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SAFFMAN-TAYLOR INSTABILITY AND INTERFACE CAPTURING BY SEMI-LAGRANGIAN LEVEL SET METHOD

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Abstract. *This work presents some results concerning the interface evolution of two imiscible fluids flowing inside an isotropic porous medium in a typical viscous instability problem, known as the Saffman-Taylor instability. This involves the application of an iterative parallelizable graph based algorithm coupled with a semi-Lagrangian level set advection method.*

1 INTRODUCTION

Many physical problems deal with the displacement of a resident fluid, by the injection of a less viscous one. In such experiment the displacing fluid tends to penetrate into the region occupied by the displaced one, in such way that, very small variations in the shape of the interface between then, become bigger and bigger, as time goes by. The amplified variation of small perturbations of the interface is known as viscous finger and this behaviour is characteristic of Saffman-Taylor instability.

The first work to treat the instability of viscous origin is due to Saffman and Taylor[1], in 1958. The authors studied the viscous instability of fluids in a porous medium and in the Hele-Shaw cells. Several works on Saffman-Taylor instability have appeared since then. There are works on experiments[2,3], on secondary oil recovery in petroleum reservoir[4,5,6], on the numerical study of the instability[7,8,9,10], and, among others, to hydrodynamics[11], for example.

We proceed to an investigation of the Saffman-Taylor instability in a isotropic porous medium through the interface capturing mathod called semi-Lagrangian level set method (SLLS). This method is a combination of a semi-Lagrangian scheme and the level set

method introduced by Osher and Sethian[12], in 1988, for problems concerning moving interfaces involving split and merge of curves, anisotropy and curvatures. The successful combination of these methods was first proposed by Strain[13,14], in two works published in 1999. The same combination is used by Min[15] in the study that involves the simulation of waves in a two-dimensional space, and by Moura Neto *at al*[16], in a study involving tracking the interface of two imiscible fluids inside a porous media. This last work compares the level set method with the semi-Lagrangian implementation with the classic up-wind implementation, and reveals the advantage of the first one. Enright *et al*[17] compare the semi-Lagrangian method with the advection schemes TVD Runge-Kutta and HJ-WENO, in the simulation of the deformation of materials due to a rotational field and Gutierrez[18] *et al* investigate the conservation of mass in the solution of Navier-Stokes equation with free surface.

The velocity field associated with the fluid flow in porous media is determined here through an iterative method introduced by Douglas Jr.[19] *et al* in the context of mixed finite element method, which was generalized by Gonçalves and Moura Neto[20] on a graph setting.

In the next section we formulate the semi-Larangian level set method and in section 3 we tackle the problem of Saffman-Taylor and its simulation through LSSL method.

2 SEMI-LAGRANGIAN LEVEL SET METHOD

2.1 Level set method

The level set method is based on a smooth representation of the interface through an scalar field ϕ, the level set function, and the evolution equation which preserves its representation.

Let Γ_t be the interface (a simple curve or a set of curves) at time t. For each fixed time t, the interface is represented by the level set function $\phi(\cdot, t)$,

$$\Gamma_t = \left\{ \mathbf{x} \in I\!R^2 \mid \phi(\mathbf{x}, t) = 0 \right\} . \tag{1}$$

The evolution equation for ϕ must be compatible with the Eq. (1). So, assume that $\mathbf{x} = \mathbf{x}(t)$ is a point in the interface, $\mathbf{x}(t) \in \Gamma_t$. Besides, assume that the interface evolves according to the velocity field $\mathbf{u}$,

$$\dot{\mathbf{x}} = \mathbf{u}(\mathbf{x}, t) , \tag{2}$$

where $\dot{\mathbf{x}}$ represents the derivative of $\mathbf{x}$ in relation to the time t. In this way, the evolution equation for ϕ, is

$$\phi_t + \mathbf{u} \cdot \nabla\phi = 0 . \tag{3}$$

To complete the description of the level set method, in an analytical level, we need to specify initial conditions for ϕ, compatible with Eq. (1). That is, it is necessary to define $\phi(\cdot, 0)$ in such way that $\Gamma_0 = \{x \in I\!R \mid \phi(\mathbf{x}, 0) = 0\}$. A simple manner to do so is to consider $\phi_0(\mathbf{x}) = \phi(\mathbf{x}, 0)$ the signed distance function to the interface: if $\mathbf{x}$ is inside

the region delimited by the interface, $\phi(\mathbf{x}, 0)$ is negative; if $\mathbf{x}$ is outside this region then $\phi(\mathbf{x}, 0)$ is positive; in the case that $\mathbf{x}$ belongs to the interface then $\phi(\mathbf{x}, 0)$ is null (Fig. 1).

