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Nonlinear Observer-based Actuator Fault Diagnosis for Lipschitz Nonlinear Systems

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***Abstract*—This paper presents a nonlinear observer-based actuator fault diagnosis framework for a class of Lipschitz nonlinear systems in the presence of actuator faults, including both multiplicative and additive types. The proposed diagnosis algorithm leverages a nonlinear observer designed using a modified linear quadratic Riccati equation that ensures exponential convergence of the state estimation error, independent of the Lipschitz constant. A deterministic threshold-based detection mechanism is then developed to detect fault occurrences. The framework further distinguishes between multiplicative and additive faults by leveraging specific control strategies and dynamic analysis. Theoretical results are validated with rigorous stability proofs, ensuring robustness and fault isolation capabilities. This framework provides a systematic solution for fault detection, isolation, and diagnosis in complex nonlinear systems.**

***Index Terms*—Nonlinear observer, fault diagnosis, Lipschitz nonlinear systems, actuator fault.**

I. INTRODUCTION

The increasing complexity and performance requirements of modern dynamical systems, particularly in industrial applications, have heightened the need for robust fault detection, isolation, and diagnosis techniques. These systems often operate in uncertain environments, subject to various disturbances, sensor noise, actuator faults, and nonlinear properties. Hence, ensuring the reliable operation of such systems is critical, as undetected faults can lead to significant performance degradation, system failure, or safety hazards [1].

The primary challenge in fault detection lies in the accurate identification and isolation of faults in the presence of potential simultaneously existed additive and multiplicative faults [2], which is sometimes hard even impossible to distinguish their corresponding fault features. Traditional fault detection methods, such as model-based approaches [3], have shown promise but often fall short when dealing with the complex nonlinearities inherent in real-world systems. In [4], a joint estimation method is proposed for additive and parametric (multiplicative) faults in a linear system, utilizing linear fractional representation. In [5], an adaptive fuzzy fault-tolerant control strategy is designed for Markov jump systems under the presence of both additive and multiplicative actuator faults, although without incorporating fault detection mechanism.

Motivated by these observations, this paper gives a nonlinear observer-based actuator fault diagnosis framework for a class of Lipschitz nonlinear in the presence of each or both additive and multiplicative fault types. The key ideas of such framework contain twofold aspect: *i)*, once fault detected, decreasing the control input to detect whether or not there exists additive fault, and *ii)*, once additive fault is identified with unbiased performance, injecting non-zero control input to detect whether or not there exists multiplicative fault. In the meantime, the proposed diagnosis algorithm leverages a nonlinear observer designed using a modified linear quadratic Riccati equation that ensures exponential convergence of the state estimation error, independent of the Lipschitz constant.

II. MODEL DESCRIPTION AND PROBLEM FORMULATION

Consider Lipschitz nonlinear systems with actuator faults as

$$\begin{cases} \dot{x} = \mathcal{A}x + \mathcal{B}\left(\mathcal{F}'u + \mathcal{F}''\right) + \mathcal{G}f\left(x\right) \\ y = \mathcal{C}x \end{cases} \tag{1}$$

where $x = \left[x_1, \cdots, x_n\right]^\top \in \mathbb{R}^n$ is system state vector, $u \in \mathbb{R}^m$ is system control input, $y \in \mathbb{R}^p$ is measurable system output, each element in $\{n, m, p, q\} \in \mathbb{N}^+$, $f(x) \in \mathbb{R}^q$ is nonlinear function vector satisfying a Lipschitz condition

$$\|f\left(x_1\right) - f\left(x_2\right)\| \leq \lambda\|x_1 - x_2\| \tag{2}$$

where $\lambda \in \mathbb{R}^+$ is known as the Lipschitz constant. $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, $\mathcal{G} \in \mathbb{R}^{n\times q}$ and $\mathcal{C} \in \mathbb{R}^{p\times n}$ are known system matrices, $\mathcal{F}' \in \mathbb{R}^{m\times m}$ and $\mathcal{F}'' \in \mathbb{R}^m$ are actuator fault matrices with $\mathcal{F}'$ and $\mathcal{F}''$ being the multiplicative and additive fault matrices, respectively. For the guarantee of observability using low-dimensional measured output y, it is reasonable to assume that $\dim\left(y\right) \geq \dim\left(f(x)\right)$. While, for simplicity, it is assumed that $\dim\left(y\right) = \dim\left(f(x)\right)$. Such assumption is reasonable because of that the case $\dim\left(y\right) > \dim\left(f(x)\right)$ can be treated by enlarging the dimension of $f(x)$ using available information to $\dim\left(y\right) = \dim\left(f(x)\right)$. In the meantime, it is assumed that the pair $\left(\mathcal{A}, \mathcal{C}\right)$ is observable, the pair $\left(\mathcal{A}, \mathcal{G}\right)$ is controllable, the square system $\left(\mathcal{A} + \delta I, \mathcal{G}, \mathcal{C}\right)$ is minimum phase one (has only stable zeros) for some constant $\delta \in \mathbb{R}^+$.

