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Caspers, Martijn

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HARMONIC ANALYSIS AND BMO-SPACES OF FREE ARAKI-WOODS FACTORS

MARTIJN CASPERS

Abstract. We consider semi-group BMO-spaces associated with arbitrary von Neumann algebras and prove interpolation theorems. This extends results by Junge-Mei for the tracial case. We give examples of multipliers on free Araki-Woods algebras and in particular we find $L_\infty \to$ BMO multipliers. We also provide L_p-bounds for a natural generalization of the Hilbert transform.

1. Introduction

Recall that the BMO-norm of a classical integrable function $f : \mathbb{R}^n \to \mathbb{C}$ is defined as

$$\|f\|_{\rm BMO} = \sup_{Q \in \mathcal{Q}} \frac{1}{|Q|} \int_Q |f(s) - \fint_Q f|^2 ds,$$

where $\fint_Q f$ is the average of f over Q and $\mathcal{Q}$ is the set of all cubes in $\mathbb{R}^n$. The importance of the BMO-norm and BMO-spaces lies in the fact that they arise as end-point estimates/spaces for the bounds of linear maps on function spaces on $\mathbb{R}^n$. This includes many singular integral operators, Calderón-Zygmund operators and Fourier multipliers. BMO-spaces are by Fefferman-Stein duality [FeSt72] dual to Hardy spaces and provide optimal bounds for the Hilbert transform. By interpolation BMO-spaces form an effective tool to obtain L_p-bounds of multipliers.

BMO-spaces can also be studied through semi-groups. Consider for example the heat semi-group $\mathcal{S} := (\Phi_t)_{t \geq 0} := (e^{-t\Delta})_{t\geq 0}$ with Laplacian Δ acting on $L_\infty(\mathbb{R}^n)$. Then alternatively the BMO-norm may be realized through an equivalent (semi-)norm

$$\|f\|_{\mathrm{bmo}_{\mathcal{S}}} = \sup_{t \geq 0} \| |\Phi_t(f)|^2 - \Phi_t(|f|^2) \|^{\frac{1}{2}}.$$

BMO-spaces associated with more general semi-groups were first studied in [StVa74], [Var85] and much more recently in [XuYa05a], [XuYa05b]. See also [Gra08], [Gra09]. These concern semi-groups on measure spaces, which from our viewpoint is the commutative situation.

The development and exploration of structural properties of C*-algebras and von Neumannn algebras led to the demand of a thorough development of harmonic analysis on non-commutative spaces. After the founding work by Eymard defining the Fourier algebra of a group [Eym64], the study of its L_∞-multipliers turned out to have tremendous impact on the structure of operator algebras (see e.g. [BrOz08]). In recent years also the L_p-theory was pursued. Under suitable Hörmander-Mikhlin type conditions several multiplier theorems were established for group von Neumann algebras [JMP14], [CPPR15], [GJP17a] and vector valued harmonic analysis [Cad17], [Par09]. On quantum spaces several surprising multiplier theorems have been achieved [CXY13],

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[Ric16]. See also [XXX16], [GJP17b]. These results naturally raise questions about end-point estimates and optimal bounds for multipliers.

Parallel to this development semi-groups on non-commutative measure spaces have played a more and more important role in recent years. They lead to strong applications in non-commutative potential theory and quantum probability, see e.g. [CiSa03], [CFK14]. Semi-groups naturally appear in approximation properties of von Neumann algebras [JoMa04], [CaSk15]. Also the approach by Ozawa-Popa [OzPo10] and Peterson [Pet09] yields new deformation-rigidity properties of von Neumann algebras through the theory of semi-groups and derivations (see also [Avs11]).

In [JuMe12] Junge and Mei pursued the theory of non-commutative semi-group BMO-spaces associated with non-commutative measure spaces. They introduce several notions of BMO starting from a Markov semi-group on a tracial von Neumann algebra. Relations between these spaces are studied and interpolation results are obtained. A crucial ingredient of their approach is formed by Markov dilations of semi-groups that allows one to 'intertwine' semi-group BMO-spaces with BMO-spaces associated with martingales and derive results from this probabilistic martingale setting.

The first aim of this paper is the study of BMO-spaces associated with an arbitrary σ-finite von Neumann algebra. We take the natural definition using a faithful normal state which is not necessarily tracial anymore as a starting point. We extend interpolation results from [JuMe12, Theorem 5.2] to the arbitrary setting under a modularity assumption on the Markov semi-group. The modularity assumption is necessary to carry out our proof through Haagerup's reduction method and due to the fact that the probabilistic martingale BMO-spaces in [JuPe14] are studied (in principle only) in the tracial setting. This culminates in Theorem 3.15, which briefly states the following. Let $\mathcal{S}$ be a modular Markov semi-group admitting a reversed Markov dilation with a.u. continuous path on a σ-finite von Neumann algebra $\mathcal{M}$. We have

$$[\mathrm{bmo}^{\circ}_{\mathcal{S}}(\mathcal{M}), L^{\circ}_p(\mathcal{M})]_{1/q} \approx_{pq} L^{\circ}_{pq}(\mathcal{M}). \tag{1.1}$$

Other interpolation theorems for Poisson semi-groups and different BMO-spaces are then discussed in Section 4. Proofs here are similar and some aspects in fact simplify.

In Section 5 we give examples of multiplier theorems of non-tracial von Neumann algebras, namely free Araki-Woods factors (see [Shl97]). The first part of Section 5 introduces a natural generalization of the (free) Hilbert transform. We get L_p-bounds through Cotlar's trick. Recently in [MeRi16] Mei and Ricard obtained the analogous result for free group factors. We also give examples of $L_\infty \to$ BMO multipliers and show that the interpolation result of (1.1) applies. We leave it as an open question whether the Hilbert transform admits a $L_\infty \to$ BMO-estimate (or even a BMO $\to$ BMO-estimate as for the classical Hilbert transform [FeSt72], [Gra09]). In Section 5.3 we construct a reversed Markov dilation for the semi-groups that we use on free Araki-Woods factors. The construction is essentially due to Ricard [Ric08] which is combined with an ultraproduct argument to go from the discrete to continuous case.

2. Preliminaries and notation

We start with some general conventions. For general operator theory we refer to [Tak02] and for operator spaces to [EfRu00], [Pis02]. Throughout the paper $\mathcal{M}$ will be a von Neumann algebra with fixed normal faithful state φ. $\mathcal{S} = (\Phi_t)_{t\geq 0}$ will be a fixed Markov semi-group, see Section 2.3 for details. $(\sigma^{\varphi}_s)_{s\in\mathbb{R}}$ denotes the modular automorphism group of φ, see [Tak03] for modular theory.

2.1. **General notation.** For the complex interpolation method we refer to the book [BeLö76]. See also [Cas13] for a short summary and the relation to non-commutative L_p-spaces. Let S be the strip of all complex numbers with imaginary part in the interval $[0,1]$. For a compatible couple of Banach spaces (X,Y) denote $\mathcal{F}(X,Y)$ for the space of functions $S \to X+Y$ that (i) are continuous on S and analytic on the interior of S, (ii) $f(s) \in X$ and $f(i+s) \in Y$, (iii) $\|f(s)\|_X \to 0$ and $\|f(i+s)\|_Y \to 0$ as $|s| \to \infty$. We write $(X,Y)_\theta$ for the interpolation space at parameter $\theta \in [0,1]$.

2.2. **L_p-spaces associated with an arbitrary von Neumann algebra.** This paper establishes results on interpolation and harmonic analysis on non-tracial von Neumann algebras. The L_p-spaces of such von Neumann algebras can be described through constructions introduced by Haagerup [Haa77], [Ter81] and Connes-Hilsum [Con80], [Hil81] (the latter in fact relies on Haagerup's construction to treat sums and products of unbounded operators). In principle we use the definition of Hilsum [Hil81], though it is easy to recast each of the statements in terms of [Haa77].

For a general von Neumann algebra $\mathcal{M}$ we let ϕ' be a fixed normal, semi-finite, faithful weight on the commutant $\mathcal{M}'$. For a normal, semi-finite weight φ on $\mathcal{M}$ we write D_φ for Connes's spatial derivative $d\varphi/d\phi'$ [Con80], [Ter81]. For every von Neumann algebra in this paper ϕ' is implicitly fixed; it can be chosen arbitrary and ϕ' will be suppressed in the notation. $L_p(\mathcal{M})$ with $\mathcal{M} \subseteq B(\mathcal{H})$ is defined as all closed densely defined operators x on $\mathcal{H}$ such that $|x|^p = D_\varphi$ for some $\varphi \in \mathcal{M}_*^+$. Then $\|x\|_p = \|\varphi\|^{1/p}$. Products and sums of elements in (different) L_p-spaces are understood as strong products and strong sums (so closure of the product and sum). We will omit these closures in the notation. L_p-spaces satisfy classical properties as Hölder estimates. In particular for all $x \in \mathcal{M}$ and $\varphi \in \mathcal{M}_*$ positive we have $D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \in L_p(\mathcal{M})$. In fact such elements are (norm) dense in $L_p(\mathcal{M})$ for $1 \leq p < \infty$.

We turn $L_p(\mathcal{M}), 1 \leq p \leq \infty$ into a compatible couple (or compatible scale) of Banach spaces. Assume $\mathcal{M}$ is σ-finite, meaning that there exists a faithful, normal state φ on $\mathcal{M}$. Then there is a contractive embedding $\kappa_p^\varphi : L_p(\mathcal{M}) \to L_1(\mathcal{M})$ determined by

$$D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \mapsto D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}.$$

Considering $L_p(\mathcal{M})$ as (non-isometric) linear subspaces of $L_1(\mathcal{M})$ we may and will interpret intersections, sum spaces and interpolation spaces of $L_p(\mathcal{M})$ and $L_r(\mathcal{M})$ within $L_1(\mathcal{M})$. Such spaces depend on φ and we will usually mark φ in the notation (we shall need a transition between the tracial and non-tracial case). For example $[L_p(\mathcal{M}), L_r(\mathcal{M})]_\theta^\varphi$ will denote the complex interpolation spaces between $L_p(\mathcal{M})$ and $L_r(\mathcal{M})$ at parameter $\theta \in [0,1]$ with respect to the embeddings of $L_p(\mathcal{M})$ and $L_r(\mathcal{M})$ in $L_1(\mathcal{M})$ through κ_p^φ and κ_r^φ.

2.3. **Semi-groups.** We recall preliminaries on semi-groups.

Definition 2.1. A map $\Phi : \mathcal{M} \to \mathcal{M}$ is called Markov if it is normal ucp (unital completely positive) and $\varphi \circ \Phi = \varphi$ (where φ is the fixed faithful normal state on $\mathcal{M}$). Through complex interpolation between $\mathcal{M}$ and $L_1(\mathcal{M})$, a Markov map has a contractive L_2-implementation given by

$$\Phi^{(2)} : D_\varphi^{\frac{1}{4}} x D_\varphi^{\frac{1}{4}} \to D_\varphi^{\frac{1}{4}} \Phi(x) D_\varphi^{\frac{1}{4}}.$$

A Markov map is called *KMS-symmetric* if $\Phi^{(2)}$ is self-adjoint. A Markov map is called *GNS-symmetric* if $\varphi(\Phi(x)^* y) = \varphi(x^* \Phi(y))$ for all $x, y \in \mathcal{M}$. Φ is called φ-modular if for every $s \in \mathbb{R}$ we have $\Phi \circ \sigma_s^\varphi = \sigma_s^\varphi \circ \Phi$.

If Φ is φ-modular then it is KMS-symmetric if and only if it is GNS-symmetric.

Definition 2.2. A family $(\Phi_t)_{t\geq 0}$ is called a semi-group if $\Phi_{s+t} = \Phi_s \circ \Phi_t$ and for every $x \in \mathcal{M}$ we have $\Phi_t(x) \to x$ in the strong topology as $t \searrow 0$. A semi-group $(\Phi_t)_{t\geq 0}$ is called *Markov*, *KMS-symmetric* or *φ-modular* if for each $t \geq 0$ the map Φ_t is respectively Markov, KMS-symmetric or φ-modular.

By interpolation between L_1 and L_∞ we may in fact define $\Phi_t^{(p)}$ as (the closure of)

$$\Phi_t^{(p)} : L_p(\mathcal{M}) \to L_p(\mathcal{M}) : D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \mapsto D_\varphi^{\frac{1}{2p}} \Phi_t(x) D_\varphi^{\frac{1}{2p}}, \tag{2.1}$$

see [JuXu07, Lemma 7.1]. If Φ_t is φ-modular then for x analytic,

$$\begin{aligned} \Phi_t^{(2)}(xD_\varphi^{\frac{1}{2}}) &= \Phi_t^{(2)}(D_\varphi^{\frac{1}{4}} \sigma_{i/4}^\varphi(x) D_\varphi^{\frac{1}{4}}) = D_\varphi^{\frac{1}{4}} \Phi_t(\sigma_{i/4}^\varphi(x)) D_\varphi^{\frac{1}{4}} \\ &= D_\varphi^{\frac{1}{4}} \sigma_{i/4}^\varphi(\Phi_t(x)) D_\varphi^{\frac{1}{4}} = \Phi_t(x) D_\varphi^{\frac{1}{2}}. \end{aligned} \tag{2.2}$$

For $1 \leq p < \infty$ let $A_p \geq 0$ be the unbounded generator of our Markov semi-group, which may be characterized by

$$\mathrm{Dom}(A_p) = \{\xi \in L_p(\mathcal{M}) \mid \lim_{t\searrow 0} t^{-1}(\Phi_t^{(p)}(\xi) - \xi) \text{ exists}\}$$

and for $\xi \in \mathrm{Dom}(A_p)$, $A_p\xi = \lim_{t\searrow 0} t^{-1}(\xi - \Phi_t^{(p)}(\xi))$. We have $\exp(-tA_p) = \Phi_t^{(p)}$. We also set,

$$L_p^\circ(\mathcal{M}) = \left\{ \xi \in L_p(\mathcal{M}) \mid \lim_{t\to\infty} \Phi_t^{(p)}(\xi) = 0 \right\}.$$

Note that as φ is a normal faithful state, we have an inclusion

$$\kappa_{r,p}^\varphi := (\kappa_r^\varphi)^{-1} \circ \kappa_p^\varphi : L^p(\mathcal{M}) \subseteq L^r(\mathcal{M}) : D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \mapsto D_\varphi^{\frac{1}{2r}} x D_\varphi^{\frac{1}{2r}}, \qquad x \in \mathcal{M}, \tag{2.3}$$

whenever $r \leq p$ and this inclusion is a contractive mapping that intertwines $\Phi_t^{(p)}$ and $\Phi_t^{(r)}$. It follows therefore that $\mathrm{Dom}(A_p) \subseteq \mathrm{Dom}(A_r)$. We also set,

$$\mathcal{M}^\circ = \{x \in \mathcal{M} \mid \Phi_t(x) \to 0 \quad \sigma\text{−weakly}\}.$$

And for notational convenience $L_\infty^\circ(\mathcal{M}) = \mathcal{M}^\circ$.

Lemma 2.3. *For $1 \leq r \leq p \leq \infty$ we have $L_p^\circ(\mathcal{M}) \subseteq L_r^\circ(\mathcal{M})$ for the inclusion* (2.3).

Proof. Assume $p \neq \infty$. Take $y \in L_p^\circ(\mathcal{M})$ then $\Phi_t^{(p)}(y) \to 0$. So $\Phi_t^{(r)}(\kappa_{r,p}^\varphi(y)) = \kappa_{r,p}^\varphi(\Phi_t^{(p)}(y)) \to 0$ which is equivalent to $\kappa_{r,p}^\varphi(y) \in L_r^\circ(\mathcal{M})$. Assume $p = \infty$. Take $y \in \mathcal{M}^\circ$ so that $\Phi_t^{(p)}(y) \to 0$ strongly. Then $\Phi_t^{(p)}(\kappa_{p,\infty}^\varphi(y)) = D_\varphi^{\frac{1}{2p}} \Phi_t(y) D_\varphi^{\frac{1}{2p}} \to 0$ by [JuSh05, Lemma 1.3]. □

Remark 2.4. Suppose that the state φ is almost periodic, meaning that its modular operator ∇_φ has a complete set of eigenspaces. In this case there is the following averaging trick in order to assure the existence of φ-modular semi-groups (see e.g. [OkTo15, Theorem 4.15] for a similar argument). By [Con73, Lemma 3.7.3] there exists a compact group $\widehat{\Gamma}$ with group homomorphism $\rho : \mathbb{R} \to \widehat{\Gamma}$ with dense range and a continuous unitary representation $s \mapsto U_s, s \in \widehat{\Gamma}$ on $B(L_2(\mathcal{M}))$ such that for $t \in \mathbb{R}$ we get $\nabla_\varphi^{it} = U_{\rho(t)}$. Let Φ be a Markov map on $\mathcal{M}$. Then the map

$$\Phi^{av} = \int_{\widehat{\Gamma}} \mathrm{ad}(U_s^*) \circ \Phi \circ \mathrm{ad}(U_s) ds$$

is also Markov. Moreover, it is φ-modular as

$$\Phi^{av} \circ \sigma_t^{\varphi} = \Phi^{av} \circ \mathrm{ad}(\nabla_{\varphi}^{it}) = \int_{\widehat{\Gamma}} \mathrm{ad}(U_s^*) \circ \Phi \circ \mathrm{ad}(U_{s+\rho(t)}) ds$$
$$= \int_{\widehat{\Gamma}} \mathrm{ad}(U_s^* U_{\rho(t)}) \circ \Phi \circ \mathrm{ad}(U_s) ds = \sigma_t^{\varphi} \circ \Phi^{av}.$$

Similarly, if $(\Phi_t)_{t \geq 0}$ is a Markov semi-group then $(\Phi_t^{av})_{t \geq 0}$ is a Markov semi-group that is moreover φ-modular.

2.4. **Markov dilations of semi-groups.** The following terminology was introduced in [JuMe12] (see also [Ana06] and [Ric08]). It forms the crucial condition that is being used in Junge and Mei their proofs of interpolation results.

