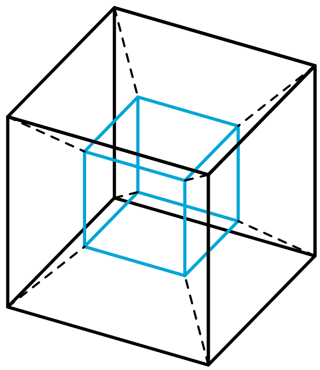
Symmetries of the Hypercube and the Hyperdiamond Using Quaternions



Thesis

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by

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Layman's Summary

This thesis explores the connection between symmetries and a special kind of number system called the quaternions. Quaternions are like regular numbers but they are especially good at describing 3-dimensional rotations, that we come by everyday. Furthermore, quaternions are even able to describe rotations in four dimensions. This sounds more complicated than it is in reality, but it still requires some technical work.

The quaternions will be used as useful tool to analyze how objects like cubes can be rotated. In particular, the symmetries of such objects are studied, in what ways can we turn the cube such that it looks exactly the same as in the starting position? Visually this can be thought of as rotating a dice for example, what are the different ways for the side with 6 eyes to point upwards?

Expanding on this, the symmetries of some fascinating 4-dimensional objects will be studied. These are shapes like the hypercube and hyperdiamond, of which mind-boggling animation can be made that can be seen in [20]. Quaternions are the perfect tool to explore these.

Why does this matter? Well, symmetries are everywhere. from physics and chemistry to computer graphics and robotics. Using traditional methods (like matrices) to do rotational calculations is often not efficient. Quaternions make rotational calculations much easier and more elegant, they require less storage, result in shorter computations and have many more technical advantages.

So, while the mathematics are complex, the key idea is simple: quaternions give us a cleaner and more powerful way to describe how things rotate, not just in our familiar 3-dimensional world, but even in four dimensions.

Summary

QUATERNIONS prove to be a useful tool when determining rotation groups of regular polytopes and other objects in three and four dimensions. In this thesis it will be studied how the unit quaternions relate to the special orthogonal groups in three and four dimension. Thereafter, this theory is applied to the normal subgroups of the binary octahedral groups as well as the cube and its dual the octahedron. The normal subgroups of the binary octahedral group form 4-dimensional objects, for example the 4-orthoplex or 16-cell and the hyperdiamond or 24-cell.

Symmetries that keep the origin in place, consist of orientation preserving and orientation flipping isometries, called rotations and reflections respectively. The special orthogonal group in n dimensions, containing all real $n \times n$ orthogonal matrices with positive determinant, is precisely the group containing all rotations in n dimensions. The map $\Phi : \mathbb{H}_1 \to SO(3)$ defined by $q \to \Phi_q$, where $\Phi_q(p) = qpq^{-1}$ for $p \in \mathbb{H}$ is a rotation of $\operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$, is a two-to-one surjective homomorphism. The map $\Psi : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ defined by $q \to \Psi_{q_1,q_2}$, where $\Psi_{q_1,q_2}(p) = q_1 p q_2^{-1}$ is a rotation of $\mathbb{H} \cong \mathbb{R}^4$, is a two-to-one surjective homomorphism.

The n-cube is the generalization of square and cube in general dimensions. Using Φ the binary octahedral group $2O$ can be established, which describes all quaternions that describe rotational symmetries of the cube and its dual the octahedron. $2O$ contains 48 unit quaternions, which is twice as many elements as the number of rotational symmetries of the cube and octahedron, due to the two-to-one relation of Φ.

The binary octahedral group $2O$ describes an object in four dimensions. Its normal subgroups do as well, the most interesting are Q_8, which describes a 4-orthoplex or 16-cell and has the hypercube or 8-cell as dual, and $2T$, which describes a hyperdiamond or 24-cell. The binary rotation groups of these regular polytopes and the object that $2O$ itself describes, are studied in particular. These binary rotation groups can all be constructed in the same manner. Write $N \in \{Q_8, 2T, 2O\}$, then $2O_N := \{(q_1, q_2) \in 2O \times 2O : q_1 N = q_2 N\} \subset \mathbb{H}_1 \times \mathbb{H}_1$ is equal to the entire binary rotation group of N. This means that for the rotation group of N, denote this as $O_N \subset SO(4)$, it holds that $O_N \cong 2O_N/\{\pm(1,1)\}$, where the isomorphism is given by the restriction of $\Psi : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ to $2O_N$. For the remaining normal subgroups $\{1\}$ and $\{\pm 1\}$, this construction only gives a subgroup of the entire binary rotation group.

Preface & Acknowledgements

I thoroughly enjoyed writing my own definitions and theorems to try and rigorously develop the theory from the ground up. I am looking forward to doing more research projects in the future, especially those which allow me really delve into a theoretical subject. The algebraic component of this thesis was particularly enjoyable and it has definitely motivated me to want to study more of this field. It was a hectic, but rewarding and insightful process until the end and it has left with a sense of contentedness.

I would really like to sincerely thank Dr. P.M Visser and Dr. J. Spandaw for inspiring me to engage with every aspect of this subject. They made sure the work I was doing was fully rigorous, like for the definitions of isometries and the orthogonal groups for example. While sometimes I could get stuck on a specific proof or definition (which in hindsight I might have worried about for too long), the feedback was always insightful. I have fond memories of our nearly weekly Wednesday meetings and discussions, where we were trying to figure out a specific problem or a new concept.

Floris van der Valk
Delft, June 2025

List of Symbols

Symbol	Description
$\mathbb{H}$	The quaternion space
$\mathbb{H}_1$	The unit quaternion group
$\operatorname{Im}\mathbb{H}$	The imaginary hyperplane in $\mathbb{H}$
$\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$	The imaginary basis vectors of $\mathbb{H}$
q^*	The conjugate of q
$2O$	The binary octahedral group
$2T$	The binary tetrahedral group
Q_8	The quaternion group containing the elements $\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$
O_H, $2O_H$	The rotation group and the binary rotation group of the hypercube and 16-cell
O_D, $2O_D$	The rotation group and the binary rotation group of the hyperdiamond
O_O, $2O_O$	The rotation group and the binary rotation group of $2O$
O	The rotation group of the octahedron and cube
S_n, $S(X)$	The permutation group on n elements and on X
V_4	The Klein four-group
C_n	The cyclic group of order n
D_4	The dihedral group of order 8
$O(n)$	The orthogonal group in n dimensions
$SO(n)$	The special orthogonal group in n dimensions
L_q, R_q	The left and right multiplication maps by $q \in \mathbb{H}$
Ψ	The two-to-one map from $\mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$
Ψ_{q_1,q_2}	The operator on $\mathbb{H}$ define by $\Psi_{q_1,q_2}(p) = q_1 p q_2^{-1}$
Φ	The two-to-one map from $\mathbb{H}_1 \to SO(3)$
Φ_q	The operator on $\operatorname{Im}\mathbb{H}$ define by $\Phi_q(p) = qpq^{-1}$
D_1, D_2	The hyperdiamonds with vertices determined by $2T$ and $2O \setminus 2T$
$\operatorname{Aut}(G)$	The automorphism group of G
$G_x, \operatorname{Stab}(G)$	The stabilizer of x
Gx	The orbit of x
$[a]$	The conjugacy class of a
$\overline{g_i}$	The coset number i
e_G	The identity element in G
id_G	The identity map on G
$\trianglelefteq$	Normal subgroup
$\rtimes_\tau$	Semidirect product with τ
$\hookrightarrow$	Injective homomorphism
$\twoheadrightarrow$	Surjective homomorphism

1

INTRODUCTION

SYMMETRIES form the backbone of many modern scientific frameworks. For example, they pop up in the physical conservation laws through time and spacial symmetries, as well as the orientation sensitive molecules in chemistry. Rotations play an important role in the theory of symmetry, but are relevant in fields that are not directly concerned with symmetries as well, such as robotics.

It is important for rotating robotic arms and other parts to receive information as efficiently as possible. It turns out that quaternions do a much better job at handling rotational calculations than the more standard matrix approach. They require less storage, result in shorter computations for combining rotations, allow for the angle and axis of rotation to be extracted easier and are simpler to normalize. These quaternions are the 4-dimensional extension of the complex numbers $\mathbb{C}$, denoted as $\mathbb{H}$. Besides the imaginary $\mathbf{i}$-axis, the quaternions have two extra imaginary axes, namely the $\mathbf{j}$-axis and the $\mathbf{k}$-axis. That the quaternions are able to represent rotations in three dimension is quite symbolic of their connection to the complex numbers, since unit complex numbers can represent rotations in two dimensional complex plane.

In this thesis it will studied exactly how quaternions relate to rotations in three dimensions, as well as in four dimensions. Furthermore, quaternions are used to explore symmetries of the binary octahedral group and its normal subgroups. Some of these form beautiful symmetrical objects, like the 4-orthoplex and the hyperdiamond, also known as the 16-cell and 24-cell. The hypercube, as the dual to the 4-orthoplex, will be a prominent object as well. For the hypercube there is a specific construction of its binary rotation group, that can be extended to general normal subgroups of the binary octahedral group. Our main goal is to find out for what normal subgroups this construction describes their entire binary rotations groups.

This thesis is structured as follows. In chapter 2 the main goal is to establish the relations between unit quaternions and the special orthogonal groups in three and four dimensions. This will be done by studying isometries in general dimensions and how they lead to the special orthogonal groups in general dimensions. Thereafter, the fascinating structure of the quaternions and in particular the unit quaternions are explored and connected to rotations in three and four dimensions.

In chapter 3, all of the binary rotation groups of the normal subgroups of the binary octahedral group are constructed. Leading up to this, the theory of symmetries for these objects are introduced side by side with the n-cube. Thereafter, the binary octahedral group is established, which consist of 48 unit quaternions. These quaternions describe the rotations of the cube, but at the same time they represent vectors in four dimensions. Using the fact that the normal subgroups of the binary octahedral group form the vertices of polytopes and other symmetrical objects, the binary rotation groups for these normal subgroups can be constructed. This is done first for the hypercube and thereafter the method for how this is done is extended to the other normal subgroups of the binary octahedral group, of which one describes the vertices of the hyperdiamond.

All figure are created with the LaTeXpackage Tikz with the help of ChatGPT by OpenAI. Some figures are inspired by already existing figures, which will be referenced in this case. In this thesis basics of linear algebra and group theory are assumed to be well understood, such that a bachelor applied mathematics student should be able to understand everything. For this purpose, some simpler and some more technical algebraic definition and theorems are treated throughout the text, of which several come from [12] or are rewritten or combined to fit the context. These definitions and theorems will be cited as such, as well when they come from a different source of course.

2

Isometries Using Quaternions

This chapter provides a basis for the study of symmetries of several objects that will be done in chapter 3. In this chapter isometries are studied thoroughly and in particular how they relate with quaternions. In section 2.1, isometries are defined, as well as what it means for an isometry to be orientation preserving. This is the property that distinguishes rotations from reflections. In section 2.2, it is shown that the orthogonal group exactly represents all isometries. The subgroup the special orthogonal group encompasses all rotations, while its complement contains the reflection. General rotations and reflections are explored more visually in section 2.3, with an emphasis on two, three and four dimensional rotations, which will reveal important properties needed in later proofs. In section 2.4, quaternion are formally introduced as well as the important unit quaternions. These unit quaternions can be used to represent rotations in three and four dimensions and open up a whole new way of looking at rotations, which is explored in section 2.5. Lastly, a brief exploration of how unit quaternions can be used to represent reflections in four dimensions is done in section 2.6.

2

2.1. ISOMETRIES OF n-DIMENSIONAL SPACE

IN this section we want to describe symmetries of n-dimensional space, which keep the origin in place. These are called isometries, which excludes translational and glide reflectional symmetries. We are less interested in these, since we will be looking at symmetries of finite objects in chapter 3 and for these translations and glide reflections are not symmetries.

Definition 2.1. *An isometry of n-dimensional space is a linear map $f : \mathbb{R}^n \to \mathbb{R}^n$, which preserves the distance between points in $\mathbb{R}^n$, that is $d(x, y) = d(f(x), f(y))$ for all $x, y \in \mathbb{R}^n$.*

Note that since f has to be linear, automatically the origin is mapped to itself, that is $f(O) = O$. So the condition that the origin is kept in place is already hidden in this definition and does not have to be stated specifically, but it is nonetheless important to be aware of.[1]

Since for $x, y \in \mathbb{R}^n$ such that $f(x) = f(y)$, definition 2.1 implies that

$$d(x, y) = d(f(x), f(y)) = 0,$$

it holds that every isometry is injective. Injectivity also means that the dimension of the image of f should match the pre-image, meaning every isometry is surjective and thus bijective as well.

Between finite dimensional vector spaces all linear transformations can be represented by a matrix and vice versa. Since we are working with transformations from n-dimensional space to itself, this means that all isometries of n-dimensional space will form a subset of the $n \times n$ matrices. Exactly what subset this is, is determined by the condition that the map should be distance preserving. As distance, we will use the standard Euclidean norm induced distance, which is just the well known square root norm. This way, if a map according to the definition is distance preserving, it implies the it is norm preserving as well. If we take the matrix R that represents a linear transformation f from the definition, this can be seen by the fact that for all $x, y \in \mathbb{R}^n$

$$d(x, y) = d(f(x), f(y)) \iff \left\|x - y\right\| = \left\|Rx - Ry\right\| = \left\|R(x - y)\right\|, \tag{2.1}$$

which is the same as saying that for $z \in \mathbb{R}^n$

$$\|z\| = \|Rz\|.$$

This clearly holds the other way around as well, as indicated by the two-way implication sign in equation 2.1. So it turns out norm and distance preserving are actually equivalent and thus norm preservation could have been used just as well in definition 2.1.

Define the inner product in the usual way such that for $x \in \mathbb{R}^n$, we have that $x \cdot x = \|x\|^2$. Since the norm is preserved by an isometry, the inner product between a vector and itself is thus preserved as well. To prove isometries preserve the inner product between any two vectors, note that for $x, y \in \mathbb{R}^n$ it holds that

$$(x+y)\cdot(x+y) = x\cdot x+2(x\cdot y)+y\cdot y \iff 2(x\cdot y) = (x+y)\cdot(x+y)-x\cdot x-y\cdot y = \left\|x + y\right\|-\|x\|-\left\|y\right\|$$

[1]Actually, if f only has the properties that it preserves distance and maps the origin to itself, then f is linear as well. We will not prove this and just define isometries as linear in the first place.

Now the entire right side in the last equation is preserved, so the left side must be preserved as well.

Furthermore, norm preservation implies that for all $z \in \mathbb{R}^n$

$$(Rz)\cdot(Rz) = z\cdot z \iff z^T R^T R z = z^T z \iff R^T R = I,$$

where in the last equivalence we use that it has to be true for all $z \in \mathbb{R}^n$ and that $R^T R$ is symmetric by $(R^T R)^T = R^T (R^T)^T = R^T R$. This is exactly the restriction that will give us the subset of matrices representing the isometries, which we will study in more detail in section 2.2.

Another distinction that can be made in the isometries is whether a isometry is orientation preserving or not. Intuitively thinking about orientation in our 3-dimensional world, we think of rotations. Two similar object have the same orientation if one is able to be rotated to the other. But our own two hands for example, are very similar yet one can not be rotated to the other. They do not have the same orientation and we need some kind of mirror to get one hand from the other. This is why orientation preserving isometries will be called rotations and the non-orientation preserving isometries will be named reflections from now on. In section 2.3 our experience of rotations compared to the mathematical definitions is discussed further. The rest of this section will be dedicated to defining orientation mathematically, such that it is inline with our experience of it.

Definition 2.2. *Let $\mathscr{B}_1 = \{b_1, \dots, b_n\}$ and $\mathscr{B}_2 = \{b'_1, \dots, b'_n\}$ be two bases of the n-dimensional vector space V. Then these bases have the same orientation if the change of basis matrix B mapping the first basis to the second basis has a positive determinant, that is* $\det(B) > 0$.

The relation between bases $\mathscr{B}_1$ and $\mathscr{B}_2$ defined by

$$\mathscr{B}_1 \sim \mathscr{B}_2 \iff \mathscr{B}_1 \text{ and } \mathscr{B}_2 \text{ have the same orientation} \tag{2.2}$$

is an equivalence relation. It is reflexive, as $\mathscr{B}_1 \sim \mathscr{B}_1$ implies that the change of basis matrix B equals I and $\det(I) = 1 > 0$. If $\mathscr{B}_1 \sim \mathscr{B}_2$ and $\mathscr{B}_2 \sim \mathscr{B}_3$ with change of basis matrix B_1 and B_2 respectively, then the change of basis matrix which maps $\mathscr{B}_1$ to $\mathscr{B}_3$ is equal to $B_2 B_1$. We see $\det(B_2 B_1) = \det(B_2)\det(B_1) > 0$, implying the relation is transitive as well. Lastly, the inverse of a change of basis matrix B is just B^{-1}, which is well-defined since $\det(B) \neq 0$. Furthermore, $\det(B^{-1}) = \frac{1}{\det(B)} > 0$, showing the relation is symmetric and it can be concluded that 2.2 is thus an equivalence relation. This means definition 2.2 is well defined and inline with what orientation intuitively encompasses.

The matrix R associated with any isometry could be interpreted as a change of basis matrix, indeed, if we choose a basis $\mathscr{B}_1$ for $\mathbb{R}^n$, then the isometry maps $\mathscr{B}_1$ to another basis $\mathscr{B}_2$. This way we call an isometry orientation preserving, if $\mathscr{B}_1$ and $\mathscr{B}_2$ have the same orientation. From the definition this is equivalent to wether $\det(R) > 0$ or not.

This definition gives a way to distinguish rotations from reflections, which is further discussed in section 2.2.

2

2.2. THE ORTHOGONAL GROUP AND THE SPECIAL ORTHOGONAL GROUP

IN the previous section it was shown that isometries of n-dimensional space can be represented by a subset of the $n \times n$ matrices, namely the matrices R with the property $R^T R = I$. However, this is precisely how orthogonal matrices are defined. Let us state this in a definition and also give a name to the set of all orthogonal matrices.

Definition 2.3. *A matrix $R \in \mathbb{R}^{n\times n}$ is orthogonal if $R^T R = I$.*

Definition 2.4. *$O(n)$ is the set of all real orthogonal $n \times n$ matrices, that is*

$$O(n) = \{R \in \mathbb{R}^{n\times n} : R^T R = I\}.$$

So $O(n)$ is precisely isomorphic to the group of the isometries of the n-dimensional space. That $O(n)$ is really a group is proved in theorem 2.8, but first a few properties of orthogonal matrices are shown that will be useful later on.

Lemma 2.5. *If $R \in \mathbb{R}^{n\times n}$ is an orthogonal matrix, then* $\det(R) = \pm 1$.

Proof. From the definition we have $1 = \det(I) = \det(R^T R) = \det(R^T)\det(R) = \det^2(R)$. So indeed $\det(R) = \pm 1$. □

Lemma 2.6. *If $R \in \mathbb{R}^{n\times n}$ is orthogonal, then R has orthonormal columns.*

Proof. Write the columns as $R = [r_1, r_2, \ldots, r_n]$. If we look at the entries of the product $R^T R$, we can write

$$(R^T R)_{ij} = r_i^T r_j = r_i \cdot r_j. \tag{2.3}$$

Now, since $R^T R = I$, it should hold that equation 2.3 is equal to 1 if $i = j$ and 0 if $i \neq j$ for all $i, j \in \{1, \ldots, n\}$. This means that the columns have to be mutually orthogonal, but also $\|r_i\| = 1$ for all $i \in \{1, \ldots, n\}$. So indeed the columns are orthonormal. □

Lemma 2.7. *If $R \in \mathbb{R}^{n\times n}$ is orthogonal, then R is invertible and $R^{-1} = R^T$.*

Proof. The inverse is well-defined, since $\det(R) \neq 0$. Next, multiply $R^T R = I$ by R^{-1} on the right on both sides, we get $(R^T R)R^{-1} = IR^{-1}$. Substitute $I = R^T R$ and use associativity to get $(R^T R)R^{-1} = R^T(RR^{-1})$, which implies $R^{-1} = R^T$. □

Lemma 2.7 shows the important result that isometries are invertible. Intuitively this is clear by the fact that a isometry can map any two points with the same distance to the origin to each other, so going one way implies that going back is also possible. This property also enables us to show that $O(n)$ is a group.

Theorem 2.8. *$O(n)$ is a group under matrix multiplication.*

Proof. Let $A, B \in O(n)$, then $AB(AB)^T = ABB^T A^T = I$, so $AB \in O(n)$ again, meaning $O(n)$ is closed. We also know that matrix multiplication is associative and $I \in O(n)$ is the identity element. Furthermore, from lemma 2.7 we know $A^{-1} = A^T$ and

$$A^{-1}(A^{-1})^T = A^T(A^T)^T = A^T A = I$$

implies that $A^{-1} \in O(n)$. So we conclude that $O(n)$ is indeed a group. □

In section 2.1, orientation was defined, which implied that exactly those isometries with positive determinant are orientation preserving. We will now introduce this subgroup of $O(n)$, which exactly contains the orientation preserving isometries of the n-dimensional space that we call rotation. Moreover, its complement contains all non-orientation preserving isometries, which we call reflections.

Definition 2.9. *$SO(n)$ is the set of all real orthogonal $n \times n$ matrices with determinant equal to 1, that is*

$$SO(n) = \{R \in \mathbb{R}^{n\times n} : R^T R = I \text{ and } \det(R) = 1\}.$$

Indeed all matrices in $SO(n)$ have positive determinant, while its complement in $O(n)$ does not and thus $SO(n)$ contains all rotations. We should formally prove that $SO(n)$ is a subgroup though.

Lemma 2.10. *$SO(n)$ is a subgroup of $O(n)$.*

Proof. Let $A, B \in SO(n)$, then $\det(AB) = \det(A)\det(B) = 1$, so $AB \in SO(n)$. Also $\det(A^{-1}) = \det(A^T) = \det(A) = 1$, so $A^{-1} \in SO(n)$. □

$O(n)$ and $SO(n)$ are called the orthogonal group and special orthogonal group of n dimensions respectively. From now on the group of all isometries of the n-dimensional space, will be denoted as $O(n)$. Moreover, the group of all rotations of n-dimensional space will be denoted as $SO(n)$.

2.3. ROTATIONS AND REFLECTIONS IN GENERAL DIMENSIONS

BEFORE the definition in sections 2.1 and 2.2 was widely known, Euler looked at rotations in a more intuitive way, resulting in his famous rotation theorem. In this section we will connect Euler's theorem with $SO(3)$, which will be very useful in proofs later on. Before that, we will look at how 2-dimensional rotations operate. After this we will even try to make sense of what 4-dimensional and general dimensional rotations look like. And lastly, we will quickly look at reflections in general dimensions.

2.3.1. ROTATIONS IN TWO DIMENSIONS

Rotations in two dimensions are relatively simple. Every rotation in $SO(2)$ is of the form

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

for some $\theta \in [0, 2\pi)$.
Any point v in $\mathbb{R}^2$ is just mapped to the point v' which lies an angle of θ further along the circle around the origin with radius equal to the length of v, as illustrated in figure 2.1.

2.3.2. ROTATIONS IN THREE DIMENSIONS

Three dimensional rotations have a very specific property, which was noticed by Euler. We will use his theorem without proving it.

Theorem 2.11. *(Euler's theorem) A rotation in three dimensional space has a line which remains stationary, called the axis of rotation. [10]*

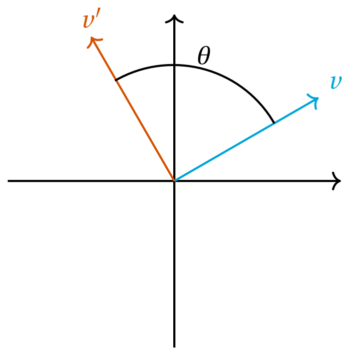

Figure 2.1: A plane rotation of the vector v over an angle of θ.

Denote the rotations of the form according to Euler's theorem as Euler rotations, which will be proved to be equivalent to $SO(3)$. But first, we have to understand Euler rotations.