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The goal is to design an appropriate nonlinear observer and decision mechanism to determine the timing of the occurrence of at least one type of fault, either multiplicative, additive, or both, and to accurately identify the specific fault(s), enabling further isolating and processing.

III. MAIN RESULTS

A. Nonlinear Observer

In the absence of actuator faults, the following nonlinear observer can be used:

$$\dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}u + \mathcal{L}\left(y - \mathcal{C}\hat{x}\right) + \mathcal{G}f\left(\hat{x}\right), \ \mathcal{L} = Q\mathcal{C}^\top \tag{3}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimation of x, and $\mathcal{L} \in \mathbb{R}^{n\times p}$ is observer gain matrix by linear LTR design [6], and matrix Q is obtained by solving the following linear quadratic Riccati equation:

$$Q\left(\mathcal{A} + \delta I\right)^\top + \left(\mathcal{A} + \delta I\right)Q - Q\mathcal{C}^\top CQ + \varepsilon\mathcal{G}\mathcal{G}^\top = 0 \tag{4}$$

with $\delta \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$. According to [7], the existence of a positive definite solution Q for (4) can be guaranteed for any $\delta \in \mathbb{R}^+$ and $\varepsilon \in \mathbb{R}^+$, provided that the pair (A, C) is observable and the pair (A, G) is controllable, as assumed in the previous subsection. It is worth emphasizing that the Riccati equation in (4) is independent of the Lipschitz coefficient λ, ensuring the applicability of the observer to a broad range of plants under consideration.

***Lemma* 1.** *[8] For the nominal nonlinear system without actuator faults, as described in (1), where both $\mathcal{C}$ and $\mathcal{G}$ are full-rank, $\operatorname{rank}\left(\mathcal{C}^\top\right) = \operatorname{rank}\left(\mathcal{G}\right)$, and $(\mathcal{A} + \delta I, \mathcal{G}, \mathcal{C})$ is minimum phase, the solution Q of the Riccati equation (4) satisfies: $\lim\limits_{\varepsilon\to\infty} \frac{Q(\varepsilon)}{\varepsilon} = 0$. Furthermore, regardless of the magnitude of the Lipschitz constant λ in (2), if the design parameter ε is chosen to be sufficiently large, the state estimate $\hat{x}$ generated by the nonlinear observer in (14) will converge to the true state x at an exponential rate.*

B. Fault Detection and Isolation

By nonlinear observer (14), it is known that $\lim\limits_{t\to\infty} \hat{x} = x$, hence, $\lim\limits_{t\to\infty} \tilde{y} \overset{\text{def}}{=} \lim\limits_{t\to\infty} \left(y - \hat{y}\right) = \lim\limits_{t\to\infty} \left(\mathcal{C}x - \mathcal{C}\hat{x}\right) = 0$. The following deterministic threshold-based logic can be designed to detect the occurrence of at least one type of fault, either multiplicative, additive, or both:

$$\begin{cases} \|\tilde{y}\| \geq \chi, & \text{at least one type or both occur} \\ \|\tilde{y}\| < \chi, & \text{fault free} \end{cases} \tag{5}$$

for some $\chi \in \mathbb{R}^+$. Next, the following theorem presents the first step in detecting the existence of an additive fault.

Theorem 1. *[Detection of additive fault] Decreasing the control input to as small as possible, stationary or non-stationary. The following threshold-based logic can be used to detect the occurrence of additive fault for some $\chi'' \in \mathbb{R}^+$:*

$$\begin{cases} \|\tilde{y}\| \geq \chi'', & \textit{additive fault occurs} \\ \|\tilde{y}\| < \chi'', & \textit{additive-fault free} \end{cases} \tag{6}$$