Note that, for $t > 0$, $\phi(\mathbf{x}, t)$ is not necessarily equal to the distance to Γ_t. If the level set function satisfies the evolution equation (3) then the interface satisfies Eq. (1). That is why, by detemining the zeros of $\phi(\cdot, t)$ for a fixed time t, we determine the interface Γ_t.

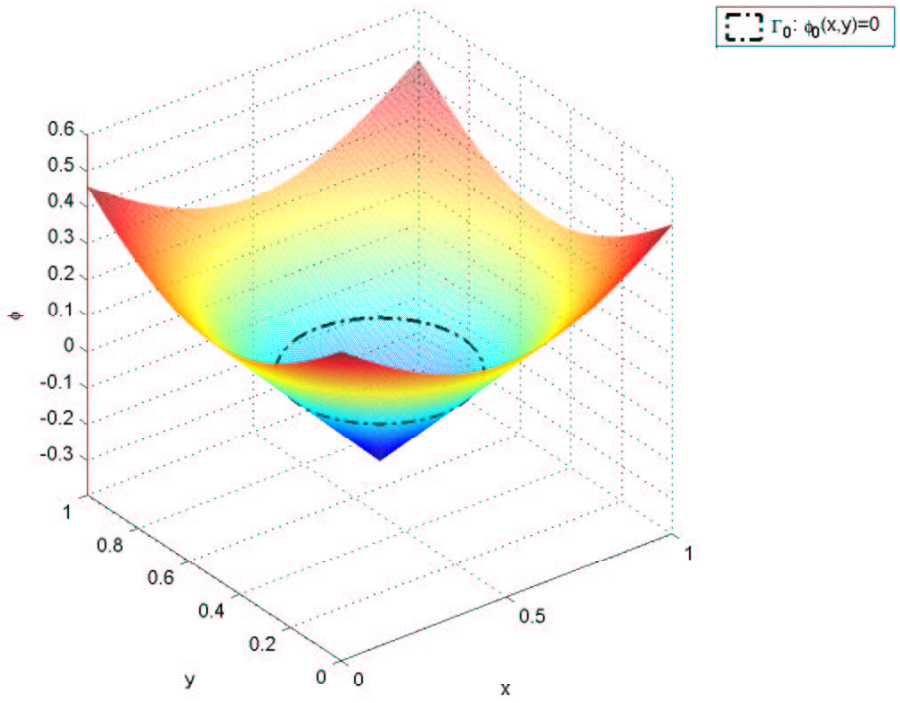

Figure 1: The initial level set function, $\phi_0(\mathbf{x})$, as a distance function with sign to the interface Γ_0.

2.2 Semi-Lagrangian scheme

The semi-Lagrangian scheme is used to solve Eq. (3). It makes use of a standard finite diference discretization for the spatial representation of the level set function and a method based on the characteristic curves for the discretization of the time variable.

Assume that the velocity field is known. The characteristic curves of level set evolution equation,

$$\frac{\partial \phi}{\partial t} + u(x, y)\frac{\partial \phi}{\partial x} + v(x, y)\frac{\partial \phi}{\partial y} = 0\,, \tag{4}$$

$0 \leq x, y \leq L$, $t \geq 0$, are solutions of the system of ordinary differential equations

$$\left.\begin{array}{l} \dot{S}_1 = u\left(S_1(t), S_2(t)\right) \\ \\ \dot{S}_2 = v\left(S_1(t), S_2(t)\right) \end{array}\right\} t \geq t^*\,, \tag{5}$$

$$(S_1(t^*), S_2(t^*)) = (x^*, y^*) \ ,$$

where (x^*, y^*) is an arbitrary point and t^* is an arbitrary time.

The solution $(S_1(t), S_2(t))$ is a trajectory of (x^*, y^*) and $\phi\left(S_1(t), S_2(t), t\right)$ is constant over the time,

$$\phi\left(S_1(t), S_2(t), t\right) = \phi\left(x^*, y^*, t^*\right) \ , \tag{6}$$

for $t \geq t^*$. In fact, the time derivative of the function on left-hand-side of Eq. (6) is null, according to Eqs. (4) and (5).

