Asymptotic Results
in
Feedback Systems

Asymptotic Results in Feedback Systems

PROEFSCHRIFT

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Contents

Copies of four papers

[1] To appear in Stochastic Processes and its Applications
[2] To appear in Journal of Applied Probability

1. Introduction.

In this thesis the term *feedback system* is used in two different ways. In the general theory of stochastic processes, a feedback system, or *feedback process*, is a process in which the state of the process at time n will be fed back as input for the calculation of the state of the process at time $n+1$. In queueing theory, the term feedback system, or *feedback queue*, stands for a queueing model where customers, after getting service at a service facility, may be fed back to the service facility before they depart from the system.

The main part of this thesis consists of four papers about feedback systems. The first two papers deal with feedback processes. A global description of the subjects studied in these two papers is given in Section 2 and 3, respectively. The last two papers deal with feedback queues. In Section 4 and 5 a summary of the results obtained in these two papers is given. The rest of Section 1 is devoted to the definition of feedback processes and feedback queues.

1.1 Feedback processes.

DEFINITION: An $\mathbf{R}^p$-valued stochastic process X_n is called a *feedback process* with input sequence $Y_n \in \mathbf{R}^q$ if there exist functions $f_n : \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^p$ such that for all $n \in \mathbf{N}$

$$X_{n+1} = f_n(X_n, Y_n). \tag{1.1}$$

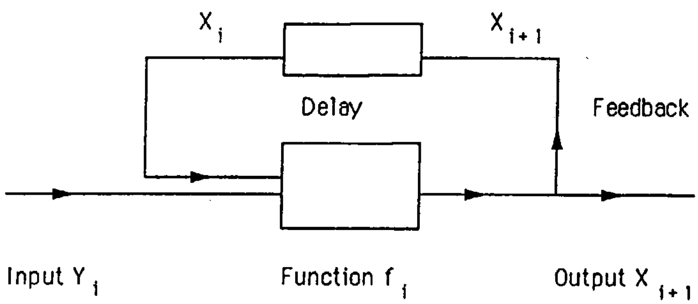

Figure 1

EXAMPLE 1: Let W_n be the waiting time of the n-th customer in a single server queueing model. Let S_n be the service time of the n-th customer and T_n be the interarrival time between the n-th and the $(n+1)$-st customer. Then

$$W_{n+1} = \max(0, W_n + (S_n - T_n))$$

is of the form (1.1) with $X_n = W_n$, $Y_n = S_n - T_n$ and $f_n(x,y) = \max(0, x+y)$.

EXAMPLE 2: Let $(Y_k)_{k\geq 1}$ be a sequence of i.i.d. random variables and X_n be the maximum value of the first n of these random variables. Then X_n is of the form (1.1) with $f_n(x,y) = \max(x,y)$.

EXAMPLE 3: Let

$$X_{n+1} = A_n X_n + B_n,$$

where (A_n, B_n) are i.i.d. random vectors. This stochastic difference equation arises in various disciplines, for example economics, physics and biology (see Kesten [13] and Vervaat [19]). In most applications X_n represents a stock of certain objects at time n, B_n the quantity that is added just before time $n+1$ (or taken away in case $B_n < 0$) and the factor A_n indicates the intrinsic decay or increase of the stock X_n between time n and $n+1$. In this case X_n is of the form (1.1) with $Y_n = (A_n, B_n)$ and $f_n(x,(y,z)) = yx + z$.

A question of interest for such feedback processes is under what conditions on the input sequence Y_n and the functions f_n the process X_n converges in distribution to a non-degenerate random vector X. For instance, in the first example above it is well-known that convergence occurs if S_n and T_n are sequences of i.i.d. random variables, independent of each other, satisfying $ES_1 < ET_1$.

Another question of interest is the following : If the X_n's do not converge in distribution, can we find norming constants a_n and b_n such that $(X_n - a_n)/b_n$ does converge in distribution to a non-degenerate random vector X? This question has been answered for the second example above and leads to the well-known extreme value theory (see Gnedenko [6]).

In this thesis we shall consider two models in applied probability of the form (1.1). The first model arises in the analysis of production networks. If the production network has p nodes, then X_n is an $\mathbf{R}^p$-valued random vector. The input sequence Y_n is a sequence of random matrices of order p, so that Y_n is $\mathbf{R}^q$-valued with $q = p^2$. The vector X_n represents the times at which the nodes become active for the n-th time. The matrix Y_n represents the transportation times between the different nodes. This model will be described in more detail in Section 2.

The second model arises from a model for storage systems and is of the form (1.1) with $f_n(x,y) = \max(x, \alpha_n x + y)$, where α_n is a sequence of parameters between zero and one. In this case the feedback function f_n depends on n. This model will be described in more detail in Section 3. For both models we study the asymptotic behaviour of X_n when n tends to infinity.

1.2 Feedback queues.

DEFINITION: A queueing system in which customers may repeatedly return to some service facility to obtain several phases of service before they leave the system is called a *feedback queue* (see Figure 2).

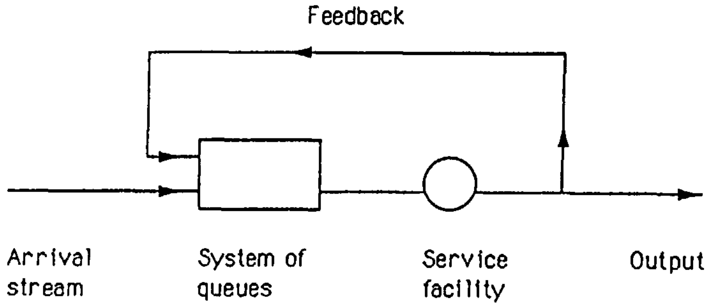

Figure 2

The interest in such feedback queues stems from so-called time-shared computer systems (see Kleinrock [14]). A simple model for a time-sharing computer system consists of a single resource (CPU) and a system of queues containing those users of the computer system (customers) awaiting service. In addition there exists a scheduling algorithm, which is a set of decision rules determining which customer will be served next and for how long. The newly arriving customers are placed in the system of queues and, when the scheduling algorithm finally permits, are taken into service. The interval of time during which a customer is permitted to remain in service (quantum) may or may not be enough to satisfy the request. If sufficient, then the customer departs from the system; if not, then he reenters the system of queues as a partially completed task and waits within the system of queues until the scheduling algorithm decides to give him a second quantum, and so on. Eventually, after a sufficient number of visits to the service facility, the customer will have gained enough service and will depart. The third and fourth model in this thesis, which will be described in more detail in Sections 4 and 5, deal with this kind of feedback systems. In Section 4 we shall use a sequence of feedback models to approximate a queueing model with so-called processor sharing service discipline. In Section 5 we consider the queue length process in feedback models where the service times of customers depend on the number of services the customer has already obtained.

2. Discrete event dynamic systems.

A common property of production processes is that machines do not act independently, i.e. some machines cannot start a new activity until certain other machines have all completed their current activities. Furthermore some machines cannot start a new activity until certain outside resources become available. Finally, endproducts can only be delivered after certain machines have completed their activities.

A mathematical description of this kind of processes is given by the relations

$$\begin{aligned} x_i(n+1) &= \max_{\substack{1\le j\le p\\ 1\le k\le m}} (a_{ij} + x_j(n), b_{ik} + u_k(n)) \\ y_i(n) &= \max_{1\le j\le p} (c_{ij} + x_j(n)), \end{aligned} \tag{2.1}$$

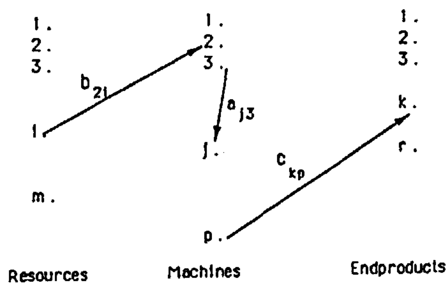

Figure 3

where the interpretation of the used quantities is:

m : Number of resources.

p : Number of machines.

r : Number of endproducts.

$x_i(n)$: Time instant at which machine i becomes active for the n-th time.

$u_i(n)$: Time instant at which resource i becomes available for the n-th time.

$y_i(n)$: Time instant at which endproduct i is delivered for the n-th time.

a_{ij} : Transportation time from machine j to machine i, including the duration of the activity of machine j.

b_{ij} : Transportation time between resource j and machine i.

c_{ij} : Transportation time between machine j and delivery point of endproduct i.

If we introduce the following notation

$$\begin{aligned} x \oplus y &= \max(x, y) \\ x \otimes y &= x + y, \end{aligned} \tag{2.2}$$

then the relations (2.1) can be written as

$$
\begin{aligned}
x_i(n+1) &= \sum_{j=1\oplus}^{p} [a_{ij} \otimes x_j(n)] \oplus \sum_{j=1\oplus}^{m} [b_{ij} \otimes u_j(n)] \\
y_i(n) &= \sum_{j=1\oplus}^{p} [c_{ij} \otimes x_j(n)],
\end{aligned} \tag{2.3}
$$

where

$$\sum_{j=1\oplus}^{p} z_j = z_1 \oplus \cdots \oplus z_p.$$

In matrix notation this becomes

$$
\begin{aligned}
x(n+1) &= [A \otimes x(n)] \oplus [B \otimes u(n)] \\
y(n) &= C \otimes x(n).
\end{aligned} \tag{2.4}
$$

A process of the form (2.4) is called a *discrete event dynamic system* (DEDS).

The algebraic structure $(R, \oplus, \otimes)$, where R denotes the real numbers extended with minus infinity, and $x \oplus y$ and $x \otimes y$ are defined as in (2.2), is called *max-algebra*. A systematic theory, analogous to conventional linear algebra, exists for the max-algebra (see Cuninghame-Green [5]). The main reason for the introduction of the notation $\oplus$ and $\otimes$ in (2.2) is to make this analogy more clear.

In the first paper in this thesis we concentrate on the equation $x(n+1) = A \otimes x(n)$ which for instance can be obtained from (2.4) if all resources are immediately available at the beginning of the process. Until now the study of DEDS in the context of max-algebra has been purely deterministic. In practice, however, processing times and/or transportation times are quite often stochastic quantities. Such stochastic fluctuations can, for instance, be caused by machine failure or depreciation. This is our motivation to study a stochastic extension of $x(n+1) = A \otimes x(n)$. In fact, we study the model

$$x(n+1) = A(n) \otimes x(n) \tag{2.5}$$

where $A(n)$, $n = 0, 1, \ldots$, is a sequence of i.i.d. real valued $p \times p$ matrices. Notice that the process is a feedback system of the form (1.1) with $X_n = x(n)$, $Y_n = A(n)$ and $f_n(x, A) = A \otimes x$. We are interested in the asymptotic behaviour of $x(n)$ for $n \to \infty$ and we shall prove that under suitable conditions on the input sequence $A(n)$ the process $x(n)$ is asymptotically normal.

3. A process between maxima and sums.

Consider the following stochastic process

$$(3.1)\qquad \begin{cases} X_{n+1} = \max(\beta X_n, \alpha\beta X_n + Y_{n+1}) \\ X_0 = 0 \end{cases}$$

where $0 \leq \alpha \leq 1$, $0 < \beta < 1$ and $\{Y_n, n \geq 1\}$ a sequence of i.i.d. $\mathbf{R}^+$-valued random variables. Such a process may be used to model a storage system in which X_n denotes the contents at time n. In the time period from n to $n+1$ the contents is depleted to βX_n. The random variable Y_{n+1} is to be regarded as an offered input, acceptance of which implies a further depletion to $\alpha\beta X_n$. The process arises, for example, in describing a system for the storage of solar energy (see Haslett [10]), X_n measuring the energy contents of the storage tank. The release of energy is proportional to the content, and the offered input, being solar radiation equivalent to an energy amount of Y_{n+1}, gives rise to a further release of energy equivalent to $(1-\alpha)\beta X_n$.

A necessary and sufficient condition for positive recurrence of the Markov chain X_n is $E\log(1+Y_1) < \infty$ (see Greenwood and Hooghiemstra [8]). In this case the limiting stationary distribution of X_n is the unique solution $X^{(\alpha,\beta)}$ of the equation

$$(3.2)\qquad X^{(\alpha,\beta)} \stackrel{d}{=} \max(\beta X^{(\alpha,\beta)}, \alpha\beta X^{(\alpha,\beta)} + Y),$$

where $X^{(\alpha,\beta)}$ and Y are independent, $Y \stackrel{d}{=} Y_1$ and $\stackrel{d}{=}$ denotes equality in distribution.

In Hooghiemstra and Keane [11] the model is studied when Y_1 is exponentially distributed. They were able to compute $EX^{(\alpha,\beta)}$ numerically with satisfactory precision for $\beta \leq 0.95$. In Greenwood and Hooghiemstra [8] a centering function $c(\beta)$ is found for which $X^{(\alpha,\beta)} - c(\beta)$ has a limiting distribution for fixed α as $\beta \to 1$ and they found the density function h of the limit law. It turns out that h satisfies the functional equation

$$h(x) = \int_{-\infty}^{x} e^{-(x-\alpha u)} h(u)du.$$

The same problem was studied by Hooghiemstra and Scheffer [12] for the case $P[Y_1 > x] = x^{-\rho}$, $\rho > 0$. They showed that for such input, the laws $(1-\beta)^{1/\rho}X^{(\alpha,\beta)}$, $\beta \in (0,1)$, form a tight family with limiting density h as $\beta \to 1$ satisfying

$$h(x) = x^{-1}\int_0^x (x-\alpha u)^{-\rho} h(u)du.$$

Instead of focussing on $X^{(\alpha,\beta)}$, the limit law of X_n defined by (3.1) and letting $\beta \to 1$, one may proceed by putting $\beta = 1$ in (3.1). This gives

$$\begin{cases} X_{n+1} = \max(X_n, \alpha X_n + Y_{n+1}) \\ X_0 = 0. \end{cases} \tag{3.3}$$

The process X_n in (3.3) will not have a proper limit distribution. So questions of interest are:

(1) How can X_n be normalized to obtain a proper limit distribution?

(2) What are the proper limit distributions for normalized sequences $(X_n - a_n)/b_n$?

(3) What classes of Y_1 form the domains of attraction of these limit distributions ?

In the two boundary cases $\alpha = 0$ and $\alpha = 1$ the answer to these questions is well-known. In fact these cases bring us back to extreme value theory (see Gnedenko [6]) and the theory of sums of i.i.d. random variables (see Gnedenko and Kolmogorov [7]), respectively.

Indeed, for $\alpha = 0$ (3.3) becomes

$$X_{n+1} = \max(Y_1, \dots, Y_{n+1}).$$

The possible limit distributions are the three extreme value distributions

$$\begin{aligned} \Phi_\rho(x) &= \exp(-x^{-\rho})\, 1_{[0,\infty)}(x), \quad \rho > 0 \\ \Psi_\rho(x) &= \exp(-(-x)^\rho) 1_{(-\infty,0]}(x) + 1_{(0,\infty)}(x), \quad \rho < 0 \\ \Lambda(x) &= \exp(-e^{-x}). \end{aligned}$$

For $\alpha = 1$ on the other hand, (3.3) becomes

$$X_{n+1} = \sum_{i=1}^{n+1} Y_i.$$

In this case the possible limit distributions are the stable distributions.

The intermediate case $0 < \alpha < 1$ is studied in Greenwood and Hooghiemstra [9]. They showed that the limiting behaviour of normed sequences formed from X_n is parallel to the extreme value case $\alpha = 0$. More specifically, if Y_i are random variables with distribution F and if a_n and b_n are norming constants such that $F^n(a_n + b_n x)$ converges in distribution to a non-degenerate limit law, then the sequence $b_n^{-1}(X_n - a_n/(1-\alpha))$ is tight on the interior of the support of this limit law. For F in the domain of attraction of the first extreme value distribution $\Phi_\rho(x)$ they showed that a limit density exists and is the unique density solution of the functional equation

$$h_\alpha(x) = \rho x^{-1} \int_0^x (x - \alpha u)^{-\rho} h_\alpha(u) du.$$

For the two remaining extreme value distributions a similar result is obtained for some specific distributions F in their domain of attraction.

Because of the discontinuity in the norming constants between $\alpha \in [0,1)$ and $\alpha = 1$ it is interesting to replace the fixed α by a sequence (α_n) tending upward to 1. In the second paper of this thesis we investigate what happens when

$$\alpha_n = 1 - n^{-\ell}, \quad 0 < \ell < \infty$$

for the three cases

$$\begin{aligned}
&(1) \quad \ell = 1, && \int y^2 dF(y) < \infty \\
&(2) \quad 0 < \ell < 1, && F(y) = (1 - y^{-\rho})1[y > 1], \quad \rho > 2 \\
&(3) \quad \ell > 1, && \int y^2 dF(y) < \infty
\end{aligned}$$

In fact we prove the existence of norming constants a_n and b_n such that

$$\frac{X_n}{a_n} \to 1 \quad \text{a.s.}$$

$$\frac{X_n - a_n}{b_n} \xrightarrow{d} Z$$

with Z standard normal ($\xrightarrow{d}$ is convergence in distribution).

4. Feedback and processor sharing queues.

As mentioned in the introduction, the interest in feedback queues stems from the principle of time sharing in computer systems. The most natural queueing model for time-shared computer systems is the M/G/1 queue with *Round Robin* service. New customers, arriving according to a Poisson process, join the end of the queue. When a customer has reached the front of the queue he receives a fixed quantum q of service. At the end of this service quantum the customer leaves the system if his total service requirement is met; if not, then he returns to the end of the queue with his remaining service requirement reduced by an amount q.

The M/G/1 *processor sharing* model is described as follows: customers, arriving according to a Poisson process, are taken into service immediately upon their arrival. They stay into service uninterruptedly until their service demand is satisfied. When there are n customers in the system each customer receives service at a rate which is $1/n$ times the rate of service a solitary customer in the system would receive.

It is commonly held that, as the length of the quantum size q in the Round Robin model tends to zero, the processor sharing model and Round Robin model become indistinguishable. Hence results for the processor sharing model may serve as approximations for corresponding Round Robin results. In Schassberger [17] this statement has been made precise.

The processor sharing model was introduced because of the mathematical difficulties arising in the analysis of the more realistic Round Robin model. The derivation of the stationary queue length distribution in the M/G/1 processor sharing model was obtained by Sakata et al. [16]. The much harder problem of calculating the stationary sojourn time distribution was solved by Yashkov [20], Ott [15] and Schassberger [17].

Recently van den Berg and Boxma [3] suggested a new approach for analyzing M/G/1 processor sharing models by way of an approximating sequence of M/M/1 feedback queues. The M/M/1 *feedback queue* is the following model. New customers, arriving according to a Poisson process, join the end of the queue. After completion of a service a customer returns to the end of the queue with probability $p(i)$ or departs from the system with probability $1-p(i)$, where i denotes the number of services the customer has already obtained. All random mechanisms, i.e. interarrival times, service times and feedback mechanism are independent.

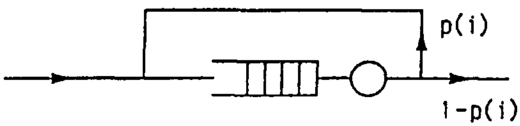

Figure 4

Let us briefly sketch the approach of van den Berg and Boxma in the case of an M/M/1 processor sharing model. So, assume that the service times $X_1, X_2, \ldots$ of customers in the processor sharing queue are exponential, say with mean 1. In the feedback queue with parameter n a customer obtains exponential service slices with mean length n^{-1}. When a customer has completed a service he departs from the system with probability n^{-1} or joins the end of the queue with probability $1-n^{-1}$. It is easily checked that for each n the total service time a customer gets in the feedback model is exponential with mean 1. Furthermore, when $n \to \infty$ the feedback queue with parameter n looks more and more like the processor sharing queue. On heuristic grounds van den Berg and Boxma concluded that performance measures such as the sojourn time in the feedback model converge to the corresponding performance measure in the processor sharing model. This is made rigorous in the third paper in this thesis. In this paper we present a *probabilistic coupling* between the M/M/1 processor sharing queue and the approximating sequence of M/M/1 feedback queues, i.e. we construct a probability space $(\Omega, \mathcal{A}, P)$ on which both the processor sharing and the feedback queues are defined. On this probability space the arrival times of the customers in the processor sharing and feedback models are the same. Furthermore the construction is done such that, if $X_{k,n}$ denotes the total service time of the k-th customer in the feedback model with parameter n, then $X_{k,n} \to X_k$, P almost surely. These two properties will be used to prove the almost sure convergence of departure times of customers in the feedback queue with parameter n to the corresponding quantity in the processor sharing model as $n \to \infty$. In fact, in the proof we introduce and use a sequence of round robin queues defined on the same probability space $(\Omega, \mathcal{A}, P)$. Finally, the theory of regenerative processes (see Smith [18] and Cohen [4]) is used to show that if the workload of the system is smaller than one, then the steady state sojourn time distribution of the feedback model converges as $n \to \infty$ to the steady state sojourn time distribution in the processor sharing model.

Van den Berg and Boxma [3] showed how to choose the feedback probabilities to obtain more general total service times. It is possible to obtain all service time distributions that are finite mixtures of phase distributions. The random variable X is said to have a distribution which is a finite mixture of phase distributions if it has Laplace-Stieltjes transform

$$Ee^{-sX} = \sum_{j=1}^{m} \alpha_j \prod_{i=1}^{r_j} (1 + \mu_{ij} s)^{-1}, \tag{4.1}$$

where $(\alpha_1, \cdots, \alpha_m)$ is a probability vector, r_i are positive integers and μ_{ij} are positive real numbers. It is known that the distributions defined by (4.1) are dense in the space of all probability measures on $(0,\infty)$ equipped with the Prohorov distance or some equivalent metric.

5. Feedback queues and multitype branching processes.

The fourth paper in this thesis is devoted to the analysis of the queue length process in feedback queues where both the probability that a customer is fed back after service completion and the distribution of the service time of a customer depend on the number of times he has already been served.

The basic feedback queueing model is the M/G/1 queue with Bernoulli feedback. In this model both the feedback probability p and the service time distribution $B(\cdot)$ of a customer are independent of the number of times he has already been served. It is easily seen that the M/G/1 queue with Bernoulli feedback has a stationary queue length process which has the same distribution as a corresponding M/G/1 queue without feedback, i.e. an M/G/1 system in which the service distribution of a customer is equal to the total service time a customer obtains in the feedback model. That is, if $\beta(\cdot)$ is the Laplace-Stieltjes transform of the service time distribution, then the Laplace-Stieltjes transform of the total service time distribution is given by

$$\frac{(1-p)\beta(\cdot)}{1-p\beta(\cdot)}.$$

Hence the generating function of the number of customers in the system is well-known from the theory for M/G/1 queues without feedback.

In van den Berg [2] a generalization of this feedback model is investigated under the condition that the service time distribution $B(\cdot)$ is exponential. In this generalized feedback model the feedback probability $p(i)$ depends on the number of services i the customer already has obtained (see Figure 4). If we assume $p(N) = 0$ for some N and call customers who are visiting the queue for the i-th time type i customers, then the stationary joint distribution of the number of type i customers in the system at an arbitrary epoch, $i = 1, \cdots, N$ is of "product form" type (see Baskett, Chandy, Muntz and Palacios [1]). More precisely, if we denote by X_i the steady-state number of type i customers at an arbitrary epoch then

$$P(X_1 = x_1, \ldots, X_N = x_N) = (1-\rho)\frac{(x_1 + \cdots + x_N)!}{x_1! \cdots x_N!}\prod_{i=1}^{N}(\lambda\beta q(i))^{x_i}$$

with $q(i) = \prod_{j=1}^{i-1} p(j)$, $\rho = \lambda\beta \sum_{i=1}^{N} q(i)$, λ the arrival rate and β the mean service time.

A natural generalization of van den Berg's model is an M/M/1 feedback model in which, in addition to the probability that a customer is fed back after service completion, also the service time of a customer depends on the number of times he has already been served. For this extended model the joint stationary distribution of the number of type i customers in the system is no longer of product form type. In fact, no results concerning the distribution of the queue length were available. The study of this model led to the results presented in the fourth paper. To illustrate the ideas of this paper we consider the following feedback model. Customers arrive at the system according to a Poisson process with intensity $\lambda > 0$. Each customer requires exactly two services. The two service times are independent, with distribution function $B_i(\cdot)$, finite mean β_i and Laplace-Stieltjes transform $\beta_i(\cdot)$, $i = 1, 2$, respectively.

Define the following embedded sequence of times t_n, called generation times:

(1) t_0 is the arrival time of the first customer.

(2) t_{n+1} is the first instant after t_n in which all customers present at t_n have been served exactly once in (t_n, t_{n+1}). If there are no customers present at t_n, t_{n+1} is the instant of the first arrival after t_n.

If we put $Z_n = (Z_n^1, Z_n^2) = (X_{t_n}^1, X_{t_n}^2)$ where X_t^i is the number of type i customers in the system at time t, then Z_n is a two-dimensional Markov chain. It turns out that Z_n is a two type branching process with immigration at state $(0,0)$. The offspring generating functions are given by

$$f^{(1)}(s_1, s_2) = s_2 \beta_1(\lambda(1 - s_1))$$
$$f^{(2)}(s_1, s_2) = \beta_2(\lambda(1 - s_1)).$$

The immigration generating function is given by $g(s_1, s_2) = s_1$. For this multitype branching process a stability condition can be given, which turns out to be $\lambda(\beta_1 + \beta_2) < 1$. If this condition is satisfied, then an expression is derived for the generating function of the stationary joint distribution of Z_n. If $\lambda(\beta_1 + \beta_2) > 1$, then an almost sure convergence result is given for the fraction of type i customers in the system at generation times.

