On the ℓs-boundedness of a family of integral operators

Gallarati, Chiara; Lorist, Emiel; Veraar, Mark

DOI
10.4171/RMI/916

Publication date
2016

Document Version
Accepted author manuscript

Published in
Revista Matemática Iberoamericana

Citation (APA)
Gallarati, C., Lorist, E., & Veraar, M. (2016). On the ℓs-boundedness of a family of integral operators. *Revista Matemática Iberoamericana*, *32*(4), 1277-1294. https://doi.org/10.4171/RMI/916

ON THE ℓ^s-BOUNDEDNESS OF A FAMILY OF INTEGRAL OPERATORS

CHIARA GALLARATI, EMIEL LORIST, AND MARK VERAAR

ABSTRACT. In this paper we prove an ℓ^s-boundedness result for integral operators with operator-valued kernels. The proofs are based on extrapolation techniques with weights due to Rubio de Francia. The results will be applied by the first and third author in a subsequent paper where a new approach to maximal L^p-regularity for parabolic problems with time-dependent generator is developed.

1. INTRODUCTION

In the influential work [34, 35], Weis has found a characterization of maximal L^p-regularity in terms of $\mathcal{R}$-sectoriality, which stands for $\mathcal{R}$-boundedness of a family of resolvents on a sector. The definition of $\mathcal{R}$-boundedness is given in Definition 3.15. It is a random boundedness condition on a family of operators which is a strengthening of uniform boundedness. Maximal regularity of solution to PDEs is important to know as it provides a tool to solve nonlinear PDEs using linearization techniques (see [4, 23, 25]). An overview on recent developments on maximal L^p-regularity can be found in [7, 21]. Maximal L^p-regularity means that for all $f \in L^p(0,T;X)$, where X is a Banach space, the solution u of the evolution problem

$$\begin{cases} u'(t) &= Au(t) + f(t), \quad t \in (0,T) \\ u(0) &= 0 \end{cases} \tag{1.1}$$

has the "maximal" regularity in the sense that u', Au are both in $L^p(0,T;X)$. Using a mild formulation one sees that to prove maximal L^p-regularity one needs to bound a singular integral with operator-valued kernel $Ae^{(t-s)A}$.

In [11] the first and third author have developed a new approach to maximal L^p-regularity for the case that the operator A in (1.1) depends on time in a measurable way. In this new approach $\mathcal{R}$-boundedness plays a central rôle again. Namely, the $\mathcal{R}$-boundedness of the family of integral operators $\{I_k : k \in \mathcal{K}\} \subseteq L^p(\mathbb{R};X)$ is required in the proofs. Here I_k is defined by

$$(I_k f)(t) = \int_{\mathbb{R}} k(t-s)T(t,s)f(s)\,ds, \tag{1.2}$$

Date: July 16, 2018.

2010 *Mathematics Subject Classification.* Primary: 42B20; Secondary: 42B25, 42B37, 46E30, 47B55.

Key words and phrases. ℓ^s-boundedness, extrapolation, integral operators, A_p-weights, Hardy-Littlewood maximal function.

The first and third author are supported by Vrije Competitie subsidy 613.001.206 and Vidi subsidy 639.032.427 of the Netherlands Organisation for Scientific Research (NWO).

where $T(t,s) \in \mathcal{L}(X)$ is a two-parameter evolution family and $\mathcal{K}$ is the class of kernels which satisfy $|k| * f \leq Mf$ for $f : \mathbb{R} \to \mathbb{R}_+$ simple and where M is the Hardy-Littlewood maximal operator. For evolution families one usually sets $T(t,s) = 0$ if $t < s$.

In this paper we give a class of examples for which we can prove the $\mathcal{R}$-boundedness of $\{I_k : k \in \mathcal{K}\}$. We now state a special case of our main result. It is valid for general families of operators $\{T(t,s) : -\infty < s \leq t < \infty\} \subseteq \mathcal{L}(L^q(\Omega, w))$. We will not use any regularity conditions for $(t,s) \mapsto T(t,s)$ below.

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let $p, q \in (1, \infty)$. Assume that for all A_q-weights w,*

$$\|T(t,s)\|_{\mathcal{L}(L^q(\Omega,w))} \leq C, \quad s,t \in \mathbb{R}, \tag{1.3}$$

where C depends on the A_q-constant of w in a consistent way. Then the family of integral operators $\{I_k : k \in \mathcal{K}\} \subseteq \mathcal{L}(L^p(\mathbb{R}; L^q(\Omega)))$ as defined in (1.2) *is $\mathcal{R}$-bounded.*

In the setting where $T(t,s) = e^{(t-s)A}$ where A is as in (1.1), the condition (1.3) also appears in [10] and [17, 18] in order to obtain $\mathcal{R}$-sectoriality of A. There (1.3) is checked by using Calderón-Zygmund and Fourier multiplier theory. Examples of such results for two-parameter evolution families will be given in [11].

As a consequence of the Kahane-Khintchine inequality (see Remark 3.16) one can see that in standard spaces such as L^p-spaces, $\mathcal{R}$-boundedness is equivalent to so-called ℓ^2-boundedness. The latter is a special case of ℓ^s-boundedness (see Definition 3.1). In L^p-spaces this boils down to classical $L^p(\ell^s)$-estimates from harmonic analysis (see [14, 15], [12, Chapter V] and [5, Chapter 3]). It follows from the work of Rubio de Francia (see [26, 27, 28] and [12]) that $L^p(\ell^s)$-estimates are strongly connected to estimates in weighted L^p-spaces.

To prove Theorem 1.1 we apply weighted techniques of Rubio de Francia. Without additional effort we actually prove the more general Corollary 3.14, which states that the family of integral operators on $L^p(v, L^q(w))$ is ℓ^s-bounded for all $p, q, s \in (1, \infty)$ and for arbitrary A_p-weights v and A_q-weights w. Both the modern extrapolation methods with A_q-weights as explained in the book of Cruz-Uribe, Martell and Pérez [5] and the factorization techniques of Rubio de Francia (see [12, Theorem VI.5.2] or [15, Theorem 9.5.8]), play a crucial rôle in our work. It is unclear how to apply the extrapolation techniques of [5] to the inner space L^q directly, but it does play a rôle in our proofs for the outer space L^p. The factorization methods of Rubio de Francia enable us to deal with the inner spaces (see the proof of Proposition 3.13).

In the literature there are many more $\mathcal{R}$-boundedness results for integral operators (e.g. [6, Section 6], [7, Proposition 3.3 and Theorem 4.12], [13], [16, Section 3], [19, Section 4], [21, Chapter 2]). However, it seems they are of a different nature and cannot be used to prove Theorems 1.1, 3.10 and Corollary 3.14.

Throughout this paper we will write $\mathcal{B}(X)$ for the space of all bounded operators on a Banach space X and denote the corresponding norm as $\|\cdot\|_{\mathcal{B}(X)}$. Let $\mathcal{L}(X) \subseteq \mathcal{B}(X)$ denote the subspace of all bounded *linear* operators. For $p \in [1, \infty]$ we let $p' \in [1, \infty]$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$.

Acknowledgement The authors thank the referees for helpful comments.

2. Extrapolation and weights

2.1. **Preliminaries on weights.** First we will introduce Muckenhoupt weights and state some of their properties. Details can be found in [15, Chapter 9] and [31, Chapter V].

A weight is a locally integrable function on $\mathbb{R}^d$ with $w(x) \in (0,\infty)$ for almost every $x \in \mathbb{R}^d$. The space $L^p(\mathbb{R}^d, w)$ is defined as all measurable functions f with

$$\|f\|_{L^p(\mathbb{R}^d,w)} = \left(\int_{\mathbb{R}^d} |f|^p w \,\mathrm{d}\mu\right)^{\frac{1}{p}} < \infty.$$

With this notion of weights and weighted L^p-spaces we can define the class of Muckenhoupt weights A_p for all $p \in (1,\infty)$ for a fixed dimension $d \in \mathbb{N}$. Let $⨍_Q = \frac{1}{|Q|}\int_Q$. For $p \in (1,\infty)$ a weight w is said to be an A_p-*weight* if

$$[w]_{A_p} = \sup_Q ⨍_Q w(x)\,\mathrm{d}x \left(⨍_Q w(x)^{-\frac{1}{p-1}}\,\mathrm{d}x\right)^{p-1} < \infty,$$

where the supremum is taken over all cubes $Q \subseteq \mathbb{R}^d$ with axes parallel to the coordinate axes. The extended real number $[w]_{A_p}$ is called the A_p-*constant*.

Recall that $w \in A_p$ if and only if the Hardy-Littlewood maximal operator M is bounded on $L^p(\mathbb{R}^d, w)$. The Hardy-Littlewood maximal operator is defined as

$$M(f)(x) = \sup_{Q \ni x} ⨍_Q |f(y)|\,\mathrm{d}y, \quad f \in L^p(\mathbb{R}^d, w)$$

with Q ranging over all cubes in $\mathbb{R}^d$ with axes parallel to the coordinate axes.