Proof. By decreasing the control input to as small as possible, one knows that $u \approx 0$. Hence, the dynamics of state observation error $\tilde{x} \overset{\text{def}}{=} x - \hat{x}$ is known to be governed by:

$$\begin{aligned} \dot{\tilde{x}} &= \left(\mathcal{A} - \mathcal{L}\mathcal{C}\right)\tilde{x} + \mathcal{B}\left(\mathcal{F}'u + \mathcal{F}''\right) - \mathcal{B}u + \mathcal{G}\left(f\left(x\right) - f\left(\hat{x}\right)\right) \\ &\approx \left(\mathcal{A} - \mathcal{L}\mathcal{C}\right)\tilde{x} + \mathcal{B}\mathcal{F}'' + \mathcal{G}\left(f\left(x\right) - f\left(\hat{x}\right)\right) \end{aligned} \tag{7}$$

Choose a radially unbounded positive definite Lyapunov function as $V = \tilde{x}^\top Q^{-1}\tilde{x}$, its time derivative along (7) can be calculated as

$$\begin{aligned} \dot{V} =& -2\delta V - \tilde{x}^\top\mathcal{C}^\top\mathcal{C}\tilde{x} - \varepsilon\tilde{x}Q^{-1}\mathcal{G}\mathcal{G}^\top Q^{-1}\tilde{x} \\ &+ 2\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\tilde{x} + 2\tilde{x}^\top Q^{-1}\mathcal{G}\left(f\left(x\right) - f\left(\hat{x}\right)\right) \\ \leq& -2\delta V - \|\tilde{y}\|^2 - \varepsilon\|\mathcal{G}^\top Q^{-1}\tilde{x}\|^2 + 2\lambda\|\tilde{x}\| \cdot \|\mathcal{G}^\top Q^{-1}\tilde{x}\| \\ &+ 2\|\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \end{aligned} \tag{8}$$

By Corollary 1, it is known that the maximum value of $-\varepsilon\|\mathcal{G}^\top Q^{-1}\tilde{x}\|^2 + 2\lambda\|\tilde{x}\| \cdot \|\mathcal{G}^\top Q^{-1}\tilde{x}\|$ is $\frac{\lambda^2\|\tilde{x}\|^2}{\varepsilon}$ at the point $\|\mathcal{G}^\top Q^{-1}\tilde{x}\| = \frac{\lambda\|\tilde{x}\|}{\varepsilon}$, therefore, one knows

$$\dot{V} \leq -2\delta V - \|\tilde{y}\|^2 + \frac{\lambda^2\|\tilde{x}\|^2}{\varepsilon} + 2\|\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \tag{9}$$

By the fact that $V \geq \sigma_{\min}\left(Q^{-1}\right)\|\tilde{x}\|^2$, one has

$$\begin{aligned} \dot{V} \leq& -\left(2\delta - \frac{\lambda^2\sigma_{\min}\left(Q^{-1}\right)}{\varepsilon}\right)V \\ &- \|\tilde{y}\|^2 + 2\|\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \end{aligned} \tag{10}$$

Invoking Lemma 1, one knows that $\frac{\lambda^2\sigma_{\min}\left(Q^{-1}\right)}{\varepsilon}$ approaches to zero with large-enough value of ε, (10) becomes as

$$\begin{aligned} \dot{V} \leq& -2\delta V + 2\|\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \\ \leq& -2\delta V + 2\sigma_{\max}\left(\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\right) \cdot \|\tilde{x}\| \\ \leq& -2\delta V + \frac{2\sigma_{\max}\left(\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\right)}{\sqrt{\sigma_{\min}\left(Q^{-1}\right)}} \cdot \sqrt{V} \end{aligned} \tag{11}$$

which is negative definite if and only if the inequality $V > \left(\frac{2\sigma_{\max}\left(\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\right)}{\delta\sqrt{\sigma_{\min}\left(Q^{-1}\right)}}\right)^2$ holds. While, if no additive fault exists, that is, $\mathcal{F}'' = 0$, (10) yields $\dot{V} \leq -2\delta V$, meaning that the error dynamics is globally exponentially stable. While, if additive fault exists, it can be known that $\|\tilde{x}\|$ is bounded by

$$0 \leq \|\tilde{x}\| \leq \frac{2\sigma_{\max}\left(\mathcal{F}''^\top\mathcal{B}^\top Q^{-1}\right)}{\delta\sigma_{\min}\left(Q^{-1}\right)} \tag{12}$$

As a consequence, these exists some proper constant $\chi'' \in \mathbb{R}^+$, such that $\|\tilde{y}\| = \|\mathcal{C}\tilde{x}\| \geq \chi''$ means that additive fault occurs, vice versa, $\|\tilde{y}\| = \|\mathcal{C}\tilde{x}\| < \chi''$ means that additive fault relieves. □