The same kind of analysis can be performed for a general kind of queueing models called multitype M/G/1 queues with Markov routing. In these models customers of different types arrive according to independent Poisson processes with rates $\lambda_1, \cdots, \lambda_N$. Type i customers have service times with distribution

$B_i(\cdot)$ with finite mean β_i. After being served, a type i customer returns to the end of the queue becoming a type j customer with probability p_{ij}, where $P = (p_{ij})$ is a substochastic matrix, i.e. (i) $p_{ij} \geq 0$ for all i and j and (ii) $\sum_j p_{ij} \leq 1$ for all i. ($1 - \sum_j p_{ij}$ is the probability that a type i customer leaves the system.) The joint queue length process at generation times is once again a multitype branching process. If $I - P$ is non-singular and the matrix M with entries $m_{ij} = \lambda_j \beta_i + p_{ij}$ is primitive (i.e. there exists an n such that M^n is strictly positive), then we obtain the same kind of results as mentioned above. The assumption that $I - P$ is non-singular is equivalent to saying that customers eventually leave the system with probability one.

A related model is one with so-called permanent customers. In this model there are, besides the ordinary customers, also a number of customers staying in the system forever. The joint queue length process at generation times is in this case a multitype branching process with immigration at each state. For such processes similar results as mentioned above are obtained.

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Copies of four papers

Asymptotic Behavior of Random Discrete Event Systems.

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Abstract

In this paper we discuss some aspects of the asymptotic behavior of Discrete Event Dynamic Systems (DEDS) in which the activity times are random variables. The main result is that a central limit theorem holds for DEDS and consequently that the cycletime of the system is asymptotically normally distributed.

1. Introduction

A large class of dynamic systems, such as the material flow in production or assembly lines, the message flow in communication networks and jobs in computer systems can be modeled by Discrete Event Dynamic Systems (DEDS). These are systems where the occurrence of events is determined by the system itself and not described by time. Examples of such events are the beginning or completion of a task in an assembly line or the arrival of a message in a communication network.

Current research on DEDS uses a number of methods. Among these are the logical approach to automata (see e.g. [11]), the perturbation analysis of trajectories (see e.g. [6]), simulation, and the temporal approach, which we shall follow in this article (see e.g. [3]).

In these models activity times at a node of, for instance, a production network, are successively determined by combining the activity times at other nodes during previous activity cycles with delay and/or transport times. The aim is then to describe the dynamic temporal behavior of the network, given the knowledge of the nature of the delay and transport times and the initial state of the system. An important aspect of the temporal approach is that it permits a conceptual simplification by use of the so-called max-algebra to describe the models, yielding an analogy to conventional system theory. The elements of this max-algebra are the real numbers (together with $-\infty$) and the only admissible operations are maximization and addition.

In [4] a systematic theory parallel to linear algebra has been developed for the max-algebra, and in [2] and [3] the use of the max-algebra in the temporal approach to DEDS has been discussed and illustrated. In §2 the models for DEDS and the max-algebra will be introduced and some results on DEDS will be recalled.

At present, only deterministic DEDS have been studied using the temporal approach via the max-algebra. However, often in practice delay and transport times are of a stochastic nature, either inherently or because of our lack of information concerning the precise nature of the system. In this article we shall concentrate on models which take this random behavior into account. In particular, we derive for a class of random discrete event systems (see §2 for a definition) the expected cycle time and we show that this cycletime is asymptotically normal (see §3). Calculations for the expectation and variance of the cycletime are given in several examples in section 4. Finally, in section 5 we consider reducible random DEDS and we compare the asymptotic behavior of random DEDS with deterministic DEDS.

2. Discrete Event Dynamic Systems

In this section we shall first introduce the models we use for Discrete Event Dynamic Systems (DEDS) and show how these models can arise. Then we introduce the concept of random DEDS. Finally, some known results on the asymptotic behavior of deterministic DEDS are recalled.

Consider a production network with the following functional description. There are a fixed number n of nodes in the network. We shall be interested in the time point at which node i ($1 \leq i \leq n$) becomes active (i.e. starts production) for the k-th time. This time point will be denoted by $x_i(k)$. In order to start the $(k+1)$-st activity at node i, it is necessary to wait until each node j has finished its k-th activity and "supplied" node i. As soon as all necessary supplies from the k-th production cycle are available at node i, it becomes active for the $(k+1)$-st time. Let $a_{ij}(k)$ denote the sum of the production time at node j in the k-th cycle and the transporttime from node j to node i. Then the above description gives rise to the formula

$$x_i(k+1) = \max_{1 \leq j \leq n} (x_j(k) + a_{ij}(k)). \tag{2.1}$$

At this stage it is both intuitive and convenient to introduce the max-algebra notation. Following [4], we define for real numbers r and s the operations $\oplus$

and $\otimes$ by:

$$r \oplus s = \max(r, s),$$
$$r \otimes s = r + s.$$

The reason for using the symbols $\oplus$ and $\otimes$ is that a number of results from convential linear algebra and system theory can be "transferred" to the max-algebra and DEDS by replacing the $+$ and $\times$ signs by $\oplus$ and $\otimes$ respectively. The formula (2.1) for the n-vector $x(k+1)$ of $(k+1)$-st activity times then becomes, in matrix notation

$$x(k+1) = A(k) \otimes x(k). \tag{2.2}$$

For many purposes the above autonomous formulation of a dynamic system is too restrictive: one must add a vector $u(k)$ of outside resource times and one should consider a general output vector $y(k)$, obtained from $x(k)$ by adding production and transport times. In this manner one arrives at the general form:

$$\begin{aligned} x(k+1) &= (A(k) \otimes x(k)) \oplus (B(k) \otimes u(k)), \\ y(k) &= C(k) \otimes x(k) \end{aligned} \tag{2.3}$$

of a linear discrete event dynamic system. The reader is referred to [2] and [3] for a detailed description and discussion.

In this paper we shall be interested in the asymptotic behavior of the time vector $x(k)$. Therefore we assume that $C(k)$ is the identity matrix in the max-algebra and that all outside resources are available at the start of the process, so that the term $B(k) \otimes u(k)$ disappears. This allows us to concentrate on the behavior of the system as a function of the matrices $A(k)$, which we shall assume to be of a stochastic nature.

DEFINITION 1. *Let $(A(k))_{k \geq 0}$ in (2.2) be a sequence of independent, identically distributed (i.i.d.) real $n \times n$ matrices and let an initial random vector $x(0)$ be given independent of $(A(k))_{k \geq 0}$. Then the system, which is described by (2.2), will be called a random discrete event dynamical system. (A similar definition can be given when the general form (2.3) is used).*

We shall assume that the matrices $A(k)$ are real-valued and finite with probability one. Our goal in this article is to show under suitable conditions that the sequence $x(k)$ is asymptotically normal and to give examples of explicit calculations (for $n = 2$) of the asymptotic mean and variance, thereby determining the average cycle time and the nature of the deviation from this average during long-term operation of the system.

Next, we briefly describe some results of [2] and [3] concerning the behavior of deterministic DEDS, which can be seen as a special case of the above definition in which the i.i.d. sequence $A(k)$ is simple a constant matrix $A = (a_{ij})$ with real entries.

An eigenvalue of the matrix A (in the max-algebra sense) is a real number λ such that the equation

$$A \otimes x = \lambda \otimes x,$$

possesses a solution $x \in \mathbb{R}^n$. Then the following results can be formulated.

1) Every (real-valued) matrix A possesses a *unique* eigenvalue $\lambda = \lambda(A)$.
2) For each sequence $\gamma = (i_1, \ldots, i_j, i_{j+1} = i_1)$ of nodes, the *average weight*

$$\omega(\gamma) = \frac{a_{i_1 i_2} \otimes \cdots \otimes a_{i_j i_1}}{j},$$

   (in this, division by j is the conventional algebraic operation, not a max-algebra operation) satisfies

$$\omega(\gamma) \leq \lambda.$$

3) There is a $\gamma = (i_1, \ldots, i_d, i_1)$ with $i_j \neq i_k$,if $j \neq k$, such that $w(\gamma) = \lambda$. Such a sequence γ is then called a critical circuit.
4) There exist d and k_0 such that for all $k \geq k_0$,

$$x(k + d) = \lambda^{\otimes d} \otimes x(k).$$

   If the critical circuit γ is unique then d equals the number of distinct nodes of γ.

REMARKS:

An interpretation of 1) - 4) is that the asymptotic behavior of the system is completely determined by the "slowest" circuit (i.e. the circuit with maximal average weight), other circuits playing no role, after finite time.

If the arc from node j to node i is absent from the system, this can be modelled mathematically by setting $a_{ij} = -\infty$. This is in a certain sense convenient, since $-\infty$ is the zero element of the max-algebra. An interpretation in the production network is that node j does not have to supply node i to start the next activity at node i. If $-\infty$ entries are allowed in A, it is necessary to place an irreducibility assumption on the underlying graph to ensure the validity of the given results.

3. Asymptotic normality

For ease of exposition we assume that $n = 2$, although the results remain in principle valid for general n. Let a random DEDS be given; our notation is as in the preceding section. We define

$$z(k) := x_2(k) - x_1(k), \quad k \geq 0. \tag{3.1}$$

PROPOSITION 1. *The process $\{z(k)\}, k \geq 1$ is a Markov chain. For fixed $z \in \mathbb{R}$, the jump probability measure $P(z, \cdot)$ of this Markov chain is the distribution of the random variable*

$$(a_{21} \oplus (a_{22} \otimes z)) - (a_{11} \oplus (a_{12} \otimes z))$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has the distribution of $A(k)$.

PROOF: We have

$$x_1(k+1) = a_{11}(k) \otimes x_1(k) \oplus a_{12}(k) \otimes x_2(k),$$
$$x_2(k+1) = a_{21}(k) \otimes x_1(k) \oplus a_{22}(k) \otimes x_2(k),$$

so that

$$z(k+1) = a_{21}(k) \oplus a_{22}(k) \otimes z(k) - a_{11}(k) \oplus a_{12}(k) \otimes z(k)$$

and the proposition follows . ∎

Now define

$$d(k) := x_1(k) - x_1(k-1), \quad k \geq 1. \tag{3.2}$$

Then we have

$$x_1(k) = x_1(0) + \sum_{j=1}^{k} d(j), \quad k \geq 1,$$

$$x_2(k) = x_2(0) + (z(k) - z(0)) + \sum_{j=1}^{k} d(j), \quad k \geq 1.$$

PROPOSITION 2. *For each $k \geq 1$, the distribution of $(d(k), z(k))$ given $z(0), d(1), z(1), \ldots, d(k-1), z(k-1)$ depends only on $z(k-1)$. If $z(k-1) = z$ this distribution is equal to the distribution of the random vector $(a_{11} \oplus (a_{12} \otimes z), a_{21} \oplus (a_{22} \otimes z) - a_{11} \oplus (a_{12} \otimes z))$ where*

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

has the distribution of $A(k)$.

PROOF: We have

$$d(k) = a_{11}(k-1) \oplus a_{12}(k-1) \otimes z(k-1) \tag{3.3}$$

which, together with the previous proposition, yields the desired result. ∎

REMARKS:

1. Proposition 2 says that the joint distribution of $(d(k), z(k))$, depends only on the value of $z(k-1)$ and *not* on other values of d or z in the past.
2. For similar results when $n \geq 3$, see [9].

It should be clear now that $x_1(k) - x_1(0)$ is equal in distribution to a sum of random variables with distributions depending only on an underlying Markov chain, and that we are in the situation studied in [8] for discrete state space chains, and in [5] for general state space chains. We shall formulate (a special case of) a theorem of [5] which we shall use. For a definition of uniform Φ-recurrence and a sketch of the proof of this theorem we refer to the appendix.

THEOREM 1. *Suppose that the Markov chain $z(k)$, $k \geq 0$, is aperiodic and uniformly Φ-recurrent, and denote by π its unique invariant probability measure . If the entries of A have finite first moments then*

$$\lim_{k \to \infty} \left(\frac{x_1(k)}{k}, \frac{x_2(k)}{k}\right) = (\mu, \mu) \tag{3.4}$$

exists almost surely for any initial activity time $(x_1(0), x_2(0))$, and we have

$$\mu := E_\pi(d(1)),$$

the expectation of $d(1)$, given that the distribution of $z(0)$ equals π. Moreover, if the entries of A and *the initial activity time $(x_1(0), x_2(0))$ have finite second moments, then*

$$0 \leq \sigma^2 := E_\pi((d(1) - \mu)^2) + 2 \sum_{l=2}^{\infty} E_\pi((d(1) - \mu)(d(l) - \mu)) < \infty$$

and if $\sigma^2 > 0$, then the sequence

$$\frac{(x_1(k), x_2(k)) - k \cdot (\mu, \mu)}{\sigma \cdot \sqrt{k}}, \quad k \geq 1,$$

converges in distribution to the random vector (N, N), where N is a standard normal random variable. ▮

4. Examples

We present three examples illustrating the theory of the preceding section. In all examples the dimension of the system equals two ($n = 2$), and the entries of the matrix $A(k)$ are i.i.d..

1) *Bernoulli delays*

Assume that for each $k \geq 0$, $A(k)$ has independent and identically distributed entries $a_{ij}(k)$, $i, j \in \{1, 2\}$, which take the values 0 or 1 only, each with probability 1/2. For any initial vector $x(0)$ there exists an index k_0 such that for $k \geq k_0$ the Markov chain $z(k)$ takes values in the finite set $\{-1, 0, 1\}$. The transition probabilities on $\{-1, 0, 1\}$ are easily seen from Proposition 1 to be given by the matrix

$$P = \begin{pmatrix} 1/4 & 1/2 & 1/4 \\ 3/16 & 5/8 & 3/16 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}.$$

The Markov chain $z(k)$ is aperiodic (all entries of P are positive) and uniformly Φ-recurrent (the state space is finite). It follows from $\pi^t P = \pi^t$ that the discrete measure π on $\mathbb{R}$ defined by

$$\pi(\{-1\}) = 3/14, \quad \pi(\{0\}) = 8/14, \quad \pi(\{1\}) = 3/14$$

will be the unique invariant probability measure.
From Proposition 2 we find

$$\mu = E_\pi(d(1)) = \frac{3}{14} \cdot \frac{1}{2} + \frac{8}{14} \cdot \frac{3}{4} + \frac{3}{14} \cdot \frac{3}{2} = \frac{6}{7}.$$

It is difficult to calculate σ^2 directly through (3.5), because $E_\pi(d(1) - \mu)(d(l) - \mu)$ involve the evolution of the Markov chain $z(k)$, $k \geq 0$ from time 0 up to time l. However, it is possible to calculate σ^2 via a detour. To this end we first note that in cases where for each k, $a_{ij}(k)$, $i, j \in \{1,2\}$, are i.i.d. it is convenient to work with $\tilde{z}(k)$ and $\tilde{d}(k)$ where

$$\tilde{z}(k) := |x_2(k) - x_1(k)|, \tag{4.1}$$

$$\tilde{d}(k) := x_1(k) \oplus x_2(k) - x_1(k-1) \oplus x_2(k-1). \tag{4.2}$$

The transition probabilities

$$P\{z(k) = i, \tilde{d}(k) = j | z(k-1) = z\}$$

for $z \in \{-1, 0, 1\}$ are given by the table

z \ (i,j)	(-1,0)	(0,0)	(1,0)	(-1,1)	(0,1)	(1,1)
-1	0	1/4	0	1/4	1/4	1/4
0	0	1/16	0	3/16	9/16	3/16
1	0	1/4	0	1/4	1/4	1/4

Of course when we add together the entries with $z(k)$ fixed and $\tilde{d}(k) = 0$ or $\tilde{d}(k) = 1$ we obtain the matrix P. We also note from the table that the row with $z = -1$ is identical to the row with $z = 1$. This implies that $\tilde{z}(k)$ itself is a Markov chain. The transition probabilities $P\{\tilde{z}(k) = i, \tilde{d}(k) = j | \tilde{z}(k-1) = z\}$ for $z \in \{0, 1\}$ are given by the simple table

z \ (i,j)	(0,0)	(0,1)	(1,0)	(1,1)
0	1/16	9/16	0	6/16
1	1/4	1/4	0	1/2

Now we shall show how we can use the more simple variables $\tilde{z}(k)$ and $\tilde{d}(k)$ to calculate σ^2. It follows from Theorem 1 , and the continuous mapping theorem (Theorem 5.1 of [1]) that

$$\frac{\max(x_1(k), x_2(k)) - \mu \cdot k}{\sigma \cdot \sqrt{k}}$$

converges in distribution to a standard normal random variable N. Consequently

$$\lim_{k \to \infty} \operatorname{var}\left(\frac{\max(x_1(k), x_2(k)) - \mu \cdot k}{\sigma \cdot \sqrt{k}}\right) = 1.$$

Also from (4.2) ,

$$\max(x_1(k), x_2(k)) = \max(x_1(0), x_2(0)) + \sum_{j=1}^{k} \tilde{d}(j).$$

These two lines together imply that

$$\sigma^2 = \lim_{k\to\infty} \frac{1}{k} \cdot \mathrm{var}(\sum_{j=1}^{k} \tilde{d}(j)), \tag{4.3}$$

because the result of Theorem 1 is independent of $x(0)$, which value we therefore take equal to 0. The right-hand side of (4.3) can be evaluated in this simple example. To this end define $\delta(k) := (\tilde{z}(k-1), \tilde{z}(k))$, $k \geq 1$. The Markov chain $\delta(k)$, $k \geq 1$, has state space $\{(0,0),(0,1),(1,0),(1,1)\}$ and, according to the simple table above, its transition matrix equals

$$P_\delta = \begin{pmatrix} 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 5/8 & 3/8 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

The invariant probability vector b^t of the matrix P_δ is : $b^t = \frac{1}{14}(5,3,3,3)$. Furthermore it is not difficult to verify that

$$\begin{aligned} f_1 &:= P\{\tilde{d}(k) = 1 | \delta(k) = (0,0)\} = 9/10 \\ f_2 &:= P\{\tilde{d}(k) = 1 | \delta(k) = (0,1)\} = 1 \\ f_3 &:= P\{\tilde{d}(k) = 1 | \delta(k) = (1,0)\} = 1/2 \\ f_4 &:= P\{\tilde{d}(k) = 1 | \delta(k) = (1,1)\} = 1. \end{aligned}$$

Now introduce the limit matrix B of P_δ, defined as the 4×4 matrix with all 4 rows equal to the equilibrium vector b^t, and define the fundamental matrix Z by

$$Z := (I - (P_\delta - B))^{-1},$$

which exists according to Theorem 4.3.1 of [7].

It follows from Theorem 4.6.3 of the same reference (reformulated in our notation) that

$$\begin{aligned} &\lim_{k\to\infty} \frac{1}{k} \mathrm{var}(\sum_{\ell=1}^{k} \tilde{d}(\ell)) \\ &= 2\{\sum_{i,j=1}^{4} f_i f_j b_i (z_{ij} - \delta_{ij})\} - \mu(\mu - 1) \quad , \end{aligned}$$

where δ_{ij} denotes the Kronecker-delta. Easy but tedious calculations show that

$$Z = \frac{1}{49}\begin{pmatrix} 64 & 9 & -12 & -12 \\ -20 & 37 & 16 & 16 \\ 15 & 9 & 37 & -12 \\ -20 & -12 & 16 & 65 \end{pmatrix},$$

and hence $\sigma^2 = 33/343$.

2) *Exponential delays*

Let for each $k \geq 0$, $A(k)$ has independent, identically distributed entries $a_{ij}(k)$, $i,j \in \{1,2\}$, with distribution

$$P\{a_{ij}(k) < x\} = (1 - e^{-\lambda x}) \cdot 1_{(0,\infty)}(x), \quad \lambda > 0.$$

As in the previous example we find that for all $z \geq 0$ and $s \in \mathbb{R}$,

$$P\{z(k) < s | z(k-1) = -z\} = P\{z(k) < s | z(k-1) = z\}$$

This follows from Proposition 1, since the i.i.d. assumptions on a_{ij} imply:

$$\begin{aligned} &a_{21} \oplus (a_{22} \otimes z) - a_{11} \oplus (a_{12} \otimes z) \\ &\stackrel{d}{=} a_{21} \oplus (a_{22} \otimes -z) - a_{11} \oplus (a_{12} \otimes -z), \end{aligned}$$

where $\stackrel{d}{=}$ denotes equality in distribution. Hence in this example $\tilde{z}(k)$ is a Markov chain with state space $[0,\infty)$. We first calculate the transition kernel

$$p(z,[0,s)) := P\{\tilde{z}(k) < s \,|\, \tilde{z}(k-1) = z\}.$$

Let us define random variables u_z and v_z such that

$$\begin{aligned} u_z &\stackrel{d}{=} \max(a_{11}(k), a_{12}(k) - z), \\ v_z &\stackrel{d}{=} \max(a_{21}(k), a_{22}(k) - z). \end{aligned}$$

Then

$$\begin{aligned} &p(z,[0,s)) = P\{|z(k)| < s \,|\, z(k-1) = z\} \\ &= \int_0^s P\{v_z < y + s\} dP\{u_z < y\} \\ &+ \int_s^\infty P\{y - s < v_z < y + s\} dP\{u_z < y\}, \end{aligned}$$

and u_z, v_z have common distribution:

$$P\{u_z < y\} = (1 - e^{-\lambda y})(1 - e^{-\lambda(y+z)}), \quad y \geq 0.$$

Hence we find

$$(4.4) \qquad \begin{aligned} p(z,[0,s)) = 1 - e^{-\lambda s} & \\ - \frac{2}{3}e^{-\lambda(s+z)} + \frac{2}{3}e^{-\lambda(2s+z)} & \\ + \frac{1}{3}e^{-\lambda(s+2z)} - \frac{1}{3}e^{-\lambda(2s+2z)}, & \quad s, z \geq 0. \end{aligned}$$

LEMMA 1. *The Markov chain $\tilde{z}(k), k \geq 0$, is aperiodic and uniformly Φ-recurrent.*

PROOF: Aperiodicity is clear. To prove uniform Φ-recurrence we use Theorem A-1 (i) of the appendix. Let $\Phi(A) := \int_A \lambda e^{-\lambda s} ds$. Then it is easily seen, using the inequalities

$$\begin{aligned} & 0 \leq \frac{2}{3}e^{-\lambda z} - \frac{1}{3}e^{-2\lambda z} \leq \frac{1}{3}, && \forall z \in [0,\infty), \\ & |1 - 2e^{-\lambda s}| \leq 1, && \forall s \in [0,\infty), \end{aligned}$$

that

$$\begin{aligned} p(z,A) &= \int_A \frac{d}{ds} p(z,[0,s))ds \\ &= \int_A \lambda e^{-\lambda s}(1 + (\frac{2}{3}e^{-\lambda z} - \frac{1}{3}e^{-2\lambda z})(1 - 2e^{-\lambda s}))ds \\ &\geq \frac{2}{3} \int_A \lambda e^{-\lambda s} ds \\ &= \frac{2}{3}\Phi(A). \end{aligned}$$

Thus, for $0 < \epsilon < \frac{2}{3}\Phi(A)$, we have ${}_AP^{(1)}(z,A) = p(z,A) > \epsilon$ uniformly in z. ∎

It is evident from the definition $\tilde{z}(k) = |z(k)|$, $k \geq 0$, that the conclusion of Lemma 1 also holds for $z(k)$. Hence the assumptions of Theorem 1 are satisfied. To calculate $\mu = E_\pi d(1)$ observe from the derivation in the previous example that

$$\mu = E_{\tilde{\pi}}(\tilde{d}(1)),$$

where $\tilde{\pi}$ is the invariant probability measure of $\tilde{z}(k)$ and with $\tilde{d}(k)$ defined by (4.2). The invariant probability measure of $\tilde{z}(k)$ is given by the unique distribution $\tilde{\pi}(\cdot)$ for which, for all $s \in [0, \infty)$,

$$\tilde{\pi}(s) = \int_0^\infty p(z, [0, s)) d\tilde{\pi}(z).$$

So, defining the Laplace-Stieltjes transform of $\tilde{\pi}(\cdot)$ by

$$\hat{\pi}(\rho) := \int_0^\infty e^{-\rho s} d\tilde{\pi}(s),$$

we get

$$\hat{\pi}(\rho) = \int_0^\infty (\int_0^\infty e^{-\rho s} \frac{d}{ds} p(z, [0, s)) ds) d\tilde{\pi}(z).$$