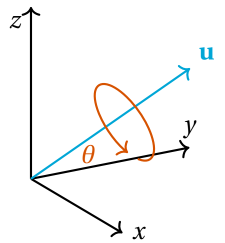

Figure 2.2: An Euler rotation with unit vector u on the axis of rotation and a rotation angle θ of the perpendicular plane

To represent an Euler rotation, one can take a unit vector $u \in \mathbb{R}^3$ on the axis of rotation. Euler's theorem implies that a three dimensional rotation can be seen as the two dimensional rotation of the perpendicular plane of u over an angle of $\theta \in [0, \pi]$. The notation used for an Euler rotation will be $R(\theta, u) : \mathbb{R}^3 \to \mathbb{R}^3$.

Intuitively one might think that the rotation angle should vary from 0 to 2π just like in two dimensions. However, a rotation $R(\theta_1, u)$ with $\theta_1 \in [\pi, 2\pi]$ is the same rotation as $R(2\pi - \theta_1, -u)$. In other words, we can flip the sign of the unit vector u, making it point in the opposite direction and this way θ represents a whole new set of rotations. The case $\theta = 0$ is an exception as it gives the same rotation for every unit vector u, namely the stationary or identity rotation.

Now we will get started on proving formally that all Euler rotations $R(\theta, u)$ are elements of $SO(3)$ and vice versa. The eigenvalues of a matrix in $SO(3)$ will play an important role, as it allows us to find the axis of rotation for this matrix, connecting it to an Euler rotation.

Lemma 2.12. *If $R \in SO(3)$, then R has eigenvalues $1, e^{i\theta}$ and $e^{-i\theta}$ for some $\theta \in [0, 2\pi)$.*[2]

Proof. Let $R \in SO(3)$, we need to find $0 \neq \lambda \in \mathbb{C}$ such that for some $\vec{v} \neq 0$

$$R\vec{v} = \lambda\vec{v} \tag{2.4}$$

or equivalently solve the characteristic equation

$$\det(R - \lambda I) = 0. \tag{2.5}$$

Taking the complex conjugate of equation 2.4 gives $R\vec{v}^* = \lambda^*\vec{v}^*$, since R is real. Now taking the transpose gives $(R\vec{v}^*)^T = (\lambda^*\vec{v}^*)^T \iff \vec{v}^{*T}R^T = \lambda^*\vec{v}^{*T}$. We can then cleverly multiply this with first equation.

$$\begin{aligned}
(\lambda^*\vec{v}^{*T})(\lambda\vec{v}) &= (\vec{v}^{*T}R^T)(R\vec{v}) \\
\lambda^*\lambda\vec{v}^{*T}\vec{v} &= \vec{v}^{*T}\vec{v} \\
\lambda^*\lambda &= 1 \\
|\lambda| &= 1
\end{aligned}$$

Here it is used that the eigenvector $\vec{v} \neq 0$ and thus $\vec{v}^{*T}\vec{v} \neq 0$. From the last line we see that every eigenvalue must be of the form $\lambda = e^{i\alpha}$ for some $\alpha \in [0, 2\pi)$.
Now consider the following equality:

$$R^T(I - R) = R^T - I = -(I - R)^T.$$

We use that I only has non-zero terms on the diagonal, so taking the transpose of R before or after subtracting I gives the same result. Next, if the determinant is taken on both sides, we get the following result.

$$\begin{aligned}
\det(R^T(I - R)) &= \det(-(I - R)^T) \\
\det(R)\det(I - R) &= \det(-I)\det(I - R) \\
\det(I - R) &= -\det(I - R).
\end{aligned}$$

Here it is used that the determinant of a transposed matrix equals the determinant of the matrix itself and $\det(R) = 1$. We see that

$$\det(I - R) = 0.$$

Comparing with equation 2.5 we see that indeed $\lambda = 1$ must be an eigenvalue, call $\lambda_1 := 1$. Now for the last part of the proof we use the fact that for a matrix the product of its eigenvalues must be equal to its determinant.

$$\begin{aligned}
\lambda_1\lambda_2\lambda_3 &= \det(R) = 1 \\
\lambda_2\lambda_3 &= 1.
\end{aligned}$$

So we can conclude that if $\lambda_2 = e^{i\theta}$ for some $\theta \in [0, 2\pi)$, then it must hold that $\lambda_3 = e^{-i\theta}$. In other words λ_2 and λ_3 are complex conjugates and this completes the proof. [1] □

[2] It is interesting to note that θ is actually the angle over which the perpendicular plane to the eigenvector corresponding to eigenvalue 1 rotates. This will not be used in this thesis though.

Theorem 2.13. *Every Euler rotation $R(\theta, u)$, with $\theta \in [0, \pi]$ and $u \in \mathbb{R}^3$, can be represented as a matrix $R \in SO(3)$.*

Proof. Let $R(\theta, u)$ be an Euler rotation. By Euler's theorem in 2.11, the unit vector u is kept stationary by $R(\theta, u)$ and its perpendicular plane is rotated with an angle θ. An Euler rotation is a linear transformation thus we can at least write $R(\theta, u)$ as a matrix, call this matrix R. Set $u_3 = u$, then with the Gram-Schmidt method we can find unit vectors u_1, u_2, such that $\{u_1, u_2, u_3\}$ form a orthonormal basis for $\mathbb{R}^3$. Define $A = [u_1, u_2, u_3]$ and note that we can always get $\det(A) = 1$ by possibly switching u_1 and u_2 from positions, which means $A \in SO(3)$. Next, we can write

$$\begin{aligned} R(u_1) &= \cos(\theta) u_1 + \sin(\theta) u_2 \\ R(u_2) &= -\sin(\theta) u_1 + \cos(\theta) u_2 \\ R(u_3) &= u_3. \end{aligned}$$

We can write this as

$$RA = AM \qquad \text{with} \qquad M = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note that $M \in SO(3)$, so $R = AMA^{-1} \in SO(3)$. [1] □

Theorem 2.14. *Every matrix $R \in SO(3)$ is a rotation $R(\theta, u)$ for some $\theta \in [0, \pi]$ and $u \in \mathbb{R}^3$.*

Proof. By 2.12, we know R has eigenvalue 1. Thus we can find a unit vector u_3 that is an eigenvector of R corresponding to the eigenvalue 1. With the Gram-Schmidt method we can find the orthonormal basis $\{u_1, u_2, u_3\}$ for $\mathbb{R}^3$, where u_1 and u_2 form a basis for the orthogonal plane to u_3, denoted by $u_3^\perp$. Define the matrix $A = [u_1, u_2, u_3]$, which lies in $SO(3)$ after possibly switching u_1 and u_2 from position. Then

$$RA = [Ru_1, Ru_2, Ru_3] = [Ru_1, Ru_2, u_3]$$

remains orthogonal. Using lemma 2.6 implies that $Ru_1, Ru_2 \in u_3^\perp$. So we can write the equations

$$\begin{aligned} Ru_1 &= au_1 + bu_2 \\ Ru_2 &= cu_1 + du_2 \\ Ru_3 &= u_3, \end{aligned}$$

as

$$RA = AM \qquad \text{with} \qquad M = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now $M = A^{-1}RA \in SO(3)$, so $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1$, thus $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SO(2)$. This means R is a plane rotation around u_3, in other words an Euler rotation. □

Combining theorem 2.13 and 2.14, we can conclude that $SO(3)$ is isomorphic to all Euler rotations.

2.3.3. ROTATIONS IN FOUR AND HIGHER DIMENSIONS

Some examples and terminology in this section are from [18].

Compared to the previous two sections, in four dimensions our intuition does not get us very far, being 3-dimensional creatures ourselves. Yet, 4-dimensional rotations are well explored and understood, because they are not as different from the rotations we experience ourselves than one might expect.

The most peculiar thing about 3-dimensional rotations, is that in some sense it is still just a 2-dimensional rotation. However, instead of just one plane rotating around a single point, there are infinitely many planes rotating around the infinitely many points that lie on the axis of rotation. Sadly this does not directly extend to four dimension. Instead a stationary rotational axis is switched for a stationary plane. Actually, in general dimensions rotations do not have an stationary axis at all. Furthermore, suppose some four dimensional creature is looking at a 3-dimensional rotation. They observe the stationary rotational axis as well, but for them their extra fourth axis is stationary as well and these together indeed span a stationary plane. So we have found that at least some rotation in four dimensions are of the form where one plane rotates around a stationary perpendicular plane. Indeed, four dimensions is the first dimension in which two planes can be completely perpendicular. These kind of four dimensional rotations are called simple rotations, hinting at other types.

But first, to illustrate the previous example, compare it to the scenario where we are looking at a 2-dimensional rotation on some surface. Then we see the perpendicular line to this surface as the axis of rotation. However, a 2-dimensional creature does not see this axis at all and for them there is just a single stationary point. So axes of rotation are not that intertwined with rotations as us three dimensional creatures might think and planes seem to hold more importance.

Still, that in four dimension a plane rotates around another plane is a difficult concept to grasp. Therefore, instead of thinking of rotations as rotating around something, rotations should be thought of as happening in a plane itself. This way rotations can be generalized easier for general dimensions.

Actually, if we think of rotations as just a planar phenomenon, then it is possible to rotate the stationary plane of a simple rotation at the same time as the already rotating perpendicular plane. These are appropriately called double rotations. Note that indeed neither of the plane rotations have an axis of rotation, as the entire perpendicular space is rotating as well and in general the intersection of two planes in four dimensions is just one point.

What is not discussed about three dimensional rotations so far are the principal rotations. In three dimensions there are three principal rotations, which are the plane rotations around the three basis axes. However, in line with our discussion about rotations as a planar phenomenon, call these rotations by their rotating plane, denote the planes as XY, XZ and YZ. Combining these three principal rotations with all possible rotation angles for each, every 3-dimensional rotation can be constructed. Note that it is a convention to take the three standard axes of the 3-dimensional space and that actually any three perpendicular directions could function as normal vectors to three planes, inducing three principal rotations.

In four dimensions the number of principal rotations is not four, what might be expected, but six. The reason for this is that a rotation needs two dimensions, so finding all principal rotation planes means finding all distinct pairs of dimension. In four dimensions, calling the axes X, Y, Z, W, these pairs are XY, XZ, XW, YZ, YW and ZW. In two dimension just one rotation plane takes up the only two available dimension and thus in two dimensions there is only one principal rotation, namely just the XY plane. For a general dimension the number of principal rotations is equal to

$$\binom{n}{2} = \frac{n(n-1)}{2}. \tag{2.6}$$

This way we have actually very informally shown the dimension of $SO(n)$, which is thus also equal to equation 2.6.

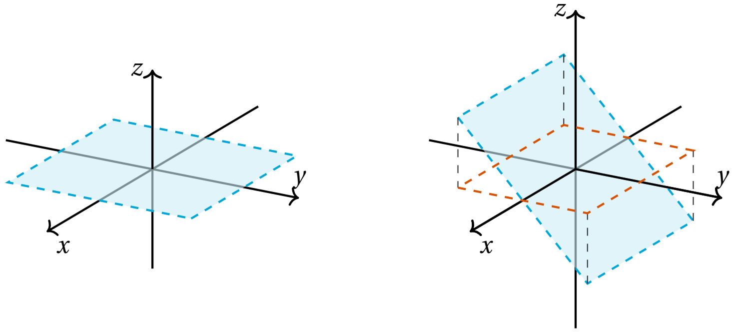

Figure 2.3: A square in the XY plane, being continuously rotated with the YZ plane, so around the x-axis. The square is projected straight onto the XY plane at every moment. The left figure is at 0 degrees rotation, while the right is at 45 degrees rotation.

Consider how a 2-dimensional creature would experience it, when one of the 3-dimensional principal rotations is continuously rotating and being projected onto its 2-dimensional world. Suppose the 2-dimensional creature is living in the XY plane and a square in its plane is rotated with the YZ plane (so around the x-axis), as illustrated in figure 2.3. Furthermore, the square is projected on to the XY plane straight down. Remember that the 2-dimensional creature can only observe the 1-dimensional projection of this image, just like we only observe 2-dimensional projection of our environment. Then, for the 2-dimensional creature, the projection of the rotation of the square with the YZ plane looks like an impossible twist of an object turning itself inside-out. However, for us the square remains totally rigid and intact.

Rotating the square with the XZ plane gives a similar result as with the YZ plane, but what about the XY plane? Well this rotation perfectly fits in the universe of the 2-dimensional creature and thus appears as a totally normal rotation, or at least exactly like 2-dimensional creatures are used to. So to summarize, two out of three principal rotations would seem impossible to the 2-dimensional creature, while the last one seems totally normal.

This brings us to how 4-dimensional rotations project to three dimensions. We are now dealing with six principal rotations now of which three are in planes that only use 'our' three dimensions, namely the XY, XZ, YZ planes. The projections of these rotations would appear normal, just like rotations in the XY plane for two dimensions. However, the other three principal rotations using the extra fourth W axis would in turn seems impossible to us. Projections of object rotating according to these rotations would make it seems like edges are changing length and shapes in general will get distorted and turned inside-out. This kind of projection can not be done justice with stationary figures, but online there are a lot of examples of these mind-bending projections, for example of the hypercube in [20].

In higher dimensions than four, one could see that rotations become more complex as the number of possible mutually orthogonal planes increases. In six dimensions already three perpendicular planes can be rotating at the same time and so on. What has not been highlighted yet, is that the planes do not even have to rotate around the same angle. Although, in four dimension for example, taking the same angles for the two planes of a double rotation create a specific type of rotation called isoclinic rotation.[9] However, this will not be explored in this thesis.

2.3.4. REFLECTIONS IN GENERAL DIMENSIONS

The fact that $SO(n)$ contains all rotations, but no reflections, automatically means that that the complements of $SO(3)$ in $O(3)$ contains all reflections, since these are the only two types of symmetries keeping the origin in place. In this section we will cover a subset of the reflections in n dimensions that can be seen as mirroring through an $(n-1)$-dimensional hyperplane, call these mirrors. Every mirror hyperplane can be specified by a normal unit vector, which is orthogonal to every vector in the mirror hyperplane. This creates a nice visualization and notation of a mirror. In 2 and 3 dimensions a reflection is visualized in figure 2.4.

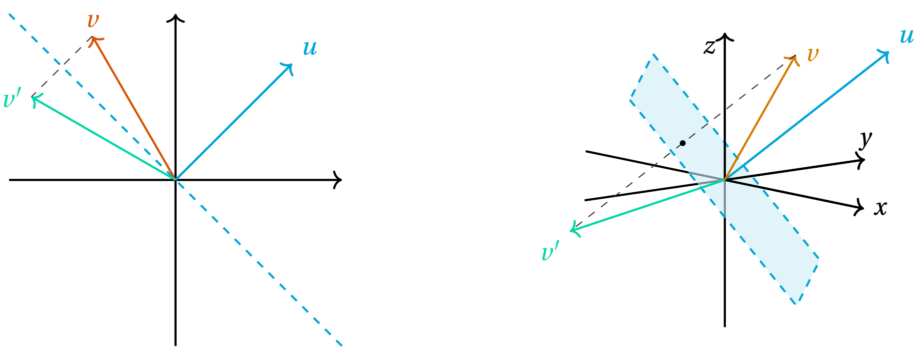

Figure 2.4: The vector v is mirrored through the perpendicular plane of the unit vector u. The image is of v is v' and the dashed line connecting these two vector intersects the plane exactly halfway.

In figure 2.4 it can be seen that the mirror of a vector $v \in \mathbb{R}^3$ can be obtained by subtracting the projection of v onto the unit normal vector of the mirror plane (line) u twice. The length of the projection of v onto u is equal to $\|v\|\cos(\theta)$, where θ is the angle between u and v. This is equal to the dot product $v \cdot u = \|v\|\|u\|\cos(\theta)$, only if $\|u\| = 1$. Taking the normal vector to a hyperplane as an unit vector is therefore a nice convention. The mirror can then be written mathematically as

2

$$v \mapsto v - 2(v \cdot u)u. \tag{2.7}$$

Reflections do no commute in general, but combining reflections are still isometries. Interestingly, a double reflection results in a net rotation, while combining a rotation with a reflection gives another reflections. This can be seen most directly by looking at the matrix representation of isometries. For reflections $S_1, S_2 \in O(n) \setminus SO(n)$ and rotation $R \in SO(n)$, it holds that

$$\begin{aligned}\det(S_1 S_2) &= \det(S_1)\det(S_2) = -1 \cdot -1 = 1,\\ \det(S_1 R) &= \det(S_1)\det(R) = -1 \cdot 1 = -1 = \det(RS_1).\end{aligned}$$

So indeed $S_1 S_2 \in SO(n)$ and $S_1 R, RS_1 \in O(n) \setminus SO(n)$. Similarly, this way we can see combined rotations are rotations, but this is already implied by the fact to $SO(n)$ is a group.

The behavior of combined reflections and rotations also follows from the their respective orientation flipping and orientation preserving properties.

2.4. QUATERNIONS

BETWEEN the unit quaternions and SO(3), as well as between the direct product of the unit quaternions and SO(4), there is a two-to-one relation, which results in very interesting representations of 3 and 4-dimensional rotations. How these are exactly linked will be explored in section 2.5. Here, the quaternions and some important and useful properties are introduced.

A quaternion q is a element of the real four dimensional vector space $\mathbb{H}$, which has a basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$. We represent q as follows.

$$q = r\mathbf{1} + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \qquad \text{with } r, a, b, c \in \mathbb{R}. \tag{2.8}$$

For convenience we will leave the $\mathbf{1}$ out from now on. Since $\mathbb{H}$ is a vector space, addition and scalar multiplication with real numbers is defined elementwise. For $q_1, q_2 \in \mathbb{R}$ with $q_1 = r_1 + a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $q_2 = r_2 + a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ we have that

$$(r_1 + a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) + (r_2 + a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = (r_1 + r_2) + (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j} + (c_1 + c_2)\mathbf{k}$$
$$s(r_1 + a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}) = sr_1 + sa_1\mathbf{i} + sb_1\mathbf{j} + sc_1\mathbf{k}, \quad s \in \mathbb{R}.$$

On top of this, for $\mathbb{H}$, multiplication rules follow from the rules defined for the basis vectors.

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k} \qquad \mathbf{ik} = -\mathbf{ki} = -\mathbf{j} \qquad \mathbf{jk} = -\mathbf{kj} = \mathbf{i}/$$

Note that in general quaternions do not commute. The linear extension of the multiplication rules for the basis vectors gives us a multiplication formula for general vectors or quaternions. In equation 2.9 the famous Hamilton quaternion multiplication rule is written out.

$$\begin{aligned}(r_1 + a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k})(r_2 + a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) = &(r_1r_2 - a_1a_2 - b_1b_2 - c_1c_2)\\ &+ (r_1a_2 + a_1r_2 + b_1c_2 - c_1b_2)\mathbf{i}\\ &+ (r_1b_2 + b_1r_2 + c_1a_2 - a_1c_2)\mathbf{j}\\ &+ (r_1c_2 + c_1r_2 + a_1b_2 - b_1a_2)\mathbf{k}\end{aligned}$$

As a convention, for a quaternion $q = r + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, r is called the *real part* and $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ the *imaginary part*. This way we define the subspace $\operatorname{Im}\mathbb{H} := \{a\mathbf{i}+b\mathbf{j}+c\mathbf{k} : a, b, c \in \mathbb{R}\} \cong \mathbb{R}^3$. Note that now $\mathbb{H} = \mathbb{R} \oplus \operatorname{Im}\mathbb{H}$, because we have $\mathbb{H} \cong \mathbb{R}^4$ of course.

We define the conjugate and norm similarly to the complex numbers.

$$q^* := r - a\mathbf{i} - b\mathbf{j} - c\mathbf{k} \tag{2.9}$$

$$\|q\|^2 := qq^* = q^*q = r^2 + a^2 + b^2 + c^2 \tag{2.10}$$

Here equation 2.9 is used to work out the product and to verify that $qq^* = q^*q$.
The most important properties of the quaternions are now defined, next some useful and insightful characteristics are shown.
With the defined norm, it can be shown that the inverse of a quaternion exists and how it can be constructed.

Lemma 2.15. *For every quaternion $q \neq 0$ there exists a inverse $q^{-1} \in \mathbb{H}$, which is given by*

$$q^{-1} = \frac{q^*}{\|q\|^2}. \tag{2.11}$$

Proof. Let $q = r + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \in \mathbb{H}$, then $\frac{q^*}{\|q\|^2} = \frac{r}{\|q\|^2} - \frac{r}{\|q\|^2}\mathbf{i} - \frac{a}{\|b\|^2}\mathbf{j} - \frac{c}{\|q\|^2}\mathbf{k} \in \mathbb{H}$. Furthermore,

$$q\frac{q^*}{\|q\|^2} = \frac{qq^*}{qq^*} = \frac{r^2 + a^2 + b^2 + c^2}{r^2 + a^2 + b^2 + c^2} = 1$$

for $q \neq 0$. In a similar way $\frac{q^*}{\|q\|^2}q = 1$, so we can take indeed take the inverse of q as $q^{-1} = \frac{q^*}{\|q\|^2}$. □

In the next lemma and corollary, we highlight some properties of the norm defined on $\mathbb{H}$ in equation 2.10.

Lemma 2.16. *For two quaternions $q_1, q_2 \in \mathbb{H}$, it holds that*

$$(q_1q_2)^* = q_2^*q_1^*. \tag{2.12}$$

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Proof. Equation 2.9 can be used to write out both sides. □

Corollary 2.17. *Let $q, q_1, q_2 \in \mathbb{H}$, it then holds that*

$$\|q_1 q_2\| = \|q_1\| \|q_2\| \tag{2.13}$$
$$\|q\| = \|q^*\| \tag{2.14}$$
$$\|q^{-1}\| = \|q\|^{-1}, \qquad q \neq 0. \tag{2.15}$$

Proof. First of all note that

$$\|q_1 q_2\|^2 = q_1 q_2 (q_1 q_2)^* = q_1 q_2 q_2^* q_1^* = \|q_1\|^2 \|q_2\|^2,$$

since the norm is a real number and in $\mathbb{R}$ multiplication is commutative. The norm is also non-negative, so we can take the square root on both sides resulting in equation 2.13.
For equation 2.14, note that $(q^*)^* = q$ such that

$$\|q\|^2 = qq^* = q^* q = q^*(q^*)^* = \|q^*\|^2.$$

Again, taking the square root gives the desired outcome.
Now for equation 2.15, we use equation 2.14.

$$\|q^{-1}\| = \left\| \frac{q^*}{\|q\|^2} \right\| = \frac{\|q^*\|}{\|q\|^2} = \frac{\|q\|}{\|q\|^2} = \|q\|^{-1}.$$

□

Lastly for this section, we define the already mentioned unit quaternions, which will play a very important role in connecting quaternions to rotations.

Definition 2.18. *The set of all quaternions with length one is denoted as $\mathbb{H}_1$, that is*

$$\mathbb{H}_1 := \{q \in \mathbb{H} : \|q\| = 1\}. \tag{2.16}$$

Lemma 2.19. *Let $q \in \mathbb{H}_1$, then $q^{-1} = q^*$.*

Proof. Using lemma 2.15 and $\|q\| = 1$ we immediately get the desired result. □

Lemma 2.20. *$\mathbb{H}_1$ is compact and connected.*

Proof. Since $\mathbb{H}$ is a four dimensional real vector space, $\mathbb{H}_1$ is isomorph with the 3-sphere $\mathbb{S}^3$. The 3-sphere is clearly connected and even path connected. $\mathbb{H}_1$ is compact since it is closed and bounded in $\mathbb{H}$ and $\mathbb{H}$ is a Hausdorff space. □

Lemma 2.21. *$\mathbb{H} \setminus \{0\}$ is group under quaternion multiplication and has $\mathbb{H}_1$ as a subgroup.*

Proof. Both $\mathbb{H} \setminus \{0\}$ and $\mathbb{H}_1$ are closed under multiplication. For $\mathbb{H} \setminus \{0\}$ this is clear and for $\mathbb{H}_1$, take $q_1, q_2 \in \mathbb{H}_1$, then $\|q_1 q_2\| = \|q_1\| \|q_2\| = 1$, so $q_1 q_2 \in \mathbb{H}_1$. Furthermore, they both contain the identity element 1 and a inverse element for every element. For $\mathbb{H}_1$ specifically this is because $\|q^*\| = 1$, so $q^* \in \mathbb{H}_1$. □

2.5. ROTATIONS USING QUATERNIONS

WE are ready to show how unit quaternion can represent rotations in three and four dimensions, although this section will therefore be a little technical.