Definition 2.5. A standard Markov dilation of a semi-group $\mathcal{S} = (\Phi_t)_{t \geq 0}$ on a von Neumann algebra $\mathcal{M}$ with normal faithful state φ consists of: (1) A von Neumann algebra $\mathcal{N}$ with faithful normal state $\varphi_{\mathcal{N}}$, (2) an increasing filtration $(\mathcal{N}_s)_{s \geq 0}$ with $\varphi_{\mathcal{N}}$-preserving conditional expectations $\mathcal{E}_s : \mathcal{N} \to \mathcal{N}_s$, (3) state preserving $*$-homomorphisms $\pi_s : \mathcal{M} \to \mathcal{N}_s$ such that

$$\mathcal{E}_s(\pi_t(x)) = \pi_s(\Phi_{t-s}(x)), \qquad s < t, x \in \mathcal{M}. \tag{2.4}$$

Definition 2.6. A reversed Markov dilation of a semi-group $\mathcal{S} = (\Phi_t)_{t \geq 0}$ on a von Neumann algebra $\mathcal{M}$ with normal faithful state φ consists of: (1) A von Neumann algebra $\mathcal{N}$ with faithful normal state $\varphi_{\mathcal{N}}$, (2) a decreasing filtration $(\mathcal{N}_s)_{s \geq 0}$ with $\varphi_{\mathcal{N}}$-preserving conditional expectations $\mathcal{E}_s : \mathcal{N} \to \mathcal{N}_s$, (3) state preserving $*$-homomorphisms $\pi_s : \mathcal{M} \to \mathcal{N}_s$ such that

$$\mathcal{E}_s(\pi_t(x)) = \pi_s(\Phi_{s-t}(x)), \qquad t < s, x \in \mathcal{M}. \tag{2.5}$$

We call a (standard or reversed) Markov dilation *modular* if in their definitions we have moreover

$$\sigma_t^{\varphi_{\mathcal{N}}} \circ \pi_s = \pi_s \circ \sigma_t^{\varphi}, s \geq 0, t \in \mathbb{R}. \tag{2.6}$$

Without loss of generality for a standard Markov dilation we may assume that $\mathcal{N}_s$ is generated by $\pi_t(x), t \leq s, x \in \mathcal{M}$ and $\mathcal{N} = (\cup_{s \geq 0} \mathcal{N}_s)''$. Then the condition (2.6) implies that $\sigma^{\varphi_{\mathcal{N}}}$ preserves $\mathcal{N}_s$ for every $s \geq 0$.

We typically denote standard/reversed Markov dilations by means of a triple $(\mathcal{N}_t, \pi_t, \mathcal{E}_t)_{t \geq 0}$. The von Neumann algebra $\mathcal{N}$ is then implicitly understood as the σ-weak closure of $\cup_{t \geq 0} \mathcal{N}_t$.

Definition 2.7. An L_∞-martingale $(x_t)_{t \geq 0}$ in a von Neumann algebra $\mathcal{N}$ with faithful normal state ψ and with filtration $(\mathcal{N}_t)_{t \geq 0}$ has a.u. continuous path if for every $T > 0, \epsilon > 0$ there exists a projection $e \in \mathcal{N}$ with $\psi(1-e) < \epsilon$ such that $[0, T] \to \mathcal{N} : t \mapsto x_t e$ is continuous.

We require Lemma 2.8 which was already observed in [JuMe12, p. 716] and [JuMe12, p. 637]. For properties of vector valued L_p-spaces we refer to [Pis96]. Let $x = (x_t)_{t \geq 0}$ be a martingale as in Definition 2.7. Let $2 < p < \infty$. Let $\sigma = \{t_1, \ldots, t_{n_\sigma}\}$ be a (finite) set of elements $0 < t_1 < \ldots < t_{n_\sigma} < \infty$. We write

$$\|x\|_{h_p^d(\sigma)} = \left(\sum_{t_i \in \sigma} \|x_{t_{i+1}} - x_{t_i}\|_{L_p}^p \right)^{\frac{1}{p}},$$

and then $\|x\|_{h_p^d} = \lim_{\sigma, \mathcal{U}} \|x\|_{h_p^d(\sigma)}$ for any ultrafilter containing the filter base of tails. This yields a norm, which is independent of the choice of ultrafilter [JuPe14]. Note that the $h_p^d(\sigma)$-norm is just the $L_p(\ell_p(\sigma))$-norm [Pis96] of the martingale difference sequence $d_i(x) = x_{t_{i+1}} - x_{t_i}$. It follows straight from the definitions that if $\mathcal{Q}$ is a von Neumann subalgebra of $\mathcal{N}$ with expectation $\mathcal{E}_\mathcal{Q}$ satisfying for all $t \geq 0$, $\mathcal{E}_\mathcal{Q} \circ \mathcal{E}_t = \mathcal{E}_t \circ \mathcal{E}_\mathcal{Q}$. Then for every martingale $x = (x_t)_{t \geq 0}$ in $\mathcal{N}$ we get

$$\| \mathcal{E}_\mathcal{Q}(x) \|_{h_p^d} \leq \| x \|_{h_p^d}. \tag{2.7}$$

Lemma 2.8. *If a martingale $x = (x_t)_{t \geq 0}$ has a.u. continuous path then $\|x\|_{h_p^d} = 0$ for all $p > 2$.*

Proof. We use the notation of Definition 2.7. By Doob's inequality [Jun02] for every $2 < p < \infty$ and $T > 0$ there exists a continuous function $f : [0, T] \to \mathcal{N}$ and an element $a \in L_p(\mathcal{N})$ such that $x_t = f(t)a$. Then taking the ultralimit over all finite subsets $\sigma \subseteq [0, T]$ we get $\|x\|_{L_p(\ell_\infty^c(\sigma))} \to 0$. By interpolation

$$\|x\|_{h_p^d(\sigma)} \leq \|d_j(x)\|^{\theta}_{L_p(\ell_\infty^c(\sigma))} \|d_j(x)\|^{1-\theta}_{L_p(\ell_2^c(\sigma))},$$

with $\theta = p/2$. Let $\sigma = \{t_1 < \ldots < t_n\}$ be a finite subset of $[0, T]$. Set $d_j(x) = x_{t_{j+1}} - x_{t_j}$. The norm $\|(d_j(x))_j\|_{L_p(\ell_2^c(\sigma)}$ can be upper estimated by the norm $\|x\|_p$ by the Burkholder-Gundy inequality [HJX10, Theorem 6.4] and in particular is uniformly bounded in σ. Then as we already showed that $\|d_j(x)\|_{L_p(\ell_\infty^c(\sigma))} \to 0$ we conclude.

□

Because modular Markov dilations are state preserving homomorphisms, they extend to maps

$$\pi_s^{(p)} : L_p(\mathcal{M}) \to L_p(\mathcal{N}_s) : D_\varphi^{\frac{1}{2p}} x D_\varphi^{\frac{1}{2p}} \to D_{\varphi_\mathcal{N}}^{\frac{1}{2p}} \pi_s(x) D_{\varphi_\mathcal{N}}^{\frac{1}{2p}}, \qquad x \in \mathcal{M}, s \geq 0.$$

These are $\mathcal{M}$-$\mathcal{M}$ bimodule maps in the sense that $\pi_s(x)\pi_s^{(p)}(y)\pi_s(z) = \pi_s^{(p)}(xyz)$ for $x, z \in \mathcal{M}$ and $y \in L_p(\mathcal{M})$.

We shall need a notion of almost uniform continuity of Markov dilations. These notions were considered in [JuMe12] (see also [JuMe10]) and play an important role for embeddings of various BMO-spaces. Our notion *differs* from what is used in [JuMe12, p. 725], which assumes a.u. continuity of two martingales $m(f)$ and $n(f)$. But actually the proof of the interpolation result in the first statement of [JuMe12, Theorem 5.2 (ii)] only uses a.u. continuity of the martingale $m(f)$, which is what we need (the second statement of [JuMe12, Theorem 5.2 (ii)] requires more).

Definition 2.9. A reversed Markov dilation $(\mathcal{N}_t, \pi_t, \mathcal{E}_t)_{t \geq 0}$ for a Markov semi-group $\mathcal{S} = (\Phi_t)_{t \geq 0}$ on a von Neumann algebra $\mathcal{M}$ has a.u. continuous path if there exists a σ-weakly dense subset $B \subseteq \mathcal{M}$ such that for all $x \in B$ the L_∞-martingale

$$m(x) = (m_t(x))_{t \geq 0} = (\pi_t \circ \Phi_t(x))_{t \geq 0}. \tag{2.8}$$

has a.u. continuous path.

Remark 2.10. In the work in progress [JRS] it is proved that Markov semi-groups on finite von Neumann algebras always admit a standard (as well as reversed) Markov dilation with a.u. continuous path.

3. Semi-group BMO for σ-finite von Neumann algebras

In this section we generalize some of the interpolation results from [JuMe12], in particular Theorem 3.8, for finite von Neumann algebras to arbitrary σ-finite von Neumann algebras.

Throughout this section we let $\mathcal{S} = (\Phi_t)_{t \geq 0}$ be a Markov semi-group on a σ-finite von Neumann algebra $\mathcal{M}$ with fixed normal faithful state φ. In order to do reduction we must assume later that $\mathcal{S}$ is φ-modular. Furthermore in order to interpret BMO-spaces (see Section 3.3) as interpolation spaces we must assume that $\mathcal{S}$ is GNS-symmetric (which in case the semi-group is φ-modular is equivalent to being KMS-symmetric).

3.1. **The Haagerup reduction method.** Let $\mathsf{G} = \cup_{n\in\mathbb{N}} \frac{1}{n}\mathbb{Z}$ equipped with the discrete topology. We set $\mathcal{R} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{G}$ which is the subalgebra of $\mathcal{M} \otimes \mathcal{B}(\ell_2(\mathsf{G}))$ generated by operators

$$l_g = 1 \otimes \lambda_g, \qquad \pi_\varphi(x) = \sum_{g\in\mathsf{G}} \sigma^\varphi_{-g}(x) \otimes e_{g,g}, \qquad g \in \mathsf{G}, x \in \mathcal{M}. \tag{3.1}$$

The map π_φ identifies $\mathcal{M}$ as a subalgebra of $\mathcal{R}$ and hence we often omit it. For every $\gamma \in \widehat{\mathsf{G}}$ there exists an automorphism $\theta_\gamma : \mathcal{R} \to \mathcal{R}$ called the *dual action* that is determined by $\theta_\gamma(\pi_\varphi(x)) = \pi_\varphi(x), \theta_\gamma(l_g) = \langle \gamma, g\rangle_{\widehat{\mathsf{G}},\mathsf{G}} l_g$ with $x \in \mathcal{M}, g \in \mathsf{G}$. There exists a normal conditional expectation $\mathcal{E}_\mathcal{M} : \mathcal{R} \to \pi_\varphi(\mathcal{M}) \simeq \mathcal{M}$ that is given by

$$\mathcal{E}_\mathcal{M}(x) = \int_{\gamma\in\widehat{\mathsf{G}}} \theta_\gamma(x) d\gamma, \qquad x \in \mathcal{R}. \tag{3.2}$$

We set $\widetilde{\varphi} = \varphi \circ \pi_\varphi^{-1} \circ \mathcal{E}_\mathcal{M}$, which is a normal faithful state on $\mathcal{R}$ that restricts to φ on $\mathcal{M}$. We define $b_n = -i \log(\lambda_{2^{-n}})$ where we use the principal branch of the logarithm so that $0 \leq \Im(\log(z)) < 2\pi$. Then set $a_n = 2^n b_n, h_n = e^{-a_n}$ and

$$\widetilde{\varphi}_n = h_n^{\frac{1}{2}} \widetilde{\varphi_n} h_n^{\frac{1}{2}}, \qquad \mathcal{R}_n := \mathcal{R}_{\widetilde{\varphi}_n},$$

Here $\mathcal{R}_{\widetilde{\varphi}_n} := \{x \in \mathcal{R} \mid \sigma_t^{\widetilde{\varphi}_n}(x) = x\}$ is the centralizer of $\widetilde{\varphi}_n$. By construction the operator h_n is boundedly invertible. Furthermore,

$$D^{it}_{\widetilde{\varphi}} h_n D^{-it}_{\widetilde{\varphi}} = h_n, \quad \text{and} \quad D^{it}_{\widetilde{\varphi}_n} h_n D^{-it}_{\widetilde{\varphi}_n} = h_n. \tag{3.3}$$

Now we recall the following theorem from [HJX10] (see also [CPPR15, Section 7] for the weight case), which is known as the reduction method.

Theorem 3.1. *With the above notation we have:*

1. *Each $\mathcal{R}_n$ is finite with normal faithful trace $\widetilde{\varphi}_n$.*
2. *There exist normal conditional expectations $\mathcal{E}_n : \mathcal{R} \to \mathcal{R}_n$ such that $\widetilde{\varphi} \circ \mathcal{E}_n = \widetilde{\varphi}$ and $\sigma_t^{\widetilde{\varphi}} \circ \mathcal{E}_n = \mathcal{E}_n \circ \sigma_t^{\widetilde{\varphi}}$ for all $t \in \mathbb{R}$.*
3. *For each $x \in \mathcal{R}$ we have $\mathcal{E}_n(x) \to x$ in the σ-strong topology.*

The following lemma is standard. We included a sketch of the proof for convenience of the reader.

Lemma 3.2. *Let Φ be a φ-modular Markov map on $\mathcal{M}$. Then there exists a unique normal $\widetilde{\varphi}$-modular extension $\widetilde{\Phi}$ on $\mathcal{R}$ such that*

$$\widetilde{\Phi}(\pi_\varphi(x)\lambda_g) = \pi_\varphi(\Phi(x))\lambda_g, \qquad x \in \mathcal{R}, g \in \widehat{\mathsf{G}}. \tag{3.4}$$

In particular we have

$$\widetilde{\Phi}(h_n^{it} \pi_\varphi(x) h_n^{-it}) = h_n^{it} \widetilde{\Phi}(\pi_\varphi(x)) h_n^{-it}, x \in \mathcal{M}. \tag{3.5}$$

Moreover if Φ is Markov then so is $\widetilde{\Phi}$ and if $(\Phi_t)_{t\geq 0}$ is a Markov semi-group then so is $(\widetilde{\Phi}_t)_{t\geq 0}$ for both $\widetilde{\varphi}$ and $\widetilde{\varphi}_n$. If $(\Phi_t)_{t\geq 0}$ is KMS-symmetric, then so is $(\widetilde{\Phi}_t)_{t\geq 0}$ for both $\widetilde{\varphi}$ and $\widetilde{\varphi}_n$.

Proof. As $\mathcal{R} = \mathcal{M} \rtimes_{\sigma^\varphi} \mathsf{G} \subseteq \mathcal{M} \otimes \mathcal{B}(\ell_2(\mathsf{G}))$. We let $\widetilde{\Phi}$ be the restriction of $\Phi \otimes \mathrm{id}_{\mathcal{B}(\ell_2(\mathsf{G}))}$ to $\mathcal{R}$. Using that Φ commutes with the modular group of φ (3.4) follows. If Φ is Markov then for $x \in \mathcal{M}, g \in \mathsf{G}$,

$$\begin{aligned} &\widetilde{\varphi} \circ \widetilde{\Phi}(\pi_\varphi(x)\lambda_g) = \widetilde{\varphi}(\pi_\varphi(\Phi(x))\lambda_g) = \varphi \circ \mathcal{E}_\mathcal{M}(\pi_\varphi(\Phi(x))\lambda_g) \\ =&\varphi(\Phi(x))\delta_{g,0} = \varphi(x)\delta_{g,0} = \widetilde{\varphi}(\pi_\varphi(x)\lambda_g). \end{aligned}$$

So $\widetilde{\Phi}$ is Markov. As h_n is contained in $1 \otimes \mathcal{L}(\mathsf{G})$ we have that $\widetilde{\Phi}(h_n^* h_n) = h_n^* h_n = \widetilde{\Phi}(h_n)^* \widetilde{\Phi}(h_n)$. So h_n is in the multiplicative domain of $\widetilde{\Phi}$ [BrOz08, Proposition 1.5.6] and so $\widetilde{\Phi}(h_n^{\frac{1}{2}} y h_n^{\frac{1}{2}}) = h_n^{\frac{1}{2}} \widetilde{\Phi}(y) h_n^{\frac{1}{2}}$. It follows that $\widetilde{\Phi}$ is Markov for $\widetilde{\varphi}_n$. Also $\widetilde{\Phi}(h_n^{-it} h_n^{it}) = 1 = h_n^{-it} h_n^{it} = \widetilde{\Phi}(h_n^{-it}) \widetilde{\Phi}(h_n^{it})$, so that also h_n^{it} is in the multiplicative domain of $\widetilde{\Phi}$ and so (3.5) follows from [BrOz08, Proposition 1.5.6].

Furthermore, since Φ_t is strongly continuous $\widetilde{\Phi}_t$ is strongly continuous. As the modular group $\sigma^{\widetilde{\varphi}}$ is determined by $\sigma_t^{\widetilde{\varphi}}(\pi_\varphi(x)) = \pi_\varphi(\sigma_t^\varphi(x)), x \in \mathcal{M}$ and $\sigma_t^{\widetilde{\varphi}}(l_g) = l_g, g \in \mathsf{G}$ it follows that $\widetilde{\Phi}_t \circ \sigma_s^{\widetilde{\varphi}} = \sigma_s^{\widetilde{\varphi}} \circ \widetilde{\Phi}_t$. From the definition one finds that $\widetilde{\varphi}(\widetilde{\Phi}_t(x) y) = \widetilde{\varphi}(x \widetilde{\Phi}_t(y))$ which for $\widetilde{\varphi}$-modular semi-groups yields that the semi-group is KMS-symmetric (see (2.2)). □

From this point let $\widetilde{\mathcal{S}} = (\widetilde{\Phi}_t)_{t \geq 0}$ be the extension of the Markov semi-group of Theorem 3.2 of a prefixed φ-modular Markov semi-group $\mathcal{S} = (\Phi_t)_{t \geq 0}$.

3.2. **Reducing Markov dilations.** In this section we show that Markov dilations and a.u. continuity behaves well with reduction.

Proposition 3.3. *Suppose that $\mathcal{S}$ is φ-modular and admits a standard (resp. reversed) φ-modular Markov dilation. Then the semi-group $\widetilde{\mathcal{S}}$ admits a standard (resp. reversed) $\widetilde{\varphi}$-modular Markov dilation. Moreover, if the reversed Markov dilation of $\mathcal{S}$ has a.u. continuous path, then the reversed Markov dilation of $\widetilde{\mathcal{S}}$ may be chosen to have a.u. continuous path.*

Proof. As before write $\mathcal{S} = (\Phi_t)_{t \geq 0}$ for the semi-group and $\widetilde{\mathcal{S}} = (\widetilde{\Phi}_t)_{t \geq 0}$ for the crossed product extension as in Lemma 3.2.

Part 1: Dilations. Let $(\mathcal{N}_s, \pi_s, \mathcal{E}_s)_{s \geq 0}$ be a φ-modular reversed Markov dilation for $\mathcal{S}$ with respect to a normal faithful state ψ on $\mathcal{N}$. Let $\mathcal{O} = \mathcal{N} \rtimes_{\sigma^{\psi_\mathcal{N}}} \mathsf{G}$ and $\mathcal{O}_s = \mathcal{N}_s \rtimes_{\sigma^{\psi_\mathcal{N}}} \mathsf{G}$ and equip it with the dual weight $\widetilde{\psi} = \psi \circ \pi_\psi^{-1} \circ \int_{\widehat{\mathsf{G}}} \theta_\gamma d\gamma$. Because $\sigma_t^\psi \circ \pi_s = \pi_s \circ \sigma_t^\varphi$ it follows that π_s extends uniquely to a normal map $\widetilde{\pi}_s : \mathcal{R} \to \mathcal{O}$ that intertwines the modular groups of σ^ψ and σ^φ. Similarly because for $\varphi_\mathcal{N}$-preserving conditional expectations we have $\mathcal{E}_s \circ \sigma_t^\psi = \sigma_t^\psi \circ \mathcal{E}_s, s \geq 0, t \in \mathbb{R}$ we get conditional expectations $\widetilde{\mathcal{E}}_s : \mathcal{O} \to \mathcal{O}_s$. In particular $(\mathcal{O}_s)_{s \geq 0}$ is filtered as the operators $\pi_\psi(x), x \in \cup_{s \geq 0} \mathcal{N}_s, l_t, t \in \mathsf{G}$ are dense in $\mathcal{O}$. We claim that $(\mathcal{O}_s, \widetilde{\pi}_s, \widetilde{\mathcal{E}}_s)_{s \geq 0}$ is a reversed Markov dilation.