Using (4.4) we find

$$\begin{aligned}
&\int_0^\infty e^{-\rho s} \frac{d}{ds} p(z, [0, s)) ds \\
&\quad = \frac{\lambda}{\lambda + \rho} + \frac{2}{3} \cdot \frac{\lambda}{\lambda + \rho} e^{-\lambda z} - \frac{4}{3} \cdot \frac{\lambda}{2\lambda + \rho} e^{-\lambda z} \\
&\quad - \frac{1}{3} \cdot \frac{\lambda}{\lambda + \rho} e^{-2\lambda z} + \frac{2}{3} \cdot \frac{\lambda}{2\lambda + \rho} e^{-2\lambda z},
\end{aligned}$$

and hence

$$\begin{aligned}
\hat{\pi}(\rho) = & \frac{\lambda}{\lambda + \rho} + \frac{2}{3} \cdot \frac{\lambda}{\lambda + \rho} \hat{\pi}(\lambda) - \frac{4}{3} \cdot \frac{\lambda}{2\lambda + \rho} \hat{\pi}(\lambda) \\
& - \frac{1}{3} \cdot \frac{\lambda}{\lambda + \rho} \hat{\pi}(2\lambda) + \frac{2}{3} \cdot \frac{\lambda}{2\lambda + \rho} \hat{\pi}(2\lambda).
\end{aligned}$$

Substituting $\rho = \lambda$ and $\rho = 2\lambda$ gives

$$\hat{\pi}(\lambda) = 53/114, \quad \hat{\pi}(2\lambda) = 17/57,$$

and so

$$\hat{\pi}(\rho) = \frac{23}{19} \cdot \frac{\lambda}{\lambda + \rho} - \frac{8}{19} \cdot \frac{\lambda}{2\lambda + \rho},$$

or equivalently,

$$\tilde{\pi}(s) = 1 - \frac{23}{19} e^{-\lambda s} + \frac{4}{19} e^{-2\lambda s}.$$

Using that, under the condition $\tilde{z}(0) = z$,

$$\tilde{d}(1) \stackrel{d}{=} \max(a_{11}(0), a_{21}(0), a_{12}(0) - z, a_{22}(0) - z),$$

we find after some calculations

$$\begin{aligned}\mu = E_{\tilde{\pi}}(\tilde{d}(1)) &= \int_0^{\infty} E(\tilde{d}(1)|\tilde{z}(0) = z)d\tilde{\pi}(z)\\ &= \int_0^{\infty} \frac{1}{\lambda}(\frac{3}{2} + \frac{2}{3}e^{-\lambda z} - \frac{1}{12}e^{-2\lambda z})d\tilde{\pi}(z)\\ &= \frac{1}{\lambda} \cdot \frac{407}{228}.\end{aligned}$$

3) *Uniform delays*

Let $a_{ij}(k)$ be mutually independent random variables uniformly distributed on the interval [0,1]. Once again we consider the Markov chain $\tilde{z}(k) := |x_2(k) - x_1(k)|$ and calculate the transition kernel

$$p(z,[0,s)) = P\{\tilde{z}(k) < s|\tilde{z}(k-1) = z\}, \quad 0 \leq z, s \leq 1.$$

Let us define u_z, v_z as in the previous example. Then one can easily check that the joint probability density ρ_z of (u_z, v_z) is given by

$$\rho_z(u,v) = \begin{cases} (2u+z)(2v+z), & 0 < u < 1-z, 0 < v < 1-z \\ 2u+z, & 0 < u < 1-z, 1-z < v < 1 \\ 2v+z, & 1-z < u < 1, 0 < v < 1-z \\ 1, & 1-z < u < 1, 1-z < v < 1. \end{cases}$$

Furthermore we have the following relation between the transition density $\frac{d}{ds}p(z,[0,s))$ and ρ_z:

$$\frac{d}{ds}p(z,[0,s)) = 2 \cdot \int_0^{1-s} \rho_z(u, u+s)du.$$

For the calculation of this density we have to consider the following cases:

case A: $0 \leq s \leq \min(z, 1-z)$

$$\begin{aligned}&\frac{d}{ds}p(z,[0,s))\\ &\quad = \frac{8}{3} - 2z + 2z^2 - \frac{2}{3}z^3 - 2s + 2sz - 2sz^2 - 2s^2 + \frac{4}{3}s^3.\end{aligned}$$

case B: $z \leq s \leq 1-z$

$$\frac{d}{ds}p(z,[0,s)) = \frac{8}{3} + 2z^2 - \frac{2}{3}z^3 - 4s - 2sz^2 + \frac{4}{3}s^3$$

case C: $1-z \leq s \leq z$

$$\frac{d}{ds}p(z,[0,s)) = 2 - 2s$$

case D: $\max(z, 1-z) \leq s \leq 1$

$$\frac{d}{ds}p(z,[0,s)) = 2 - 4s + 2s^2 + 2z - 2zs.$$

LEMMA 2. *The Markov chain $\tilde{z}(k), k \geq 0$, is aperiodic and uniformly Φ-recurrent.*

PROOF: Aperiodicity is clear. For the proof of uniform Φ-recurrence we will use the following

CLAIM. *Define*

$$f(s) := \begin{cases} 2(1-s), & 0 \leq s \leq -\frac{1}{2} + \frac{1}{2}\sqrt{3} \\ \frac{8}{3} - 4s + \frac{4}{3}s^3, & -\frac{1}{2} + \frac{1}{2}\sqrt{3} \leq s \leq 1. \end{cases}$$

Then for all $z \in [0,1]$ we have $\frac{d}{ds}p(z,[0,s)) \geq f(s)$.

The claim follows from easy, but tedious calculations in the four cases A, B, C and D.

At this point we have found a function $f(s)$, continuous on $[0,1]$, and positive on $(0,1)$, such that $f(s) \leq \frac{d}{ds}p(z,[0,s))$ for all $z \in [0,1]$, and all $s \in (0,1)$. Now define on the Borel subsets of $(0,1)$,

$$\Phi(A) := \int_A f(s)ds.$$

Using this measure Φ it is easily checked that the Markov chain $\tilde{z}(k), k \geq 1$, is uniformly Φ-recurrent. ∎

We now proceed as in example 2 and calculate the invariant probability measure $\tilde{\pi}$ of $\tilde{z}(k)$ and from this the expectation $E_{\tilde{\pi}}(\tilde{d}(1))$. Let us denote the density of the stationary distribution of $\tilde{z}(k)$ by g. To obtain a numerical approximation for g we will subdivide cases A, B, C, D once more. We want to distinguish between $s \leq \frac{1}{2}$ and $s \geq \frac{1}{2}, z \leq \frac{1}{2}$ and $z \geq \frac{1}{2}$. Hence we get eight subcases as shown in Figure 1.

Furthermore, for convenience of notation we replace z by $1-z$ if $\frac{1}{2} \leq z \leq 1$ and s by $1-s$ if $\frac{1}{2} \leq s \leq 1$.

If we use these replacements, calculations yield the following eight polynomials for the densities of the transition probabilities :

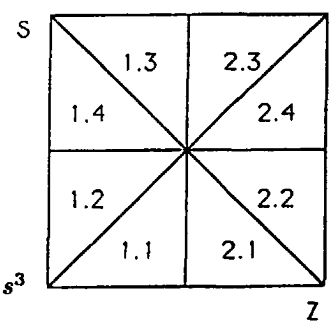

Figure 1

1.1 $\frac{8}{3} - 2z + 2z^2 - \frac{2}{3}z^3 - 2s + 2sz - 2sz^2 - 2s^2 + \frac{4}{3}s^3$

1.2 $\frac{8}{3} + 2z^2 - \frac{2}{3}z^3 - 4s - 2sz^2 + \frac{4}{3}s^3$

1.3 $2sz + 2s^2$

1.4 $-\frac{2}{3}z^3 + 2sz^2 + 4s^2 - \frac{4}{3}s^3$

2.1 $2 + \frac{2}{3}z^3 - 2s + 2sz - 2sz^2 - 2s^2 + \frac{4}{3}s^3$

2.2 $2 - 2s$

2.3 $2s - 2zs + 2s^2$

2.4 $2s$

In order to find the stationary density g of $\tilde{z}(k)$, i.e. the normalized solution of

$$g(s) = \int_0^1 \frac{d}{ds} p(z, [0, s)) g(z) dz \quad ,$$

we put

$$g_0(s) := g(s), \quad 0 \leq s \leq \frac{1}{2},$$
$$g_1(s) := g(1-s), \quad 0 \leq s \leq \frac{1}{2}.$$

Then, if we denote by $P_{ij}(z, s)$, $i = 1, 2$, $j = 1, 2, 3, 4$ the polynomial of subcase $i.j$, g_0 and g_1 must, for $0 \leq s \leq \frac{1}{2}$, satisfy the equations

(4.5)
$$\begin{aligned} g_0(s) = &\int_0^s P_{12}(z,s) g_0(z) dz + \int_s^{\frac{1}{2}} P_{11}(z,s) g_0(z) dz \\ &+ \int_0^s P_{22}(z,s) g_1(z) dz + \int_s^{\frac{1}{2}} P_{21}(z,s) g_1(z) dz, \end{aligned}$$

(4.6)
$$\begin{aligned} g_1(s) = &\int_0^s P_{14}(z,s) g_0(z) dz + \int_s^{\frac{1}{2}} P_{13}(z,s) g_0(z) dz \\ &+ \int_0^s P_{24}(z,s) g_1(z) dz + \int_s^{\frac{1}{2}} P_{23}(z,s) g_1(z) dz. \end{aligned}$$

Now assume

(4.7)
$$g_0(s) = \sum_{n \geq 0} c_n s^n,$$
$$g_1(s) = \sum_{n \geq 0} d_n s^n.$$

From (4.5), (4.6) and (4.7) and the formulas for

$P_{ij}(z,s)$ we can deduce the following recurrence relations for c_n and d_n:

$$\begin{aligned} c_n &= c_{n-2}(\frac{-2}{n(n-1)}) + c_{n-3}(\frac{2}{(n-1)(n-2)}) \\ &+ d_{n-3}(\frac{2}{(n-1)(n-2)}) + d_{n-4}(\frac{-2(n+1)}{n(n-1)(n-3)}), \\ d_n &= c_{n-3}(\frac{2}{(n-1)(n-2)}) + c_{n-4}(\frac{-2(n+1)}{n(n-1)(n-3)}) \\ &+ d_{n-3}(\frac{-2}{(n-1)(n-2)}), \quad n \geq 4. \end{aligned}$$

Using $\tilde{c}_n := c_n \cdot n!$ and $\tilde{d}_n := d_n \cdot n!$ instead of c_n and d_n gives, for $n \geq 4$,

$$\begin{aligned} \tilde{c}_n &= -2\tilde{c}_{n-2} + 2n\tilde{c}_{n-3} \\ &\quad + 2n\tilde{d}_{n-3} - 2(n+1)(n-2)\tilde{d}_{n-4}, \\ \tilde{d}_n &= 2n\tilde{c}_{n-3} - 2n\tilde{d}_{n-3} - 2(n+1)(n-2)\tilde{c}_{n-4}. \end{aligned} \tag{4.8}$$

For the terms $\tilde{c}_n$ and $\tilde{d}_n$, $n = 0,1,2,3$ we find

$$\begin{aligned} \tilde{c}_0 &= \frac{2}{3}(4 - 3\beta^{(1)} + 3\beta^{(2)} - \beta^{(3)}), & \tilde{d}_0 &= 0, \\ \tilde{c}_1 &= -2 + 2\beta^{(1)} - 2\beta^{(2)}, & \tilde{d}_1 &= 2\beta^{(1)}, \\ \tilde{c}_2 &= -4 + 2\tilde{c}_0, & \tilde{d}_2 &= 4, \\ \tilde{c}_3 &= 8 - 2\tilde{c}_1 + 6\tilde{c}_0, & \tilde{d}_3 &= 6\tilde{c}_0, \end{aligned} \tag{4.9}$$

where $\beta^{(n)}$ is the n-th moment of the stationary distribution of $\tilde{z}(k)$, i.e.

$$\beta^{(n)} := \int_0^1 z^n g(z)dz. \tag{4.10}$$

To calculate the mean cycletime μ we use that, under the condition $\tilde{z}(0) = z$,

$$\tilde{d}(1) \stackrel{d}{=} \max(a_{11}(0), a_{21}(0), a_{12}(0) - z, a_{22}(0) - z),$$

with $\tilde{d}(1)$ as defined in (4.2). Hence,

$$P(\tilde{d}(1) < y | \tilde{z}(0) = z) = \begin{cases} y^2(y+z)^2, & 0 \leq y \leq 1 - z, \\ y^2, & 1 - z \leq y \leq 1, \end{cases}$$

and so

$$\begin{aligned} E(\tilde{d}(1) \mid \tilde{z}(0) = z) &= \int_0^1 y \, d\, P(\tilde{d}(1) < y \mid \tilde{z}(0) = z) \\ &= \int_0^{1-z} (4y^4 + 6y^3 z + 2y^2 z^2) dy \\ &+ \int_{1-z}^1 2y^2 dy \\ &= \frac{4}{5} - \frac{1}{2}z + \frac{2}{3}z^2 - \frac{1}{3}z^3 + \frac{1}{30}z^5 \quad . \end{aligned}$$

It follows that

$$\mu = \frac{4}{5} - \frac{1}{2}\beta^{(1)} + \frac{2}{3}\beta^{(2)} - \frac{1}{3}\beta^{(3)} + \frac{1}{30}\beta^{(5)}.$$

From (4.7) and (4.10) we have

$$\beta^{(i)} = \int_0^{\frac{1}{2}} s^i \sum_{n \geq 0} c_n s^n ds + \int_0^{\frac{1}{2}} (1-s)^i \sum_{n \geq 0} d_n s^n ds.$$

Using this relation together with the recurrence relations (4.8) and the initial conditions (4.9) we can calculate $\beta^{(i)}$ numerically for $i = 1, 2, 3, 5$. This yields

$$\beta^{(1)} = .284,\ \beta^{(2)} = .124,\ \beta^{(3)} = .067,\ \beta^{(5)} = .027.$$

So for the mean cycletime μ we obtain

$$\mu = .719.$$

An alternative way of calculating μ can be obtained from discrete approximations of the uniform distribution. Let us assume that for each k, $a_{ij}(k)$ are mutually independent with distribution

$$P(a_{ij}(k) = \ell/(m-1)) = 1/m,$$
$$\text{for } \ell = 0, 1, \ldots, m-1, \quad m \in \mathbb{N}, m \geq 2.$$

Then $\tilde{z}(k)$, $k \geq 0$, is a Markov chain with state space $S = \{\ell/(m-1), \ell = 0, 1, \ldots, m-1\}$. The transition probabilities $p_{j\ell}$ of this Markov chain are given by

$$p_{j\ell} := P(\tilde{z}(k) = \ell/(m-1) | \tilde{z}(k-1) = j/(m-1))$$
$$= \begin{cases} \displaystyle\sum_{h=0}^{m-1} b_{jh}^2 & , \quad \ell = 0, \\ 2 \cdot \displaystyle\sum_{h=0}^{m-\ell-1} b_{jh} \cdot b_{j,h+\ell}, & \ell \neq 0, \end{cases}$$

where

$$b_{jh} := P(x_1(k) = x + \frac{m-h-1}{m-1} |$$
$$\max(x_1(k-1), x_2(k-1)) = x, \tilde{z}(k-1) = j/(m-1)).$$

Easy calculations show that

$$b_{j,h} = \begin{cases} 1/m, & j > h, \\ \dfrac{j+1+2(m-h-1)}{m^2}, & j \leq h. \end{cases}$$

Further

$$f_{j\ell} := E(\tilde{d}(1) | \tilde{z}(0) = j/(m-1), \tilde{z}(1) = \ell/(m-1))$$
$$= \frac{1}{m-1} (\sum_{h=\ell}^{m-1} h \cdot b_{j,m-h-1} \cdot b_{j,m-h+\ell-1}) /$$
$$(\sum_{h=0}^{m-\ell-1} b_{jh} \cdot b_{j,\ell+h}).$$

Now fix m; from the transition probabilities $p_{j\ell}$ it is possible to calculate numerically the stationary distribution $\{\tilde{\pi}_j : j = 0, \ldots, m-1\}$ of $\tilde{z}(k)$ and from this

$$\mu = \sum_{j=0}^{m-1} \sum_{\ell=0}^{m-1} \tilde{\pi}_j p_{j\ell} f_{j\ell}.$$

In the next table we show for increasing m the approximate values of μ.

m	2	5	10	15	20	25	50	75
μ	.8571	.7661	.7414	.7337	.7299	.7276	.7232	.7217

5. Expectation of cycle times

In this section the emphasis will be on mean cycle times. As in the previous sections we shall consider 2-dimensional systems.

In subsection 5.1 we assume that $a_{21}(k) = -\infty$. We show an example where the state space of the Markov chain $z(k) = x_2(k) - x_1(k)$ becomes countably infinite and where no invariant probability measure exists for $z(k)$.

In subsection 5.2 we show that a main result from the theory of deterministic DEDS, i.e. the fact that the slowest circuit in the network determines the asymptotic behavior of the system, does not necessarily remain true for random DEDS.

5.1 Reducible systems

In this subsection we assume that $P(a_{21}(k) = -\infty) = 1$ while the other entries are real-valued and finite with probability one . The system description then becomes

$$\begin{aligned} x_1(k+1) &= a_{11}(k) \otimes x_1(k) \oplus a_{12}(k) \otimes x_2(k) \\ x_2(k+1) &= a_{22}(k) \otimes x_2(k). \end{aligned} \tag{5.1}$$

Instead of $d(k)$ given by (3.2), we use once again (see (4.2)),

$$\tilde{d}(k) = x_1(k) \oplus x_2(k) - x_1(k-1) \oplus x_2(k-1). \tag{5.2}$$

The reason for this is that $\lim_{k\to\infty} \frac{x_1(k)}{k}$ is not necessarily equal to $\lim_{k\to\infty} \frac{x_2(k)}{k}$ for systems of the form (5.1), because Theorem 1 is no longer valid.

In example 1 the state space of $z(k) = x_2(k) - x_1(k)$ remains finite, so that $\mu = E_\pi(\tilde{d}(1))$ can be computed as before . In example 2 the state space of $z(k)$ becomes countable infinite. Depending on a parameter p the Markov chain $z(k)$ will be positive recurrent, null-recurrent or transient. Only in the first case it is possible to obtain the mean cycle time from the invariant probability measure.

Example 1

Consider the following distributions of the transition times.

$$\begin{aligned} &P(a_{11}(k) = 0) = P(a_{11}(k) = 1) = \frac{1}{2} \\ &P(a_{12}(k) = 0) = P(a_{12}(k) = 1) = \frac{1}{2} \\ &P(a_{21}(k) = -\infty) = 1 \\ &P(a_{22}(k) = 1) = P(a_{22}(k) = 2) = \frac{1}{2}. \end{aligned}$$

In the stationary situation the state space of $z(k)$ equals $\{0, 1, 2\}$ and the Markov transition matrix is given by:

$$P = \begin{pmatrix} 3/8 & 1/2 & 1/8 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & 1/4 \end{pmatrix} \quad ,$$

which has as invariant probability vector:

$$\pi = (4/14, 7/14, 3/14).$$

We find in this case: $E_\pi(\tilde{d}(1)) = 3/2$. Note that this value is equal to $E(a_{22})$. This is not surprising, since in equilibrium $P(x_2(k) \geq x_1(k)) = 1$ and hence from (5.2): $\tilde{d}(k) = x_2(k) - x_2(k-1) = a_{22}(k-1)$.

Example 2

Consider the following distributions of the transition times, $p \in (0, 1)$

$$P(a_{11}(k) = 0) = 1 - P(a_{11}(k) = 1) = 1 - p,$$
$$P(a_{12}(k) = 0) = P(a_{12}(k) = 1) = 1/2$$
$$P(a_{21}(k) = -\infty) = 1$$
$$P(a_{22}(k) = 0) = P(a_{22}(k) = 1) = 1/2.$$

In the stationary situation the Markov chain $z(k)$, $k \geq 0$, has state space $\{1, 0, -1, -2, \ldots\}$, and transition matrix given by

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \cdots \\ \frac{1-p}{4} & \frac{1}{2} & \frac{1+p}{4} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1-p}{2} & \frac{1}{2} & \frac{p}{2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1-p}{2} & \frac{1}{2} & \frac{p}{2} & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

The Markov chain is positive recurrent for $p < \frac{1}{2}$, null recurrent for $p = \frac{1}{2}$ and transient (drifts away to $-\infty$) for $p > \frac{1}{2}$. Only in the first case a unique invariant probability distribution exists. Some calculations show that this distribution is given by:

$$\pi_1 = \frac{1-2p}{2(3-p)},$$
$$\pi_0 = \frac{3}{1-p}\pi_1,$$
$$\pi_{-j} = \frac{2+p}{(1-p)^2}\left(\frac{p}{1-p}\right)^{j-1}\pi_1, \quad j = 1, 2, \ldots$$

Using this distribution we find

$$E_\pi(\tilde{d}(1)) = \frac{1}{2} \quad , \tag{5.3}$$

which value is independent of p.

For $p > \frac{1}{2}$ the chain is transient. The Markov chain $z(k)$ drifts away to $-\infty$ and hence for k large enough $\tilde{d}(k) = x_1(k) - x_1(k-1)$. Consequently

$$E(\tilde{d}(k)) = E(x_1(k) - x_1(k-1)) = E(a_{11}(k)) = p \quad . \tag{5.4}$$

This also holds for $p = \frac{1}{2}$, because for a null-recurrent Markov chain we have for all states $\lim_{k\to\infty} P(z(k) = j) = 0$ and hence $\lim_{k\to\infty} P(z(k) \in \{1, 0, -1\}) = 0$. The following intuitive explanation of the answer in (5.3) can be given. Since μ can be expected to be increasing in p, we conclude from (5.4) that $\mu \leq 1/2$ for $p < \frac{1}{2}$. However, μ can not be smaller than $E(x_2(k) - x_2(k-1))$ and hence also $\mu \geq 1/2$.

5.2 The slowest circuit

In section 2 it was pointed out that for deterministic DEDS the asymptotic behavior of the system is completely determined by the slowest circuit in the network. For random DEDS this is in general not the case. We shall show that the mean cycle time of the process is at least equal to the maximum of the average weights of the circuits. With some examples it will be shown that equality holds only for very few cases.

Let again $A(k) = \begin{pmatrix} a_{11}(k) & a_{12}(k) \\ a_{21}(k) & a_{22}(k) \end{pmatrix}$ be a sequence of i.i.d. real-valued random matrices and let $d(k) = x_1(k) - x_1(k-1)$.

PROPOSITION 3. *Suppose that the Markov chain $z(k)$, $k \geq 0$, is aperiodic and uniformly Φ-recurrent, and that the entries of $A(k)$ have finite first moment. The mean cycle time μ satisfies,*

$$\mu \geq \max\{E(a_{11}), E(a_{22}), E(\frac{a_{12} \otimes a_{21}}{2})\} \tag{5.5}$$

where a_{ij} denotes a random variable with the same distribution as $a_{ij}(k)$.

PROOF: We have $d(k+1) = a_{11}(k) \oplus (a_{12}(k) \otimes z(k))$ (see (3.3)) and hence $\mu \geq E(a_{11})$. According to (3.4), we also have $\mu = E_\pi(x_2(k+1) - x_2(k))$, and so by symmetry $\mu \geq E(a_{22})$. From the system equations it can be derived that

$$\begin{aligned} x_1(k+1) - x_1(k-1) = & (a_{11}(k) \otimes a_{11}(k-1)) \\ & \oplus (a_{11}(k) \otimes a_{12}(k-1) \otimes z(k-1)) \\ & \oplus (a_{12}(k) \otimes a_{21}(k-1)) \\ & \oplus (a_{12}(k) \otimes a_{22}(k-1) \otimes z(k-1)) \end{aligned}$$

This implies that $\mu \geq E(\frac{a_{12} \otimes a_{21}}{2})$ ∎

We shall now present some examples. Only in the first example equality in (5.5) holds. In all examples we assume the entries of $A(k)$ to be mutually independent.