Next we will summarize a few basic properties of weights which we will need. The proofs can be found in [15, Theorems 9.1.9 and 9.2.5], [15, Theorem 9.2.5 and Exercise 9.2.4], [15, Proposition 9.1.5].

Proposition 2.1. *Let $w \in A_p$ for some $p \in [1,\infty)$. Then we have*

1. *If $p \in (1,\infty)$ then $w^{-\frac{1}{p-1}} \in A_{p'}$ with $[w^{-\frac{1}{p-1}}]_{A_{p'}} = [w]_{A_p}^{\frac{1}{p-1}}$.*
2. *For every $p \in (1,\infty)$ and $\kappa > 1$ there is a constant $\sigma = \sigma_{p,\kappa,d} \in (1,p)$ and a constant $C_{p,d,\kappa} > 1$ such that $[w]_{A_{\frac{p}{\sigma}}} \leq C_{p,\kappa,d}$ whenever $[w]_{A_p} \leq \kappa$. Moreover, $\kappa \mapsto \sigma_{p,\kappa,d}$ and $\kappa \mapsto C_{p,\kappa,d}$ can be chosen to be decreasing and increasing, respectively.*
3. *$A_p \subseteq A_q$ and $[w]_{A_q} \leq [w]_{A_p}$ if $q > p$.*
4. *For $p \in (1,\infty)$, there exists a constant $C_{p,d}$ such that*

$$\|M\|_{\mathcal{B}(L^p(\mathbb{R}^d,w))} \leq C_{p,d} \cdot [w]_{A_p}^{\frac{1}{p-1}}.$$

2.2. **Extrapolation.** The celebrated result of Rubio de Francia (see [26, 27, 28], [12, Chapter IV]) allows one to extrapolate from weighted L^p-estimates for a single p to weighted L^q-estimates for all q. The proofs and statement have been considerably simplified and clarified in [5] and can be formulated as follows (see [5, Theorem 3.9]).

Theorem 2.2. *Let $f, g : \mathbb{R}^d \to \mathbb{R}_+$ be a pair of nonnegative, measurable functions and suppose that for some $p_0 \in (1,\infty)$ there exists an increasing function α on $\mathbb{R}_+$ such that for all $w_0 \in A_{p_0}$*

$$\|f\|_{L^{p_0}(\mathbb{R}^d,w_0)} \leq \alpha([w_0]_{A_{p_0}})\|g\|_{L^{p_0}(\mathbb{R}^d,w_0)}.$$

Then for all $p \in (1, \infty)$ *there is a constant* $c_{p,d}$ *s.t. for all* $w \in A_p$,

$$\|f\|_{L^p(\mathbb{R}^d,w)} \leq 4\alpha\Big(c_{p,d}[w]_{A_p}^{\frac{p_0-1}{p-1}+1}\Big)\|g\|_{L^p(\mathbb{R}^d,w)}.$$

Note that for certain weights the above L^p-norms are allowed to be infinite. Estimates as in the above result with increasing function α will appear frequently. In this situation we say that

$$\|f\|_{L^{p_0}(\mathbb{R}^d,w_0)} \leq C\|g\|_{L^{p_0}(\mathbb{R}^d,w_0)}$$

with an A_{p_0}-*consistent* constant C. This means that for two weights $w_0, w_1 \in A_p$ we have $C([w_0]_{A_p}) \leq C([w_1]_{A_p})$ whenever $[w_0]_{A_p} \leq [w_1]_{A_p}$. Note that the L^p-estimate obtained in Theorem 2.2 is again A_p-consistent for all $p \in (1, \infty)$.

Take $n \in \mathbb{N}$ and let for $i = 1, \cdots, n$ the triple $(\Omega_i, \Sigma_i, \mu_i)$ be a σ-finite measure space. Define the product measure space

$$(\Omega, \Sigma, \mu) = (\Omega_1 \times \cdots \times \Omega_n, \Sigma_1 \times \cdots \times \Sigma_n, \mu_1 \times \cdots \times \mu_n)$$

Then of course (Ω, Σ, μ) is also σ-finite. For $\overline{q} \in (1, \infty)^n$ we write

$$L^{\overline{q}}(\Omega) = L^{q_1}(\Omega_1, \cdots L^{q_n}(\Omega_n)). \tag{2.1}$$

Next we extend Theorem 2.2 to values in the above mixed $L^{\overline{q}}(\Omega)$ spaces. For the case $\Omega = \mathbb{N}$ this was already done in [5, Corollary 3.12].

Theorem 2.3. *Let* $f, g : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$ *be a pair of nonnegative, measurable functions and suppose that for some* $p_0 \in (1, \infty)$ *there exists an increasing function* α *on* $\mathbb{R}_+$ *such that for all* $w_0 \in A_{p_0}$

$$\|f(\cdot, s)\|_{L^{p_0}(\mathbb{R}^d,w_0)} \leq \alpha([w_0]_{A_{p_0}})\|g(\cdot, s)\|_{L^{p_0}(\mathbb{R}^d,w_0)} \tag{2.2}$$

for all $s \in \Omega$. *Then for all* $p \in (1, \infty)$ *and* $\overline{q} \in (1, \infty)^n$ *there exist* $c_{p,\overline{q},d} > 0$ *and* $\beta_{p_0,p,\overline{q}} > 0$ *such that for all* $w \in A_p$,

$$\|f\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))} \leq 4^n\alpha\Big(c_{p,\overline{q},d}[w]_{A_p}^{\beta_{p_0,p,\overline{q}}}\Big)\|g\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))}. \tag{2.3}$$

Proof. We will prove this theorem by induction. The base case $n = 0$ is just weighted extrapolation, as covered in Theorem 2.2.

Now take $n \in \mathbb{N} \cup \{0\}$ arbitrary and assume that the assertion holds for all pairs $f, g : \mathbb{R}^d \times \Omega \to \mathbb{R}_+$ of nonnegative, measurable functions. Let $(\Omega_0, \Sigma_0, \mu_0)$ be a σ-finite measure space and take nonnegative, measurable functions $f, g : \mathbb{R}^d \times \Omega_0 \times \Omega \to \mathbb{R}_+$. Assume that (2.2) holds for p_0, all $w \in A_{p_0}$ and all $s \in \Omega_0 \times \Omega$.

Now take $(s_0, s_1, \cdots, s_n) \in \Omega_0 \times \Omega$ arbitrary. Let $\overline{q} \in (1, \infty)^n$ be given and take $r \in (1, \infty)$ arbitrary. Define $\overline{r} = (r, q_1, \cdots, q_n)$ and the pair of functions $F, G : \mathbb{R}^d \to [0, \infty]$ as

$$F(x) = \|f(x, \cdot)\|_{L^{\overline{r}}(\Omega\times\Omega_0)} \qquad G(x) = \|g(x, \cdot)\|_{L^{\overline{r}}(\Omega\times\Omega_0)}$$

By our induction hypothesis we know for all $p \in (1, \infty)$ there exist $c_{p,\overline{q},d}$ and $\beta_{p_0,p,\overline{q}}$ such that for all $w \in A_p$

$$\|f(\cdot, s_0, \cdot)\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))} \leq 4^n\alpha(c_{p,\overline{q},d}[w]_{A_p}^{\beta_{p_0,p,\overline{q}}})\|g(\cdot, s_0, \cdot)\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))}$$

Now taking $p = r$ we obtain

$$\|F\|_{L^r(\mathbb{R}^d,w)} = \Big(\int_{\Omega_0}\int_{\mathbb{R}^d}\|f(x, s_0, \cdot)\|_{L^{\overline{q}}(\Omega)}^r w(x)\,\mathrm{d}x\,\mathrm{d}\mu_0\Big)^{\frac{1}{r}}$$

$$\leq 4^n \alpha(c_{r,\overline{q},d}[w]_{A_r}^{\beta_{p_0,r,\overline{q}}}) \Big(\int_{\Omega_0} \int_{\mathbb{R}^d} \|g(x,s_0,\bullet)\|^r_{L^{\overline{q}}(\Omega)} w(x)\,\mathrm{d}x\,\mathrm{d}\mu_0 \Big)^{\frac{1}{r}}$$
$$= 4^n \alpha(c_{r,\overline{q},d}[w]_{A_r}^{\beta_{p_0,r,\overline{q}}}) \|G\|_{L^r(\mathbb{R}^d,w)}$$

using Fubini's theorem in the first and third step. So with Theorem 2.2 using $p_0 = r$ we obtain for all $p \in (1,\infty)$ that there exist $c_{r,p,\overline{q},d} > 0$ and $\beta_{p_0,p,\overline{r}} > 0$ such that for all $w \in A_p$,

$$\begin{aligned} \|f\|_{L^p(\mathbb{R}^d,w;L^{\overline{r}}(\Omega_0\times\Omega))} &= \|F\|_{L^p(\mathbb{R}^d,w)} \\ &\leq 4^{n+1}\alpha\Big(c_{r,p,\overline{q},d}[w]_{A_p}^{\beta_{p_0,p,\overline{r}}}\Big)\|G\|_{L^p(\mathbb{R}^d,w)} \\ &= 4^{n+1}\alpha\Big((c_{r,p,\overline{q},d}[w]_{A_p}^{\beta_{p_0,p,\overline{r}}}\Big))\|g\|_{L^p(\mathbb{R}^d,w;L^{\overline{r}}(\Omega_0\times\Omega))}. \end{aligned}$$

This proves (2.3) for $n+1$. ∎

Remark 2.4. Note that in the application of Theorem 2.3 it will often be necessary to use an approximation by simple functions to check the requirements, since point evaluations in (2.2) are not possible in general. Furthermore note that in the case that $f = Tg$ with T a bounded linear operator on $L^p(\mathbb{R}^d, w)$ for all $w \in A_p$ this theorem holds for all UMD Banach function spaces, which is one of the deep results of Rubio de Francia and can be found in [29, Theorem 5].