For $g \in \mathsf{G}$ let $l_g^\mathcal{R} \in \mathcal{R}$ and $l_g^\mathcal{O} \in \mathcal{O}$ be the operators l_g of (3.1) in these respective von Neumann algebras. It follows from the relations $\widetilde{\Phi}_t \circ \pi_\varphi = \pi_\varphi \circ \Phi_t$, $\widetilde{\pi}_s \circ \pi_\varphi = \pi_\psi \circ \pi_s$ and $\pi_\psi \circ \mathcal{E}_s = \widetilde{\mathcal{E}}_s \circ \pi_\psi$ that for $x \in \mathcal{M}$, $t < s$,

$$\begin{aligned} \widetilde{\pi}_s \circ \widetilde{\Phi}_{s-t}(\pi_\varphi(x) l_g^\mathcal{R}) &= \widetilde{\pi}_s \circ \pi_\varphi(\Phi_{s-t}(x)) l_g^\mathcal{O} = \pi_\psi \circ \pi_s \circ \Phi_{s-t}(x) l_g^\mathcal{O} \\ &= \pi_\psi \circ \mathcal{E}_s \circ \pi_t(x) l_g^\mathcal{O} = \widetilde{\mathcal{E}}_s \circ \widetilde{\pi}_t(\pi_\varphi(x) l_g^\mathcal{R}). \end{aligned} \tag{3.6}$$

Therefore (2.4) follows by density.

This proves the first statement for reversed Markov dilations, for standard Markov dilations the proof is similar.

Part 2: A.u. continuity of the paths. Suppose now that the reversed Markov dilation $(\mathcal{N}_s, \pi_s, \mathcal{E}_s)_{s \geq 0}$ in part 1 has a.u. continuous path. By Definition 2.9 there exists a σ-weakly dense subspace $B \subseteq \mathcal{M}$ such that for $x \in B, T > 0, \epsilon > 0$ there is a projection $e \in \mathcal{N}$ with $\psi(1 - e) < \epsilon$ such that the map $[0, T] \ni t \mapsto m_t(x) e$ is continuous with $m_t(x) := \pi_t(\Phi_t(x))$. Let $\widetilde{e} = \pi_\psi(e)$. Then we have for $g \in \mathsf{G}$ that

$$\begin{aligned}\widetilde{\pi}_t(\widetilde{\Phi}_t(l_g\pi_\varphi(x)))\widetilde{e} &= l_g\widetilde{\pi}_t(\widetilde{\Phi}_t(\pi_\varphi(x)))\pi_\psi(e)\\ &= l_g\pi_\psi\left(\pi_t(\Phi_t(x))\right)\pi_\psi(e) = l_g\pi_\psi\left(\pi_t(\Phi_t(x))e\right).\end{aligned}$$

So if we put $\widetilde{m}_t(y) := \widetilde{\pi}_t(\widetilde{\Phi}_t(y))$ we see that

$$[0,T] \ni t \mapsto \widetilde{m}_t(l_g\pi_\varphi(x))\widetilde{e} = l_g\pi_\psi\left(m_t(x)e\right).$$

is continuous as $[0,T] \ni t \mapsto m_t(x)e$ is continuous. As the span of $l_g\pi_\varphi(x), x \in B, g \in \mathsf{G}$ is σ-weakly dense in $\mathcal{O}$ this concludes the second claim.

□

3.3. **Semi-group BMO and interpolation structure.** For $x \in \mathcal{M}$ we define the column BMO-semi-norm

$$\|x\|_{\mathrm{bmo}^c_{\mathcal{S}}} = \sup_t \|\Phi_t(x)^*\Phi_t(x) - \Phi_t(x^*x)\|^{\frac{1}{2}}.$$

Then set the row BMO-semi-norm and the BMO-semi-norm by

$$\|x\|_{\mathrm{bmo}^r_{\mathcal{S}}} = \|x^*\|_{\mathrm{bmo}^c_{\mathcal{S}}}, \qquad \|x\|_{\mathrm{bmo}_{\mathcal{S}}} = \max(\|x\|_{\mathrm{bmo}^c_{\mathcal{S}}}, \|x\|_{\mathrm{bmo}^r_{\mathcal{S}}}).$$

As proved in [JuMe12, Proposition 2.1] these assignments are indeed semi-norms. To proceed further to interpolation we need to treat normed spaces instead and we need to identify BMO-spaces as subspaces of $L_1(\mathcal{M})$. We can do this using GNS-symmetry and modularity of Markov semi-groups. Note that in [JuMe12] KMS/GNS-symmetry is also part of the standard assumptions on the semi-groups.

Lemma 3.4. *We have,*

$$\{x \in \mathcal{M} \mid \|x\|_{\mathrm{bmo}_{\mathcal{S}}} = 0\} \supseteq \{x \in \mathcal{M} \mid \forall t \geq 0 : \Phi_t(x) = x\}.$$

Moreover, if $\mathcal{S}$ is GNS-symmetric then we have equality of these sets. In particular on $\mathcal{M}^\circ$ the $\mathrm{bmo}_{\mathcal{S}}$*-semi-norm is actually a norm.*

Proof. $\supseteq$. For each t the space of fixed points for Φ_t is a $*$-algebra, see [JuXu07, Remark 7.3]. This shows that if $\Phi_t(x) = x$ we also have that

$$\Phi_t(x^*x) - \Phi_t(x)^*\Phi_t(x) = x^*x - x^*x = 0,$$

and similarly with x replaced by x^*. That is $\|x\|_{\mathrm{bmo}_{\mathcal{S}}} = 0$.
$\subseteq$. Assume $\mathcal{S}$ is GNS-symmetric. If $\|x\|_{\mathrm{bmo}_{\mathcal{S}}} = 0$ (in particular both the row and column BMO-semi-norm is 0) then by [BrOz08, Proposition 1.5.6] we see that x is in the multiplicative domain of Φ_t for every $x \in \mathcal{M}$. We then get for $y \in \mathcal{M}$ that

$$\varphi(yx) = \varphi(\Phi_t(yx)) = \varphi(\Phi_t(y)\Phi_t(x)) = \varphi(y\Phi_{2t}(x)),$$

where the last equality uses Φ_t is GNS-symmetric. This implies that $\Phi_{2t}(x) = x$ for all $t \geq 0$.

Finally, take $x \in \mathcal{M}^\circ$ so $\Phi_t(x) \to 0$ σ-weakly. Then, if $\|x\|_{\mathrm{bmo}_{\mathcal{S}}} = 0$ we get by this lemma that for all $t \geq 0$ we have $\Phi_t(x) = x$ so that $x = 0$.

□

If $\mathcal{S}$ is φ-modular GNS-symmetry of $\mathcal{S}$ is equivalent to KMS-symmetry. We prefer to include the KMS-symmetry as part of our statements as all embeddings and interpolation structures are defined with respect to symmetric embeddings. Assume now that $\mathcal{S}$ is φ-modular and KMS-symmetric. We write $\mathrm{bmo}^\circ_{\mathcal{S}}$ for the completion of $\mathcal{M}^\circ$ equipped with respect to the $\mathrm{bmo}^\circ_{\mathcal{S}}$-norm. We denote $\mathrm{bmo}^\circ_{\mathcal{S}}(\mathcal{M})$ in case we explicitly want to distinguish the von Neumann algebra.

We now turn $\mathrm{bmo}_{\mathcal{S}}^{\circ}$ and into the framework of compatible couples of Banach spaces, see [BeLö76]. Here we really need to restrict ourselves to $\mathrm{bmo}_{\mathcal{S}}^{\circ}$ and not just $\mathcal{M}$ with the $\mathrm{bmo}_{\mathcal{S}}$-norm.

Lemma 3.5. *Suppose that $A_2 \geq 0$ is a positive self-adjoint operator on a Hilbert space H so that $\Phi_t^{(2)} = \exp(-tA_2)$ is a semi-group of positive contractions on H. Suppose that for $\xi \in H$ we have that $\Phi_t^{(2)}\xi \to 0$ weakly as $t \to \infty$. Then in fact $\Phi_t^{(2)}\xi \to 0$ in the norm of H.*

Proof. Take a spectral resolution $A_2 = \int_0^\infty \lambda dE_A(\lambda)$. Let p_0 be the kernel projection of A_2 and let $p_1 = 1 - p_0$. Then $\Phi_t^{(2)}\xi \to 0$ weakly implies that $p_0\xi = 0$. Now let p be a spectral projection of A_2 of an interval $[\lambda_0, \infty]$ such that $\|(1-p)\xi\|_H \leq \epsilon$. Choose $t_0 \geq 0$ such that for $t \geq t_0$ we have $\|\exp(-tA_2)p\xi\|_H \leq \epsilon$. Then we see $\|\exp(-tA_2)\xi\|_H \leq \|\exp(-tA_2)p\xi\|_H + \|\exp(-tA_2)(1-p)\xi\|_H \leq 2\epsilon$. $\square$

Lemma 3.6. *Let $\mathcal{S} = (\Phi_t)_{t\geq 0}$ be a φ-modular, KMS-symmetric, Markov semi-group. Consider BMO-spaces as defined above. We have $\mathrm{bmo}_{\mathcal{S}}^{\circ} \subseteq L_1^{\circ}(\mathcal{M})$ through an extension of the embedding $x \mapsto D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}$. Moreover, for $x \in \mathcal{M}^{\circ} \subseteq \mathrm{bmo}_{\mathcal{S}}^{\circ}$ we have*

$$\|D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}\|_1 \leq \|x\|_{\mathrm{bmo}_{\mathcal{S}}^c} \qquad \text{and} \qquad \|D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}\|_1 \leq \|x\|_{\mathrm{bmo}_{\mathcal{S}}^r}.$$

Proof. $\Phi_t^{(2)}$ is a semi-group of positive contractions on $L_2(\mathcal{M})$. Further, φ-modularity of $\mathcal{S}$ implies that $\Phi_t^{(2)}(xD_\varphi^{\frac{1}{2}}) = \Phi_t(x)D_\varphi^{\frac{1}{2}}$ by (2.2). Take $x \in \mathcal{M}^{\circ}$ so that for $y \in \mathcal{M}$

$$\lim_{t\to\infty} \langle \Phi_t(x)D_\varphi^{\frac{1}{2}}, yD_\varphi^{\frac{1}{2}} \rangle = \lim_{t\to\infty} \varphi(y^*\Phi_t(x)) \to 0.$$

This shows that $\Phi_t(x)D_\varphi^{\frac{1}{2}} \to 0$ weakly. By Lemma 3.5 then $\|\Phi_t(x)D_\varphi^{\frac{1}{2}}\|_2 \to 0$. Writing $x = u|x|$ for the polar decomposition we therefore see that

$$\begin{aligned} &\|D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}\|_1^2 = \|D_\varphi^{\frac{1}{2}} u|x| D_\varphi^{\frac{1}{2}}\|_1^2 \leq \|D_\varphi^{\frac{1}{2}} u\|_2^2 \||x| D_\varphi^{\frac{1}{2}}\|_2^2 \leq \||x|D_\varphi^{\frac{1}{2}}\|_2^2 \\ &= \limsup_{t\to\infty} \varphi(x^*x) - \|\Phi_t(x)D_\varphi^{\frac{1}{2}}\|_2^2 = \limsup_{t\to\infty} \varphi(\Phi_t(x^*x) - \Phi_t(x^*)\Phi_t(x)) \\ &\leq \limsup_{t\to\infty} \|\Phi_t(x^*x) - \Phi_t(x^*)\Phi_t(x)\| = \sup_{t\geq 0} \|\Phi_t(x^*x) - \Phi_t(x^*)\Phi_t(x)\| \end{aligned}$$

This shows that $\|D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}\|_1 \leq \|x\|_{\mathrm{bmo}_{\mathcal{S}}^c}$, which yields the claim. As $\|D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}\|_1 = \|D_\varphi^{\frac{1}{2}} x^* D_\varphi^{\frac{1}{2}}\|_1$ we also get that $\|D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}\|_1 \leq \|x\|_{\mathrm{bmo}_{\mathcal{S}}^r}$. Note that $D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}} \in L_1^{\circ}(\mathcal{M})$ by Lemma 2.3. Then in particular, as by construction $\mathcal{M}^{\circ}$ is dense in $\mathrm{bmo}_{\mathcal{S}}^{\circ}$, we get $\mathrm{bmo}_{\mathcal{S}}^{\circ} \subseteq L_1^{\circ}(\mathcal{M})$ through the embedding of the lemma. $\square$

We denote the embedding extending $\mathcal{M}^{\circ} \ni x \mapsto D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}} \in L_1^{\circ}(\mathcal{M})$ of Lemma 3.6 by

$$\kappa_{\mathrm{bmo}}^{\varphi} : \mathrm{bmo}_{\mathcal{S}}^{\circ} \hookrightarrow L_1^{\circ}(\mathcal{M}). \tag{3.7}$$

This shows in particular that $(\mathrm{bmo}_{\mathcal{S}}^{\circ}, L_1^{\circ}(\mathcal{M}))$ forms a compatible couple of Banach spaces.

Remark 3.7. We did not consider compatible couples for the case that φ is an arbitrary normal, semi-finite, faithful weight; neither this seems obvious. For L_p-spaces such interpolation structures were explored in [Ter82] and [Izu97].

We recall the following tracial theorem which we will generalize to the non-tracial setting in this paper.

Theorem 3.8 (Theorem 5.2.(ii) of [JuMe12])**.** *Assume that $\mathcal{M}$ is finite and φ is a normal faithful tracial state. Suppose that $\mathcal{S}$ is a KMS-symmetric Markov semi-group. Assume that $\mathcal{S}$ admits a reversed Markov dilation with a.u. continuous path. Then for all $1 \leq p < \infty, 1 < q < \infty$ we have*

$$[\mathrm{bmo}^{\circ}_{\mathcal{S}}(\mathcal{M}), L^{\circ}_{p}(\mathcal{M})]^{\varphi}_{\frac{1}{q}} \approx_{pq} L^{\circ}_{pq}(\mathcal{M}).$$

Here $\approx_{pq}$ means complete isomorphism of operator spaces with complete norm of the isomorphism and its inverse bounded by a constant times pq.

Remark 3.9. As noted already in [JuMe12, p. 716, after Lemma 4.1], in Theorem 3.8 the condition that $\mathcal{S}$ has a.u. continuous path may be replaced by the weaker condition (see Lemma 2.8) that there exists a σ-weakly dense subset $B \subseteq \mathcal{M}$ such that for every $2 \leq p < \infty$ the martingale $m(x), x \in B$ defined in (2.8) has the property that $\| m(x) \|_{h^d_p} = 0$.

3.4. **Interpolation for σ-finite BMO-spaces.** We explicitly record the following lemma here, which is an immediate consequence of complex interpolation, c.f. [BeLö76]. Recall that a subspace Y of a Banach space X is called 1-complemented if there is a norm 1 projection $p : X \to Y$.

Lemma 3.10. *Let (X_1, X_2) be a compatible couple of Banach spaces. Let $Y_i \subseteq X_i$ be 1-complemented subspaces. Then $(Y_1, Y_2)_\theta$ is a 1-complemented subspace of $(X_1, X_2)_\theta$.*

Next we prove that the inclusions of BMO-spaces we need to consider in the proof of Theorem 3.15 are 1-complemented. Both proofs are based on finding Stinespring dilations of the semi-group and the conditional expectation that 'commute' in some sense, c.f. (3.8) and (3.13).

Proposition 3.11. *Let $\mathcal{S}$ be a φ-modular Markov semi-group. We have that* $\mathrm{bmo}^{\circ}_{\mathcal{S}}(\mathcal{M})$ *is an isometric 1-complemented subspace of* $\mathrm{bmo}^{\circ}_{\widetilde{\mathcal{S}}}(\mathcal{R})$.

Proof. As $\widetilde{\mathcal{S}}$ restricts to $\mathcal{S}$ on $\mathcal{M}$ it follows straight from the definition of BMO-spaces that $\mathrm{bmo}^{\circ}_{\mathcal{S}}(\mathcal{M})$ is an isometric subspace of $\mathrm{bmo}^{\circ}_{\widetilde{\mathcal{S}}}(\mathcal{R})$.

We now prove that the conditional expectation $\mathcal{E}_{\mathcal{M}}$ provides a norm 1 projection $\mathrm{bmo}^{\circ}_{\widetilde{\mathcal{S}}}(\mathcal{R}) \to \mathrm{bmo}^{\circ}_{\mathcal{S}}(\mathcal{M})$. For every t we may take a Stinespring dilation for the ucp map Φ_t. That is, there exist a Hilbert space H, a contractive map $V_t : L_2(\mathcal{M}) \to H$ and a representation $\pi_t : \mathcal{M} \to B(H)$ such that $\Phi_t(x) = V_t^* \pi_t(x) V_t$. We take amplifications $\widetilde{V}_t = 1_{\ell_2(\mathsf{G})} \otimes V_t : \ell_2(\mathsf{G}) \otimes L_2(\mathcal{M}) \to \ell_2(\mathsf{G}) \otimes H$ and $\widetilde{\pi}_t = 1_{B(\ell_2(\mathsf{G}))} \otimes \pi_t$. Then $\widetilde{\Phi}_t(x) = \widetilde{V}_t^* \widetilde{\pi}_t(x) \widetilde{V}_t, x \in \mathcal{R}$.