Example 3

Let

$$\begin{aligned} P(a_{11} = 2) &= P(a_{11} = 3) = 1/2 \\ P(a_{12} = 2) &= P(a_{12} = 3) = 1/2 \\ P(a_{21} = 0) &= P(a_{21} = 1) = 1/2 \\ P(a_{22} = 0) &= P(a_{22} = 1) = 1/2 \end{aligned}$$

It can be computed that $\mu = E(a_{11}) = \frac{5}{2}$. This can be explained as follows. The Markov chain $z(k)$ has as its state space in the stationary situation $\{-1, -2, -3\}$ and thus $a_{11}(k) \oplus a_{12}(k) \otimes z(k)$ is always equal to $a_{11}(k)$ in stationary situation. Then, according to (3.3), $d(k+1) = a_{11}(k)$ for all k and thus $\mu = E(a_{11})$. The above property holds in general if
$P(a_{11} \geq a_{22}; a_{11} \geq \frac{a_{12} \otimes a_{21}}{2}) = 1$. ∎

Example 4

Let

$$\begin{aligned} P(a_{11} = 1) &= P(a_{11} = 2) = 1/2 \\ P(a_{12} = 1) &= P(a_{12} = 2) = 1/2 \\ P(a_{21} = 0) &= P(a_{21} = 1) = 1/2 \\ P(a_{22} = 0) &= P(a_{22} = 1) = 1/2 \end{aligned}$$

Just as in the previous example we have that $E(a_{11}) > E(a_{22})$ and $E(a_{11}) > E(\frac{a_{12} \otimes a_{21}}{2})$. Some computations show however that $\mu = \frac{53}{34}$, which is larger than $E(a_{11}) = \frac{3}{2}$. This can be explained by the fact that in this case $P(a_{11} \geq \frac{a_{12} \otimes a_{21}}{2}) < 1$. The state space of $z(k)$ is equal to $\{-2, -1, 0\}$ in the stationary situation and thus $P(a_{12}(k) \otimes z(k) > a_{11}(k)) > 0$. This implies that $\mu > E(a_{11})$. ∎

Example 5

Let

$$\begin{aligned} P(a_{11} = 0) = P(a_{11} = 1) = 1/2 \\ P(a_{12} = 1) = P(a_{12} = 2) = 1/2 \\ P(a_{21} = 1) = P(a_{21} = 2) = 1/2 \\ P(a_{22} = 0) = P(a_{22} = 1) = 1/2. \end{aligned}$$

In this case the circuit with maximal average weight is : $1 \to 2 \to 1$ and this average weight equals

$$\frac{1}{2} \cdot E(a_{12} \otimes a_{21}) = \frac{3}{2}.$$

Furthermore $P(\frac{a_{12} \otimes a_{21}}{2} \geq a_{11}; \frac{a_{12} \otimes a_{21}}{2} \geq a_{22}) = 1$ (cf. ex. 3). The Markov chain z(k) has in the stationary situation as state space $\{-2, -1, 0, 1, 2\}$, while its invariant probability distribution π is given by

$$\begin{aligned} \pi(-2) = \pi(2) = 1/24 \\ \pi(-1) = \pi(1) = 1/4 \\ \pi(0) = 5/12. \end{aligned} \tag{5.6}$$

From (5.6) we conclude that with positive probability the term $a_{11}(k) \otimes a_{12}(k-1) \otimes z(k-1)$ is larger than the term $a_{12}(k) \otimes a_{21}(k-1)$. Hence μ will be larger than $E(\frac{a_{12} \otimes a_{21}}{2})$. Some calculations show that $\mu = 77/48$ which is indeed larger than $3/2$. The crucial point in this example is that the Markov chain $z(k)$ can, with positive probability, get into states, in which the "faster" circuits do influence the behavior of the system. ∎

In the previous examples it was shown that in contrast to the theory on deterministic DEDS the asymptotic behavior of random DEDS is not necessarily determined by the slowest circuit only.

Appendix

1. *Uniform Φ-recurrence*

A Markov chain $(X_k)_{k \geq 0}$ with state space $\mathbb{R}$ is called uniformly Φ- recurrent if there exists a σ-finite measure Φ on the Borel sets $\mathcal{B}$ of $\mathbb{R}$ such that for each $A \in \mathcal{B}$ with $\Phi(A) > 0$

$$\sum_{m=1}^{k} {}_A P^m(x, A) \to 1 \qquad (k \to \infty)$$

uniformly in x, where for $A, B \in \mathcal{B}$, ${}_BP^m(x, A)$ is defined as the taboo probability

$${}_BP^m(x,A) := P\{X_m \in A, X_i \notin B, 1 \le i \le m-1 | X_0 = x\}.$$

THEOREM A-1.

(i) *Suppose a Markov chain with state space $\mathbb{R}$ satisfies the following condition: for each Borel set A with $\Phi(A) > 0$ there exist $k > 0, \epsilon > 0$ such that $\sum_{m=1}^{k} {}_AP^m(x, A) > \epsilon$ for all $x \in \mathbb{R}$. Then the chain is uniformly Φ-recurrent.*

(ii) *A uniformly Φ-recurrent chain has an invariant probability measure π. Moreover, there exist a finite constant a and a number $\rho < 1$ such that for each initial probability measure μ,*

$$\| (\mu - \pi)P^k \| \le a \cdot \rho^k,$$

if the chain is aperiodic. Here $\| \cdot \|$ denotes the total variation norm.

PROOF: See [10]. ∎

2. Central limit theorem for stationary mixing processes

For a stationary sequence $\xi_1, \xi_2, \ldots$ of random variables on some basic space $(\Omega, \mathcal{F}, P)$ we define $\mathcal{F}_{1,n}$ as the σ-field generated by $\xi_1, \ldots, \xi_n$ and $\mathcal{F}_{n,\infty}$ as the σ-field generated by $\xi_n, \xi_{n+1}, \ldots$. Let $\phi : \mathbb{N} \to [0, \infty)$ be a given function. We call the sequence $\xi_1, \xi_2, \ldots$, ϕ-mixing if $n \ge 1, k \ge 1, E_1 \in \mathcal{F}_{1,n}$ and $E_2 \in \mathcal{F}_{n+k,\infty}$ together imply

$$|P(E_1 \cap E_2) - P(E_1) \cdot P(E_2)| \le \phi(k) \cdot P(E_1).$$

THEOREM A-2. *Suppose that $\{\xi_k\}$ is ϕ-mixing with $\sum_{k=1}^{\infty} \sqrt{\phi(k)} < \infty$ and that $E\xi_1 = 0, E\xi_1^2 < \infty$. Then the series*

$$\sigma^2 = E(\xi_1^2) + 2 \sum_{l=2}^{\infty} E(\xi_1 \xi_l)$$

converges absolutely; if $\sigma^2 > 0$ then $X_k := S_k / \sigma\sqrt{k}$, where $S_k = \xi_1 + \xi_2 + \cdots + \xi_k$, converges in distribution to a standard normal random variable N.

PROOF: See [1], Theorem 20.1.

Theorem 20.1 actually quotes that $S_{[kt]}/\sigma\sqrt{k}$, $0 \leq t \leq 1$, converges in distribution to standard Brownian motion on [0,1]. Theorem A-2 follows after applying the continuous mapping theorem (Theorem 5.1 of [1]) to the projection at time $t = 1$. Observe that via the continuous mapping theorem many other similar results can be obtained. ∎

3. Proof of Theorem 1 of §3

The pair $(x_1(k), x_2(k))$ can be written as

$$(x_1(k), x_2(k)) =$$
$$(x_1(0) + \sum_{j=1}^{k} d(j), x_2(0) + z(k) - z(0) + \sum_{j=1}^{k} d(j)).$$

Since the Markov chain $z(k)$ is uniformly Φ-recurrent we have $k^{-1}(z(k) - z(0)) \to 0$, almost surely, and hence (3.4) is a direct consequence of Theorem 1 of [5]. In order to prove the remainder of the theorem we would like to apply Theorem A.2 to the sequence $\{d(k) - \mu\}$, $k = 1, 2, \cdots$. This involves two difficulties:

(i) The sequence $\{d(k)\}_{k \geq 1}$ is not stationary, because the initial distribution of $z(0)$ is in general not equal to the invariant measure π.

(ii) What conditions on $z(k)$ should be imposed to ensure the ϕ-mixing condition with a function ϕ that decreases so rapidly that $\sum \sqrt{\phi(k)} < \infty$?

The answer to both questions was given in the paper of Grigorescu and Oprişan [5]. Theorem A-1 (ii) shows that, if $z(k)$ is aperiodic and uniformly Φ-recurrent, it has a unique stationary probability measure π, such that for each bounded measurable function f on $\mathbb{R}$ and for all $y \in \mathbb{R}$, there exist a constant $C > 0$ and a real number $\rho \in (0,1)$ with

$$\left| \int_{\mathbb{R}} f(x) P^k(y, dx) - \int_{\mathbb{R}} f(x)\pi(dx) \right| \leq (\sup f) \cdot C \cdot \rho^k.$$

According to the proof of [5] on page 68 this shows that , if the initial distribution of $z(k)$ equals π, the stationary sequence $\{d(k)\}_{k \geq 1}$ is ϕ-mixing with $\phi(k) = C \cdot \rho^k$. This answers question (ii), since obviously $\sum \sqrt{\phi(k)} < \infty$.

The answer to question (i) is rather technical. Although $d(k)$ is not stationary, we have seen above that the distribution of $z(k)$ converges geometrically fast to its stationary distribution π. Hence the trick is to introduce a sequence $\{p_k\}$ of integers going to infinity slowly enough to allow $p_k/\sqrt{k} \to 0$. The section

$$(\sigma\sqrt{k})^{-1} \cdot \sum_{l \leq p_k} d(l)$$

will not influence the asymptotic behaviour of

$$(\sigma\sqrt{k})^{-1} \sum_{l=1}^{k} d(l),$$

whereas for $l > p_k$ the distribution of $z(l)$ is sufficiently close to the stationary distribution π to ensure Theorem A-2 to hold through. Precise mathematical details are in the proof on page 70 of [5].

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Strong law and central limit theorem for a process between maxima and sums.

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1. Introduction. Consider the iterative scheme

$$\begin{cases} X_1 = 0 \\ X_{n+1} = \max\,(X_n, \alpha_n X_n + Y_n), \quad n \geq 1, \end{cases} \tag{1.1}$$

where $Y_1, Y_2, \dots$ are i.i.d. random variables in $\mathbb{R}$ with common distribution F and $\alpha_1, \alpha_2, \dots$ are real numbers in $(0,1)$ tending upward to 1. In this paper we prove that for several choices of F and (α_n) there exist norming constants a_n and b_n such that

$$\frac{X_n}{a_n} \to 1 \quad \text{a.s.}, \tag{1.2}$$

$$\frac{X_n - a_n}{b_n} \xrightarrow{d} Z, \tag{1.3}$$

with Z standard normal ($\xrightarrow{d}$ denotes convergence in distribution).

Our results extend earlier work on (1.1) by Greenwood and Hooghiemstra [7] for $\alpha_n \equiv \alpha \in (0,1)$ constant. The two boundary cases $\alpha = 0$ and $\alpha = 1$ correspond to classical situations. Indeed, for $\alpha = 0$,

$$X_{n+1} = \max\,(Y_1, \dots, Y_n),$$

so that (1.3) holds with norming constants satisfying $\lim_{n\to\infty} F^n(a_n + b_n x) = G(x)$ and with limit law $P(Z \leq x) = G(x)$ where G is one of the three types of extreme-value distributions (Gnedenko [5], de Haan [8]). On the other hand, for $\alpha = 1$ and when $F[0,\infty) = 1$,

$$X_{n+1} = Y_1 + \cdots + Y_n,$$

so that (1.3) holds with norming constants satisfying $\lim_{n\to\infty} F^{n*}(a_n + b_n x) = G(x)$ and with limit law $P(Z \leq x) = G(x)$ where $*$ is convolution and G is one of the stable laws (Gnedenko and Kolmogorov [6]). For intermediate $\alpha \in (0,1)$, Greenwood and Hooghiemstra found that the norming constants are the same as for maxima (except for a factor $1/(1-\alpha)$) but the limit laws are different. In fact, $G(x)$ appears as the unique solution of some integral equation.

Because of the discontinuity in the norming constants between $\alpha \in (0,1)$ and $\alpha = 1$ it is interesting to replace the fixed α by a sequence (α_n) tending upward to 1. In this paper we investigate what happens when

$$\alpha_n = 1 - n^{-l} \quad (0 < l < \infty) \tag{1.4}$$

for the three following cases

(i) $l = 1$, $\int y^2 dF(y) < \infty$,
(ii) $0 < l < 1$, $F(y) = (1 - y^{-\frac{1}{k}})\, 1[y > 1]$, $0 < k < \frac{1}{2}$,
(iii) $l > 1$, $\int y^2 dF(y) < \infty$.

In each of these cases we find that the limit law is standard normal. In case (iii) the process is essentially sums : we find that $a_n = \mu n + O(n^{2-l})$ and $b_n = \sigma n^{1/2}$ with $\mu = E \max(0, Y_1)$ and $\sigma^2 = \operatorname{var}(\max(0, Y_1))$. In case (ii), on the other hand, we find that the norming constants interpolate between maxima and sums and take the form $a_n = An^r$ and $b_n = Bn^s$ with $r = k + l(1-k)$, $s = r - l/2$ and A, B constants. Case (i) is the dividing line : here it turns out that $a_n = A'n$ and $b_n = B'n^{1/2}$ with A' and B' depending on F in a more complicated way than in case (iii). Note the norming in case (ii) by n^s instead of $n^{\frac{1}{2}}$. Since $k < \frac{1}{2}$ and $0 < l < 1$, we have $s = k + \frac{l}{2}(1-2k) < \frac{1}{2}$. Also note that for $l \to 0$ we have $s \to k$, which is intuitively right because n^k is the norming constant for maxima.

Our method of proof is flexible. Our aim is not so much to solve (1.1) but rather to present a general approach to the study of limiting behavior for iterative schemes with an i.i.d. feedback. Section 2 contains the skeleton of the proof. Here we formulate the basic ingredients, collect what technical facts need to be verified in order to get (1.2) and (1.3), and show how a_n, b_n and Z come out. This uses several standard tools, among which recursive inequalities, central limit theorem for martingale difference arrays, invariance principle and continuous mapping theorem. In sections 3, 4 and 5 we then apply the general skeleton to the three cases (i),(ii) and (iii), and do explicit computations.

The interest in iterative schemes of the type (1.1) stems from a storage problem for solar energy described by Haslett [9]. For earlier studies we refer to

Daley and Haslett [4], Haslett [10], Hooghiemstra and Keane [11], Hooghiemstra and Scheffer [12], Greenwood and Hooghiemstra [7].

The initial condition in (1.1) plays no role, because if (X_n') is the process with $X_1' = x \in \mathbb{R}$ and the same input (Y_n), then it is clear that $|X_n' - X_n|$ decreases a.s. and the limiting behavior is the same.

There is an interesting link between our result for case (ii) and a recent result for sums of order statistics by Csörgö and Mason [3]. Let

$$Y_1^{(n)} \leq Y_2^{(n)} \leq \cdots \leq Y_n^{(n)}$$

be the order statistics of $Y_1, Y_2, \ldots, Y_n$ and let

$$\tilde{X}_{n+1} = \sum_{i=0}^{c_n} Y_{n-i}^{(n)}$$

where c_n is any sequence of positive integers such that $c_n \to \infty$ and $c_n/n \to 0$ as $n \to \infty$. If $F(y) = (1 - y^{\frac{1}{k}}) 1[y > 1]$, $0 < k < \frac{1}{2}$, then $\tilde{X}_n$ satisfies (1.3) with Z standard normal and with

$$a_n = n\frac{1}{1-k}\left(\frac{c_n}{n}\right)^{1-k},$$

$$b_n = n^{1/2}\left(\frac{2k^2}{(1-2k)(1-k)}\right)^{1/2}\left(\frac{c_n}{n}\right)^{1/2-k}.$$

To see the link with our situation, iterate (1.1) and write

$$X_{n+1} = \max_e \sum_{i=1}^{n} e_i Y_i \prod_{j=i+1}^{n} \alpha_j^{e_j} \tag{1.5}$$

where the maximum runs over $e = (e_1, \ldots, e_n) \in \{0,1\}^n$ and the empty product equals 1. If α_n is given by (1.4), then $\alpha_n^{cn^l} \to e^{-c}$ for every $c > 0$. Hence the maximum in (1.5) is attained roughly at e given by

$$e_i = 1[Y_i = Y_{n-m}^{(n)} \text{ for some } 0 \leq m < cn^l],$$

i.e. place 1's at the positions of the cn^l largest order statistics and 0's elsewhere. Indeed, then these order statistics have a coefficient in (1.5) of order 1 (namely somewhere between e^{-c} and 1, the precise value depending on their position), while allowing more 1's in e cannot increase the sum in (1.5) very much because the product over the α's decreases rapidly. This indicates that X_n is of the same order in n as $\tilde{X}_n$ when $c_n = cn^l$. But for this choice of c_n the norming constants of $\tilde{X}_n$ are $a_n = \tilde{A}n^r$ and $b_n = \tilde{B}n^s$ with exactly the same exponents $r = l + k(1-l)$ and $s = r - l/2$ as we find in case (ii) (though with different constants $\tilde{A}$ and $\tilde{B}$). Hence our conclusion is that in (1.1) the process (X_n) roughly only makes a jump when Y_n is one of the cn^l largest order statistics.

2. General skeleton of proof. This section consists of two parts. In part I we prove an almost sure result for the iteration scheme (1.1) for the case where (α_n) is arbitrary and (Y_n) is an arbitrary i.i.d. sequence with finite expectation. This result will be used in sections 3, 4 and 5 to prove (1.2) when $\alpha_n = 1 - n^{-l}$ and (i) $l = 1$, (ii) $0 < l < 1$, (iii) $l > 1$. In part II we formulate a functional central limit theorem (FCLT) for martingale difference arrays, proved by McLeish [13]. This theorem will be the basis for the proof of (1.3) given in sections 3, 4 and 5. A rough sketch of how this FCLT is applied in order to obtain (1.3) concludes this section.

Part I: Consider the iteration scheme (1.1) with $E|Y_1| < \infty$. Denote by ψ the function

$$\psi(x) := E \max(0, Y_1 - x), \quad x \in \mathbb{R}. \tag{2.1}$$

Define furthermore

$$\begin{cases} a_1 := 0 \\ a_{n+1} := a_n + \psi((1-\alpha_n)a_n), \quad n \geq 1, \end{cases} \tag{2.2}$$

and

$$Z_n := \max(0, Y_n - (1-\alpha_n)a_n) - (a_{n+1} - a_n), \quad n \geq 1. \tag{2.3}$$

Observe that both (a_n) and (Z_n) are defined in terms of the input quantities F and (α_n).

Theorem 2.1. *Let $\alpha_1, \alpha_2, \ldots$ be real numbers in* [0,1]. *If (i) $a_n \to \infty$ and (ii) $a_n^{-1} \sum_{k=1}^{n-1} Z_k \to 0$ a.s., then $X_n / a_n \to 1$ a.s.*

PROOF: Define for $n \geq 1$,

$$Z'_n := \max(0, Y_n - (1-\alpha_n)X_n) - (a_{n+1} - a_n), \tag{2.4}$$

and note that according to (1.1)

$$\sum_{k=1}^{n-1} Z'_k = X_n - a_n.$$

Define

$$W_n := \frac{1}{a_n} \sum_{k=1}^{n-1} Z_k, \quad W'_n := \frac{1}{a_n} \sum_{k=1}^{n-1} Z'_k.$$

In order to show that $W_n' \to 0$ a.s. we note that

$$W_{n+1}' - W_{n+1} = \frac{a_n}{a_{n+1}}(W_n' - W_n) + \frac{1}{a_{n+1}}(Z_n' - Z_n),$$

and also that $Z_n' - Z_n$ lies between 0 and $-(1-\alpha_n)(X_n - a_n) = -(1-\alpha_n)a_n W_n'$. Hence $W_{n+1}' - W_{n+1}$ lies between $\frac{a_n}{a_{n+1}}(W_n' - W_n)$ and

$$\frac{a_n}{a_{n+1}}(W_n' - W_n) - (1-\alpha_n)\frac{a_n}{a_{n+1}}W_n' = \frac{a_n}{a_{n+1}}\left[\alpha_n(W_n' - W_n) - (1-\alpha_n)W_n\right].$$

Therefore

$$|W_{n+1}' - W_{n+1}| \leq \frac{a_n}{a_{n+1}} \max\left(|W_n' - W_n|, |W_n|\right). \tag{2.5}$$

Now (2.5) together with $a_n \to \infty$ and $W_n \to 0$ implies $W_n' \to 0$. ∎

In sections 3, 4 and 5 we shall use Theorem 2.1 to prove (1.2).

Part II: Definition: Let $\{\xi_{n,j} : n \geq 1, 1 \leq j \leq n\}$ be a triangular array of random variables and let $\{\mathcal{F}_{n,j} : n \geq 1, 0 \leq j \leq n\}$ be a triangular array of σ-fields with $\mathcal{F}_{n,0} \subseteq \mathcal{F}_{n,1} \subseteq \cdots \subseteq \mathcal{F}_{n,n}$ for each n. Then $\{\xi_{n,j}\}$ is called a *martingale difference array* with respect to $\{\mathcal{F}_{n,j}\}$ if $E(\xi_{n,j} \mid \mathcal{F}_{n,j-1}) = 0$ for each n and j.

Theorem 2.2. *(McLeish [13], Corollary (3.8)) Let $\{\xi_{n,j}\}$ be a martingale difference array with respect to $\{\mathcal{F}_{n,j}\}$ and let $k_n(t)$ be a sequence of integer-valued, non-decreasing, right-continuous functions defined on $t \in [0,1]$ such that $k_n(0) = 0$ and $k_n(1) = n$ for all n. If there exists a constant $\sigma^2 > 0$ such that for all $t \in [0,1]$,*

(i) $\sum_{j=1}^{k_n(t)} E(\xi_{n,j}^2 \mid \mathcal{F}_{n,j-1}) \to \sigma^2 t$ in probability,

(ii) $\sum_{j=1}^{k_n(t)} E(\xi_{n,j}^2 1[|\xi_{n,j}| > \varepsilon] \mid \mathcal{F}_{n,j-1}) \to 0$ in probability, for all $\varepsilon > 0$,

then the random function $W_n(t) := \sigma^{-1} \sum_{j=1}^{k_n(t)} \xi_{n,j}$, $t \in [0,1]$, converges weakly to standard Brownian motion W on $D[0,1]$. Here $D[0,1]$ denotes the space of cadlag functions on $[0,1]$ endowed with the Skorohod J_1-topology.

In sections 3, 4 and 5 we shall use Theorem 2.2 to prove (1.3). For each of the three cases indicated in the introduction the proof consists of the following steps.

1. The triangular array

$$\xi_{n,j} = \frac{1}{c_n}(X_{j+1} - X_j - \psi((1-\alpha_j)X_j))$$

is a martingale difference array with respect to the σ-fields $\{\mathcal{F}_{n,j} : n \geq 1, 0 \leq j \leq n\}$, where $\mathcal{F}_{n,j-1} \equiv \mathcal{F}_{j-1} = \sigma(X_1, \dots, X_j)$ is the σ-field generated by $X_1, \dots, X_j$. The norming constants c_n and the functions $k_n(t)$ are chosen such that for some σ the assumptions (i) and (ii) of Theorem 2.2 are satisfied. Hence the process $W_n(t) = \sigma^{-1}\sum_{j=1}^{k_n(t)} \xi_{n,j}$ converges to standard Brownian motion.

2. A process $\tilde{W}_n(t)$ is constructed such that $\rho(W_n, \tilde{W}_n) \to 0$ in probability, where ρ denotes the supremum metric on $D[0,1]$. The process $\tilde{W}_n(t)$ turns out to be more convenient than $W_n(t)$ and converges to standard Brownian motion as well. This is a direct consequence of Theorem 4.1 of Billingsley [1] and the fact that convergence in supremum metric implies convergence in Skorohods J_1-topology.

3. A continuous functional Φ is constructed such that $|\Phi(\tilde{W}_n(1)) - \frac{X_n - a_n}{b_n}| \to 0$ in probability, with a_n as defined in (2.2) and b_n some multiple of c_n. The continuous mapping theorem (Theorem 5.1 of Billingsley [1]) then implies that $\frac{X_n - a_n}{b_n}$ converges in distribution to a standard normal random variable.

3. The case l = 1. In this section we consider the process (1.1) with $\alpha_n = 1 - n^{-1}$. Recall the definition of $\psi(\cdot)$, (a_n) and (Z_n) in (2.1), (2.2) and (2.3) respectively. (Throughout sections 3, 4 and 5 we repeatedly use these definitions without further reference.) We shall prove the following two theorems.

Theorem 3.1. *If* $E\,|Y_1| < \infty$ *then*

$$\frac{X_n}{n} \to A \quad \text{a.s.}$$

where A *is the unique solution of the equation*

$$A = \psi(A) = E\max(0, Y_1 - A). \tag{3.1}$$

Theorem 3.2. *Suppose that the distribution function* F *of the sequence* $Y_1, Y_2, \dots$ *satisfies*
(i) $F(0) < 1$, *(ii)* $\int y^2 dF(y) < \infty$, *(iii)* F *is continuous in a neighbourhood of* A. *Then*

$$\frac{(2c+1)^{\frac{1}{2}}}{\sigma n^{\frac{1}{2}}}(X_n - nA) \xrightarrow{d} Z,$$

with Z standard normal and

$$c := P(Y_1 > A) = 1 - F(A), \tag{3.2}$$

$$\sigma^2 := \text{var}\,(\max(0,\, Y_1 - A))\,. \tag{3.3}$$

Note that the function $\psi(\cdot)$ is continuous and non-increasing on $\mathbb{R}$ with $\psi(-\infty) = \infty$ and $\psi(\infty) = 0$. Hence the existence and unicity of A is guaranteed. Furthermore $A = 0$ iff $F(0) = 1$ in which case Theorem 3.1 is trivial. Thus from now on we assume $A > 0$. The proof of Theorem 3.1 will be given by two lemmas (Lemma 3.3 and 3.4). After the proof of Theorem 3.1, we give the proof of Theorem 3.2.

Lemma 3.3. $\lim_{n\to\infty} a_n/n = A$.

PROOF: Define

$$d_n := \frac{a_n}{n} - A.$$

Then

$$d_{n+1} = \frac{a_n + \psi(\frac{a_n}{n})}{n+1} - A = \frac{n}{n+1} d_n + \frac{1}{n+1}\phi(d_n)$$

where

$$\phi(d) := \psi(d + A) - A = E \max\ (0, Y_1 - A - d) - E \max\ (0, Y_1 - A).$$

It is easily seen that d and $\phi(d)$ have opposite signs and that $|\phi(d)| \le |\,d\,|$. It follows that

$$|d_{n+1}| \le \frac{n}{n+1}|d_n|,$$

and hence $d_n \to 0$. ∎

Lemma 3.4. $a_n^{-1} \sum_{k=1}^{n-1} Z_k \to 0$ a.s.