As an application of Theorem 2.3 we will present a short proof of the boundedness of the Hardy-Littlewood maximal operator on mixed $L^{\overline{q}}$-spaces.

Definition 2.5. *Let $p \in (1,\infty)$ and $w \in A_p$. For $f \in L^p(\mathbb{R}^d, w; X)$ with $X = L^{\overline{q}}(\Omega)$ we define the maximal function $\widetilde{M}$ as*

$$\widetilde{M} f(x,s) = \sup_{Q \ni x} \fint_Q |f(y,s)|\,\mathrm{d}y$$

with Q all cubes in $\mathbb{R}^d$ as before.

We can see that $\widetilde{M}$ is measurable, as the value of the supremum in the definition stays the same if we only consider rational cubes. We will show that the maximal function is bounded on the space $X = L^{\overline{q}}(\Omega)$. Note that if $\Omega = \mathbb{N}$, the result below reduces to the weighted version of the Fefferman-Stein theorem [1].

Theorem 2.6. *$\widetilde{M}$ is bounded on $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$ for all $p \in (1,\infty)$ and $w \in A_p$.*

Proof. Let M be the Hardy-Littlewood maximal operator and assume that $f \in L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$ is simple. By Proposition 2.1 and the definition of the Hardy-Littlewood maximal operator we know that

$$\|\widetilde{M} f(\bullet,s)\|_{L^p(\mathbb{R}^d,w)} = \|M f(\bullet,s)\|_{L^p(\mathbb{R}^d,w)} \leq C_{p,d}\cdot [w]_{A_p}^{\frac{1}{p-1}} \|f(\bullet,s)\|_{L^p(\mathbb{R}^d,w)}$$

Then by Theorem 2.3 we get that

$$\|\widetilde{M} f\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))} \leq \alpha_{p,\overline{q},d}([w]_{A_p})\|f\|_{L^p(\mathbb{R}^d,w;L^{\overline{q}}(\Omega))}$$

with $\alpha_{p,\overline{q},d}$ an increasing function on $\mathbb{R}_+$. With a density argument we then get that $\widetilde{M}$ is bounded on $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$. ∎

Remark 2.7. Using deep connections between harmonic analysis with weights and martingale theory, Theorem 2.6 was obtained in [2] and [29, Theorem 3] for UMD Banach function spaces in the case $w = 1$. It has been extended to the weighted setting in [32]. As our main result Theorem 3.10 is formulated for iterated $L^{\overline{q}}(\Omega)$-spaces we prefer the above more elementary treatment.

3. Main result

In this section we present the proofs of Theorems 1.1 and 3.10 and Corollary 3.14 which are our main results. In Subsection 3.1 we will first obtain a preliminary result which is one of the ingredients in the proofs.

3.1. ℓ^s-boundedness.

In this section we will introduce ℓ^s-boundedness and present some simple examples. For this we will use the notion of a Banach lattice (see [22]). An example of a Banach lattice is L^p or any Banach function space (see [36, Section 63]). In our main results only iterated L^p-spaces will be needed.

Although ℓ^s-boundedness is used implicitly in the literature for operators on L^p-spaces, on Banach functions spaces it was introduced in [34] under the name $\mathcal{R}_s$-boundedness. An extensive study can be found in [20, 33].

Definition 3.1. *Let X and Y be Banach lattices and let $s \in [1, \infty]$. Then we call a family of operators $\mathscr{T} \subseteq \mathcal{B}(X, Y)$ ℓ^s-bounded if there exists a constant C such that for all integers N, for all sequences $(T_n)_{n=1}^N$ in $\mathscr{T}$ and $(x_n)_{n=1}^N$ in X,*

$$\Big\| \Big(\sum_{n=1}^{N} |T_n x_n|^s \Big)^{\frac{1}{s}} \Big\|_Y \leq C \Big\| \Big(\sum_{n=1}^{N} |x_n|^s \Big)^{\frac{1}{s}} \Big\|_X$$

with the obvious modification for $s = \infty$. The least possible constant C is called the ℓ^s-bound *of $\mathscr{T}$ and is denoted by $\mathcal{R}^{\ell^s}(\mathscr{T})$ and often abbreviated as $\mathcal{R}^s(\mathscr{T})$.*

Example 3.2. Take $p \in (1, \infty)$ and let $\mathscr{T} \subseteq \mathcal{B}(L^p(\mathbb{R}^d))$ be uniformly bounded by a constant C. Then $\mathscr{T}$ is ℓ^p-bounded with $\mathcal{R}^p(\mathscr{T}) \leq C$.

The following basic properties will be needed later on.

Proposition 3.3. *Let $\mathscr{T} \subseteq \mathcal{L}(X, Y)$, where X and Y are Banach function spaces.*

1. *Let $1 \leq s_0 < s_1 \leq \infty$ and assume that X and Y have an order continuous norm. If $\mathscr{T} \subseteq \mathcal{L}(X, Y)$ is ℓ^{s_j}-bounded for $j = 0, 1$, then $\mathscr{T}$ is ℓ^s-bounded for all $s \in [s_0, s_1]$ and with $\theta = \frac{s - s_0}{s_1 - s_0}$, the following estimate holds:*
$$\mathcal{R}^s(\mathscr{T}) \leq \mathcal{R}^{s_0}(\mathscr{T})^{1-\theta} \mathcal{R}^{s_1}(\mathscr{T})^{\theta} \leq \max\{\mathcal{R}^{s_0}(\mathscr{T}), \mathcal{R}^{s_1}(\mathscr{T})\}$$
2. *If $\mathscr{T}$ is ℓ^s-bounded, then the adjoint family $\mathscr{T}^* = \{T^* \in \mathcal{L}(Y^*, X^*) : T \in \mathscr{T}\}$ is $\ell^{s'}$-bounded and $\mathcal{R}^{s'}(\mathscr{T}^*) = \mathcal{R}^s(\mathscr{T})$.*

Proof. (1) follows from Calderón's theory of complex interpolation of vector-valued function spaces (see [3] and [20, Proposition 2.14]). For (2) we refer to [20, Proposition 2.17] and [24, Proposition 3.4]. ∎

Remark 3.4. Below we will only need Proposition 3.3 in the case $X = Y = L^{\overline{q}}(\Omega)$. To give the details of the proof of Proposition 3.3 in this situation one first needs to know that $X^* = L^{\overline{q}'}(\Omega)$ which can be obtained by elementary arguments (see Proposition A.1 below). As a second step one needs to show that $X(\ell^s_N)^* = X^*(\ell^{s'}_N)$ and this is done in Lemma A.2.

Example 3.5. Let $1 \le s_0 \le q \le s_1 \le \infty$. Let $X = L^q(\Omega)$ and let $\mathscr{T} \subset \mathcal{L}(X)$ be ℓ^{s_j}-bounded for $j \in \{0,1\}$. Then for $s \in [s_0, q]$, $\mathcal{R}^s(\mathscr{T}) \le \mathcal{R}^{s_0}(\mathscr{T})$ and for $s \in [q, s_1]$, $\mathcal{R}^s(\mathscr{T}) \le \mathcal{R}^{s_1}(\mathscr{T})$. Indeed, note that by Example 3.2,

$$\mathcal{R}^q(\mathscr{T}) = \sup_{T \in \mathscr{T}} \|T\| \le \mathcal{R}^{s_j}(\mathscr{T}), \quad j \in \{0,1\}.$$

Now the estimates follow from Proposition 3.3 by interpolating with exponents (s_0, q) and (q, s_1).

In particular, it follows that the function $s \mapsto \mathcal{R}^s(\mathscr{T})$, is decreasing on $[s_0, q]$ and increasing on $[q, s_1]$.

3.2. **Convolution operators.** Let $\mathcal{K}$ be the following class of kernels

$$\mathcal{K} = \{k \in L^1(\mathbb{R}^d) : \text{for all simple } f : \mathbb{R}^d \to \mathbb{R}_+ \text{ one has } |k| * f \le Mf \text{ a.e.}\}.$$

There are many examples of classes of functions k with this property (see [14, Chapter 2] and [24, Proposition 4.5 and 4.6]). It follows from [24, Lemma 4.3] that every $k \in \mathcal{K}$ satisfies $\|k\|_{L^1(\mathbb{R}^d)} \le 1$.