For $\gamma \in \widehat{\mathsf{G}}$ set $W_\gamma : \ell_2(\mathsf{G}) \to \ell_2(\mathsf{G})$ by $(W_\gamma \xi)(s) = \langle \gamma, s \rangle \xi(s)$. Define a partial isometry

$$W : \ell_2(\mathsf{G}) \to L_2(\widehat{\mathsf{G}}, \ell_2(\mathsf{G})) : \xi \mapsto (\gamma \mapsto W_\gamma \xi).$$

We may naturally view W as a map $\ell_2(\mathsf{G}) \to \ell_2(\mathsf{G}) \otimes L_2(\widehat{\mathsf{G}})$. We extend this map to a map $\widetilde{W} : \ell_2(\mathsf{G}) \otimes L_2(\mathcal{M}) \to \ell_2(\mathsf{G}) \otimes L_2(\mathcal{M}) \otimes L_2(\widehat{\mathsf{G}})$ as $\widetilde{W} = \Sigma_{23}(W \otimes 1_{B(L_2(\mathcal{M}))})$, where Σ_{23} flips the second and third tensor coordinate. Then for $x \in \mathcal{R}$ we get that

$$\widetilde{W}^*(x \otimes 1_{B(L_2(\widehat{\mathsf{G}}))}) \widetilde{W} = \int_{\gamma \in \widehat{\mathsf{G}}} W_\gamma^* x W_\gamma d\gamma = \int_{\gamma \in \widehat{\mathsf{G}}} \theta_\gamma(x) d\gamma = \mathcal{E}_{\mathcal{M}}(x).$$

That is, $\widetilde{W}$ is a Stinespring dilation for the conditional expectation $\mathcal{E}_{\mathcal{M}}$. We also set $\widetilde{W}^H = \Sigma_{23}(W \otimes 1_H)$ as a map $\ell_2(\mathsf{G}) \otimes H \to \ell_2(\mathsf{G}) \otimes H \otimes L_2(\widehat{\mathsf{G}})$. Note that

$$\widetilde{W}^H \widetilde{V}_t = (\widetilde{V}_t \otimes 1_{L_2(\widehat{\mathsf{G}})}) \widetilde{W}. \tag{3.8}$$

Further, for $x \in \mathcal{R}$,

$$\widetilde{\pi}_t \left(\widetilde{W}^*(x \otimes 1_{B(L_2(\widehat{\mathsf{G}}))}) \widetilde{W} \right) = (\widetilde{W}^H)^* (\widetilde{\pi}_t(x) \otimes 1_{B(L_2(\widehat{\mathsf{G}}))}) \widetilde{W}^H. \tag{3.9}$$

Now take $x \in \mathcal{R}^\circ$. As $\mathcal{E}_\mathcal{M}$ is given by (3.2) and θ_γ commutes with $\widetilde{\Phi}_t$, we get that,

$$\begin{aligned}&\Phi_t\left(\mathcal{E}_\mathcal{M}(x)^*\mathcal{E}_\mathcal{M}(x)\right) - \Phi_t\left(\mathcal{E}_\mathcal{M}(x)\right)^*\Phi_t\left(\mathcal{E}_\mathcal{M}(x)\right)\\ =&\Phi_t\left(\mathcal{E}_\mathcal{M}(x)^*\mathcal{E}_\mathcal{M}(x)\right) - \mathcal{E}_\mathcal{M}\left(\widetilde{\Phi}_t\left(x\right)^*\right)\mathcal{E}_\mathcal{M}\left(\widetilde{\Phi}_t\left(x\right)\right).\end{aligned} \tag{3.10}$$

Now using the Stinespring dilations and (3.9)

$$\begin{aligned}(3.10) =&\widetilde{V}_t^*\widetilde{\pi}_t\left(\widetilde{W}^*(x^*\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}\widetilde{W}^*(x\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}\right)\widetilde{V}_t\\ &-\widetilde{W}^*(\widetilde{V}_t^*\widetilde{\pi}_t(x)^*\widetilde{V}_t\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}\widetilde{W}^*(\widetilde{V}_t^*\widetilde{\pi}_t(x)\widetilde{V}_t\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}\\ =&\widetilde{V}_t^*(\widetilde{W}^H)^*(\widetilde{\pi}_t(x)^*\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}^H(\widetilde{W}^H)^*(\widetilde{\pi}_t(x)\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}^H\widetilde{V}_t\\ &-\widetilde{W}^*(\widetilde{V}_t^*\widetilde{\pi}_t(x)^*\widetilde{V}_t\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}\widetilde{W}^*(\widetilde{V}_t^*\widetilde{\pi}_t(x)\widetilde{V}_t\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}\end{aligned}$$

Finally we use (3.8) to find,

$$\begin{aligned}(3.10) =&\widetilde{W}^*(\widetilde{V}_t^*\widetilde{\pi}_t(x)^*(\widetilde{V}_t\widetilde{V}_t^*-1)^{\frac{1}{2}}\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}^H\\ &\times\quad(\widetilde{W}^H)^*((\widetilde{V}_t\widetilde{V}_t^*-1)^{\frac{1}{2}}\widetilde{\pi}_t(x)\widetilde{V}_t\otimes 1_{B(L_2(\widehat{\mathsf{G}}))})\widetilde{W}.\end{aligned}$$

So that

$$\begin{aligned}&\|\Phi_t\left(\mathcal{E}_\mathcal{M}(x)^*\mathcal{E}_\mathcal{M}(x)\right) - \Phi_t\left(\mathcal{E}_\mathcal{M}(x)\right)^*\Phi_t\left(\mathcal{E}_\mathcal{M}(x)\right)\|\\ \leq&\|\widetilde{V}_t^*\widetilde{\pi}_t(x)^*(\widetilde{V}_t\widetilde{V}_t^*-1)\widetilde{\pi}_t(x)\widetilde{V}_t\otimes 1_{B(L_2(\widehat{\mathsf{G}}))}\|\\ =&\|\widetilde{V}_t^*\widetilde{\pi}_t(x)^*(\widetilde{V}_t\widetilde{V}_t^*-1)\widetilde{\pi}_t(x)\widetilde{V}_t\| = \|\widetilde{\Phi}_t\left(x^*x\right) - \widetilde{\Phi}_t\left(x\right)^*\widetilde{\Phi}_t\left(x\right)\|.\end{aligned}$$

Taking the supremum over $t \geq 0$ we get that $\|\mathcal{E}_\mathcal{M}(x)\|_{\mathrm{bmo}^c_\mathcal{S}} \leq \|x\|_{\mathrm{bmo}^c_{\widetilde{\mathcal{S}}}}$. By taking adjoints we get the row estimate. By density of $\mathcal{M}^\circ$ in $\mathrm{bmo}_{\widetilde{\mathcal{S}}}(\mathcal{R})$ we conclude the proof. □

Proposition 3.12. *Suppose that $\mathcal{S}$ admits a φ-modular standard or reversed Markov-dilation. We have that $\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}_n)$ is an isometric 1-complemented subspace of $\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R})$.*

Proof. The conditional expectation of $\mathcal{R}$ onto $\mathcal{R}_n$ is given by

$$\mathcal{F}_n(x) = 2^n\int_0^{2^{-n}}\sigma_s^{\widetilde{\varphi}_n}(x)ds, \qquad x\in\mathcal{R}, \tag{3.11}$$

see [HJX10]. Let $(\mathcal{N}_t,\pi_t,\mathcal{E}_t)_{t\geq 0}$ be a φ-modular Markov dilation for $\mathcal{S}$ to a von Neumann algebra $\mathcal{N}$ with normal faithful state ψ. By Proposition 3.3 we see that $\widetilde{\mathcal{S}}$ admits a $\widetilde{\varphi}$-modular Markov dilation $(\mathcal{O}_t,\widetilde{\pi}_t,\widetilde{\mathcal{E}}_t)_{t\geq 0}$. Moreover, the proof shows that we may take $\mathcal{O}_t = \mathcal{N}_t\rtimes_{\sigma^\psi}\mathsf{G}$, $\widetilde{\mathcal{E}}_t = \mathcal{E}_t\rtimes_{\sigma^\psi}\mathsf{G}$ and $\widetilde{\pi}_t = \pi_t\rtimes\mathsf{G}$. Let k_n be the element in $\mathcal{O} = \cup_{t\geq 0}\mathcal{O}_t$ that satisfies $\forall t\geq 0, \widetilde{\pi}_t(h_n) = k_n$ (formally, it is defined as follows: let $d_n = -i\log(\lambda_{2^{-n}})\in\mathcal{O}$ (principal branch of the log), then set $e_n = 2^n d_n$ and $k_n = e^{-d_n}$). Set $\widetilde{\psi}_n = k_n\widetilde{\psi}k_n$. The conditional expectation (3.11) may be lifted to the $\mathcal{O}$-level by setting

$$\mathcal{F}_n^\mathcal{O}(x) = 2^n\int_0^{2^{-n}}\sigma_s^{\widetilde{\psi}_n}(x)ds, \qquad x\in\mathcal{O},$$

We get for $x\in\mathcal{R}, t\geq 0$ that

$$\begin{aligned}\mathcal{F}_n^\mathcal{O}\circ\widetilde{\pi}_t(x) =&2^n\int_0^{2^{-n}}\sigma_s^{\widetilde{\psi}_n}(\widetilde{\pi}_t(x))ds = 2^n\int_0^{2^{-n}}k_n^{it}\sigma_s^{\widetilde{\psi}}(\widetilde{\pi}_t(x))k_n^{-it}ds\\ =&2^n\int_0^{2^{-n}}\widetilde{\pi}_t\left(h_n^{it}\sigma_s^{\widetilde{\varphi}}(x)h_n^{-it}\right)ds = \widetilde{\pi}_t\circ\mathcal{F}_n(x).\end{aligned} \tag{3.12}$$

Recall that $\widetilde{\mathcal{E}}_t$ commutes with $\sigma_s^{\widetilde{\psi}}$ and has k_n in it range so that $\widetilde{\mathcal{E}}_t(k_n^{is} x k_n^{-is}) = k_n^{is} \widetilde{\mathcal{E}}_t(x) k_n^{-is}$. Essentially the same computation as before shows that for $x \in \mathcal{O}$ we get that

$$
\begin{aligned}
\mathcal{F}_n^{\mathcal{O}} \circ \widetilde{\mathcal{E}}_t(x) =& 2^n \int_0^{2^{-n}} \sigma_s^{\widetilde{\psi}_n}(\widetilde{\mathcal{E}}_t(x)) ds = 2^n \int_0^{2^{-n}} k_n^{is} \sigma_s^{\widetilde{\psi}}(\widetilde{\mathcal{E}}_t(x))) k_n^{-is} ds \\
=& 2^n \int_0^{2^{-n}} \widetilde{\mathcal{E}}_t \left(k_n^{is} \sigma_s^{\widetilde{\psi}}(x) k_n^{-is} \right) ds = \widetilde{\mathcal{E}}_t \circ \mathcal{F}_n^{\mathcal{O}}(x).
\end{aligned} \tag{3.13}
$$

For $x \in \mathcal{R}, t \geq 0$ we get using (2.4) in the second equality and (3.13) in the third,

$$
\begin{aligned}
&\|\widetilde{\Phi}_t(\mathcal{F}_n(x)^* \mathcal{F}_n(x)) - \widetilde{\Phi}_t(\mathcal{F}_n(x))^* \widetilde{\Phi}_t(\mathcal{F}_n(x))\| \\
=&\|\widetilde{\pi}_{2t} \left(\widetilde{\Phi}_t(\mathcal{F}_n(x)^* \mathcal{F}_n(x)) - \widetilde{\Phi}_t(\mathcal{F}_n(x))^* \widetilde{\Phi}_t(\mathcal{F}_n(x)) \right) \| \\
=&\|\widetilde{\mathcal{E}}_{2t} \left(\widetilde{\pi}_t(\mathcal{F}_n(x)^* \mathcal{F}_n(x)) \right) - \widetilde{\mathcal{E}}_{2t}(\widetilde{\pi}_t(\mathcal{F}_n(x)))^* \widetilde{\mathcal{E}}_{2t}(\widetilde{\pi}_t(\mathcal{F}_n(x)))\| \\
=&\|\widetilde{\mathcal{E}}_{2t} \left(\mathcal{F}_n^{\mathcal{O}}(\widetilde{\pi}_t(x)^*) \mathcal{F}_n^{\mathcal{O}}(\widetilde{\pi}_t(x)) \right) - \widetilde{\mathcal{E}}_{2t}(\mathcal{F}_n^{\mathcal{O}}(\widetilde{\pi}_t(x)))^* \widetilde{\mathcal{E}}_{2t}(\mathcal{F}_n^{\mathcal{O}}(\widetilde{\pi}_t(x)))\|.
\end{aligned} \tag{3.14}
$$

Next write P_{2t} and $P_n^{\mathcal{O}}$ for the L_2-implementation of $\widetilde{\mathcal{E}}_{2t}$ and $\mathcal{F}_n^{\mathcal{O}}$ respectively. Then P_{2t} and $P_n^{\mathcal{O}}$ are commuting projections and $\widetilde{\mathcal{E}}_{2t}(x) = P_{2t} x P_{2t}$, $\mathcal{F}_n^{\mathcal{O}}(x) = P_n^{\mathcal{O}} x P_n^{\mathcal{O}}$. Therefore we may estimate (3.14) as

$$
\begin{aligned}
&\|\widetilde{\Phi}_t(\mathcal{F}_n(x)^* \mathcal{F}_n(x)) - \widetilde{\Phi}_t(\mathcal{F}_n(x))^* \widetilde{\Phi}_t(\mathcal{F}_n(x))\| \\
=&\|P_{2t} P_n^{\mathcal{O}} \widetilde{\pi}_t(x)^* P_n^{\mathcal{O}} \widetilde{\pi}_t(x) P_n^{\mathcal{O}} P_{2t} - P_{2t} P_n^{\mathcal{O}} \widetilde{\pi}_t(x)^* P_n^{\mathcal{O}} P_{2t} P_n^{\mathcal{O}} \widetilde{\pi}_t(x) P_n^{\mathcal{O}} P_{2t}\| \\
=&\|P_n^{\mathcal{O}} P_{2t} \widetilde{\pi}_t(x)^* (1 - P_{2t}) P_n^{\mathcal{O}} (1 - P_{2t}) \widetilde{\pi}_t(x) P_{2t} P_n^{\mathcal{O}}\| \\
\leq&\|P_{2t} \widetilde{\pi}_t(x)^* (1 - P_{2t}) \widetilde{\pi}_t(x) P_{2t}\|.
\end{aligned}
$$

By the same computation replacing $\mathcal{F}_n(x)$ by just x one gets that

$$
\|\widetilde{\Phi}_t(x^* x) - \widetilde{\Phi}_t(x)^* \widetilde{\Phi}_t(x)\| = \|P_{2t} \widetilde{\pi}_t(x)^* (1 - P_{2t}) \widetilde{\pi}_t(x) P_{2t}\|.
$$

So that in all we conclude that

$$
\|\widetilde{\Phi}_t(\mathcal{F}_n(x)^* \mathcal{F}_n(x)) - \widetilde{\Phi}_t(\mathcal{F}_n(x))^* \widetilde{\Phi}_t(\mathcal{F}_n(x))\| \leq \|\widetilde{\Phi}_t(x^* x) - \widetilde{\Phi}_t(x)^* \widetilde{\Phi}_t(x)\|
$$

Taking the supremum over all $t \geq 0$ gives $\|\mathcal{F}_n(x)\|_{\mathrm{bmo}^c_{\widetilde{\mathcal{S}}}(\mathcal{R}_n)} \leq \|x\|_{\mathrm{bmo}^c_{\widetilde{\mathcal{S}}}(\mathcal{R})}$, which concludes the proof for the column estimate. The row estimate follows by taking adjoints. □

Let $\widetilde{\Phi}_t^{(p)}$ and $\widetilde{\Phi}_t^{(p,n)}$ be the semi-groups acting on $L_p(\mathcal{R})$ through interpolation with respect to $\widetilde{\varphi}$ and $\widetilde{\varphi}_n$, see (2.1). Note that the definition of the subspace $L_p^{\circ}(\mathcal{R})$ of $L_p(\mathcal{R})$ depends on the choice of the state. As we are dealing with different states, namely $\widetilde{\varphi}$ and $\widetilde{\varphi}_n$ these spaces may in principle be different. We distinghuish this in the notation by writing $L_p^{\circ}(\mathcal{R}, \widetilde{\varphi})$ and $L_p^{\circ}(\mathcal{R}, \widetilde{\varphi}_n)$. The following proposition shows that the spaces are equal however, so that after it we continue writing $L_p^{\circ}(\mathcal{R})$.

Proposition 3.13. *Using the notation introduced before Theorem 3.1. Let $1 \leq p < \infty$. We have*

$$
D_{\widetilde{\varphi}}^{\frac{1}{2p}} = h_n^{-\frac{1}{2p}} D_{\widetilde{\varphi}_n}^{\frac{1}{2p}} = D_{\widetilde{\varphi}_n}^{\frac{1}{2p}} h_n^{-\frac{1}{2p}}. \tag{3.15}
$$

Furthermore, we have for $y \in \mathcal{R}$,

$$
\kappa_p^{\widetilde{\varphi}_n}(y) = h_n^{\frac{1}{2} - \frac{1}{2p}} \kappa_p^{\widetilde{\varphi}}(y) h_n^{\frac{1}{2} - \frac{1}{2p}}. \tag{3.16}
$$

We have $\widetilde{\Phi}^{(p,n)}_t = \widetilde{\Phi}^{(p)}_t$ *so that in particular* $L^\circ_p(\mathcal{R}, \widetilde{\varphi}) = L^\circ_p(\mathcal{R}, \widetilde{\varphi}_n)$. *The same statements hold if* $\mathcal{R}$ *is replaced by* $\mathcal{R}_n$.

Proof. (3.15) is an elementary property of spatial derivatives, see [Ter81, Section III]. (3.16) follows as for $y = D_{\widetilde{\varphi}}^{\frac{1}{2p}} x D_{\widetilde{\varphi}}^{\frac{1}{2p}}, x \in \mathcal{R}$ we get

$$\kappa_p^{\widetilde{\varphi}}(D_{\widetilde{\varphi}}^{\frac{1}{2p}} x D_{\widetilde{\varphi}}^{\frac{1}{2p}}) = D_{\widetilde{\varphi}}^{\frac{1}{2}} x D_{\widetilde{\varphi}}^{\frac{1}{2}} = h_n^{-\frac{1}{2}} D_{\widetilde{\varphi}_n}^{\frac{1}{2}} x D_{\widetilde{\varphi}_n}^{\frac{1}{2}} h_n^{-\frac{1}{2}} = h_n^{-\frac{1}{2}+\frac{1}{2p}} \kappa_p^{\widetilde{\varphi}_n}(y) h_n^{-\frac{1}{2}+\frac{1}{2p}}.$$

By using the definitions and Lemma 3.2 we see that for $x \in \mathcal{R}$ we get

$$\begin{aligned}
&\widetilde{\Phi}^{(p,n)}_t(D_{\widetilde{\varphi}}^{\frac{1}{2p}} x D_{\widetilde{\varphi}}^{\frac{1}{2p}}) = \widetilde{\Phi}^{(p,n)}_t(D_{\widetilde{\varphi}_n}^{\frac{1}{2p}} h_n^{-\frac{1}{2p}} x h_n^{-\frac{1}{2p}} D_{\widetilde{\varphi}_n}^{\frac{1}{2p}}) = D_{\widetilde{\varphi}_n}^{\frac{1}{2p}} \widetilde{\Phi}^{(p)}_t(h_n^{-\frac{1}{2p}} x h_n^{-\frac{1}{2p}}) D_{\widetilde{\varphi}_n}^{\frac{1}{2p}} \\
=& D_{\widetilde{\varphi}_n}^{\frac{1}{2p}} h_n^{-\frac{1}{2p}} \widetilde{\Phi}^{(p)}_t(x) h_n^{-\frac{1}{2p}} D_{\widetilde{\varphi}_n}^{\frac{1}{2p}} = D_{\widetilde{\varphi}}^{\frac{1}{2p}} \widetilde{\Phi}^{(p)}_t(x) D_{\widetilde{\varphi}}^{\frac{1}{2p}} = \widetilde{\Phi}^{(p)}_t(D_{\widetilde{\varphi}}^{\frac{1}{2p}} x D_{\widetilde{\varphi}}^{\frac{1}{2p}}).
\end{aligned}$$

This shows by density that $\widetilde{\Phi}^{(p,n)}_t = \widetilde{\Phi}^{(p)}_t$. □

Now consider compatible couples $[\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L^\circ_p(\mathcal{R})]^{\widetilde{\varphi}}$ and $[\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L^\circ_p(\mathcal{R})]^{\widetilde{\varphi}_n}$ with respect to respective states $\widetilde{\varphi}$ and $\widetilde{\varphi}_n$. Note that $\mathcal{R}^\circ$ is by definition contained in $\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R})$. Let $\kappa^{\widetilde{\varphi}}_{[\mathrm{bmo},p;q]}$ and $\kappa^{\widetilde{\varphi}_n}_{[\mathrm{bmo},p;q]}$ be the respective natural identifications of $[\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L^\circ_p(\mathcal{R})]^{\widetilde{\varphi}}_{1/q}$ and $[\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L^\circ_p(\mathcal{R})]^{\widetilde{\varphi}_n}_{1/q}$ as subspaces of $L_1(\mathcal{R})$.