Definition 2.22. *For $q \in \mathbb{H}_1$ the linear operators L_q and R_q, $\mathbb{H} \to \mathbb{H}$, when working on a $p \in \mathbb{H}$, are given by*

$$L_q p = qp \qquad \text{and} \qquad R_q p = pq. \tag{2.17}$$

It can be easily checked that L_q and R_q are indeed linear. For $p, p_1, p_2 \in \mathbb{H}$ and $r \in \mathbb{R}$ it holds that

$$\begin{aligned}
&L_q(p_1 + p_2) = q(p_1 + p_2) = qp_1 + qp_2 = L_q(p_1) + L_q(p_2),\\
&rL_q(p) = rqp = qrp = L_q(rp).
\end{aligned}$$

That R_q is linear is analogue to this. Linearity will be used in the proof of theorem 2.23.

Theorem 2.23. *The mappings $f, g : \mathbb{H}_1 \to SO(4)$, given by $q \mapsto L_q$ and $q \mapsto R_{q^{-1}}$ respectively, are homomorphisms.*

Proof. First we have to prove that L_q and R_q are in fact elements of $SO(4)$. For this, note that L_q and R_q are linear and norm preserving by equation 2.13 and the fact that $\|q\| = 1$. So at least $L_q, R_q \in O(4)$. Next note that for $q = 1 \in \mathbb{H}_1$, $L_q = R_q = I$ and $\det(I) = 1$. The fact that the mappings $q \mapsto L_q$ and $q \mapsto R_q$ are continuous and $\mathbb{H}_1$ is connected thus implies that $L_q, R_q \in SO(4)$ for every $q \in \mathbb{H}_1$.
Next we see that for $q_1, q_2 \in \mathbb{H}_1$, we have that

$$\begin{aligned}
&f(q_1q_2) = L_{q_1q_2} = L_{q_1}L_{q_2} = f(q_1)f(q_2),\\
&g(q_1q_2) = R_{(q_1q_2)^{-1}} = R_{q_2^{-1}q_1^{-1}} = R_{q_1^{-1}}R_{q_2^{-1}} = g(q_1)g(q_2),
\end{aligned}$$

concluding the proof. □

Definition 2.24. *Define the homomorphism $\Psi : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ by*

$$(q_1, q_2) \mapsto \Psi_{q_1,q_2} := L_{q_1}R_{q_2^{-1}}. \tag{2.18}$$

So $\Psi_{q_1,q_2}(p) = q_1 p q_2^{-1}$ for $p \in \mathbb{H}$.
Specifically for the case $q_1 = q_2$, also define the homomorphism $\Phi : \mathbb{H}_1 \to SO(4)$ by

$$q \mapsto \Phi_q := \Psi_{q,q} = L_q R_{q^{-1}}. \tag{2.19}$$

So $\Phi_q(p) = qpq^{-1}$ for $p \in \mathbb{H}$. [15]

Note that Ψ is indeed a homomorphism since it is a composition of two homomorphisms and thus Φ is as well.

The operator Φ_q acting on a $p \in \mathbb{H}$ preserves the real part of p. We can see this since $\Phi_q(r) = qrq^{-1} = r$ implies that every $r \in \mathbb{R} \subset \mathbb{H}$ is not changed by applying Φ_q. Furthermore, that $\Phi_q \in SO(4)$ means that $\mathbb{R}^\perp = \mathrm{Im}\mathbb{H}$ is mapped to itself. This is true since we have seen that any element in $SO(4)$ is inner product preserving, so $\mathbb{R}^\perp$ is still perpendicular to $\mathbb{R}$ after a 4-dimensional rotation. Thus we can view Φ_q as an operator on

$\mathbb{R}^3 \cong \mathrm{Im}\mathbb{H}$ by letting Φ_q only act on $v \in \mathbb{H}$ with zero real part. This already gives a hint at the connection between unit quaternion conjugation and $SO(3)$, but before we continue shaping this idea, the discovery that Φ_q preserves the real part is stated in lemma 2.25.

2

Lemma 2.25. [3] *Let $q = \mathbb{H}_1$ and $p \in \mathbb{H}$, then $\Phi_q(p)$ preserves the real part of p, that is*

$$Re(p) = Re(qpq^{-1})$$

A quaternion $v \in \mathbb{H}$ with zero real part is of the form $v = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Define $\vec{v}$ to be the three-element vector notation of this,

$$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2.20}$$

This way we can see a general quaternion $q = r + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ as $q = r + v$, where we thus have a real part $r \in \mathbb{R}$ and imaginary part $v = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, but now we see v as a vector $\vec{v} \in \mathrm{Im}\mathbb{H} \cong \mathbb{R}^3$. This is a very important observation, as now a 3-dimensional vector can be represented by a purely imaginary quaternion and this way the quaternions can be linked to $SO(3)$. In the next two theorems, this map is explicitly shown and it is proven that the unit quaternions have a two-to-one relation with $SO(3)$. However, first two useful lemmas will be proved.

Lemma 2.26. *For $q_1, q_2 \in \mathbb{H}$, written as $q_1 = r_1 + v_1$ and $q_2 = r_2 + v_2$, with $r_1, r_2 \in \mathbb{R}$ and $v_1, v_2 \in Im\mathbb{H}$ it holds that*

$$q_1 q_2 = (r_1 r_2 - v_1 \cdot v_2) + (r_1 v_2 + r_2 v_1 + v_1 \times v_2). \tag{2.21}$$

Proof. Write out

$$q_1 q_2 = (r_1 + v_1)(r_2 + v_2) = r_1 r_2 + r_1 v_2 + r_2 v_2 + v_1 v_2. \tag{2.22}$$

Then write $v_1 = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $v_2 = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$ such that

$$\begin{aligned} v_1 v_2 &= (a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k})(a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}) \\ &= -a_1 a_2 + a_1 b_2\mathbf{k} - a_1 c_2\mathbf{j} - b_1 a_2\mathbf{k} - b_1 b_2 + b_1 c_2\mathbf{i} + c_1 a_2\mathbf{j} - c_1 b_2\mathbf{i} - c_1 c_2. \\ &= -(a_1 a_2 + b_1 b_2 + c_1 c_2) + (b_1 c_2 - c_1 b_2)\mathbf{i} + (c_1 a_2 - a_1 c_2)\mathbf{j} + (a_1 b_2 - b_1 a_2)\mathbf{k}. \end{aligned}$$

We recognize this exactly as $v_1 v_2 = -v_1 \cdot v_2 + v_1 \times v_2$ if we interpret them as vectors. Substituting this back in equation 2.22, already gives the required result. □

Lemma 2.27. *Define the map $F : \mathbb{H} \to \mathbb{H}$ with $F(q) = r + Rv$, where $q = r + v \in \mathbb{H}$ and for some $R \in SO(3)$, a rotation of $Im\mathbb{H}$. Then F is an automorphism of the real algebra $\mathbb{H}$.*

Proof. First note that R is inner product preserving and linear. Furthermore, it can be shown that $Rv_1 \times Rv_4 = R(v_1 \times v_4)$. Putting this al together and using lemma 2.26, we get

$$\begin{aligned} F(q_1)F(q_2) &= (r_1 + Rv_1)(r_2 + Rv_2) \\ &= (r_1 r_2 - Rv_1 \cdot Rv_2) + (r_1 Rv_2 + r_2 Rv_1 + Rv_1 \times Rv_2) \\ &= (r_1 r_2 - v_1 \cdot v_2) + R(r_1 v_2 + r_2 v_1 + v_1 \times v_2) = F(q_1 q_2). \end{aligned}$$

[3] For a directer proof see appendix A.1.

Next, take $q_3 = r_3 + v_3 \in \mathbb{H}$, then $F(r_3 + R^T v) = r_3 + RR^T v_3 = r_3 + v_3 = q_3$, since R^T is the well defined inverse of R, so F is surjective. The injectivity is clear by the uniqueness of the rotation R, concluding the proof. □

Theorem 2.28. *If the operator Φ_q, for some $q \in \mathbb{H}_1$, works on some $v \in \mathbb{H}$ with zero real part, then $\Phi_q(v) = qvq^{-1}$ can be written as $R_q\vec{v}$, with $R_q \in SO(3)$. Furthermore, if we write $q = r + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then R_q is given by*

$$\begin{bmatrix} r^2+a^2-b^2-c^2 & 2(ab-cr) & 2(ac+br) \\ 2(ab+cr) & r^2-a^2+b^2-c^2 & 2(bc-ar) \\ 2(ac-br) & 2(bc+ar) & r^2-a^2-b^2+c^2 \end{bmatrix}. \tag{2.23}$$

Proof. We want to write out qvq^{-1}. First multiply q and v.

$$\begin{aligned} qv &= (r + a\mathbf{i} + b\mathbf{j} + c\mathbf{k})(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \\ &= (rx\mathbf{i} + ry\mathbf{j} + rz\mathbf{k}) + (-ax + ay\mathbf{k} - az\mathbf{j}) + (-bx\mathbf{k} - by + bz\mathbf{i}) + (cx\mathbf{j} - cy\mathbf{i} - cz) \\ &= (-ax - by - cz) + (rx + bz - cy)\mathbf{i} + (ry - az + cx)\mathbf{j} + (rz + ay - bx)\mathbf{k} \end{aligned}$$

Now we have multiply this by q^{-1}, to do this, divide the multiplication up into parts.

$$\begin{aligned} qvr &= (-arx - bry - crz) + (r^2x + brz - cry)\mathbf{i} + (r^2y - arz + crx)\mathbf{j} + (r^2z + ary - brx)\mathbf{k}, \\ qv(-a\mathbf{i}) &= (a^2x + aby + acz)\mathbf{i} + (arx + abz - acy) + (ary - a^2z + acx)\mathbf{k} + (-arz - a^2y + abx)\mathbf{j}, \\ qv(-b\mathbf{j}) &= (abx + b^2y + bcz)\mathbf{j} + (-brx - b^2z + bry)\mathbf{k} + (bry - abz + bcx) + (brz - aby - b^2x)\mathbf{i}, \\ qv(-c\mathbf{k}) &= (acx + bcy + c^2z)\mathbf{k} + (crx + bcz - c^2y)\mathbf{j} + (-cry + acz - c^2x)\mathbf{i} + (crz + acy - bcx). \end{aligned}$$

Adding all of this up, makes the real parts cancel and some other terms combine. Next, we first write the expression in terms of $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$, then we order it for x, y and z.

$$\begin{aligned} &(r^2x + brz - cry + a^2x + aby + acz + brz - aby - b^2x - cry + acz - c^2x)\mathbf{i} \\ &+(r^2y - arz + crx - arz - a^2y + abx + abx + b^2y + bcz + crx + bcz - c^2y)\mathbf{j} \\ &+(r^2z + ary - brx + ary - a^2z + acx - brx - b^2z + bry + acx + bcy + c^2z)\mathbf{k} \\ &= \left[(r^2 + a^2 - b^2 - c^2)x + 2(ab - cr)y + 2(ac + br)z\right]\mathbf{i} \\ &+ \left[(r^2 - a^2 + b^2 - c^2)y + 2(ab + cr)x + 2(bc - ar)z\right]\mathbf{j} \\ &+ \left[(r^2 - a^2 - b^2 + c^2)z + 2(ac - br)x + 2(ar + bc)y\right]\mathbf{k}. \end{aligned}$$

We can write this as

$$\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \end{bmatrix} \begin{bmatrix} r^2+a^2-b^2-c^2 & 2(ab-cr) & 2(ac+br) \\ 2(ab+cr) & r^2-a^2+b^2-c^2 & 2(bc-ar) \\ 2(ac-br) & 2(ar+bc) & r^2-a^2-b^2+c^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Call this 3×3 matrix R_q, writing out or letting a computer calculate the determinant we get $\det R_q = 1$ and $R_q^T R_q = I$, so $R_q \in SO(3)$.[4] If we now see $\mathbf{i}, \mathbf{j}, \mathbf{k}$ as basis vectors of a three dimensional space, we indeed see that applying the operator Φ_q to v can be represented as $R_q\vec{v}$. □

[4]This will also follow from theorem 2.29.

Theorem 2.29. *The homomorphism* $\Phi : \mathbb{H}_1 \to SO(3)$ *is surjective and two-to-one.*

Proof. Let $R \in SO(3)$, with its rotational axis determined by the unit vector $u \in \mathbb{R}^3$ and over rotation angle $\theta \in [0,\pi]$. We want to find some $q = r + a\mathbf{i} + b\mathbf{j} + c\mathbf{k} \in \mathbb{H}_1$ such that R_q in theorem 2.28 is the same rotation as R. Note that from this theorem we also know that at least any Φ_q is a rotation or $R_q \in SO(3)$.
To prove surjectivity, we will show that the unit quaternion $q_1 := \cos(\frac{1}{2}\theta) + \sin(\frac{1}{2}\theta)u$ is a rotation with rotational axis u and rotation angle θ. Now in general note that if we use the vector notation $q = r + v$, so with $v = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$, then

$$\Phi_q(v) = qvq^{-1} = q(q-r)q^{-1} = qqq^{-1} - qrq^{-1} = q - r = v.$$

For $q_1 = r_1 + v_1$, with $r_1 = \cos(\frac{1}{2}\theta)$ and $v_1 = \sin(\frac{1}{2}\theta)u$, we can see that similarly $\Phi_{q_1}(u) = u$. This means that u is fixed by Φ_{q_1} and thus by the rotation R_{q_1}, meaning u is the axis of rotation.
Now consider the case where $u = \mathbf{i}$, define $q_2 := \cos(\frac{1}{2}\theta) + \sin(\frac{1}{2}\theta)\mathbf{i}$. Then

$$\begin{aligned}
\Phi_{q_2}(\mathbf{i}) &= \mathbf{i}, \\
\Phi_{q_2}(\mathbf{j}) &= (\cos(\tfrac{1}{2}\theta) + \sin(\tfrac{1}{2}\theta)\mathbf{i})\,\mathbf{j}\,(\cos(\tfrac{1}{2}\theta) - \sin(\tfrac{1}{2}\theta)\mathbf{i}) \\
&= (\cos^2(\tfrac{1}{2}\theta) - \sin^2(\tfrac{1}{2}\theta))\mathbf{j} + 2\cos(\tfrac{1}{2}\theta)\sin(\tfrac{1}{2}\theta)\mathbf{k} \\
&= \cos(\theta)\mathbf{j} + \sin(\theta)\mathbf{k}, \\
\Phi_{q_2}(\mathbf{k}) &= -\sin(\theta)\mathbf{j} + \cos(\theta)\mathbf{k}.
\end{aligned}$$

If we see $\operatorname{Im}\mathbb{H}$ as the three dimensional space again, we can write this as R_{q_2} given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

This is a rotation by θ around $\mathbf{i}$, so our propositions holds at least for q_2. To prove this holds for the arbitrary q_1, we make use of the automorphism from lemma 2.27$F : \mathbb{H} \to \mathbb{H}$ with $F(q) = r + R_F v$, where $q = r + v \in \mathbb{H}$ and $R_F \in SO(3)$ such that it is a rotation of $\operatorname{Im}\mathbb{H}$ with$R_F\mathbf{i} = u$. Then, the fact that R_F maps $\mathbf{i}$ to u implies that $F(\Phi_{q_2}(F^{-1}(q)))$ is a rotation of θ around u, since Φ_{q_2} is a rotation of θ around $\mathbf{i}$ and u is fixed and thus the axis of rotation. But then

$$\begin{aligned}
F(\Phi_{q_2}(F^{-1}(q))) &= F(q_2F^{-1}(q)q_2^{-1}) = F(q_2)F(F^{-1}(q))F(q_2^{-1}) \\
= (\cos(\tfrac{1}{2}\theta) + R_F\sin(\tfrac{1}{2}\theta)\mathbf{i})q(\cos(\tfrac{1}{2}\theta) &- R_F\sin(\tfrac{1}{2}\theta)\mathbf{i}) = q_1qq_1^{-1} = \Phi_{q_1}(q).
\end{aligned}$$

So we found a unit quaternion representing R, which was arbitrary. This proves that Φ is surjective. That Φ is two-to-one follows from the fact that the kernel is equal to $\ker(\Phi) = \{\pm 1\}$. We prove this as follows.
We can calculate $\operatorname{tr}(R_q) = 3r^2 - a^2 - b^2 - c^2 = 4r^2 - 1$ for general $q \in \mathbb{H}$. However, if $q \in \ker(\Phi)$, then Φ_q must be the identity matrix and thus $\operatorname{tr}(R_q) = \operatorname{tr}(I) = 3$. Equating these two gives

$$4r^2 - 1 = 3 \iff r^2 = 1 \iff r = \pm 1.$$

The fact that we must have $q \in \mathbb{H}_1$, implies that the imaginary part should be zero, so $q \in \ker(\Phi) \iff q = \pm 1$, concluding the proof. □

Another result from the proof of theorem 2.29, is that q describes the rotation in $SO(3)$ with axis of rotation u and rotation angle 2θ, since any unit quaternion $q \in \mathbb{H}_1$ can be written as $q = \cos(\theta) + \sin(\theta)u$, for some $u \in \operatorname{Im}\mathbb{H} \cap \mathbb{H}_1$ and $\theta \in [0, \frac{1}{2}\pi]$. The two-to-one relation between $\mathbb{H}_1$ and $SO(3)$ can be seen very directly this way. From $qvq^{-1} = (-q)v(-q)^{-1}$ for $v \in \operatorname{Im}\mathbb{H}$, it follows that $-q$ has to be the other unit quaternions describing the same rotation as q. Indeed $-q = -\cos(\theta) - \sin(\theta)u = \cos(\pi-\theta) + \sin(\pi-\theta)(-u)$, meaning $-q$ is a rotation in $SO(3)$ with rotation axis $-u$ and rotation angle $2(\pi-\theta) = 2\pi - 2\theta$. From section 2.3 we know this is the same rotation as the rotation of q.

This way θ is the angle between $1 \in \mathbb{H}$ and q and if $\theta > \frac{1}{2}\pi$, then it is the angle between $-1 \in \mathbb{H}$ and q.

To show what relation the unit quaternions and $SO(4)$ possess, the just proven theorem 2.29 is used.

Theorem 2.30. *The homomorphism* $\Psi : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ *is surjective and two-to-one.*

Proof. Let $R \in SO(4)$ and let $q_1 := R1$. Since R is norm preserving $\|q_1\| = \|1\| = 1$, thus $q_1 \in \mathbb{H}_1$. Note that $L_{q_1^{-1}}(R1) = q_1^{-1}q_1 = 1$. Define $R_1 := L_{q_1^{-1}} \circ R$ and note that $R_1 \in SO(4)$. R_1 preserves 1 and thus also all of $\mathbb{R}$, but then $\mathbb{R}^\perp = \operatorname{Im}\mathbb{H}$ is preserved as well, by inner product preservation. Call $\hat{R}_1$ the restriction of R_1 to $\operatorname{Im}\mathbb{H}$, then $\hat{R}_1 \in SO(3)$, since the norm is still preserved on just the $\operatorname{Im}\mathbb{H}$ part. Then however, by theorem 2.29, there must exist a $q_2 \in \mathbb{H}_1$ such that for all $p \in \operatorname{Im}\mathbb{H}$, it holds that $\hat{R}_1 p = q_2 p q_2^{-1} = \Phi_{q_2}(p)$. We can write

$$Rp = L_{q_1}\hat{R}_1 p = q_1 q_2 p q_2^{-1} \quad \text{for } p \in \operatorname{Im}\mathbb{H} \tag{2.24}$$

Furthermore,

$$Rr = q_1 r = q_1 q_2 r q_2^{-1} \quad \text{for } r \in \mathbb{R} \tag{2.25}$$

So equation 2.24 holds for $p \in \mathbb{H}$ as well. Finally call $q_3 := q_1 q_2 \in \mathbb{H}_1$, then we have indeed found two unit quaternion such that $R = \Psi_{q_3,q_2}$, proving that Ψ is surjective.
The fact that Ψ is two-to-one, follows from the fact that $\ker(\Psi) = \{\pm(1,1\}$. We prove this as follows.
$(q_1, q_2) \in \ker(\Psi)$ is equivalent to $\Psi_{q_1,q_2}(p) = p$ for all $p \in \mathbb{H}$. Taking specifically $p = 1$, we get $q_1 1 q_2^{-1} = 1 \iff q_1 = q_2$. But this means our problem reduces to finding q_1 such that $\Psi_{q_1,q_1}(p) = \Phi_{q_1}(p) = p$ for $p \in \mathbb{H}$. From theorem 2.29 we see this only holds for $q_1 = \pm 1$, which corresponds to the pairs $(1,1)$ and $(-1,-1)$. [15] □

Similarly to the remark after theorem 2.29, we have $q_1 p q_2^{-1} = (-q_1)p(-q_2)^{-1}$. From theorem 2.30 it follows that these are the only two pairs describing the corresponding rotation in $SO(4)$. Theorems 2.29 and 2.30 allow 3 and 4-dimensional rotations to be represented as quaternions, giving new insights in rotations. From the proofs the result can be summarized with

$$\mathbb{H}_1/\{\pm 1\} \cong SO(3), \quad (\mathbb{H}_1 \times \mathbb{H}_1)/\{\pm(1,1)\} \cong SO(4). \tag{2.26}$$

The three and six dimensional nature of $SO(3)$ and $SO(4)$ respectively, that was found in section 2.3.4, can also be seen in this construction from the 3-dimensional unit quaternions. Informally, $SO(3) \times SO(3)$ is linked this way with $SO(4)$ which is quite unique compared to rotation in other dimensions.

As a last remark for this section, note that the quaternions are often seen as an extension of the complex numbers, which actually directly inspired them. That quaternions are able to represent rotations this way really emphasizes this, since unit complex numbers can represent 2-dimensional rotations, drawing a direct parallel.

2.6. REFLECTIONS USING QUATERNIONS

IN section 2.3.4 we have seen that in an n-dimensional space a specific subset of the reflections, which we call mirrors, can be characterized by a $(n-1)$-hyperplane, which in turn is fully described by the perpendicular unit vector to this plane. Applying this mirror on a vector means that the vector is mirrored through the mirror hyperplane or equivalently we can subtract its projection on to the normal vector of the mirror hyperplane twice. This holds for four dimensions as well, so for $x \in \mathbb{H} \cong \mathbb{R}^4$ the mirror through the 3-dimensional hyperplane with perpendicular unit vector $q \in \mathbb{H}_1$, is given by

$$x \mapsto x - 2(x \cdot q)q$$

We can rewrite this in a simpler way, but for this we need the following two equalities. Write $x = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ and $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, then

$$\mathrm{Re}(x^* q) = \mathrm{Re}((x_0 - x_1\mathbf{i} - x_2\mathbf{j} - x_3\mathbf{k})(q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k})) = x_0q_0 + x_1q_1 + x_2q_2 + x_3q_3 = x \cdot q,$$
$$\mathrm{Re}(x^* q) = \frac{x^* q + (x^* q)^*}{2} = \frac{x^* q + xq^*}{2}.$$

Combining all of this gives

$$x - 2(x \cdot q)q = x - 2q\mathrm{Re}(x^* q) = x - q(x^* q + q^* x) = -qx^* q.$$

We see that similarly to quaternion rotations, $\pm q$ both represents the same mirror. However, in this case this is not really a property of quaternion mirror, since there are always two choices for a perpendicular unit vector to a $(n-1)$-hyperplane. In fact, the unit vector in just a representation of the line through the origin that they lie on in the first place. The map

$$x \mapsto -qx^* q$$

is nonetheless an interesting description of mirrors, as the unit quaternions thus also describe all four dimensional mirrors twice. Furthermore, we can prove in a different way than in section 2.3.4, that two mirrors combined result in a rotation. Define $\sigma_q(x) := -qx^* q$, then for $q_1, q_2 \in \mathbb{H}_1$ and $x \in \mathbb{H}$, reflecting according to q_2 after q_1 results in

$$\sigma_{q_2}(\sigma_{q_1}(x)) = \sigma_{q_2}(-q_1x^* q_1) = q_2(q_1x^* q_1)^* q_2 = q_2q_1^* xq_1^* q_2 = q_3xq_4^{-1} = \Psi_{q_3,q_4}(x),$$

where $q_3 := q_2q_1^*$ and $q_4 := q_2^* q_1$, since q_1 and q_2 are unit quaternions. Indeed, this is a rotation.

