi}(f)$. Furthermore, if $f \in L_{\Phi}$, then $M_{\Phi}(f) = \sup\,\{|f^*(f)| - M_{\Psi}(f^*)\}$ for all $f^* \in L_{\Phi}^{*}$, and if $g \in L_{\Psi}$, then $M_{\Psi}(g) = \sup\,\{|g^*(g)| - M_{\Phi}\,(g^*)\}$ for all $g^* \in L_{\Psi}^{*}$. If $g \in L_{\Psi}$, there exists a relation between $\|g\|_{M\Psi}$ and M_{Ψ}, viz. $\|g\|_{M\Psi} = \inf k^{-1}$ for all $k > 0$ such that $M_{\Psi}(kg) \leqslant 1$. It turns out that this relation remains true for any $f^* \in L_{\Phi}^{*}$, i.e. $\|f^*\|_{M\Psi} = \inf k^{-1}$ for all $k > 0$ such that $M_{\Psi}(kf^*) \leqslant 1$. Similarly $\|g^*\|_{M\Phi} = \inf k^{-1}$ for all $k > 0$ such that $M_{\Phi}(kg^*) \leqslant 1$. The idea to extend M_{Φ} and M_{Ψ} as indicated above is suggested by H. Nakano's work on modulared spaces ([2], Ch. XI), but Nakano had no imbedding problem to consider.

Finally, suggested by a theorem of I. Amemiya in modulared spaces, we can prove that for any $f^* \in L_{\Phi}^{*}$ we have $\|f^*\|_{\Psi} = \inf_{k>0} \{M_{\Psi}(kf^*) + 1\}/k$.

The proofs of these results will appear in a joint publication with A. C. Zaanen.

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STELLINGEN

behorende bij W. A. J. Luxemburg, Banach Function Spaces,
Delft 12 October 1955.

I

Als men eigenschap (P2) van de metrische functie $\varrho(f)$ vervangt door de eigenschappen:

(P2a): als $0 \leqslant f(x) \leqslant g(x)$, dan is $\varrho(f) \leqslant \varrho(g)$,

(P2b): als $0 \leqslant f_n(x) \uparrow f(x)$ en $\sup_n \varrho(f_n) < \infty$, dan is $\varrho(f) < \infty$,

die samen zwakker zijn dan (P2), dan is in het algemeen de Banachruimte X niet meer perfect. In dit geval kan men echter wel laten zien dat X en zijn tweede geassocieerde ruimte X'' uit dezelfde elementen bestaan.

De norm $\|f\|_L = \inf_{0 \leqslant f_n \uparrow |f|} \{\lim \|f_n\|\}$, die in dit verband is ingevoerd door G. G. Lorentz, is dezelfde als de norm in de tweede geassocieerde ruimte X'' van X.

Hoofdstuk 1, par. 1 van dit proefschrift en G. G. Lorentz, Spaces of Measurable Functions (nog niet gepubliceerd).

II

De Banachruimte X is dan en slechts dan reflexief indien X een absoluut continue norm heeft en de eenheidsbol van X compact is in de $\sigma(X,X')$ topologie.

Hoofdstuk 1, par. 2-3 van dit proefschrift.

III

De door H. Nakano ontwikkelde dualiteitstheorie voor lineaire topologische vectorruimten is van een te algemeen karakter; die van J. Dieudonné (N. Bourbaki) verdient de voorkeur.

H. Nakano, Topology and linear Topological Spaces, Tokyo 1951. J. Dieudonné, La dualité dans les espaces vectoriels topologiques, Ann. École Norm. (3), 59 (1942), 107-139.

IV

De n - dimensionale Euclidische ruimte R_n is de vereniging van een verzameling met Lebesgue maat nul en een verzameling van de eerste categorie.

Wisk. Opg. XX no. 15.

V

In een separabele (complexe) Hilbertruimte H geldt de volgende aanvulling van de stelling van Riesz-Fischer: als $f \in H$ en a_k is een rij (complexe) getallen zodanig dat $\Sigma\ |a_k|^2 = \|f\|^2$, dan bestaat er een orthonormale basis φ_n in H zodanig dat $(f, \varphi_k) = a_k$.

VI

De stelling, die T. Takahashi geeft over gegeneraliseerde convergentie van rijen van functies, is foutief.

T. Takahashi, Note on the mean convergence of a sequence of functions, Proc. Phys. Math. Soc. of Japan (III), 15 (1933), 339-343.