To keep the presentation as simple as possible we only consider the iterated space $X = L^{\overline{q}}(\Omega)$ with $\overline{q} \in (1,\infty)^n$ below (see (2.1)). For a kernel $k \in L^1(\mathbb{R}^d)$, $p \in (1,\infty)$ and $w \in A_p$ define the convolution operator T_k on $L^p(\mathbb{R}^d, w; X)$ as $T_k f = k * f$. Of course by the definition of $\widetilde{M}$ we also have $|k * f| \le \widetilde{M} f$ almost everywhere for all simple $f : \mathbb{R}^d \to X$.

Proposition 3.6. *Let $\overline{q} \in (1,\infty)^n$ and $X = L^{\overline{q}}(\Omega)$. For all $s \in [1,\infty]$ and $p \in (1,\infty)$ and $w \in A_p$, the family of convolution operators $\mathscr{T} = \{T_k : k \in \mathcal{K}\}$ on $L^p(\mathbb{R}^d, w; X)$ is ℓ^s-bounded and there is an increasing function $\alpha_{p,\overline{q},s,d}$ such that $\mathcal{R}^s(\mathscr{T}) \le \alpha_{p,\overline{q},s,d}([w]_{A_p})$.*

Proof. Let $1 < s < \infty$. Assume that $f_1, \cdots, f_N$ are simple. Take $t \in \Omega$ and $i \in \{1, \cdots, N\}$ arbitrary. Note that we have $f_i(\bullet, t) \in L^p(\mathbb{R}^d, w)$. Then since $|T_{k_i} f_i(x,t)| \le \widetilde{M} f_i(x,t)$ for almost all $x \in \mathbb{R}^d$, the result follows from Theorem 2.6 using the vector $(q_1, \cdots, q_n, s)$ and the measure space

$$(\Omega \times \{1, \cdots, N\}, \Sigma \times P(\{1, \cdots, N\}), \mu \times \lambda)$$

with λ the counting measure. Now the result follows by the density of the simple functions in $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$.

The proof of the cases $s = 1$ and $s = \infty$ follow the lines of [24, Theorem 4.7], where the unweighted setting is considered. In the case $s = \infty$ also assume that $f_1, \cdots, f_N$ are simple. With the boundedness of $\widetilde{M}$ from Theorem 2.6 we have

$$\begin{aligned}
\int_{\mathbb{R}^d} \Big\| \sup_{1 \le n \le N} |T_{k_n} f_n(x)| \Big\|_{L^{\overline{q}}(\Omega)}^p w(x)\,\mathrm{d}x &\le \int_{\mathbb{R}^d} \Big\| \sup_{1 \le n \le N} \widetilde{M} f_n(x) \Big\|_{L^{\overline{q}}(\Omega)}^p w(x)\,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \Big\| \widetilde{M}\Big(\sup_{1 \le n \le N} |f_n| \Big)(x) \Big\|_{L^{\overline{q}}(\Omega)}^p w(x)\,\mathrm{d}x \\
&\le \alpha_{p,\overline{q},d}([w]_{A_p})^p \int_{\mathbb{R}^d} \Big\| \Big(\sup_{1 \le n \le N} |f_n| \Big)(x) \Big\|_{L^{\overline{q}}(\Omega)}^p w(x)\,\mathrm{d}x
\end{aligned}$$

with $\alpha_{p,\overline{q},d}$ an increasing function on $\mathbb{R}_+$. The claim now follows by the density of the simple functions in $L^p(\mathbb{R}^d, w; L^{\overline{q}}(\Omega))$.

For $s=1$ we use duality. For $f \in L^p(\mathbb{R}^d, w; X)$ and $g \in L^{p'}(\mathbb{R}^d, w'; X^*)$, let

$$\langle f, g\rangle = \int_{\mathbb{R}^d} \langle f(x), g(x)\rangle_{X,X^*}\, dx.$$

It follows from Proposition A.1 that in this way $L^p(\mathbb{R}^d, w; X)^* = L^{p'}(\mathbb{R}^d, w'; X^*)$. Moreover, one has $T_k^* = T_{\tilde{k}}$ with $\tilde{k}(x) = k(-x)$. Now since $k \in \mathcal{K}$ if and only if $\tilde{k} \in \mathcal{K}$ we know by the second case that the adjoint family $\mathscr{T}^* = \{T^* : T \in \mathscr{T}\}$ is ℓ^∞-bounded on $L^{p'}(\mathbb{R}^d, w'; X^*)$. Now the result follows from Proposition 3.3. ∎

Remark 3.7. Proposition 3.6 is an extension of [24, Theorem 4.7] to the weighted setting. The result remains true for UMD Banach function spaces X and can be proved using the same techniques of [24] where one needs to apply the weighted extension of [29, Theorem 3] which is obtained in [32].

The endpoint case $s=1$ of Proposition 3.6 plays a crucial rôle in the proof of Theorems 1.1 and 3.10. Quite surprisingly the case $s=1$ plays a central rôle in the proof of [24, Theorem 7.2] as well, where it is used to prove $\mathcal{R}$-boundedness of a family of stochastic convolution operators.

3.3. **Integral operators with operator valued kernel.** In this section (Ω, Σ, μ) is a σ-finite measure space such that $L^q(\Omega)$ is separable for some (for all) $q \in (1,\infty)$.

Definition 3.8. *Let $\mathcal{J}$ be an index set. For each $j \in \mathcal{J}$, let $T_j : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{L}(L^q(\Omega))$ be such that for all $\phi \in L^q(\Omega)$, $(x,y) \mapsto T_j(x,y)\phi$ is measurable and $\|T_j(x,y)\| \le 1$. For $k \in \mathcal{K}$ define the operator I_{k,T_j} on $L^p(\mathbb{R}^d, v; L^q(\Omega))$ as*

$$I_{k,T_j} f(x) = \int_{\mathbb{R}^d} k(x-y) T_j(x,y) f(y)\, \mathrm{d}y \tag{3.1}$$

and denote the family of all such operators by $\mathcal{I}_T$.

In the above definition we consider a slight generalization of the setting of Theorem 1.1: We allow different operators T_j for $j \in \mathcal{J}$ in the ℓ^s-boundedness result of Theorem 3.10.

We first prove that the family of operators $\mathcal{I}_T$ is uniformly bounded.

Lemma 3.9. *Let $1 < p, q < \infty$ and write $X = L^q(\Omega)$. Assume that for all $\phi \in X$ and $j \in \mathcal{J}$, $(x,y) \mapsto T_j(x,y)\phi$ is measurable and $\|T_j(x,y)\| \le 1$. Then there exists an increasing function $\alpha_{p,d}$ on $\mathbb{R}_+$ such that for all $I_{k,T_j} \in \mathcal{I}_T$,*

$$\left\| I_{k,T_j} \right\|_{\mathcal{L}(L^p(\mathbb{R}^d, v; X))} \le \alpha_{p,d}([v]_{A_p}), \quad v \in A_p.$$

Proof. Let $f \in L^p(\mathbb{R}^d, v; X)$ arbitrary. Then by Minkowski's inequality for integrals in (i), the properties of $k \in \mathcal{K}$ in (ii) and boundedness of M on $L^p(\mathbb{R}^d, v)$ in (iii), we get

$$\begin{aligned}
\|I_{k,T_j} f\|_{L^p(\mathbb{R}^d, v; X)} &= \left(\int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}^d} k(x-y) T_j(x,y) f(y)\, \mathrm{d}y \right\|_X^p v(x)\, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\overset{(i)}{\le} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |k(x-y)| \|T_j(x,y) f(y)\|_X \, \mathrm{d}y \right)^p v(x)\, \mathrm{d}x \right)^{\frac{1}{p}} \\
&\le \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |k(x-y)| \|f(y)\|_X \, \mathrm{d}y \right)^p v(x)\, \mathrm{d}x \right)^{\frac{1}{p}}
\end{aligned}$$

$$\overset{(ii)}{\leq} \left(\int_{\mathbb{R}^d} (M(\|f\|_X)(x))^p v(x)\,\mathrm{d}x\right)^{\frac{1}{p}} \overset{(iii)}{\leq} \alpha_{p,d}([v]_{A_p})\,\|f\|_{L^p(\mathbb{R}^d,v;X)}$$

with $\alpha_{p,d}$ an increasing function on $\mathbb{R}_+$. This proves the lemma. ∎

Theorem 3.10. *Let $1 < p, q < \infty$ and write $X = L^q(\Omega)$. Assume the following conditions*

(1) For all $\phi \in X$ and $j \in \mathcal{J}$, $(x, y) \mapsto T_j(x, y)\phi$ is measurable.