It is interesting to note that according to the Cartan–Dieudonné theorem, every isometry in $O(n)$ is a composition of at most n mirrors through hyperplanes, although we will not go into detail about this. [7]

3

Symmetries Using Quaternions

We have seen in chapter 2 that for every dimension $O(n)$ is the group of all symmetries keeping the origin in place, called isometries, and $O(n)$ is equivalent to the group of all $n \times n$ orthogonal matrices, which all have determinant ± 1. The subgroup $SO(n)$ of $O(n)$ containing all isometries which are also orientation preserving, turns out to be exactly the orthogonal $n \times n$ matrices with determent $+1$. To reiterate, the elements of $SO(n)$ are called rotations and the elements of $O(n) \setminus SO(n)$ are called reflections.

These isometries are the symmetries of the entire space, but what are symmetries of objects in this space? First of all, symmetries of an object should be a subgroup of $O(n)$, since, as briefly discussed, finite object do not have translational or glide reflectional symmetries. What subgroup is of course determined by exactly what object we are looking at, some objects may not even have symmetries, besides the identity, which of course should be a symmetry of all objects. In this chapter, we will look at the symmetries of the regular polytopes the cube, hypercube and hyperdiamond, as well as the symmetries of the binary octahedral group interpreted as vertex set.

In section 3.1 the n-cube is introduced and its geometric structure and symmetries are studied. Then the quaternions will be applied as a tool to define the binary rotation groups of the cube and hypercube in section 3.2. The structure of these rotation groups is studied further in section 3.3. Finally, the binary rotation groups of the normal subgroups of the binary octahedral group are analyzed in section 3.4 and 3.5.

3.1. THE n-CUBE

THE n-cube is the analogue of the square and the cube in n-dimensional space. The square and cube are therefore just the 2-cube and the 3-cube. We want to formally define the n-cube in a n-dimensional vector space. There are many ways to do this, but for us the most convenient way is to keep the center of the n-cube at the origin. This will be very similar to how the square and cube are placed in the following figures.

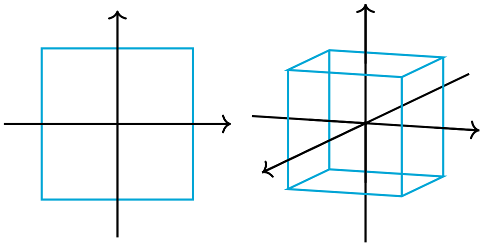

Figure 3.1: The square with its center at the origin of a 2-dimensional space on the left and a cube with its center at the origin of a 3-dimensional space on the right.

This way the theory about origin preserving isometries that we have build up so far, can be immediately implemented.

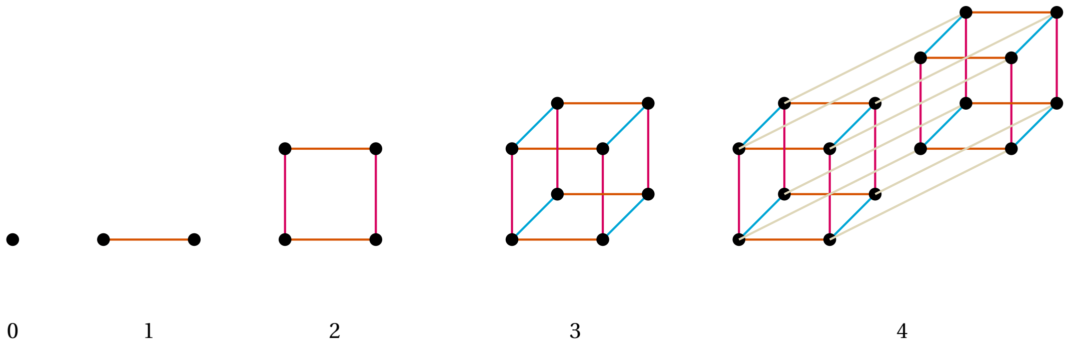

Figure 3.2: The construction of the 4-cube from lower-dimensional cubes. The number corresponds to the dimension and the colour represents the extension to dimension 1 for orange, dimension 2 for magenta, dimension 3 for blue and dimension 4 for beige. [14]

To accomplish this, we will define the n-cube by the placement of its vertices, namely place the vertices at all n-dimensional vectors of the form

$$(\pm 1, \ldots, \pm 1)$$

in n-dimensional vector space. From this, it is clear that every n-cube thus has 2^n vertices. We can also see this if we look at how an n-cube is build from a $(n-1)$-cube. This is illustrated in figure 3.2. We start with $n = 0$, then the 0-cube is defined to be just a vertex. For $n = 1$, connect this 0-cube with an edge to another 0-cube, creating the 1-cube.

Similarly, for $n = 2$, add an additional edge to every vertex of the 1-cube, connecting it to another 1-cube, creating the well known square. Next, to form a cube, again, take two 2-cubes and add an edge to each vertex to connect the 2-cube. Lastly, we will look at the 4-dimensional cube, because for higher dimensions the figures get very messy. On top of this, humans as 3-dimensional creatures can not imagine extra dimensions, so figure 3.2 just gives a nice visualization of the structure of the 4-cube, but does not display what the 4-cube really entails. Nevertheless, connecting two 3-cubes to each other forms the 4-cube. While not totally in line with other definitions, we will call the 4-cube the hypercube in this thesis and it will be studied thoroughly.

By this construction of the n-cube, it can also be seen that every vertex of an n-cube has n edges connected to itself. Looking at the vector positions we chose for the vertices, these are exactly the edges connecting a vertex straight to each axis, which lies halfway towards the vertices on the opposite side of these axes.

From figure 3.2, we also see that at least for these cases an n-cube indeed has 2^n vertices. What is also interesting to note is that every n-cube seems to contain $2n$ $(n-1)$-cubes. Indeed, the 1-cube contains two 0-cubes, the square contains four 1-cubes and the cube contains six squares. That the hypercube contains 8 cubes, is a bit harder to see. However, if we draw the hypercube as in figure 3.3, one can count six enclosed areas on the sides, a smaller cube in the middle and the outer vertices form a bigger cube as well, indeed adding up to eight cubes. Theorem 3.1 encompasses all these intuitive propositions and more.

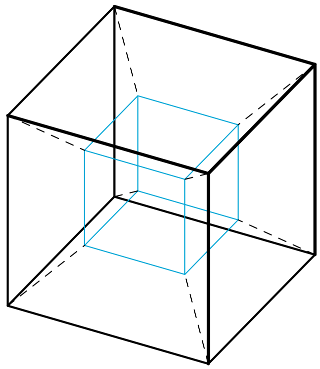

Figure 3.3: The hypercube projected specifically so it is easier to count the eight contained cube.

Theorem 3.1. *For $k \geq n$, the number of k-cubes contained in an n-cube is given by*

$$2^{n-k}\frac{n!}{k!\,(n-k)!}. \tag{3.1}$$

Proof. Any vertex of the n-cube is connected to n edges. Choosing any combination of k of these edges induces the formation of a k-cube contained in the n-cube. So each vertex is exactly part of $\binom{n}{k}$ k-cubes. This would make for a total of $2^n\binom{n}{k}$ k-cubes, however, in this count every k-cube is counted 2^k times, since it has this many vertices. So we can conclude that the total number of k-cubes is equal to

$$\frac{2^n\binom{n}{k}}{2^k} = 2^{n-k}\frac{n!}{k!(n-k)!}$$

[8] □

We can create the following table for what kind of k-cubes certain n-cubes contain.

3

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	Total
0	1	-	-	-	-	-	-	-	-	-	-	1
1	2	1	-	-	-	-	-	-	-	-	-	3
2	4	4	1	-	-	-	-	-	-	-	-	9
3	8	12	6	1	-	-	-	-	-	-	-	27
4	16	32	24	8	1	-	-	-	-	-	-	81
5	32	80	80	40	10	1	-	-	-	-	-	243
6	64	192	240	160	60	12	1	-	-	-	-	729
7	128	448	672	560	280	84	14	1	-	-	-	2187
8	256	1024	1792	1792	1120	448	112	16	1	-	-	6561
9	512	2304	4608	5376	4032	2016	672	144	18	1	-	19683
10	1024	5120	11520	15360	13440	8064	3360	960	180	20	1	59049

Table 3.1: The number of k-cubes (horizontally) in an n-cube (vertically). The final column shows the total number of all k-cubes, which is 3^n.[1]

3.1.1. THE n-ORTHOPLEX

The dual of a regular regular polytope in n dimensions is created by taking the centers of the $(n-1)$-dimensional faces of the regular polytope as vertices of the dual, which are connected to each other if their corresponding faces connect in the regular polytope. This way the cube has a dual as well, called the octahedron. The octahedron is constructed by taking the center of the six square faces of the cube, creating a double pyramid type shape, as illustrated in figure 3.4. There it can also be seen that the dual to the octahedron is a cube again.

Note that the vertices of the octahedron lie exactly on the axes if the center of the cube lies at the origin. Particularly, with the way we placed the vertices of the n-cube in the previous section, the coordinates of the vertices of the octahedron are exactly $(\pm1,0,0),(0,\pm1,0),(0,0,\pm1)$. This can be directly extended to general dimensions where the dual of the n-cube is called the n-orthoplex and its vertices lie exactly on the axes, with specific coordinates $(\pm1,0,\ldots,0),\ldots,(0,\ldots,0,\pm1)$. It is important to note that the n-orthoplex has the exact same symmetry group as the n-cube, as is true for general regular polytopes and their duals. Note that the 4-orthoplex is also called the 16-cell.

The vertices of the hypercube can be split in those with an even number of minus signs

[1] This can be seen as an application of the Al-Khwarizmi's binomial theorem that states $\sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k = (x+y)^n$. Taking $x=2$ and $y=1$ yields our result.

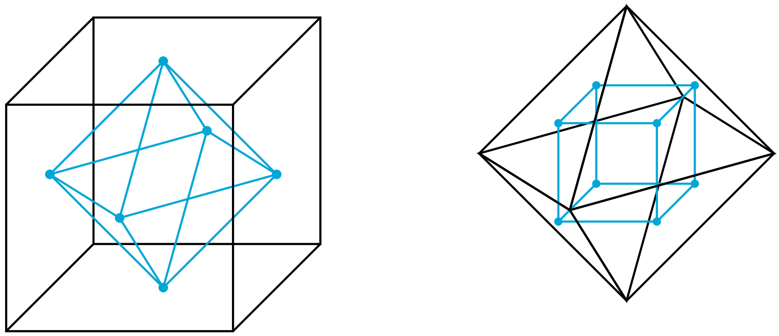

Figure 3.4: On the left is the dual octahedron to the cube, constructed with vertices at the centers of the square faces of the cube. On the right is the dual of the octahedron, which is again a cube, constructed with vertices at the centers of the triangular faces of the octahedron.

and those with an odd number of minus signs. These two sets of 8 vertices exactly describe two 4-orthoplexes and is the only way to split the vertices of the hypercube such that it results in two 4-orthoplexes. This can be proved with the fact that there lie two vertices of the 4-orthoplex on every axis in the standard configuration as a dual to the standard hypercube. This means that all vectors to these vertices are orthogonal, except for the pairs on the same axes, which are each others opposites. Take the vertex $(1,1,1,1)$ of the hypercube, which is orthogonal to the six vertices of the hypercube with two minus signs and its opposite is $(-1,-1,-1,-1)$. By just taking one point the selection of candidates for a 4-orthoplex is already fixed and we obtained the two sets with an even and odd number of minus signs. Now all that is left to prove is that these two sets really form two 4-orthoplexes. Looking at the set with an even number of minus signs, we indeed see that every vertex in this set has its unique opposite vertex and is orthogonal to the remaining vertices. The distance ratio is consistent with the standard configuration, which is $2:\sqrt{2}$, between opposite vertices and orthogonal vertices. Indeed $d((1,1,1,1),(-1,-1,-1,-1)) = \sqrt{2^2+2^2+2^2+2^2} = 4$ and $d((1,1,1,1),(-1,-1,1,1)) = \sqrt{2^2+2^2+0+0} = \sqrt{8} = 2\sqrt{2}$. This clearly holds for the other combination too and the other distances can be checked as well. We also find the interesting fact that this 4-orthoplex has edges twice as long as the standard configuration. With similar arguments the set of eight hypercube vertices with an odd number of minus signs describes a 4-orthoplex as well.

3.1.2. SYMMETRIES OF THE SQUARE AND THE CUBE

Before we try to figure out the symmetries of the square and the cube, we have to discuss what a symmetry of an object really means mathematically. In the introduction of this chapter it was mentioned that all symmetries of an object should be a subgroup of $O(n)$, but do all symmetries of an object really form a group? Let us check this briefly. First, applying a symmetry yields the same object, meaning that applying multiple symmetries combine to another symmetry of the object, so they are closed. That symmetries of an object are associative and have an inverse are properties they inherit from $O(n)$. Lastly, the identity, which just keeps everything in place, is a symmetry of any object. So the symmetries indeed form a group, call it the symmetry group of an object.

Furthermore, just like $SO(n)$ is a subgroup of $O(n)$, just the rotational symmetries of an object form a subgroup of the entire symmetry group. This is called the rotational group of an object, which is just as and if not, maybe even more interesting then the total symmetry group. The rotational group is indeed a group, since we have seen that combining rotations always results in a new rotation and the inverse rotations are symmetries as well.

In this thesis an object will be defined by the positions of its vertices, which we already did for the n-cube at the start of section 3.1. It of course depends on the way these vertices are connected what kind of symmetries the object has, however, in this thesis we will treat only very symmetrical object for which any rotation or reflection mapping vertices to each other will be a symmetry. So, when trying to find symmetries of an object, we will be looking for rotations and reflections, which map the set of vertices defining a very symmetrical object to itself.

We would like to determine a formula for the number of symmetries of an n-cube, which will be discussed further in section 3.1.4. Before we can find this, we need to look at the cases $n = 2, 3$ intuitively first, which the rest of this section will be dedicated to.

A square can be rotated 90, 180, 270 and 360 degrees, where a rotation by 360 degrees corresponds to the identity. Besides these four rotations, four reflection axis can be determined as well, as illustrated in figure 3.5. This gives rise to the well known dihedral group $D_4 := \{e, r, r^2, r^3, \sigma, r\sigma, r^2\sigma, r^3\sigma\}$. The construction of this group makes use of the fact that all four reflections can be written as a rotation plus one specifically chosen reflection, usually the horizontal reflection axis. We will not go into more detail about the dihedral group, for us the main take away from this example is that there are eight symmetries of which half rotations.

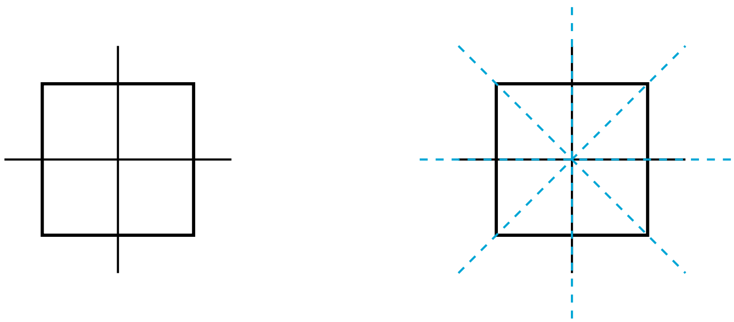

Figure 3.5: The square with its center at the origin of a 2-dimensional plane and its axis of reflection.

From Euler's theorem 2.11, we know that to find the rotational symmetries of the cube, we should be looking for rotational axis. It turns out there are four types of rotations of the cube. See figure 3.6 as a reference.

- The identity, which keeps everything in place.
- Rotations around the three axes through the centers of opposite sides, these are

the dashed lines. Rotations can go over 90, 180 or 270 degrees for a total of $3 \cdot 3 = 9$ rotations of this type.

- Rotations around the four axes corresponding to the diagonals, which go through opposite vertices, these are the full lines. Rotation can go over 120 or 240 degrees for a total of $4 \cdot 2 = 8$ rotations of this type.

- Rotations around the six axes through the centers of opposite edges, these are the dotted lines. Rotations can go over 180 degrees for a total of $6 \cdot 1 = 6$ rotations of this type.

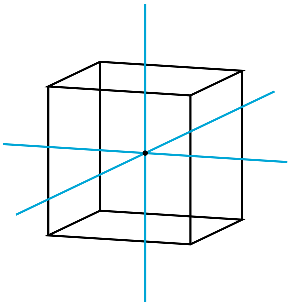

Figure 3.6: The cube with its center at the origin of a 3-dimensional space with axes of the space on the left and with its axes of rotation on the right.

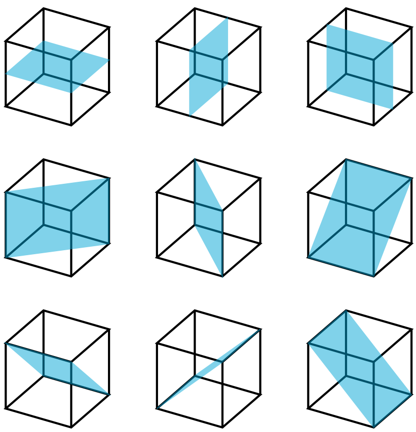

Figure 3.7: The cube with its center at the origin of a 3-dimensional space with its nine planes of reflection.

Reflections of the cube are characterized by nine different reflection planes of which there are two types, see figure 3.7 for a reference.

- Reflections through the plane containing the centers of four faces, these are top three planes. Before reflecting the cube can be separately rotated according to the second rotation type over 0, 90, 180 or 270 degrees, for a total of $3 \cdot 4 = 12$ reflections of this type.

- Reflections through the plane containing two opposite edges, these are the bottom six planes. Before reflecting the cube can be separately rotated according to the fourth rotation type over 0 or 180 degrees, for a total of $6 \cdot 2 = 12$ reflections of this type.

The structure of the rotational group of the cube, denote this group as O, can be determined very visually. Namely, every rotation permutes the diagonals of the cube in a unique way. If we name the diagonals as in figure 3.8, then O is isomorphic to the permutation group $S(\{1,2,3,4\})$, that is $O \cong S_4$. Indeed, for each rotation it can be visually checked in what way the diagonals are permuted. To prove this formally, it is very useful to use the fact that O is acting the four diagonals. Let us quickly define this and introduce a necessary theorem.

Definition 3.2. *Let G be a group and X be a set. We say G acts on X if for every $g \in G$ and $x \in X$, an element $g \circ x \in X$ is given such that $e \circ x = x$ for all $x \in X$ and $(gh) \circ x = g \circ (h \circ x)$ for $g, h \in G$ and $x \in X$.*

Theorem 3.3. *Let G act on X. Then for all $g \in G$ the map $\epsilon_g : X \to X$, defined by $\epsilon_g(x) := g \circ x$ is bijective and the map $f : G \to S(X)$, defined by $f(g) := \epsilon_g$ is a homomorphism.*

Proof. Since G acts on X, we have that for all $x \in X$

$$\epsilon_e(x) = e \circ x = x.$$

This means $\epsilon_e = \mathrm{id}_X \in S(X)$. We also have that for all $g_1, g_2 \in G$ and all $x \in X$

$$\epsilon_{g_1 g_2}(x) = (g_1 g_2) \circ x = g_1 \circ (g_2 \circ x) = \epsilon_{g_1}(\epsilon_{g_2}(x)).$$

This shows that

$$\epsilon_{g_1 g_2} = \epsilon_{g_1} \circ \epsilon_{g_2} \tag{3.2}$$

and in particular that for all $g \in G$

$$\epsilon_g \circ \epsilon_{g^{-1}} = \epsilon_{g^{-1}} \circ \epsilon_g = \epsilon_e = \mathrm{id}_X,$$

which implies that ϵ_g is bijective and thus $\epsilon_g \in S(X)$. Furthermore, this means that f is well-defined and equation 3.2 shows that $f(g_1 g_2) = f(g_1) \circ f(g_2)$, so f is a homomorphism. [12] □

Lemma 3.4. *The rotational group of the cube O is isomorphic to S_4.*

Proof. The rotational group of the cube O acts on the set of the diagonals of the cube. Let $f : O \to S_4$ be the group homomorphism induced by this action, according to theorem 3.3. Take any rotation of the fourth type (through the centers of two opposite edges), by inspection it can be seen that this rotation fixes two diagonals, while switching the diagonals that the two opposite edges are attached to from position. Note that this type of rotation indeed has order 2. We have seen there are six rotations of the fourth type and thus all six uniquely map to the six transpositions in S_4. The set of all transpositions generate S_4 and therefore f must be surjective. Then it follows from the fact the $\#O = \#S_4$ that f is bijective and thus an isomorphism. $\square$

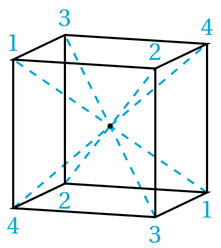

Figure 3.8: The cube with its diagonals numbered 1 to 4.

Furthermore, it turns out that the entire symmetry group of the cube is isomorphic to the direct product $S_4 \times C_2$. Intuitively this can be seen as two times the permutation of the diagonals, where one set corresponds to just the rotations and the other set with the case where orientation of the diagonals is flipped as well. This will not be proved this formally.

3.1.3. ORBIT STABILIZER THEOREM

The orbit stabilizer theorem is very useful when it comes to counting symmetries, which will be done a lot in section 3.1.4 and in other parts of this chapter. That is why this short section is dedicated to formally introducing this theorem. We will make use of the already defined action of a group on a set in definition 3.2.

Definition 3.5. *Let G act on X, then the orbit of an element x, denoted as Gx is defined as*

$$Gx := \{g \circ x : g \in G\} \tag{3.3}$$

So the orbit of an element $x \in X$ are the elements that x can reach with one operation. It turns out that the orbit of x actually induces a equivalence relation. Call two elements $x, y \in X$ equivalent if $y \in Gx$, so if there exist a $g \in G$ such that $y = g \circ x$. Let us check if this is indeed an equivalence relation. It is reflexive, since $x = e \circ x$. Next, $y = g \circ x$ implies $g^{-1}y = x$, so it is symmetric as well. That the relation is transitive follows from the fact that if for $x, y, z \in X$ and x, y and y, z are equivalent, it holds that for some $g_1, g_2 \in G$

$$y = g_1 \circ x, \; z = g_2 \circ y \implies z = g_2 \circ (g_1 \circ x) = (g_2 g_1) \circ x.$$

So x, z are equivalent as well, proving transitivity and thus we can conclude the relation is indeed an equivalence relation.

This means that the orbits are precisely the equivalence classes of this relation, which have the very helpful property that every equivalence class is disjoint and every element is in exactly one equivalence class. These properties thus also hold for orbits. This will be used in the proof of the orbit stabilizer theorem, but first the thus far missing stabilizer is introduced.

Definition 3.6. *Let G act on X. Then the stabilizer of $x \in X$ is defined as*

$$G_x := \{g \in G : g \circ x = x\}.$$

Sometimes the stabilizer $x \in X$ is written as Stab(x).