***Corollary* 1.** *The maximum value of $-\varepsilon\|\mathcal{G}^\top Q^{-1}\tilde{x}\|^2 + 2\lambda\|\tilde{x}\| \cdot \|\mathcal{G}^\top Q^{-1}\tilde{x}\|$ is $\frac{\lambda^2\|\tilde{x}\|^2}{\varepsilon}$ when $\|\mathcal{G}^\top Q^{-1}\tilde{x}\| = \frac{\lambda\|\tilde{x}\|}{\varepsilon}$.*

Proof. Denote $\mathcal{X} = \|\mathcal{G}^\top Q^{-1}\tilde{x}\| \geq 0$, $-\varepsilon\|\mathcal{G}^\top Q^{-1}\tilde{x}\|^2 + 2\lambda\|\tilde{x}\| \cdot \|\mathcal{G}^\top Q^{-1}\tilde{x}\| \overset{\text{def}}{=} \mathcal{F}\left(\mathcal{X}\right) = -\varepsilon\mathcal{X}^2 + 2\lambda\|\tilde{x}\|\mathcal{X}$. It is known that $\frac{\mathrm{d}\mathcal{F}(\mathcal{X})}{\mathrm{d}\mathcal{X}} = -2\varepsilon\mathcal{X} + 2\lambda\|\tilde{x}\|$, hence, let $-2\varepsilon\mathcal{X} + 2\lambda\|\tilde{x}\| = 0$,

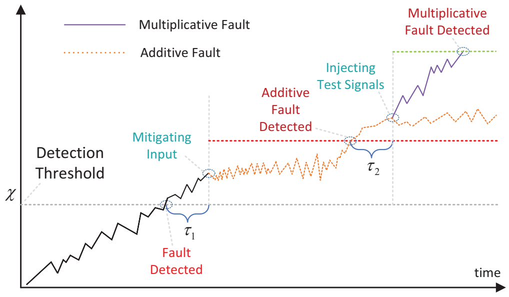

Fig. 1. A schematic diagram of the proposed algorithm for illustration.

it yields $\mathcal{X} = \frac{2\lambda\|\tilde{x}\|}{2\varepsilon}$. Furthermore, $\frac{\mathrm{d}''\mathcal{F}}{\mathrm{d}\mathcal{X}} = -2\varepsilon < 0$, as a consequence, $\mathcal{X} = \frac{2\lambda\|\tilde{x}\|}{2\varepsilon}$ is the maximum value point, that is, the maximum value point is $\mathcal{X} = \|\mathcal{G}^\top Q^{-1}\tilde{x}\| = \frac{\lambda\|\tilde{x}\|}{\varepsilon}$, and $\mathcal{F}_{\max}(\mathcal{X}) = -\varepsilon\left(\frac{\lambda\|\tilde{x}\|}{\varepsilon}\right)^2 + 2\lambda\|\tilde{x}\|\frac{\lambda\|\tilde{x}\|}{\varepsilon} = \frac{\lambda^2\|\tilde{x}\|^2}{\varepsilon}$. □

Once the occurrence of additive fault is detected, some previous efforts have been made to identify the unbiased value of such fault [9], [10], which are not discussed in details here. By these advances in identification algorithms, it can be achieved that $\hat{\mathcal{F}}'' \approx \mathcal{F}''$ with $\hat{\mathcal{F}}''$ being the estimated value of $\mathcal{F}''$. Next, the following theorem presents the second step in detecting the existence of a multiplicative fault.

Theorem 2. *[Detection of multiplicative fault] Upon converged identification of $\hat{\mathcal{F}}''$, injecting non-zero control input, stationary or non-stationary. The following threshold-based logic can be used to detect the occurrence of multiplicative fault for some $\chi' \in \mathbb{R}^+$:*

$$\begin{cases} \|\tilde{y}\| \geq \chi', & \textit{multiplicative fault occurs} \\ \|\tilde{y}\| < \chi', & \textit{multiplicative-fault free} \end{cases} \tag{13}$$

Proof. By proper unbiased identification of additive fault, one knows that $\hat{\mathcal{F}}'' \approx \mathcal{F}''$ has been achieved, that is, the nonlinear observer for additive fault identification is obtained as

$$\dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}\left(u + \hat{\mathcal{F}}''\right) + \mathcal{L}\left(y - \mathcal{C}\hat{x}\right) + \mathcal{G}f\left(\hat{x}\right), \ \mathcal{L} = Q\mathcal{C}^\top \tag{14}$$