Proposition 3.14. *Let* $\mathcal{S}$ *be a* φ*-modular KMS-symmetric semi-group. We have a complete isometry*

$$\sigma_{p,q,n} : [\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L^\circ_p(\mathcal{R})]^{\widetilde{\varphi}}_{1/q} \to [\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L^\circ_p(\mathcal{R})]^{\widetilde{\varphi}_n}_{1/q} \tag{3.17}$$

Moreover, the isometry is explicitly given by

$$\kappa^{\widetilde{\varphi}_n}_{[\mathrm{bmo},p;q]} \circ \sigma_{p,q,n}(y) = h_n^{\frac{1}{2}-\frac{1}{2pq}} \kappa^{\widetilde{\varphi}}_{[\mathrm{bmo},p;q]}(y) h_n^{\frac{1}{2}-\frac{1}{2pq}}.$$

Proof. We use short hand notation $X = \kappa^{\widetilde{\varphi}}_{\mathrm{bmo}}(\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}))$, $X_n = \kappa^{\widetilde{\varphi}_n}_{\mathrm{bmo}}(\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}))$, $Y = \kappa^{\widetilde{\varphi}}_p(L^\circ_p(\mathcal{R}))$ and $Y_n = \kappa^{\widetilde{\varphi}_n}_p(L^\circ_p(\mathcal{R}))$. The norm on X and X_n is just the norm of $\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R})$ through the respective embeddings $\kappa^{\widetilde{\varphi}}_{\mathrm{bmo}}$ and $\kappa^{\widetilde{\varphi}_n}_{\mathrm{bmo}}$. Similarly the norms on Y and Y_n is just the norm of $L_p(\mathcal{R})$. Let σ_n be the map $[\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L^\circ_p(\mathcal{R})]^{\widetilde{\varphi}}_{\frac{1}{q}} \to L^\circ_1(\mathcal{R})$ defined by

$$\sigma_n\left(\kappa^{\widetilde{\varphi}}_{[\mathrm{bmo},p;q]}(y)\right) = h_n^{\frac{1}{2}-\frac{1}{2pq}} \kappa^{\widetilde{\varphi}}_{[\mathrm{bmo},p;q]}(y) h_n^{\frac{1}{2}-\frac{1}{2pq}},$$

i.e. the mapping (3.17) on the L_1-level.

Take $f \in \mathcal{F}(X, Y)$ which we view as a function on the strip $S \to X + Y$, where $X + Y$ is a (non-isometric) subspace of $L_1(\mathcal{R})$, see Section 2.2. Define

$$(U_n f)(z) = h_n^{\frac{iz}{2p}+\frac{1}{2}} f(z) h_n^{\frac{iz}{2p}+\frac{1}{2}} \in L^\circ_1(\mathcal{R}), \qquad z \in S.$$

We claim that $U_n f \in \mathcal{F}(X_n, Y_n)$. Take $s \in \mathbb{R}$ so that by definition of $\mathcal{F}(X, Y)$, $f(s) = \kappa^{\widetilde{\varphi}}_{\mathrm{bmo}}(x)$ for some $x \in \mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R})$. Then by (3.7)

$$(U_n f)(s) = h_n^{\frac{is}{2p}+\frac{1}{2}} f(s) h_n^{\frac{is}{2p}+\frac{1}{2}} = \kappa^{\widetilde{\varphi}_n}_{\mathrm{bmo}}(h_n^{\frac{is}{2p}} x h_n^{\frac{is}{2p}}).$$

Further, for $y \in \mathcal{R}^\circ$ it follows from the definition of the BMO-norm and Lemma 3.2 that

$$\begin{aligned} \|h_n^{\frac{is}{2p}} y h_n^{\frac{is}{2p}}\|^2_{\mathrm{bmo}^c_{\widetilde{\mathcal{S}}}} &= \sup_{t \geq 0} \|\widetilde{\Phi}_t((h_n^{\frac{is}{2p}} y h_n^{\frac{is}{2p}})^*(h_n^{\frac{is}{2p}} y h_n^{\frac{is}{2p}})) - \widetilde{\Phi}_t(h_n^{\frac{is}{2p}} y h_n^{\frac{is}{2p}})^* \widetilde{\Phi}_t(h_n^{\frac{is}{2p}} y h_n^{\frac{is}{2p}}))\| \\ &= \sup_{t \geq 0} \|h_n^{-\frac{is}{2p}} \widetilde{\Phi}_t(y^* y) h_n^{\frac{is}{2p}} - h_n^{-\frac{is}{2p}} \widetilde{\Phi}_t(y)^* \widetilde{\Phi}_t(y) h_n^{\frac{is}{2p}}\| = \|y\|^2_{\mathrm{bmo}^c_{\widetilde{\mathcal{S}}}}. \end{aligned}$$

The same holds for the row BMO-space so that $\|h_n^{\frac{is}{2p}} y h_n^{\frac{is}{2p}}\|^2_{\mathrm{bmo}_{\widetilde{\mathcal{S}}}} = \|y\|^2_{\mathrm{bmo}_{\widetilde{\mathcal{S}}}}$. And by density this in fact holds for all $y \in \mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R})$. This shows that for $s \in \mathbb{R}$ we get $(U_n f)(s) \in X_n$ and

$$\|(U_n f)(s)\|_{\mathrm{bmo}_{\widetilde{\mathcal{S}}}} = \|f(s)\|_{\mathrm{bmo}_{\widetilde{\mathcal{S}}}}. \tag{3.18}$$

Next consider $i + s \in i + \mathbb{R}$. By definition of $\mathcal{F}(X, Y)$ we have $f(i + s) \in Y$, so write $f(i + s) = \kappa_p^{\widetilde{\varphi}}(x)$ for $x \in L_p^\circ(\mathcal{R})$. Then from (3.3) and (3.16)

$$(U_n f)(i + s) = h_n^{\frac{is}{2p} - \frac{1}{2p} + \frac{1}{2}} f(s) h_n^{\frac{is}{2p} - \frac{1}{2p} + \frac{1}{2}} = h_n^{\frac{is}{2p}} \kappa_p^{\widetilde{\varphi}_n}(x) h_n^{\frac{is}{2p}} = \kappa_p^{\widetilde{\varphi}_n}(h_n^{\frac{is}{2p}} x h_n^{\frac{is}{2p}}). \tag{3.19}$$

Proposition 3.13 shows then that $(U_n f)(i + s) \in Y_n$ and

$$\|(U_n f)(i + s)\|_p = \|f(i + s)\|_p. \tag{3.20}$$

We get from the equations (3.18) and (3.20) that $U_n f \in \mathcal{F}(X_n, Y_n)$ as the fact that h_n is boundedly invertible implies that $U_n f$ is continuous on the strip S and analytic on its interior. Moreover, $\|U_n f\|_{\mathcal{F}(X_n, Y_n)} \leq \|f\|_{\mathcal{F}(X,Y)}$. So the assignment $f \mapsto U_n f$ is a contraction. Consider for $f \in \mathcal{F}(X_n, Y_n)$ the function

$$(V_n f)(z) = h_n^{-\frac{iz}{2p} - \frac{1}{2}} f(z) h_n^{-\frac{iz}{2p} - \frac{1}{2}}, \qquad z \in S.$$

Then exactly as in the previous paragraph one proves that $V_n f \in \mathcal{F}(X, Y)$ and $\|V_n f\|_{\mathcal{F}(X,Y)} \leq \|f\|_{\mathcal{F}(X_n, Y_n)}$. Moreover $V_n = U_n^{-1}$ and hence $\mathcal{F}(X, Y)$ and $\mathcal{F}(X_n, Y_n)$ are isometrically isomorphic.

Now take $x \in [X, Y]_{1/q}$. Let $\epsilon > 0$. Take $f \in \mathcal{F}(X, Y)$ such that $f(\frac{i}{q}) = x$ and $\|x\|_{[X,Y]_{1/q}} \leq \|f\|_{\mathcal{F}(X,Y)} + \epsilon$. Then $\sigma_n(x) = (U_n f)(\frac{i}{q})$ so that $\|\sigma_n(x)\|_{[X_n, Y_n]_{1/q}} \leq \|U_n f\|_{\mathcal{F}(X_n, Y_n)} = \|f\|_{\mathcal{F}(X,Y)} \leq \|x\|_{[X,Y]_{1/q}} + \epsilon$. This shows that the map (3.17) is well-defined and contractive. Repeating this argument for V_n instead of U_n shows that in fact (3.17) is an isometric isomorphism. That the map is completely isometric follows by repeating the argument on matrix levels. □

Theorem 3.15. *Let $(\mathcal{M}, \varphi)$ be a von Neumann algebra with normal faithful state. Let $\mathcal{S} = (\Phi_t)_{t \geq 0}$ be a φ-modular KMS-symmetric Markov semi-group. Assume that $\mathcal{S}$ admits a reversed Markov dilation with a.u. continuous path. Then we have, for all $1 \leq p < \infty, 1 < q < \infty$,*

$$[\mathrm{bmo}^\circ_{\mathcal{S}}(\mathcal{M}), L_p^\circ(\mathcal{M})]_{1/q} \approx_{pq} L_{pq}^\circ(\mathcal{M}).$$

Proof. Because $\mathcal{S} = (\Phi_t)_{t \geq 0}$ is φ-modular it may be extended to Markov semi-group $\widetilde{\mathcal{S}} = (\widetilde{\Phi}_t)_{t \geq 0}$ on $\mathcal{R}$, see Lemma 3.2. By Proposition 3.3 $\widetilde{\mathcal{S}}$ also has a reversed Markov dilation with a.u. continuous path. We claim that this map preserves $\mathcal{R}_n := \mathcal{R}_{\widetilde{\varphi}_n}$, which was defined as the centralizer of $\widetilde{\varphi}_n$. Let $x \in \mathcal{R}_n$. Then, by applying [Tak03, Theorem VIII.3.3] twice and Lemma 3.2 we get,

$$\sigma_s^{\widetilde{\varphi}_n}(\widetilde{\Phi}_t(x)) = h_n^{is} \sigma_s^{\widetilde{\varphi}}(\widetilde{\Phi}_t(x)) h_n^{-is} = h_n^{is} \widetilde{\Phi}_t(\sigma_s^{\widetilde{\varphi}}(x)) h_n^{-is} = \widetilde{\Phi}_t(h_n^{is} \sigma_s^{\widetilde{\varphi}}(x) h_n^{-is}) = \widetilde{\Phi}_t(x).$$

So that $x \in \mathcal{R}_n$. Denote the restriction of $\widetilde{\Phi}_t$ to $\mathcal{R}_n$ by $\widetilde{\Phi}_{n,t}$. In all, we obtained Markov semi-groups $\widetilde{\mathcal{S}} = (\widetilde{\Phi}_t)_{t \geq 0}$ and $\widetilde{\mathcal{S}}_n = (\widetilde{\Phi}_{n,t})_{t \geq 0}$ with respect to the respective states $\widetilde{\varphi}$ and $\widetilde{\varphi}|_{\mathcal{R}_n}$. Note that by Lemma 3.2

$$\widetilde{\varphi}_n \circ \widetilde{\Phi}_t(x) = \widetilde{\varphi}(h_n^{\frac{1}{2}} \widetilde{\Phi}_t(x) h_n^{\frac{1}{2}}) = \widetilde{\varphi}(\widetilde{\Phi}_t(h_n^{\frac{1}{2}} x h_n^{\frac{1}{2}}) = \widetilde{\varphi}(h_n^{\frac{1}{2}} x h_n^{\frac{1}{2}}) = \widetilde{\varphi}_n(x).$$

This shows that $\widetilde{\Phi}_t : \mathcal{R}_n \to \mathcal{R}_n$ is also Markov with respect to $\widetilde{\varphi}_n$, which is tracial on $\mathcal{R}_n$.

As the semi-groups $\mathcal{S}$ and $\widetilde{\mathcal{S}}_n$ are restrictions of $\widetilde{\mathcal{S}}$ we have isometric inclusions of the corresponding BMO-spaces

$$\mathrm{bmo}^\circ_{\mathcal{S}}(\mathcal{M}) \subseteq \mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), \qquad \mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}_n}(\mathcal{R}_n) \subseteq \mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), \qquad n \in \mathbb{N}.$$

Moreover, these inclusions are 1-complemented by Lemmas 3.11 and 3.12. Lemma 3.2 also shows that $\widetilde{\mathcal{S}}$ admits a reversed Markov dilation with a.u. continuous path. Moreover, this dilation may be chosen to be a dilation with respect to $\widetilde{\varphi}_n$. Let $m(x) = (m_t(x))_{t \geq 0}$ be the martingale with x in the set B described in Definition 2.9 for this Markov dilation. By Lemma 2.8 we see that for every $2 \leq p < \infty$ we have $\| m(x) \|_{h_p^d} = 0$ and then by (2.7) we see that $\| m(\mathcal{F}_n(x)) \|_{h_p^d} = 0$. This shows that $\mathcal{F}_n(B)$ is a σ-weakly dense subset of $\mathcal{R}_n$ such that the martingale $m(x), x \in \mathcal{F}_n(B)$ has vanishing h_p^d-norm. Therefore, by Remark 3.9 the Theorem 3.8 applies to the von Neumann algebra $\mathcal{R}_n$ with normal tracial state $\widetilde{\varphi}_n$ with Markov semi-group $\widetilde{\mathcal{S}}_n$.

So Theorem 3.8 yields

$$[\mathrm{bmo}^\circ_{\mathcal{S}_n}(\mathcal{R}_n), L_q^\circ(\mathcal{R}_n)]^{\widetilde{\varphi}_n}_{1/p} \approx_{pq} L_{pq}(\mathcal{R}_n)^\circ.$$

Now we have isometries

$$\begin{array}{ccc}
[\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}_n}(\mathcal{R}_n), L_p^\circ(\mathcal{R}_n)]^{\widetilde{\varphi}_n}_{1/q} & \xrightarrow{\text{Lemma 3.10}} & [\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L_p^\circ(\mathcal{R})]^{\widetilde{\varphi}_n}_{1/q} \\
\approx_{pq} \Big\uparrow & & \Big\downarrow \sigma^{-1}_{p,q,n} \quad \text{Prop. 4.4} \\
L^\circ_{pq}(\mathcal{R}_n) & & [\mathrm{bmo}^\circ_{\mathcal{S}}(\mathcal{R}), L_p^\circ(\mathcal{R})]^{\widetilde{\varphi}}_{1/q}.
\end{array}$$

Furthermore, for $x \in \mathcal{R}_n$,

$$\begin{aligned}
&\kappa^{\widetilde{\varphi}}_{[\mathrm{bmo},p;q]} \circ \sigma^{-1}_{p,q,n}(D_{\widetilde{\varphi}}^{\frac{1}{2pq}} x D_{\widetilde{\varphi}}^{\frac{1}{2pq}}) = h_n^{-\frac{1}{2}+\frac{1}{2pq}} \kappa^{\widetilde{\varphi}_n}_{[\mathrm{bmo},p;q]}(D_{\widetilde{\varphi}}^{\frac{1}{2pq}} x D_{\widetilde{\varphi}}^{\frac{1}{2pq}}) h_n^{-\frac{1}{2}+\frac{1}{2pq}} \\
=& h_n^{-\frac{1}{2}+\frac{1}{2pq}} \kappa^{\widetilde{\varphi}_n}_{[\mathrm{bmo},p;q]}(D_{\widetilde{\varphi}_n}^{\frac{1}{2pq}} h_n^{-\frac{1}{2pq}} x h_n^{-\frac{1}{2pq}} D_{\widetilde{\varphi}_n}^{\frac{1}{2pq}}) h_n^{-\frac{1}{2}+\frac{1}{2pq}} \\
=& h_n^{-\frac{1}{2}+\frac{1}{2pq}} D_{\widetilde{\varphi}_n}^{\frac{1}{2}} h_n^{-\frac{1}{2pq}} x h_n^{-\frac{1}{2pq}} D_{\widetilde{\varphi}_n}^{\frac{1}{2}}) h_n^{-\frac{1}{2}+\frac{1}{2pq}} \\
=& D_{\widetilde{\varphi}}^{\frac{1}{2}} x D_{\widetilde{\varphi}}^{\frac{1}{2}}.
\end{aligned}$$

It follows that for each $n \in \mathbb{N}$ we have an isometric embedding,

$$j_n : L^\circ_{pq}(\mathcal{R}_n) \to [\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L_p^\circ(\mathcal{R})]^{\widetilde{\varphi}}_{1/q},$$

and these embeddings are compatible with the inclusions $L^\circ_{pq}(\mathcal{R}_n) \subseteq L^\circ_{pq}(\mathcal{R}_{n+1})$ with respect to $\widetilde{\varphi}$. This shows that $\cup_{n \in \mathbb{N}} L^\circ_{pq}(\mathcal{R}_n, \widetilde{\varphi})$ can isometrically be identified with a subspace of $[\mathrm{bmo}_{\widetilde{\mathcal{S}}}(\mathcal{R}), L_p^\circ(\mathcal{R})]^{\widetilde{\varphi}}_{1/q}$. As $\cup_{n \in \mathbb{N}} L^\circ_{pq}(\mathcal{R}_n)$ is dense in $L^\circ_{pq}(\mathcal{R})$, c.f. [Gol84, Theorem 8], we see that $L^\circ_{pq}(\mathcal{R})$ is isometrically contained in the space $[\mathrm{bmo}_{\widetilde{\mathcal{S}}}(\mathcal{R}), L_p^\circ(\mathcal{R})]^{\widetilde{\varphi}}_{1/q}$. By [BeLö76, Theorem 4.2.2.(a)] we have that $\mathcal{R}^\circ$ is dense in $[\mathrm{bmo}^\circ_{\widetilde{\mathcal{S}}}(\mathcal{R}), L_q^\circ(\mathcal{R})]^{\widetilde{\varphi}}_{1/p}$. Further as $\mathcal{R}^\circ$ is also contained in $L^\circ_{pq}(\mathcal{R})$ we must have an isomorphism

$$[\mathrm{bmo}^\circ_{\mathcal{S}_n}(\mathcal{R}), L_q^\circ(\mathcal{R})]_{1/p} \approx_{pq} L^\circ_{pq}(\mathcal{R}). \tag{3.21}$$

Now again by Lemma 3.10 we see that the space $[\mathrm{bmo}^\circ_{\mathcal{S}}(\mathcal{M}), L^\circ_p(\mathcal{M})]^\varphi_{1/q}$ is a 1-complemented subspace of the left hand side of (3.21) and hence of $L^\circ_{pq}(\mathcal{R})$. Further by [BeLö76, Theorem 4.2.2.(a)] the space $[\mathrm{bmo}^\circ_{\mathcal{S}_n}(\mathcal{M}), L^\circ_p(\mathcal{M})]^\varphi_{1/q}$ contains $\mathcal{M}^\circ$ densely. Since in turn $\mathcal{M}^\circ$ is dense in $L^\circ_{pq}(\mathcal{M})$ which is included in $L^\circ_{pq}(\mathcal{R})$ isometrically, we conclude that $[\mathrm{bmo}^\circ_{\mathcal{S}_n}(\mathcal{M}), L_q(\mathcal{M})^\circ]^\varphi_{1/p} \approx_{pq} L^\circ_{pq}(\mathcal{M})$. Isomorphisms holds for complete bounds by considering matrix levels. □

4. Other BMO-spaces associated with Markov semi-groups

As in the rest of this paper let $\mathcal{M}$ be von Neumann algebra with faithful normal state φ. Let $\mathcal{S} = (\Phi_t)_{t\geq 0}$ be a Markov semi-group on $\mathcal{M}$. Define a semi-norm (see [JuMe12, Proposition 2.1]),

$$\|x\|_{\mathrm{BMO}^c_{\mathcal{S}}} = \sup_{t\geq 0} \|\Phi_t(|x-\Phi_t(x)|^2)\|^{\frac{1}{2}},$$

and then set

$$\|x\|_{\mathrm{BMO}^r_{\mathcal{S}}} = \|x^*\|_{\mathrm{BMO}^c_{\mathcal{S}}}, \qquad \|x\|_{\mathrm{BMO}_{\mathcal{S}}} = \max(\|x\|_{\mathrm{BMO}^c_{\mathcal{S}}}, \|x\|_{\mathrm{BMO}^r_{\mathcal{S}}}).$$

Lemma 4.1. *For $x \in \mathcal{M}$ we have $\|x\|_{\mathrm{BMO}_{\mathcal{S}}} = 0$ if and only if for all $t \geq 0, \Phi_t(x) = x$.*

Proof. The if part is obvious. Conversly, if $\|x\|_{\mathrm{BMO}_{\mathcal{S}}} = 0$ then for all $t \geq 0$ we have $\|\Phi_t(|x - \Phi_t(x)|^2)\| = 0$ and so $0 = \varphi(\Phi_t(|x-\Phi_t(x)|^2)) = \varphi(|x-\Phi_t(x)|^2)$. As φ is faithful $x = \Phi_t(x)$. □

We see that on $\mathcal{M}^\circ$ the BMO-semi-norm is actually a norm and its completion will be denoted by $\mathrm{BMO}^\circ_{\mathcal{S}}$ or $\mathrm{BMO}^\circ_{\mathcal{S}}(\mathcal{M})$. Note that we do not need to assume KMS-symmetry here.