Equation (6) is the basis of the method for advancing ϕ over time. Assume that ϕ is known at time $k\Delta t$, ϕ^k, and that we wish to determine ϕ^{k+1}. Given any point $(x, y) \in [0, L]\times[0, L]$, we solve Eq. (5) getting back in time in order to determine (x^*, y^*, t^*) knowing previously the final point

$$(S_1((k+1)\Delta t), S_2((k+1)\Delta t)) = (x, y)$$

and assuming that $t^* = k\Delta t$. From Eq. (6), we get

$$\phi^{k+1}(x, y) = \phi(x, y, (k+1)\Delta t) = \phi(x^*, y^*, k\Delta t) = \phi^k(x^*, y^*) \, . \tag{7}$$

This is the semi-Lagrangian scheme for Eq. (4). A numerical procedure is used to determine (x^*, y^*), according to Eq. (5).

In order to get an approximation of second order in the characteristic determination, which influences the approximation of ϕ, we use an implicit method for solving the Eq. (5),

$$\begin{aligned} \tfrac{S_1^{k+1} - S_1^k}{\Delta t} &= u\left(\tfrac{S_1^{k+1}+S_1^k}{2}, \tfrac{S_2^{k+1}+S_2^k}{2}\right) \ , \\ \tfrac{S_2^{k+1} - S_2^k}{\Delta t} &= v\left(\tfrac{S_1^{k+1}+S_1^k}{2}, \tfrac{S_2^{k+1}+S_2^k}{2}\right) \ , \end{aligned} \tag{8}$$

with $(S_1^{k+1}, S_2^{k+1}) = (x_i, y_j)$, a point of grade. So, substituting in Eq. (8) the values of (S_1^{k+1}, S_2^{k+1}) e $(S_1^k, S_2^k) = (x^*, y^*)$ which is an unknown, we get

$$\begin{aligned} x^* &= x_i - \Delta t \, u\left(\tfrac{x_i+x^*}{2}, \tfrac{y_j+y^*}{2}\right) \\ y^* &= y_j - \Delta t \, v\left(\tfrac{x_i+x^*}{2}, \tfrac{y_j+y^*}{2}\right) \end{aligned} \tag{9}$$

Given (x_i, y_j) and Δt, we solve the Eq. (9) for (x^*, y^*) through the iteration,

$$\begin{aligned} x^*_{r+1} &= x_i - \Delta t \, u\left(\tfrac{x_i+x^*_r}{2}, \tfrac{y_j+y^*_r}{2}\right) \\ y^*_{r+1} &= y_j - \Delta t \, v\left(\tfrac{x_i+x^*_r}{2}, \tfrac{y_j+y^*_r}{2}\right) \end{aligned} \tag{10}$$

where r is the iteration counter, $r = 0, 1, 2, \ldots$, and we repeat it until we get the convergence. The value of initialization can be $(x_0^*, y_0^*) = (x_i, y_j)$.

The values of u and v are known only at the points of grade (ih, jh), $i, j = 0, 1, \ldots, N$, that correspond to the edges of the cells. We compute (u, v) at others points through a quasi-linear approximation. As we need to calculate (u, v) in (x^ξ, y^ξ) we localize inside the square with vertex (x_i, y_j), (x_{i+1}, y_j), (x_i, y_{j+1}), (x_{i+1}, y_{j+1}), and consider

$$\begin{aligned} u(x^\xi, y^\xi) \;\approx\; & \left(1 - \frac{y^\xi - y_i}{h}\right)\left[\left(1 - \frac{x^\xi - x_i}{h}\right) u_{i,j} + \frac{x^\xi - x_i}{h}\, u_{i+1,j}\right] \\ & + \frac{y^\xi - y_i}{h}\left[\left(1 - \frac{x^\xi - x_i}{h}\right) u_{i,j+1} + \frac{x^\xi - x_i}{h}\, u_{i+1,j+1}\right]. \end{aligned} \tag{11}$$

An analogous approximation for $v(x^\xi, y^\xi)$ is considered; it is enough to substitute u by v in Eq. (11). See the Fig. 2.