The dynamics of state observation error $\tilde{x} = x - \hat{x}$ is known to be governed by:

$$\begin{aligned} \dot{\tilde{x}} = &\left(\mathcal{A} - \mathcal{L}\mathcal{C}\right)\tilde{x} + \mathcal{B}\left(\mathcal{F}' - I\right)u + \mathcal{B}\mathcal{F}'' - \mathcal{B}\hat{\mathcal{F}}'' \\ &+ \mathcal{G}\left(f\left(x\right) - f\left(\hat{x}\right)\right) \\ \approx &\left(\mathcal{A} - \mathcal{L}\mathcal{C}\right)\tilde{x} + \mathcal{B}\left(\mathcal{F}' - I\right)u + \mathcal{G}\left(f\left(x\right) - f\left(\hat{x}\right)\right) \end{aligned} \tag{15}$$

Differentiating both sides of Lyapunov function $V = \tilde{x}^\top Q^{-1}\tilde{x}$ along (15) yields

$$\begin{aligned} \dot{V} = &-2\delta V - \tilde{x}^\top\mathcal{C}^\top\mathcal{C}\tilde{x} - \varepsilon\tilde{x}Q^{-1}\mathcal{G}\mathcal{G}^\top Q^{-1}\tilde{x} \\ &+ 2u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\tilde{x} + 2\tilde{x}^\top Q^{-1}\mathcal{G}\left(f\left(x\right) - f\left(\hat{x}\right)\right) \\ \leq &-2\delta V - \|\tilde{y}\|^2 - \varepsilon\|\mathcal{G}^\top Q^{-1}\tilde{x}\|^2 + 2\lambda\|\tilde{x}\| \cdot \|\mathcal{G}^\top Q^{-1}\tilde{x}\| \\ &+ 2\|u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \end{aligned} \tag{16}$$

According to Corollary 1, it is known that the maximum value of $-\varepsilon\|\mathcal{G}^\top Q^{-1}\tilde{x}\|^2 + 2\lambda\|\tilde{x}\| \cdot \|\mathcal{G}^\top Q^{-1}\tilde{x}\|$ is $\frac{\lambda^2\|\tilde{x}\|^2}{\varepsilon}$ at the point $\|\mathcal{G}^\top Q^{-1}\tilde{x}\| = \frac{\lambda\|\tilde{x}\|}{\varepsilon}$, (16) becomes

$$\begin{aligned} \dot{V} \leq &-2\delta V - \|\tilde{y}\|^2 + \frac{\lambda^2\|\tilde{x}\|^2}{\varepsilon} \\ &+ 2\|u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \\ \leq &-\left(2\delta - \frac{\lambda^2\sigma_{\min}\left(Q^{-1}\right)}{\varepsilon}\right)V - \|\tilde{y}\|^2 \\ &+ 2\|u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \end{aligned} \tag{17}$$

Choosing large-enough ε gives $\frac{\lambda^2\sigma_{\min}\left(Q^{-1}\right)}{\varepsilon} \approx 0$, it yields

$$\dot{V} \leq -2\delta V + 2\|u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\| \cdot \|\tilde{x}\| \tag{18}$$

If no multiplicative fault exists, that is, $\mathcal{F}' = I$, one has $\dot{V} \leq -2\delta V$ and knows that the error dynamics is globally exponentially stable, resulting in $\lim\limits_{t\to+\infty} V = 0$, hence, $\lim\limits_{t\to+\infty}\{\tilde{x}, \tilde{y}\} = 0$. While, if multiplicative fault exists as $\mathcal{F}' \neq I$, it follows

$$\begin{aligned} \dot{V} \leq &-2\delta V + 2\sigma_{\max}\left(u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\right) \cdot \|\tilde{x}\| \\ \leq &-2\delta V + \frac{2\sigma_{\max}\left(u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\right)}{\sqrt{\sigma_{\min}\left(Q^{-1}\right)}} \cdot \sqrt{V} \end{aligned} \tag{19}$$

which is negative definite function if and only if the inequality $V > \left(\frac{2\sigma_{\max}\left(u^\top\left(\mathcal{F}'-I\right)^\top\mathcal{B}^\top Q^{-1}\right)}{\delta\sqrt{\sigma_{\min}\left(Q^{-1}\right)}}\right)^2$ holds, that is, V is ultimately bounded by $\left(\frac{2\sigma_{\max}\left(u^\top\left(\mathcal{F}'-I\right)^\top\mathcal{B}^\top Q^{-1}\right)}{\delta\sqrt{\sigma_{\min}\left(Q^{-1}\right)}}\right)^2$, the state observation error is known to be ultimately bounded by

$$0 \leq \|\tilde{x}\| \leq \frac{2\sigma_{\max}\left(u^\top\left(\mathcal{F}' - I\right)^\top\mathcal{B}^\top Q^{-1}\right)}{\delta\sigma_{\min}\left(Q^{-1}\right)} \tag{20}$$