PROOF: Define

$$Z_n'' := \max(0, Y_n - A) - A.$$

The random variables (Z_n'') are i.i.d. and have zero expectation (see(3.1)). By the law of large numbers and since $a_n \sim An$,

$$\frac{1}{a_n} \sum_{k=1}^{n-1} Z_k'' \to 0 \quad \text{a.s.}$$

The recursion $a_{n+1} = a_n + \psi(a_n/n)$, together with Lemma 3.3 and the continuity of ψ, gives that $a_{n+1} - a_n$ tends to A. Hence $Z_n'' - Z_n \to 0$ a.s. and the lemma follows. ∎

PROOF OF THEOREM 3.1: Lemma 3.3 implies $a_n \to \infty$. Together with Lemma 3.4 this proves the assumptions in Theorem 2.1 and hence we conclude that

$$X_n/a_n \to 1 \quad \text{a.s.}$$

Via Lemma 3.3 this proves the claim. ∎

To prove Theorem 3.2 we shall follow the three main steps as indicated at the end of Section 2. So let $\mathcal{F}_{j-1}$ denote the σ-field generated by $X_1, \dots, X_j$ and let

$$\xi_{n,j} = \frac{1}{n^{1/2}} \left(X_{j+1} - X_j - \psi(\tfrac{X_j}{j}) \right).$$

Lemma 3.5. *The random function* $W_n(t) := \sigma^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \xi_{n,j}, \quad t \in [0,1]$, *converges weakly to standard Brownian motion* W *on* $D[0,1]$. *Here* σ^2 *is as defined in (3.3).*

PROOF: We check the assumptions of Theorem 2.2.

(i) $E(\xi_{n,j}^2 \,|\, \mathcal{F}_{j-1}) = \frac{1}{n} \operatorname{var}(X_{j+1}|X_j) = \frac{1}{n} \operatorname{var}(\max(0, Y_j - \frac{X_j}{j}|X_j)$.

Using continuity in x of the function $\operatorname{var}(\max(0, Y_1 - x))$ and Theorem 3.1, we have

$$\sum_{j=1}^{\lfloor nt \rfloor} E(\xi_{n,j}^2 \,|\, \mathcal{F}_{j-1}) \to t \operatorname{var}(\max(0, Y_1 - A)) = \sigma^2 t \quad \text{a.s.}$$

(ii) From the iteration scheme (1.1) we obtain

$$\sum_{j=1}^{n} E\left(\xi_{n,j}^2 \, 1[|\xi_{n,j}| > \varepsilon] \,|\, \mathcal{F}_{j-1}\right) = \frac{1}{n} \sum_{j=1}^{n} \psi^2(\tfrac{X_j}{j}) \, 1[\psi(\tfrac{X_j}{j}) > \varepsilon\sqrt{n}]$$
$$+ \frac{1}{n} \sum_{j=1}^{n} \int_{X_j/j}^{\infty} \left(y - \tfrac{X_j}{j} - \psi(\tfrac{X_j}{j}) \right)^2 1[|y - \tfrac{X_j}{j} - \psi(\tfrac{X_j}{j})| > \varepsilon\sqrt{n}] \, dF(y).$$

The first term goes to zero a.s. as a direct consequence of Theorem 3.1. To prove the same for the second term we use that

$$y - \tfrac{X_j}{j} - \psi(\tfrac{X_j}{j}) \to y - 2A \quad \text{a.s.}$$

The assumption $\int y^2 dF(y) < \infty$ implies

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} (y - 2A)^2 \, 1[|y - 2A| > \varepsilon\sqrt{n}] \, dF(y) \to 0,$$

and this finishes the proof of Lemma 3.5. ∎

The random function $W_n(t)$ can be rewritten as

$$W_n(t) = \frac{1}{\sigma n^{1/2}} \left(X_{\lfloor nt \rfloor + 1} - A\lfloor nt \rfloor - \sum_{j=1}^{\lfloor nt \rfloor} (\psi(\tfrac{X_j}{j}) - A) \right).$$

The continuity of F in a neighbourhood of A implies that $\psi(\cdot)$ is differentiable in a neighbourhood of A and hence by the mean value theorem

$$\psi(\tfrac{X_j}{j}) - A = (\tfrac{X_j}{j} - A)\psi'(\theta_j) \quad \text{a.s.}$$

where θ_j lies between A and X_j/j. From Theorem 3.1 and (3.2) we have

$$\psi'(\theta_j) = -(1 - F(\theta_j)) \to -(1 - F(A)) = -c \quad \text{a.s.} \tag{3.4}$$

Now introduce

$$\tilde{W}_n(t) := \frac{1}{\sigma n^{1/2}} \left(X_{\lfloor nt \rfloor + 1} - A\lfloor nt \rfloor + c\sum_{j=1}^{\lfloor nt \rfloor} (\tfrac{X_j}{j} - A) \right), \quad t \in [0,1].$$

Lemma 3.6. *Let ρ denotes the supremum metric on $D[0,1]$, then $\rho(W_n, \tilde{W}_n) \to 0$ in probability. Consequently $\tilde{W}_n$ converges weakly to standard Brownian motion W on $D[0,1]$.*

PROOF: By definition

$$\rho(W_n, \tilde{W}_n) = \sup_{0 \le t \le 1} |W_n(t) - \tilde{W}_n(t)| = \frac{1}{\sigma n^{1/2}} \sup_{0 \le t \le 1} |\sum_{j=1}^{\lfloor nt \rfloor} (\tfrac{X_j}{j} - A)(1 - F(\theta_j) - c)|.$$

Abbreviate $\delta_j := 1 - F(\theta_j) - c$. According to (3.4), $\delta_j \to 0$ a.s. Fix $\epsilon > 0$ arbitrary and let $n(\omega)$ be such that $|\delta_j| \le \epsilon$ for $j \ge n(\omega)$. Here ω is a realization of our experiment and we throw away the null set of ω's for which $\delta_j \not\to 0$. Next define

$$S_n := \sum_{j=1}^{n} (\max(0, Y_j - A) - A),$$

$$S_n' := \sum_{j=1}^{n} \left(\max(0, Y_j - \tfrac{X_j}{j}) - A \right) = X_{n+1} - nA.$$

In analogy with the proof of (2.5) the following recursive inequality holds

$$|S_{n+1}' - S_{n+1}| \le \max(|S_n' - S_n|, |S_n|).$$

Hence

$$|\sum_{j=1}^{\lfloor nt\rfloor}(\tfrac{X_j}{j}-A)(1-F(\theta_j)-c)| = |\sum_{j=1}^{\lfloor nt\rfloor}\tfrac{\delta_j}{j}(X_j-(j-1)A)-A\sum_{j=1}^{\lfloor nt\rfloor}\tfrac{\delta_j}{j}|$$
$$\leq \sum_{j=2}^{\lfloor nt\rfloor}|\tfrac{\delta_j}{j}S'_{j-1}|+A|\sum_{j=1}^{\lfloor nt\rfloor}\tfrac{\delta_j}{j}|.$$

For the first term we have (by induction on the recursive inequality)

$$\sum_{j=2}^{\lfloor nt\rfloor}|\tfrac{\delta_j}{j}S'_{j-1}| \leq \sum_{j=2}^{n(\omega)}|\tfrac{\delta_j}{j}S'_{j-1}|+\epsilon\sum_{j=1}^{n}\tfrac{1}{j}\{|S_j|+|S'_j-S_j|\}$$
$$\leq \sum_{j=2}^{n(\omega)}|\tfrac{\delta_j}{j}S'_{j-1}|+2\epsilon\sum_{j=1}^{n}\tfrac{1}{j}\max_{1\leq i\leq j}|S_i|.$$

We conclude

$$\rho(W_n,\tilde{W}_n) \leq \frac{1}{\sigma n^{1/2}}\sum_{j=2}^{n(\omega)}|\tfrac{\delta_j}{j}S'_{j-1}|+2\epsilon\sum_{j=1}^{n}\tfrac{1}{j}\max_{1\leq i\leq j}\tfrac{|S_i|}{\sigma n^{1/2}}+\frac{A}{\sigma n^{1/2}}|\sum_{j=1}^{\lfloor nt\rfloor}\tfrac{\delta_j}{j}|$$

The first term tends to zero. The last term is at most

$$\frac{A}{\sigma n^{1/2}}(\max_{1\leq j\leq\lfloor nt\rfloor}|\delta_j|)\log\lfloor nt\rfloor = o(1).$$

As to the middle term, the random walk S_n has zero expectation and finite variance σ^2 and therefore by the invariance principle (see Billingsley [1]).

$$\sum_{j=1}^{n}\tfrac{1}{j}\max_{1\leq i\leq j}\tfrac{|S_i|}{\sigma n^{1/2}} \xrightarrow{d} \int_0^1 \max_{0\leq u\leq t}|W(u)|\tfrac{dt}{t}$$

with W standard Brownian motion. The integral is a finite random variable by the law of the iterated logarithm. This completes the proof. ∎

Next we define the map $\Phi : D[0,1] \to D[0,1]$ by

$$\Phi(X)(t) := X(t) - \frac{1}{t^c}\int_0^t cu^{c-1}X(u)du.$$

This map is continuous with respect to the Skorohod J_1-topology. The function Φ has been chosen such that $\Phi(\tilde{W}_n)(1)$ is close to $\frac{1}{\sigma n^{1/2}}(X_n - nA)$. This is made precise in the following lemma.

Lemma 3.7.
(i) $V(t) := \Phi(W)(t)$ *is a Gaussian process,*
(ii) $E\,V(1) = 0$, $\operatorname{var} V(1) = \frac{1}{2c+1}$,
(iii) $|\Phi(\tilde{W}_n)(1) - \frac{1}{\sigma n^{1/2}}(X_n - nA)| \to 0$ a.s.

PROOF:
(i) Every linear functional of a Gaussian process is Gaussian.
(ii) Compute

$$\begin{aligned}
E\,V(1) &= E\,W(1) - \int_0^1 cu^{c-1} E\,W(u)du = 0, \\
\operatorname{var} V(1) &= E\left(W(1) - \int_0^1 cu^{c-1} W(u)\,du\right)^2 \\
&= 2\int_0^1\int_0^u E\,\left(W(1) - cu^{c-1}W(u)\right)\left(W(1) - ct^{c-1}W(t)\right)\,dtdu \\
&= \tfrac{1}{2c+1}.
\end{aligned}$$

In the last equality we use that for standard Brownian motion $EW(s)W(t) = \min(s,t)$.
(iii) For a piecewise constant function $x_n(t)$ with values $x_n(\frac{k}{n})$ on the intervals $[\frac{k}{n}, \frac{k+1}{n})$, $k = 0, \dots, n-1$, and with $x_n(0) = 0$, we have

$$\Phi(x_n)(\tfrac{k}{n}) = x_n(\tfrac{k}{n}) - \tfrac{1}{k^c}\sum_{j=1}^{k-1}((j+1)^c - j^c)x_n(\tfrac{j}{n})$$

Assume that the function $x_n(\frac{k}{n})$ is of the form

$$x_n(\tfrac{k}{n}) = s_n(\tfrac{k}{n}) + c\sum_{j=1}^{k-1}\tfrac{1}{j}s_n(\tfrac{j}{n}). \tag{3.5}$$

Then

$$\begin{aligned}
&\Phi(x_n)(\tfrac{k}{n}) - s_n(\tfrac{k}{n}) = \\
&\quad = c\sum_{j=1}^{k-1}\tfrac{1}{j}s_n(\tfrac{j}{n}) - \tfrac{1}{k^c}\sum_{j=1}^{k-1}((j+1)^c - j^c)\left(s_n(\tfrac{j}{n}) + c\sum_{i=1}^{j-1}\tfrac{1}{i}s_n(\tfrac{i}{n})\right) \\
&\quad = \tfrac{1}{k^c}\sum_{j=1}^{k-1}\left(\tfrac{c(j+1)^c}{j} - (j+1)^c + j^c\right)s_n(\tfrac{j}{n})
\end{aligned}$$

where the last equality follows from interchanging summations. The function $\tilde{W}_n(.)$ is a piecewise constant function of the form (3.5) with $s_n(\frac{k}{n}) = \frac{1}{\sigma n^{1/2}}(X_k - Ak)$. Hence

$$|\Phi(\tilde{W}_n)(1) - \frac{1}{\sigma n^{1/2}}(X_n - An)| =$$

$$= \frac{1}{\sigma n^{c+1/2}} |\sum_{j=1}^{n-1} \left(\frac{c(j+1)^c}{j} - (j+1)^c + j^c\right)(X_j - Aj)|$$

$$\leq \frac{K}{\sigma n^{c+1/2}} |\sum_{j=1}^{n-1} j^{c-1}(\frac{X_j - Aj}{j})| \to 0 \quad \text{a.s.}$$

where K is some constant depending on c and where the limit zero follows with the fact that Theorem 3.1 implies $\frac{X_j - Aj}{j} \to 0$ a.s. ∎

PROOF OF THEOREM 3.2: Combining Lemmas 3.5, 3.6 and 3.7 with the continuous mapping theorem (see Billingsley [1], Theorem 5.1) we conclude that

$$\frac{1}{\sigma n^{1/2}}(X_n - An) \xrightarrow{d} V(1) \stackrel{d}{=} N(0, \frac{1}{2c+1}). \quad ∎$$

4. The case 0 < l < 1. In this section we consider the recursion (1.1) with $\alpha_n = 1 - n^{-l}$ and $0 < l < 1$. We consider only the special case where the sequence $Y_1, Y_2, \cdots$ has distribution function

$$F(y) = (1 - y^{-\frac{1}{k}})\, 1_{[1,\infty)}(y) \quad , 0 < k < \tfrac{1}{2}.$$

We shall prove the following two theorems.

Theorem 4.1. *Let $r := k + l(1-k)$. Then*

$$\frac{X_n}{n^r} \to A := (\frac{k}{r(1-k)})^k \quad \text{a.s.}$$

Theorem 4.2. *Let r and A be as defined in Theorem 4.1. Then*

$$(\frac{c+s}{s})^{\frac{1}{2}}(\frac{X_n - An^r}{\sigma n^s}) \xrightarrow{d} Z,$$

with Z standard normal and

$s := r - l/2,$

$c := A^{-1/k} = r(1-k)/k,$

$$\sigma^2 := (2s)^{-1} E \max{}^2(0, Y_1 - A) = (2s)^{-1} \frac{2k^2}{(1-k)(1-2k)} A^{-(1-2k)/k}.$$

To prove Theorem 4.1 we need the following technical lemma.

Lemma 4.3. *Let the sequence* (c_n) *satisfy the recurrence relation*

$$c_{n+1} = (\tfrac{n}{n+1})^r (c_n + \tfrac{r}{n} c_n^{-\frac{1-k}{k}}).$$

Then, independent of the initial value $c_1 > 0$, $\lim_{n\to\infty} c_n = 1$.

PROOF: Put

$$f_n(x) := (\tfrac{n}{n+1})^r (x + \tfrac{r}{n} x^{-\frac{1-k}{k}}).$$

Then $c_{n+1} = f_n(c_n)$. Denote by p_n the unique positive fixed point of f_n. One easily shows that the sequence (p_n) is monotone decreasing with $\lim_{n\to\infty} p_n = 1$. Now we distinguish two cases.

Case 1: There exists n_0 such that $c_{n_0} > p_{n_0}$.

From the fact that (p_n) is monotone decreasing and that $f_n(x)$ is monotone increasing in x on $[p_n, \infty)$, it follows that $c_n > p_n$ for all $n \geq n_0$. Furthermore, $f_n(x) < x$ on (p_n, ∞), hence (c_n) is decreasing for $n \geq n_0$ and has limit greater or equal to 1. Put $q_n := c_n - 1$, $d_n := (\frac{n}{n+1})^r$. Then

$$\begin{aligned} q_{n+1} = f_n(c_n) - 1 &= f_n(c_n) - f_n(p_n) + p_n - 1 \\ &\leq d_n(c_n - p_n) + p_n - 1 \\ &= d_n q_n + (1 - d_n)(p_n - 1) \end{aligned}$$

Iteration gives

$$q_{n+1} \leq (\prod_{i=n_0}^{n} d_i) q_{n_0} + \sum_{i=n_0}^{n} (\prod_{j=i+1}^{n} d_j)(1 - d_i)(p_i - 1).$$

Clearly $\prod_{i=n_0}^{n} d_i \to 0$. Moreover, one easily checks that $\sum_i (1-d_i)(p_i - 1) < \infty$ (both $1 - d_i$ and $p_i - 1$ are $O(\frac{1}{i})$ as $i \to \infty$) and hence also the second term tends to zero. This completes the proof of the lemma in case 1.

Case 2: $c_n \leq p_n$ for all n.

In this case

$$p_{n+1} - c_{n+1} \leq p_n - c_{n+1} = f_n(p_n) - f_n(c_n) \leq (\tfrac{n}{n+1})^r (p_n - c_n)$$

and iteration gives $p_{n+1} - c_{n+1} \leq (\frac{1}{n+1})^r (p_1 - c_1) \to 0$. ∎

PROOF OF THEOREM 4.1: For $\psi(\cdot)$ we have

$$\psi(x) = \begin{cases} \frac{1}{1-k} & , \text{ if } x < 1, \\ \frac{k}{1-k} x^{-(1-k)/k} & , \text{ if } x \geq 1. \end{cases} \tag{4.1}$$

From the definition of (a_n) one easily shows that $a_n/n^l \geq 1$ for n sufficiently large. Hence (a_n) satisfy

$$a_{n+1} = a_n + \tfrac{k}{1-k} n^{l\frac{1-k}{k}} a_n^{-\frac{1-k}{k}}.$$

If we put $c_n = a_n/An^r$, then c_n satisfies the recurrence relation in Lemma 4.3 and hence $a_n/An^r \to 1$. Theorem 4.1 follows from Theorem 2.1 if we can prove

$$\frac{1}{a_n} \sum_{k=1}^{n-1} Z_k \to 0 \quad \text{a.s.} \tag{4.2}$$

However, by direct computation

$$E(Z_n^2) = O(n^{-\frac{1-2k}{k}(r-l)}) = O(n^{2r-l-1})$$

hence

$$\frac{E(Z_n^2)}{a_n^2} = O(n^{-1-l}).$$

This implies

$$\sum_{n=1}^{\infty} \frac{E(Z_n^2)}{a_n^2} < \infty$$

which is a sufficient condition for the a.s.-convergence in (4.2) (see Breiman[2], Theorem 3.27). ∎

The proof of Theorem 4.2 will be given through a series of lemmas, similarly as in section 3. Again we denote by $\mathcal{F}_{j-1}$ the σ-field generated by $X_1, \cdots, X_j$. This time we define

$$\xi_{n,j} := n^{-s} \left(X_{j+1} - X_j - \psi(\frac{X_j}{j^l}) \right).$$

Lemma 4.4. *The random function W_n, defined by*

$$W_n(t) = \sigma^{-1} \sum_{j=1}^{k_n(t)} \xi_{n,j}, \tag{4.3}$$

where $k_n(t) = \lfloor nt^{\frac{1}{2s}} \rfloor$, $t \in [0,1]$, converges weakly to standard Brownian motion W on $D[0,1]$. Here σ is as defined in Theorem 4.2.

PROOF: We check the assumptions of Theorem 2.2.

(i) For almost all ω,

$$E(\xi_{n,j}^2 | \mathcal{F}_{j-1}) = n^{-2s} \operatorname{var}(X_{j+1} | X_j)$$
$$= \frac{1}{n^{2r-l}} \operatorname{var}\left(\max(0, Y_j - \frac{X_j}{j^l}) | X_j \right).$$

Using the explicit form of F and the almost sure behaviour of X_j given in Theorem 4.1 it is readily seen that the main contribution to $E(\xi_{n,j}^2|\mathcal{F}_{j-1})$ behaves as

$$\frac{1}{n^{2r-l}} j^{-(r-l)(\frac{1-2k}{k})} E \max{}^2(0, Y_1 - A).$$

Via the identity $(r-l)(\frac{2k-1}{k}) = 2r - l - 1 = 2s - 1$ we hence obtain

$$\sum_{j=1}^{k_n(t)} E(\xi_{n,j}^2 | \mathcal{F}_{j-1}) \to t \, \frac{E \max{}^2(0, Y_1 - A)}{2s} = \sigma^2 t \quad \text{a.s.}$$

(ii) From (1.1)

$$\sum_{j=1}^{n} E \ (\xi_{n,j}^2 \, 1\,[|\xi_{n,j}| > \varepsilon] \,|\, \mathcal{F}_{j-1})$$
$$= n^{-2s} \sum_{j=1}^{n} F(\tfrac{X_j}{j^l}) \psi^2(\tfrac{X_j}{j^l}) \, 1[\psi(\tfrac{X_j}{j^l}) > \varepsilon j^{r-\frac{l}{2}}]$$
$$+ n^{-2s} \sum_{j=1}^{n} \int_{\frac{X_j}{j^l}}^{\infty} \left(y - \tfrac{X_j}{j^l} - \psi(\tfrac{X_j}{j^l}) \right)^2 1(|y - \tfrac{X_j}{j^l} - \psi(\tfrac{X_j}{j^l})| > \varepsilon j^{r-\frac{l}{2}}) dF(y)$$
$$\leq n^{-2s} \sum_{j=1}^{n} \psi^2(\tfrac{X_j}{j^l}) + n^{-2s} \sum_{j=1}^{n} \int_{\varepsilon j^{r-\frac{l}{2}}}^{\infty} y^2 dF(y) \quad \text{a.s.}$$

The first term goes to 0 almost surely because $2(r-l)(\frac{k-1}{k}) < 2s - 1$ (see (i) and (4.1)). The second term is equal to

$$n^{-2s} (1 - 2k)^{-1} \varepsilon^{-\frac{1-2k}{k}} \sum_{j=1}^{n} j^{-s(\frac{1-2k}{k})}$$

which goes to 0 almost surely because $s(\frac{2k-1}{k}) < 2s - 1$. ∎

The definition of $W_n(t)$ in (4.3) can be rewritten as

(4.4)
$$W_n(t) = \frac{1}{\sigma n^s}\left[\sum_{j=1}^{k_n(t)} (X_{j+1} - X_j - \psi(Aj^{r-l})) + \sum_{j=1}^{k_n(t)} \left(\psi(Aj^{r-l}) - \psi(\tfrac{X_j}{j^l})\right)\right].$$

By the mean value theorem

$$\psi(\tfrac{X_j}{j^l}) - \psi(Aj^{r-l}) = \tfrac{X_j - Aj^r}{j^l}\psi'(\theta_j), \tag{4.5}$$

where θ_j lies between $\frac{X_j}{j^l}$ and Aj^{r-l}. Now introduce

$$\tilde{W}_n(t) := \frac{1}{\sigma n^s}\left[\sum_{j=1}^{k_n(t)} (X_{j+1} - X_j - \psi(Aj^{r-l})) + A^{-\frac{1}{k}} \sum_{j=1}^{k_n(t)} \frac{X_j - Aj^r}{j}\right]. \tag{4.6}$$

REMARK: The righthand side of (4.6) is what we obtain if we replace $\psi'(\theta_j) = -(1 - F(\theta_j))$ by $\psi'(Aj^{r-l})$, use (4.1) and substitute (4.5) into definition (4.4).

Lemma 4.5. *$\rho(W_n, \tilde{W}_n) \to 0$ in probability.*

PROOF: The proof runs parallel to that of Lemma 3.6 and can be easily reconstructed by the reader. ∎

The above lemma implies that

$$\frac{1}{\sigma n^s}\left(X_{k_n(t)+1} - A(k_n(t))^r + A^{-\frac{1}{k}} \sum_{j=1}^{k_n(t)} \tfrac{X_j - Aj^r}{j}\right) \xrightarrow{d} W, \tag{4.7}$$

because $\sum_{j=1}^{n} \psi(Aj^{r-l}) = An^r + o(n^s)$.

The same continuous functional Φ as in section 3, but with $c = A^{-\frac{1}{k}}$, can be applied to finish the proof of Theorem 4.2 (see above Lemmas 3.6 and 3.7). However, note that (4.7) contains one small additional complication, namely instead of $k_n(t) = \lfloor nt \rfloor$ we now have

$$k_n(t) = \lfloor nt^{\frac{1}{2s}} \rfloor, \quad t \in [0,1].$$

An obvious way out is to define the Gaussian process $\tilde{W}$ by

$$\tilde{W}(t) := W(t^{2s}), \quad t \in [0,1].$$

Then (4.7) implies that

$$\frac{1}{\sigma n^{s}}\left(X_{\lfloor nt\rfloor+1}-A\lfloor nt\rfloor^{r}+A^{-\frac{1}{s}}\sum_{j=1}^{\lfloor nt\rfloor}\frac{X_j-Aj^r}{j}\right)\xrightarrow{d}\tilde{W}.$$

Hence by the continuous mapping theorem

$$\frac{1}{\sigma n^{s}}(X_{\lfloor nt\rfloor+1}-A\lfloor nt\rfloor^{r})\xrightarrow{d}\Phi(\tilde{W})(t)).$$

Finally, a straightforward calculation shows that

$$\mathrm{var}\left(\Phi(\tilde{W})(1)\right)=E\left(\tilde{W}(1)-\int_0^1 cu^{c-1}\tilde{W}(u)du\right)^2=\frac{s}{s+c}. \quad ∎$$

5. The case l > 1. In this section we consider the recursion (1.1) with $\alpha_n = 1-n^{-l}$ and $l>1$. We shall prove the following two theorems.

