

Initial Orbit Determination Using Angle Measurements

Comparing IOD methods on very short arc
observations of GEO objects from the
MeerLICHT telescope

by

Bert Van den Abbeele

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Student number:	4451597	
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Thesis committee:	Dr. ir. Ernst Schrama,	TU Delft, supervisor & chair
	Dr. Thomas Wijnen,	TU Delft, supervisor
	Dr. Alessandra Menicucci	TU Delft, examiner
	Dr. Marco Langbroek	Leiden University, external committee member

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Preface

There is a need for automated tracking of objects in Earth orbit. The Royal Netherlands Air Force (RNLAf) commissioned a study to track satellites in low Earth orbit (LEO) using optical telescopes. Now, in a follow up study, the RNLAf wants to assess the suitability of the BlackGEM telescopes for orbit determination and tracking of satellites in high orbits around Earth. An array of BlackGEM telescopes is currently being constructed in Chile. This thesis will compare three initial orbit determination (IOD) methods on very short arc data of objects in geostationary Earth orbit (GEO) obtained from the MeerLICHT telescope, a prototype of the BlackGEM telescopes.

This report compares the performance of the methods of Gauss and Gooding and a newly developed circular orbit method on data from the MeerLICHT telescope with observation arcs of 1 to 5 minutes. Next to that, two techniques to reduce the measurement error that is specific to the used system are developed and analysed.

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*Bert Van den Abbeele
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List of Abbreviations

AOP	Argument of Pericentre
CCD	Charge-Coupled Device
FITS	Flexible Image Transport System
GEO	Geostationary Earth Orbit
GTO	Geostationary Transfer Orbit
IOD	Initial Orbit Determination
LEO	Low Earth Orbit
RA	Right Ascension
RAAN	Right Ascension of the Ascending Node
RNLAF	Royal Netherlands Air Force
TA	True Anomaly
TLE	Two-Line Element

List of Symbols

α	Reciprocal of the Semi-Major Axis
α	Right Ascension
Δt	Time Step
δ	Declination
θ	Local Sidereal Time
θ	True Anomaly
λ	Longitude
μ	Gravitational Parameter
ρ	Slant Range Between Observer and Target Object
$\hat{\rho}$	Line of Sight Vector (of Unit Length) of the Observations
χ	Universal Anomaly
ϕ	Latitude
Ω	Right Ascension of the Ascending Node
ω	Argument of Pericentre
ω	Angular Velocity
a	Semi-Major Axis
$C(z)$	Stumpff Function
e	Eccentricity
f, g	Lagrange Coefficients
f	Earth Flattening Factor
h	Elevation
h	Magnitude of Angular Momentum
$\hat{h}$	Direction of the Angular Momentum
$\vec{h}$	Angular Momentum Vector
i	Inclination
M	Mean Anomaly
$\vec{R}$	Position Vector of the Observer
$\vec{r}$	Position Vector of the Target Object
r_e	Equatorial Radius of Earth
$S(z)$	Stumpff Function
$\vec{v}$	Velocity Vector of the Target Object

Introduction

With the ever increasing satellite population in orbit around Earth, it remains very important to be able to pick up new objects and include them in the catalogue as fast as possible. The Royal Netherlands Air Force (RNLAf) has commissioned a study regarding detection and tracking of new (classified) objects. In a first phase, the Feasibility study On Optical Tracking of Orbiting Satellites (FOTOS) focused on objects in low-Earth orbit (LEO) using an all-sky camera [34, 35]. Now, this study is extended for objects in geostationary orbits (GEO) using data from the MeerLICHT telescope.

After a literature study, several research questions were formulated, which are listed below. The goal was to identify the best performing IOD method and use it to generate orbit estimates of all observations. Orbit estimates of the same object would then be linked together by a to be developed algorithm.

- I. Using simulated observations, which initial orbit determination proves most suitable for linking tracks of satellites in geostationary orbits, geostationary transfer orbits and/or high Earth orbits?
 - i. Which initial orbit determination method achieves the highest accuracy in estimating the satellite state vector?
 - ii. Which initial orbit determination method is the most robust against modelled measurement errors in the simulated orbit?
 - iii. For each initial orbit determination method, what is the effect of the time between orbit estimates on the ability to link them together?
- II. Using real observations, which initial orbit determination methods are able to successfully link tracks of the same satellite?