Furthermore, let A_2 be the closed densely defined operator such that $\exp(-tA_2) = \Phi^{(2)}_t, t \geq 0$, see Section 2.3. The Poisson semi-group $\mathcal{P} = (\Psi_t)_{t\geq 0}$ is defined as the unique Markov semi-group such that $\Psi^{(2)}_t = \exp(-tA_2^{\frac{1}{2}}), t \geq 0$ (see [Sau99]). Therefore we obtain BMO-spaces

$$\mathrm{bmo}^\circ_{\mathcal{P}} = \mathrm{bmo}^\circ_{\mathcal{P}}(\mathcal{M}), \qquad \mathrm{BMO}^\circ_{\mathcal{P}} = \mathrm{BMO}^\circ_{\mathcal{P}}(\mathcal{M}),$$

together with their obvious row and column counterparts. Then [JuMe12, Theorem 5.2] proves the following tracial interpolation result.

Theorem 4.2. *Let $\mathcal{M}$ be a von Neumann algebra with faithful normal tracial state φ. Let $\mathcal{S} = (\Phi_t)_{t\geq 0}$ be a KMS-symmetric Markov semi-group for $(\mathcal{M}, \varphi)$. Assume that $\mathcal{S}$ admits a standard Markov dilation. Then,*

$$[X, L^\circ_p(\mathcal{M})]_{1/q} \approx_{pq} L^\circ_{pq}(\mathcal{M}),$$

where X is any of the spaces $\mathrm{BMO}^\circ_{\mathcal{S}}(\mathcal{M}), \mathrm{bmo}^\circ_{\mathcal{P}}(\mathcal{M})$ or $\mathrm{BMO}^\circ_{\mathcal{P}}(\mathcal{M})$.

We may generalize this to the non-tracial setting in the following way. The proof follows closely the lines of Theorem 3.15. We give the main differences. Firstly, we have that $\mathrm{BMO}^\circ_{\mathcal{S}}(\mathcal{M})$ embeds contractively into $L_1(\mathcal{M})$ as for $x \in \mathcal{M}^\circ$ with polar decomposition $x = u|x|$ we get that

$$\begin{aligned}
\|D_\varphi^{\frac{1}{2}} x D_\varphi^{\frac{1}{2}}\|_1^2 &= \|D_\varphi^{\frac{1}{2}} u|x| D_\varphi^{\frac{1}{2}}\|_1^2 \leq \|D_\varphi^{\frac{1}{2}} u\|_2^2 \||x| D_\varphi^{\frac{1}{2}}\|_2^2 \leq \|D_\varphi^{\frac{1}{2}} x^* x D_\varphi^{\frac{1}{2}}\|_1 = \varphi(x^*x) \\
&= \lim_{t\to\infty} \varphi(x^*x + \Phi_t(x)^*\Phi_t(x) - \Phi_t(x)^*x - x^*\Phi_t(x)) \\
&= \lim_{t\to\infty} \varphi(\Phi_t(x^*x + \Phi_t(x)^*\Phi_t(x) - \Phi_t(x)^*x - x^*\Phi_t(x))) \\
&\leq \limsup_{t\to\infty} \varphi(\Phi_t(x^*x + \Phi_t(x)^*\Phi_t(x) - \Phi_t(x)^*x - x^*\Phi_t(x))) \leq \|x\|^2_{\mathrm{BMO}^c_{\mathcal{S}}}.
\end{aligned}$$

A similar argument holds for the row estimate, which yields a version of Lemma 3.6 for $\mathrm{BMO}^\circ_{\mathcal{S}}$. Similarly the spaces $\mathrm{BMO}^\circ_{\mathcal{P}}$ and $\mathrm{bmo}^\circ_{\mathcal{P}}$ embed contractively into $L_1(\mathcal{M})$. The same statements hold for the completely bounded norms by considerig matrix amplifications. Let X be any of these spaces. We denote the embedding of the complex interpolation spaces by

$$\kappa^{\varphi}_{[X,p;q]} : [X, L^{\circ}_p(\mathcal{M})]^{\varphi}_{1/q} \to L^{\circ}_1(\mathcal{M}).$$

Lemma 4.3. *Let $\mathcal{M}_1$ be a von Neumann subalgebra of $\mathcal{M}$ that is invariant under the semi-group $\mathcal{S}$ and which admits a φ-preserving conditional expectation $\mathcal{E}$. Then we have 1-complemented inclusions*

$$\mathrm{BMO}^{\circ}_{\mathcal{S}}(\mathcal{M}_1) \subseteq \mathrm{BMO}^{\circ}_{\mathcal{S}}(\mathcal{M}), \quad \mathrm{BMO}^{\circ}_{\mathcal{P}}(\mathcal{M}_1) \subseteq \mathrm{BMO}^{\circ}_{\mathcal{P}}(\mathcal{M}). \tag{4.1}$$

Moreover, we have a 1-complemented inclusion $\mathrm{bmo}^{\circ}_{\mathcal{P}}(\mathcal{M}_1) \subseteq \mathrm{bmo}^{\circ}_{\mathcal{P}}(\mathcal{M})$ *and if $\mathcal{S}$ admits a standard Markov dilation we have a 1-complemented inclusion* $\mathrm{bmo}^{\circ}_{\widetilde{\mathcal{P}}}(\mathcal{R}_n) \subseteq \mathrm{bmo}^{\circ}_{\widetilde{\mathcal{P}}}(\mathcal{R})$.

Proof. It is immediate that (4.1) are isometric inclusions. Also for any $t \geq 0$ by the Kadison-Schwarz inequality,

$$\begin{aligned} &\| \Phi_t(|\mathcal{E}(x) - \Phi_t(\mathcal{E}(x))|^2) \|^2 = \| \Phi_t(\mathcal{E}(x - \Phi_t(x))^* \mathcal{E}(x - \Phi_t(x))) \|^2 \\ \leq &\| \Phi_t(\mathcal{E}((x - \Phi_t(x))^*(x - \Phi_t(x)))) \|^2 \leq \| \Phi_t((x - \Phi_t(x))^*(x - \Phi_t(x))) \|^2. \end{aligned}$$

Taking the supremum over $t \geq 0$ we see that $\| \mathcal{E}(x) \|_{\mathrm{BMO}^{\circ}_{\mathcal{S}}(\mathcal{M}_1)} \leq \| x \|_{\mathrm{BMO}^{\circ}_{\mathcal{S}}(\mathcal{M})}$. The same argument applies to the Poisson semi-group $\mathcal{P}$ so that (4.1) follows. Accoding to [Ana06] a standard Markov dilation for $\mathcal{S}$ yields a Markov dilation for $\mathcal{P}$. The proof of the remaining statements are then similar to Lemmas 3.11 and 3.12. □

The proof of the following proposition is similar to the one of Proposition 4.2.

Proposition 4.4. *Let $\mathcal{S}$ be a φ-modular semi-group. Let X be any of the spaces* $\mathrm{BMO}^{\circ}_{\widetilde{\mathcal{S}}}, \mathrm{bmo}^{\circ}_{\widetilde{\mathcal{P}}}$ *or* $\mathrm{BMO}^{\circ}_{\widetilde{\mathcal{P}}}$. *We have a complete isometry*

$$\sigma_{X,p,q,n} : [X, L^{\circ}_p(\mathcal{R})]^{\widetilde{\varphi}}_{1/q} \to [X, L^{\circ}_p(\mathcal{R})]^{\widetilde{\varphi}_n}_{1/q} \tag{4.2}$$

Moreover, the isometry is explicitly given by

$$\kappa^{\widetilde{\varphi}_n}_{[X,p;q]} \circ \sigma_{X,p,q,n}(y) = h_n^{\frac{1}{2} - \frac{1}{2pq}} \kappa^{\widetilde{\varphi}}_{[X,p;q]}(y) h_n^{\frac{1}{2} - \frac{1}{2pq}}.$$

We now get the following theorem. The KMS-symmetry is only needed because Theorem 4.2 assumes it.

Theorem 4.5. *Following the notation introduced above. Assume moreover that $\mathcal{S}$ is a φ-modular KMS-symmetric Markov semi-group that admits a standard Markov dilation. Then, for all $1 \leq p < \infty, 1 < q < \infty$,*

$$[X, L^{\circ}_p(\mathcal{M})]_{1/q} \approx_{pq} L^{\circ}_{pq}(\mathcal{M}),$$

where X is any of the spaces $\mathrm{BMO}^{\circ}_{\mathcal{S}}, \mathrm{bmo}^{\circ}_{\mathcal{P}}$ *or* $\mathrm{BMO}^{\circ}_{\mathcal{P}}$.

Proof. We sketch the proof. First we observe that again $\mathcal{S}$ may be extended to a Markov semi-group $\widetilde{\mathcal{S}}$ on $\mathcal{R}$ which has a standard Markov dilation, c.f. Proposition 3.3. Again $\widetilde{\mathcal{S}}$ restricts to $\mathcal{R}_n$ as a Markov semi-group with respect to $\widetilde{\varphi}_n$. Depending on which space X is (as in the statement of the theorem) we define the following. Let Y be either $\mathrm{BMO}^{\circ}_{\mathcal{S}}(\mathcal{R}), \mathrm{bmo}^{\circ}_{\mathcal{P}}(\mathcal{R})$ or $\mathrm{BMO}^{\circ}_{\mathcal{P}}(\mathcal{R})$. Let Y_n be either $\mathrm{BMO}^{\circ}_{\mathcal{S}}(\mathcal{R}_n), \mathrm{bmo}^{\circ}_{\mathcal{P}}(\mathcal{R}_n)$ or $\mathrm{BMO}^{\circ}_{\mathcal{P}}(\mathcal{R}_n)$. We may therefore apply the tracial Theorem

4.2 to interpolate for each n and find $[Y_n, L_p^{\circ}(\mathcal{R}_n)]_{1/q}^{\widetilde{\varphi_n}} \approx_{pq} L_{pq}^{\circ}(\mathcal{R}_n)$. One now checks that there is a diagram

$$\begin{array}{ccc} [Y_n, L_p^{\circ}(\mathcal{R}_n)]_{1/q}^{\widetilde{\varphi}_n} & \xrightarrow{\subseteq} & [Y, L_p^{\circ}(\mathcal{R})]_{1/q}^{\widetilde{\varphi}_n} \\ \approx_{pq} \Big\uparrow & & \Big\downarrow \sigma_{X,p,q,n}^{-1} \\ L_{pq}^{\circ}(\mathcal{R}_n) & & [Y, L_p^{\circ}(\mathcal{R})]_{1/q}^{\widetilde{\varphi}}. \end{array}$$

that is compatible with respect to the interpolation structure of $\widetilde{\varphi}$. The remainder of the proof is then exactly the same as in Theorem 3.15. □

5. Fourier multipliers on free Araki-Woods factors

We recall the definiton of free Araki-Woods factors from [Shl97]. Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and let $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. For $\xi \in \mathcal{H}_{\mathbb{C}}$ with $\xi = \xi_1 + i\xi_2$ and $\xi_1, \xi_2 \in \mathcal{H}_{\mathbb{R}}$ we set $\overline{\xi} = \xi_1 - i\xi_2$. Let $(V_t)_{t\in\mathbb{R}}$ be a strongly continuous 1-parameter group of orthogonal transformations on $\mathcal{H}_{\mathbb{R}}$ and use the same notation for its extension to a strongly continuous unitary 1-parameter group on $\mathcal{H}_{\mathbb{C}}$. Through Stone's theorem we have $V_t = A^{it}$ where A is a positive (possibly) unbounded self-adjoint operator on $\mathcal{H}_{\mathbb{C}}$. We define a new innerproduct on $\mathcal{H}_{\mathbb{C}}$ by setting $\langle \xi, \eta \rangle_A = \langle \frac{2A}{1+A}\xi, \eta\rangle$. Let $\mathcal{H}$ be the completion of $\mathcal{H}_{\mathbb{C}}$ with respect to the latter inner product. We have that the embedding $\mathcal{H}_{\mathbb{R}} \hookrightarrow \mathcal{H}$ is isometric [Shl97, p. 332]. We construct a Fock space,

$$\mathcal{F} = \mathbb{C}\Omega \oplus \bigoplus_{k=1}^{\infty} \mathcal{H}^{\otimes k}.$$

We denote φ_Ω for the vector state $x \mapsto \langle x\Omega, \Omega\rangle$. For $\xi \in \mathcal{H}$ let $a(\xi)$ be the creation operator on $\mathcal{F}$ defined by

$$a(\xi) : \eta_1 \otimes \ldots \otimes \eta_k \mapsto \xi \otimes \eta_1 \otimes \ldots \otimes \eta_k.$$

Let $a^*(\xi)$ be its adjoint which is the annihilation operator

$$a^*(\xi) : \eta_1 \otimes \ldots \otimes \eta_k \mapsto \langle \eta_1, \xi\rangle_A \eta_2 \otimes \ldots \otimes \eta_k.$$

For $\xi \in \mathcal{H}$ define the self-adjoint operator $s(\xi) = a(\xi) + a^*(\xi)$. Let,

$$\mathcal{M} := \mathcal{A}_0'' \text{ with } \mathcal{A}_0 := \Gamma(\mathcal{H}_{\mathbb{R}}, (V_t)_t) := \langle s(\xi) \mid \xi \in \mathcal{H}_{\mathbb{R}}\rangle,$$

where $\langle s(\xi) \mid \xi \in \mathcal{H}_{\mathbb{R}}\rangle$ stands for the $*$-algebra generated by these operators. The von Neumann algebra $\mathcal{M}$ is called the *free Araki-Woods* algebra. The vacuüm vector Ω is separating and cyclic for this algebra. Set $\varphi_\Omega(\,\cdot\,) = \langle\,\cdot\,\Omega, \Omega\rangle$. Therefore *if* for $\xi \in \mathcal{F}$ there is an operator $W(\xi)$ such that $W(\xi)\Omega = \xi$, then this operator is unique. For various calculations and to define suitable Fourier multipliers in the first place we need the following Wick theorem.

Theorem 5.1 (See Proposition 2.7 of [BKS07] or Lemma 3.2 of [HoRi11])**.** *Suppose that* $\xi_1, \ldots, \xi_n \in \mathcal{H}_{\mathbb{C}}$ *then,*

$$W(\xi_1 \otimes \ldots \otimes \xi_n) = \sum_{j=0}^{n} a(\xi_1) \ldots a(\xi_j) a^*(\overline{\xi_{j+1}}) \ldots a^*(\overline{\xi_n}). \tag{5.1}$$

The linear span of operators of the form (5.1) form a $*$-algebra which we shall denote by $\mathcal{A}$ (in fact, this follows from (5.5) below). Moreover $\mathcal{A}$ is dense in $\mathcal{M}$.

If T is a contractive operator on $\mathcal{H}_{\mathbb{R}}$ such that for every $t \in \mathbb{R}$ we have $TV_t = V_tT$ then there exists a unique normal ucp map (see [Hia01], [Was17, Proposition 3.3] for the even more general result for the q-Araki-Woods case),

$$\Gamma(T) : \mathcal{M} \to \mathcal{M} : W(\xi_1 \otimes \ldots \otimes \xi_n) \mapsto W(T\xi_1 \otimes \ldots \otimes T\xi_n).$$

This assignment is called *second quantization.* We are now ready to define the Hilbert transform.

Definition 5.2. Fix spaces $\mathcal{H}_{\mathbb{C}}^{\pm} \subseteq \mathcal{H}_{\mathbb{C}}$ that are closed in $\mathcal{H}_{\mathbb{C}}$ and such that $\mathcal{H}_{\mathbb{C}}^{+} \cap \mathcal{H}_{\mathbb{C}}^{-} = \{0\}$. So as Banach spaces $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}^{+} \oplus \mathcal{H}_{\mathbb{C}}^{-}$. Assume moreover that $\mathcal{H}_{\mathbb{C}}^{+}$ and $\mathcal{H}_{\mathbb{C}}^{-}$ are orthogonal in $\mathcal{H}$ for the inner product $\langle \, , \, \rangle_A$. Set $\epsilon = (\mathcal{H}_{\mathbb{C}}^{+}, \mathcal{H}_{\mathbb{C}}^{-})$. The mapping $H_\epsilon : \mathcal{A} \to \mathcal{A}$ defined as the linear extension of

$$H_\epsilon : W(\xi_1 \otimes \ldots \otimes \xi_n) = \pm W(\xi_1 \otimes \ldots \otimes \xi_n),$$

with $\xi_1 \in \mathcal{H}_{\mathbb{C}}^{\pm}, \xi_2, \ldots, \xi_n \in \mathcal{H}_{\mathbb{C}}$ and $H_\epsilon(1) = 0$ will be called the *Hilbert transform* (which only depends on the decomposition $\mathcal{H}_{\mathbb{C}} = \mathcal{H}_{\mathbb{C}}^{+} \oplus \mathcal{H}_{\mathbb{C}}^{-}$).