Because of non-zero control input u has been injected, it reaches a conclusion that these exists some proper constant $\chi' \in \mathbb{R}^+$, such that $\|\tilde{y}\| = \|\mathcal{C}\tilde{x}\| \geq \chi'$ means that multiplicative fault occurs, vice versa, $\|\tilde{y}\| = \|\mathcal{C}\tilde{x}\| < \chi'$ means that multiplicative fault relieves. □

Finally, upon detecting the occurrence of a multiplicative fault, the following nonlinear observer can be employed to estimate the unknown $\mathcal{F}'$:

$$\dot{\hat{x}} = \mathcal{A}\hat{x} + \mathcal{B}\hat{\mathcal{F}}'u + \mathcal{L}\left(y - \mathcal{C}\hat{x}\right) + \mathcal{G}f\left(\hat{x}\right), \quad \mathcal{L} = Q\mathcal{C}^\mathrm{T}, \tag{21}$$

where $\hat{\mathcal{F}}'$ represents the estimation of $\mathcal{F}'$. This estimation can be achieved using an unbiased identification algorithm, as described in [9], [11]. To better understand the workflow and principles of the proposed algorithm, an Algorithm 1 and a diagram in Fig. 1 are provided, where MFO and AFO are short for 'multiplicative fault occurs' and 'additive fault occurs'.

Algorithm 1 Detection Algorithm for Potential Coexistence of Additive and Multiplicative Faults

```
Input: system output $y$, observed output $\hat{y}$, control input $u$,
    threshold values $\chi$, $\chi'$ and $\chi''$
Output: $\|\tilde{y}\| \geq \chi \Leftrightarrow$ fault occurs: $\begin{cases} \|\tilde{y}\| \geq \chi' \Leftrightarrow \text{MFO} \\ \|\tilde{y}\| \geq \chi'' \Leftrightarrow \text{AFO} \end{cases}$
  $\tilde{y} \overset{\text{def}}{=} y - \hat{y}$
  if $\|\tilde{y}\| \geq \chi \wedge$ duration time $> \tau_1$ then
    print FAULT OCCURS
    decreasing control input $u$
    if $\|\tilde{y}\| \geq \chi'' \wedge$ duration time $> \tau_2$ then
      print ADDITIVE FAULT OCCURS
      implement unbiased identification
    else
      print NO ADDITIVE FAULT OCCURS
      injecting non-zero control input $u$
      if $\|\tilde{y}\| \geq \chi'$ then
        print MULTIPLICATIVE FAULT OCCURS
        implement unbiased identification
      end if
    end if
  end if
```

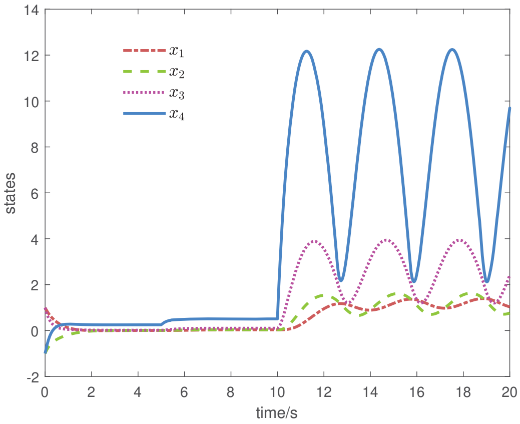

Fig. 2. State response.

IV. SIMULATION EXAMPLE

Consider the Lipschitz nonlinear systems in the form of (1) with $\mathcal{A} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$, $\mathcal{B} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^\top$, $\mathcal{C} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $\mathcal{G} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$, $f(x) = \begin{bmatrix} -0.5 \sin(x_1) \\ -0.2 \cos(x_3) \end{bmatrix}$, $x_0 = \begin{bmatrix} 1 & -1 & 1 & -1 \end{bmatrix}^\top$. Choose $\delta = 1$ and $\varepsilon = 10^4$. Q can be obtained using the `LQR` function in Matlab by solving the Riccati function (4), as given in (22). The gain matrix in nonlinear observer is therefore calculated and obtained as $\mathcal{L} = Q\mathcal{C}^\top = \begin{bmatrix} -0.658572 & 13.122352 \\ -8.982355 & 86.314918 \\ 98.104488 & -0.658572 \\ 8.671102 & -65.585651 \end{bmatrix}$, the initial value $\hat{x}_0 = \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix}^\top$. $\chi = 0.0002$, $\chi' = 0.005$, $\chi'' = 0.0003$. Initial control input $u(t) = 1$, after $\|\tilde{y}\| \geq \chi$ is detected, $u(t)$ is changed as $u(t) = 10^{-8} \sin(t)$, obeying the principle in Theorem 1. Because of that identification algorithm is not involved in this work, for simulation purpose, $\mathcal{F}''$ is set to 0 manually after $t \geq 10$, and the control input is changed as $u(t) = 10\,|\sin(t)|$. The fault values in the simulations are: for $0 \leq t < 5$, $\mathcal{F}' = 1$, $\mathcal{F}'' = 0$; for $5 \leq t < 10$, $\mathcal{F}' = 1$, $\mathcal{F}'' = 2$; for $10 \leq t < 20$, $\mathcal{F}' = 5$, $\mathcal{F}'' = 0$; and for $20 \leq t$, $\mathcal{F}' = 1$, $\mathcal{F}'' = 0$.