Over the course of this thesis, the research goals have been adapted. The development of a linking algorithm was abandoned in favour of analysing the newly developed circular orbit method, as well as a more elaborate study of the effect of the measurement reduction techniques that were developed. Now, this thesis will build towards a set of recommendations to optimise the MeerLICHT or BlackGEM telescope system for satellite orbit determination in GEO that can be extended to general high Earth orbits.

Observations are available from the MeerLICHT telescope in Sutherland, South Africa. The telescope is a prototype of the BlackGEM telescope, an array of which is being constructed in Chile at the time of writing [5]. The methods of Gauss [10], Gooding [21] and a new method, derived as part of this thesis, calculating a circular orbit will be compared on the basis of simulations using two line element (TLE) data, as well as real observations of very short arcs between 1 and 5 minutes. The former two methods use three line of sight observations to determine the orbit, the circular orbit method requires only two. In addition, two error reduction techniques specific to the MeerLICHT telescope and data reduction routine from the FOTOS study are implemented to test their effect on the orbit error. One method aims to reduce the error on the determination of the endpoints of the streak by the FOTOS method, the other uses a model of the motion of the shutter wheel of the telescope to increase the accuracy of the timestamp of the observations. This report will formulate recommendations on the use of IOD methods on data from the MeerLICHT telescope as well as the measurement error reduction

techniques to implement to arrive at a robust observation strategy to detect and recover objects in GEO that can be extended to other high Earth orbits.

This report will start with an in-depth discussion of the IOD methods used. Next, the different observation errors will be discussed and the reduction measures explained. In a separate chapter, the IOD methods will be validated. Then, simulations using TLE sets provided by the 18th Space Control Squadron [1] will be used to assess the inherent error of the IOD methods and their response to input error. Representative error magnitudes will provide insight in the theoretical limits of the MeerLICHT system and the improvement that can be expected from the error reduction measures. Then, the effect of the error reduction will be investigated on the observation data. Based on these results, a decision will be made on which error reductions to include in the analysis of the results of the observation data. The performance of the IOD methods will be compared in terms of accuracy at the time of the observations, as well as after propagation of the orbit estimate to another epoch where observations are available. The latter does not depend on any errors in the TLE data and provides immediate insight into the capability of each method to predict future positions for obtaining additional observations.

Methods of Initial Orbit Determination

The problem of initial orbit determination is to estimate the orbit of a tracked object using a limited number of measurements. With an adequately accurate initial state estimate, an object's position at a future epoch can be predicted and additional observations can be obtained at that epoch. With that new information, the object's orbit can be refined using more accurate orbit determination methods which require more observations. They can take into account perturbations such as Earth's gravity field, third body gravitation and solar radiation pressure, and allow the estimation of constants such as the ballistic coefficient. Examples of these are a least squares fit or Kalman filter.

The problem of initial orbit determination is old and has been studied extensively. The first methods to be developed were the ones by Laplace in 1780 [4] and Gauss a few years later [23]. Since then, numerous new methods have been developed and a wide range of improvements have been proposed to existing methods. Escobal [14] developed the double r iteration method, which was the first to be specifically aimed at efficient evaluation by a computer. Later, Gooding [21] presented a method based on Escobal [14] which solves Lambert's problem, obtaining an orbit from two position vectors and the travel time between them [10, 19, 26], on the first and last observation using guesses for the slant ranges at those points. The solution propagated to the second observation then serves as a measure for the error. Modifications to the Gooding algorithm were proposed by Henderson and Mortari [24], who further refined the final outcome with a genetic algorithm and expanded Gooding's original method to allow more observations to be used. Ansalone and Curti [2] and Hinagawa et al. [25] developed methods bearing a strong resemblance to Gooding [21], using genetic algorithms to produce estimates for the slant ranges at the first and last observations. DeMars et al. [12] uses an admissible region defined by a combination of range and range rate ensuring a bound orbit around Earth, in which he creates a grid of hypothetical solutions, all of which are propagated to subsequent observations until only the best fit remains. Karimi and Mortari [28] developed several new methods based on the classical algorithms and investigate the optimal number of observations that should be used. Armellin et al. [3] also makes use of Lambert's problem. They generate a solution on the first and second, and second and third observation and require the velocity vector of both solutions to coincide at the middle observation. Feng et al. [15] model data from spacecraft star sensors containing accidental satellite observations and propose a machine learning method for initial orbit determination. Finally, Danchick [11] adapts the classical Laplace method for space based observations and combines several improvements to the method from other authors, reducing the inherent error in the original method and providing a new algorithm to identify the correct solution if multiple ones exist.