(2) For all $s \in (1,\infty)$, $\mathscr{T} = \{T_j(x, y) : x, y \in \mathbb{R}^d, j \in \mathcal{J}\}$ is ℓ^s-bounded,

Then for all $v \in A_p$ and all $s \in (1,\infty)$, the family of operators $\mathcal{I}_T \subseteq L^p(\mathbb{R}^d, v; X)$ as defined in (3.1)*, is ℓ^s-bounded with $\mathcal{R}^s(\mathcal{I}_T) \leq C$ where C depends on $p, q, d, s, [v]_{A_p}$ and on $\mathcal{R}^\sigma(\mathscr{T})$ for $\sigma \in (1,\infty)$ and is A_p-consistent.*

Example 3.11. When $\Omega = \mathbb{R}^e$ with μ the Lebesgue measure and $q_0 \in (1,\infty)$, then the weighted boundedness of each of the operators $T_j(x, y)$ on $L^{q_0}(\mathbb{R}^e, w)$ for all A_{q_0}-weights w in an A_{q_0}-consistent way, is a sufficient condition for the ℓ^s-boundedness which is assumed in Theorem 3.10. Indeed, this follows from [5, Corollary 3.12] (also see Theorem 2.3).

Usually, the weighted boundedness is simple to check with [12, Theorem IV.3.9] or [15, Theorem 9.4.6], because often for each $x, y \in \mathbb{R}^d$ and $j \in \mathcal{J}$, $T_j(x, y)$ is given by a Fourier multiplier operator in $\mathbb{R}^e$.

Example 3.12. Let $q \in (1,\infty)$. Let $T(t) = e^{t\Delta}$ for $t \geq 0$ be the heat semigroup, where Δ is the Laplace operator on $\mathbb{R}^e$. Then it follows from the weighted Mihlin multiplier theorem [12, Theorem IV.3.9]) that for all $w \in A_q$, $\|T(t)\|_{\mathcal{L}(L^q(\mathbb{R}^e,w))} \leq C$, where C is A_q-consistent. Therefore, as in Example 3.11, $\{T(t) : t \in \mathbb{R}_+\}$ is ℓ^s-bounded on $L^q(\mathbb{R}^d, w)$ by an A_q-consistent $\mathcal{R}^s$-bound.

In order to give an example of an operator $I_{k,T}$ as in (3.1), we could let $T(x, y) = T(\phi(x, y))$, where $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is measurable. Other examples can be given if one replaces the heat semigroup by a two parameter evolution family $T(t, s)$. As explained in the introduction, this is the setting of [11] (see Theorem 1.1).

To prove Theorem 3.10 we will first show a result assuming ℓ^s-boundedness for a fixed $s \in (1,\infty)$. Here we can also include $s = 1$.

Proposition 3.13. *Let $1 \leq s < q < \infty$ and write $X = L^q(\Omega)$. Assume the following conditions*

(1) For all $\phi \in X$ and all $j \in \mathcal{J}$, $(x, y) \mapsto T_j(x, y)\phi$ is measurable.

(2) $\mathscr{T} = \{T_j(x, y) : x, y \in \mathbb{R}^d, j \in \mathcal{J}\}$ is ℓ^s-bounded.

Then for all $p \in (s,\infty)$ and all $v \in A_{\frac{p}{s}}$ the family of operators $\mathcal{I}_T \subseteq L^p(\mathbb{R}^d, v; X)$ defined as in (3.1)*, is ℓ^s-bounded and there exist an increasing function $\alpha_{s,p,q,d}$ such that*

$$\mathcal{R}^s(\mathcal{I}_T) \leq \mathcal{R}^s(\mathscr{T})\alpha_{s,p,q,d}([v]_{A_{\frac{p}{s}}}).$$

Proof. Without loss of generality we can assume $\mathcal{R}^s(\mathscr{T}) = 1$. We start with a preliminary observation. By [12, Theorem VI.5.2] or [15, Theorem 9.5.8], the ℓ^s-boundedness is equivalent to the following: for every $u \geq 0$ in $L^{\frac{q}{q-s}}(\Omega)$ there exists a $U \in L^{\frac{q}{q-s}}(\Omega)$ such that

$$\begin{aligned} &\|U\|_{L^{\frac{q}{q-s}}(\Omega)} \leq \|u\|_{L^{\frac{q}{q-s}}(\Omega)}, \\ &\int_\Omega |T_j(x, y)\phi|^s u\,d\mu \leq \int_\Omega |\phi|^s U\,d\mu, \quad x, y \in \Omega,\ j \in \mathcal{J}\ \phi \in L^q(\Omega). \end{aligned} \tag{3.2}$$

For $n = 1, \cdots, N$ take $I_{k_n,T_{j_n}} \in \mathcal{I}_T$ and let $I_n = I_{k_n,T_{j_n}}$ where $j_1, \ldots, j_N \in \mathcal{J}$. Take $f_1, \cdots, f_N \in L^p(\mathbb{R}^d, v; X)$ and note that

$$\Big\|\Big(\sum_{n=1}^N |I_n f_n|^s\Big)^{\frac{1}{s}}\Big\|_{L^p(\mathbb{R}^d,v;X)} = \Big\|\sum_{n=1}^N |I_n f_n|^s\Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}\big(\mathbb{R}^d,v;L^{\frac{q}{s}}(\Omega)\big)}.$$

Let $r \in (1, \infty)$ be such that $\frac{1}{r} + \frac{s}{q} = 1$ and fix $x \in \mathbb{R}^d$. As $L^r(\Omega) = L^{\frac{q}{s}}(\Omega)^*$, we can find a function $u \in L^r(\Omega)$, which will depend on x, with $u \geq 0$ and $\|u\|_{L^r(\Omega)} = 1$ such that

$$\Big\|\sum_{n=1}^N |I_n f_n(x)|^s\Big\|_{L^{\frac{q}{s}}(\Omega)} = \sum_{n=1}^N \int_\Omega |I_n f_n(x)|^s u \,\mathrm{d}\mu. \tag{3.3}$$

By the observation in the beginning of the proof, there is a function $U \geq 0$ in $L^r(\Omega)$ (which depends on x again) such that (3.2) holds. Since $\|k_n\|_{L^1(\mathbb{R}^d)} \leq 1$, Hölder's inequality yields

$$|I_n f_n(x)|^s \leq \int_{\mathbb{R}^d} |k_n(x-y)||T_{j_n}(x,y) f_n(y)|^s \,\mathrm{d}y. \tag{3.4}$$

Applying (3.4) in (i), estimate (3.2) in (ii), and Hölder's inequality in (iii), we get:

$$\begin{aligned}
\sum_{n=1}^N \int_\Omega |I_n f_n(x)|^s u \,\mathrm{d}\mu &\overset{(i)}{\leq} \sum_{n=1}^N \int_\Omega \int_{\mathbb{R}^d} |k_n(x-y)||T_{j_n}(x,y) f_n(y)|^s \,\mathrm{d}y \, u \,\mathrm{d}\mu \\
&= \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)| \int_\Omega |T_{j_n}(x,y) f_n(y)|^s \, u \,\mathrm{d}\mu \,\mathrm{d}y \\
&\overset{(ii)}{\leq} \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)| \int_\Omega |f_n(y)|^s \, U \,\mathrm{d}\mu \,\mathrm{d}y \\
&= \int_\Omega \sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)||f_n(y)|^s \,\mathrm{d}y \, U \,\mathrm{d}\mu \\
&\overset{(iii)}{\leq} \Big\|\sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(x-y)||f_n(y)|^s \,\mathrm{d}y\Big\|_{L^{\frac{q}{s}}(\Omega)}.
\end{aligned}$$

Combining (3.3) with the above estimate and applying the ℓ^1-boundedness result of Proposition 3.6 to $|f_n|^s \in L^{\frac{p}{s}}\big(\mathbb{R}^d, v; L^{\frac{q}{s}}(\Omega)\big)$ (here we use $v \in A_{\frac{p}{s}}$), we get

$$\begin{aligned}
\Big\|\Big(\sum_{n=1}^N |I_n f_n|^s\Big)^{\frac{1}{s}}\Big\|_{L^p(\mathbb{R}^d,v;X)} &\leq \Big\|\sum_{n=1}^N \int_{\mathbb{R}^d} |k_n(\bullet - y)||f_n(y)|^s \,\mathrm{d}y\Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}\big(\mathbb{R}^d,v;L^{\frac{q}{s}}(\Omega)\big)} \\
&\leq \alpha_{p,q,s,d}([v]_{A_{\frac{p}{s}}}) \Big\|\sum_{n=1}^N |f_n|^s\Big\|^{\frac{1}{s}}_{L^{\frac{p}{s}}\big(\mathbb{R}^d,v;L^{\frac{q}{s}}(\Omega^e,w)\big)} \\
&= \alpha_{p,q,s,d}([v]_{A_{\frac{p}{s}}}) \Big\|\Big(\sum_{n=1}^N |f_n|^s\Big)^{\frac{1}{s}}\Big\|_{L^p(\mathbb{R}^d,v;X)}
\end{aligned}$$

with $\alpha_{p,q,s,d}$ an increasing function on $\mathbb{R}_+$. This proves the ℓ^s-boundedness. ■

Next we prove Theorem 3.10. For a constant ϕ depending on a parameter $t \in I \subset \mathbb{R}$, we write $\phi \propto t$ if $\phi_t \leq \phi_s$ whenever $t \leq s$ and $s, t \in I$.