The simulation results in Fig. 2 shows the state response under all control inputs. The trajectory of $|\tilde{y}|$ and the corresponding fault alarm signals are provided in Fig. 3(a). To avoid false alarms during the initial period, the fault alarm is not triggered before $t = 2$s, as shown in Fig. 3(b). As observed in Fig. 3(c), the fault occurrence can be detected within 0.02s, and the additive fault can be detected within 0.01s after the initial fault detection. Additionally, Fig. 3(d) demonstrates that the occurrence of a multiplicative fault can be detected within 0.003s, both indicating rapid detection performance, which is suitable for online implementation. The effectiveness of proposed algorithm is thus well demonstrated.

V. CONCLUSIONS

This work has developed a nonlinear observer-based framework for actuator fault diagnosis in Lipschitz nonlinear systems, effectively addressing simultaneous multiplicative and additive actuator faults. The results demonstrate that the proposed observer ensures exponential convergence of estimation errors by employing a modified Riccati equation, which is independent of the Lipschitz constant, thus enhancing stability and robustness. By two skilled handling, including decreasing control input and injecting non-zero control input, potential additive and multiplicative faults can be sequentially detected if exist. The proposed actuator fault diagnosis algorithm offers a comprehensive solution to fault diagnosis challenges, combining theoretical rigor with practical applicability, and pave the way for future work to extend the framework to account for external disturbances.

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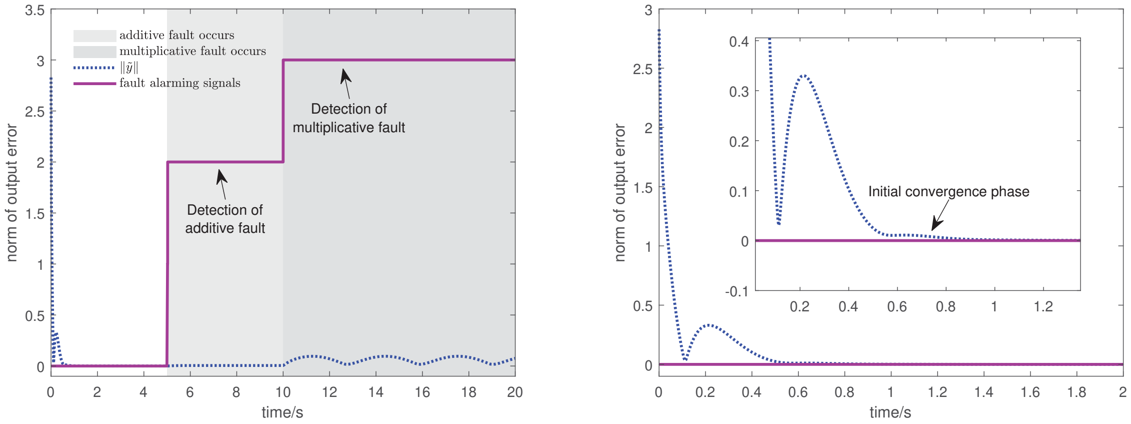
REFERENCES