Theorem 5.1. *If* $E|Y_1|<\infty$ *then*

(5.1) $$\frac{X_n}{n}\to\psi(0)=E\max(0,Y_1)\quad \text{a.s.}$$

Theorem 5.2. *Suppose that the distribution function F of the sequence* $Y_1, Y_2,\cdots$ *satisfies*
(i) $F(0)<1$, *(ii)* $\int y^2dF(y)<\infty$, *(iii) F is continuous in a neighbourhood of 0. Then*

$$\frac{1}{\sigma n^{1/2}}\left(X_n-\sum_{j=1}^{n}\psi(\frac{a_j}{j^l})\right)\xrightarrow{d}Z$$

with Z standard normal and

(5.2) $$\sigma^2 := \mathrm{var}(\max(0,Y_1)).$$

Furthermore,

(5.3) $$\sum_{j=1}^{n}\psi(\tfrac{a_j}{j^l})=\psi(0)n+\tfrac{\psi'(0)\psi(0)}{2-l}n^{2-l}+o(n^{2-l}).$$

The proof of Theorem 5.1 follows from Lemmas 5.3 and 5.4, the proof of Theorem 5.2 from Lemmas 5.5 and 5.6. Note that the second term in (5.3) is only important when $1<l<3/2$.

Lemma 5.3. $\lim_{n\to\infty} \frac{a_n}{n} = \psi(0)$.

PROOF: Because ψ is non-increasing,

$$0 \le a_{n+1} = a_n + \psi(\frac{a_n}{n^l}) \le a_n + \psi(0).$$

From this inequality and the initial value $a_1 = 0$ it is easy to show that $0 \le \frac{a_n}{n} \le \psi(0)$. Since $l > 1$ we obtain that $\frac{a_n}{n^l} \to 0$, and so by continuity of ψ we have $\psi(\frac{a_n}{n^l}) \ge \psi(0) - \epsilon$ for n sufficiently large. Hence from the above inequality $\frac{a_n}{n} \ge \psi(0) - \epsilon$ for n sufficiently large. ∎

Lemma 5.4. $a_n^{-1} \sum_{k=1}^{n-1} Z_k \to 0 \quad$ a.s.

PROOF: Recall that

$$Z_k = \max(0, Y_k - \frac{a_k}{k^l}) - (a_{k+1} - a_k).$$

Lemma 5.3 implies that $\frac{a_k}{k^l} \to 0$ and that $a_{k+1} - a_k \to \psi(0)$. If we define

$$Z_k'' := \max(0, Y_k) - \psi(0)$$

then by the law of large numbers

$$\frac{1}{n} \sum_{k=1}^{n-1} Z_k'' \to 0 \quad \text{a.s.}$$

The lemma follows since $a_n \sim n\psi(0)$ with $\psi(0) > 0$ and since $Z_k - Z_k'' \to 0 \quad$ a.s. ∎

PROOF OF THEOREM 5.1: The assumptions of Theorem 2.1 are fulfilled because of the preceeding two lemmas. ∎

Again we define $\mathcal{F}_{j-1} = \sigma(X_1, \ldots, X_j)$ but this time

$$\xi_{n,j} := \frac{1}{n^{1/2}} \left(X_{j+1} - X_j - \psi(\frac{X_j}{j^l}) \right).$$

Lemma 5.5. *The random function $W_n(t) := \sigma^{-1} \sum_{j=1}^{\lfloor nt \rfloor} \xi_{n,j}$, $t \in [0,1]$, converges weakly to standard Brownian motion W on $D[0,1]$. Here σ^2 is as defined in (5.2).*

PROOF: The proof of Lemma 5.5 is completely similar to that of Lemma 3.5 and therefore is omitted. ∎

The random function $W_n(t)$ can be rewritten as

$$W_n(t) = \frac{1}{\sigma n^{1/2}} \left[X_{\lfloor nt \rfloor + 1} - \sum_{j=1}^{\lfloor nt \rfloor} \psi(\frac{a_j}{j^l}) - \sum_{j=1}^{\lfloor nt \rfloor} \left(\psi(\frac{X_j}{j^l}) - \psi(\frac{a_j}{j^l}) \right) \right].$$

In Lemma 5.6 we shall prove that

$$\frac{1}{\sigma n^{1/2}} \sup_{0 \le t \le 1} \sum_{j=1}^{\lfloor nt \rfloor} \left(\psi(\frac{X_j}{j^l}) - \psi(\frac{a_j}{j^l}) \right) \to 0 \quad \text{in probability,}$$

so that according to Theorem 4.2 of Billingsley [1],

$$\frac{1}{\sigma n^{1/2}} \left(X_{\lfloor nt \rfloor + 1} - \sum_{j=1}^{\lfloor nt \rfloor} \psi(\frac{a_j}{j^l}) \right)$$

converges to standard Brownian motion. Theorem 5.2 will then follow by taking the projection at $t = 1$.

Lemma 5.6. $\frac{1}{n^{1/2}} \sup\limits_{0 \le t \le 1} |\sum_{j=1}^{\lfloor nt \rfloor} \psi(\frac{X_j}{j^l}) - \psi(\frac{a_j}{j^l})| \to 0$ *in probability.*

PROOF: By continuity of F in a neighbourhood of 0 the mean value theorem yields

$$\psi(\frac{X_j}{j^l}) - \psi(\frac{a_j}{j^l}) = \frac{X_j - a_j}{j^l} \psi'(\theta_j),$$

where θ_j lies between $\frac{X_j}{j^l}$ and $\frac{a_j}{j^l}$. Recall the definition of Z_j and Z'_j in (2.3) and (2.4), respectively, and put

$$S_j := \sum_{k=1}^{j-1} Z_k,$$

$$S'_j := \sum_{k=1}^{j-1} Z'_k = X_j - a_j.$$

Since $|\psi'(t)| = |1 - F(t)| \le 1$, we have

$$\sup_{0 \le t \le 1} |\sum_{j=1}^{\lfloor nt \rfloor} \psi(\frac{X_j}{j^l}) - \psi(\frac{a_j}{j^l})| = \sup_{0 \le t \le 1} |\sum_{j=1}^{\lfloor nt \rfloor} \frac{X_j - a_j}{j^l} \psi'(\theta_j)|$$

$$\le \sum_{j=1}^{n} j^{-l} |X_j - a_j| = \sum_{j=1}^{n} j^{-l} |S'_j|$$

$$\le \sum_{j=1}^{n} j^{-l} |S_j| + \sum_{j=1}^{n} j^{-l} |S'_j - S_j|.$$

From the recursive inequality $|S'_{j+1} - S_{j+1}| \le \max(|S'_j - S_j|, |S_j|)$ we obtain by induction (see the proof of Lemma 3.6)

$$\begin{aligned}\sup_{0\le t\le 1} \Big|\sum_{j=1}^{\lfloor nt\rfloor} \psi(\frac{X_j}{j^l}) - \psi(\frac{a_j}{j^l})\Big| &\le \sum_{j=1}^{n} j^{-l}|S_j| + \sum_{j=1}^{n-1} j^{-l} \max_{1\le i\le j}|S_i| \\ &\le 2\sum_{j=1}^{n} j^{-l} \max_{1\le i\le j}|S_i|.\end{aligned} \tag{5.3}$$

From the Lindeberg condition (compare Theorem 2.2) we easily obtain that the random function $S_{\lfloor nt\rfloor}/(\sigma n^{1/2})$ converges to Brownian motion W. Now take ε arbitrary and write

$$\begin{aligned}&\frac{1}{\sigma n^{1/2}} \sum_{j=1}^{n} j^{-l} \max_{1\le i\le j}|S_i| \\ &= \frac{1}{\sigma n^{1/2}} \left(\sum_{j=1}^{\lfloor n\varepsilon\rfloor} j^{-l} \max_{1\le i\le j}|S_i| + \sum_{j=\lfloor n\varepsilon\rfloor+1}^{n} j^{-l} \max_{1\le i\le j}|S_i| \right).\end{aligned} \tag{5.4}$$

The first term on the right hand side of (5.4) goes to zero in probability as $n \to \infty$ and afterwards $\varepsilon \downarrow 0$, because for $j \le \lfloor n\varepsilon\rfloor$,

$$\frac{1}{\sigma n^{1/2}} \max_{1\le i\le j}|S_i| \le \frac{1}{\sigma n^{1/2}} \max_{i\le \lfloor n\varepsilon\rfloor}|S_i| \xrightarrow{d} \max_{0\le t\le \varepsilon}|W(t)|.$$

Note that we use $l > 1$ to ensure that $\sum_{j=1}^{\lfloor n\varepsilon\rfloor} j^{-l} < \infty$. The second term on the right hand side of (5.4) can be rewritten as

$$n^{1-l} \sum_{j=\lfloor n\varepsilon\rfloor+1}^{n} \frac{\max_{1\le i\le j}|S_i|/(\sigma n^{1/2})}{(j/n)^l} \frac{1}{n},$$

which also converges to zero in probability since $l > 1$ and since

$$\sum_{j=\lfloor n\varepsilon\rfloor+1}^{n} \frac{\max_{1\le i\le j}|S_i|/(\sigma n^{1/2})}{(j/n)^l} \frac{1}{n} \xrightarrow{d} \int_{\varepsilon}^{1} \frac{\max_{0\le u\le t}|W(u)|}{u^l} du.$$

This finishes the proof of Lemma 5.6. ∎

PROOF OF THEOREM 5.2: The weak convergence to Z follows immediately from Lemmas 5.5 and 5.6. To prove the last statement of Theorem 5.2 we use the mean value theorem. Indeed

$$\sum_{j=1}^{n} \psi(\frac{a_j}{j^l}) = \psi(0)n + \sum_{j=1}^{n} \frac{a_j}{j^l}\psi'(\theta_j)$$

where θ_j lies between 0 and $\frac{a_j}{j^l}$. Using continuity of ψ' in a neighbourhood of 0 and the expansion $a_j = \psi(0)j + o(j)$, we conclude that

$$\sum_{j=1}^{n} \psi(\frac{a_j}{j^l}) = \psi(0)n + \frac{\psi'(0)\psi(0)}{2-l}n^{2-l} + o(n^{2-l}). \quad \blacksquare \tag{5.5}$$

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The $M/G/1$ processor sharing queue as the almost sure limit of feedback queues.

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Abstract. In the paper a probabilistic coupling between the $M/G/1$ processor sharing queue and the $M/M/1$ feedback queue, with general feedback probabilities, is established. This coupling is then used to prove the almost sure convergence of sojourn times in the feedback model to sojourn times in the $M/G/1$ processor sharing queue. Using the theory of regenerative processes it follows that for stable queues the stationary distribution of the sojourn time in the feedback model converges in law to the corresponding distribution in the processor sharing model. The results do not depend on Poisson arrival times, but are also valid for general arrival processes.

1. Introduction. Recently Van den Berg and Boxma [1] suggested a new approach for analyzing $M/G/1$ processor sharing models. They considered an approximating sequence of feedback queues, and concluded on heuristic grounds that performance measures such as the sojourn time in the feedback model converge to the corresponding performance measure in the processor sharing queue. In this paper we present a probabilistic coupling between the $M/G/1$ processor sharing queue and the approximating $M/M/1$ feedback queues, which shows that the sojourn time of the k-th customer in the feedback model converges almost surely to the corresponding quantity in the processor sharing model. From this result we conclude the distributional convergence of the steady state sojourn times.

The feedback model is a single server queue, having Poisson arrivals with intensity λ. Customers receive exponential service slices with mean length n^{-1}. When a customer has completed his k-th service he departs from the system with probability $1 - p_k$ or joins the end of the queue with probability p_k. All random mechanisms i.e., the arrival process, the service slices and feedback probabilities are independent of each other. Also the sequence of service slices is assumed to be an independent sequence. We denote by q_k the probability that a customer pays exactly $k,(k \geq 1)$ visits to the queue,

$$q_k := (1 - p_k) \prod_{i=1}^{k-1} p_i, \tag{1.1}$$

and by $G(z) := \sum_{k=1}^{\infty} q_k z^k$ its generating function. The total service X obtained by an arbitrary customer has Laplace transform

$$Ee^{-sX} = \sum_{k=1}^{\infty} q_k Ee^{-s(n^{-1}Y_1 + \cdots + n^{-1}Y_k)} = G(\frac{n}{n+s}),$$

where $Y_1, Y_2, \ldots$ is a sequence of independent, exponential random variables with mean 1.

The $M/G/1$ processor sharing queue (see Kleinrock [2], Yashkov [7], Ott [3] or Schassberger [5]) needs no introduction. To fix notation we assume that the service times in the processor sharing queue are given by the i.i.d. sequence $X_1, X_2, \ldots$ with finite mean EX_1. To keep the exposition as clear as possible we start with the case of exponential service times $X_1, X_2, \ldots$, with mean 1; this corresponds in the feedback queue to the case where $p_k = 1 - n^{-1}$ for all k. The total service time X in the feedback queue is also exponentially distributed with mean 1, since $G(z) = z/(z + n(1-z))$ and so

$$Ee^{-sX} = G(\frac{n}{n+s}) = (1+s)^{-1}.$$

In §2 we introduce a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ on which the arrival and service times of both the $M/M/1$ processor sharing and the $M/M/1$ feedback queues are defined. Theorem 1 shows the connection between the above mentioned queues and a third queue in which the exponential slices of mean length n^{-1} are replaced by deterministic slices of length n^{-1} (actually this queue is a so-called Round Robin queue and the connection with the feedback model is a special version of the strong law of large numbers). After the formal introduction of the Round Robin queue, in §3, the proof that sojourn times in the feedback model have the corresponding quantity in the processor sharing queue as almost sure limit is given. The Round Robin queue is used as an intermediate stage in the proof. The step to convergence of stationary sojourn times is made with the use of regenerative processes, similar to Schassberger [5]. In contrast to his paper we obtain convergence of means during a busy period by using Fatou's lemma twice, instead of dominated convergence (which does not work here).

Finally we extend our results to more general service time distributions in §4. The most general service time distribution that we can obtain is a finite mixture of phase distributions given in (4.1). The proof for the general service times is completely parallel to the one for exponential distributions. We also note that neither the coupling nor the proofs in §3 depend on the arrival process, so that the construction and the theorems carry over to general independent or even dependent arrival processes.

2. The almost sure construction. In this section a probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is constructed on which the arrival and service times of both the $M/M/1$ processor sharing queue and the $M/M/1$ feedback queues are defined.

Let us introduce for each $k \geq 1$ a probability space $(\Omega_k, \mathcal{A}_k, P_k)$, on which are defined a random variable A_k, exponentially distributed with parameter λ, $0 < \lambda < 1$, and a sequence $Y_{k,1}, Y_{k,2}, \cdots$, of independent random variables with common exponential distribution with parameter 1. The sequence $\{Y_{k,j}\}_{j=1}^{\infty}$ is independent of A_k. The random variables $A_1, A_2, \cdots$ will denote the inter-arrival times; $n^{-1}Y_{k,j}, j = 1, 2, \cdots$, will denote the successive service slices of the k-th customer in the feedback queue.

Next, we define for each $k \geq 1$ a probability space (Σ_k, B_k, Q_k) on which X_k is defined as an exponential random variable with parameter 1, and where $N_{k,n}$, $n = 1, 2, \cdots$, is defined as a sequence of geometric variables with parameter n^{-1}, i.e.,

$$Q_k(N_{k,n} = j) = n^{-1}(1 - n^{-1})^{j-1}, j = 1, 2, \cdots$$

The space (Σ_k, B_k, Q_k) and the variables X_k, $N_{k,n}$ are constructed to satisfy $n^{-1}N_{k,n} \to X_k$, Q_k almost surely. This is possible according to the representation theorem (cf. Skorohod [6], or Pollard [4], p. 58), since $n^{-1} N_{k,n} \to X_k$ in distribution. The random variable X_k denotes the service time of the k-th customer in the processor sharing model; $N_{k,n}$ denotes the number of slices assigned to the k-th customer in the feedback queue.

The probability space $(\Omega, \mathcal{A}, \mathbf{P})$ is defined as the product space

$$(\Omega, \mathcal{A}, \mathbf{P}) := \left(\prod_{k=1}^{\infty} (\Omega_k, \mathcal{A}_k, P_k) \right) \times \left(\prod_{k=1}^{\infty} (\Sigma_k, B_k, Q_k) \right).$$

The random variables introduced above are considered as random variables on

$(\Omega, \mathcal{A}, \mathbf{P})$ by taking the composition with the correct projection map of Ω on Ω_k or Σ_k.

Define

$$X_{k,n} = n^{-1} \sum_{j=1}^{N_{k,n}} Y_{k,j}, \tag{2.1}$$

i.e., $X_{k,n}$ denotes the total service time of customer k in the feedback model.

Theorem 1.

(i) For each $k \geq 1$ and each $n \geq 1$,

$$\mathbf{P}\{X_{k,n} > x\} = e^{-x}, \quad x > 0, \tag{2.2}$$

hence the distribution of $X_{k,n}$ is for each n identical to that of X_k.

(ii) For each $k \geq 1$,

$$\lim_{n\to\infty} X_{k,n} = X_k, \quad \mathbf{P} \quad a.s. \tag{2.3}$$

(iii) For each $k \geq 1$,

$$\lim_{n\to\infty} \sup_{0\leq t\leq 1} \Big| \sum_{j=1}^{[tN_{k,n}]} n^{-1}(Y_{k,j} - 1)\Big| = 0, \quad \mathbf{P} \quad a.s. \tag{2.4}$$

PROOF:

(i) This has been shown in the introduction by means of generating functions.

(ii) In the proof of (ii) and (iii) we drop the subscript k. We first show that for fixed $t \in [0,1]$,

$$\lim_{n\to\infty} n^{-1} \sum_{j=1}^{[tN_n]} (Y_j - 1) = 0, \quad \mathbf{P} \quad a.s., \tag{2.5}$$

if Y_j is a sequence of independent exponentially distributed random variables with parameter 1, and N_n, independent of $\{Y_j\}$, has a geometric distribution with parameter $p = n^{-1}$.

Let $\varepsilon > 0$ be arbitrary and define

$$A(n,\varepsilon) := \{\Big| \sum_{j=1}^{[tN_n]} n^{-1}(Y_j - 1)\Big| \geq \varepsilon\}.$$

Using the Markov inequality

$$\begin{aligned}\mathbf{P}(A(n,\varepsilon)) &= \mathbf{P}\{(\sum_{j=1}^{[tN_n]}(Y_j-1))^4 \geq n^4\varepsilon^4\}\\ &\leq (n\varepsilon)^{-4}E\{\sum_{j=1}^{[tN_n]}(Y_j-1)\}^4.\end{aligned}$$

Write $Z_j := Y_j - 1$; it is easy to check that

$$EZ_j = 0, \qquad EZ_j^2 = 1, \qquad EZ_j^4 = 9.$$

Hence

$$\begin{aligned}E\left(\sum_{j=1}^{[tN_n]} Z_j\right)^4 &= E_{N_n}\{[tN_n]EZ_1^4 + \binom{[tN_n]}{2}\binom{4}{2}(EZ_1^2)^2\}\\ &= 9E[tN_n] + 3E\{[tN_n]([tN_n]-1)\} \leq Kn^2,\end{aligned}$$

for some constant K, since $EN_n = n$ and $EN_n^2 = 2n^2 - n$. Apply Borel-Cantelli's lemma to obtain (2.5). Relation (2.3) and hence (ii) follows from (2.5) with $t = 1$ and the identity

$$n^{-1}\sum_{j=1}^{N_n} Y_j = \sum_{j=1}^{N_n} n^{-1}(Y_j - 1) + \frac{N_n}{n},$$

since by construction $n^{-1}N_n \to X$, $\mathbf{P}$ a.s.

(iii) For $t \in [0,1]$ we define

$$\begin{aligned}g_n(t) &:= n^{-1}\sum_{j=1}^{[tN_n]} Y_j,\\ g(t) &:= tX.\end{aligned}$$

We showed that for fixed t and $\omega \notin N$ with $\mathbf{P}(N) = 0$

$$\lim_{t\to\infty} g_n(t) = g(t).$$

The null set N may depend on t. Nevertheless, the statement implies $g_n(t) \to g(t)$ for $\mathbf{P}$ almost all $\omega \in \Omega$ and all rational $t \in [0,1]$, since the countable union of null sets is a null set. Both g_n and g are non-decreasing functions of t on [0,1], therefore continuity of g and convergence on the rationals entails pointwise convergence of $g_n \to g$, for $\mathbf{P}$ almost all $\omega \in \Omega$. It is well known that with the mentioned two properties (monotonicity of g_n and g and continuity of the limit) pointwise convergence on a closed, bounded interval implies uniform convergence. We hence conclude that $\mathcal{A}$ contains a $\mathbf{P}$ null set N so that for $\omega \notin N$,

$$\lim_{n\to\infty} \sup_{[0,1]} |g_n(t) - g(t)| = 0.$$

Observe that $n^{-1}N_n \to X$, $\mathbf{P}$ a.s. implies

$$\sup_{[0,1]} \left|\frac{[tN_n]}{n} - tX\right| \to 0, \quad \mathbf{P}\ a.s.$$

Relation (2.4) follows from the above two limit statements and the triangle inequality. ▮

3. Convergence of sojourn times. Denote by D_k the departure time of the k-th customer in the processor sharing queue and by $D_{k,n}$ the departure time of the k-th customer in the feedback queue. In this section we shall prove the almost sure convergence of $D_{k,n}$ to D_k for all k. The method we use is to introduce a Round Robin queue on the same probability space $(\Omega, \mathcal{A}, \mathbf{P})$. If we denote by $\tilde{D}_{k,n}$ the departure time of the k-th customer in the Round Robin queue we will successively prove: $\tilde{D}_{k,n} \to D_k$ and $D_{k,n} - \tilde{D}_{k,n} \to 0$, $\mathbf{P}$ a.s. The almost sure convergence of $D_{k,n}$ to D_k is an immediate consequence of these two results.

First we introduce the Round Robin queue with time slices n^{-1}. Each customer present in the system receives successively a deterministic time slice n^{-1} of service time, and then goes back to the end of the queue until he has obtained his entire service time. Assume that the Round Robin queue has the same arrival process $A_1, A_1 + A_2, \cdots$ as both the processor sharing and the feedback queue. Furthermore, assume that the k-th customer demands $N_{k,n}$ (the number of cycles in the feedback queue) time slices of service. This fixes the desired coupling between the Round Robin queue and the other models. Since $n^{-1}N_{k,n} \to X_k$, $\mathbf{P}$ a.s., the total amount of service demanded by each customer in the Round Robin queue approaches the amount in the processor sharing model. The proof of the following theorem is reminiscent of the proof of Theorem 4.1 of Schassberger [5].

Theorem 2.

For each $k \geq 1$ we have **P** *almost surely*

(i) $\tilde{D}_{k,n} \to D_k$,

(ii) $D_{k,n} - \tilde{D}_{k,n} \to 0$,

(iii) $D_{k,n} \to D_k$.

PROOF:

(i) Customers arrive at $A_1, A_1 + A_2, \cdots$. The first customer cannot remain in the processor sharing queue for ever, since he then would receive a service of A_2 in the period $[A_1, A_1 + A_2)$, at least $A_3/2$ in the period $[A_1 + A_2, A_1 + A_2 + A_3)$, and so forth, and we have

$$\sum_{j=2}^{\infty} \frac{A_j}{j-1} \to \infty,$$

P almost surely. Therefore

$$A_1 + \cdots + A_{m-1} \leq D_1 < A_1 + \cdots + A_m, \tag{3.1}$$

for some $m = m(\omega) \geq 2$. Let E_m be the set of all ω satisfying (3.1). For **P** almost every $\omega \in E_m$, $n^{-1}N_{k,n}(\omega) \to X_k(\omega)$, for $k = 1, 2, \cdots, m$. Let $\omega \in E_m$ with this property. Denote by $x_1(t)$ the amount of service obtained by customer 1 in the processor sharing queue up to time t and by $x_{1,n}(t)$ the corresponding amount in the Round Robin queue. Since customer 1 is delayed by at most $(m-1)$ other customers we have for $t = \min(D_1, \tilde{D}_{1,n})$,

$$|x_{1,n}(t) - x_1(t)| \leq \frac{m-1}{n} + \sum_{k=2}^{m-1} |X_k(\omega) - n^{-1}N_{k,n}(\omega)| \to 0.$$

Hence

$$|\tilde{D}_{1,n} - D_1| \leq m\left(\frac{m-1}{n} + \sum_{k=2}^{m-1} |X_k(\omega) - n^{-1}N_{k,n}(\omega)|\right) \to 0.$$

We conclude that $\tilde{D}_{1,n} \to D_1$ for **P** almost all $\omega \in E_m$. Since the countable union of null sets is a null set we have $\tilde{D}_{1,n} \to D_1$, **P** a.s. Repetition of the argument yields for each $k \geq 1 : \tilde{D}_{k,n} \to D_k$, **P** a.s.