Theorem 3.7. *(Orbit-stabilizer theorem) Let G act on X, then for all $x \in X$*

$$\#Gx = [G : G_x]. \tag{3.4}$$

Proof. Let $x \in X$. First we prove that G_x is a subgroup of G. We have $e \in G_x$, so $G_x \neq \varnothing$. Let $g_1, g_2 \in G_x$, then

$$(g_1 g_2) \circ x = g_1 \circ (g_2 \circ x) = g_1 \circ x = x,$$

implying G_x is closed. G_x also contains all inverse, since for $g \in G_x$

$$g^{-1} \circ x = g^{-1} \circ (g \circ x) = x.$$

Next, define the map $f : G/G_x \to Gx$ by $f(aG_x) = a \circ x$. This is well defined, since

$$aG_x = bG_x \iff b^{-1}a \in G_x \iff (b^{-1}a) \circ x = 0 \iff a \circ x = b \circ x. \tag{3.5}$$

But we can conclude more from equation 3.5. It states that if a and b are in different left cosets, then $a \circ x$ and $b \circ x$ can not give the same value, so f is injective. Furthermore, if we choose a particular left coset of G_x, then for every element c in this left coset, $c \circ x$ gives the same value. Since every element of G is in one left coset this means that the image of f is equal to the entire orbit of x by definition, so f is surjective as well and thus bijective. From this we can conclude that $\#Gx = \#G/G_x = [G : G_x]$. [12] □

In particular from theorem 3.7, we will mostly use that for any $x \in X$

$$\#G = \#Gx \cdot \#G_x. \tag{3.6}$$

Choosing a convenient element x can make calculating the size of a group relatively simple. For example in section 3.1.2, we assumed that the 24 rotations of the cube we found were only ones, but now we can prove that these are the only ones with the orbit stabilizer theorem. Let O work on the faces of the cube, then there is only one orbit containing all six cube, since they can all be rotated to each other. Furthermore, each square is fixed by four rotations around the axis through its center and the center of the opposite face (of which one the identity). This gives us $\#O = 6 \cdot 4 = 24$, as expected.

3.1.4. NUMBER OF SYMMETRIES OF THE n-CUBE

Now it is time to look at the symmetries of a general n-cube. There are multiple counting arguments for what number of symmetries an n-cube has [8]. A relatively simple argument is that every vertex has to be mapped to one of the 2^n vertices. If we choose one of these vertices, then the connected n edges can be permuted in any way, giving $n!$ possibilities. This implies that the total number of symmetries is $2^n n!$. There is also a fun recursive argument making use of the orbit-stabilizer theorem and is more rigorous. Let the symmetry group of the n-cube act on the $2n$ $(n-1)$ cubes. We know that it contains this number of $(n-1)$-cubes from theorem 3.1. Then the orbit of a particular $(n-1)$-cube is equal to all of the $(n-1)$-cubes, as any hyperface can get mapped to every hyperface. So the size of the orbit is $2n$. Moreover, the size of the stabilizer is equal to the number of symmetries that keep the $(n-1)$-cubes, for which we can use the orbit stabilizer again on its $2(n-1)$ $(n-2)$-cubes. This recursion also gives that the number of symmetries of an n-cube is equal to

$$2n \cdot 2(n-1) \cdot \ldots \cdot 2 \cdot 1 = 2^n n!, \tag{3.7}$$

where it is used that the 0-cube has just one symmetry, namely the identity. The recursion can also stopped at the square, which we know has 8 symmetries.

For the square and the cube, we have seen that half of the symmetries are rotations, while the other half are reflections. This is actually the case for all finite objects with at least one reflectional symmetry. This can be seen using the first isomorphism theorem. Call the symmetry group of an object with at least one reflectional symmetry G and define the homomorphism $\det : G \to \{\pm 1\}$, which maps the isometries in G to the determinant of their corresponding matrix. This is indeed a homomorphism by the properties of combining rotations and reflections, as we have seen. Furthermore, it is surjective, since we have at least one reflections getting mapped to -1 and the identity gets mapped to 1 of course. But then the first isomorphism theorem tells us that $G/\ker(\det) = G/G^+ \cong \{\pm 1\}$, where G^+ denotes the rotation group of the object. In particular, the index of the rotation group in the symmetry group is 2 and we can conclude that indeed half the symmetries of an object are rotations. For completeness, this means the number of rotational symmetries of an n-cube is equal to

$$2^{n-1} n!. \tag{3.8}$$

3.2. ROTATIONAL SYMMETRIES OF THE CUBE AND HYPERCUBE USING QUATERNIONS

ROTATIONS in three and four dimensions are nicely represented as quaternions, although they are both double covers in different ways. In this section we will look at which quaternions exactly make up the rotation groups of the standard cube and the standard hypercube, making use of the map in theorem 2.29 and 2.30.

In section 3.1 the n-cube was already defined in n-dimensional space by the exact placement of its vertices. For three and four dimensions this can be directly extended to the 4-dimensional quaternion space. Just place the vertices of the cube in $\operatorname{Im}\mathbb{H}$ at $\{\pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$, such that the map Φ can really rotate this cube. Thereafter, place the vertices of the hypercube at $\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$, which Ψ can obviously rotate.

To actually figure out the rotation groups of the cube and the hypercube, we will instead look at the duals of the cube and hypercube, the octahedron and the 4-orthoplex respectively. From section 3.1.1, we know that the vertices of the octahedron are the centers of the faces of the cube, so they are $\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. Similarly, the vertices of the 4-orthoplex are $Q_8 := \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$. The rotations of the regular polytopes defined by these vertices are easier to analyze with quaternion, while the rotation groups of the n-cube coincides with that of the n-orthoplex, which we saw in section 3.1.1 as well.

3.2.1. ROTATION GROUP OF THE CUBE USING QUATERNIONS

Define $2O$ to be the group of all unit quaternions that map to a rotation of the cube through the map Φ from theorem 2.29, that is

$$2O := \Phi^{-1}(O). \tag{3.9}$$

By definition this indeed is a group and a subgroup of $\mathbb{H}_1$.

The naming of $2O$ very deliberately means two times the rotational group of the cube O, as we have seen that for every rotation there are two unit quaternions representing this rotation. This is also why $2O$ is often called the binary octahedral group. We actually defined $2O$ to be rotations of the cube, not the octahedron, but of course their symmetry groups coincide, as discussed in section 3.1.1. The fact that $2O$ has twice as many elements as O, is directly inherited from the relation between rotations and unit quaternions, $SO(3) \cong \mathbb{H}_1/\{\pm 1\}$. This means we can write

$$O \cong 2O/\{\pm 1\}.$$

Let us state the most important property of $2O$ in a theorem.

Theorem 3.8. *The map $\Phi_q : \mathbb{H} \to \mathbb{H}$ by $\Phi_q(p) = qpq^{-1}$ is a rotation of the octahedron with vertices $\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, as well as the cube with vertices $\{\pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$ for all $q \in 2O$.*

When looking for the elements of $2O$, we can use the remark after theorem 2.29 and the rotations in O we already found in section 3.1.2. This tells us that for a certain rotation in O with rotation angle $\theta \in [0, \pi]$ and rotation axis u, the according quaternion is given by $q = \pm\left(\cos\left(\frac{1}{2}\theta\right) + \sin\left(\frac{1}{2}\theta\right) u\right)$. This gives us that $2O$ contains the following 48 unit quaternions:

$$\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}, \qquad \frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{i}, \pm 1 \pm \mathbf{j}, \pm 1 \pm \mathbf{k}, \pm\mathbf{i} \pm \mathbf{j}, \pm\mathbf{i} \pm \mathbf{k}, \pm\mathbf{j} \pm \mathbf{k}\}, \qquad \frac{1}{2}\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}. \tag{3.10}$$

The binary octahedral does not only contains quaternions representing rotations in $SO(3)$ that are symmetries of the cube and octahedron. Because at the same time, these quaternions are also vectors or vertices in 4-dimensional space. This way subsets of $2O$ describe geometric shapes in four dimensions. The standard hypercube and 4-orthoplex can already be recognized in the right 16 quaternions and left 8 quaternions respectively. These regular polytopes have rotational symmetries as well, however, since these are 4-dimensional objects, unit quaternion pairs are needed to describe their rotations in $SO(4)$. It is important to keep these different representations of unit quaternions separate from each other.

For the construction of the rotation groups for some of these 4-dimensional objects, we will need to fully understand the structure of $2O$ itself. The normal subgroups and conjugacy classes of $2O$ with give us a big part in this understanding. Let us define these concepts formally.

Definition 3.9. *Let G be a group, then a subgroup $N \subset G$ is called a normal subgroup if for every $n \in N$ it holds that for all $g \in G$*

$$gng^{-1} \in N.$$

We denote this as $N \trianglelefteq G$.

Theorem 3.10. *Let G be a group with normal subgroup $N \trianglelefteq G$, then $gN = Ng$ for $g \in G$ or in other words the left and right cosets of N coincide.*

Proof. Let $n \in N$ and $g \in G$, then

$$gn = (gng^{-1})g \in Ng,$$

since $gng^{-1} \in N$. So $gN \subseteq Ng$, but similarly $ng = g(g^{-1}ng) \in Ng$, which implies $Ng \subseteq gN$. We can conclude $gN = Ng$. [12] □

This proof also tells us that when $N \trianglelefteq G$, we can talk about just the cosets, instead of the left and right cosets.

Let a group G act on itself by conjugation, which is a special case of an action. The theory from section 3.1.3 still applies but now we say conjugacy classes instead of orbits.

Definition 3.11. *Let G be a group, then the conjugacy class of $a \in G$ is*

$$[a] := \{b \in G : \exists g \in G : a = gbg^{-1}\}.$$

Furthermore, $a, b \in G$ are called conjugate if they are in the same conjugacy class.

Elements of the same conjugacy class have the same order[2], which can be seen by

$$(gag^{-1})^n = e \iff ga^n g^{-1} = e \iff a^n = e.$$

Conjugacy classes give a nice way to prove that certain subgroups are normal. This is exactly what we will be doing, so before figuring out the conjugacy classes of $2O$, the next theorem shows us how to prove a subgroup is normal.

Theorem 3.12. *Let G be a group, then a subgroup $N \subset G$ is normal if and only if N is a union of conjugacy classes*

Proof. Let N be a normal subgroup of G and let $n \in N$. Then for every $b \in [n]$, there exists a $g \in G$ such that $b = gng^{-1}$ by definition. But since N is normal, we have that $b = gng^{-1} \in N$. Because $n \in [n]$ by $ene^{-1} = n$, this means $N = \bigcup_{n \in N}[n]$.
Conversely, let $N = \bigcup_{a \in I}[a]$ be a subgroup of G, where I is a subset of G. Then for every $n \in N$ and $g \in G$, $gng^{-1} \in [a]$ for some $a \in I$. But $[a]$ is a subset of N, so $gng^{-1} \in N$, proving N is normal. □

Remark 3.13. *Be aware that theorem 3.12 does not say any union of conjugacy classes is normal. It implies that a subset N of G is normal if and only if it is a subgroup and it is a union of conjugacy classes.*

To find the conjugacy classes of $2O$, one more trick will be used. Since all elements of $2O$ are unit quaternions, lemma 2.25 can be used. It states that the conjugation with a unit quaternion preserves the real part. So within the group $2O$, we know that elements of the same conjugacy class must have the same real part. However, this does not always work the other way around, as two elements with the same real part are not always part

Conjugacy Class Number	Elements	Order	Size	Real Part
1	1	1	1	1
2	-1	2	1	-1
3	$\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$	4	6	0
4	$\frac{1}{2}(1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})$	6	8	$\frac{1}{2}$
5	$\frac{1}{2}(-1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})$	3	8	$-\frac{1}{2}$
6	$\frac{1}{\sqrt{2}}(1\pm\mathbf{i}), \frac{1}{\sqrt{2}}(1\pm\mathbf{j}), \frac{1}{\sqrt{2}}(1\pm\mathbf{k})$	8	6	$\frac{1}{\sqrt{2}}$
7	$\frac{1}{\sqrt{2}}(-1\pm\mathbf{i}), \frac{1}{\sqrt{2}}(-1\pm\mathbf{j}), \frac{1}{\sqrt{2}}(-1\pm\mathbf{k})$	8	6	$-\frac{1}{\sqrt{2}}$
8	$\frac{1}{\sqrt{2}}(\pm\mathbf{i}\pm\mathbf{j}), \frac{1}{\sqrt{2}}(\pm\mathbf{i}\pm\mathbf{k}), \frac{1}{\sqrt{2}}(\pm\mathbf{j}\pm\mathbf{k})$	4	12	0

Table 3.2: Conjugacy classes of $2O$ with their order, number of elements and real part.

of the same conjugacy class. All conjugacy classes are written in table 3.2 and a python code finding these classes can be found in appendix A.2.

To prove that the conjugacy classes do not fall apart further in a more mathematical fashion, like they already do for elements with zero real part, which splits in the two conjugacy classes 3 and 8, we can look at the geometry of the conjugacy classes. Conjugacy classes 1 and 2 contain only one element, so it is clear they do not split further. The conjugacy classes 4 and 5 are geometrically cubes in the imaginary hyperplane at 'height' $\frac{1}{2}$ and $-\frac{1}{2}$ respectively, determined by the real part. Of course from theorem 3.8 we know conjugation with elements of $2O$ is a rotation of these cubes and thus the conjugacy classes do not split further. Similarly, conjugacy class 3 is the standard octahedron in $\operatorname{Im}\mathbb{H}$ and theorem 3.8 tells us that conjugation with $2O$ is a rotation of the octahedron as well. Conjugacy classes 6 and 7 are octahedrons as well, but at height 1 and -1 respectively and are also kept in place by conjugation. Lastly, conjugacy class 8 is a bit unique, as it contains no less than 12 elements. It corresponds to a cuboctahedron, which has the same rotation group as the cube and octahedron. This shape looks like a cube with its corner cut off and is illustrated in figure **??**. This can be seen by the fact that the 12 vertices of the cuboctahedron lie at the center of each edge of the standard cube scaled down by a factor of $\frac{1}{\sqrt{2}}$. The cuboctahedron also entirely lies in the imaginary hyperplane, with its center at the origin, so the cuboctahedron corresponding to conjugacy class 8 is kept in place by conjugation with $2O$ as well.

Lemma 3.14. *Q_8 is a normal subgroup of $2O$.*

Proof. First check that Q_8 is even a subgroup. We can see this by the fact that $Q_8 = 2O \cap \mathbb{Z}^4$ implies Q_8 is closed. Also inverses of all elements of Q_8 are in Q_8 again, as these are just conjugates, so Q_8 is indeed a subgroup. That Q_8 is normal follows from the fact that it is the union of conjugacy classes 1,2 and 3 by theorem 3.12. □

Lemma 3.15. *The set $2T$ defined by $2T := Q_8 \cup \frac{1}{2}\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$ is a normal subgroup of $2O$.*

[2] This also follows from the fact that conjugation defines an automorphism on G.

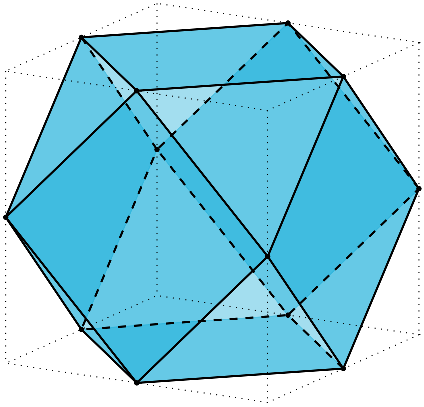

Figure 3.9: The cuboctahedron, which looks like a cube with its corners cut off, such that the remaining vertices are all of the same length, this bigger cube is dotted in the figure. The cuboctahedron has six squares and eight triangles as faces, where the last are highlighted with a slightly lower opacity. It has the same symmetry group as the cube and octahedron, which we can see with the orbit stabilizer theorem applied to one of its squares and the fact that every symmetry of the cube is one of the cuboctahedron as well.

Proof. Similarly to Q_8, $2T = 2O \cap \mathbb{Q}^4$ and therefore $2T$ is closed. It can also be seen that all inverses, which are just the conjugates, are in $2T$ again, proving $2T$ is a subgroup. It is normal since it is the union of conjugacy classes 1,2,3,4 and 5 by theorem 3.12.[3] □

Trying the combinations of unions of the conjugacy classes in table 3.2, we find the the only other none trivial normal subgroup is $\{\pm 1\}$. The trivial normal subgroups are given by $\{1\}$ and $2O$ itself.

The cosets of Q_8 in $2O$ have a very interesting structure and they are written out in table 3.3. Notably every coset is a rotated copy of Q_8 or in other words a rotated 4-orthoplex with its center at the origin. Furthermore, note that $2T$ is the union of cosets 1,5 and 6, and its other coset in $2O$ is the union of cosets 2,3 and 4, which is similarly a rotated copy of $2T$. The group $2T$, called the binary tetrahedral group $2T$, actually contains the vertices of the very special regular polytope the hyperdiamond, which will be discussed in more detail in section 3.4.

In table 3.3, call a representatives of a coset g_i, according to the number of the coset that they are in. Since Q_8 is a normal subgroup of $2O$ by lemma 3.14, the quotient group $2O/Q_8$ of the cosets of Q_8 can be constructed. $2O/Q_8$ has $[2O : Q_8] = 6$ elements, namely the six cosets, which we can represent with $\overline{g_i} := g_i Q_8$. It can be quickly checked what the inverses of $\overline{g_i}$ are, since for unit quaternion this is of course just conjugation. In the last column of table 3.3 these inverses are given and it is interesting to note that all cosets have themselves as inverse, except that $\overline{g_5}^{-1} = \overline{g_6}$ and $\overline{g_6}^{-1} = \overline{g_5}$. From this information we suspect that $2O/Q_8 \cong S_3$, since it seems to have the exact same structure. We know this for sure by the fact that $2O/Q_8$ is not abelian and S_3 is the only non-abelian group

[3] This can of course also be seen by the fact that the index of $2T$ in $2O$ is 2.

Coset number	Elements	Inverse Coset
1	$Q_8 = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$	1
2	$\frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{i}\} \cup \frac{1}{\sqrt{2}}\{\pm\mathbf{j} \pm \mathbf{k}\}$	2
3	$\frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{j}\} \cup \frac{1}{\sqrt{2}}\{\pm\mathbf{i} \pm \mathbf{k}\}$	3
4	$\frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{k}\} \cup \frac{1}{\sqrt{2}}\{\pm\mathbf{i} \pm \mathbf{j}\}$	4
5	$\frac{1}{2}\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$ with even number of minus signs	6
6	$\frac{1}{2}\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$ with odd number of minus signs	5

Table 3.3: The six cosets of Q_8 in $2O$.

with six elements. An explicit isomorphism is given by

$$\overline{g_1} \mapsto e,\ \overline{g_2} \mapsto (12),\ \overline{g_3} \mapsto (13),\ \overline{g_4} \mapsto (23),\ \overline{g_5} \mapsto (132),\ \overline{g_6} \mapsto (123).$$

3.2.2. ROTATION GROUP OF THE HYPERCUBE USING QUATERNIONS

Similarly to the cube, we want to construct the binary rotation group of the hypercube, denote this as $2O_H \subset \mathbb{H}_1 \times \mathbb{H}_1$. However, this time we do not even know the non-binary rotation group itself, denoted as $O_H \subset SO(4)$. What is also different is that we need 4-dimensional rotations now, so we will need to use theorem 2.30. This also states that 4-dimensional rotation have a two-to-one relation with unit quaternion pairs, so we are trying to find $192 \cdot 2 = 384$ unit quaternion pairs, since the total number of rotational symmetries of the hypercube is 192, given by equation 3.8. In this section we will find these 384 quaternion pairs such that $O_H = \Psi(2O_H)$.

Rotating just the imaginary hyperplane can produce rotational symmetries of the hypercube, similar to how rotating just the XY plane is a rotation of the cube for certain angles. Actually, for the cube this results in exactly the 3-dimensional extension of the four rotations of the square. Similarly for the hypercube, rotating just on of its cube faces in $\operatorname{Im}\mathbb{H}$ also gives rotations of the hypercube itself, so this way we already found $\#2O = 48$ rotations of the hypercube, namely the pairs (q, q) with $q \in 2O$. This feeds the suspicion that the binary rotation group of the hypercube $2O_H$ is very closely related with the binary octahedral group $2O$.

It turns out that $2O_H$ is a subgroup of $2O \times 2O$ with index 6, since $\#(2O \times 2O) = 48 \cdot 48 = 2304 = 384 \cdot 6$. The quaternion pairs of $2O_H$ are given by the union of the direct products of the cosets six cosets of Q_8 in $2O$. Indeed, this leaves us with $8 \cdot 8 \cdot 6 = 384$ pairs. $2O_H$ can this way be defined as

$$2O_H := \{(q_1, q_2) \in 2O \times 2O : q_1 Q_8 = q_2 Q_8\}, \tag{3.11}$$

Note that this highlights the fact that $2O_H \subset 2O \times 2O$. This definition is equivalent to

$$2O_H = \{(g_i q_1, g_i q_2) : \overline{g_i} \in 2O/Q_8,\ q_1, q_2 \in Q_8\}, \tag{3.12}$$

which is the characterization that will be used in some proofs.

Let us formally prove that indeed all elements of $2O_H$ defined this way are rotations of the hypercube.

Theorem 3.16. *The map* $\Psi_{q_1,q_2} : \mathbb{H} \to \mathbb{H}$ *by* $\Psi_{q_1,q_2}(p) = q_1 p q_2^{-1}$ *is a rotation of* Q_8 *for all* $(q_1, q_2) \in 2O_H$.

Proof. Take an arbitrary element of $2O_H$, which can be written as $(g_i q_1, g_i q_2)$, for some $q_1, q_2 \in Q_8$ and $\overline{g_i} \in 2O/Q_8$. We want to prove that $\Psi_{g_i q_1, g_i q_2}$ maps Q_8 to Q_8. We only have to prove this, because theorem 2.30 tells us that since $g_i q_1, g_i q_2 \in \mathbb{H}_1$, we at least know that $\Psi_{g_i q_1, g_i q_2}$ is a rotation of $\mathbb{H}$. Let $p \in Q_8$, then proving that Q_8 gets mapped to Q_8 is equivalent to just proving $\Psi_{g_i q_1, g_i q_2}(p) \in Q_8$, since a rotation maps points uniquely. Write

$$\Psi_{g_i q_1, g_i q_2}(p) = (g_i q_1) p (g_i q_2)^{-1} = g_i q_1 p q_2^{-1} g_i^{-1}.$$

Now since $p, q_1, q_2 \in Q_8$, $q_1 p q_2 \in Q_8$ as well. But $g_i \in 2O$, which means by theorem 3.8 that if $q_1 p q_2 \in \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$, then $g_i q_1 p q_2^{-1} g_i^{-1}$ is mapped to $\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ again. If $q_1 p q_2^{-1} \in \{\pm 1\}$, then $g_i q_1 p q_2^{-1} g_i^{-1} = g_i (\pm 1) g_i^{-1} = \pm 1$. So $\Psi_{g_i q_1, g_i q_2}(p) \in Q_8$, concluding the proof. □

As discussed, the points of Q_8 can be seen as the vertices of the 4-orthoplex in the four dimensional space $\mathbb{H}$. Since the rotation groups of the 4-orthoplex and its dual the hypercube coincide, this means that $2O_H$ only contains quaternions that represent rotations of the hypercube. Furthermore, from theorem 2.30, we know that $q_1 p q_2^{-1} = -q_1 p (-q_2)^{-1}$, and thus (q_1, q_2) and $(-q_1, -q_2)$ represent the same rotation. However, up to this sign the map $\Psi : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ maps unit quaternion pairs to unique rotations. We have seen that $\#2O_H = 384$, which is exactly twice the number of rotations of the hypercube, and together with theorem 3.16 every one of these rotations must thus be represented twice in $2O_H$. So $2O_H$ is indeed the binary rotation group of the hypercube.

Combining 4-dimensional rotations as quaternions is still a bit tricky, as we want a second rotation to "sandwich" the entire previous rotation, like

$$\Psi_{q_1,q_2}(\Psi_{r_1,r_2}(p)) = q_1 \Psi_{r_1,r_2}(p) q_2^{-1}.$$

However, the standard multiplication for an direct product already correctly combines quaternion pairs. This can be seen by the fact that for $(q_1, q_2), (r_1, r_2) \in 2O_H$ we have that $(q_1, q_2)(r_1, r_2) = (q_1 r_1, q_2 r_2)$ and this way

$$\Psi_{q_1 r_1, q_2 r_2}(p) = (q_1 r_1) p (q_2 r_2)^{-1} = q_1 r_1 p r_2^{-1} q_2^{-1} = \Psi_{q_1,q_2}(\Psi_{r_1,r_2}(p))$$

as required. Let us check if $2O_H$ really forms a group this way.