VII

In de verzamelingstheoretische topologie verdient het begrip convergentie volgens gerichte verzamelingen (Moore-Smith convergentie) de voorkeur boven het daarmede equivalent zijnde begrip convergentie volgens filters.

N. Bourbaki, Livre III Topologie Générale, Chap. I
J. L. Kelley, General Topology, Ch. 2.

VIII

Laat $\mu(\Delta) = \infty$, laat de maat μ totaal σ-eindig en atoomvrij op Δ zijn, en laat Φ een willekeurige Young functie zijn. Indien voor iedere karakteristieke functie χ_E van de μ-meetbare deelverzamelingen E van Δ in $L_\Phi(\Delta,\mu)$ geldt dat $\|\chi_E\|_\Phi = \|\chi_E\|_{M\Phi}$, dan is de Orlicz ruimte L_Φ of wel de ruimte L_1 of wel de ruimte L_∞. Geldt daarentegen $\|\chi_E\|_\Phi = 2\ \|\chi_E\|_{M\Phi}$, dan is $\Phi(u) = u^2/2$ voor alle $u \geqslant 0$.

Hoofdstuk 2, par. 2, St. 3 van dit proefschrift.

IX

In Hoofdstuk 2 van dit proefschrift hebben we stellingen uit de theorie der L_p-ruimten uitgebreid voor Orliczruimten; op dezelfde manier kunnen we bekende stellingen uit de theorie der H_p-ruimten uitbreiden.

X

Laat Φ een willekeurige Young functie zijn. Dan geldt de volgende uitbreiding van een bekende stelling van S. Bernstein: Als $t_n(x)$ een trigonometrisch polynoom van de n^e graad is, dan geldt in $L_\Phi\,[0, 2\pi]$ dat $\|t_n'\|_{M\Phi} \leqslant n\,\|t_n\|_{M\Phi}$ en $\|t_n'\|_\Phi \leqslant n\,\|t_n\|_\Phi$.

Vgl. A. Zygmund, Trigonometrical Series, par. 7.31.

XI

De constante p^p, $0 < p < 1$, in de ongelijkheid

$$\Sigma\left(\frac{a_n + a_{n+1} + \ldots}{n}\right)^p > p^p\,\Sigma\, a_n^p\,,\ a_n > 0,\ n = 1,2,\ldots$$

kan vervangen worden door de constante $\operatorname{Max}\left(p^p,\ \frac{1}{2}\left(\frac{1+p}{1-p}\right)^{1-p}\right)$.

Vgl. G. H. Hardy - J. E. Littlewood, - G. Polya, Inequalities, Th. 345.

XII

De door N. G. de Bruijn gegeven schatting $|a_n| \leqslant n|a_1|$ voor de coëfficienten van de Taylorontwikkeling $w = f(z) = a_1z + a_2z^2 + \ldots + a_nz^n + \ldots$ van een in $|z| < 1$ regulier analytische functie, die de eenheidscirkel univalent afbeeldt op een gebied G, dat stervormig is t.o.v. een niet in G in het eindige gelegen punt, kan in zoverre aangevuld worden, dat bovendien $\sup_n |a_n|$ eindig is.

N. G. de Bruijn, Ein Satz über schlichte Funktionen, Indag. Math, 3 (1941), 8-10.

XIII

Een in $|z| < 1$ regulier analytische oneven functie $w = f(z) = a_1z + a_3z^3 + \ldots + a_{2n+1}\,z^{2n+1} + \ldots$, die de eenheidscirkel univalent afbeeldt op een gebied G, dat stervormig is t.o.v. een in G gelegen punt, voldoet aan de ongelijkheden $|a_{2n+1}| \leqslant |a_1|$.

XIV

In de leerboeken over functietheorie behoort zeker plaats ingeruimd te worden voor de middelwaardestelling voor analytische functies.

Vgl. W. A. J. LUXEMBURG, De middelwaardestelling voor analytische functies, Handelingen van het XXXIIIe Nederlands Natuur en Geneeskundig Congres, Leiden 1953.

XV

De wijze, waarop S. Goldstein de theorie der kanonieke transformaties behandelt, is niet bevredigend.

S. Goldstein, Classical Mechanics, Ch. 8.

XVI

De afleiding, die J. M. Luttinger en C. Kittel geven van de formule voor de frequente van de ferromagnetische resonantieabsorptie, is niet gerechtvaardigd.

J. M. LUTTINGER - C. KITTEL, Eine Bemerkung zur Quantentheorie der ferromagnetischen Resonanz, Helv. Phys. Acta, 21 (1948), 480-482.