Remark 5.3. Let $(\sigma_t)_{t\in\mathbb{R}}$ be the modular automorphism group of φ_Ω. We have

$$\sigma_t(W(\xi_1 \otimes \ldots \otimes \xi_n)) = W((A^{it}\xi_1) \otimes \ldots \otimes (A^{it}\xi_n)), \tag{5.2}$$

see [Shl97]. Suppose that the spaces $\mathcal{H}_{\mathbb{C}}^{\pm}$ are invariant subspaces for all $A^{it}, t \in \mathbb{R}$. It follows that σ_t and H_ϵ with $\epsilon = (\mathcal{H}_{\mathbb{C}}^{+}, \mathcal{H}_{\mathbb{C}}^{-})$ commute for all $t \in \mathbb{R}$.

5.1. **L^p-boundedness and Cotlar's trick.** To define a fixed Hilbert transform we prefix a decomposition $\mathcal{H}_{\mathbb{C}}^{+} \oplus \mathcal{H}_{\mathbb{C}}^{-}$ of the Hilbert space $\mathcal{H}_{\mathbb{C}}$. Here the $\mathcal{H}_{\mathbb{C}}^{\pm}$ are closed in $\mathcal{H}_{\mathbb{C}}$ and orthogonal in $\mathcal{H}$. Set $\epsilon = (\mathcal{H}_{\mathbb{C}}^{+}, \mathcal{H}_{\mathbb{C}}^{-})$ as before. We write $\mathcal{E}_\Omega^\perp(x) = x - \varphi_\Omega(x)$. This is the orthocomplement of the projection onto $\mathbb{C}1 \subseteq \mathcal{M}$ with respect to the inner product of the vacuüm state.

Proposition 5.4 (Cotlar formula for the Hilbert transform)**.** *The following relation holds true:*

$$\mathcal{E}_\Omega^\perp \left(H_\epsilon(x) H_\epsilon(y)^* \right) = \mathcal{E}_\Omega^\perp \left(H_\epsilon(x H_\epsilon(y)^*) + H_\epsilon(y H_\epsilon(x)^*)^* - H_\epsilon(H_\epsilon(xy^*)^*)^* \right), \tag{5.3}$$

for all $x, y \in \mathcal{A}$.

Proof. By linearity we may assume that x and y are Wick operators of elementary tensors. So say $x = W(\xi_1 \otimes \ldots \otimes \xi_m)$ and $y = W(\eta_1 \otimes \ldots \otimes \eta_n)$. Moreover, assume that $\xi_1 \in \mathcal{H}_{\mathbb{C}}^{\epsilon_x}, \eta_1 \in \mathcal{H}_{\mathbb{C}}^{\epsilon_y}$ for signs $\epsilon_x, \epsilon_y = \pm 1$. By (5.1) we get,

$$\begin{aligned} &xy^* = \\ &\left(\sum_{r=0}^{m} a(\xi_1) \ldots a(\xi_r) a^*(\overline{\xi_{r+1}}) \ldots a^*(\overline{\xi_m}) \right) \left(\sum_{s=0}^{n} a(\overline{\eta_n}) \ldots a(\overline{\eta_{s+1}}) a^*(\eta_s) \ldots a^*(\eta_1). \right) \end{aligned} \tag{5.4}$$

We rename vectors by setting $(\mu_1, \ldots, \mu_{n+m}) = (\xi_1, \ldots, \xi_m, \overline{\eta_n}, \ldots, \overline{\eta_1})$. In the first equality in the next computation we collect the terms in (5.4) by separating the ones where no annihilation operator is on the left of a creation operator (first summand of (5.5)) and the ones where such a combination does occur (second summand of (5.5)). The second equation is the Wick formula

(5.1),

$$
\begin{aligned}
xy^* = & \sum_{r=0}^{n+m} a(\mu_1)\dots a(\mu_r)a^*(\overline{\mu_{r+1}})\dots a^*(\overline{\mu_{n+m}}) \\
& + \left(\sum_{r=0}^{m-1} a(\xi_1)\dots a(\xi_r)a^*(\overline{\xi_{r+1}})\dots a^*(\overline{\xi_{m-1}})\right) a^*(\overline{\xi_m})a(\overline{\eta_n}) \\
& \times \quad \left(\sum_{s=0}^{n-1} a(\overline{\eta_{n-1}})\dots a(\overline{\eta_{s+1}})a^*(\eta_s)\dots a^*(\eta_1)\right) \\
= & W(\xi_1 \otimes \dots \otimes \xi_m \otimes \overline{\eta_n} \otimes \dots \otimes \overline{\eta_1}) \\
& + \langle \overline{\eta_n}, \overline{\xi_m}\rangle_A W(\xi_1 \otimes \dots \otimes \xi_{m-1}) W(\eta_1 \otimes \dots \otimes \eta_{n-1})^*.
\end{aligned} \tag{5.5}
$$

Now we separate cases. Note that we may assume that $n, m \neq 0$ because otherwise the proposition is trivial.

Case 1: Assume $m > n > 0$. Applying the equation (5.5) inductively on the length of m we see that,

$$
xy^* = \sum_{k=0}^{n} \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}}\rangle_A W(\xi_1 \otimes \dots \otimes \xi_{m-k} \otimes \overline{\eta_{n-k}} \otimes \dots \otimes \overline{\eta_1}). \tag{5.6}
$$

Here the $k = 0$ term is understood as $W(\xi_1 \otimes \dots \otimes \xi_m \otimes \overline{\eta_n} \otimes \dots \otimes \overline{\eta_1})$. In particular as $m > n$ we see that $\mathcal{E}_\Omega^\perp(xy^*) = xy^*$ and similarly

$$
\begin{aligned}
\mathcal{E}_\Omega^\perp(H_\epsilon(xH_\epsilon(y)^*)) = & H_\epsilon(xH_\epsilon(y)^*), \\
\mathcal{E}_\Omega^\perp(H_\epsilon(yH_\epsilon(x)^*)^*) = & H_\epsilon(yH_\epsilon(x)^*)^*, \\
\mathcal{E}_\Omega^\perp(H_\epsilon(H_\epsilon(xy^*)^*)^*) = & H_\epsilon(H_\epsilon(xy^*)^*)^*.
\end{aligned}
$$

So to prove the Cotlar identity (5.3) we can ignore the projection $\mathcal{E}_\Omega^\perp$ in this case. Now for the right hand side of the Cotlar identity (5.3) we argue that we get the Equation (5.7) below. Firstly, because $H_\epsilon(y) = \epsilon_y y$ we find that $xH_\epsilon(y)^*$ equals ϵ_y times the expression (5.6). Then, as $m > n$,

$$
H_\epsilon(xH_\epsilon(y)^*) = \sum_{k=0}^{n} \epsilon_y \epsilon_x \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}}\rangle_A W(\xi_1 \otimes \dots \otimes \xi_{m-k} \otimes \overline{\eta_{n-k}} \otimes \dots \otimes \overline{\eta_1}).
$$

Secondly, $yH_\epsilon(x)^* = \epsilon_x yx^*$. Then, $yH_\epsilon(x)^*$ is ϵ_x times the adjoint of the expression (5.6). Then, we get the following two summands, where the second line appears as if $k = n$ then there is no more tensor η_1 appearing in the decomposition of $yH_\epsilon(x)^*$ in terms of Wick words,

$$
\begin{aligned}
H_\epsilon(yH_\epsilon(x)^*) = & \sum_{k=0}^{n-1} \epsilon_x \epsilon_y \prod_{l=0}^{k-1} \langle \overline{\xi_{m-l}}, \overline{\eta_{n-l}}\rangle_A W(\eta_1 \otimes \dots \otimes \eta_{n-k} \otimes \overline{\xi_{m-k}} \otimes \dots \otimes \overline{\xi_1}) \\
& + \epsilon_x \prod_{l=0}^{n-1} \langle \overline{\xi_{m-l}}, \overline{\eta_{n-l}}\rangle_A H_\epsilon\left(W(\overline{\xi_{m-n}} \otimes \dots \otimes \overline{\xi_1})\right).
\end{aligned}
$$

By a similar argument we get also that,

$$H_\epsilon(H_\epsilon(xy^*)^*) = \sum_{k=0}^{n-1} \epsilon_x \epsilon_y \prod_{l=0}^{k-1} \langle \overline{\xi_{m-l}}, \overline{\eta_{n-l}} \rangle_A W(\eta_1 \otimes \ldots \otimes \eta_{n-k} \otimes \overline{\xi_{m-k}} \otimes \ldots \otimes \overline{\xi_1}) + \epsilon_x \prod_{l=0}^{n-1} \langle \overline{\xi_{m-l}}, \overline{\eta_{n-l}} \rangle_A H_\epsilon \left(W(\overline{\xi_{m-n}} \otimes \ldots \otimes \overline{\xi_1}) \right).$$

Then,

$$\begin{aligned} & H_\epsilon(xH_\epsilon(y)^*) + H_\epsilon(yH_\epsilon(x)^*)^* - H_\epsilon(H_\epsilon(xy^*)^*)^* \\ = & \epsilon_x \epsilon_y \sum_{k=0}^{n} \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}} \rangle_A W(\xi_1 \otimes \ldots \otimes \xi_{m-k} \otimes \overline{\eta_{n-k}} \otimes \ldots \otimes \overline{\eta_1}). \end{aligned} \tag{5.7}$$

On the other hand, from (5.4) we conclude that,

$$H_\epsilon(x)H_\epsilon(y)^* = \epsilon_x \epsilon_y xy^*, \tag{5.8}$$

which equals (5.7) by (5.5).

Case 2: Assume $m = n > 0$. Because as $H_\epsilon(1) = 0$ we find the following decomposition (so the summand $k = n$ vanishes),

$$H_\epsilon(xH_\epsilon(y)^*) = \sum_{k=0}^{n-1} \epsilon_y \epsilon_x \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}} \rangle_A W(\xi_1 \otimes \ldots \otimes \xi_{m-k} \otimes \overline{\eta_{n-k}} \otimes \ldots \otimes \overline{\eta_1}).$$

Further, again as as $H_\epsilon(1) = 0$,

$$H_\epsilon(yH_\epsilon(x)^*) = \sum_{k=0}^{n-1} \epsilon_x \epsilon_y \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}} \rangle_A W(\eta_1 \otimes \ldots \otimes \eta_{n-k} \otimes \overline{\xi_{m-k}} \otimes \ldots \otimes \overline{\eta_1}),$$

and

$$H_\epsilon(H_\epsilon(xy^*)^*) = \sum_{k=0}^{n-1} \epsilon_x \epsilon_y \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}} \rangle_A W(\eta_1 \otimes \ldots \otimes \eta_{n-k} \otimes \overline{\xi_{m-k}} \otimes \ldots \otimes \overline{\xi_1}).$$

Then,

$$\begin{aligned} & H_\epsilon(xH_\epsilon(y)^*) + H_\epsilon(yH_\epsilon(x)^*)^* - H_\epsilon(H_\epsilon(xy^*)^*)^* \\ = & \epsilon_x \epsilon_y \sum_{k=0}^{n-1} \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}} \rangle_A W(\xi_1 \otimes \ldots \otimes \xi_{m-k} \otimes \overline{\eta_{n-k}} \otimes \ldots \otimes \overline{\eta_1}). \end{aligned} \tag{5.9}$$

This expression is in the range of the projection $\mathcal{E}_\Omega^\perp$. On the other hand, by (5.5) and using $n = m$ we get

$$\begin{aligned} & \mathcal{E}_\Omega^\perp \left(H_\epsilon(x)H_\epsilon(y)^* \right) = \epsilon_x \epsilon_y \mathcal{E}_\Omega^\perp \left(xy^* \right) \\ = & \epsilon_x \epsilon_y \sum_{k=0}^{n-1} \prod_{l=0}^{k-1} \langle \overline{\eta_{n-l}}, \overline{\xi_{m-l}} \rangle_A W(\xi_1 \otimes \ldots \otimes \xi_{m-k} \otimes \overline{\eta_{n-k}} \otimes \ldots \otimes \overline{\eta_1}), \end{aligned} \tag{5.10}$$

which concludes the proof of Case 2 as this equals (5.9).

Case 3: Assume $n > m$. The proof can be obtained by a mutatis mutandis copy of Case 1. We sketch a second way to finish the proof. By Case 1:

$$H_\epsilon(y)H_\epsilon(x)^* = H_\epsilon(yH_\epsilon(x)^*) + H_\epsilon(xH_\epsilon(y)^*)^* - H_\epsilon(H_\epsilon(yx^*)^*)^*. \tag{5.11}$$

Then one verifies that $H_\epsilon(H_\epsilon(yx^*)^*) = H_\epsilon(H_\epsilon(xy^*)^*)^*$. So that taking adjoints of (5.11) we see,

$$H_\epsilon(x)H_\epsilon(y)^* = H_\epsilon(yH_\epsilon(x)^*)^* + H_\epsilon(xH_\epsilon(y)^*) - H_\epsilon(H_\epsilon(xy^*)^*)^*.$$

□

We write D for the operator D_{φ_Ω}.

Lemma 5.5. *For $x \in \mathcal{A}$ we have that*

$$\varphi_\Omega(H_\epsilon(x)^*H_\epsilon(x)) = \varphi_\Omega(x^*x).$$

So certainly for every $1 \leq p < \infty$ we get that $\| D^{\frac{1}{2p}} \varphi_\Omega(H_\epsilon(x)^*H_\epsilon(x)) D^{\frac{1}{2p}} \|_p = \| D^{\frac{1}{2p}} \varphi_\Omega(x^*x) D^{\frac{1}{2p}} \|_p$.

Proof. As x is in the algebra $\mathcal{A}$ we may take a decomposition $x = x^+ + x^-$ with $x^\pm$ in the linear span of Wick operators $W(\xi_1 \otimes \ldots \otimes \xi_n), \xi_1 \in \mathcal{H}_\mathbb{C}^\pm$. We have

$$\varphi_\Omega(H_\epsilon(x)^*H_\epsilon(x)) = \langle x^+\Omega, x^+\Omega\rangle + \langle x^-\Omega, x^-\Omega\rangle - \langle x^+\Omega, x^-\Omega\rangle - \langle x^-\Omega, x^+\Omega\rangle.$$

As $\mathcal{H}_\mathbb{C}^+$ and $\mathcal{H}_\mathbb{C}^-$ are orthogonal for the inner product of $\mathcal{H}$ we find that

$$\varphi_\Omega(H_\epsilon(x)^*H_\epsilon(x)) = \langle x^+\Omega, x^+\Omega\rangle + \langle x^-\Omega, x^-\Omega\rangle = \varphi_\Omega(x^*x).$$

□

Theorem 5.6. *For every $1 < p < \infty$ and every choice of $\epsilon = (\mathcal{H}_\mathbb{C}^+, \mathcal{H}_\mathbb{C}^-)$ as in Definition 5.2 such that A^{it} leaves $\mathcal{H}_\mathbb{C}^\pm$ invariant for all $t \in \mathbb{R}$ the map H_ϵ extends to a bounded map $L_p(\mathcal{M}) \to L_p(\mathcal{M})$ that is determined by*

$$H_\epsilon : D^{\frac{1}{2p}} x D^{\frac{1}{2p}} \mapsto D^{\frac{1}{2p}} H_\epsilon(x) D^{\frac{1}{2p}}, \qquad x \in \mathcal{A}. \tag{5.12}$$

Moreover, let c_p be the norm of (5.12). Then for $p \geq 2$ a power of 2 we have $c_p \leq p^{\gamma/2}$ with $\gamma = {}^3\log(2)$. Further, for $C = 2^{\gamma/2}$ we have $c_p \leq C p^{\gamma/2}$ for $p \geq 2$ arbitrary.

Proof. For $p = 2$ the map H_ϵ defines a contraction on $L_2(\mathcal{M})$ and so the statement is true.

The space $D^{\frac{1}{2p}} \mathcal{A} D^{\frac{1}{2p}}$ is dense in $L_p(\mathcal{M})$. As $H_\epsilon : \mathcal{A} \to \mathcal{A}$ commutes with the modular automorphism group of φ_Ω, c.f. Remark 5.3, it follows from a computation similar to (2.2) that,

$$H_\epsilon(D^{\frac{1}{p}} x) = D^{\frac{1}{p}} H_\epsilon(x). \tag{5.13}$$

We now estimate c_{2p} in terms of c_p. By Cotlar's identity (5.3) and Lemma 5.5 we get that,

$$\begin{aligned}
&\| D^{\frac{1}{2p}} H_\epsilon(x) \|_{2p}^2 \\
=&\| D^{\frac{1}{2p}} H_\epsilon(x) H_\epsilon(x)^* D^{\frac{1}{2p}} \|_p \\
\leq&\| D^{\frac{1}{2p}} \mathcal{E}_\Omega^\perp \left(H_\epsilon(x) H_\epsilon(x)^* \right) D^{\frac{1}{2p}} \|_p + \| D^{\frac{1}{2p}} \varphi_\Omega \left(H_\epsilon(x) H_\epsilon(x)^* \right) D^{\frac{1}{2p}} \|_p \\
\leq&\| D^{\frac{1}{2p}} H_\epsilon(x H_\epsilon(x)^*) D^{\frac{1}{2p}} \|_p + \| D^{\frac{1}{2p}} H_\epsilon(x H_\epsilon(x)^*)^* D^{\frac{1}{2p}} \|_p \\
&\qquad + \| D^{\frac{1}{2p}} H_\epsilon(H_\epsilon(xx^*)^*)^* D^{\frac{1}{2p}} \|_p + \| D^{\frac{1}{2p}} \varphi_\Omega \left(xx^* \right) D^{\frac{1}{2p}} \|_{2p} \\
\leq& c_p \| D^{\frac{1}{2p}} x \|_{2p} \| H_\epsilon(x)^* D^{\frac{1}{2p}} \|_{2p} + c_p \| D^{\frac{1}{2p}} x \|_{2p} \| H_\epsilon(x)^* D^{\frac{1}{2p}} \|_{2p} \\
&\qquad + c_p^2 \| D^{\frac{1}{2p}} x \|_{2p}^2 + \| D^{\frac{1}{2p}} x \|_{2p}^2 \\
=& 2 c_p \| D^{\frac{1}{2p}} x \|_{2p} \| H_\epsilon(x)^* D^{\frac{1}{2p}} \|_{2p} + (c_p^2 + 1) \| D^{\frac{1}{2p}} x \|_{2p}^2 .
\end{aligned}$$

By density we conclude that $c_{2p} \leq c_p + \sqrt{2c_p^2 + 1}$. In particular $c_{2p} \leq (1+\sqrt{2}) c_p$ from which it follows that for p a power of 2 we get that $c_p \leq p^\gamma$ with $\gamma = \frac{\log(2)}{\log(1+\sqrt{3})}$. For other $p \geq 2$ the result follows by interpolation, see [BeLö76], [Ter82]. □

Remark 5.7. We do not know what the optimal constants are for the norm of H_ϵ on $L_p(\mathcal{M})$. We also leave it as an open question whether the Hilbert transform is a bounded map $L_\infty \to$ BMO or even BMO $\to$ BMO as for the classical Hilbert transform.