Theorem 3.17. $2O_H$ *is a group and* $2O_H / \{\pm(1,1)\} \cong O_H$.

Proof. First note that (1,1) is the identity element and the operation $2O_H$ is clearly associative. Next we prove that every element has an inverse in $2O_H$. Let $(q_1, q_2) \in 2O_H$, then $(q_1^{-1}, q_2^{-1}) \in 2O \times 2O$ is clearly the only candidate for the inverse. All that is left to show is that this inverse is an element of $2O_H$. Since q_1 and q_2 are in the same coset of Q_8 in $2O$, their inverses are in the same possible different coset as well, because $2O/Q_8$ is well-defined, so $(q_1^{-1}, q_2^{-1}) \in 2O_H$.
Finally, we show that $2O_H$ is closed under multiplication. Let $(p_1, p_2), (r_1, r_2) \in 2O_H$, then $(p_1, p_2), (r_1, r_2) = (p_1 r_1, p_2 r_2)$. Again since $2O/Q_8$ is well-defined, $p_1 r_1$ and $p_2 r_2$ are in the same coset, so $(p_1 r_1, p_2 r_2) \in 2O_H$. So $2O_H$ is a group and as discussed, its size being twice that of O_H and the two-to-one relation of Ψ proves that $2O_H / \{\pm(1,1)\} \cong O_H$. □

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It is interesting to note that the way that $2O_H$ is defined, means we can see $2O_H$ as $\bigcup_{g_i \in 2O/Q_8} \overline{g_i} \times \overline{g_i}$, which looks exactly like the six cosets of $\overline{g_1} \times \overline{g_1} = Q_8 \times Q_8$. We will make this proposition formal in the next theorem.

Theorem 3.18. *$Q_8 \times Q_8$ is a normal subgroup of $2O_H$.*

Proof. Let $(q_1, q_2) \in Q_8 \times Q_8$ and $(p_1, p_2) \in 2O_H$. We have to prove that

$$(p_1, p_2)(q_1, q_2)(p_1, p_2)^{-1} \in Q_8 \times Q_8.$$

From the proof of lemma 3.17 we know $(p_1, p_2)^{-1} = (p_1^{-1}, p_2^{-1})$, this gives

$$(p_1, p_2)(q_1, q_2)(p_1^{-1}, p_2^{-1}) = (p_1 q_1 p_1^{-1}, p_2 q_2 p_2^{-1}).$$

Since $p_1, p_2 \in 2O$ and Q_8 is a normal subgroup of $2O$ by lemma 3.14, $p_1 q_1 p_1^{-1}, p_2 q_2 p_2^{-1} \in Q_8$, concluding the proof. $\square$

For an overview of the cosets of $Q_8 \times Q_8$ in $2O_H$, table 3.4 is made. Note that $2O_H/(Q_8 \times Q_8)$ contains six elements as $\#Q_8 \times Q_8 = 8 \cdot 8 = 64 = \frac{384}{6}$, so $[2O_H : Q_8 \times Q_8] = 6$. The elements of the cosets follow directly from the construction of $2O_H$ as $\bigcup_{g_i \in 2O/Q_8} \overline{g_i} \times \overline{g_i}$ and the cosets of Q_8 in $2O$ in table 3.3.

Coset Number	Elements
1	$Q_8 \times Q_8$
2	$(\frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{i}\} \cup \frac{1}{\sqrt{2}}\{\pm \mathbf{j} \pm \mathbf{k}\}, \frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{i}\} \cup \frac{1}{\sqrt{2}}\{\pm \mathbf{j} \pm \mathbf{k}\})$
3	$(\frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{j}\} \cup \frac{1}{\sqrt{2}}\{\pm \mathbf{i} \pm \mathbf{k}\}, \frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{j}\} \cup \frac{1}{\sqrt{2}}\{\pm \mathbf{i} \pm \mathbf{k}\})$
4	$(\frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{k}\} \cup \frac{1}{\sqrt{2}}\{\pm \mathbf{i} \pm \mathbf{j}\}, \frac{1}{\sqrt{2}}\{\pm 1 \pm \mathbf{k}\} \cup \frac{1}{\sqrt{2}}\{\pm \mathbf{i} \pm \mathbf{j}\})$
5	(q_1, q_2) with $q_1, q_2 \in \frac{1}{2}\{\pm 1 \pm \mathbf{k} \pm \mathbf{i} \pm \mathbf{j}\}$ such that q_1 and q_2 have an even number of minus signs
6	(q_1, q_2) with $q_1, q_2 \in \frac{1}{2}\{\pm 1 \pm \mathbf{k} \pm \mathbf{i} \pm \mathbf{j}\}$ such that q_1 and q_2 have an odd number of minus signs

Table 3.4: Cosets of $Q_8 \times Q_8$ in $2O_H$.

The inverse cosets are very similar to table 3.3, since every coset has itself as inverse, except for coset 5 and 6, which are each others inverse. This can be quickly verified by taking conjugates of the elements of the cosets. Due to the non-abelian nature of the cosets, $2O_H/(Q_8 \times Q_8) \cong S_3$, as it is the only non-abelian group of order six. Call the cosets $\overline{h_i}$ according to their numbering, then an explicit isomorphism is given by

$$\overline{h_1} \mapsto e,\ \overline{h_2} \mapsto (12),\ \overline{h_3} \mapsto (13),\ \overline{h_4} \mapsto (23),\ \overline{h_5} \mapsto (132),\ \overline{h_6} \mapsto (123).$$

$Q_8 \times Q_8$ is geometrically the direct product of the two 4-orthoplexes described by Q_8 at the same position. The other five cosets have the same geometric structure, but each coset is the direct product of two 4-orthoplexes that are rotated copies of Q_8. Note that for each coset the two 4-orthoplexes are the same, just like $Q_8 \times Q_8$. This way the cosets in $2O_H/Q_8 \times Q_8$ can be seen as 'rotated' copies of $Q_8 \times Q_8$.

Class Number	Elements	Order	Size
1	$(1,1)$	1	1
2	$(-1,1)$	2	1
3	$(1,-1)$	2	1
4	$(-1,-1)$	2	1
5	$(1,\{\pm\mathbf{i},\pm\mathbf{j},\pm\mathbf{k}\})$	4	6
6	$(\{\pm\mathbf{i},\pm\mathbf{j},\pm\mathbf{k}\},1)$	4	6
7	$(-1,\{\pm\mathbf{i},\pm\mathbf{j},\pm\mathbf{k}\})$	4	6
8	$(\{\pm\mathbf{i},\pm\mathbf{j},\pm\mathbf{k}\},-1)$	4	6
9	$(1\pm\mathbf{i},1\pm\mathbf{i}),(1\pm\mathbf{j},1\pm\mathbf{j}),(1\pm\mathbf{k},1\pm\mathbf{k})$	8	12
10	$(1\pm\mathbf{i},-1\pm\mathbf{i}),(1\pm\mathbf{j},-1\pm\mathbf{j}),(1\pm\mathbf{k},-1\pm\mathbf{k})$	8	12
11	$(-1\pm\mathbf{i},1\pm\mathbf{i}),(-1\pm\mathbf{j},1\pm\mathbf{j}),(-1\pm\mathbf{k},1\pm\mathbf{k})$	8	12
12	$(-1\pm\mathbf{i},-1\pm\mathbf{i}),(-1\pm\mathbf{j},-1\pm\mathbf{j}),(-1\pm\mathbf{k},-1\pm\mathbf{k})$	8	12
13	$(\pm\mathbf{i},\pm\mathbf{i}),(\pm\mathbf{j},\pm\mathbf{j}),(\pm\mathbf{k},\pm\mathbf{k})$	4	12
14	$(\pm\mathbf{i},\{\pm\mathbf{j},\pm\mathbf{k}\}),(\pm\mathbf{j},\{\pm\mathbf{i},\pm\mathbf{k}\}),(\pm\mathbf{k},\{\pm\mathbf{i},\pm\mathbf{j}\})$	4	24
15	$(\pm\mathbf{i}\pm\mathbf{j},\pm\mathbf{i}\pm\mathbf{j}),(\pm\mathbf{i}\pm\mathbf{k},\pm\mathbf{i}\pm\mathbf{k}),(\pm\mathbf{j}\pm\mathbf{k},\pm\mathbf{j}\pm\mathbf{k})$	4	48
16	$(1\pm\mathbf{i},\pm\mathbf{j}\pm\mathbf{k}),(1\pm\mathbf{j},\pm\mathbf{i}\pm\mathbf{k}),(1\pm\mathbf{k},\pm\mathbf{i}\pm\mathbf{j})$	8	24
17	$(-1\pm\mathbf{i},\pm\mathbf{j}\pm\mathbf{k}),(-1\pm\mathbf{j},\pm\mathbf{i}\pm\mathbf{k}),(-1\pm\mathbf{k},\pm\mathbf{i}\pm\mathbf{j})$	8	24
18	$(\pm\mathbf{j}\pm\mathbf{k},1\pm\mathbf{i}),(\pm\mathbf{i}\pm\mathbf{k},1\pm\mathbf{j}),(\pm\mathbf{i}\pm\mathbf{j},1\pm\mathbf{k})$	8	24
19	$(\pm\mathbf{j}\pm\mathbf{k},-1\pm\mathbf{i}),(\pm\mathbf{i}\pm\mathbf{k},-1\pm\mathbf{j}),(\pm\mathbf{i}\pm\mathbf{j},-1\pm\mathbf{k})$	8	24
20	$(1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k},1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})$ such that both elements have the same number of minus signs	6	32
21	$(1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k},-1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})$ such that both elements have the same number of minus signs	6	32
22	$(-1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k},1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})$ such that both elements have the same number of minus signs	6	32
23	$(-1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k},-1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k})$ such that both elements have the same number of minus signs	3	32

Table 3.5: The conjugacy classes of $2O_H$. The quaternions are not all normalized for the sake of clarity, but in reality they of course should be. See A.2.1 for code providing these.

To possibly discover more normal subgroup and for the general structure of $2O_H$, its conjugacy classes are given in table 3.5.

It is interesting to note that $2O\times 2O$ is exactly six times the size of $2O_H$ and of course $2O_H\subset 2O\times 2O$. This look similar to the normal subgroups Q_8 and $Q_8\times Q_8$ in $2O$ and $2O_H$ respectively, so perhaps this case is similar. Wether $2O_H$ is normal in $2O\times 2O$ or not, can be proved or disproved if we know the conjugacy classes of $2O\times 2O$. Because then it can be seen if $2O_H$ is a union of certain conjugacy classes, using theorem 3.12.

The conjugacy classes can be constructed relatively simply from the conjugacy classes of $2O$ in table 3.2. Lemma 3.19 shows us how.

Lemma 3.19. *If $[a]$ and $[b]$ are two conjugacy classes of G, then $[a] \times [b]$ is a conjugacy class of $G \times G$. Moreover, every conjugacy class of $G \times G$ is of this form.*

Proof. Let $[a], [b]$ be two conjugacy classes of G. Let $(h_1, h_2) \in [a] \times [b]$ and $(g_1, g_2) \in G \times G$. Then

$$(g_1, g_2)(h_1, h_2)(g_1^{-1}, g_2^{-1}) = (g_1 h_1 g_2^{-1}, g_2 h_2 g_2^{-1}) \in [a] \times [b],$$

so $[a] \times [b]$ is a conjugacy class of $G \times G$.
Denote the number of conjugacy classes in G as m. Since this holds for every conjugacy class, we can construct $m \cdot m$ conjugacy classes of $G \times G$. But since every element of G is in only one conjugacy class, every element of $G \times G$ is contained in of the $m \cdot m$ combined conjugacy classes and thus we have already constructed every conjugacy class of G. □

Lemma 3.19 tells us that $2O \times 2O$ has $8 \cdot 8 = 64$ conjugacy classes of the from $[a] \times [b] \subset 2O \times 2O$ with every combination of conjugacy classes $[a], [b] \subset 2O$. Sadly, this way we can see that $2O_H$ is not a union of conjugacy classes and thus not a normal subgroup. The counter example goes as follows.

Take the element $(\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}, \frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})) \in 2O_H \subset 2O \times 2O$, which indeed lies in $2O_H$ as both elements of the pair obviously lie in the same coset in $2O/Q_8$. Now there can only be one direct product of two conjugacy classes of $2O$ which contains this element and indeed, this has to be the direct product of conjugacy class 4 with itself, see table 3.2. However, this direct product also contains the element $(\frac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k}), \frac{1}{2}(1+\mathbf{i}+\mathbf{j}-\mathbf{k}) \notin 2O_H$, since the two elements of the pair are not in the same coset of $2O$ as $\frac{1}{2}(1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$ has an even number of minus signs, while $\frac{1}{2}(1+\mathbf{i}+\mathbf{j}-\mathbf{k})$ has an odd number of minus signs. It can be concluded by theorem 3.12 that $2O_H$ is not a union of conjugacy classes in $2O \times 2O$ and thus not normal.

3.3. EXACT SEQUENCE STRUCTURES OF THE CUBE AND THE HYPERCUBE

WE would like to know wether the rotation group of the hypercube $2O_H$ is direct or semidirect product of other groups and what structure certain normal groups of $2O_H$ and their quotient group have. For this we need the theory of exact splitting sequences.

3.3.1. EXACT SPLITTING SEQUENCES

Definition 3.20. *Let $G_0, G_1, \ldots, G_n$ be groups and let $g_1, g_2, \ldots, g_n$ be homomorphisms such that g_i is a homomorphism between G_{i-1} and G_i. Then the sequence*

$$G_0 \xrightarrow{g_1} G_1 \xrightarrow{g_2} \ldots \xrightarrow{g_n} G_n \tag{3.13}$$

is called an exact sequence if the image of every homomorphism is the kernel of the next homomorphism, that is $g_i(G_{i-1}) = \ker(g_{i+1})$.

Specifically we are interested in short exact sequences, which have the following form,

$$1 \to G_0 \xrightarrow{g} G_1 \xrightarrow{p} G_2 \to 1, \tag{3.14}$$

where 1 denotes the group with just the identity element. Therefore the first and last homomorphisms in the sequence are just the trivial homomorphisms, which are the only possible homomorphisms. For the sequence in equation 3.14 to be exact we need that the kernel of g is equal to the image of the leftmost map, which is just the identity in G_0. This then implies that g is injective. Furthermore, the image of the homomorphism p must be equal to the kernel of the rightmost homomorphism, but this just maps all of G_2 to the identity, implying that p must be surjective. This means that the definition for short sequences to be exact, implies that g and p are injective and surjective respectively. From now on short sequences will be denoted as

$$K \overset{g}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} Q, \tag{3.15}$$

where the leftmost and rightmost homomorphisms and groups are left out and the hooked arrow denotes injectivity, while the double headed arrow denotes surjectivity. If we see the image of K as an embedding of the group K as a subgroup in G, then we can see K itself as the kernel of p, this is why G_0 is written as K.

This is an important observation, because this means $G/K \cong Q$ by the first isomorphism theorem and is why we write G_2 as Q. Furthermore, this isomorphism is given by the p induced natural homomorphism $\overline{p} : G/K \to Q$. All of this is illustrated in figure 3.10

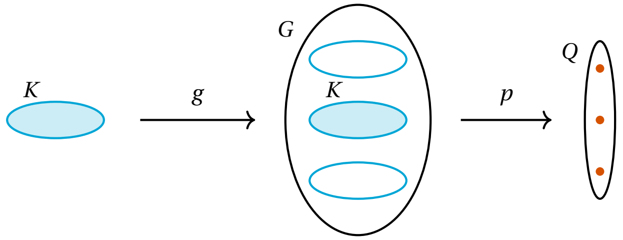

Figure 3.10: A visualization of a short exact sequence, where the homomorphism g is seen as an embedding of K in G, K and its coset are represented by the blue ovals contained in G. Furthermore, p maps the elements of a coset K to the same element in Q, these elements are represented as the orange points.

Unfortunately, in general it is not true that G is isomorphic to the direct product of K and Q. There is, however, a more powerful version of the short exact sequence, namely the splitting short exact sequence. For these kind of sequences there exists a semidirect product for which G is isomorphic to this semi direct product of K and Q. But before giving the conditions for a sequence to split, first the semidirect product is introduced. For this we need the automorphism group of a group. This is the group containing all automorphism from a group G to itself and is a subgroup of $S(G)$, denote it as $\mathrm{Aut}(G)$.

Definition 3.21. *Let K and Q be groups and let $\tau : Q \to Aut(K)$ be a homomorphism. The semidirect product of K and Q with respect to τ is the group $K \times Q$ with the operation*

$$(k_1, q_1)(k_2, q_2) = (k_1\tau(q_1)(k_2), q_1 q_2). \tag{3.16}$$

The notation for this is $K \rtimes_\tau Q$. [12]

Definition 3.22. *A short exact sequence*

$$K \overset{g}{\hookrightarrow} G \overset{p}{\twoheadrightarrow} Q,$$

is called split if there exists a homomorphism $f : Q \to G$ such that $p \circ f = id_Q$.

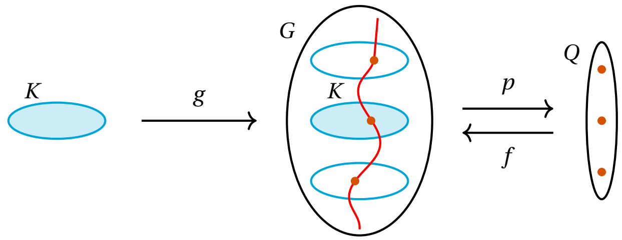

Figure 3.11: A visualization of a split short exact sequence. The homomorphism f exactly maps to the well-chosen points of in G that form a isomorphic subgroup to Q This is represented by the orange line intersecting every coset in a single point, represented by the orange points.

We can interpret the definition of split in the following way. The difficulty in finding f is not so much the fact that $p \circ f = \mathrm{id}_Q$, but that it must be a homomorphism. Because for f to be homomorphism, we need to be able to find one element in each coset of K, such that together they form a group isomorphic to Q. This is hard to find and definitely not true for general short exact sequences, but in turn we get the powerful splitting lemma. All of this is illustrated in figure 3.11, which is an extension of figure 3.10.

Lemma 3.23. *(Splitting lemma) If a short sequence is split, then there exists a homomorphism $\tau : Q \to Aut(K)$ such that $G \cong K \rtimes_\tau Q$.*

Proof. First we will define a homomorphism $\tau : Q \to \mathrm{Aut}(K)$. Since the short sequence is split there exists a homomorphism $f : Q \to G$ with $p \circ f = \mathrm{id}_Q$. Then for $q \in Q$ and $k \in K$ we know that

$$f(q)g(k)f(q^{-1}) \in \ker(p),$$

since indeed

$$p(f(q)g(k)f(q^{-1})) = p(f(q))p(g(k))p(f(q^{-1})) = qe_Q q^{-1} = e_Q,$$

where it is used that $g(k) \in \ker(p)$ by definition. More generally $g(K) = \ker(p)$, so for some $k' \in K$

$$f(q)g(k)f(q^{-1}) = g(k').$$

Note that k' is unique since g is injective and is determined by the choices of q and k. Now define $\tau_q : K \to K$ such that $\tau_q(k) = k'$ or in other words such that

$$f(q)g(k)f(q)^{-1} = g(\tau_q(k)), \tag{3.17}$$

where it is used that $f(q^{-1}) = f(q)^{-1}$. Define $\tau : Q \to \mathrm{Aut}(K)$ with $\tau(q) := \tau_q$, we claim this is well-defined, that is $\tau_q \in \mathrm{Aut}(K)$. Furthermore, we claim that τ is a homomorphism. First we check if τ_q is even a homomorphism for $q \in Q$. Let $k_1, k_2 \in K$, then from equation 3.17 it follows that

$$f(q)g(k_1k_2)f(q)^{-1} = g(\tau_q(k_1k_2)).$$

The left side is equal to

$$\begin{aligned} f(q)g(k_1)g(k_2)f(q)^{-1} &= f(q)g(k_1)f(q)^{-1}f(q)g(k_2)f(q)^{-1} \\ &= g(\tau_q(k_1))g(t_q(k_2)) \\ &= g(\tau_q(k_1)\tau_q(k_2)). \end{aligned}$$

It follows that $\tau_q(k_1k_2) = \tau_q(k_1)\tau_q(k_2)$ since g is injective and thus τ_q is a homomorphism. Next, look at the case $q = e_Q$, then for $k \in K$ equation 3.17 reads

$$f(e_Q)g(k)f(e_Q)^{-1} = g(k) = g(\tau_{e_Q}(k)).$$

So $\tau_{e_Q} = \mathrm{id}_K \in \mathrm{Aut}(K)$. Next we prove that τ is a homomorphism. Let $q_1, q_2 \in Q$ and $k \in K$, then equation 3.17 gives us

$$f(q_1q_2)g(k)f(q_1q_2)^{-1} = g(\tau_{q_1q_2}(k)).$$

The left side is equal to

$$f(q_1)f(q_2)g(k)f(q_2)^{-1}f(q_1)^{-1} = f(q_1)g(\tau_{q_2}(k))f(q_1)^{-1} = g(\tau_{q_1}(\tau_{q_2}(k))).$$

Again by injectivity of g, we get that $\tau_{q_1} \circ \tau_{q_2} = \tau_{q_1q_2}$, so τ is indeed a homomorphism. Furthermore, in particular we have that for $q \in Q$, it holds that $\tau_q \circ \tau_{q^{-1}} = \tau_{q^{-1}} \circ \tau_q = \tau_{e_Q}$, which implies that τ_q is invertible and is thus bijective. Together with the fact that τ_q is a homomorphism we can conclude $\tau_q \in \mathrm{Aut}(K)$. To summarize, we have found a well-defined homomorphism $\tau : Q \to \mathrm{Aut}(K)$.
What is left to show is that there actually exists a isomorphism between $K \rtimes_\tau Q$ and G. Define $j : K \rtimes_\tau Q \to G$ with $j(k,q) = g(k)f(q)$. First we show this is a homomorphism. Let $k_1, k_2 \in K$ and $q_1, q_2 \in Q$, then

$$\begin{aligned} j((k_1,q_1)(k_2,q_2)) &= j((k_1\tau(q_1)(k_2), q_1q_2)) = g(k_1\tau_{q_1}(k_2))f(q_1q_2) \\ &= g(k_1)g(\tau_{q_1}(k_2))f(q_1)f(q_2) = g(k_1)f(q_1)g(k_2)f(q_1)^{-1}f(q_1)f(q_2) \\ &= g(k_1)f(q_1)g(k_2)f(q_2) = j((k_1,q_1))j((k_2,q_2)). \end{aligned}$$

That j is injective follow from

$$j(k,q) = e_G \implies g(k)f(q) = e_G \implies p(g(k)f(q)) = p(e_G) \implies e_Q q = e_Q \implies q = e_Q,$$

since $g(k) \in \ker(p)$. Now using $q = e_Q$, we get that $g(k) = e_G$, which can only hold if $k = e_K$, since g is injective. Lastly, we have to prove that j is surjective. Let $h \in G$, then we have to find $k \in K$ and $q \in Q$ such that

$$\begin{aligned} j(k,q) = h \implies g(k)f(q) = h &\implies p(g(k)f(q) = p(h) \\ &\implies p(g(k))p(f(q)) = p(h) \implies q = p(h). \end{aligned}$$

Set $q := p(h)$, we now need to find $k \in K$ such that $g(k)f(p(h)) = h$ or equivalently $g(k) = hf(p(h))^{-1}$. Using that $g(K) = \ker(p)$, we can argue that for such a k to exist we must have that $hf(p(h))^{-1} \in \ker(p)$ as well. Conversely, if $hf(p(h))^{-1} \in \ker(p)$, then there must be some $k \in K$ such that $g(k) = hf(p(h))^{-1}$. So all that we have to check is wether $hf(p(h))^{-1} \in \ker(p)$ or not.

$$p(hf(p(h))^{-1}) = p(hf(p(h)^{-1})) = p(h)p(f(p(h)^{-1})) = p(h)p(h)^{-1} = e_Q.$$

So $h = g(k)f(p(h))$ with $p(h) \in Q$ and for some $k \in K$, proving j is surjective and thus j is an isomorphism $K \rtimes_\tau Q \to G$, concluding the proof. [5] □

3.3.2. SEQUENCE STRUCTURE OF THE BINARY OCTAHEDRAL GROUP

In this section we look at the split short exact sequence

$$V_4 \overset{g}{\hookrightarrow} S_4 \overset{p}{\twoheadrightarrow} S_3, \tag{3.18}$$

which corresponds to the cube and its normal subgroup the Klein four-group V_4. This is a very visual example, but most importantly it might gives us a way to expand this relatively simple case to the short exact sequences

$$\begin{aligned} Q_8 &\hookrightarrow 2O \twoheadrightarrow S_3, \\ Q_8 \times Q_8 &\hookrightarrow 2O_H \twoheadrightarrow S_3, \end{aligned}$$

since $2O$ is the double cover of $O \cong S_4$ and $2O/Q_8 \cong S_3$, just like $2O_H/(Q_8 \times Q_8) \cong S_3$, as we have seen in section 3.2.1 and 3.2.2.