Proof of Theorem 3.10. Fix $q \in (1,\infty)$, $p = q$, $v \in A_q$ and $\kappa = 2[v]_{A_q} \geq 2$. The case $p \neq q$ will be considered at the end of the proof.

Step 1. First we prove the theorem for very small $s \in (1, q)$. Proposition 2.1 gives $\sigma_1 = \sigma_{q,\kappa,d} \in (1, q)$ and $C_{q,\kappa,d}$ such that for all $s \in (1, \sigma_1]$ and all weights $u \in A_q$ with $[u]_{A_q} \leq \kappa$,

$$[u]_{A_{\frac{q}{s}}} \leq [u]_{A_{\frac{q}{\sigma}}} \leq C_{q,\kappa,d}.$$

Moreover, $\sigma_1 \propto \kappa^{-1}$ and $C \propto \kappa$.

By Proposition 3.13, $\mathcal{I}_T \subseteq \mathcal{L}(L^q(\mathbb{R}^d, v; X))$ is ℓ^s-bounded for all $s \in (1, \sigma_1)$ and

$$\mathcal{R}^s(\mathcal{I}_T) \leq \mathcal{R}^s(\mathscr{T})\alpha_{s,q,d}([v]_{A_{\frac{q}{s}}}) \leq \mathcal{R}^s(\mathscr{T})\beta_{q,s,d,\kappa}, \tag{3.5}$$

with $\beta_{q,s,d,\kappa} = \alpha_{q,s,d}(C_{q,\kappa,d})$. Note that $\beta \propto \kappa$ and $\beta \propto s'$.

Step 2. Now we use a duality argument to prove the theorem for large $s \in (q,\infty)$. By Proposition 2.1, $v' \in A_{q'}$ and $\tilde{\kappa} = 2[v']_{A_{q'}} = 2[v]_{A_q}^{\frac{1}{q-1}} = 2(\kappa^{\frac{1}{q-1}})$. Note that we can identify $X^* = L^{q'}(\Omega)$ and $L^q(\mathbb{R}^d, v; X)^* = L^{q'}(\mathbb{R}^d, v'; X^*)$ by Proposition A.1. Define $\mathcal{I}_T^* = \{I^* : I \in \mathcal{I}_T\}$.

It is standard to check that for $I_{k,T_j} \in \mathcal{I}_T$ the adjoint I^*_{k,T_j} satisfies

$$I^*_{k,T_j} g(x) = \int_{\mathbb{R}^d} \tilde{k}(y - x)\tilde{T}_j(x, y)g(y)\,\mathrm{d}y = I_{\tilde{k},\tilde{T}_j} g(x)$$

with $\tilde{k}(x) = k(-x)$ and $\tilde{T}_j(x, y) = T_j^*(y, x)$. As already noted before we have $\tilde{k} \in \mathcal{K}$. Furthermore, by Proposition 3.3 the adjoint family $\mathscr{T}^*$ is $\mathcal{R}^{s'}$-bounded with $\mathcal{R}^{s'}(\mathscr{T}^*) = \mathcal{R}^s(\mathscr{T})$. Therefore, it follows from Step 1 that there is a $\sigma_2 = \sigma_{q',\tilde{\kappa},d} \in (1, q')$ such that for all $s' \in (1, \sigma_2]$, $\mathcal{I}_T^*$ is $\ell^{s'}$-bounded on $L^{q'}(\mathbb{R}^d, v'; X^*)$ and using Proposition 3.3 again, we obtain $\mathcal{I}_T$ is ℓ^s-bounded and

$$\mathcal{R}^s(\mathcal{I}_T) = \mathcal{R}^{s'}(\mathcal{I}_T^*) \leq \mathcal{R}^{s'}(\mathscr{T}^*)\beta_{q',s',d,\tilde{\kappa}} = \mathcal{R}^s(\mathscr{T})\beta_{q',s',d,\tilde{\kappa}}. \tag{3.6}$$

Therefore, Proposition 3.3 yields that $\mathcal{I}_T$ is ℓ^s-bounded on $L^q(\mathbb{R}^d, v; X)$ for all $s \in [\sigma_2', \infty)$.

Step 3. We can now finish the proof in the case $p = q$ by an interpolation argument. In the previous steps 1 and 2 we have found $1 < \sigma_1 < q < \sigma_2' < \infty$ such that $\mathcal{I}_\alpha$ is ℓ^s-bounded for all $s \in (1, \sigma_1] \cup [\sigma_2', \infty)$ with

$$\mathcal{R}^s(\mathcal{I}_T) \leq \mathcal{R}^s(\mathscr{T})\gamma_{q,s,d,\kappa}. \tag{3.7}$$

where $\gamma_{q,s,d,\kappa} = \beta_{q,s,d,\kappa}$ if $s \leq \sigma_1$ and $\gamma_{q,s,d,\kappa} = \beta_{q',s',d,\tilde{\kappa}}$ if $s \geq \sigma_2'$. Clearly, $\gamma := \gamma_{q,s,d,\kappa}$ satisfies $\gamma \propto \kappa$, $\gamma \propto s'$ for $s \in (1, \sigma_1]$ and $\gamma \propto s$ for $s \in [\sigma_2', \infty)$. Moreover, $\sigma_1 \propto \frac{1}{\kappa}$ and $\sigma_2' \propto \kappa$.

Now Proposition 3.3 yields the ℓ^s-boundedness and the required estimates for the remaining $s \in [\sigma_1, \sigma_2']$ and by (3.7) we find

$$\begin{aligned}\mathcal{R}^s(\mathcal{I}_T) &\leq \max\{\mathcal{R}^{\sigma_1}(\mathcal{I}_T), \mathcal{R}^{\sigma_2'}(\mathcal{I}_T)\} \\ &\leq \max\{\mathcal{R}^{\sigma_1}(\mathscr{T}), \mathcal{R}^{\sigma_2'}(\mathscr{T})\}\gamma.\end{aligned}$$

where $\gamma = \max\{\gamma_{q,\sigma_1,d,\kappa}, \gamma_{q',\sigma_2,d,\tilde{\kappa}}\}$. By Example 3.5, $\mathcal{R}^{\sigma_1}(\mathscr{T}) \propto \kappa$ and $\mathcal{R}^{\sigma_2'}(\mathscr{T}) \propto \kappa$. Also $\gamma \propto \kappa$ in the above. Therefore, the obtained $\mathcal{R}^s$-bound is A_q-consistent.

Step 4. Next let $p, q \in (1,\infty)$. Fix $s \in (1,\infty)$. For $n = 1, \cdots, N$ take $I_{k_n,T_{j_n}} \in \mathcal{I}_T$ and let $I_n = I_{k_n,T_{j_n}}$. Take $f_1, \cdots, f_N \in L^p(\mathbb{R}^d, v; X) \cap L^q(\mathbb{R}^d, v; X)$ and let

$$F = \Big\| \Big(\sum_{n=1}^{N} |I_n f_n|^s \Big)^{\frac{1}{s}} \Big\|_X \quad \text{and} \quad G = \Big\| \Big(\sum_{n=1}^{N} |f_n|^s \Big)^{\frac{1}{s}} \Big\|_X.$$

By the previous step we know that for all $v \in A_q$,

$$\|F\|_{L^q(\mathbb{R}^d,v)} \leq C \|G\|_{L^q(\mathbb{R}^d,v)},$$

where C depends on d, s, q, and $[v]_{A_p}$ and is A_p-consistent. Therefore, by Theorem 2.2 we can extrapolate to obtain for all $p \in (1,\infty)$ and $v \in A_p$,

$$\|F\|_{L^p(\mathbb{R}^d,v)} \leq \tilde{C} \|G\|_{L^p(\mathbb{R}^d,v)},$$

where $\tilde{C}$ depends on C, p and $[v]_{A_p}$ and is again A_p-consistent. This implies the required $\mathcal{R}^s$-boundedness for all $p, q \in (1,\infty)$ with constant $\tilde{C}$. ∎

Corollary 3.14. *Let $\Omega \subseteq \mathbb{R}^d$ be an open set. Let $1 < p, q, q_0 < \infty$. Assume the following conditions*

(1) For all $\phi \in L^q(\Omega)$ and $j \in \mathcal{J}$, $(x,y) \mapsto T_j(x,y)\phi$ is measurable.

(2) For all $w \in A_{q_0}$, $\sup_{j \in \mathcal{J}, x,y \in \Omega} \|T_j(x,y)\|_{\mathcal{L}(L^{q_0}(\Omega,w))} \leq C$, where C is A_{q_0}-consistent.