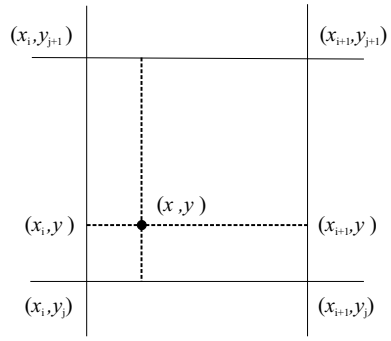

Figure 2: Vector field interpolation of (u, v). The values of (u, v) at (x_i, y_j) and (x_{i+1}, y_j) are used for the linear interpolation at (x^ξ, y_j). Analogously, we obtain the values at (x^ξ, y_{j+1}), using the values at (x_i, y_{j+1}) and (x_{i+1}, y_{j+1}). Finally, these values are linearly interpolated in y.

To improve the accuracy of ϕ, the value used on the right-hand-side of Eq. (7) is computed by means of a quasi-cubic approximation that is determined in the following manner. We do a linear interpolation of ϕ in x when $y = y_{j-1}$, a cubic interpolation of ϕ in x when $y = y_j$, a cubic interpolation of ϕ in x when $y = y_{j+1}$ and a linear interpolation in x when $y = y_{j+2}$. Finally, with these four interpolated points, we do a cubic interpolation in y.

3 SIMULATION OF SAFFMAN-TAYLOR INSTABILITY

3.1 Formulation of Saffman-Taylor problem

Consider a consolidated porous medium, initially saturated by two imiscible fluids, F_1 and F_2, whose interface is known. Suppose that the displaced fluid F_2 moves inside the

porous medium by the action of the displacing fluid F_1. Such situation is illustrated, for example, in Fig. 3 where the displacement is horizontal from left to right.

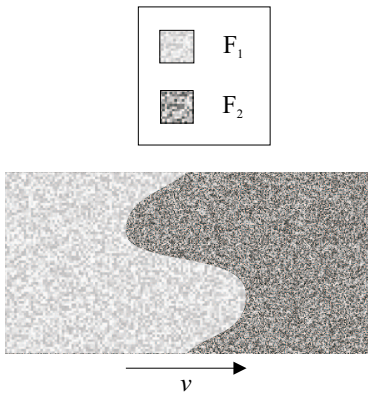

Figure 3: Porous medium saturated by two imiscible fluids, F_1 and F_2, that are moving horizontally with velocity v.

Let Ω_1 and Ω_2 be the subregions of the saturated porous medium occupied, respectively, by fluids F_1 and F_2. To complete, denote by Γ the interface between them, represented by γ as a graph of a function,

$$\Gamma(t) = \left\{(\gamma(y,z,t),y,z)\,|(y,z) \in I\!R^2\right\} \text{ and } \ t \geq 0\,. \tag{12}$$

So, the porous layer, Ω, completely saturated by the two fluids, satisfy the decomposition $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$.

Negleting gravitational and capilarity effects, we introduce the mathematical model for the Saffman-Taylor problem given by the following nondimensional system of equations: for $\mathbf{x} \in \Omega_1$,

$$\mathbf{u}_1 = -\varepsilon_1 \nabla p_1 \tag{13}$$

and

$$\nabla \cdot \mathbf{u}_1 = 0\,; \tag{14}$$

for $\mathbf{x} \in \Omega_2$,

$$\mathbf{u}_2 = -\varepsilon_1\varepsilon_2 \nabla p_2 \tag{15}$$

and

$$\nabla \cdot \mathbf{u}_2 = 0\,; \tag{16}$$

with interface conditions

$$(\mathbf{u}_1 - \mathbf{u}_2)|_\Gamma \cdot \mathbf{n} = 0 \tag{17}$$

and

$$(p_1 - p_2)|_\Gamma = 0\,; \tag{18}$$

and boundary conditions

$$\mathbf{u}_1 \longrightarrow \mathbf{e}_1 \quad \text{when} \quad x \longrightarrow -\infty \tag{19}$$

and

$$\mathbf{u}_2 \longrightarrow \mathbf{e}_1 \quad \text{when} \quad x \longrightarrow +\infty\,. \tag{20}$$

Here, $\mathbf{u}_\kappa$ and p_κ ($\kappa = 1, 2$) are, respectively, the Darcy velocity and pressure of fluid F_κ. These are nondimensional quantities, as well as, the parameters $\varepsilon_1 = K/L^2$, where L is a characteristic space length, K is the permeability tensor and $\varepsilon_2 = \mu_1/\mu_2$, the viscous ratio among fluid F_1 and F_2.