The methods in this report have been selected because they require a minimum number of observations. The minimum number of observations for calculating a Keplerian orbit is three. Multiple methods exist using many more measurements and determining the best fitting orbit solution through those measurements, including most of the methods listed above. Given the sparsity of data obtained from the MeerLICHT telescope, these methods were deemed inappropriate. It must be noted that this decision severely limits the number of candidate IOD methods. In the end, the methods of Gauss [23] and Gooding [20, 21] were chosen. The former was selected since it was also used for the FOTOS study and can be used to compare results. In addition, many references use the method of Gauss to benchmark

new methods, which makes it easier to confirm obtained results with literature. Gooding's method was selected because it uses less assumptions than the methods of Gauss and Laplace. Finally, a new circular orbit method with a minimum of assumptions was developed. This was done because a circular orbit is often a good initial estimate and this is especially the case for GEO satellites. The circular orbit method only estimates four out of six Kepler elements, since the eccentricity is 0 by definition and it follows that the argument of pericentre is undefined. As a result, a circular orbit method only requires two observations of right ascension and declination to formulate a solution instead of three.

This chapter will discuss the theory behind the methods of initial orbit determination used in the analysis in this report. An adequate understanding of the theory is essential for a proper discussion of their shortcomings and performance. First, [section 2.2](#) will deal with the method of Gauss. Next, Gooding's method will be discussed in [section 2.3](#). Finally, the new circular orbit method will be detailed in [section 2.4](#). First, [section 2.1](#) will cover the calculation of the observer position in the inertial frame from geodesic coordinates and the line of sight vector of observations from right ascension and declination.

2.1. Observer Position and Line of Sight Vector

The position of MeerLICHT is given in the header of every FITS image file and is provided in geodetic coordinates in [Table 2.1](#). Instead of the longitude of the observer, the local sidereal time θ is used, which is the angle from the direction of the vernal equinox to the observer and defines the position of the observer with respect to the stars. The observer position can be calculated from local sidereal time and geodetic latitude using [Equation 2.1](#).

$$\vec{R} = \left[\frac{r_e}{\sqrt{1 - (2f - f^2) \sin^2 \phi}} + h \right] \cos \phi (\cos \theta \hat{i} + \sin \theta \hat{j}) + \left[\frac{r_e(1 - f)^2}{\sqrt{1 - (2f - f^2) \sin^2 \phi}} + h \right] \sin \theta \hat{k} \quad (2.1)$$

where r_e is the equatorial radius of Earth; f is the flattening factor of Earth; $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the axes of the coordinate frame and θ is the local sidereal time of the observation site.

The unit vector pointing from the observer to the object is a direction cosine vector and is calculated using [Equation 2.2](#).

$$\hat{\rho} = \cos \delta \cos \alpha \hat{i} + \cos \delta \sin \alpha \hat{j} + \sin \delta \hat{k} \quad (2.2)$$

where α is the right ascension and δ is the declination.

The observer position and the line of sight vector of the observations are the only known variables together with the time between observations. The IOD methods discussed in this chapter start from this information to generate a state vector at one of the observations.

Table 2.1: MeerLICHT telescope position.

Variable	Value
Latitude, ϕ	-32.3799 [°]
Longitude, λ	20.8112 [°]
Elevation, h	1803.0 [m]

2.2. Method of Gauss

The classical method developed by Gauss to determine the orbit of Ceres around the Sun has been adapted to accommodate orbit determination of Earth orbiting objects [23]. The method as described in Curtis [10] has been used in the FOTOS study for initial orbit determination.

The method of Gauss requires three measurements of right ascension and declination to calculate the position at each observation and a full state vector at the middle observation time. This is different from the other classical method of Laplace, which uses the first and third observation to generate derivatives to the line of sight vector at the second observation. As such, the position vector at the first and third observations is not immediately available from the Laplace method.

2.2.1. Calculation

The method of Gauss starts from Equation 2.3 to calculate an object's position at three observation times from three line of sight vectors.

$$\begin{aligned}\vec{r}_1 &= \vec{R}_1 + \rho_1 \hat{\rho}_1 \\ \vec{r}_2 &= \vec{R}_2 + \rho_2 \hat{\rho}_2 \\ \vec{r}_3 &= \vec{R}_3 + \rho_3 \hat{\rho}_3\end{aligned}\tag{2.3}$$

Here, $\vec{r}$ is the position vector of the object; $\vec{R}$ is the position vector of the observer; ρ is the slant range between observer and target and $\hat{\rho}$ is the line of sight vector of unit length. The geometry is shown in Figure 2.1.