First check if the short sequence in equation 3.18 is exact in the first place, let alone split. For this purpose, g and p should be defined. The most obvious choice for g, is to map V_4 to $\{e, (12)(34), (13)(24), (14)(23)\} \subset S_4$, which is clearly injective, call this subgroup of S_4 also V_4 for simplicity. Then it must hold that $\ker(p) = V_4$. Intuitively, from figure 3.11, we have seen that for the sequence to split, the cosets of V_4 should be mapped to S_4/V_4 such that we can define a map backwards to well chosen representatives. The cosets of V_4 in S_4 are written down in table 3.6.

Representative	Coset
e	$e, (12)(34), (13)(24), (14)(23)$
(12)	(12), (34), (1423), (1324)
(13)	(13), (1432), (24), (1234)
(23)	(23), (1243), (1324), (14)
(123)	(123), (243), (142), (134)
(132)	(132), (143), (234), (124)

Table 3.6: Cosets of V_4 in S_4.

Notice how the representative are chosen as exactly the six elements of S_3 (ignoring the (4) that is not written down). This means that writing S_4/V_4 as S_3 is justified. This could have also been seen by the fact that the cosets in S_4/V_4 do note commute and $[S_4 : V_4] = 6$ and S_3 is the only non-abelian group of six elements.

Define $p : S_4 \to S_3$ such that every element in S_4 is mapped to the element in S_3 associated with the coset representative of its coset. Then clearly p is a surjective homomorphism. This can be done visually as S_4 is the rotation group of the cube and S_4 acting on the cube can be interpreted as a permutation of its four diagonals, explored in section 3.1.2, also see figure 3.12.

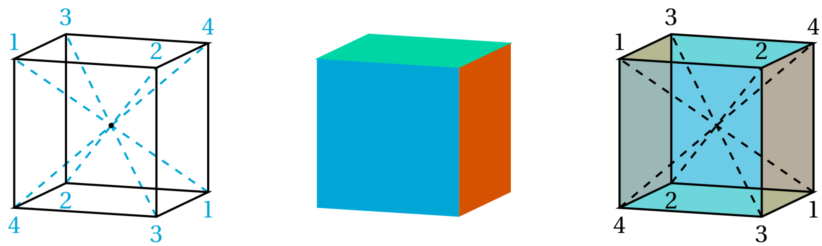

Figure 3.12: On the left, the cube with its four numbered diagonal. In the middle, the colouring of the cubes faces with the colours blue, orange and green, denoted b, o and g. The three unseen faces have the same colour as their opposite face. On the right the other two figures are combined

Similarly, S_3 acting on the cube can be seen in the following way. Colour the opposite faces of the cube the same colour, say blue, red and yellow as in figure 3.12 and let S_3 permutes these colours.

Now it can be checked which rotation of the cube corresponds to which permutation of the diagonals and to which permutation of the colours. How the diagonals are permuted is clear for each element in S_4. So let us look at the colour permutation in detail.

It can be shown that the four rotations corresponding to the elements of one coset of V_4 in S_4, permute the colours of the faces in the same way. For example, take the rotations corresponding to (13) and (1234) in S_4. To switch diagonals 1 and 3, we have to rotate around the axis through the middles of the two edges connecting the diagonals 1 and 3, see figure 3.13.

For the second permutation (1234), it can be seen that a rotation around the axis through the middles of the blue faces by 90 degrees suffices. For this rotation it can be easily seen that the blue sides are kept in places, while the green and orange sides switch places. For the first rotation (13), this might be a bit harder to imagine. Here the two blue sides change places, but this still means the same opposite sides stay blue. Meanwhile green and orange are switched. So indeed $(13), (1234) \in S_4$ permute the colours in the same way. This can be written as $(go)(b) \in S(\{b, g, o\})$, where $S(\{b, g, o\})$ is of course isomorphic to S_3.

So far the example, let us look at if the sequence is split. Define $f : S_3 \to S_4$ by mapping each element to the same permutation keeping 4 in place, so (12)(3) is mapped to (12)(3)(4). Then indeed $p \circ f(\sigma) = \mathrm{id}_{S_3}$ and f is a homomorphism, thus the sequence in equation 3.14 is indeed split. As discussed, we see that a sequence being split comes down to wether or not we have the ability to choose 'nice' representatives which form the quotient group together and in this case S_4 has a subgroup isomorphic to S_3 with one element in each coset of V_4.

Figure 3.13: To permute the diagonals 1 and 3 of the cube numbered as in figure 3.12, the cube has to be rotated 180 degrees around the axis through the middles of the edges connect diagonals 1 and 3. This axis is shown on the left. To permute the diagonals according to (1234), the cube has to be rotated 90 degrees around the axis through the centers of the blue faces in figure 3.12. This axis is shown on the right

We can now actually explicitly work out what colour in $S(\{b,o,g\})$ the numbers 1,2 and 3 in S_3 are linked to. To do this, first take $(12)(3) \in S_3$, so $f((12)(3)) = (12)(3)(4)$. This is, similarly to (13), the rotation around the axis through the midpoints of the edges connecting diagonals 1 and 2 over 180 degrees. In this rotation blue and green switch places, but orange stays in place, so we can conclude that 3 must represent red. Furthermore, taking $(13)(2) \in S_3$, of which we have already seen the corresponding rotation and we know it switches green and orange, but keeps blue in place. So 2 must represent green, leaving blue to be represented by 1.

Now the structure of the group S_4 can be described as $S_4 \cong V_4 \rtimes_\tau S_3$ for some homomorphism $\tau : S_3 \to \mathrm{Aut}(V_4)$, by lemma 3.23. From the proof the proof of the splitting lemma, it can be seen that τ is the homomorphism that for $q \in S_3$ maps $\tau(q) \mapsto \tau_q$, where $\tau_q(k)$ satisfies

$$f(q)g(k)f(q)^{-1} = g(\tau_q(k)). \tag{3.19}$$

For every $q \in S_3$ a table can be made to see what automorphism τ_q describes, this is done in table 3.7. It is interesting to note that τ is an isomorphism, since coincidentally $\mathrm{Aut}(V_4) \cong S_3$ and from the table it follows that τ is bijective, although this is definitely not true in general split exact short sequences.

	$\tau_{e_{S_3}}$	$\tau_{(12)}$	$\tau_{(13)}$	$\tau_{(23)}$	$\tau_{(123)}$	$\tau_{(132)}$
e_{V_4}	e_{V_4}	e_{V_4}	e_{V_4}	e_{V_4}	e_{V_4}	e_{V_4}
(12)(34)	(12)(34)	(12)(34)	(14)(23)	(13)(24)	(14)(23)	(13)(24)
(13)(24)	(13)(24)	(14)(23)	(13)(24)	(12)(34)	(12)(34)	(14)(23)
(14)(23)	(14)(23)	(13)(24)	(12)(34)	(14)(23)	(13)(24)	(12)(34)

Table 3.7: Mappings of the automorphism $\tau_q : V_4 \to V_4$ for $q \in S_3$ horizontally, permuting $k \in V_4$ vertically.

3.3.3. SEQUENCE STRUCTURE OF THE BINARY ROTATION GROUP OF THE HYPERCUBE

In the previous section we have seen that

$$V_4 \overset{g}{\hookrightarrow} O \overset{p}{\twoheadrightarrow} S_3, \tag{3.20}$$

is a split exact short sequence, since $O \cong S_4$. It was also shown that this implies that $O \cong V_4 \rtimes_\tau S_3$, for the homomorphism $\tau : S_3 \to \mathrm{Aut}(V_4)$, which is given explicitly at the end of the previous section. We are hoping that this gives us a way to extend this to describe the structures of $2O$ and $2O_H$. We have already seen that the two short sequences

$$Q_8 \hookrightarrow 2O \twoheadrightarrow S_3, \tag{3.21}$$

$$Q_8 \times Q_8 \hookrightarrow 2O_H \twoheadrightarrow S_3, \tag{3.22}$$

are exact, by normality of Q_8 and $Q_8 \times Q_8$ and the structure of the cosets of both. However, disappointingly there is an argument that these sequences are not split. For this, we use the almost trivial fact that -1 is the only element of $\mathbb{H}_1$ that has order 2. All other elements, besides the identity, have at least order 3. With this information, try to find an homomorphism $f : S_3 \to 2O$. Now for any homomorphism $\hat{f} : G_1 \to G_2$, it holds that for $g \in G_1$, $\hat{f}(g)$ has an order that divides the order of g. This follows from the equality

$$g^m = e_{G_1} \iff \hat{f}(g^m) = e_{G_2} \iff \hat{f}(g)^m = e_{G_2}.$$

In our situation this means that the three elements of order 2, $(12), (13), (23) \in S_3$, have to be mapped to some element in $2O$, which has an order dividing 2. But this can only be ± 1 and thus f can not be injective, meaning we can not satisfy $p \circ f = \mathrm{id}_{S_3}$.

The only elements in $2O_H$ that have order 2 or lower are the four elements $(\pm 1, \pm 1)$, but these are all in the same coset of $Q_8 \times Q_8$, namely in $Q_8 \times Q_8$ itself. Since $p : 2O_H \twoheadrightarrow S_3$ maps every coset to one element as discussed, we can not find a homomorphism $f : S_3 \to 2O_H$ satisfying $p \circ f = \mathrm{id}_{S_3}$ for this sequence either.

It seems like we can not find a 'nice' element for every coset, as in figure 3.11, creating an isomorphic subgroup to S_3 of neither $2O$ nor $2O_H$. Actually, $2O$ and $2O_H$ both do not have any subgroup isomorphic to S_3.

3.4. ROTATION GROUP OF THE HYPERDIAMOND

THE hyperdiamond or 24-cell, is one of the six regular polytopes in four dimensions and is probably the most unique regular polytope, as its type only exist in four dimensions and it is thus the only one of its type. The hyperdiamond has 24 octahedrons as hyperfaces, 96 triangle as faces, 96 edges and 24 vertices [19]. Its vertices in $\mathbb{H}$ are given by the set

$$2T := \frac{1}{2}\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\} \cup Q_8, \tag{3.23}$$

which we have already shortly seen in lemma 3.15, where it was proved that $2T$ is a normal subgroup of $2O$ with index 2. So while $2T$ represents some rotations of the cube, it also contains all vertices of the hyperdiamond at the same time. For the rotation group of the hyperdiamond, the notation $O_D \subset SO(4)$ will be used, but since we are working with quaternions, we will mostly be look at the binary rotation group of the hyperdiamond, $2O_D \subset \mathbb{H}_1 \times \mathbb{H}_1$. So we want to find unit quaternion pairs such that $O_D = \Psi(2O_D)$.
$2T$ is actually often called the binary tetrahedral group, as it is a double cover of the rotation group of the tetrahedron. This is where it gets its 'T' from, but we will not go into more detail about this in this thesis.

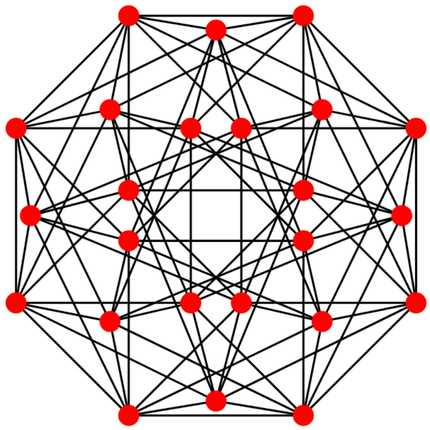

Figure 3.14: The hyperdiamond as a graph.[19]

$2T$ can be split in a standard hypercube and 4-orthoplex. This directly follows from the cosets of Q_8 in $2T$, see table 3.8. Furthermore, we know that the vertices of the hypercube also describe two 4-orthoplexes, as discussed in section 3.1.1. So $2T$ also describes three 4-orthoplexes. That $Q_8 \trianglelefteq 2T$ follows straight from its normality in $2O$.

Coset Number	Elements
1	Q_8
2	$\frac{1}{2}\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$ with even number of + signs
3	$\frac{1}{2}\{\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$ with odd number of + signs

Table 3.8: Cosets of Q_8 in $2T$.

It is interesting to note that the short sequence

$$Q_8 \overset{g}{\hookrightarrow} 2T \overset{p}{\twoheadrightarrow} C_3 \tag{3.24}$$

is exact and it splits. For $g : Q_8 \to 2T$, just choose the obvious embedding. If we write $C_3 = \{1, c, c^2\}$ choose $p : 2T \to C_3$ such that

$$p(\overline{g}_1) = 1,\ p(\overline{g}_2) = c,\ p(\overline{g}_3) = c^2.$$

Then indeed $\ker(p) = g(Q_8)$ and p is surjective. To prove that the short sequence splits, define $f : C_3 \to 2T$ by

$$f(1) = 1,\ f(c) = \frac{1}{2}(-1 + \mathbf{i} + \mathbf{j} + \mathbf{k}),\ f(c^2) = \frac{1}{2}(-1 - \mathbf{i} - \mathbf{j} - \mathbf{k}). \tag{3.25}$$

Then this is indeed a homomorphism, since

$$\left(\frac{1}{2}(-1+\mathbf{i}+\mathbf{j}+\mathbf{k})\right)^2 = \frac{1}{2}(-1-\mathbf{i}-\mathbf{j}-\mathbf{k})$$
$$\left(\frac{1}{2}(-1+\mathbf{i}+\mathbf{j}+\mathbf{k})\right)^3 = 1.$$

Furthermore, $p(f(c)) = c$, which is enough to prove $p \circ f = \mathrm{id}_{C_3}$. So the sequence is split and $2T \cong Q_8 \rtimes_\tau C_3$ for some $\tau : C_3 \to \mathrm{Aut}(Q_8)$. Note that the choice for f is not unique.

Denote a general element of C_3 as c^m, then from the proof of lemma 3.23 we know that $\tau : C_3 \to \mathrm{Aut}(Q_8)$ is given by $\tau(c^m) = \tau_{c^m}$, where for $q \in Q_8$ we have that $\tau_{c^m}(q)$ satisfies

$$f(c^m)g(q)f(c^m)^{-1} = g(\tau(q)).$$

We can make a table to see exactly which automorphisms on Q_8 $\tau_1, \tau_c, \tau_{c^2}$ describe, see table 3.9.

	τ_1	τ_c	τ_{c^2}
1	1	1	1
-1	-1	-1	-1
$\mathbf{i}$	$\mathbf{i}$	$\mathbf{k}$	$\mathbf{j}$
$-\mathbf{i}$	$-\mathbf{i}$	$-\mathbf{k}$	$-\mathbf{j}$
$\mathbf{j}$	$\mathbf{j}$	$\mathbf{i}$	$\mathbf{k}$
$-\mathbf{j}$	$-\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{k}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$\mathbf{i}$
$-\mathbf{k}$	$-\mathbf{k}$	$-\mathbf{j}$	$-\mathbf{i}$

Table 3.9: Mappings of the automorphism $\tau_{c^m} : C_3 \to \mathrm{Aut}(Q_8)$ for $c^m \in C_3$ horizontally, permuting $q \in Q_8$ vertically.

The two cosets of $2T$ in $2O$ are written out in table 3.10. The first coset is just $2T$ itself, so a hyperdiamond by definition, but the second coset is actually a rotated copy of $2T$ and thus also a hyperdiamond. It is clear that $2O/2T \cong C_2$, since it has only two elements.

Coset Number	Elements
1	$\frac{1}{2}(\pm1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k}) \cup \{\pm1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$
2	$\frac{1}{\sqrt{2}}\{\pm1\pm\mathbf{i}\} \cup \frac{1}{\sqrt{2}}\{\pm1\pm\mathbf{j}\} \cup \frac{1}{\sqrt{2}}\{\pm1\pm\mathbf{k}\}$ $\cup \frac{1}{\sqrt{2}}\{\pm\mathbf{i}\pm\mathbf{j}\} \cup \frac{1}{\sqrt{2}}\{\pm\mathbf{i}\pm\mathbf{k}\} \cup \frac{1}{\sqrt{2}}\{\pm\mathbf{j}\pm\mathbf{k}\}$

Table 3.10: The coset of $2T$ in $2O$

The short exact sequence

$$2T \hookrightarrow 2O \twoheadrightarrow C_2 \tag{3.26}$$

is not split for a similar reason as why the short sequences in equations 3.21 and 3.22 are not split. Namely, $-1 \in 2O$ is the only element with order 2, which is even true for all of

3

$\mathbb{H}_1$, and $-1 \in 2T$ itself already.

Let us look at the rotational symmetries of $2T$. Again, we are looking for pairs keeping the vertex set of $2T$ in place through the map Ψ. Since $2T$ is a group, all pairs in $(q_1, q_2) \in 2T \times 2T$ exactly keep $2T$ in place as rotations, we have explicitly $q_1 2T q_2^{-1} \in 2T$. This gives us already $24 \cdot 24 = 576$ pairs and $576/2 = 288$ rotations. The size of O_D can be calculated with the orbit stabilizer theorem. Take one of the octahedron hyperfaces of the hyperdiamond, then equation 3.8 tells us that the rotation group of the octahedron has 24 elements, giving the size of its stabilizer. Now an octahedron hyperface can be rotated to any other octahedron hyperface, meaning that the orbit is just all 24 octahedrons [19]. Then the orbit stabilizer theorem gives us $24 \cdot 24 = 576$ rotations. This is exactly the number of unit quaternion pairs we already found with $2T \times 2T$, but of course there are 2 unit quaternion pairs describing each rotation. So we need exactly twice as many unit quaternion pairs to construct all of $2O_D$. An obvious candidate for the remaining quaternions, is the direct product of the other coset of $2T$, which also has 576 elements. Note that this is very similar to how we constructed $2O_H$ from the cosets of Q_8 in $2O$. Let g_1 and g_2 denote the cosets of $2T$ in $2O$, then we can define

$$2O_D := \{(q_1, q_2) \in 2O \times 2O : q_1 2T = q_2 2T\}. \tag{3.27}$$

Since we only have two cosets we can write $2O_D = (2T \times 2T) \cup ((2O \setminus 2T) \times (2O \setminus 2T))$.

Theorem 3.24. *The map $\Psi_{q_1,q_2} : \mathbb{H} \to \mathbb{H}$ by $\Psi_{q_1,q_2}(p) = q_1 p q_2^{-1}$ is a rotation of $2T$ for all $(q_1, q_2) \in 2O_D$.*

Proof. We already showed this for $2T \times 2T$, so all that is left is to show $(2O\setminus 2T) \times (2O\setminus 2T)$ are rotations of $2T$ as well. Let $(q_1, q_2) \in (2O \setminus 2T) \times (2O \setminus 2T)$ and $p \in 2T$, then $\Psi_{q_1,q_2} = q_1 p q_2^{-1}$. Now $q_1 p \in 2O\setminus 2T$ and $q^{-1} \in 2O\setminus 2T$, as $2O/2T \cong C_2$, meaning the second coset is its own inverse. For the same reason this means $(q_1 p) q_2^{-1} \in 2T$ and since rotations are bijections, this means all of $2T$ is mapped to itself. ☐

For similar arguments as for $2O_H$ in theorem 3.17, $2O_D$ really is a group with the operation defined by the direct product. Furthermore, since it contains twice the number of elements as O_D and $\Psi : \mathbb{H}_1 \to SO(4)$ is unique up to a sign, we can conclude

$$O_D \cong 2O_D/\{\pm(1,1)\}. \tag{3.28}$$

This raises the question if this kind of structure works for the other normal subgroups of $2O$ as well, since it already works for Q_8 and $2T$. This is explored in the next section 3.5.

$2O_D$ has 48 conjugacy classes given by the code in appendix A.2.1.

Lastly for this section it will be shown that $2O_H$ is not normal in $2O_D$, which would be a reasonable hypothesis since $2O_H$ is a subgroup with index $[2O_D : 2O_H] = 3$. Take $\left(\frac{1}{\sqrt{2}}(1-\mathbf{j}), \frac{1}{\sqrt{2}}(-1-\mathbf{j})\right) \in 2O_H$ and $\left(\frac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k}), -\mathbf{j}\right) \in 2O_D$, then

$$\left(\frac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k}), -\mathbf{j}\right)^{-1} = \left(\frac{1}{2}(1-\mathbf{i}-\mathbf{j}-\mathbf{k}), \mathbf{j}\right).$$

Next calculate the conjugation

$$\left(\tfrac{1}{2}(1+\mathbf{i}+\mathbf{j}+\mathbf{k}),-\mathbf{j}\right)\left(\tfrac{1}{\sqrt{2}}(1-\mathbf{j}),\tfrac{1}{\sqrt{2}}(-1-\mathbf{j})\right)\left(\tfrac{1}{2}(1-\mathbf{i}-\mathbf{j}-\mathbf{k}),\mathbf{j}\right)=\left(\tfrac{1}{\sqrt{2}}(1-\mathbf{k}),\tfrac{1}{\sqrt{2}}(-1-\mathbf{j})\right),$$

which is not an element of $2O_H$, since the two elements are not in the same coset of $2O/Q_8$.

3.5. ROTATION GROUP OF THE BINARY OCTAHEDRAL GROUP

LET us try to extend the method with which we found the binary rotation groups of the normal subgroups Q_8 and $2T$ of $2O$, to the the biggest normal subgroup of $2O$, which is $2O$ itself. Now $2O$ is not a regular polytope itself, but it is highly symmetric. We have seen from the coset of Q_8 and $2T$, that $2O$ can geometrically be interpreted as six 4-simplexes or two hyperdiamonds. Furthermore, the cosets of Q_8 in $2T$ can split up one or both of these hyperdiamonds in a hypercube and 4-orthoplex.

So, since $2O$ is such an symmetric object and contains all kinds of other symmetric objects, it is still very interesting to investigate the rotation group of $2O$, denote this as $O_O \subset SO(4)$. Our suspicion is that the binary rotation group of $2O$, denote this as $2O_O \subset \mathbb{H}_1 \times \mathbb{H}_1$, is equal to just $2O \times 2O$, since unlike before we only have one coset. Indeed for $q_1, q_2 \in 2O \times 2O$ it holds that $q_1 2O q_2^{-1} \in 2O$, since $2O$ is a group. This gives us already $48 \cdot 48 = 2304$ rotations. Now all that is left to prove is that these are all rotations of $2O$ such that $O_O = \Psi(2O_O)$.

Counting the size of the rotation group of $2O$[4] is a bit harder than for Q_8 and $2T$ as we can not immediately apply the orbit stabilizer theorem. For example, we are not sure if the six 4-orthoplexes form one orbit, maybe some rotation of $2O$ even maps them to other 4-orthoplexes that are not even part of these six from the start. In this section we will show that we can apply the orbit stabilizer theorem to the hyperdiamond described by $2T$ and how to do this.

Denote the hyperdiamond in $2O$ described by $2T$ as D_1 and denote its complementary hyperdiamond described by $2O \setminus 2T$ as D_2. Since $2O_D \subset 2O \times 2O \subseteq 2O_O$, all unit quaternions pairs in $2O_D$ are rotations of $2O$ as well and these rotations map D_1 to itself. This implies that D_2 is also mapped to itself by these rotations and thus they are rotational symmetries of D_2 as well. Since D_2 is of course a hyperdiamond, these are all rotational symmetries of D_2. We have seen that $\#2O_D = 1152$ which is half the size of $2O \times 2O$. Our hypothesis is that $2O \times 2O$ is the entire binary rotation group of $2O$. If this is the case, then $2O_D$ must be normal in $2O_O$ since it then must have index 2, so if we can prove this we are done.