(ii) According to (i),

$$\tilde{D}_{1,n} < A_1 + \cdots + A_m,$$

for $\mathbf{P}$ almost all $\omega \in E_m$ and n sufficiently large. From Theorem 1 we have

$$\max_{1 \le k \le m} \sup_{0 \le t \le 1} \left| \sum_{j=1}^{[tN_{k,n}]} n^{-1}(Y_{k,j} - 1) \right| \to 0, \quad \mathbf{P} \ a.s.$$

This implies that $(\tilde{D}_{1,n} - D_{1,n}) \to 0$, $\mathbf{P}$ a.s. The same argument leads to the statement: for each $k \ge 1, (\tilde{D}_{k,n} - D_{k,n}) \to 0, \mathbf{P}$ a.s.

(iii) Follows from (i) and (ii). ∎.

Next we use the theory of regenerative processes to show that for $\rho = \lambda < 1$ the steady state sojourn time distribution of the feedback model converges as $n \to \infty$ to the steady state sojourn time distribution of the processor sharing model. We follow the same line of thought as Schassberger [5], Theorem 4.2. Let $C_n(y)$ be the number of customers served during the first busy period in the feedback model having sojourn time less than or equal to y. The corresponding number in the processor sharing queue is denoted by $C(y)$. Similarly C_n and C denote the total number of customers served in the first busy period in the feedback and the processor sharing queue, respectively.

Lemma 1.

(i) $C_n \to C$, $\mathbf{P}$ a.s.

(ii) For each $y > 0$, $C_n(y) \to C(y)$, $\mathbf{P}$ a.s.

(iii) $\mathbf{E}C_n = \mathbf{E}C < \infty$,

(iv) For each $y > 0$, $\mathbf{E}C_n(y) \to \mathbf{E}C(y)$.

PROOF:

(i) and (ii). For those ω for which a finite number of customers are served during the first busy period in the processor sharing model (a set of probability one), $C_n(\omega) \to C(\omega)$ and $C_n(y,\omega) \to C(y,\omega)$, by part (iii) of Theorem 2.

(iii) Both the processor sharing queue and the feedback queue are examples of $M/M/1$ queues with different service disciplines. Statement (iii) follows since for these systems the number of customers served during a busy period does not depend on the service discipline.

(iv) Clearly $0 \le C_n(y) \le C_n$. By Fatou's lemma and (i) … (iii), we have

$$\liminf_{n \to \infty} \mathbf{E}C_n(y) \ge \mathbf{E}(\liminf_{n \to \infty} C_n(y)) = \mathbf{E}C(y).$$

On the other hand (apply Fatou's lemma on $C_n - C_n(y) \geq 0$),

$$\liminf_{n\to\infty} \mathsf{E}(C_n - C_n(y)) \geq \mathsf{E}(\liminf_{n\to\infty}(C_n - C_n(y)) = \mathsf{E}C - \mathsf{E}C(y),$$

and hence

$$\limsup_{n\to\infty} \mathsf{E}C_n(y) \leq \mathsf{E}C(y).$$

Therefore

$$\limsup_{n\to\infty} \mathsf{E}C_n(y) \leq \mathsf{E}C(y) \leq \liminf_{n\to\infty} \mathsf{E}C_n(y). \quad \blacksquare$$

Theorem 3.

For $\rho = \lambda < 1$, the steady state sojourn time of the $M/M/1$ processor sharing queue with mean service time 1, is the distributional limit of the steady state sojourn time of the M/M/1 feedback queue, with feedback mechanism given by $G(z) = z/(z - n(z-1))$ and exponential slices with mean length n^{-1}.

PROOF: This follows directly from $\mathsf{E}C_n(y)/\mathsf{E}C_n \to \mathsf{E}C(y)/\mathsf{E}C$ and the general theory of regenerative processes. $\blacksquare$

4. General service and or interarrival times. Van den Berg and Boxma [1] showed how to choose the feedback mechanism $G(z)$ to obtain more general total service times. It is possible to obtain all service time distributions that are finite mixtures of phase distributions. The random variable X is said to have a distribution which is finite mixture of phase distributions if it has Laplace transform

$$Ee^{-sX} = \sum_{j=1}^{m} \alpha_j \prod_{i=1}^{r_j} (1 + \mu_{ij} s)^{-1}, \tag{4.1}$$

where $(\alpha_1, \ldots, \alpha_m)$ is a probability vector, $r_1, \ldots, r_m$ are positive integers and μ_{ij} positive real numbers. (It is known that the subclass of distributions defined in (4.1) is dense in the space of all probability measures on $(0, \infty)$ equipped with the Prohorov distance, or some other equivalent metric.) Further we define for each $n \geq 1$ the integer valued random variable N_n by its generating function

$$G_n(z) = Ez^{N_n} = \sum_{k=1}^{\infty} z^k P\{N_n = k\} = \sum_{j=1}^{m} \alpha_j \prod_{i=1}^{r_j} \frac{(1-\nu_{ij})z}{1-\nu_{ij} z} \tag{4.2,}$$

where $\nu_{ij} := 1 - (n\mu_{ij})^{-1}$.
The following two lemmas originate from [1].

Lemma 2.

If $Y_1, Y_2, \ldots$ is an i.i.d. sequence of exponential random variables with mean 1, and $X_n := n^{-1} \sum_{j=1}^{N_n} Y_j$ then

$$Ee^{-sX_n} = \sum_{j=1}^{m} \alpha_j \prod_{i=1}^{r_j} (1 + \mu_{ij} s)^{-1},$$

i.e., for each $n \geq 1$ the distribution of X_n is identical to that of X in (4.1).

Lemma 3.

For the random variables X and N_n defined in (4.1) and (4.2) we have

$$n^{-1} N_n \to X,$$

in distribution.

Consider the $M/G/1$ processor sharing queue with service time distribution given in (4.1). It is clear from Lemma 2 and 3 that the $M/M/1$ feedback queue with feedback mechanism $G_n(z)$ given in (4.2) and exponential service slices of mean length n^{-1} converges in the sense of Theorem 2 and 3 to the above defined $M/G/1$ processor sharing queue. Indeed Lemma 2 and 3 show that we can repeat the almost sure construction of §2 for the more general service time distribution (4.1). The only other thing we needed for the proof of Theorem 1 was the first and second moment (especially the polynomial behaviour of these moments as functions of n) of N_n. However it is easy to check from (4.1) and (4.2) that

$$EN_n = nEX,$$
$$EN_n^2 = n^2 EX^2 - nEX.$$

Hence we have proved the following theorem.

Theorem 4.

For $\rho = EX_1/EA_1 < 1$, the steady-state sojourn time of the $M/G/1$ processor sharing queue with service time distribution given in (4.1) is the distributional limit as $n \to \infty$ of the corresponding random variable in the $M/M/1$ feedback queue with exponential slices of mean length n^{-1} and with the number of feedbacks governed by N_n given in (4.2).

Finally we note that the special (exponential) distribution of the interarrival times A_j, $j \geq 1$, is irrelevant. Theorem 4 is valid for general independent interarrival times, and even for dependent arrival processes.

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Multitype branching processes in M/G/1-queueing theory.

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1. Introduction. Consider a single server queueing system with infinite waiting room. Customers arrive at the system according to a Poisson process with intensity $\lambda > 0$. Each customer requires two services: a customer who enters the queue will return to the queue (feedback) after his first service for a second one. Fed back customers return instantaneously, joining the end of the queue. The service discipline is First Come First Served. The two service times of a customer are mutually independent random variables having distribution functions $B_i(t) = 1 - e^{-\mu_i t}$, $i = 1, 2$. These service times are also independent of the service times of other customers and of the arrival process. We are interested in the stationary joint distribution of type i customers in the system, where type i customers are those customers who visit the queue for the i-th time, $i = 1, 2$.

The state of the model at time t is given by $y(t) = \big(y_1(t), \cdots, y_{k(t)}(t)\big)$, where $k(t)$ is the number of customers in the system at time t and $y_j(t)$, $j = 1, \ldots, k(t)$ is the type of the customer who is in j-th position in FCFS order in the queue at time t . So the state space S consists of all finite sequences of 1's and 2's. It is easily checked that $y(t)$ is a continuous-time Markov chain and the stability condition for the system turns out to be $\frac{\lambda}{\mu_1} + \frac{\lambda}{\mu_2} < 1$. To find the equilibrium state probabilities we must solve the global balance equations, i.e. we must find the non-negative vector $(\pi(y) : y \in S)$ with $\sum_{y \in S} \pi(y) = 1$ such that

(1.1)
$$\lambda \pi(\emptyset) = \mu_2 \pi(2)$$
$$(\lambda + \mu_{y_1}) \pi(y_1, \cdots, y_n) = \mu_2 \pi(2, y_1, \cdots, y_n) + \lambda \pi(y_1, \cdots, y_{n-1}) 1[y_n = 1] + \mu_1 \pi(1, y_1, \cdots, y_{n-1}) 1[y_n = 2], \quad n \geq 1.$$

For general values of λ, μ_1 and μ_2 these equations seem to be difficult to solve. Only in the case $\mu_1 = \mu_2$ a solution of (1.1) is easily found. In that case

(1.2)
$$\pi(y_1, \cdots, y_n) = c \left(\frac{\lambda}{\mu_1} \right)^n$$

is a solution of (1.1). The constant c equals the probability of an empty system, i.e. $c = 1 - 2\frac{\lambda}{\mu_1}$.

The problem that arises in this simple model is typical for M/M/1 queues with different services for different types of customers. To describe the state of the system it is necessary not only to take into account the number of customers of different types in the system but also their order in the queue. This leads to a Markov chain with an intractable state space for which the balance equations are difficult to solve.

To avoid these difficulties we will introduce an imbedded sequence of times, called generation times, such that the number of customers of different types at these times only depends on the number of customers of different types at the previous generation time and not on the order of the customers in the queue at that time. In fact we will prove that the number of customers of different types at successive generation times is a multitype branching process with immigration. Consequently the theory of multitype branching processes can be used for the analysis of these models.

In Section 2 we will recall some of the terminology and theory of multitype branching processes. Furthermore a proof is given of a multitype version of a theorem of Pakes for singletype branching processes with state dependent immigration. In Section 3 we will introduce the notion of generation times. For various multitype M/G/1 queues these generation times are used to analyse the models. In Section 4 we do moment calculations, both for branching processes with immigration in each state and for branching processes with immigration only in state zero. In Section 5 we try to extend the results to M/G/1 queues with a countable number of customer types. Finally in Section 6 conclusions are given and some open questions are posed.

2. Multitype branching processes.

2.1. MTBP without immigration. We start this section with recalling some terminology and stating some theorems about multitype branching processes (see Athreya and Ney [1]).

Assume we have a finite number N of particle types. To define the particle production we need N generating functions, each in N variables,

$$f^{(i)}(s_1, \cdots, s_N) = \sum_{j_1, \cdots, j_N \geq 0} p^{(i)}(j_1, \cdots, j_N) s_1^{j_1} \cdots s_N^{j_N}, \tag{2.1}$$

$$0 \leq s_k \leq 1, \quad k = 1, \cdots, N.$$

where $p^{(i)}(j_1, \cdots, j_N)$ is the probability that a type i particle produces j_1 particles of type 1, j_2 of type 2, $\cdots$, j_N of type N, respectively. Let m_{ij} be the expected number of type j offspring of a single type i particle, i.e. $m_{ij} = \frac{\partial f^{(i)}}{\partial s_j}(1, \cdots, 1)$. An essential role is played by the mean matrix $M = (m_{ij}; i, j = 1, \cdots, N)$.

Throughout this section we assume that the matrix M is primitive, i.e. there is an n such that all entries of the matrix M^n are strictly positive. As a consequence of the Perron-Frobenius theorem for non-negative, primitive matrices (see Seneta [9]), there exists a positive real eigenvalue λ_{max} of M for which :

(1) $|\lambda| < \lambda_{max}$, for all other eigenvalues λ of M,

(2) the associated left eigenvector $v = (v_1, \cdots, v_N)$ with $\sum_{i=1}^N v_i = 1$ is unique and strictly positive.

Denote with $Z_n^{(i)}$ the number of type i particles at generation n and with Z_{ij} the number of type-j offspring of a type-i particle.

Theorem 2.1. *Assume we start the branching process with a single type j particle and let q_j be the probability of eventual extinction of the process. Furthermore assume that $EZ_{ij} \log Z_{ij} < \infty$ for all i, j. If $\lambda_{max} > 1$, then*

$$\lim_{n\to\infty} \frac{(Z_n^{(1)}, \cdots, Z_n^{(N)})}{\lambda_{max}^n} = (v_1 W, \cdots, v_N W) \quad a.s. \tag{2.2}$$

where W is a nonnegative random variable with $P(W = 0) = q_j$.

Corollary 2.2. *If the assumptions of Theorem 2.1 are satisfied, then conditioned on non-extinction of the branching process, i.e. conditioned on $W \neq 0$,*

$$\lim_{n\to\infty} \frac{Z_n^{(i)}}{Z_n^{(1)} + \cdots + Z_n^{(N)}} = v_i \quad a.s. \tag{2.3}$$

2.2. MTBP with state dependent immigration. Consider the multitype branching process with an independent immigration component at state zero. So in addition to the generating functions $f^{(i)}(s_1, \cdots, s_N), i = 1, \cdots, N$, representing the offspring distributions , an additional generating function $g(s_1, \cdots, s_N)$ is given, representing the immigration distribution whenever the branching process reaches state $(0, \cdots, 0)$, i.e.

$$g(s_1, \cdots, s_N) = \sum_{j_1, \cdots, j_N \geq 0} q(j_1, \cdots, j_N) s_1^{j_1} \cdots s_N^{j_N} \tag{2.4}$$

where $q(j_1, \cdots, j_N)$ is the probability that the group of immigrants consists of j_1 particles of type 1, j_2 of type 2,$\cdots$, j_N of type N, respectively. In this subsection we will prove the following multitype version of a theorem of Pakes (see Pakes [7]):

Theorem 2.3. *Let $Z_n = (Z_n^{(1)}, \cdots, Z_n^{(N)})$ be a multitype branching process with immigration at state zero with offspring generating functions*

$$f^{(i)}(s_1, \cdots, s_N), \quad i = 1, \cdots, N$$

and immigration generating function $g(s_1, \cdots, s_N)$. Assume the Markov chain Z_n to be aperiodic and irreducible. Let the mean matrix M corresponding to the branching process be primitive and its maximal eigenvalue $\lambda_{max} < 1$. Finally assume $Z_0 = (0, \cdots, 0)$. Then a necessary and sufficient condition for positive recurrence of Z_n is

$$\sum_{\substack{j_1, \cdots, j_N \geq 0 \\ j_1 + \cdots + j_N > 0}} q(j_1, \cdots, j_N) \log(j_1 + \cdots + j_N) < \infty. \tag{2.5}$$

When this condition is satisfied, the generating function $P(s_1, \cdots, s_N)$ of the stationary distribution $\{\pi(j_1, \cdots, j_N)\}$ of Z_n satisfies

$$P(s_1, \cdots, s_N) = 1 - \pi(0, \cdots, 0) \sum_{n=0}^{\infty} (1 - g(f_n(s_1, \cdots, s_N))) \tag{2.6}$$

where $f_n(s_1, \cdots, s_N)$ is defined inductively by

(2.7)
$$\begin{cases} f_0(s_1, \cdots, s_N) = (s_1, \cdots, s_N) \\ f_n(s_1, \cdots, s_N) = (f^{(1)}(f_{n-1}(s_1, \cdots, s_N)), \cdots, f^{(N)}(f_{n-1}(s_1, \cdots, s_N))). \end{cases}$$

and

$$\pi(0, \ldots, 0) = [1 + \sum_{n=0}^{\infty} (1 - g(f_n(0, \cdots, 0)))]^{-1}. \tag{2.8}$$

PROOF: Because of the assumption that Z_n is aperiodic and irreducible all states of the Markov chain are equivalent. Hence we restrict our attention to the state $(0, \cdots, 0)$. Let T be the recurrence time of state zero.

Lemma 2.4. *For $n \geq 1$ we have $P(T > n) = 1 - g(f_{n-1})$, where $f_{n-1} := f_{n-1}(0, \cdots, 0)$.*

PROOF: Let Y_n be the multitype branching process with the same offspring generating functions as Z_n. Furthermore $Y_0 = (0, \ldots, 0)$ and only at time zero there is an immigration with generating function $g(s_1, \ldots, s_N)$. Then the process Y_n has the same recurrence time of state zero as the process Z_n and the generating function of Y_n equals $g(f_{n-1}(s_1, \cdots, s_N))$. Hence $P(T > n) = P(Y_n \neq 0) = 1 - g(f_{n-1})$. ∎

From the fact that for multitype branching processes with $\lambda_{max} < 1$ we have $f_n(s_1, \cdots, s_N) \to (1, \cdots, 1)$ (see Athreya and Ney [1]) and hence $P(T > n) \to 0$ we conclude that the Markov chain Z_n is recurrent.

The expected recurrence time of state zero equals

$$\sum_{n=1}^{\infty} nP(T = n) = \sum_{n=0}^{\infty} P(T > n) = 1 + \sum_{n=1}^{\infty}(1 - g(f_{n-1})),$$

and hence Z_n is positive recurrent with $\pi(0, \cdots, 0) = [1 + \sum_{n=0}^{\infty}(1 - g(f_n))]^{-1}$ iff $\sum(1 - g(f_n)) < \infty$. See Kaplan [4] for the proof that this condition is equivalent with condition (2.5). In fact Kaplan concludes that if (2.5) is satisfied $\sum_{n=0}^{\infty}(1 - g(f_n(s_1, \cdots, s_N))) < \infty$ for all $(s_1, \cdots, s_N)$ with $0 \leq s_i \leq 1$, $i = 1, \cdots, N$.

The only thing that remains to prove is equation (2.6). Let P be the transition matrix of the Markov chain Z_n, i.e.

$$p_{i_1, \cdots, i_N; j_1, \cdots, j_N} = P(Z_{n+1} = (j_1, \cdots, j_N) | Z_n = (i_1, \cdots, i_N))$$

and define

$$P_{i_1, \cdots, i_N}(s_1, \cdots, s_N) = \sum_{j_1, \cdots, j_N \geq 0} p_{i_1, \cdots, i_N; j_1, \cdots, j_N} s_1^{j_1} \cdots s_N^{j_N}.$$

Then

$$\begin{aligned} P_{i_1, \cdots, i_N}(s_1, \cdots, s_N) &= g(s_1, \cdots, s_N) 1[(i_1, \cdots, i_N) = (0, \cdots, 0)] \\ &+ [f^{(1)}(s_1, \cdots, s_N)]^{i_1} \cdots [f^{(N)}(s_1, \cdots, s_N)]^{i_N} 1[(i_1, \cdots, i_N) \neq (0, \cdots, 0)] \end{aligned}$$

Now we use

$$\pi_{j_1, \cdots, j_N} = \sum_{i_1, \cdots, i_N} \pi_{i_1, \cdots, i_N} p_{i_1, \cdots, i_N; j_1, \cdots, j_N}$$

to conclude

$$
\begin{aligned}
P(s_1,\cdots,s_N) &= \sum_{j_1,\cdots,j_N} \pi_{j_1,\cdots,j_N} s_1^{j_1}\cdots s_N^{j_N} \\
&= \sum_{j_1,\cdots,j_N}\sum_{i_1,\cdots,i_N} \pi_{i_1,\cdots,i_N} p_{i_1,\cdots,i_N;j_1,\cdots,j_N} s_1^{j_1}\cdots s_N^{j_N} \\
&= \sum_{i_1,\cdots,i_N} \pi_{i_1,\cdots,i_N} P_{i_1,\cdots,i_N}(s_1,\cdots,s_N) \\
&= P(f_1(s_1,\cdots,s_N)) + \pi(0,\cdots,0)[g(s_1,\cdots,s_N)-1].
\end{aligned}
\tag{2.9}
$$

Iteration of this equation, together with $f_n(s_1,\cdots,s_N)\to 1$ and $\sum(g(f_n(s_1,\cdots,s_N))-1)<\infty$ yields (2.6). ∎

2.3. MTBP with immigration at each state. In this subsection we consider the same process as in the previous subsection except that there is immigration in every state and not only in state zero.

Theorem 2.5. *Let $(Z_n^{(1)},\cdots,Z_n^{(N)})$ be a multitype branching process with immigration at each state with offspring generating functions $f^{(i)}(s_1,\cdots,s_N)$, $i=1,\cdots,N$ and immigration generating function $g(s_1,\cdots,s_N)$. Let the mean matrix M corresponding to the branching process be primitive and its maximal eigenvalue $\lambda_{max}<1$. Assume the Markov chain Z_n is irreducible and aperiodic. Then a necessary and sufficient condition for the existence of a stationary distribution $\{\pi(j_1,\cdots,j_N)\}$ for the process $(Z_n^{(1)},\cdots,Z_n^{(N)})$ is*

$$
\sum_{\substack{j_1,\cdots,j_N\geq 0\\ j_1+\cdots+j_N>0}} q(j_1,\cdots,j_N)\log(j_1+\cdots+j_N)<\infty.
$$

When this condition is satisfied, the generating function $P(s_1,\cdots,s_N)$ of the distribution
$\{\pi(j_1,\cdots,j_N)\}$ satisfies

$$
P(s_1,\cdots,s_N)=\prod_{n=0}^{\infty} g(f_n(s_1,\cdots,s_N)) \tag{2.10}
$$

where $f_n(s_1,\cdots,s_N)$ is defined inductively by

$$
\begin{cases}
f_0(s_1,\cdots,s_N)=(s_1,\cdots,s_N)\\
f_n(s_1,\cdots,s_N)=(f^{(1)}(f_{n-1}(s_1,\cdots,s_N)),\cdots,f^{(N)}(f_{n-1}(s_1,\cdots,s_N))).
\end{cases}
$$

PROOF: See Quine [8].

The formula (2.10) is derived by iteration of

$$P(s_1,\cdots,s_N)=g(s_1,\cdots,s_N)P(f_1(s_1,\cdots,s_N)). \tag{2.11}$$

We will use (2.11) in Section 4 for moment calculations.

3. M/G/1 queueing models. In this section we will apply the theory of multitype branching processes to various M/G/1 queueing models. First we introduce the notion of generation times.

DEFINITION: The random times t_n, called generation times, are defined recursively :
(i) t_0 is the arrival time of the first customer,
(ii) t_{n+1} is the instant in which all customers, if any, present at t_n have obtained exactly one service in (t_n, t_{n+1}). If there are no customers present at t_n, then t_{n+1} is the instant of the first arrival after t_n.

3.1. The ordinary M/G/1 queue. Consider an ordinary single server queue in which customers arrive according to a Poisson process with rate λ and in which the service times are i.i.d. with distribution $B(\cdot)$, finite mean β and Laplace-Stieltjes transform $\beta(\cdot)$. Let X_t be the number of customers in the system at time t and $Z_n = X_{t_n}, n = 0, 1, 2, \cdots$.

Theorem 3.1. *The process Z_n is a single type branching process with immigration at state zero with offspring generating function $f(s) = \beta(\lambda(1-s))$ and immigration generating function $g(s) = s$. If $\lambda\beta < 1$ the generating function $P(s)$ of the stationary distribution (π_j) of Z_n satisfies*

$$P(s) = 1 - \pi_0 \sum_{n=0}^{\infty}(1 - f_n(s))$$

and

$$\pi_0 = [1 + \sum_{n=0}^{\infty}(1 - f_n(0))]^{-1}.$$

PROOF: Follows directly from the fact that the arrival process is Poisson, the definition of generation times and (the singletype version of) Theorem 2.3. ∎

REMARK: In Neuts [6] the M/G/1 queue is analyzed at random times t'_n, where the only difference between t_n and t'_n is that if the system is empty at time t'_n, then t'_{n+1} is defined as the instant of the first departure after t'_n. Hence in Neuts [6] the immigration generating function is given by $g(s) = \beta(\lambda(1-s))$ instead of $g(s) = s$.

3.2. The M/G/1 queue with permanent customers. Consider a single server queue with two types of customers:

(i) ordinary customers who arrive according to a Poisson process with rate λ.

(ii) permanent customers who immediately return to the end of the queue after having received a service.

The service times of the customers are independent, those of the ordinary customers with distribution function $B_1(\cdot)$, finite mean β_1 and Laplace-Stieltjes transform $\beta_1(\cdot)$, and those of the permanent customers with distribution function $B_2(\cdot)$, finite mean β_2 and Laplace-Stieltjes transform $\beta_2(\cdot)$.

Assume that there is only one permanent customer. Let Z_n denote the number of ordinary customers at the n-th service completion epoch of the permanent customer.

Theorem 3.2. *The process Z_n is a singletype branching process with immigration at each state. The offspring generating function is given by $f(s) = \beta_1(\lambda(1-s))$, the immigration generating function by $g(s) = \beta_2(\lambda(1-s))$. A stationary distribution of Z_n exists iff $\lambda\beta_1 < 1$. In that case the generating function $P(s)$ of this stationary distribution satisfies*

$$P(s) = \prod_{n=0}^{\infty} \beta_2(\lambda(1 - f_n(s))), \quad |s| \leq 1.$$

PROOF: Follows directly from the fact that the arrival process of ordinary customers is Poisson and Theorem 2.5. ∎

REMARK:

(i) The result of Theorem 3.2 was obtained earlier by Boxma and Cohen [3] without remarking that Z_n is a branching process with immigration.

(ii) A similar argument yields $P(s) = \prod_{n=0}^{\infty} \beta_2^K(\lambda(1 - f_n(s)))$ in the case that there are K permanent customers.