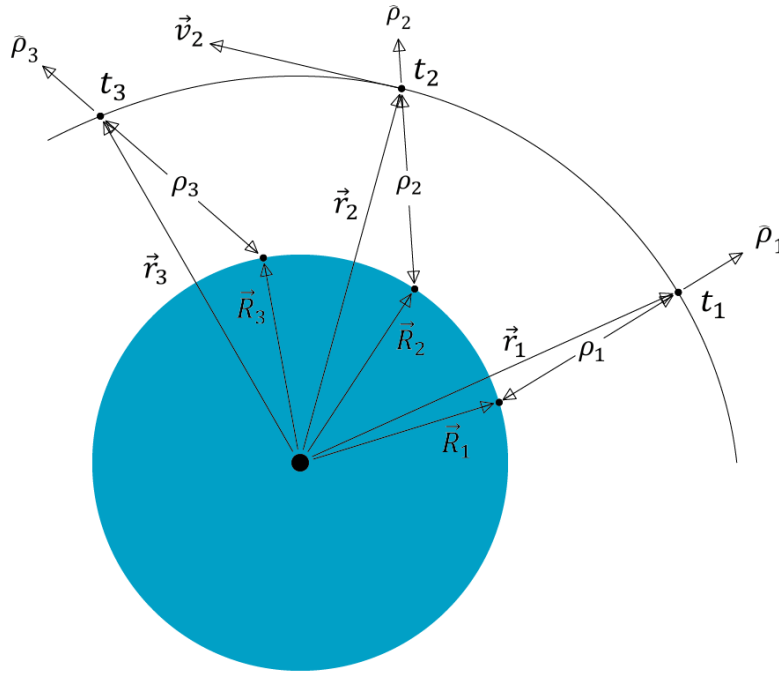


Figure 2.1: Geometry of the classical Gauss problem

In Equation 2.3, the observer position $\vec{R}_i$ and line of sight $\hat{\rho}_i$ to the observed object are known; the object's position $\vec{r}_i$ and distance from the observer ρ are unknown. Now, it is assumed that the orbit of the object is Keplerian and free of perturbations. This implies that the three positions $\vec{r}_1$, $\vec{r}_2$ and $\vec{r}_3$ must lie in the same plane which also contains the origin, which is the centre of mass of Earth. If this is the case, either position vector can be written as a linear combination of the other two. Equation 2.4 illustrates this for $\vec{r}_2$, where c_1 and c_3 are scalar constants.

$$\vec{r}_2 = c_1 \vec{r}_1 + c_3 \vec{r}_3\tag{2.4}$$

If the position and velocity vectors of an object are known at some time t_0 , its position and velocity at any other time can be calculated using the Lagrange coefficients. Equation 2.5 gives the positions $\vec{r}_1$ and $\vec{r}_2$ as function of $\vec{r}_2$ and the Lagrange coefficients.

$$\begin{aligned}\vec{r}_1 &= f_1 \vec{r}_2 + g_1 \vec{v}_2 \\ \vec{r}_3 &= f_3 \vec{r}_2 + g_3 \vec{v}_2\end{aligned}\quad (2.5)$$

In the equation, f_i and g_i are the Lagrange coefficients evaluated for some time step Δt_i between observations 2 and $i \in \{1, 3\}$.

The f and g coefficients around time t_2 with position $\vec{r}_2$ and velocity $\vec{v}_2$ can be written as function of time step Δt in the form of an infinite series, the terms of which up to and including order 4 in δt are given in Equation 2.6. Note that these coefficients are also calculated using the assumption of a Keplerian orbit.

$$\begin{aligned}f(\Delta t) &= 1 - \frac{\mu}{2r_2^3}\Delta t^2 + \frac{\mu}{2} \frac{\vec{r}_2 \cdot \vec{v}_2}{r_2^5}\Delta t^3 + \frac{\mu}{24} \left[-2\frac{\mu}{r_2^6} + 3\frac{v_2^2}{r_2^5} - 15\frac{(\vec{r}_2 \cdot \vec{v}_2)^2}{r_2^7} \right] \Delta t^4 + \mathcal{O}(\Delta t^5) \\ g(\Delta t) &= \Delta t - \frac{1}{6} \frac{\mu}{r_2^3}\Delta t^3 + \frac{\mu}{4} \frac{\vec{r}_2 \cdot \vec{v}_2}{r_2^5}\Delta t^4 + \mathcal{O}(\Delta t^5)\end{aligned}\quad (2.6)$$