Since $2O_D \subset 2O_O$, the stabilizer of D_1 and D_2 in $2O_O$ is equal to their coinciding binary rotation group, because if it was bigger, then there exists a rotation of the hyperdiamond not in $2O_D$. Call the index of the stabilizer of D_1 and D_2 in $2O_O$ k, then k is thus equal to the size of the orbit containing D_1 and also equal to the size of the orbit containing D_2, which we have not shown are the same orbit yet. Furthermore k also gives us

[4] This is also related to the root system F_4 and its Weyl group, since $2O$ can be seen as a normalized version of F_4. This will not be covered in this thesis.

the size of $2O_O$ by

$$\#2O_O = \#\mathrm{Stab}(D_1) \cdot k = \#2O_D \cdot k, \tag{3.29}$$

using the orbit stabilizer theorem. We already know that $\#2O_O = \#(2O \times 2O) \cdot l$ for some $l \in \mathbb{Z}_{>0}$, since $2O \times 2O$ is at least a subgroup of $2O_O$. This tells us $k = 2l$ and thus $k \geq 2$. We want to prove that $k = 2l = 2$.

Proving that D_1 and D_2 are the only possible hyperdiamonds that are contained in $2O$ is sufficient. Rotations from $2O_O \setminus 2O_D$ can not map D_1 to itself and thus D_1 has to be mapped to some other hyperdiamond. Indeed, if we prove that D_2 is the only other possible possible inscription of a hyperdiamond in $2O$, then the size of the orbit of D_1 can not be greater than 2, so we must have $k = 2$. We will do show this in a similar way to the simpler example of the two 4-orthoplexes in a cube in section 3.1.1. Here the orthogonality between certain edges as a characteristic of the 4-orthoplex was used to prove there is only one way to describe two 4-orthoplexes in the hypercube. However, instead of using orthogonality, here the distance between vertices is used as the characteristic of the hyperdiamond. Take the vertex $1 \in 2O$, then the distance from 1 to all other vertices in $2O$ can be calculated, see table 3.11. Note that since all elements of $2O$ lie in $\mathbb{H}_1$, the distance is fully determined by the real part.

Elements	Real Part	Distance	Number
1	1	0	1
-1	-1	2	1
$\frac{1}{2}\{1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$	$\frac{1}{2}$	1	8
$\frac{1}{2}\{-1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}\}$	$-\frac{1}{2}$	$\sqrt{3}$	8
$\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$	0	$\sqrt{2}$	6
$\frac{1}{\sqrt{2}}\{\pm\mathbf{i} \pm \mathbf{j}, \pm\mathbf{i} \pm \mathbf{k}, \pm\mathbf{j} \pm \mathbf{k}\}$	0	$\sqrt{2}$	12
$\frac{1}{\sqrt{2}}\{1 \pm \mathbf{i}, 1 \pm \mathbf{j}, 1 \pm \mathbf{k}\}$	$\frac{1}{\sqrt{2}}$	$0.765\ldots$	6
$\frac{1}{\sqrt{2}}\{-1 \pm \mathbf{i}, -1 \pm \mathbf{j}, -1 \pm \mathbf{k}\}$	$-\frac{1}{\sqrt{2}}$	$1.848\ldots$	6

Table 3.11: Distance of the elements of $2O$ to $1 \in 2O$.

Note that the elements with zero real part have been split on purpose such that the union of the first five rows equals $2T$, while the last three rows equal $2O \setminus 2T$. Now looking at just D_1, we can read the characteristic distances between the vertex equivalent to 1 in a general hyperdiamond on the 3-sphere and the other vertices of this general hyperdiamond. Comparing the distances to the vertices of D_2, we see that the choice is fully fixed for 1, except for the six elements at distance $\sqrt{2}$. At first sight it seems like we might be able to choose some of the twelve additional elements in row six in table 3.11, to create a different hyperdiamond than D_1, which still contains 1. However, in D_1 the distance between any of the six vertices with zero real part is $\sqrt{2}$. Then it can be checked that this is the only choice of six vertices from the eighteen vertices at distance $\sqrt{2}$ to the element 1 that satisfy this property. This means there only exists one hyperdiamond containing 1 in $2O$, namely D_1, and consequently this fixes D_2 as the only other hyperdiamond in $2O$. Since we know that the size of the orbit of D_1 is at least 2 from equation 3.29 and D_1 and D_2 are the only possible hyperdiamond in $2O$, they must be in the same orbit, which is immediately the only orbit of hyperdiamonds in $2O$. We are finally able to apply the orbit stabilizer theorem, giving that $\#2O_O = \#2O_D \cdot k = 1152 \cdot 2 = 2304$. This means the rotations of $2O$ that we found are all of the rotations of $2O$, so we can conclude

$$2O_O \cong 2O \times 2O \tag{3.30}$$

and thus

$$O_O \cong (2O \times 2O)/\{\pm(1,1)\}. \tag{3.31}$$

It is interesting to note that from the fact that the stabilizer of D_1 and D_2 have index 2 in O_O, it follows that half of the rotations fix D_1 and D_2, while the other half maps D_1 to D_2 and vice versa. This is also checked in appendix A.2.2.

3

Unfortunately, the method we used to construct the binary rotation groups of the hypercube, hyperdiamond and the binary octahedral group, does not work for the remaining normal subgroups of $2O$, namely $\{\pm 1\}$ and $\{1\}$. This does not come as surprise, as the rotation group of $\{1\}$ is just $SO(4)$, so its binary rotation group is all of $\mathbb{H}_1 \times \mathbb{H}_1$. For $\{\pm 1\}$, every rotation of the imaginary hyperplane $\mathrm{Im}\,\mathbb{H}$ is a rotational symmetry, these are the pairs (q,q) with $q \in \mathbb{H}_1$. Furthermore, rotations of the form $(q,-q)$ with $q \in \mathbb{H}_1$ keep $\{\pm 1\}$ fixed as well, by $q\{\pm 1\}(-q)^{-1} = \{\mp 1\}$. These rotations can be seen as all rotation flipping the real axis in $\mathbb{H}$. The total binary rotation group of $\{\pm 1\}$ is thus equal to $\{(q_1,q_2) \in \mathbb{H}_1 \times \mathbb{H}_1 : q_1 = \pm q_2\}$.

As a last remark we highlight that the short exact sequences

$$2O_D \hookrightarrow 2O_O \twoheadrightarrow C_2,$$

does not split, again for the simple reason that $2O_O \setminus 2O_D$ contains no elements with an order that divides 2. For $2O_H$ we do not even have an exact short sequence as it is not normal in $2O_O$.

The findings in this chapter can be summarized in one theorem.

Theorem 3.25. *Let N be a normal subgroup of the binary octahedral group $2O$, then*

$$2O_N := \{(q_1,q_2) \in 2O \times 2O : q_1 N = q_2 N\} \tag{3.32}$$

is a subgroup of the binary rotation group of N. Furthermore, for $N = Q_8, 2T, 2O$ it holds that $2O_N$ is equal to the entire binary rotation group of N and for the rotation group of N, denote this as $O_N \subset SO(4)$, it holds that $O_N \cong 2O_N/\{\pm(1,1)\}$, where the isomorphism is given by the restriction of $\Psi : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ to $2O_N$.

4

Conclusion

With the fact that $SO(3) \cong \mathbb{H}_1/\{\pm 1\}$ and $SO(4) \cong \mathbb{H}_1 \times \mathbb{H}_1/\{\pm(1,1)\}$ the binary octahedral group $2O$ was able to be constructed with 48 unit quaternions such that $O \cong 2O/\{\pm 1\}$, where the isomorphism is given by the restriction of $\Phi : \mathbb{H}_1 \to SO(3)$ on $2O$. The five normal subgroups of $2O$ are $\{1\}, \{\pm 1\}, Q_8, 2T$ and $2O$ itself. Q_8 consists of the vertices of a 4-orthoplex or 16-cell, which is dual to the hypercube or 8-cell, and $2T$ consists of the vertices of a hyperdiamond or 24-cell. This way $2O$ is equal to the union of six rotated copies of Q_8 or the standard 16-cell, but $2O$ is also equal to the union of two rotated copies of $2T$ or the standard 24-cell.

For the normal subgroups $Q_8, 2T, 2O \trianglelefteq 2O$ it was proved that the subgroup of $\mathbb{H}_1 \times \mathbb{H}_1$ defined by

$$2O_N := \{(q_1, q_2) \in 2O \times 2O : q_1 N = q_2 N\}$$

with $N \in \{Q_8, 2T, 2O\}$ is the entire binary rotation group of N. Let $O_N \subset SO(4)$ denote the rotation group of N, then $O_N \cong 2O_N/\{\pm(1,1)\}$, where the isomorphism is given by the restriction of $\Psi : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ to $2O_N$. $Q_8 \times Q_8$ is normal in the binary rotation group of the hypercube and $2T \times 2T$ is normal in the binary rotation group of the hyperdiamond. This way the binary rotation group of the hypercube is equal to the union of the direct products of six rotated copies of the standard 16-cell with themselves. Furthermore, the binary rotation group of the hyperdiamond is equal to the union of the direct products of two rotated copies of the standard 24-cells with themselves.

For the remaining two normal subgroups of $2O$, namely $\{\pm 1\}$ and $\{1\}$, this construction gives a subgroup of the entire binary rotation group, but not entire binary rotation group, which are $\{(q_1, q_2) \in \mathbb{H}_1 \times \mathbb{H}_1 : q_1 = \pm q_2\}$ and $\mathbb{H}_1 \times \mathbb{H}_1$ respectively.

It was found that the short exact sequences

$$V_4 \hookrightarrow O \twoheadrightarrow S_3$$
$$Q_8 \hookrightarrow 2T \twoheadrightarrow C_3$$

split, while the short exact sequences

$$
\begin{aligned}
Q_8 \hookrightarrow 2O \twoheadrightarrow S_3 \\
2T \hookrightarrow 2O \twoheadrightarrow C_2 \\
Q_8 \times Q_8 \hookrightarrow 2O_H \twoheadrightarrow S_3 \\
2T \times 2T \hookrightarrow 2O_D \twoheadrightarrow C_2
\end{aligned}
$$

do not split.

This thesis has shown that quaternions offer a powerful tool to analyze rotation groups in three and four dimensions. There are a lot more symmetric objects in these dimensions that were not touched in this thesis. Some notable shapes are the remaining polytopes in three and four dimensions. These are the tetrahedron, dodecahedron and icosahedron and their 4-dimensional analogues the 5-cell, 120-cell and the 600-cell respectively. Furthermore, the surface was only scratched with study of the hypercube, hyperdiamond and binary octahedral group. There are more inscriptions, geometric descriptions of symmetries and besides, in this thesis it was not looked so much at the non-binary rotation groups, let alone the entire symmetry groups.

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A

PYTHON CODE

A.1. EXTRA PROOFS

Here are two proofs, which in the text follow from more technical argumuntents, but are proved more directly here.

Lemma A.1. *Let $q = \mathbb{H}_1$ and $p \in \mathbb{H}$, then $\Phi_q(p)$ preserves the real part of p, that is*

$$Re(p) = Re(qpq^{-1})$$

Proof. Write $q = r + a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ and $p = r' + a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k}$. Then

$$\mathrm{Re}(qpq^{-1}) = \mathrm{Re}(qrq^{-1}) + \mathrm{Re}(q(a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k})q^{-1}).$$

Since the real part commutes in $\mathbb{H}$, we have that $\mathrm{Re}(qrq^{-1}) = r$. So what is left is to show that $\mathrm{Re}(q(a'\mathbf{i} + b'\mathbf{j} + c'\mathbf{k})q^{-1}) = 0$. This can be proved by writing everything out, which is exactly shown in the proof of theorem 2.28. □

Theorem A.2. *$2O$ is a subgroup of $\mathbb{H}_1$.*

Proof. First, it is clear that all conjugates and thus inverses of the the elements of $2O$ are also in $2O$. That $2O$ is closed can be seen from the fact that for $q_1, q_2 \in 2O$, we have that

$$q_1\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}q_1^{-1} = \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\} = q_2\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}q_2^{-1}.$$

So $q_1 q_2$ is a rotation as well, by

$$q_1 q_2\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}(q_1 q_2)^{-1} = q_1 q_2\{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}q_2^{-1}q_1^{-1} = \{\pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}.$$

□

A

A.2. CODE

All calculations can be checked with the following code defining classes for quaternions and quaternion pairs. At the bottom some objects are defined that can be used in the function rot_check_object, which checks if two quaternions are a rotation of the object according to the map $\Psi_{q_1,q_2} : \mathbb{H}_1 \times \mathbb{H}_1 \to SO(4)$ from theorem 2.30.

```
import math

class Quaternion:
    def __init__(self, w, x, y, z):
        self.w = w
        self.x = x
        self.y = y
        self.z = z

    def __add__(self, other):
        return Quaternion(
            self.w + other.w,
            self.x + other.x,
            self.y + other.y,
            self.z + other.z)

    def __sub__(self, other):
        return Quaternion(
            self.w - other.w,
            self.x - other.x,
            self.y - other.y,
            self.z - other.z)

    def __neg__(self):
        return Quaternion(-self.w, -self.x, -self.y, -self.z)

    def __mul__(self, other):
        w1, x1, y1, z1 = self.w, self.x, self.y, self.z
        w2, x2, y2, z2 = other.w, other.x, other.y, other.z

        w = w1*w2 - x1*x2 - y1*y2 - z1*z2
        x = w1*x2 + x1*w2 + y1*z2 - z1*y2
        y = w1*y2 - x1*z2 + y1*w2 + z1*x2
        z = w1*z2 + x1*y2 - y1*x2 + z1*w2

        return Quaternion(w, x, y, z)

    def norm(self):
```

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```
        return math.sqrt(self.w**2 + self.x**2 + self.y**2 + self.z**2)

    def conjugate(self):
        return Quaternion(self.w, -self.x, -self.y, -self.z)

    def inverse(self):
        n = self.norm()**2
        return Quaternion(self.w/n, -self.x/n, -self.y/n, -self.z/n)

    def normalize(self):
        n = self.norm()
        return Quaternion(self.w/n, self.x/n, self.y/n, self.z/n)

    def __repr__(self):
        return f"{self.w}+{self.x}i+{self.y}j+{self.z}k"

    def __eq__(self, other):
        return (math.isclose(self.w, other.w) and
                math.isclose(self.x, other.x) and
                math.isclose(self.y, other.y) and
                math.isclose(self.z, other.z))

    def __round__(self,d):
        return Quaternion(round(self.w,d),round(self.x,d),
                          round(self.y,d),round(self.z,d))

    def __hash__(self):
        return hash((
            round(self.w, 10),
            round(self.x, 10),
            round(self.y, 10),
            round(self.z, 10)
        ))

    def __pow__(self, n):
        if not isinstance(n, int):
            raise TypeError("Exponent must be an integer")
        if n < 0:
            raise ValueError("Negative exponents not supported
                             (use inverse if needed)")
        result = Quaternion(1, 0, 0, 0)
        for _ in range(n):
            result = result * self
        return result
```

A

```
    def order(self): # only works if order <100
        for i in range(1,100):
            if self**i == one:
                return i
                break


class Quaternion_pair:
    def __init__(self, q1, q2):
        self.q1 = q1
        self.q2 = q2

    def __repr__(self):
        return f"({self.q1}, {self.q2})"

    def __add__(self,other):
        return Quaternion_pair(self.q1+other.q1,self.q2+other.q2)

    def __sub__(self,other):
        return Quaternion_pair(self.q1-other.q1,self.q2-other.q2)

    def __neg__(self):
        return Quaternion_pair(-self.q1,-self.q2)

    def __mul__(self, other):
        return Quaternion_pair(self.q1*other.q1, self.q2*other.q2)

    def __eq__(self, other):
        return self.q1 == other.q1 and self.q2 == other.q2

    def __hash__(self):
        return hash((self.q1, self.q2))

    def order(self): # gives order of both quaternions in the pair,
                     # order of the pair is the smallest common multiple
        return (self.q1.order(), self.q2.order())

    def normalize(self):
        return Quaternion_pair(self.q1.normalize(), self.q2.normalize())

    def conjugate(self):
        return Quaternion_pair(self.q1.conjugate(), self.q2.conjugate())

    def inverse(self):
        return Quaternion_pair(self.q1.inverse(), self.q2.inverse())
```

```
    def __round__(self,d):
        return Quaternion_pair(round(self.q1,d), round(self.q2,d))

    def __getitem__(self, index):
        if index == 0:
            return self.q1
        elif index == 1:
            return self.q2

one = Quaternion(1,0,0,0)
i = Quaternion(0,1,0,0)
j = Quaternion(0,0,1,0)
k = Quaternion(0,0,0,1)

octahedron = {one,-one,i,-i,j,-j,k,-k}

hypercube_not_normalized = {one+i+j+k, one+i+j-k, one+i-j+k, one-i+j+k,
                            one+i-j-k, one-i+j-k, one-i-j+k, one-i-j-k,
                            -one+i+j+k, -one+i+j-k, -one+i-j+k, -one-i+j+k,
                            -one+i-j-k, -one-i+j-k, -one-i-j+k, -one-i-j-k}

hypercube = set()
for q in hypercube_not_normalized:
    hypercube.add(q.normalize())

hyperdiamond = octahedron.union(hypercube)

D2 = {one+i,one-i,-one+i,-one-i,one+j,one-j,
      -one+j,-one-j,one+k,one-k,-one+k,-one-k,
      i+j,i-j,-i+j,-i-j,i+k,i-k,-i+k,-i-k,j+k,j-k,-j+k,-j-k}
      # rotated hyperdiamond to create binary ocatahedral group 2O

D2_normalized = set()
for q in D2:
    D2_normalized.add(q.normalize())

twoO = D2_normalized.union(hyperdiamond) # binary octahedral group 2O

def rot_check_object(q1,q2,objec):
    # checks if quaternion pair (q1,q2) is a rotation of
    # the 4D object (define as a set of quaternions)
    # for 3D rotations, use function with q1=q2
    if round(q1.norm(),1)!=1 or round(q2.norm(),1) != 1:
        print("The quaternion pair should consist of unit quaternions")
```

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```
    else:
        l = set()
        for elt in objec:
            l.add(q1*elt*q2.conjugate())

        if l == objec:
            print((round(q1,1),round(q2,1), " is a rotation of your object")
        else:
            print((round(q1,1),round(q2,1), " is NOT a rotation of
            your object, the rotated object is: ", l)
```

A.2.1. CONJUGACY CLASSES OF $2O_H$ AND $2O_D$

For example, the following code calculates all the conjugacy classes of the binary rotation group of the hypercube $2O_H$ or the hyperdiamond $2O_D$, using the cosets of $2O$ to define $2O_H$ and $2O_D$. Change 'rot_hypercube' into 'rot_hyperdiamond' for the lower to get the conjugacy classes of $2O_D$. For the check if the defined set really are rotations, the previous code is needed to define 'octahedron' and 'hyperdiamond'.

```
    ## Cosets of 2O (not normalized)
c1 = [one,-one,i,-i,j,-j,k,-k]
c2 = [one+i,one+-i,-one+i,-one+-i,j+k,j+-k,-j+k,-j+-k]
c3 = [one+j,one+-j,-one+j,-one+-j,i+k,i+-k,-i+k,-i+-k]
c4 = [one+k,one+-k,-one+k,-one+-k,i+j,i+-j,-i+j,-i+-j]
c5 = [Quaternion(1, 1, 1, 1),
      Quaternion(1, 1, -1, -1),
      Quaternion(1, -1, 1, -1),
      Quaternion(1, -1, -1, 1),
        Quaternion(-1, 1, 1, -1),
        Quaternion(-1, 1, -1, 1),
        Quaternion(-1, -1, 1, 1),
        Quaternion(-1, -1, -1, -1)]
c6 = [Quaternion(1, 1, 1, -1),
      Quaternion(1, 1, -1, 1),
      Quaternion(1, -1, 1, 1),
      Quaternion(1, -1, -1, -1),
        Quaternion(-1, 1, 1, 1),
        Quaternion(-1, 1, -1, -1),
        Quaternion(-1, -1, 1, -1),
        Quaternion(-1, -1, -1, 1)]

## Construction of binary rotation group of the hypercube 2O_H
hypercube_rot = set()
for q1 in c1:
    for q2 in c1:
```

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```
        hypercube_rot.add(Quaternion_pair(q1, q2).normalize())
for q1 in c2:
    for q2 in c2:
        hypercube_rot.add(Quaternion_pair(q1, q2).normalize())
for q1 in c3:
    for q2 in c3:
        hypercube_rot.add(Quaternion_pair(q1, q2).normalize())
for q1 in c4:
    for q2 in c4:
        hypercube_rot.add(Quaternion_pair(q1, q2).normalize())
for q1 in c5:
    for q2 in c5:
        hypercube_rot.add(Quaternion_pair(q1, q2).normalize())
for q1 in c6:
    for q2 in c6:
        hypercube_rot.add(Quaternion_pair(q1, q2).normalize())

## Construction of binary rotation group of the hyperdiamond 2O_D
twoT = [one,-one,i,-i,j,-j,k,-k,
      Quaternion(1, 1, 1, 1),
      Quaternion(1, 1, -1, -1),
      Quaternion(1, -1, 1, -1),
      Quaternion(1, -1, -1, 1),
        Quaternion(-1, 1, 1, -1),
        Quaternion(-1, 1, -1, 1),
        Quaternion(-1, -1, 1, 1),
        Quaternion(-1, -1, -1, -1),
      Quaternion(1, 1, 1, -1),
      Quaternion(1, 1, -1, 1),
      Quaternion(1, -1, 1, 1),
      Quaternion(1, -1, -1, -1),
        Quaternion(-1, 1, 1, 1),
        Quaternion(-1, 1, -1, -1),
        Quaternion(-1, -1, 1, -1),
        Quaternion(-1, -1, -1, 1)]
twoTc = [one+i,one+-i,-one+i,-one+-i,j+k,j+-k,-j+k,-j+-k,
         one+j,one+-j,-one+j,-one+-j,i+k,i+-k,-i+k,-i+-k,
         one+k,one+-k,-one+k,-one+-k,i+j,i+-j,-i+j,-i+-j]

hyperdiamond_rot = set()
for q1 in twoT:
    D1.add(q1.normalize())
    for q2 in twoT:
        hyperdiamond_rot.add(Quaternion_pair(q1, q2).normalize())
for q1 in twoTc:
```

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```
    D2.add(round(q1.normalize(),1))
    for q2 in twoTc:
        hyperdiamond_rot.add(Quaternion_pair(q1, q2).normalize())



## Check rotation if these really are rotations of the hypercube
# for rot in hypercube_rot:
#     rot_check_object(rot[0], rot[1], octahedron)
# for rot in hyperdiamond_rot:
#     rot_check_object(rot[0], rot[1], hyperdiamond)

## Find all conjugacy classes in the list d
d = []
count = 0
for h in hypercube_rot:
    l=set()
    for g in hypercube_rot:
        l.add(round(g*h*g.conjugate(),1))
    if l not in d:
        count+=1
        d.append(l)
        print("Nr.", count, ":", len(l), "pairs in this conjugacy class")
        for quat in l:
            print(quat)
        print("\n")
```

A.2.2. The Two Types of Rotations of $2O_O$

This code shows that half of the elements of $2O\times 2O$ map D_1 and D_2 to themselves, while the other half maps them to each other.

```
tlist = []
flist = []
D1 = hyperdiamond
D2 = D2_normalized
for q1 in twoO:
    for q2 in twoO:
        st = set()
        rho = Quaternion_pair(q1, q2).normalize()
        # rot_check_object(rho[0], rho[1], hyperdiamond)

        for vert in D1:
            st.add(round(rho[0]*vert*rho[1].conjugate(),1))

        if st==D2:
            tlist.append(1)
```

```
        elif st == D1:
            flist.append(2)
        else:
            print("Does not map D1 to itself or D1 to D2")

print(len(flist), "rotations map D1 to D1", len(tlist), "rotations map D1 to D2")
```