Then for all $v \in A_p$ all $w \in A_q$ and all $s \in (1,\infty)$, the family of operators $\mathcal{I}_T \subseteq L^p(\mathbb{R}^d, v; L^q(\Omega, w))$ as defined in (3.1), is ℓ^s-bounded with $\mathcal{R}^s(\mathcal{I}_T) \leq \tilde{C}$ where $\tilde{C}$ depends on $p, q, d, s, [v]_{A_p}, [w]_{A_q}$ and on $\mathcal{R}^\sigma(\mathscr{T})$ for $\sigma \in (1,\infty)$ and is A_p- and A_q-consistent.

Proof. In the case $\Omega = \mathbb{R}^e$, note that Example 3.11 yields that for each $q \in (1,\infty)$ and each $w \in A_q$ and $s \in (1,\infty)$, $\mathscr{T}$ considered on $L^q(\Omega, w)$ is ℓ^s-bounded. Moreover, $\mathcal{R}^s(\mathscr{T}) \leq K$, where K depends on q, s, e and $[w]_{A_q}$ in an A_q-consistent way. Therefore, the result follows from Theorem 3.10.

In the case $\Omega \subseteq \mathbb{R}^e$, we reduce to the case $\mathbb{R}^e$ by a restriction-extension argument. For convenience we sketch the details. Let $E : L^q(\Omega, w) \to L^q(\mathbb{R}^e, w)$ be the extension by zero and let $R : L^q(\mathbb{R}^e, w) \to L^q(\Omega, w)$ be the restriction to Ω. For every $x, y \in \mathbb{R}^d$ and $j \in \mathcal{J}$, let $\tilde{T}_j(x,y) = E T_j(x,y) R \in \mathcal{L}(L^q(\mathbb{R}^e, w))$ and let $\tilde{\mathscr{T}} = \{\tilde{T}_j(x,y) : x, y \in \mathbb{R}^d\}$. Since $\|\tilde{T}_j(x,y)\|_{\mathcal{L}(L^q(\mathbb{R}^e,w))} \leq \|T_j(x,y)\|_{\mathcal{L}(L^q(\Omega,w))} \leq C$, it follows from the case $\Omega = \mathbb{R}^e$ that $\mathcal{I}_{\tilde{T}} \subseteq L^p(\mathbb{R}^d, v; L^q(\mathbb{R}^e, w))$ is ℓ^s-bounded with $\mathcal{R}^s(\mathcal{I}_{\tilde{T}}) \leq \tilde{C}$. Now it remains to observe that the restriction of $I_{k,\tilde{T}_j}$ to $L^p(\mathbb{R}^d, v; L^q(\Omega, w))$ is equal to I_{k,T_j} and hence $\mathcal{R}^s(\mathcal{I}_T) \leq \mathcal{R}^s(\mathcal{I}_{\tilde{T}}) \leq \tilde{C}$. ∎

Next we will prove Theorem 1.1. In order to do so we recall the definition of $\mathcal{R}$-boundedness.

Definition 3.15. *Let X and Y be Banach spaces and let $(\varepsilon_n)_{n\geq 1}$ be a Rademacher sequence on a probability space $(A, \mathscr{A}, \mathbb{P})$. A family of operators $\mathscr{S} \subseteq \mathcal{B}(X,Y)$ is said to be* $\mathcal{R}$-bounded *if there exists a constant C such that for all integers N, for all sequences $(S_n)_{n=1}^N$ in $\mathscr{S}$ and $(x_n)_{n=1}^N$ in X,*

$$\Big\| \sum_{n=1}^{N} \varepsilon_n S_n x_n \Big\|_{L^2(A;Y)} \leq C \Big\| \sum_{n=1}^{N} \varepsilon_n x_n \Big\|_{L^2(A;Y)}$$

The least possible constant C is called the $\mathcal{R}$-bound *of $\mathscr{S}$ and is denoted by $\mathcal{R}(\mathscr{S})$.*

Remark 3.16. For $X = Y = L^{\overline{q}}(\Omega)$ with $q \in (1,\infty)^n$, the notions ℓ^2-boundedness and $\mathcal{R}$-boundedness of any family $\mathscr{S} \subseteq \mathcal{B}(X,Y)$ coincide and $C^{-1}\mathcal{R}^2(\mathscr{S}) \le \mathcal{R}(\mathscr{S}) \le C\mathcal{R}^2(\mathscr{S})$, where C is a constant which only depends on $\overline{q}$. This assertion follows from the Kahane-Khintchine inequalities (see [8, 1.10 and 11.1]).

Proof of Theorem 1.1. The result follows directly from Corollary 3.14 and Remark 3.16 with $X = L^p(\mathbb{R}; L^q(\Omega))$. ∎

Appendix A. Duality of iterated $L^{\overline{q}}$-spaces

Let $(\Omega_i, \Sigma_i, \mu_i)$ for $i = 1, \dots n$ be σ-finite measure spaces. The dual of the iterated space $L^{\overline{q}}(\Omega)$ as defined in (2.1), is exactly what one would expect. In a general setting one can prove that $L^p(\Omega; X)^* = L^{p'}(\Omega, X^*)$ for reflexive Banach function spaces X from which the duality for $L^{\overline{q}}(\Omega)$ follows, as is done in [9, Chapter IV] using the so-called Radon-Nikodym property of Banach spaces. Here we present an elementary proof just for $L^{\overline{q}}(\Omega)$.

Proposition A.1. *Let $\overline{q} \in (1,\infty)^n$. For every bounded linear functional Φ on $L^{\overline{q}}(\Omega)$ there exists a unique $g \in L^{\overline{q}'}(\Omega)$ such that:*

$$\Phi(f) = \int_\Omega f g \,\mathrm{d}\mu \tag{A.1}$$

for all $f \in L^{\overline{q}}$ and $\|\Phi\| = \|g\|_{L^{\overline{q}'}(\Omega)}$, i.e. $L^{\overline{q}}(\Omega)^* = L^{\overline{q}'}(\Omega)$.

Proof. We follow the strategy of proof from [30, Theorem 6.16]. The uniqueness proof is as in [30, Theorem 6.16]. Also by repeatedly applying Hölder's inequality we have for any g satisfying (A.1) that

$$\|\Phi\| \le \|g\|_{L^{\overline{q}'}(\Omega)}. \tag{A.2}$$

So it remains to prove that g exists and that equality holds in (A.2). As in [30, Theorem 6.16] one can reduce to the case $\mu(\Omega) < \infty$. Define $\lambda(E) = \Phi(\chi_E)$ for $E \in \Sigma$. Then one can check that λ is a complex measure which is absolutely continuous with respect to μ. So by the Radon-Nikodym Theorem [30, Theorem 6.10] we can find a $g \in L^1(\Omega)$ such that for all measurable $E \subseteq \Omega$

$$\Phi(\chi_E) = \int_E g \,\mathrm{d}\mu = \int_\Omega \chi_E g \,\mathrm{d}\mu$$

and from this we get by linearity $\Phi(f) = \int_\Omega f g \,\mathrm{d}\mu$ for all simple functions f. Now take a $f \in L^\infty(\Omega)$ arbitrary and let f_i be simple functions such that $\|f_i - f\|_{L^\infty(\Omega)} \to 0$ for $i \to \infty$. Then since $\mu(\Omega) < \infty$ we have $\|f_i - f\|_{L^{\overline{q}}(\Omega)} \to 0$ for $i \to \infty$. Hence

$$\Phi(f) = \lim_{i\to\infty} \Phi(f_i) = \lim_{i\to\infty} \int_\Omega f_i g \,\mathrm{d}\mu = \int_\Omega f g \,\mathrm{d}\mu. \tag{A.3}$$

We will now prove that $g \in L^{\overline{q}'}(\Omega)$ and that equality holds in (A.2). Take $k \in \mathbb{N}$ arbitrary. Let $E_k^1 = \{s \in \Omega : \frac{1}{k} \le |g(s)| \le k\}$ and define for $i = 2, \cdots, n$

$$E_k^i = \left\{ s \in \Omega : \|g_k(s_1, \cdots, s_{i-1}, \bullet)\|_{L^{q_i'}(\Omega_i, \cdots L^{q_n'}(\Omega_n))} \ge \frac{1}{k} \right\}$$

Now take $g_k = g\prod_{i=1}^n \chi_{E_k^i}$ and let α be its complex sign function, i.e. $|\alpha| = 1$ and $\alpha|g_k| = g_k$. Take

$$f(s) = \overline{\alpha}|g_k(s)|^{q_n'-1}\prod_{i=2}^{n}\|g_k(s_1,\cdots,s_{i-1},\bullet)\|_{L^{q_i'}(\Omega_i,\cdots L^{q_n'}(\Omega_n))}^{q_{i-1}'-q_i'}$$

where we define $0\cdot\infty = 0$. Then $f \in L^\infty(\Omega)$ and one readily checks that

$$\int_\Omega f g_k \,\mathrm{d}\mu = \|g_k\|_{L^{\overline{q}'}(\Omega)}^{q_1'} \quad\text{and}\quad \|f\|_{L^{\overline{q}}(\Omega)} = \|g_k\|_{L^{\overline{q}'}(\Omega)}^{\frac{q_1'}{q_1}}. \tag{A.4}$$