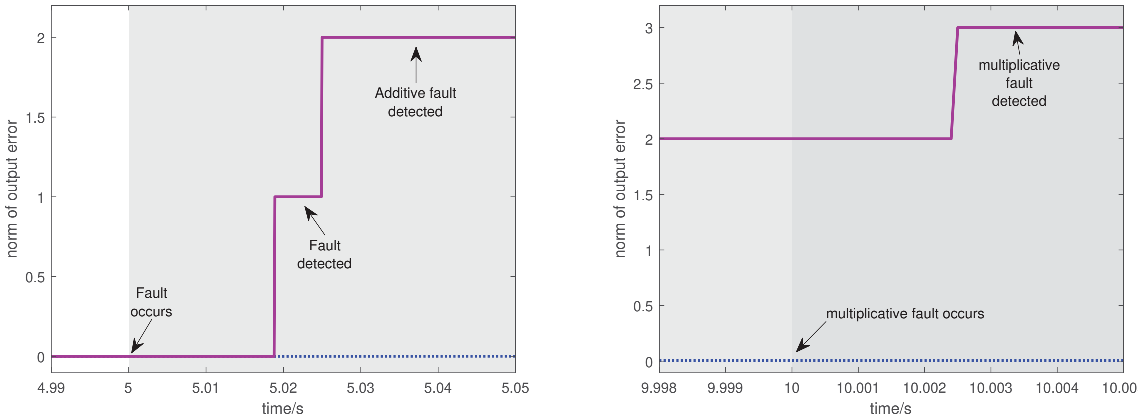
[1] S. Yin, S. X. Ding, X. Xie, and H. Luo, "A review on basic data-driven approaches for industrial process monitoring," *IEEE Transactions on Industrial electronics*, vol. 61, no. 11, pp. 6418–6428, 2014.

$$Q = \begin{bmatrix} 13.122352 & 86.314918 & -0.658572 & -65.585651 \\ 86.314918 & 1225.543746 & -8.982355 & -1063.105481 \\ -0.658572 & -8.982355 & 98.104488 & 8.671102 \\ -65.585651 & -1063.105481 & 8.671102 & 937.222406 \end{bmatrix} \tag{22}$$



(a) Trajectory of $\|\tilde{y}\|$ and four levels of alarming signals.

(b) Zoom-in view of initial convergence time of Fig. 3(b) from 0 to 2s.



(c) Zoom-in view of Fig. 3(b) from 4.99s to 5.05s.

(d) Zoom-in view of Fig. 3(b) from 9.998s to 10.005s.

Fig. 3. Simulation results. (Purple line is equal 0: no fault; Purple line is equal 1: fault detected; Purple line is equal 2: additive fault detected; Purple line is equal 3: multiplicative fault detected.)

[2] T. Höfling and T. Pfeufer, “Detection of additive and multiplicative faults-parity space vs. parameter estimation,” *IFAC Proceedings Volumes*, vol. 27, no. 5, pp. 515–520, 1994.

[3] S. Gao, Q. Zhai, and K. Zhao, “Fault diagnosis of virtually-coupled trains by adaptive observer with pattern-matched detection and reinforced identification,” *International Journal of Adaptive Control and Signal Processing*, vol. 38, pp. 2443–2464, 2024.

[4] K. Classens, S. Verbeek, W. M. Heemels, and T. Oomen, “Joint estimation of additive and parametric faults: A model-based fault diagnosis approach towards predictive maintenance,” *IFAC-PapersOnLine*, vol. 55, no. 6, pp. 304–309, 2022.

[5] H. Yang, Y. Jiang, and S. Yin, “Adaptive fuzzy fault-tolerant control for markov jump systems with additive and multiplicative actuator faults,” *IEEE Transactions on Fuzzy Systems*, vol. 29, no. 4, pp. 772–785, 2020.

[6] G. Stein and M. Athans, “The lqg/ltr procedure for multivariable feedback control design,” *IEEE Transactions on Automatic Control*, vol. 32, no. 2, pp. 105–114, 1987.

[7] F. L. Lewis, *Applied optimal control and estimation*. Prentice Hall PTR, 1992.

[8] M.-S. Chen and C.-C. Chen, “Robust nonlinear observer for lipschitz nonlinear systems subject to disturbances,” *IEEE Transactions on Automatic Control*, vol. 52, no. 12, pp. 2365–2369, 2007.

[9] S. Gao and K. Zhao, “Intermittent fault diagnosis of dynamic systems with model uncertainty and disturbance: An adaptive nondeterministic observer approach,” *IEEE Transactions on Reliability*, DOI: 10.1109/TR.2024.3431709, 2025.

[10] S. Gao, K. Zhao, T. Wen, H. Wang, and L. Zhang, “Adaptive diagnosis observer for dynamical systems with dual actuator faults: A sign-corrected regressor design,” *Journal of the Franklin Institute*, vol. 362, no. 2, p. 107462, 2025.

[11] T. Wen, S. Gao, J. Wang, and C. Roberts, “Signed-data reinforced observer-based fault diagnosis for virtually-coupled electric multiple units trains,” *Control Engineering Practice*, vol. 147, p. 105921, 2024.