Equations (13) and (15) represent the momentum equations in the regions Ω_1 and Ω_2, respectively. Besides, equations (14) and (16) are mass balances in each region, for the two incompressible fluids.

3.2 Instability condition

A linear stability analysis is performed on the Saffman-Taylor problem in order to get an instability condition. This initiates with selecting a plane wave solution of the problem that corresponds to the basic state of the stability analysis. In this stage, the first component of the interface is given by

$$\gamma(t) = \gamma_0 t\,,$$

for a constant γ_0.

A family of trigonometric perturbations, parametrizable by $\epsilon \in I\!R$,

$$\Gamma^\epsilon(t) = \left\{(\gamma^\epsilon(y,z,t), y, z)\,|(y,z) \in I\!R^2\right\} \text{ for } t \geq 0\,,$$

are introduced in the interface.

Finally, we look for interface solution of the form

$$\gamma'(y,z,t) = \mathrm{e}^{\mathrm{i}(\xi_2 y + \xi_3 z)}\mathrm{e}^{-\beta t}\,, \tag{21}$$

in the system of equations for perturbations, which yields the following expression of β

$$\beta = \alpha\left(\frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}\right) \tag{22}$$

So, when β assumes strictly negative values, it means that the perturbation initially applied to the interface grows exponentially with time, see Eq. (21). This characterizes an unstable evolution of the interface between the fluids. On the other hand, the

front displacement is said stable when β assumes positive values, since the perturbation introduced at the interface disappears with time.

Note that α is considered strictly positive, as well as the viscosities. That's why, the front evolution is instable when $\mu_1 < \mu_2$, or $\varepsilon_2 < 1$, as one can see in Eq. (22). That is, when the displacing fluid is less viscous than the displaced one, the interface between then evolves in an unstable manner.

3.3 Simulations

Now, we show the results of simulations of the Saffman-Taylor instability, using the SLLS method. In these simulations the velocity field was determined by an iterative method based on graphs (Gonçalves and Moura Neto[20]) for the resolution of a class of problems that can be reduced to the linear form of the fundamental equilibrium equations (Strang[21]).

The results that are presented refer to a viscous ratio equal to 0.5. The permeability used is 10^{-11} m^2. The experiment consists of a constant injection in the normal direction of the left face of porous layer with intensity 10^{-7} m/s, and recovery at the right face with the same direction and intensity. We point out that the time scale considered is equivalent to a time of order 10^7 s.

The sequence of figures, Figures 4 to 7, show the results obtained in simulations of the plane interface modified by trigonometric perturbations of small amplitude, with odd wave numbers varying from 1 to 7, centered in relation to the y axis. That is, the initial perturbations are the form

$$\gamma^{\epsilon}(y,0) = \gamma_0(y) + \epsilon\cos(2\pi\lambda y)$$

where λ, the wave number, assumes the values $\lambda = 1, 3, 5, 7$, and $\gamma_0(y)$, the nonperturbed interface is given by $\gamma_0(y) = 1/4$ and $\epsilon = 1/64$. This sequence of figures exhibits the front instabilities evolution and correspond to the injection problem with viscous ratio $\varepsilon_2 = 0.5$.

Figure 8 compares the amplitude variation of the trigonometric perturbations for the five first wave numbers of the unstable problem, based on the linear stability analysis and the non-linear evolution obtained by the proposed numerical tools, using a meshgrid of size 65×129 nodes. Here, the amplitude variation is the difference between the maximum value and the minimum value of the function γ, at fixed time. All results shown in this figure correspond to the solution from the begining up to the nondimensional time 0.25. According to the perturbations to the interface introduced initially, the amplitude variation, in linear approach, is equal to $2\epsilon\exp(2\pi\lambda t/3)$. Note that the amplitude variation grows, clearly, in the linear approach as well as in the non-linear one, specially for wave numbers greater than 4. However, except for $\lambda = 1$ the linear case presents a much larger rate of variation increase. Note also that, the greater is the frequency of the introduced perturbation in the interface, the greater is the variation of its amplitude, with time.

We conclude that the SLLS method is a simple numerical tool that can be applied to the study of Saffmant-Taylor problem and is capable of capturing the instabilities of