So from (A.4) we obtain

$$\|g_k\|_{L^{\overline{q}'}(\Omega)}^{q_1'} = \int_\Omega f g_k \,\mathrm{d}\mu = \Phi(f) \le \|f\|_{L^{\overline{q}}(\Omega)}\|\Phi\| = \|g_k\|_{L^{\overline{q}'}(\Omega)}^{\frac{q_1'}{q_1}}\|\Phi\|$$

which means $\|g_k\|_{L^{\overline{q}'}(\Omega)} \le \|\Phi\|$. Since this holds for all $k \in \mathbb{N}$ we obtain by Fatou's lemma that $\|g\|_{L^{\overline{q}'}(\Omega)} \le \|\Phi\|$, which proves that $g \in L^{\overline{q}'}(\Omega)$ and $\|g\|_{L^{\overline{q}'}(\Omega)} = \|\Phi\|$. From this we also get (A.3) for all $f \in L^{\overline{q}'}(\Omega)$ by Hölders inequality and the dominated convergence theorem. This proves the required result. ∎

To obtain the duality result in Proposition 3.3 for $s = 1$ and $s = \infty$, one also needs the following end-point duality result. Let $X(\ell_N^s)$ be the space of all N-tuples $(f_n)_{n=1}^N \in X^N$ with

$$\|(f_n)_{n=1}^N\|_{X(\ell_N^s)} = \Big\|\Big(\sum_{n=1}^N |f_n|^s\Big)^{1/s}\Big\|_X$$

with the usual modification if $s = \infty$.

Lemma A.2. *Define $X = L^{\overline{q}}(\Omega)$. Take $s \in [1,\infty]$ and $N \in \mathbb{N}$. Then for every bounded linear functional Φ on $X(\ell_N^s)$ there exists a unique $g \in X^*(\ell_N^{s'})$ such that*

$$\Phi(f) = \sum_{i=1}^N \langle f_i, g_i\rangle_{X,X^*}$$

for all $f \in X(\ell_N^s)$ and $\|\Phi\| = \|g\|_{X^*(\ell_N^{s'})}$, i.e. $X(\ell_N^s)^* = X^*(\ell_N^{s'})$.

Also this result can be proved with elementary arguments. Indeed, for $r_1, r_2 \in [1,\infty]$ we have $X(\ell_N^{r_1}) = X(\ell_N^{r_2})$ as sets and the following inequalities hold for all $f \in X(\ell_N^r)$ and $r \in [1,\infty]$

$$\begin{aligned} \|f\|_{X(\ell_N^r)} &\le \|f\|_{X(\ell_N^1)} \le N^{1-\frac{1}{r}}\|f\|_{X(\ell_N^r)} \\ \|f\|_{X(\ell_N^\infty)} &\le \|f\|_{X(\ell_N^r)} \le N^{\frac{1}{r}}\|f\|_{X(\ell_N^\infty)}. \end{aligned} \tag{A.5}$$

Now the lemma readily follows from $X(\ell_N^r)^* = X^*(\ell_N^{r'})$ for $r \in (1,\infty)$ and letting $r \downarrow 1$ and $r \uparrow \infty$.

References

[1] K.F. Andersen and R.T. John. Weighted inequalities for vector-valued maximal functions and singular integrals. *Studia Math.*, 69(1):19–31, 1980/81.
[2] J. Bourgain. Extension of a result of Benedek, Calderón and Panzone. *Ark. Mat.*, 22(1):91–95, 1984.
[3] A.-P. Calderón. Intermediate spaces and interpolation, the complex method. *Studia Math.*, 24:113–190, 1964.

[4] Ph. Clément and S. Li. Abstract parabolic quasilinear equations and application to a groundwater flow problem. *Adv. Math. Sci. Appl.*, 3(Special Issue):17–32, 1993/94.

[5] D. V. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer Basel AG, Basel, 2011.

[6] R. Denk, G. Dore, M. Hieber, J. Prüss, and A. Venni. New thoughts on old results of R. T. Seeley. *Math. Ann.*, 328(4):545–583, 2004.

[7] R. Denk, M. Hieber, and J. Prüss. R-boundedness, Fourier multipliers and problems of elliptic and parabolic type. *Mem. Amer. Math. Soc.*, 166(788), 2003.

[8] J. Diestel, H. Jarchow, and A. Tonge. *Absolutely summing operators*, volume 43 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[9] J. Diestel and J. J. Uhl, Jr. *Vector measures*. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis, Mathematical Surveys, No. 15.

[10] A. Fröhlich. The Stokes operator in weighted L^q-spaces. II. Weighted resolvent estimates and maximal L^p-regularity. *Math. Ann.*, 339(2):287–316, 2007.

[11] C. Gallarati and M.C. Veraar. Maximal regularity for non-autonomous equations with measurable dependence on time. See arxiv preprint server, http://arxiv.org/abs/1410.6394.

[12] J. García-Cuerva and J.L. Rubio de Francia. *Weighted norm inequalities and related topics*, volume 116 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1985. Notas de Matemática [Mathematical Notes], 104.

[13] M. Girardi and L. Weis. Criteria for R-boundedness of operator families. In *Evolution equations*, volume 234 of *Lecture Notes in Pure and Appl. Math.*, pages 203–221. Dekker, New York, 2003.

[14] L. Grafakos. *Classical Fourier analysis*, volume 249 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2008.

[15] L. Grafakos. *Modern Fourier analysis*, volume 250 of *Graduate Texts in Mathematics*. Springer, New York, second edition, 2009.

[16] B.H. Haak and P.C. Kunstmann. Admissibility of unbounded operators and wellposedness of linear systems in Banach spaces. *Integral Equations Operator Theory*, 55(4):497–533, 2006.

[17] R. Haller, H. Heck, and M. Hieber. Muckenhoupt weights and maximal L^p-regularity. *Arch. Math. (Basel)*, 81(4):422–430, 2003.

[18] H. Heck and M. Hieber. Maximal L^p-regularity for elliptic operators with VMO-coefficients. *J. Evol. Equ.*, 3(2):332–359, 2003.

[19] T. Hytönen and M.C. Veraar. R-boundedness of smooth operator-valued functions. *Integral Equations Operator Theory*, 63(3):373–402, 2009.

[20] P. Kunstmann and A. Ullmann. $\mathcal{R}_s$-sectorial operators and generalized Triebel-Lizorkin spaces. *J. Fourier Anal. Appl.*, 20(1):135–185, 2014.

[21] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.

[22] J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces II: Function spaces*, volume 97 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, Berlin, 1979.

[23] A. Lunardi. *Analytic semigroups and optimal regularity in parabolic problems*. Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995.

[24] J.M.A.M. van Neerven, M.C. Veraar, and L.W. Weis. On the R-boundedness of stochastic convolution operators. online first in Positivity 2014.

[25] J. Prüss. Maximal regularity for evolution equations in L_p-spaces. *Conf. Semin. Mat. Univ. Bari*, (285):1–39 (2003), 2002.

[26] J.L. Rubio de Francia. Factorization and extrapolation of weights. *Bull. Amer. Math. Soc. (N.S.)*, 7(2):393–395, 1982.

[27] J.L. Rubio de Francia. A new technique in the theory of A_p weights. In *Topics in modern harmonic analysis, Vol. I, II (Turin/Milan, 1982)*, pages 571–579. Ist. Naz. Alta Mat. Francesco Severi, Rome, 1983.

[28] J.L. Rubio de Francia. Factorization theory and A_p weights. *Amer. J. Math.*, 106(3):533–547, 1984.

[29] J.L. Rubio de Francia. Martingale and integral transforms of Banach space valued functions. In *Probability and Banach spaces (Zaragoza, 1985)*, volume 1221 of *Lecture Notes in Math.*, pages 195–222. Springer, Berlin, 1986.

[30] W. Rudin. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition, 1987.

[31] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[32] S.A. Tozoni. Vector-valued extensions of operators on martingales. *J. Math. Anal. Appl.*, 201(1):128–151, 1996.

[33] A. Ullmann. *Maximal functions, functional calculus, and generalized Triebel-Lizorkin spaces for sectorial operators*. PhD thesis, University of Karlsruhe, 2010.

[34] L.W. Weis. A new approach to maximal L_p-regularity. In *Evolution equations and their applications in physical and life sciences (Bad Herrenalb, 1998)*, volume 215 of *Lecture Notes in Pure and Appl. Math.*, pages 195–214. Dekker, New York, 2001.

[35] L.W. Weis. Operator-valued Fourier multiplier theorems and maximal L_p-regularity. *Math. Ann.*, 319(4):735–758, 2001.

[36] A.C. Zaanen. *Integration*. North-Holland Publishing Co., Amsterdam; Interscience Publishers John Wiley & Sons, Inc., New York, 1967.

E-mail address: `C.Gallarati@tudelft.nl`
E-mail address: `EmielLorist@gmail.com`
E-mail address: `M.C.Veraar@tudelft.nl`

Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands