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DEPARTMENT OF AEROSPACE ENGINEERING

Report LR-303

OVERALL BUCKLING OF Z-STIFFENED PANELS IN COMPRESSION

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SUMMARY

The overall buckling mode of Z-stiffened panels in compression is composed of the usual bending of the stiffeners in the plane normal to the panel and of sideways bending of the stiffener flanges. Thereby the stiffener cross-section is being distorted as well as the plate between the stiffeners. The paper presents a simple analytical method for establishing critical load and mode, thereby using some slight approximations. Deformation by shear load is taken into account. The necessity to account for shear is not specific for this particular structure; the fact that in general shear lowers the critical load of stiffened panels (in the order of 10% at low slenderness) is not common knowledge.

Numerical illustrations refer to three configurations having everything equal except the shape of the flanges, one of them having symmetric flanges. The critical load appears to depend very much on the stiffness of restraint (α), offered by the panel plate to the root of the stiffener web. This stiffness is much better with bonded than with riveted stiffeners. With symmetric flanges Euler buckling and overall flange buckling are uncoupled. With asymmetric flanges sideways flange bending dominates the Euler component of the mode more and more with decreasing restraint of the web root. Comparison with solutions obtained from exact or nearly exact methods confirm the accuracy of the method.

Formulae for the stiffness of restraint (α), the shear stiffness and the structural coefficients occurring in the main equations are given in appendices.

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NOTATION

a	$[12\,(1-\nu^2)\,k]^{\frac{1}{4}}\,(\frac{in}{t})^{\frac{1}{2}}\,\lambda$
b, b_1, c, r, s, t, t_1, H	defined by Fig. 8
e	qH
f	A/A_f
i	radius of gyration
k	buckling coefficient, defined by (3.12)
k_s	coefficient of shear buckling, defined by (3.12)
ℓ	half-wave length
n	L/ℓ
p	critical stress
q	$(A_1 + \frac{1}{2}A_w)/A$
u, v, w, x, y, z, U, V, W, φ_1, φ_2	defined by Fig. 1
A	$A_f + A_w + A_1$, total area per stiffener pitch
A_f, A_w, A_1	cross-sectional area of resp. flange, web, plate and flange
B, B_1	bending stiffness of stiffener wall and plate resp.
E	modulus of elasticity
F	defined by (3.13)
I	moment of inertia
I_f	$\int y^2\,dA$ of flange
I_s	defined by (3.5)
L	bay length

S	$\int y\,dA$ of flange
T	torsional stiffness of flange
X, Y, Z	defined by (3.26)
α	coefficient of restraint of web root, defined by (3.18)
λ	$\pi H/L$
ν	Poisson's ratio
ζ	z/H
( )'	d( )/dx

1. INTRODUCTION

The special feature of the Z-stiffened panel is that the principal axis of inertia is not perpendicular to the panel plate. Due to this pecularity the mode in overall buckling by axial compression is not the pure Euler mode: the deflexion of the stiffeners in the normal direction is accompanied by sideways bending of the stiffener flange. This additional deformation causes decrease of the buckling stress as compared to the Euler stress. The mode has some resemblance to torsional buckling but is different. Torsional buckling is a mode where the cross section does not deform or deforms so little that it may be assumed to be rigid. With this type of panel sideways flange bending is being restrained by the stiffener web, which thereby deforms out-of-plane and causes moreover deformation of the plate between the stiffeners. These latter phenomena have much in common with the local buckling mode but again there is an essential difference between the two modes. Local buckling has a short wave mode; then the in-plane deformations are negligibly small. The present structure has a long wave mode with consequently non-negligible deformation in the plane of the composing plate strips.

So it would be incorrect to classify this case under one of the headings: Euler, torsional or local buckling. The pecularity of the mode being the sideways flexure of the stiffener flange the mode might be called "overall flange buckling". This name is certainly appropriate with the special case where the neutral axis of the flange section is in the plane of the stringer web. Sideways bending of the flange is in that case uncoupled from the pure Euler mode. The flange buckles as a column upon the elastic foundation supplied by out-of-plane bending of the stringer web. Clearly "overall flange buckling" is here a mode in its own right independent of the Euler mode.

With configurations like Z-stiffened panels, where flange bending causes in-plane deformation of the web the same mechanism is active, which speaks for the same name in these cases.

The problem of panel buckling in whatever mode is formulated in most general terms when allowing for both in-plane and out-of-plane deformations of every plate strip of the assembly. The general solution for the behaviour of a plate strip - assuming sinusoidal deformation longitudinally - contains 4 integrations constants for in-plane and 4 for out-of-plane deformations. At the junction of plate strips the 8 integration constants are related to those of the adjoining strip by 4 conditions of compatibility and 4 equilibrium conditions. The resulting transcendental equations require a powerful computer for numerical evaluation. This exact method has been developed and applied by W.H. Wittrick and F.W. Williams[1,2].

The aim of the present paper is to present a simpler method, apt even for numerical evaluation by means of the pocket-computer, which nevertheless yields a very good approximation of buckling stress and mode and which may be useful for parameter studies.

So as to be able to introduce reasonable schematizations and assumptions one should imagine how the mode looks like. This will be discussed in chapter 2. The derivation of the equations and their way of solution is given in chapters 3 and 4. The various types of restraint at the root of the web are discussed in chapter 5. Chapter 6 gives numerical results for some related structures together with their discussion. The Appendices supply the formulae needed for numerical application.

2. THE BUCKLING MODE AND ITS SCHEMATIZATION

The structure is assumed to be infinitely long and to be supported at regular intervals L. The compressive stress p is constant. Then the buckling mode is sinusoidal with the argument $\pi x/\ell$, where ℓ is the half-wave length $\ell = L/n$, n being integer.

The cross section of the panel shows a large number of identical stiffeners at constant pitch. The longitudinal edges of the panel are supported in some way. Then the mode has lateral curvature of the panel plate. However, the bending stiffness of the plate is very much smaller than the bending stiffness of the stiffeners, which means that the energy of lateral bending is negligibly small compared to that of longitudinal bending. Then the edge support affects the buckling stress negligibly little and the problem of the panel of finite width may be replaced by the problem of the infinitely wide plate, where all stiffeners behave equally. Then the analysis is being confined to the part of the structure comprising one single stiffener pitch.

A qualitative picture of the buckling mode is given in Fig. 1. Sideways bending of the flange over the distance V yields positive and negative strains in the flange as indicated by the symbols + and - respectively. Then the top of the web has negative strain as well. Buckling occurs under constant normal force, so the negative strain of the top has to be compensated by positive strain at the root of the web. The web appears to be curved in its plane such that the stiffeners move downward the distance -W; $V/W < 0$. Only when the neutral axis of the flange falls in the plane of the web ($y = 0$ e.g. symmetric flange) flange buckling is not coupled with web bending ($W = 0$); overall flange buckling and Euler column buckling are uncoupled.

Sideways bending V of the flange is being restrained by the web, which thereby is being bent out-of-plane, $v(z)$. Its rotation φ_2 at the top forces torsion upon the flange. The web is being restrained at its root by the lower flange of the stiffener and the panel plate. The stiffness of this restraint is of paramount importance for the restraint of the flange against the deflexion V. Fig. 1 assumes that the stiffeners deflect in the same sense. Then the mode of the plate has S-shape. The stiffness of the plate against this deformation is greater than the stiffness against a deformation where the rotations φ_1 have opposite sign at successive stiffeners (Fig. 2). In the latter case all deformations of successive stiffeners are opposite, also their vertical displacements W. When the stiffeners force the alternating deflexions W upon the plate, the plate reacts by applying restraining forces K to the stiffeners, which reduce their deflexion. So this mode comprises two effects: lesser restraint against rotation φ_1 and restraint against deflexion W not occurring in the mode of Fig. 1. Beforehand one cannot decide which one of the two modes is critical. With the numerical example, discussed in chapter 6, the case of alternating stiffener deflexions turned out to be not critical. This mode will not be considered further in the sequel.

Out-of-plane deformations occur with flange, web and plate. As to the flange its deflexion is mainly due to the rotation φ_2 of its root: $\varphi_2\, y$, and to the translation W. In addition a small contribution to its deflexion occurs due to the curvature near the root, but this deflexion appears to be negligible in comparison to $W - \varphi_2\, y$. Therefore it will be assumed that the out-of-plane deformation of the flange is just torsion. For convenience the eventually existing flange lip will be assumed to have its centre of gravity in the edge of the flange.

The out-of-plane deformations of web and plate are governed by the differential equation (for the web)

$$B\nabla^4 v + p\, t\, \frac{\partial^2 v}{\partial x^2} = 0. \tag{2.1}$$

The half-wave length of v is ℓ in the direction of x and in the direction of z it is of the order H (web height).
Therefore

$$\nabla^4 v = \frac{\partial^4 v}{\partial z^4}\left[1 + 0(\frac{H}{\ell})^2\right]^2.$$

Since $(H/\ell)^2$ is of the order 10^{-2} the plate equation may be simplified to

$$B\,\frac{\partial^4 v}{\partial z^4} + p\, t\, \frac{\partial^2 v}{\partial x^2} = 0. \tag{2.2}$$

It replaces the plate equation by a strip theory; the load pt v" is transmitted by web strips to their edges z = 0 and z = H.

The in-plane deformations are of course curvature of flange and web on account of the deflexions V and W. The author owes to W.T. Koiter the hint that the deflexion due to shear is non-negligible in stiffened panels because their shear load carrying area is only a small fraction of the total area of the cross section. Therefore the deflexion W has to be subdivided into a part W_b due to bending and a part W_s due to shear.

The normal stresses of flange and web are assumed to have linear distribution. The same assumption for the plate means constant normal stress because successive stiffeners have equal strains.

Shear stresses are proportional to the gradient of the normal stresses and follow from axial equilibrium. The shear stiffness derives from equating the elastic energy of the shear stresses and the work done by the shear load (Appendix B). The normal stress distribution depends on V/W, therefore as well the shear stresses and the shear stiffness.

Summarizing the simplifications and assumptions are:

1. Flange deflexion in the y-z-plane other than due to rotation and translation is negligibly small.
2. The flange lip is concentrated in the edge of the flange.
3. The plate equation of web and plate may be simplified to Eq. (2.2).
4. Normal stresses are linear in y and z.
5. Finite shear stiffness is taken into account, thereby deriving the shear stiffness on the basis of assumption 4.

3. DERIVATION OF THE EQUATIONS

3.1. Equilibrium of the element dx of the panel

The normal stresses originate from the deflexions W_b and V. The assumption of linear distribution means that the longitudinal displacements u are linear in y and z.

With $u = u_1$ at the root of the web ($y = z = 0$) the web has

$$u = u_1 - W_b' \, z \tag{3.1}$$

and the flange

$$u = u_1 - W_b' H - V' y. \qquad (3.2)$$

Then the condition that at buckling the normal force remains constant, therefore that the integral of the stress increments over the total cross section vanishes, yields

$$u_1' = [(1 - q) H W_b + \frac{S}{A} V]'', \qquad (3.3)$$

where $q = (A_1 + \frac{1}{2}A_w)/A$.
Hence the normal stresses of flange, web and plate are resp.

$$\left.\begin{aligned} \sigma_f &= E [-q H W_b + (\frac{S}{A} - y) V]'' \\ \sigma_w &= E [\{(1 - q) H - z\} W_b + \frac{S}{A} V]'' \\ \sigma_1 &= E [(1 - q) H W_b + \frac{S}{A} V]''. \end{aligned}\right\} \qquad (3.4)$$

The bending moment in the x-z-plane is

$$M = \int \sigma z \, d A.$$

After some manupulation M can be written as

$$M = [E I W_b + E I_s V]'', \qquad (3.5)$$

where $I_s = S H q$.

Next we consider the equilibrium of the element dx of the panel (fig. 3).

The compressive force P gives the shear load

$$D = PW'. \qquad (3.6)$$

The equilibrium condition is

$$M' + D = 0. \tag{3.7}$$

The deflexion by shear W_s follows from

$$D = K\, W_s', \tag{3.8}$$

where K is the shear stiffness.
(3.6) and (3.8) yield

$$W_s' = \frac{P}{K - P} W_b', \tag{3.9}$$

whereupon (3.5,7,8,9) give the relation between V and W_b.

$$V''' = -\frac{I}{I_s}\left(W_b''' + \frac{P/EI}{1 - P/K} W_b'\right). \tag{3.10}$$

Since

$$W_b(x) = W_b \sin \pi x/\ell, \quad V(x) = V \sin \pi x/\ell \tag{3.11}$$

and introducing the buckling coefficients

$$k = P/P_E = P/\left(\frac{\pi^2 EI}{L^2}\right) \quad \text{and} \quad k_s = K/P_E \text{ [1]} \tag{3.12}$$

(3.10) yields

$$V/W_b = -\frac{I}{I_s}\left(1 - \frac{k/n^2}{1 - k/k_s}\right) = -\frac{I}{I_s} F. \tag{3.13}$$

Because $k < 1$ (3.13) shows that $V/W_b < 0$, as has already been concluded from qualitative considerations.

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1) As appears from (3.6,8) K is the buckling load at infinite bending stiffness ($W_b = 0$).

With

$$u_1(x) = U \cos \pi x/\ell \tag{3.14}$$

(3.3) and (3.13) yield

$$U = (1 - q - \frac{i^2}{q\,H^2} F)\, \frac{\pi H}{\ell}\, W_b .$$

Also from (3.9,12,13)

$$V/W = - \frac{I}{I_s} F\, (1 - k/k_s) . \tag{3.15}$$

3.2. Equilibrium of the flange, in-plane

Fig. 4 depicts the in-plane forces and moments acting upon the element dx, where the moment M_f is taken with respect to $y = 0$, the junction between flange and web. The forces exerted by the web on the flange are R and Q per unit length. R has been accounted for implicity when considering the equilibrium of the element of the total structure. Only Q has to be established.

The conditions of equilibrium are:

$$-Q + D' = 0,$$

$$M_f' + D + p\, A_f\, V' = 0.$$

Hence

$$Q = -M_f'' - p\, A_f\, V''.$$

Using (3.4)

$$M_f = - \int_{A_f} \sigma\, y\, dA = E\, [(q\, H\, W_b - \frac{S}{A} V)\, S + I_f\, V]''$$

and

$$Q = E\,[-\,q\,HS\,W_b - (I_f - \frac{S^2}{A})\,V]'''' - p\,A_f\,V''. \tag{3.16}$$

3.3. Equilibrium of the flange, out-of-plane

The point y of the flange has the displacement in the direction of z $W - \varphi_2\,y$. This generates upon the element dA per unit length of x the load (Fig. 5)

$$-\,p\,dA\,(W - \varphi_2\,y)''$$

The flange is assumed to be completely rigid in the y-z-plane. Only the moment with respect to y = 0 has to be considered; it affects the deformation of the web. The torsional stiffness T however, is being taken into account (Fig. 5). Then the moment per unit length is

$$m_2 = [p\,S\,W + (T - p\,I_f)\,\varphi_2]'' \tag{3.17}$$

3.4. Equilibrium of the web

For webs and other plate strips plate theory is being simplified to strip theory, the equation (2.2). Fig. 6 shows the load system per unit length of x. The web is restrained at its root by the lower stringer flange and the plate. The restraining moment m_1 is formulated as

$$m_1 = \alpha\,\frac{B}{H}\,\varphi_1, \tag{3.18}$$

where α is non-dimensional. Appendix A gives the formulae for the coefficient of restraint α of riveted and bonded joint between flange and plate.

Since $v(x,z) = v(z)\sin \pi x/\ell$ the partial differential equation (2.2) becomes an ordinary one, the non-dimensional form of which is

$$\frac{d^4v}{d\zeta^4} - a^4 v = 0, \tag{3.19}$$

where $\zeta = z/H$ and

$$a = [12\ (1 - \nu^2)\ k]^{\frac{1}{4}}\ (\frac{in}{t})^{\frac{1}{2}}\ \frac{\pi H}{L}. \tag{3.20}$$

Its general solution is

$$v = C_1 \cos a\zeta + C_2 \sin a\zeta + C_3 \cosh a\zeta + C_4 \sinh a\zeta. \tag{3.21}$$

The boundary conditions are

$$\zeta = 0:\ v = 0, \tag{3.22}$$

from (3.18) $\zeta = 0$:

$$\frac{d^2v}{d\zeta^2} - \alpha \frac{dv}{d\zeta} = 0, \tag{3.23}$$

from (3.17) $\zeta = 1$:

$$\frac{d^2v}{d\zeta^2} + \frac{1}{B}\ [p\ S\ W + (T - p\ I_f)\ \frac{1}{H}\frac{dv}{d\zeta}]\ (\frac{\pi H}{\ell})^2 = 0,$$

from (3.16) $\zeta = 1$:

$$\frac{d^3v}{d\zeta^3} - \frac{E}{HB}\ \ [q\ S\ H\ W_b + (I_f - \frac{S^2}{A})\ V]\ (\frac{\pi H}{\ell})^4 + p\ A_f\ \frac{H}{B}\ (\frac{\pi H}{\ell})^2\ V = 0.$$

The latter two boundary conditions can be written, thereby using (3.13,15,20), as $\zeta = 1$:

$$\frac{d^2v}{d\zeta^2} + X\frac{dv}{d\zeta} - Y\ v = 0, \tag{3.24}$$

$$\frac{d^3v}{d\zeta^3} + a^4\ Z\ v = 0, \tag{3.25}$$

where

$$\left.\begin{aligned} X &= 12\,(1-\nu^2)\,\lambda^2\,n^2\left(\frac{T}{Et^3\,H} - k\,\lambda^2\,\frac{i^2}{t^2}\,\frac{I_f}{tH^3}\right) \quad (\lambda = \frac{\pi H}{L}) \\ Y &= 12\,(1-\nu^2)\,\lambda^4\,n^4\,\frac{S^2 e}{At^3\,H^2}\,(\frac{1}{F} - 1) \qquad (e = qH) \\ Z &= \left\{\frac{S^2}{AH^3 t}\left[1 + \frac{1}{F}\,(\frac{e}{i})^2\right] - \frac{I_f}{tH^3}\right\}\frac{H^2}{i^2}\,\frac{n^2}{k} + \frac{A_f}{tH}. \end{aligned}\right\} \quad (3.26)$$

4. SOLUTION OF THE EQUATIONS

The 4 boundary conditions are homogeneous and linear equations in v and its derivatives, therefore by (3.21) homogeneous and linear in the four integration constants C_i. The equations are

$$C_1 + C_3 = 0 \qquad (4.1)$$

and eliminating C_3

$$2a + \alpha\,\frac{C_2 + C_4}{C_1} = 0, \qquad (4.2)$$

$$\frac{C_2}{C_1}\left[X\,\frac{1}{a}\cos a - (1 + \frac{Y}{a^2})\sin a\right] + \frac{C_4}{C_1}\left[X\,\frac{1}{a}\cosh a + (1 - \frac{Y}{a^2})\sinh a\right] =$$

$$= \cos a + \cosh a + \frac{X}{a}(\sin a + \sinh a) + \frac{Y}{a^2}(\cos a - \cosh a), \qquad (4.3)$$

$$\frac{C_2}{C_1}(-\cos a + Z\,a\sin a) + \frac{C_4}{C_1}(\cosh a + Z\,a\sinh a) =$$

$$= \sinh a - \sin a + Z\,a\,(\cosh a - \cos a). \qquad (4.4)$$

With given structural dimensions the problem is to solve for the smallest root k of these transcendental equations in a (n,k). After making a guess of n and k so as to determine a and F k can be solved from these equations. Repeating this procedure for other assumed values of k the root can be established by interpolation. This is a rather tedious method. A much simpler one is the following.

Having chosen n and some values of k the corresponding values of F can be solved as indicated in Appendix B. Then a and the coefficients X, Y, Z occurring in (4.3,4) are known and these equations can be solved for C_2/C_1 and C_4/C_1. Their substitution into (4.2) yields α, the stiffness of the restraint at the root of the web at which the assumed value of k is the exact solution.

$$\alpha = - 2a\ C_1/(C_2 + C_4). \tag{4.5}$$

In this way k as function of α is being obtained from which the value of k pertaining to a specific α can be established.

5. THE RESTRAINT AT THE ROOT OF THE WEB

The magnitude of α has a dominant effect upon k over the range $\alpha < 10$, as appears from the numerical example considered in chapter 6 (Figs. 9 and 10).

With stiffeners bonded to the plate the combined thicknesses of plate and flange yield a very great bending stiffness of this part of the stiffener pitch resulting in large value of α, $\alpha > 10$. Appendix A section 1 gives the formula for α. Thereby the assumptions have been made

1. The deformation of the glue larger is neglected and in general the load diffusion at the edges of the thickened part of the plate is instantaneous.
2. The effect of p upon the stiffness of the plate is negligible.

The overestimation of α resulting from these assumptions has little effect upon k, because in this range of α $dk/d\alpha$ is small. It appears from Appendix A section 2 that neglect of the load $pt\partial^2w/\partial x^2$ affects the stiffness negligibly when stiffener pitch is small, as occurring in heavily stiffened panels, and when rotations φ_1 of successive stiffeners are in the same sense.

With riveted flange-plate joint α is in the range of $\alpha < 10$, where $dk/d\alpha$ is large. So accurate prediction of α is needed. Unfortunately the riveted joint is not well suited for accurate analysis. The following assumptions have been made (Appendix A section 2).

1. The local joints around rivets may be replaced by a contineous joint along the rivet line. This line contact is the only contact between flange and plate.
2. At the rivet line the deflexion w and slope $\partial w/\partial y$ of flange and plate are equal, thus neglecting the deformation of the rivets.
3. The effect of the compressive stress p upon the stiffness of plate and flange is negligibly, as proven in Appendix A.2.)

When $\varphi_1 > 0$ the first assumption is inadequate, it yields penetration of the edge of the flange into the plate. It means that a second line of contact will occur near the root of the web. Then assumption 2 has to be supplemented with the assumption that the deflexions w of flange and plate in $y = 0$ are equal (Fig. 7b). This condition results in a considerable increase of α in comparison to the case $\varphi_1 < 0$ (Fig. 7a), where flange and plate lose contact at the web root. With the numerical example the case $\varphi_1 < 0$ yields $\alpha_a = 2.3$ ($k = 0.59$) and the case $\varphi_1 > 0$ $\alpha_b = 6.4$ ($k = 0.69$). Formulae (A.4) and (A.9) give α_a and α_b respectively.

In multi-bay panels successive bays have opposite sign of φ_1. In spite of their unequal restraint they have a common buckling load due to unequal half-wave lengths. Bays with $\varphi_1 < 0$ have $\ell_a < L$; where $\varphi_1 > 0$ $\ell_b > L$, such that $\ell_a + \ell_b = 2L$. Numerically it means that for some values of ℓ_a and corresponding values of ℓ_b k_a and k_b pertaining to α_a

and α_b respectively have to be established so as to find the half-wave length at which $k_a = k_b$.

The restraint is weaker when successive stiffeners are buckling in the opposite sense, because then the lateral half-wave length is equal to the stiffener pitch in stead of half pitch length (Fig. 2). However, as stated in chapter 2, the opposite deflexions W of successive stiffeners are being resisted by the plate. Due to this additional restraint this mode is not critical. However, when the deflexion W is absent this mode is the critical one. Such is the situation when the flange at the top is symmetrical. This case is analyzed in Appendix A section 3, however, without taking into account the effect of edge contact where $\varphi_1 > 0$ (Fig. 7b). Due to the greater lateral wave length the effect of p upon the stiffness cannot be neglected. α is given by (A.11).

6. NUMERICAL EXAMPLE AND DISCUSSION OF RESULTS

The numerical application refers to the panel of Fig. 8. This structure is representative of heavily loaded panels: the cross-sectional area of the stiffeners is about equal to that of the plate; the plate thickness is greater than that of the stiffeners because the width of the plate strip between successive stiffeners is greater than the widths of the stiffener strips and high local buckling stress is to be achieved.

Fig. 8 also shows its model as used in the analysis, the dimensions of which are:

$H = 38.08$, $b = 18.08$, $b_1 = 27.04$, $2s = 70$, $t = 1.92$, $t_1 = 2.4$,
$c = 0.3894$, $L = 540$.

It yields $i = 14.80$ ($L/i = 36.5$). Hence $a = 1.118057\ k^{\frac{1}{4}}\ n^{\frac{1}{2}}$,
$a_1 = 0.91914\ k^{\frac{1}{4}}\ n^{\frac{1}{2}}$.

Appendix A gives the coefficients of restraint of the web for bonded joint $\alpha = 30.3$;
for riveted joint (Z-stiffener) $\alpha_a = 2.31$ (Fig. 7a, $\varphi_1 < 0$) or
$\alpha_b = 6.39$ (Fig. 7b, $\varphi_1 > 0$);

for riveted joint (symmetrical flange) $\alpha_a = 1.50$.

Three flange configurations with equal cross-sectional area have been analyzed.

I Lipped flange. The cross-section of the lip is c t b.

II Unlipped flange. Its thickness being (1 + c) t.

III Symmetrical flange, again with thickness (1 + c) t.

Appendix C expresses the structural coefficients X, Y, Z occurring in the Eqs (4.3,4) in the dimensions of the panels and gives their numerical values.

The behaviour of panel III is least complex. Here overall Euler buckling and "overall flange buckling" are uncoupled.

With Euler buckling $k_s = 7.60$ and 9.82 for riveted and bonded joint respectively. Then the buckling coefficient derives from

$$\frac{1}{k} = \frac{1}{k_E} + \frac{1}{k_s}.$$

With $n = 1$, $k_E = 1$. Then $k = 0.884$ and 0.908 respectively for riveted and bonded joint. So shear distinctly reduces the buckling stress particularly in this range of lower slenderness.

With panel III overall flange buckling is not accompanied by deflexion W of the stiffener. Neither does it involve stiffener shear. The mode is just column buckling of the flange on elastic foundation supplied by the web. Due to the kind of joint between flange and web bending of the flange entails torsion. The stiffness of the elastic foundation is strongly affected by the stiffness of restraint at the web root (α). It is well-known that the critical wave length of columns on elastic foundation decreases with increasing foundation stiffness. Fig. 9 gives k as function of α for the wave numbers n = 1, 2, 3. Only when α is close to zero the mode n = 1 is critical. Over the range of α of practical interest n = 2 is critical; with $\alpha = 1.50$, $k = 0.565$.

For two values of α, a large and a small one, Fig. 11 depicts the mode. When α is small the slope φ_1 at the web root gives the major contribution to the deflexion V of the flange, whereas with very stiff restraint this contribution is almost negligible and the deformation of the web is dominant. This deformation is a measure of the elastic support offered to the flange and consequently of the buckling load.

The numbers in rectangular frame are representative for the curvature or bending moment at root and top of the web. The difference with non-symmetric flanges is negative curvature at the top. The reason is that the symmetric flange has much smaller I_f than the flanges of panels I and II. Moreover its torsional stiffness is greater than that of panel I. Hence $T - p\, I_f > 0$, which stabilizes the flange against torsion; the web forces torsion upon the flange as appears from its negative curvature.

The panels I and II with their asymmetric flanges have coupled deflexions V and W. When the elastic foundation of the flange, offered by the web and its restraint at the root, is very stiff, this stiffness keeps the sideways deflexion V down. Therefore with increasing α $|V/W|$ decreases. This conclusion is confirmed by the modes shown in Fig. 11 and by Table 1. The curvature at the top of the web is positive due to the large I_f and because $p\, t\, (d^2W/dx^2)$ contributes to the bending moment of the web. The smaller curvature with panel II is due to its smaller I_f and greater torsional stiffness T.

Figs 9 and 10 and Table 1 give k as function of α, also when the effect of shear is being neglected. With $\alpha > 10$ the effect of restraint approaches the effect of infinite rigidity. It concurs with the small value of φ_1 in Fig. 11. Over the range $\alpha < 10$ α appears to have the dominant effect on k. Since with decreasing α V dominates more and more over W, the effect of shear upon k must decrease. In the range of $\alpha > 10$ k is being overestimated some 10% when neglecting the effect of shear.

The large difference of α between riveted and bonded stiffeners yields much difference of k. Considerable gain of strength is being achieved when replacing the riveted by a bonded joint. Bonded ($\alpha = 30$) has with panel I k = 0.750 and with panel II k = 0.804. Riveted without "penetration" ($\alpha = 2.3$) has with the panels I and II respectively k = 0.59 and 0.66; with prevented penetration (α = 6.4) k = 0.69 and 0.75 respectively.

The better buckling strength of panel II is due to the smaller eccentricity of the flange with respect to the web and its greater torsional stiffness. With riveted joint k appears to depend very much on the more or less doubtful assumptions on the behaviour of the joint.

With these asymmetric stiffeners coupling of stiffener bending W with flange sideways bending V is so strong that the mode n = 2 never becomes critical; the component W of the overall mode dictates the wave length.

This method can be checked with some available exact or nearly exact solutions. Using finite strip method J.H. van der Sloot (Fokker Aircraft Company) got the results shown in the table. Between brackets are the values of k read from Figs 9 and 10 at the indicated values of α, which are usually less than 5% greater. These rather small differences of k may be partly attributed to small differences of dimensions of the two models.

	I	II
bonded joint	α = 12.7 k = 0.71 (0.73)	α = 15.5 k = 0.75 (0.79)
riveted joint	α = 2.7 k = 0.59 (0.61)	α = 4.4 k = 0.66 (0.715)

However, one case enables comparison of results referring to identical models and assumptions. W.H. Wittrick has kindly performed a check with his exact method[1,2] and applying the programme VIPASA. The model is the case of unlipped flange and stiffener bonded to plate. For n = 1 the exact solution is k_{ex} = 0.797 whereas the present method yields k = 0.804. For n = 2, k_{ex} = 0.947 and the present method gives k = 0.940. Also the comparison of the modes in these two cases is excellent (Table 2).

Overestimation and underestimation both are possible. Underestimation because $\Delta^4 w$ is being approximated by $\partial^4 w/\partial y^4$ (more important with $n = 2$); overestimation by neglecting the effect of p on the stiffness of the plate and bending of the flange in the plane of the cross-section (both of little importance).

ACKNOWLEDGEMENT

The author gratefully acknowledges the valuable contribution to this investigation by Dr. W.H. Wittrick, who offered kindly to apply his exact method of solution to a model with identical characteristics and thus enabled a final check of the accuracy of the present approximative solution.

Table 1

Lipped flange (model I)

n		k	k_s	- V/W	α
1	shear neglected	0.81		0.889	28.14
		0.75		1.170	6.52
		0.65		1.637	2.63
		0.55		2.105	1.446
		0.4		2.807	0.607
		0.25		3.509	0.132
	bonded flange to plate	0.76	9.19	0.736	73.3
		0.75	9.20	0.788	26.4
		0.71	9.21	0.996	7.36
		0.59	9.09	1.614	2.12
		0.4	7.77	2.566	0.644
		0.25	4.62	3.256	0.143
	riveted flange to plate	0.71	7.07	0.887	10.06
		0.59	6.85	1.515	2.26
		0.4	5.76	2.482	0.708
		0.25	3.58	3.182	0.147
2	riveted flange to plate	0.9	3.97	2.565	12.25
		0.85	3.70	2.609	4.87
		0.75	3.15	2.687	1.164
		0.7	2.88	2.722	0.412
		0.68	2.58	2.649	0.187

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2. Williams, F.W. and Wittrick, W.H.: "Computational Procedures for a Matrix Analysis of the Stability and Vibration of Thin Flat-Walled Structures in Compression", International Journal of Mechanical Sciences, Vol. 11, pp. 979 - 998, 1969.

Table 2

Comparison with exact mode (model II)

n = 1, k = 0.797 (0.804)					
		$x = \frac{1}{2}\ell$			x = 0
y	z	φ	w	v	u
$-b_1$	0	-0.0020 (0.0008)	-3.9351 (-3.74)	0.0447 (0)	-0.1647 (-0.174)
0	0	0.00404 (0.0055)	-3.9412 (-3.82)	0.0422 (0)	-0.1751 (-0.174)
0	½H	0.07112 (0.0729)	-3.9394 (-3.82)	0.8243 (0.813)	0.2257 (0.214)
0	H	0.09830 (0.1)	-3.9252 (-3.82)	2.4962 (2.520)	0.6328 (0.601)
b	H	0.10000 (0.1)	-5.7220 (-5.63)	2.5121 (2.520)	0.3628 (0.336)
n = 2, k = 0.947 (0.940)					
$-b_1$	0	-0.00136 (0.0010)	-0.6797 (-0.633)	0.0160 (0)	-0.0250 (-0.028)
0	0	0.00512 (0.0067)	-0.7094 (-0.725)	0.0153 (0)	-0.0319 (-0.028)
0	½H	0.07663 (0.0781)	-0.7065 (-0.725)	0.8871 (0.897)	0.1030 (0.108)
0	H	0.09901 (0.1)	-0.6955 (-0.725)	2.6169 (2.653)	0.2496 (0.244)
b	H	0.10000 (0.1)	-2.4958 (-2.533)	2.6156 (2.653)	-0.2949 (-0.314)

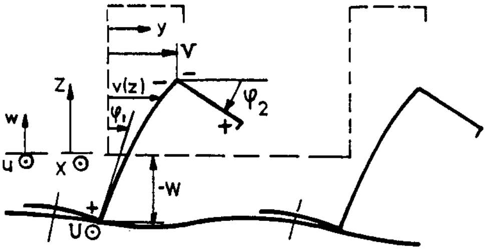


Fig. 1. Overall flange buckling.

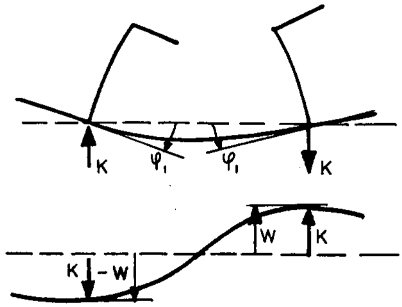


Fig. 2. Plate deformation with alternate stiffener deflexion.

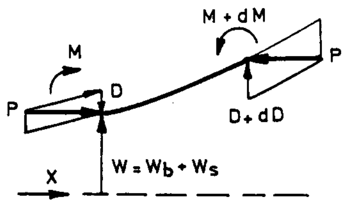


Fig. 3. Equilibrium of the element dx.

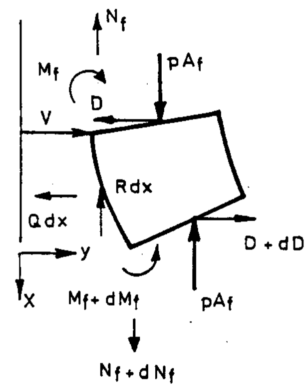


Fig. 4. Flange, in-plane equilibrium.

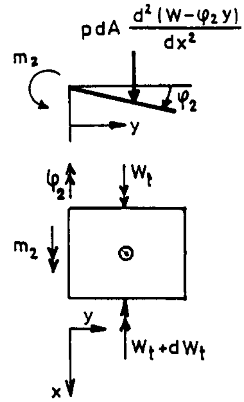


Fig. 5. Flange, out-of-plane equilibrium.

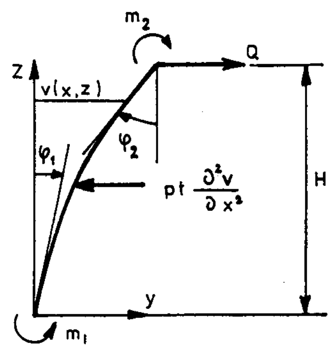
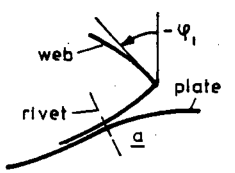
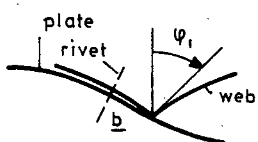


Fig. 6. Web, out-of-plane equilibrium.



a. Equality of slope at rivet line



b. As a plus equality of deflexion at root of web.

Fig. 7. Riveted joint.

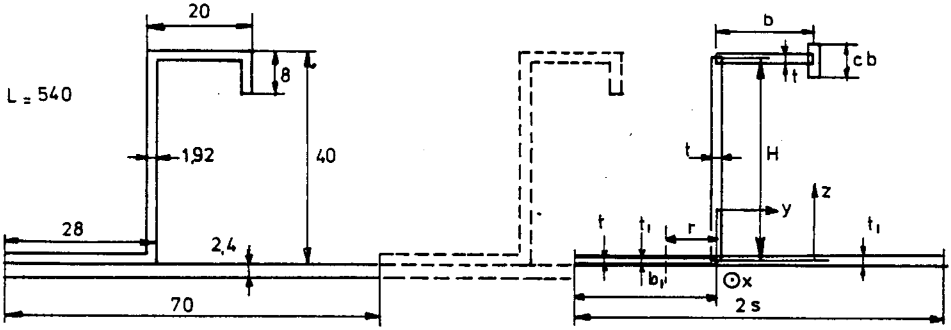


Fig. 8. Basic panel and its model.

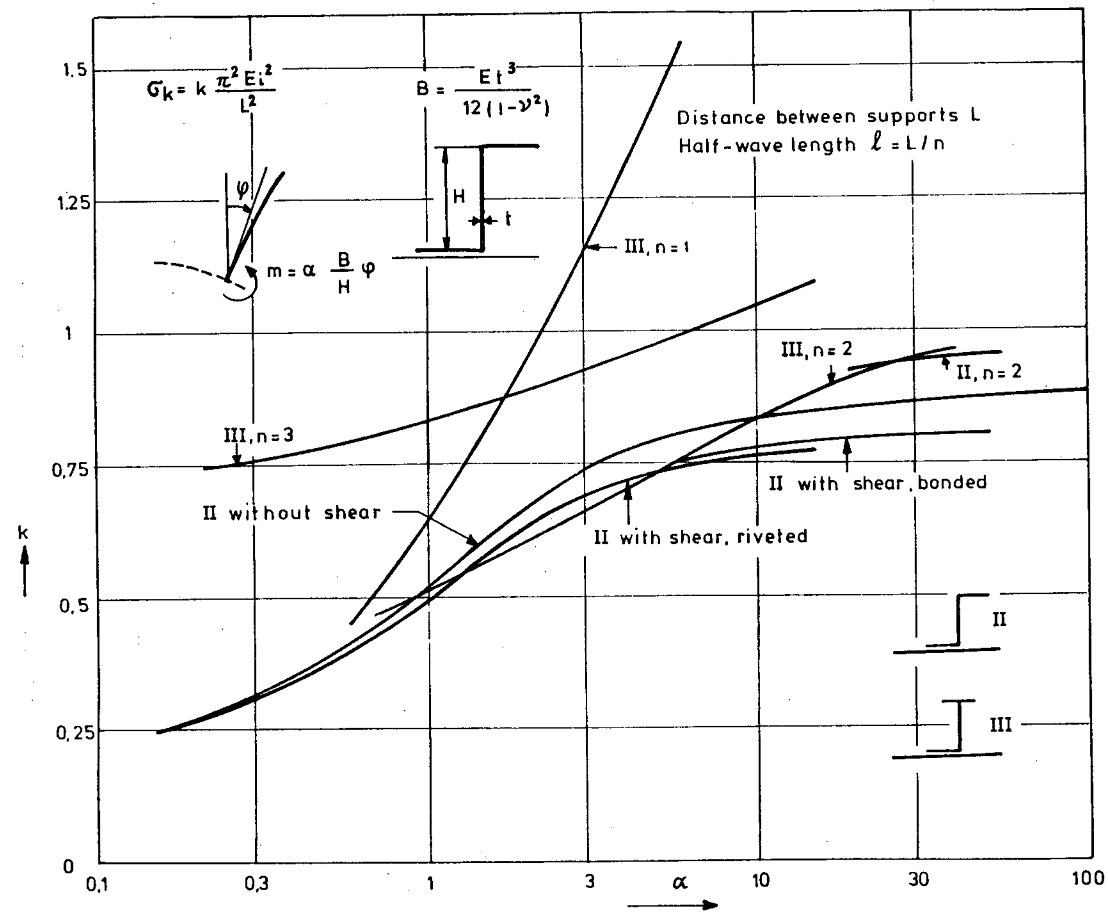


Fig. 9. Models II and III; k as function of α and n.

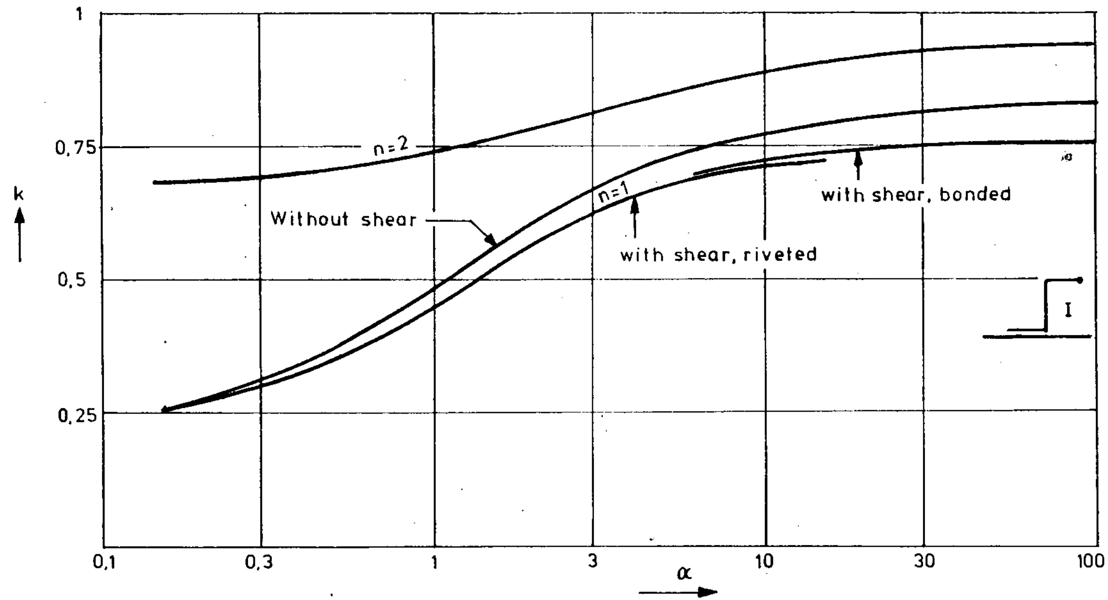


Fig. 10. Model I; k as function of α and n.

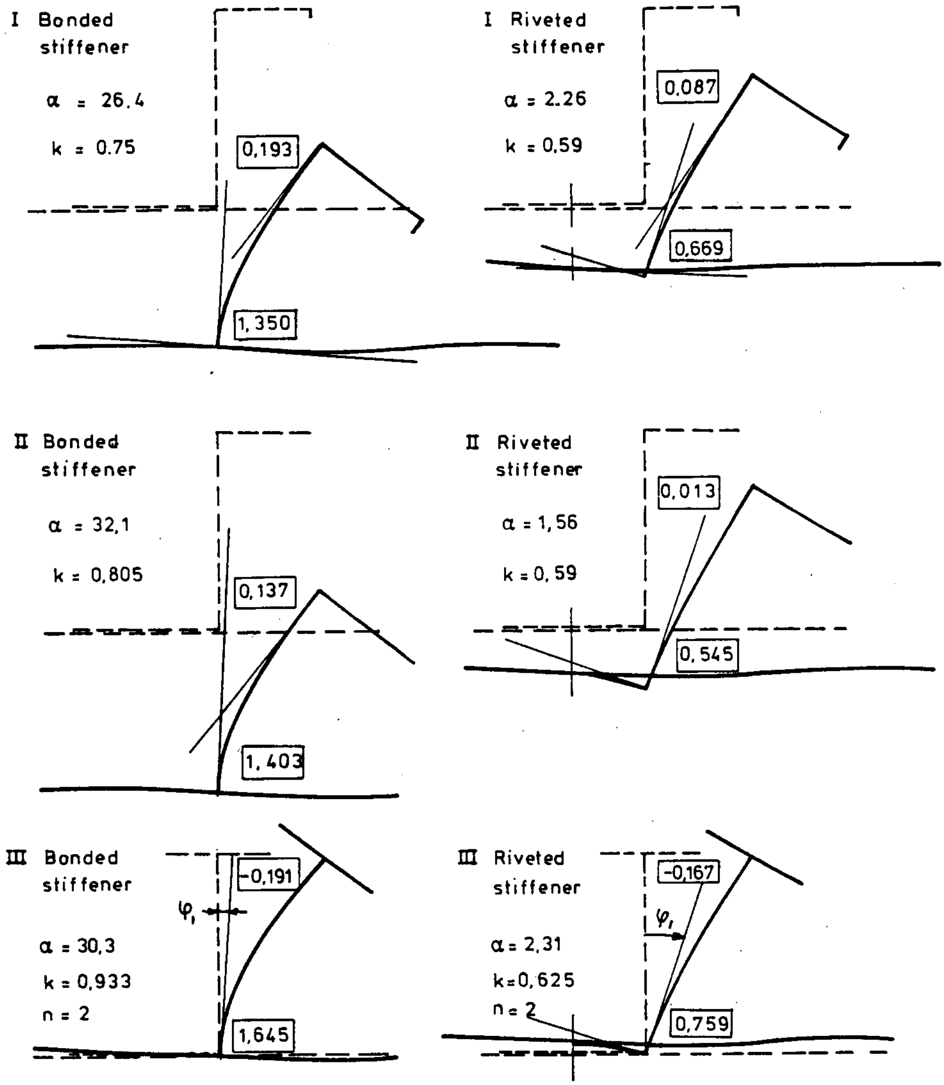


Fig. 11. Overall flange buckling modes.

APPENDIX A: THE RESTRAINT AT THE ROOT OF THE WEB

1. Bonded stiffeners

Stiffener flange and plate are integral, causing a very great stiffness of this part of the stiffener pitch. It will be assumed that deformation of the glue layer and load diffusion effects at the edges of the flange are negligible. These assumptions overestimate the stiffness. However, in this range of α $dk/d\alpha$ is so small that overestimation affects k negligibly.

When all stiffeners deflect in the same direction the plate shows an S-curve between two successive stiffeners (Fig. A1). The bending stiffness per unit length of the plate is B_1 and of the thicker part B_0. The stiffness ratio $\beta = B_1/B_0 = (1 + t/t_1)^{-3}$.

As will be shown in section 2 the effect of the compressive stress p on the stiffness of the plate is negligible. So the load $p\partial^2 w/\partial x^2$ may be disregarded. Then equilibrium of the plate strip 2s requires (Fig. A1)

$$m_0 = m - 2\ Ks;$$

the restraining moment against rotation φ_1 at the root of the web is per unit length

$$m_1 = m - m_0 = 2\ Ks$$

and the stiffness can be established by means of the elementary "beam formulae", yielding for the coefficient of restraint α, defined by

$$\alpha = \frac{H}{B}\frac{m_1}{\varphi_1}, \tag{A.1}$$

$$\alpha = 6\,\frac{H}{s}\left(\frac{t_1}{t}\right)^3 (\beta + \gamma)\,(1 + \gamma)^3 \left[\beta\,(1 + \gamma)^4 + (1 - \beta)\,(\gamma^4 - \beta)\right]^{-1}, \tag{A.2}$$

where

$$\gamma = (2s - b_1)/b_1.$$

With the dimensions of the numerical example (Fig. 8) $\beta = 0.1715$, $\gamma = 1.589$, $\alpha = 30.3$.

2. Riveted stiffeners

2.1. Anti-symmetric plate deflexion

The joint between stiffener and plate is thought to be a line-joint in the center of the flange. Since successive stiffeners have equal rotation φ_1 and deflexion W the plate deformation is anti-symmetric with respect to the point half-way between successive rivet rows (Fig. A2).

It is being assumed that the deflexions and slopes of flange and plate at their junction are equal.

The moment m_1 applied by the plate is transmitted to the stiffener somewhere near the center of the flange and has to pass through the flange over the distance r before reaching the web. So the flexibility of the flange reduces the stiffness of restraint.

The rotation of the rivet line ψ_1 can be deduced from (A.2) when putting $\beta = 1$, which yields the stiffness of the plate

$$m_1/\psi_1 = 6\ B_1/s. \qquad \text{(A.3)}$$

The additional rotation due to flange bending is

$$\psi_2 = m_1\ r/B.$$

Then $\varphi_1 = \psi_1 + \psi_2 = m_1\ (\frac{s}{6B_1} + \frac{r}{B})$,

$$\alpha = 6\,\frac{H}{s}\left(\frac{t_1}{t}\right)^3\left(1 + \frac{6\,r\,t_1^{\,3}}{s\,t^3}\right)^{-1}. \tag{A.4}$$

With the dimensions of the numerical example and assuming $r = \frac{1}{2}b_1$

$$\frac{6\,r\,t_1^{\,3}}{s\,t^3} = 4.53,$$

which means that at the assumed geometry the contribution of the flange to the flexibility of the restraint is 4.5 times the contribution of the plate. Instead of the stiffness 6 B_1/s it is 1.086 B_1/s, $\alpha = 2.31$. The difference in stiffness with bonded stiffeners is enormous: it is only 1/13-th of the latter one.

As stated before it has been assumed that the compressive stress affects the stiffness negligibly. This will be verified.

Similar to Eqs (3.20,21) and accounting for anti-symmetry with respect to the center $y = 0$ of the plate

$$w = C_5 \sin a_1\,\xi + C_6 \sinh a_1\,\xi,$$

where

$$a_1 = [12\,(1 - \nu^2)\,k]^{\frac{1}{4}}\left(\frac{in}{t_1}\right)^{\frac{1}{2}}\frac{\pi s}{L}. \qquad \xi = y/s \tag{A.5}$$

The boundary conditions are

$$y = s: \quad w = 0, \quad \partial w/\partial y = \psi_1. \tag{A.6,7}$$

Then the stiffness of restraint offered to the stiffener by the two adjoining plate strips is

$$4\,B_1/s\;a_1\,(\mathrm{cotgh}\,a_1 - \mathrm{cotg}\,a_1)^{-1}.$$

Without p the stiffness is given by (A.3). Therefore p causes a reduction of stiffness

$$\eta = \frac{2}{3} a_1 (\text{cotgh } a_1 - \text{cotg } a_1)^{-1}. \qquad (A.8)$$

In the numerical example $a_1 = 0.92\, k^{\frac{1}{4}}\, n^{\frac{1}{2}}$. With n = 1, k = 0.8 $\eta = 0.996$.
η decreases with increasing a_1, therefore with increasing n and s. The table illustrates their effect.

n	k	s	2s/H	a_1	η
1	0.8	35	1.84	0.87	0.996
2	0.8	35	1.84	1.23	0.985
1	0.8	50	2.6	1.23	0.985
2	0.8	50	2.6	1.74	0.940

In view of the much greater importance of flange flexibility η can be taken to be unity.

The assumption of equal deflexions and slopes of plate and stiffener flange at the rivet line has the consequence that the edge of the flange where it meets the web penetrates into the plate when the rotation $\varphi_1 > 0$. In a multi-bay panel φ_1 of successive bays has opposite sign. Therefore the assumption applies to half of the bays, those where $\varphi_1 < 0$. Those bays where $\varphi_1 > 0$ have to be reconsidered. Introducing the further assumption that flange and plate have equal deflexions in y = 0 (edge of the flange, Fig. 7b)

$$\alpha = 3\, H/r\, [4\, (\kappa + 1) + \mu\, (4\kappa + 1)] \times$$
$$\times \left[3\, (\kappa + 1) + \mu\, (\kappa^2 + 5\kappa + 1) + \frac{\mu^2 \kappa^3}{\kappa + 1}\right]^{-1}, \qquad (A.9)$$

where $\kappa = 2s/r-1$, $\mu = B/B_1$.

With the numerical example $\kappa = 4.18$, $\mu = 0.512$ and $\alpha = 6.39$.

2.2. Symmetric plate deflexion

Stiffness of restraint is smaller with symmetric than with anti-symmetric plate deflexion. Successive stiffeners are now buckling in opposite sense. Taking into account the effect of the compressive stress p the symmetric solution of Eq. (3.19) is

$$w = C_7 \cos a_1 \xi + C_8 \cosh a_1 \xi .$$

The boundary conditions are again (A.6,7) yielding the stiffness of restraint offered by the two plate strips

$$4 B_1/s \; a_1 \; (\mathrm{tgh}\; a_1 + \mathrm{tg}\; a_1)^{-1} .$$

Without p the stiffness is $2 B_1/s$. Therefore p causes a reduction of stiffness

$$\eta = 2 a_1 \; (\mathrm{tgh}\; a_1 + \mathrm{tg}\; a_1)^{-1} . \qquad \text{(A.10)}$$

This symmetric plate deflexion applies only to the case III of symmetric stiffener flange, where the critical mode has $n = 2$. With the numerical example $n = 2$, $k \simeq 0.6$ follows from (A.5) $a_1 = 1.144$, $\eta = 0.759$. So the effect of p may not be disregarded.

Bending of the flange adds $\psi_2 = m_1 \, r/B$ to $\psi_1 = m_1 \, s/(2 \, \eta \, B_1)$;

$$\alpha = 2 \, \eta \, \frac{H}{s} \left(\frac{t_1}{t}\right)^3 \left(1 + 2 \, \eta \, \frac{r}{s} \frac{t_1^3}{t^3}\right)^{-1} . \qquad \text{(A.11)}$$

With the numerical example $\alpha = 1.50$.

APPENDIX B: THE SHEAR STIFFNESS

The shear stiffness will be established by equating the elastic energy of shear stresses and the work done by the shear load. Per unit length of x:

$$A_e = \int \frac{\tau^2}{2G} dA = A_D = \frac{D^2}{2P_s} = \frac{D^2}{2k_s P_E} \tag{B.1}$$

where P_s is the shear stiffness and as well the critical load in the absence of bending.

With thin-walled structures the shear stresses due to shear load can be considered to be constant through the wall-thickness. Then τ follows from the equilibrium condition (Fig. B1)

$$(\tau t)_s - (\tau t)_{s0} = - \int_{s_0}^{s} \frac{\partial \sigma}{\partial x} t \, ds. \tag{B.2}$$

In this problem σ depends on the two curvatures W_b'', V'' (Eqs 3.4), whereas V and W_b are related by (3.13) such that V/W_b, therefore σ and τ and consequently P_s depend on the buckling coefficient k.

With lipped and unlipped flange the shear flows in the web are equal:

$$\frac{(\tau t)_w}{E} = - H W_b''' [q A_f + (q - \tfrac{1}{2}) A_w + (1 - q) A_w \zeta - \tfrac{1}{2} A_w \zeta^2] + - \frac{S}{A} V''' (A_1 + A_w \zeta). \tag{B.3}$$

$$\frac{D}{E} = H \int_0^1 \frac{(\tau t)_w}{E} d\zeta = - H^2 W_b''' [q (1 - q) A - \frac{1}{6} A_w] - S q H V''' =$$

$$= - I W_b''' - I_s V'''$$

and using (3.13)

$$D = - EI W_b''' (1 - F). \tag{B.4}$$

Expressing k_s into F by means of the definition (3.13) of F

$$\frac{1}{k_s} = \frac{1}{k} - \frac{1}{n^2\,(1 - F)} \tag{B.5}$$

the work done by D is

$$A_D = \tfrac{1}{2}\,\frac{(EI\;W_b''')^2}{P_E}\left[\frac{(1 - F)^2}{k} - \frac{1 - F}{n^2}\right] =$$

$$= K\,\frac{1 + c}{2\,(1 + \nu)}\,f\,\frac{b}{H}\left(\frac{iL}{\pi b^2}\right)^2\left[\frac{(1 - F)^2}{k} - \frac{1 - F}{n^2}\right], \tag{B.6}$$

where

$$K = \frac{E^2 H}{2Gt}\,t^2\,b^4\,(W_b''')^2.$$

The formulae needed for numerical calculation of A_e are the following.

For the case of lipped flange

$$(\tau t)_f/E = tb^2\,W_b'''\,(T_0 + T_1\,\xi + T_2\,\xi^2)_f. \qquad \xi = y/b \tag{B.7}$$

$$\left.\begin{aligned} T_0 &= e/b\,[-1 + (i/e)^2\,(f - 1)\,F]\,(1 + c) \\ T_1 &= e/b\,[1 + (i/e)^2\,F], \quad T_2 = -e/b\,(i/e)^2\,f\,\frac{1 + c}{1 + 2c}\,F \end{aligned}\right\} \tag{B.8}$$

With the unlipped flange, but such flange thickness t_2 that $t_2 b = (1 + c)\,tb$

$$T_0 = (T_0)_{lipped}, \quad T_1 = (1 + c)\,(T_1)_{lipped},$$
$$T_2 = (1 + 2c)\,(T_2)_{lipped}. \tag{B.8a}$$

Lipped flange:

$$Ae_f = K\,b/H\,g(T), \quad g(T) = T_0\,(T_0 + T_1 + \tfrac{2}{3}\,T_2) +$$

$$+ T_1 \left(\frac{1}{3} T_1 + \frac{1}{2} T_2\right) + \frac{1}{5} T_2^{\,2}. \tag{B.9}$$

Unlipped flange:

$$Ae_f = \frac{1}{1 + c} K\ b/H\ g(T).$$

Web (lipped and unlipped):

$$(\tau t)_w/E = tb^2\ W_b'''\ (T_0 + T_1\ \zeta + T_2\ \zeta^2)_w, \tag{B.10}$$

$$T_0 = \frac{e}{b} \frac{A_1}{bt} \left[- \frac{1 - q}{q} + (i/e)^2 F\right], \quad T_1 = \frac{A_w}{A_1} T_0, \quad T_2 = \frac{1}{2} \frac{H}{b} \frac{A_w}{bt}, \tag{B.11}$$

$$Ae_w = K\ g(T). \tag{B.12}$$

Bonded joint (approximate):

$$Ae_1 = \frac{K}{3} \left(\frac{se}{b^2}\right)^2 \frac{A_1}{A_w} \left(- \frac{1 - q}{q} + (i/e)^2\ F\right)^2. \tag{B.13}$$

Riveted joint:

$$Ae_1 = K \frac{s}{H} \left(\frac{e}{b}\right)^2 \left[\left(\frac{r}{s} + \frac{1}{6} \frac{t}{t_1}\right) \left(\frac{A_1}{bt}\right)^2 - \left(\frac{r}{s} + \frac{2}{3} \frac{t}{t_1}\right) \frac{A_1}{bt} \frac{r}{b} + \right.$$
$$\left. + \frac{2}{3} \left(\frac{r}{s} + \frac{t}{t_1}\right) \left(\frac{r}{b}\right)^2\right] \left(- \frac{1 - q}{q} + (i/e)^2\ F\right)^2. \tag{B.14}$$

A_D is a function of n, k and F, whereas A_e is function of F only. A_D and A_e both are quadratic in F. Having chosen a value of n and of k F can be solved from the quadratic equation

$$A_e = Ae_f + Ae_w + Ae_1 = A_D. \tag{B.15}$$

Next the shear buckling coefficient k_s follows from (B.5).

APPENDIX C: THE STRUCTURAL COEFFICIENTS X, Y, Z

The structural coefficients X, Y, Z occurring in (3.24,25) and defined by (3.26) are expressed in the structural dimensions of the three types of stiffener: Z-stiffener with lipped flange (I), Z-stiffener with thickened unlipped flange (II) and I-stiffener with thickened symmetric flange.

Lipped flange

$$X = 2\,(1 - \nu)\,\frac{b}{H}\,\lambda^2\,n^2\left[1 + c - 6\,(1 + \nu)\,\left(\frac{ib}{tH}\right)^2\left(\frac{1}{3} + c\right)\lambda^2\,k\right]$$

$$Y = 12\,(1 - \nu^2)\,\frac{(\frac{1}{2} + c)^2}{1 + c}\,\frac{1}{f}\,\frac{b^3\,e}{t^2H^2}\,\lambda^4\,n^4\left(\frac{1}{F} - 1\right)$$

$$Z = \frac{b^3}{Hi^2}\left\{-\frac{1}{3} - c + \frac{(\frac{1}{2} + c)^2}{1 + c}\,\frac{1}{f}\left[1 + \frac{1}{F}\left(\frac{e}{i}\right)^2\right]\right\}\frac{n^2}{k} + \frac{b}{H}\,(1 + c)$$

Unlipped flange

$$X = 2\,(1 - \nu)\,\frac{b}{H}\,(1 + c)\,\lambda^2\,n^2\left[(1 + c)^2 - 2\,(1 + \nu)\,\left(\frac{ib}{tH}\right)^2\lambda^2\,k\right]$$

$$Y = 3\,(1 - \nu^2)\,(1 + c)\,\frac{1}{f}\,\frac{b^3\,e}{t^2H^2}\,\lambda^4\,n^4\left(\frac{1}{F} - 1\right)$$

$$Z = \frac{b^3}{Hi^2}\,(1 + c)\left\{-\frac{1}{3} + \frac{1}{4f}\left[1 + \frac{1}{F}\left(\frac{e}{i}\right)^2\right]\right\}\frac{n^2}{k} + \frac{b}{H}\,(1 + c)$$

Symmetric flange

$$X = 2\,(1 - \nu)\,\frac{b}{H}\,(1 + c)\,\lambda^2\,n^2\left[(1 + c)^2 - \frac{1}{2}\,(1 + \nu)\,\left(\frac{ib}{tH}\right)^2\lambda^2\,k\right]$$

$$Y = 0$$

$$Z = -\frac{1}{12}\,\frac{b^3}{Hi^2}\,(1 + c)\,\frac{n^2}{k} + \frac{b}{H}\,(1 + c)$$

With the numerical example where $q = 0.75155$, $f = 7.0756$:

Lipped flange:

$$X = (0.045328 - 0.120872\ k)\ n^2,$$

$$Y = 0.066970\ n^4\ (\frac{1}{F} - 1),$$

$$Z = (-\ 0.45518 + 0.21328\ \frac{1}{F})\ \frac{n^2}{k} + 0.65966.$$

Unlipped flange:

$$X = (0.087500 - 0.077457\ k)\ n^2,$$

$$Y = 0.040859\ n^4\ (\frac{1}{F} - 1),$$

$$Z = (-\ 0.29344 + 0.130125\ \frac{1}{F})\ \frac{n^2}{k} + 0.65966$$

Symmetric flange:

$$X = (0.087500 - 0.019365\ k)\ n^2,$$

$$Y = 0$$

$$Z = -\ 0.082058\ \frac{n^2}{k} + 0.65966.$$

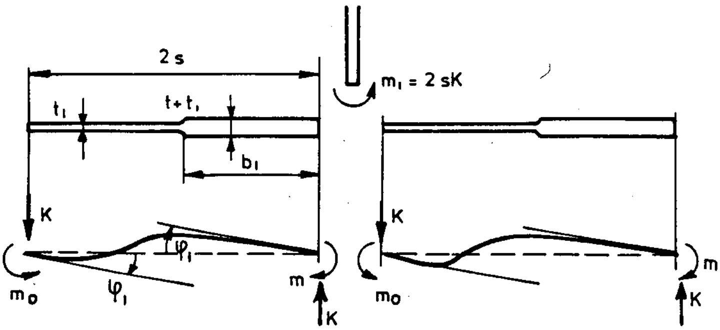


Fig. A1. Bonded joint.

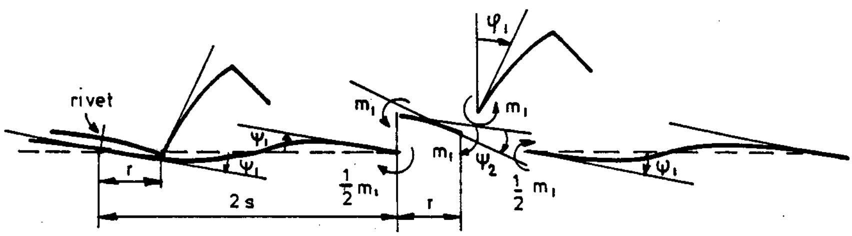


Fig. A2. Riveted joint, anti-symmetric mode.

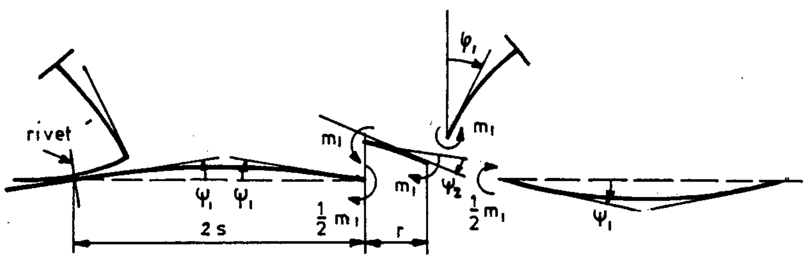


Fig. A3. Riveted joint, symmetric mode.

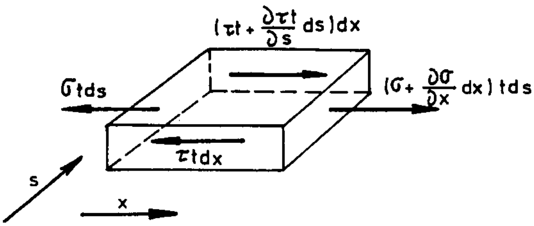


Fig. B1. $\frac{\partial \tau t}{\partial s} + \frac{\partial \sigma}{\partial x} t = 0.$