(iii) In van den Berg [2] the extension of an M/G/1 queue with Bernoulli feedback and additional permanent customers is considered. If there is only one permanent customer and the feedback parameter is given by p the offspring generating function is given by $f(s) = (1-p+ps)\beta_1(\lambda(1-s))$.The stationary distribution of Z_n exists iff $\lambda\beta_1 < 1-p$.

(iv) It is also possible to give a multitype variant of Theorem 3.2. Assume ordinary customers arrive according to independent Poisson processes with rates $\lambda_1, \cdots, \lambda_N$. Service times of type i customers have distribution function $B_i(\cdot)$, finite mean β_i and Laplace-Stieltjes transform $\beta_i(\cdot)$, $1 = 1, \cdots, N$. Service times of the permanent customer have distribution function B, finite mean β and Laplace-Stieltjes transform $\beta(\cdot)$. Then the offspring generating functions are given by $\beta_i(\sum_{j=1}^N \lambda_j(1-s_j))$ and the immigration generating function is given by $\beta(\sum_{j=1}^N \lambda_j(1-s_j))$. The stability condition is given by $\sum_{i=1}^N \lambda_i\beta_i < 1$.

3.3. The multitype M/G/1 queue with Markov routing. Consider the following multitype M/G/1 queue. Customers of different types arrive according to independent Poisson processes with rates $\lambda_1, \cdots, \lambda_N$. Type i customers have service times with distribution $B_i(\cdot)$, finite mean β_i and Laplace-Stieltjes transform $\beta_i(\cdot)$. After being served a type i customer returns to the end of the queue becoming a type j customer with probability p_{ij}, where $P = (p_{ij})$ is a substochastic matrix, i.e. (i) $p_{ij} \geq 0$ for all i and j and (ii) $\sum_{j=1}^N p_{ij} \leq 1$ for all i. With probability $1 - \sum_j p_{ij}$ a type i customer leaves the system after being served. Throughout this section it is assumed that all β_i's and at least one of the λ_i's are greater than zero.

Assumption 1: The matrix $I - P$ is non-singular

REMARK: Assumption 1 is equivalent to saying that all customers eventually leave the system with probability one.

Let $(X_t^1, \cdots, X_t^N)$ denote the number of customers of different types at time t and $Z_n = (Z_n^{(1)}, \cdots, Z_n^{(N)}) = (X_{t_n}^1, \cdots, X_{t_n}^N)$ the number of customers of different types at generation times.

Theorem 3.3. *The process Z_n is a multitype branching process with immigration at state zero where the immigration generating function is given by*

$$g(s_1, \cdots, s_N) = \sum_{i=1}^N \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_N} s_i,$$

and the offspring generating functions are given by

$$f^{(i)}(s_1,\cdots,s_N) = (1-\sum_{j=1}^{N} p_{ij} + \sum_{j=1}^{N} p_{ij}s_j)\beta_i(\sum_{j=1}^{N}\lambda_j(1-s_j)).$$

PROOF: Follows directly from the fact that the arrival processes are independent Poisson processes and the definition of generation times. The form of the immigration generating function follows from the fact that the probability that an arbitrary arrival is of type i equals $\lambda_i/(\lambda_1+\cdots+\lambda_N)$. ∎

The entry m_{ij} of the mean matrix M of the branching process is given by

$$m_{ij} = \lambda_j\beta_i + p_{ij}.$$

Assumption 2: The matrix M is primitive.

Lemma 3.4. *Let λ_{max} be the maximal eigenvalue of the matrix M and $b = (I-P)^{-1}\beta$, $\beta = (\beta_1,\ldots,\beta_N)$. Then*
(1) $\lambda_{max} > 1$ iff $\sum_{i=1}^{N}\lambda_i b_i > 1$,
(2) $\lambda_{max} = 1$ iff $\sum_{i=1}^{N}\lambda_i b_i = 1$,
(3) $\lambda_{max} < 1$ iff $\sum_{i=1}^{N}\lambda_i b_i < 1$.

PROOF: In the proof relations between vectors have to be read coordinatewise, i.e. for example for two vectors x and y, $x < y$ means $x_i < y_i$ for all $i = 1,\ldots,N$.

For an arbitrary vector x we have

$$Mx = Px + (\sum_{i=1}^{N}\lambda_i x_i)\beta$$

and hence

$$(I-M)b = (I-P)b - (\sum_{i=1}^{N}\lambda_i b_i)\beta$$

or

$$Mb = b - (1-\sum_{i=1}^{N}\lambda_i b_i)\beta.$$

We conclude
(1) $Mb > b$ iff $\sum_{i=1}^{N}\lambda_i b_i > 1$,
(2) $Mb = b$ iff $\sum_{i=1}^{N}\lambda_i b_i = 1$,

(3) $Mb < b$ iff $\sum_{i=1}^{N} \lambda_i b_i < 1$.

From the fact that $I - P$ is non-singular and P is a substochastic matrix it follows that $P^k \to 0$ and hence $b = \sum_{k=0}^{\infty} P^k \beta$. This implies $b \geq \beta > 0$. Let v' be the strict positive left eigenvector of M corresponding to the eigenvalue λ_{max} then it is easily seen that

(1) $\lambda_{max} > 1$ iff $v'Mb > v'b$ iff $Mb > b$,

(2) $\lambda_{max} = 1$ iff $v'Mb = v'b$ iff $Mb = b$,

(3) $\lambda_{max} < 1$ iff $v'Mb < v'b$ iff $Mb < b$, and the lemma follows. ∎

REMARK: The component b_i of the vector b represents the mean total service time of an arriving type i customer.

We are now ready to prove the following theorem

Theorem 3.5. *Let $Z_n = (Z_n^{(1)}, \cdots, Z_n^{(N)})$ be the joint queue length process at generation times in a multitype M/G/1 queue with Markov routing satisfying assumptions 1 and 2. Let b be as in Lemma 3.4 and assume $\sum_{i=1}^{N} \lambda_i b_i > 1$. Then*

$$\lim_{n\to\infty} \frac{Z_n^{(i)}}{Z_n^{(1)} + \cdots + Z_n^{(N)}} = v_i \quad \text{a.s.},$$

with v the positive left eigenvector with $\sum_{i=1}^{N} v_i = 1$ corresponding to the greatest eigenvalue λ_{max}.

PROOF: The theorem is a direct consequence of Corollary 2.2 because it is easily verified that the assumptions of Theorem 2.1 are satisfied (see Lemma 3.4). Note that we do not have to worry about extinction of the branching process because $\lambda_{max} > 1$ implies that with probability 1 we have an infinite busy period after finite time and hence a "non-extincting" branching process.

Theorem 3.6. *Let Z_n be the joint queue length process at generation times in a multitype M/G/1 queue with Markov routing satisfying assumptions 1 and 2. Let b be as in Lemma 3.4 and assume $\sum_{i=1}^{N} \lambda_i b_i < 1$. Then the generating function $P(s_1, \cdots, s_N)$ of the stationary distribution $\pi(j_1, \cdots, j_N)$ of Z_n satisfies*

$$P(s_1, \cdots, s_N) = 1 - \pi(0, \cdots, 0) \sum_{n=0}^{\infty} (1 - g(f_n(s_1, \cdots, s_N)))$$

with

$$\pi(0, \cdots, 0) = 1 + \sum_{n=0}^{\infty} (1 - g(f_n(0, \cdots, 0))).$$

PROOF: Follows directly from Theorem 3.3, Lemma 3.4 and Theorem 2.3 . ∎

EXAMPLE 1: Feedback queue

Customers arrive at the system according to a Poisson process with intensity λ. When a newly arriving customer, called a type 1 customer, has received his service, he departs from the system with probability $1 - p_1$ and is fed back to the end of the queue with probability p_1; in the latter case he becomes a type 2 customer. Similarly, when a customer has received its i-th service, he leaves with probability $1 - p_i$ and cycles back with probability p_i, becoming a type $i + 1$ customer. To avoid the problems that occur in dealing with an infinite number of customer types, we assume $p_N = 0$ for some $N \geq 1$.

This feedback queue is a special case of a multitype M/G/1 queue with Markov routing where $\lambda_1 = \lambda$, $\lambda_2 = \cdots = \lambda_N = 0$ and $p_{i,i+1} = p_i, i = 1, \cdots, N-1$, $p_{ij} = 0$ otherwise. It is easily checked that $b_1 = \sum_{i=1}^N \beta_i \prod_{j=1}^{i-1} p_j$ and hence the stability condition for the system equals $\lambda \sum_{i=1}^N \beta_i \prod_{j=1}^{i-1} p_j < 1$.

EXAMPLE 2: Multitype M/G/1 queue

Customers of different types arrive according to independent Poisson processes with rates λ_i, $i = 1, \cdots, N$ and leave the system after getting service. This is a special case of a multitype M/G/1 queue with Markov routing where $p_{ij} = 0$, for all i and j. In this case $b_i = \beta_i$ and hence the stability condition for the system is $\rho := \sum_{i=1}^N \lambda_i \beta_i < 1$. The eigenvector v equals $v_i = \frac{\lambda_i}{\lambda_1 + \cdots + \lambda_N}$ and hence the fraction of type i customers in the system at generation times converges almost surely to $\frac{\lambda_i}{\lambda_1 + \cdots + \lambda_N}$ when $\rho > 1$.

4. Moment calculations. In this section we will calculate first and second moments of the stationary distribution of multitype branching processes with immigration at state zero and immigration at each state. For the calculation of the first moment we assume the existence of offspring and immigration means. For the calculation of the second moments we assume the existence of offspring and immigration covariances.

4.1. MTBP with immigration at state zero. From formula (2.9) we obtain

$$\frac{\partial P}{\partial s_i}(1,\cdots,1) = \pi(0,\cdots,0)\frac{\partial g}{\partial s_i}(1,\cdots,1) + \sum_k \frac{\partial P}{\partial s_k}(1,\cdots,1)\frac{\partial f_k}{\partial s_i}(1,\cdots,1).$$

$$\begin{aligned}\frac{\partial^2 P}{\partial s_i \partial s_j}(1,\cdots,1) &= \pi(0,\cdots,0)\frac{\partial^2 g}{\partial s_i \partial s_j}(1,\cdots,1) \\ &+ \sum_k \frac{\partial P}{\partial s_k}(1,\cdots,1)\frac{\partial^2 f_k}{\partial s_i \partial s_j}(1,\cdots,1) \\ &+ \sum_{k,l} \frac{\partial^2 P}{\partial s_k \partial s_l}(1,\cdots,1)\frac{\partial f_k}{\partial s_i}(1,\cdots,1)\frac{\partial f_l}{\partial s_j}(1,\cdots,1).\end{aligned}$$

Of course we have

$$\begin{aligned}E(X_i) &= \frac{\partial P}{\partial s_i}(1,\cdots,1) \\ E(X_i X_j) &= \frac{\partial^2 P}{\partial s_i \partial s_j}(1,\cdots,1) + \frac{\partial P}{\partial s_i}(1,\cdots,1)\delta_{ij}\end{aligned}$$

where δ denotes Kronecker delta.

EXAMPLE: Consider the feedback queue of Example 1 in the previous section. For the vector of means we have

$$\begin{pmatrix} EX_1 \\ \vdots \\ EX_N \end{pmatrix} = \pi(0,\cdots,0)(I-M)^{-1}\begin{pmatrix} \frac{\partial g}{\partial s_1} \\ \vdots \\ \frac{\partial g}{\partial s_N} \end{pmatrix} = \pi(0,\cdots,0)(I-M)^{-1}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

with

$$M = \begin{pmatrix} \lambda\beta_1 & p_1 & 0 & \dots & 0 \\ \lambda\beta_2 & 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda\beta_{N-1} & 0 & 0 & \dots & p_{N-1} \\ \lambda\beta_N & 0 & 0 & \dots & 0 \end{pmatrix}.$$

The number $\pi(0,\cdots,0)$ has to be calculated numerically. Fortunately we have geometrical convergence of $f_n(s_1,\cdots,s_N)$ to $(1,\cdots,1)$ hence a finite sum in (2.8) provides a good approximation of $\pi(0,\cdots,0)$.

4.2. MTBP with immigration at each state. From formula (2.11) we obtain

$$\frac{\partial P}{\partial s_i}(1,\cdots,1) = \frac{\partial g}{\partial s_i}(1,\cdots,1) + \sum_k \frac{\partial P}{\partial s_k}(1,\cdots,1)\frac{\partial f_k}{\partial s_i}(1,\cdots,1).$$

$$\begin{aligned}\frac{\partial^2 P}{\partial s_i \partial s_j}(1,\cdots,1) = {}& \frac{\partial^2 g}{\partial s_i \partial s_j}(1,\cdots,1) + \sum_k \frac{\partial P}{\partial s_k}(1,\cdots,1)[\frac{\partial^2 f_k}{\partial s_i \partial s_j}(1,\cdots,1) \\ & + \frac{\partial g}{\partial s_i}(1,\cdots,1)\frac{\partial f_k}{\partial s_j}(1,\cdots,1) + \frac{\partial g}{\partial s_j}(1,\cdots,1)\frac{\partial f_k}{\partial s_i}(1,\cdots,1) \\ & + \sum_{k,l} \frac{\partial^2 P}{\partial s_k \partial s_l}(1,\cdots,1)\frac{\partial f_k}{\partial s_i}(1,\cdots,1)\frac{\partial f_l}{\partial s_j}(1,\cdots,1).\end{aligned}$$

EXAMPLE: Consider the multitype M/G/1 model with one permanent customer of remark (iv) in Section 3.2. In this case we have

$$\begin{pmatrix} EX_1 \\ \vdots \\ EX_N \end{pmatrix} = (I-M)^{-1} \begin{pmatrix} \frac{\partial g}{\partial s_1} \\ \vdots \\ \frac{\partial g}{\partial s_N} \end{pmatrix}$$

with

$$\frac{\partial g}{\partial s_j} = \lambda_j \beta$$

and

$$m_{ij} = \lambda_j \beta_i.$$

5. Branching processes with countably many types. Consider once more the feedback system of Example 1 in Section 3. For convenience we assumed the existence of an N such that $p_N = 0$. In this section we investigate what happens if we drop this assumption. This brings us to the theory of branching processes with countably many types.

So we now have an infinite sequence $B_i(\cdot)$ of service time distributions with means $0 < \beta_i < \infty$ and Laplace-Stieltjes transforms $\beta_i(\cdot)$ and an infinite sequence of probabilities $p_i > 0$, representing the probability that a type i customer, after receiving his service, cycles back, becoming a type $(i+1)$ customer.

A main role in the analysis is played once again by the mean matrix $M = (m_{ij})$, where m_{ij} is the expected number of type j offspring of a type i particle. We have

$$m_{i1} = \lambda\beta_i$$
$$m_{i,i+1} = p_i$$
$$m_{ij} = 0, \quad \text{otherwise.}$$

The following lemma is easily checked to be true.

Lemma 5.1. *All entries $m_{ij}^{(n)}$ of the matrix M^n are finite. Furthermore the matrix M is irreducible, i.e. for all i and j there exists an n such that $m_{ij}^{(n)}$ is strictly positive.*

It is well-known (see Seneta [9]) that for non-negative matrices M, satisfying the properties mentioned in Lemma 5.1 there exists a unique non-negative number R which is the common radius of convergence of the power series $\sum_{n=0}^{\infty} m_{ij}^{(n)} s^n$ for each pair i, j.

Lemma 5.2. *If*

$$1 < \lambda \sum_{i=1}^{\infty} \beta_i \prod_{j=1}^{i-1} p_j < \infty, \tag{5.1}$$

then there exist strictly positive vectors $u' = (u_1, u_2, \ldots)$ and $v' = (v_1, v_2, \ldots)$ and a number $r > 1$ such that

(1) $v'M = rv'$,

(2) $Mu = ru$,

(3) $\sum u_i v_i < \infty$.

PROOF: If r is a solution of the equation

$$\lambda \sum_{i=1}^{\infty} \frac{\beta_i}{r^{i-1}} \prod_{j=1}^{i-1} p_j = r \tag{5.2}$$

then it is easily checked that

$$v_1 = 1,$$
$$v_{i+1} = \left(\prod_{j=1}^{i} p_j\right) / r^i,$$

and

$$u_i = 1,$$

$$u_{i+1} = \frac{r^i}{(\prod_{j=1}^{i} p_j)} - \lambda \sum_{j=1}^{i} \beta_j r^{i-j} / (\prod_{k=j}^{i} p_k),$$

satisfy (1) and (2). Furthermore condition (5.1) is sufficient for the existence of an $r > 1$ satisfying (5.2) and for the finiteness of $\sum u_i v_i$.

From the existence of u, v and r satisfying (1), (2) and (3) in Lemma 5.2 we conclude that $R = r^{-1}$ (see Seneta [9], Theorem 6.4). The number r will take the place of λ_{max} in the case of branching processes with countably many types (see Moy [5]). In Moy [5] a theorem similar to Theorem 2.1 is proven in the case that $R < 1$ and that there exist u, v and r satisfying (1), (2) and (3). The only difference is mean square convergence instead of almost sure convergence. Furthermore an extra condition is imposed so that all $Z_n^{(i)}, i = 1, 2, \ldots$ have finite second moments, where $Z_n^{(i)}$ is once again the number of type i particles at time n, see Moy [5].

We do not know analogous results to Theorem 2.3 and Theorem 2.5 in the case of countably many customer types.

6. Conclusions and open questions. A general approach is given for the analysis of the joint queue length process in M/G/1 queues with multiple customer types and different service times for different types of customers. The used method is to define an imbedded sequence of times, called generation times. The joint queue length process at these generation times behaves as a multitype branching process with immigration at state zero. Also in the case that some customers stay in the system for ever, so called permanent customers, the introduction of generation times is useful. In this case the process behaves as a multitype branching process with immigration at each state.

The main question that remains unanswered in this paper is how to use knowledge about the joint queue length process at generation times to analyze the joint queue length process at arbitrary times. For example we expect that for multitype M/G/1 queues with Markov routing satisfying the assumptions in section 3.3 and satisfying $\sum \lambda_i b_i > 1$ a result similar to Theorem 3.5 holds at arbitrary times, i.e. we expect

$$\lim_{t \to \infty} \frac{X_t^i}{X_t^1 + \cdots + X_t^N} \to v_i, \quad \text{a.s.}$$

A second open problem is how to relate in the same model, if $\sum \lambda_i b_i < 1$ the joint stationary queue length distribution at generation times to the joint stationary queue length distribution at arbitrary times. In the queueing model of Section 3.2 with one permanent customer for example this relation is known, see v.d Berg [2]. If $R(s)$ denotes the generating function of the distribution of the number of ordinary customers at an arbitrary epoch and $Q(s)$ denotes the generating function of the distribution of the number of customers at an arbitrary epoch in the corresponding M/G/1 queue (i.e. the same model without a permanent customer), then

$$R(s) = \frac{P(s)}{\beta_2(\lambda(1-s))} \frac{1-\beta_2(\lambda(1-s))}{\beta_2\lambda(1-s)} Q(s),$$

with $P(s)$ given by Theorem 3.2. For the multitype M/G/1 queue such a relation is unknown. The last open question, already mentioned in the previous section, is how to extend the results of Section 2 to branching processes with countably many types.

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ASYMPTOTISCHE RESULTATEN IN SYSTEMEN MET TERUGKOPPELING

SAMENVATTING

In dit proefschrift bekijken we twee soorten systemen met terugkoppeling, te weten processen met terugkoppeling en wachtrijmodellen met terugkoppeling. Stochastische processen van de vorm

$$X_{n+1} = f_n(X_n, Y_n),$$

met $f_1, f_2, \ldots$ een rij functies en $Y_1, Y_2, \ldots$ een rij van stochastische variabelen worden *processen met terugkoppeling* genoemd. Wachtrijmodellen waarin klanten, na bediend te zijn, terug kunnen keren naar de bedieningsfaciliteit voor ze het systeem verlaten, worden *wachtrijmodellen met terugkoppeling* genoemd.

In het eerste gedeelte van dit proefschrift worden twee processen met terugkoppeling bestudeerd. Het eerste proces is een model voor productieprocessen. Een gemeenschappelijke eigenschap van productieprocessen is dat machines niet onafhankelijk van elkaar werken. Dit impliceert dat sommige machines geen nieuwe activiteit kunnen starten totdat sommige andere machines hun huidige activiteit beëindigd hebben. Dit leidt tot de volgende wiskundige beschrijving van het proces:

$$x_i(n+1) = \max_{1 \le j \le p} (a_{ij}(n) + x_j(n)), \quad i = 1, \ldots, p.$$

Hierin is p het aantal machines, $x_i(n)$ het tijdstip waarop machine i voor de n-de keer actief wordt en $a_{ij}(n)$ de som van de n-de procestijd van machine j en de n-de transporttijd van machine j naar machine i. In het eerste artikel in dit proefschrift bestuderen we het asymptotisch gedrag van $x_i(n)$ voor $n \to \infty$ in het geval de tijden $a_{ij}(n)$ stochastisch zijn.

Het tweede proces dat bestudeerd wordt is van de vorm

$$X_{n+1} = \max(X_n, \alpha_n X_n + Y_n)$$

met $\alpha_1, \alpha_2, \ldots$ een rij parameters tussen 0 en 1, en $Y_1, Y_2, \ldots$ een rij van onafhankelijke, gelijkverdeelde, niet-negatieve stochastische variabelen. Twee randgevallen voor de rij van parameters, namelijk $\alpha_n = 0$ voor alle n en $\alpha_n = 1$

voor alle n, leiden tot de extreme waarden theorie respectievelijk de theorie van sommen van onafhankelijke stochasten. Gezien het totaal verschillende asymptotische gedrag van X_n voor $n \to \infty$ in deze twee gevallen is het interessant naar rijen $\alpha_1, \alpha_2, \ldots$ tussen 0 en 1 te kijken. In het tweede artikel in dit proefschrift bestuderen we het asymptotisch gedrag van X_n voor rijen $\alpha_1, \alpha_2, \ldots$ die naar 1 stijgen voor $n \to \infty$.

In het tweede deel van dit proefschrift worden wachtrijmodellen met terugkoppeling bestudeerd. Het basismodel dat bestudeerd wordt, is een model bestaande uit één bediende en een wachtrij waarbij klanten arriveren volgens een Poisson proces. Nadat een klant een negatief exponentiëel verdeelde bedieningstijd met gemiddelde β heeft ontvangen keert hij terug naar het einde van de wachtrij met kans $p(i)$ en verlaat hij het systeem met kans $1 - p(i)$, waarbij i het aantal bedieningen is dat de klant reeds gehad heeft. Door op geschikte wijze de gemiddelde bedieningsduur β en de terugkeerkansen $p(i)$ te variëren blijkt een rij van modellen gevonden te kunnen worden die steeds meer op het M/G/1-"processor sharing" model gaan lijken. Dit is een één-bediende model met een willekeurige verdeling van de bedieningsduur waarin alle klanten tegelijk en met dezelfde snelheid bediend worden zodanig dat de totale bedieningssnelheid van de bediende constant blijft. In het derde artikel in dit proefschrift wordt een bewijs gegeven van de convergentie in verdeling van de stationaire verdeling van de verblijftijd in de modellen met terugkoppeling naar de stationaire verdeling van de verblijftijd in het "processor sharing" model.

In het vierde artikel in dit proefschrift worden modellen met terugkoppeling bestudeerd waarin behalve de terugkeerkans $p(i)$ ook de verdeling $B_i(\cdot)$ van de bedieningsduur van een klant afhangt van het aantal keren i dat de klant reeds bediend is. We zijn voor dit soort modellen geïnteresseerd in de gemeenschappelijke stationaire verdeling van het aantal klanten van type i in het systeem. Een klant is van type i als hij voor de i-de keer in de wachtrij staat. Het vinden van deze verdeling op een willekeurig tijdstip blijkt moeilijk te zijn. Daarom definiëren we een ingebedde rij tijdstippen, die we generatietijdstippen noemen. Het aantal klanten in de wachtrij van verschillende types op generatietijdstippen blijkt namelijk een vertakkingsproces met meerdere type deeltjes te zijn. De algemene theorie van dit soort vertakkingsprocessen stelt ons in staat een uitdrukking te vinden voor de genererende functie van de gemeenschappelijke stationaire verdeling van het aantal klanten in de wachtrij van verschillende types op generatietijdstippen.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 7 augustus 1962 geboren te Utrecht. In 1980 behaalde hij het Gymnasium-β diploma aan het Sint Bonifatius College te Utrecht. Vervolgens ging hij wiskunde studeren aan de Rijksuniversiteit Utrecht. In juni 1986 studeerde hij af in de Toegepaste Wiskunde met de bijvakken Informatica en Capita Selecta van de Wiskunde. Het afstudeerwerk, uitgevoerd onder leiding van Prof. dr. ir. J.W. Cohen, omvatte de analyse van een twee-dimensionaal wachtrijsysteem. Gedurende de periode januari 1984 - juni 1986 was hij als student assistent actief bij de subfaculteit wiskunde. Van november 1986 tot november 1990 was hij als A.I.O. werkzaam aan de Technische Universiteit Delft. Het onderzoek dat hij gedurende die periode verrichtte onder begeleiding van Prof.dr. M.S. Keane en Dr. G. Hooghiemstra heeft geleid tot de totstandkoming van dit proefschrift.

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