5.2. **Khintchine type BMO inequalities and multipliers.** We provide examples of $L_\infty \to$ BMO-multipliers on free Araki-Woods factors. Earlier results on non-commutative $L_\infty \to$ BMO-multipliers in the tracial setting were obtained by Mei [Mei17] but here we do not need to appeal to lacunary sets. We use the Markov semi-group $\mathcal{S} = (\Psi_t = \Phi_{e^{-t}})_{t \geq 0}$ that is determined by

$$\Phi_r : W(\xi_1 \otimes \ldots \otimes \xi_n) \mapsto r^n W(\xi_1 \otimes \ldots \otimes \xi_n), \qquad 0 \leq r \leq 1. \tag{5.14}$$

This semi-group is well-known to be Markov, KMS-symmetric and φ_Ω-modular.

Proposition 5.8. *Suppose that $\mathcal{H}$ is infinite dimensional. Let $(e_k)_k$ be a set of vectors in $\mathcal{H}_\mathbb{C}$ that are orthogonal in $\mathcal{H}$. For $i = (i_1, \ldots, i_n)$ a multi-index set $e_i = e_{i_1} \otimes \ldots \otimes e_{i_n}$ and $\overline{e}_i = \overline{e}_{i_1} \otimes \ldots \otimes \overline{e}_{i_n}$. Take F a set of multi-indices such that $\langle e_{i_1}, e_{j_1} \rangle_A = \langle \overline{e}_{i_1}, \overline{e}_{j_1} \rangle_A = 0$ if $i \neq j$. We have, for any $x = \sum_{i \in F} c_i W(e_i)$ with $c_i \in \mathbb{C}$ a finite sum of Wick operators whose frequency support lies in F, that,*

$$\| x \|_{\mathrm{bmo}_{\mathcal{S}}}^2 \leq 2 \max \left\{ \| \sum_i |c_i|^2 W(e_i)^* W(e_i) \|, \| \sum_i |c_i|^2 W(e_i) W(e_i)^* \| \right\}.$$

Proof. Using the definition of x and the triangle inequality,

$$\begin{aligned}
&\| \Phi_r(x)^* \Phi_r(x) - \Phi_r(x^* x) \| \\
\leq&\| \sum_{i=j} \overline{c_i} c_j (r^{|i|+|j|} - \Phi_r)(W(e_i)^* W(e_j)) \| + \| \sum_{i \neq j} \overline{c_i} c_j (r^{|i|+|j|} - \Phi_r)(W(e_i)^* W(e_j)) \|
\end{aligned} \tag{5.15}$$

If $i \neq j$ we get that $\langle e_{i_1}, e_{j_1} \rangle_A = 0$, so that by (5.5) we see that $W(e_i)^* W(e_j) = W(e_i^* \otimes e_j)$ where $e_i^* = \overline{e}_{i_n} \otimes \ldots \otimes \overline{e}_{i_1}$ so that,

$$(r^{|i|+|j|} - \Phi_r)(W(e_i)^* W(e_j)) = (r^{|i|+|j|} - r^{|i|+|j|})(W(e_i^* \otimes e_j)) = 0.$$

Therefore we continue (5.15) by using that Φ_r is a ucp map and that the expression $\sum_i |c_i|^2 W(e_i)^* W(e_i)$ is a summation of positive elements,

$$\begin{aligned}\|\Phi_r(x)^*\Phi_r(x) - \Phi_r(x^*x)\| =& \| \sum_i |c_i|^2 (r^{2|i|} - \Phi_r)(W(e_i)^* W(e_i))\| \\ \leq& 2\| \sum_i |c_i|^2 (W(e_i)^* W(e_i))\|.\end{aligned}$$

This shows that

$$\|x\|_{\mathrm{bmo}^c}^2 \leq 2\| \sum_i |c_i|^2 W(e_i)^* W(e_i)\|.$$

Then in the same way using the orthogonality $\langle \overline{e}_{i_1}, \overline{e}_{j_1}\rangle = 0, i \neq j$ we get that $\|x\|_{\mathrm{bmo}^r}^2 = \|x^*\|_{\mathrm{bmo}^c}^2 \leq 2\| \sum_i |c_i|^2 W(e_i) W(e_i)^*\|$. ☐

Let δ_F be the indicator function on a set F. The following Corollary 5.9 is a consequence of our interpolation result of Theorem 3.15. We call n the length of a multi-index $i = (i_1, \ldots, i_n)$.

Corollary 5.9. *Take F a set of multi-indices of length at most n such that $\langle e_{i_1}, e_{j_1}\rangle_A = \langle \overline{e}_{i_1}, \overline{e}_{j_1}\rangle_A = 0$ if $i \neq j$. The projection $P_F : W(e_i) \mapsto \delta_F(i) W(e_i)$ extends to a bounded map $\mathcal{M} \to \mathrm{bmo}_{\mathcal{S}}(\mathcal{M})$. Consequently, P_F determines a bounded map $P_F^{(p)} : L_p(\mathcal{M}) \to L_p(\mathcal{M})$ given by $P_F^{(p)} : D^{\frac{1}{2p}} x D^{\frac{1}{2p}} \mapsto D^{\frac{1}{2p}} P_F(x) D^{\frac{1}{2p}}$.*

Proof. Let $x = \sum_i c_i W(e_i) \in \mathcal{A}$. As for any $i \in F$ its length is bounded by n we have that $\|W(e_i)^* W(e_i)\| \leq C$ for some constant C. We get that $\sum_{i\in F} |c_i|^2 \|W(e_i)^* W(e_i)\| \leq C \sum_{i \in F} |c_i|^2 = \|x\|_2^2$. Proposition 5.8 shows therefore that we get the first inequality in

$$\|P_F(x)\|_{\mathrm{bmo}_{\mathcal{S}}} \leq \sqrt{2} \|P_F(x)\|_2 \leq \sqrt{2}\|x\|_2 \leq \sqrt{2}\|x\|_\infty.$$

In Section 5.3 we show that $(\Psi_t)_{t\geq 0}$ has a Markov dilation with a.u. continuous path. We then get $L_p \to L_p$ boundedness of P_F by interpolation, see Theorem 3.15. ☐

5.3. **A Markov dilation for the radial semi-group of free Araki-Woods factors.** In this Section we show that the radial semi-group on free Araki-Woods factors has a good reversed Markov dilation. The first step in the proof of Proposition 5.10 is due to Ricard (see the final remarks of [Ric08]). We need to find a suitable analogue for semi-groups which we do by an ultraproduct argument. Similar techniques were used in [Arh16], [Arh17] though in this case through quantization we can give a shorter argument directly on the Hilbert space level, see also the comment below this proposition.

Proposition 5.10. *For $t \geq 0$ consider the Markov semi-group $\Psi_t = \Phi_{e^{-t}}$ where $\Phi_r, 0 < r \leq 1$ is the Markov map on $\mathcal{M}$ determined by (5.14). Ψ_t admits a φ_Ω-modular Markov dilation with a.u. continuous path.*

Proof. For $t \geq 0$. Set $T_t \in B(\mathcal{H})$ by $T_t \xi = e^{-t}\xi$. The proof splits in steps.

Step 1: Constructing a dilation for subsemi-groups of $(T_t)_{t\geq 0}$. Firstly for each $t \geq 0$ we may find a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ with orthogonal projection $P_{\mathcal{H}} : \mathcal{K}_t \to \mathcal{H}$ and a unitary $U_t \in B(\mathcal{K})$ such that for every $l \in \mathbb{N}$,

$$P_{\mathcal{H}} U_t^l|_{\mathcal{H}} = T_t^l = T_{tl}. \tag{5.16}$$

Indeed, the Hilbert space $\mathcal{K} = \ell_2(\mathbb{Z}) \otimes \mathcal{H}$ with unitary $U_t, t = -\log(r)$ acting on the first tensor leg by

$$\left(\begin{array}{cccccccccccccc}
\ddots & \vdots & \vdots & & & & & & & & & & & \\
\ldots & 1 & 0 & 0 & \ldots & & & & & & & & & \\
\ldots & 0 & 1 & 0 & \ldots & & & & & & & & & \\
\ldots & 0 & 0 & 1 & \ldots & & & & & & & & & \\
 & \vdots & \vdots & \vdots & \ddots & & & & & & & & & \\
 & & & & \ldots & 1 & 0 & 0 & 0 & \ldots & & & & \\
 & & & & \ldots & 0 & \sqrt{1-r^2} & r & 0 & \ldots & & & & \\
 & & & & \ldots & 0 & r & \sqrt{1-r^2} & 0 & \ldots & & & & \\
 & & & & \ldots & 0 & 0 & 0 & 1 & \ldots & & & & \\
 & & & & & & & & \ddots & \vdots & \vdots & & & \\
 & & & & & & & & \ldots & 1 & 0 & 0 & \ldots & \\
 & & & & & & & & \ldots & 0 & 1 & 0 & \ldots & \\
 & & & & & & & & \ldots & 0 & 0 & 1 & \ldots & \\
 & & & & & & & & & \vdots & \vdots & \vdots & \ddots &
\end{array} \right),$$

where the bottom left entry r is located at position $(0,0)$. So U_t acts as a shift operator on $\ell_2(\mathbb{Z}\backslash\{0,1\}) \otimes \mathcal{H}$. $\mathcal{H}$ is a subspace of $\mathcal{K}$ by the embedding

$$J : \xi \mapsto \delta_0 \otimes \xi \in \ell_2(\mathbb{Z}) \otimes \mathcal{H} = \ell_2(\mathbb{Z}, \mathcal{H}).$$

(5.16) is then elementary to check (see also [Pis96, Theorem 1.1]). We let $P_{t,n}$ be the orthogonal projection onto the closed linear span of $\left\{ U_t^l \xi \mid l \geq n, \xi \in \mathcal{H} \right\}$. We get that for $\xi, \eta \in \mathcal{H}$ we have for $l, n \geq 0$,

$$\langle \xi, U_t^{n+l} \eta \rangle = \langle \xi, P_{\mathcal{H}} U_t^{n+l} P_{\mathcal{H}} \eta \rangle = \langle T_t^{n+l} \xi, \eta \rangle = \langle T_t^n \xi, U_t^l \eta \rangle = \langle U_t^n T_t^n \xi, U_t^{n+l} \eta \rangle. \tag{5.17}$$

Which shows that $P_{t,n} P_{\mathcal{H}} \xi = U_t^n T_t^n \xi$. Moreover from (5.17) we get for $n \geq k$ that $\langle U_t^k \xi, U_t^{n+l} \eta \rangle = \langle U_t^n T_t^{n-k} \xi, U_t^{n+l} \eta \rangle$. So we find that $P_{t,n} U_t^k \xi = U_t^n T_t^{n-k} \xi$. So if we put $J_{t,n} = U_t^n J : \mathcal{H} \to \mathcal{K}$ we get that

$$P_{t,n} J_{t,k} = J_{t,n} T_t^{n-k}, \qquad n \leq k. \tag{5.18}$$

This is a discrete Hilbert space version of the reversed Markov dilation property (2.5).

Step 2: Constructing a Markov dilation. We shall now construct a continuous version of (5.18). To do so, for $t \geq 0$ let $\mathcal{K}_t := \mathcal{K}$ be the Hilbert space as in the previous paragraph and let $J_{t,n} : \mathcal{H} \to \mathcal{K}_t$ be the injection as before. Also let $P_{t,n}$ and U_t be as before.

Set groups $\mathsf{G}_m = 2^{-m}\mathbb{Z}$ and $\mathsf{G} = \cup_{m \geq 1} \mathsf{G}_m$. The group G is understood as a topological group with the Euclidian topology inherited from $\mathbb{R}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Consider $\mathcal{K}_{\mathcal{U}} = \prod_{m,\mathcal{U}} \mathcal{K}_{2^{-m}}$. Let $K : \mathcal{H} \to \mathcal{K}_{\mathcal{U}}$ be the embedding sending ξ to the constant family $(\xi)_{\mathcal{U}}$. Let $P_{\mathcal{H}} = K^*$ be the projection onto $\mathcal{H}$. For $t \in \mathsf{G}$ we define the unitary $V_{t,m}$ on $\mathcal{K}_{2^{-m}}$ by

$$V_{t,m} = \begin{cases} U_{2^{-m}}^{t 2^m} & \text{if } t \in \mathsf{G}_m, \\ \mathrm{Id}_{\mathcal{K}_{2^{-m}}} & \text{otherwise.} \end{cases}$$

Then for $t \in \mathsf{G}$ set $V_t = (V_{t,m})_{\mathcal{U}}$ which is a unitary on $\mathcal{K}_{\mathcal{U}}$. We claim that the assignment

$$\mathsf{G} \ni t \mapsto V_t P_{\mathcal{K}} \tag{5.19}$$

is strong-$*$ continuous. Indeed, let $\xi \in \mathcal{H}$ be a unit vector. Let $t, s \in \mathsf{G}$ and assume that $s \geq t \geq 0$. Then let M be such that for any $m > M$ we have $s, t \in \mathsf{G}_m$. Fix such $m > M$. We get that

$$\begin{aligned} U_{2^{-m}}^{t2^m} \xi =& (0, \ldots, 0, \sqrt{1-r^2}\xi, \sqrt{1-r^2} r \xi, \ldots, \\ & \ldots, \sqrt{1-r^2} r^{t2^m-3}\xi, \sqrt{1-r^2} r^{t2^m-2}\xi, \sqrt{1-r^2} r^{t2^m-1}\xi, r^{t2^m}\xi, 0, 0, \ldots). \end{aligned}$$

Recalling $r = e^{-2^{-m}}$ this shows that we get from a small elementary computation,

$$\begin{aligned} & \| (U_{2^{-m}}^{s2^m} - U_{2^{-m}}^{t2^m})\xi \|_2^2 \\ =& (r^{s2^m} - r^{t2^m})^2 + (1-r^2) \sum_{l=1}^{t2^m} (r^{s2^m - l} - r^{t2^m - l})^2 + (1-r^2) \sum_{l=t2^m+1}^{s2^m} (r^{s2^m-l})^2 \\ =& (e^{-2s} - e^{-2t})^2 + (e^{-2s} - e^{-2r})^2 (e^{-2t} - 1) + e^{-2(s-t)} (e^{-2(s-t)} - 1), \end{aligned} \tag{5.20}$$

which converges to 0 as $s \to t$. This shows that the unitary group $t \mapsto V_t P_{\mathcal{H}}$ is strong-$*$ continuous. We extend (5.19) to a strongly continuous map $\mathbb{R} \ni t \to V_t P_{\mathcal{H}}$. This shows that we get an isometric embedding for every $t \in \mathbb{R}$,

$$K_t : \mathcal{H} \to \mathcal{K}_{\mathcal{U}} : \xi \mapsto V_t K \xi.$$

For $t \in \mathsf{G}$ and $m \in \mathbb{N}_{\geq 1}$ we define

$$Q_{t,m} = \begin{cases} P_{2^{-m}, s2^m} & \text{if } t \in \mathsf{G}_m, \\ 0 & \text{otherwise.} \end{cases}$$

Then set $Q_t = (Q_{t,m})_{m,\mathcal{U}}$. We claim that the mapping $\mathsf{G} \ni t \mapsto Q_t$ is decreasing and strongly continuous. Indeed we have for $t \in \mathsf{G}_m$ that $P_{2^{-m}, t2^m} = P_{2^{-m},0} U_{2^{-m}}^{-t2^m}$. Set $P = (P_{2^{-m},0})_\omega$. So that for $t \in \mathsf{G}$ we have $Q_t = P V_t^*$. A computation similar to (2.2) shows that the function $\mathsf{G} \ni t \mapsto P V_t^*$ is weakly continuous. But as Q_t is decreasing this convergence actually holds in the strong topology (see [Mur90, Theorem 4.1.1]) and by self-adjointness in the strong-$*$-topology. Therefore we obtain a decreasing strong-$*$ continuous map $\mathbb{R} \ni t \mapsto Q_t$.

For $s, t \in \mathsf{G}, s \geq t$ and any m large such that $s, t \in \mathsf{G}_m$. We get that for $\xi \in \mathcal{H}$,

$$Q_{s,m} V_{t,m} \xi = P_{2^{-m}, s2^m} U_{2^{-m}}^{t2^m} \xi = U_{2^{-m}}^{s2^m} T_{2^{-m}}^{(s-t)2^m} \xi = V_{s,m} T_{s-t} \xi.$$

This shows that for $s, t \in \mathsf{G}, s \geq t$ we get that $Q_s V_t J = V_s J T_{s-t}$. By strong continuity we get $Q_s V_t J = V_s J T_{s-t}$ for all $s \geq t \geq 0$. So by definition

$$Q_s J_t = J_s T_{s-t} \qquad \text{for all } s \geq t \geq 0. \tag{5.21}$$

We finish the proof by quantization. Let $(V_t^{\mathcal{K}})_{t \in \mathbb{R}} = (\mathrm{Id}_{B(\ell_2(\mathbb{Z}))} \otimes V_t)_{t \in \mathbb{R}}$ be the orthogonal transformation group on $\mathcal{K}_{\mathbb{R}} = \ell_2(\mathbb{Z}) \otimes \mathcal{H}_{\mathbb{R}}$. We set $\mathcal{N} = \Gamma(\mathcal{K}, (V_t^{\mathcal{K}})_{t \in \mathbb{R}})$ and $\mathcal{N}_s = \Gamma(Q_s \mathcal{K}), s \geq 0$. By second quantization we get a conditional expectation $\mathcal{E}_s := \Gamma(Q_s) : \mathcal{N} \to \mathcal{N}_s$ and a normal injective $*$-homomorphism $\pi_s = \Gamma(J_s)$. By (5.21) they satisfy

$$\mathcal{E}_s \circ \pi_t = \pi_s \circ \Psi_{s-t}, \qquad 0 \leq t \leq s.$$

It is clear from Remark 5.3 that this dilation is modular.

Step 3: A.u. continuity. Suppose that ξ_i is a net of vectors in $\mathcal{H}$ converging in norm to $\xi \in \mathcal{H}$. Then we have for creation operators $a(\xi_t) \to a(\xi)$ in norm. By the Wick Theorem 5.1 we see that $W(\xi_t) \to W(\xi)$ in norm. Now for $x = W(\xi) \in \mathcal{A}, \xi \in \mathcal{H}$ the martingale $m_t(x) = \mathcal{E}_t(\pi_t(x)) = W(Q_t J_t \xi)$ is norm continuous.

□

Remark 5.11. Proposition 5.10 could potentially also be derived from a suitable analogue of [NFBK10, Theorem 7.1], provided that in this theorem one can keep track of the location of a specified real Hilbert subspace.

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TU Delft, EWI/DIAM, P.O.Box 5031, 2600 GA Delft, The Netherlands
E-mail address: m.p.t.caspers@tudelft.nl