If it is assumed that the time Δt between observations is relatively small compared to the orbital period, it is acceptable to truncate Equation 2.6 after the second term on the right hand side, which results in Equation 2.7.

$$\begin{aligned}f(\Delta t) &= 1 - \frac{\mu}{2r_2^3}\Delta t^2 + \mathcal{O}(\Delta t^3) \\ g(\Delta t) &= \Delta t - \frac{1}{6} \frac{\mu}{r_2^3}\Delta t^3 + \mathcal{O}(\Delta t^4)\end{aligned}\quad (2.7)$$

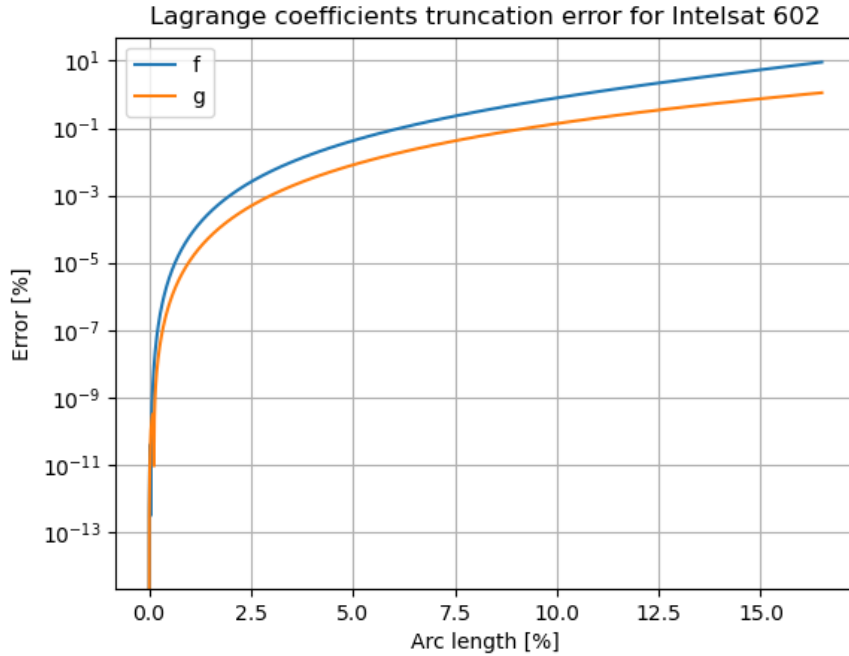


Figure 2.2: Relative error in the approximated Lagrange coefficients from Equation 2.7

Figure 2.2 plots the error of the approximated coefficients as function of arc length, expressed as percentage of the orbital period. An observation set of Intelsat 602 was used to generate the graph. Note that the error is negligible for short orbital arcs, but increases with increasing arc length.

It can be observed that the coefficients in Equation 2.7 only depend on the orbital radius at time t_2 and the time step Δt . Equation 2.3, Equation 2.3 Equation 2.5 and Equation 2.7 constitute a system of 18 equations in 18 unknowns: $\vec{r}_1$, $\vec{r}_2$, $\vec{r}_3$, $\vec{v}_2$, ρ_1 , ρ_2 , ρ_3 , c_1 , c_3 and r_2 . Orbital radius r_2 is an unknown in addition to $\vec{r}_2$ and is not calculated from $\vec{r}_2$ because the equations do not allow to solve for $\vec{r}_2$ without r_2 being known first. The complete solution procedure can be found in Curtis [10].

2.2.2. Selecting the Correct Solution

The solution in Curtis [10] culminates into an eighth order equation in r_2 of the form $f(x) = x^8 + ax^6 + bx^3 + c = 0$ which can have at most three distinct positive real roots for the case where $b > 0$. The other constants a and c are always negative and if b is negative as well, the equation always has exactly one positive root. Evidently, if $b > 0$, only one root can reflect the true orbit. To resolve this ambiguity, the approach as laid out by Wie and Ahn [33] is followed. Depending on the relative magnitude of the constants, either $g(x) = x^8 + c = 0$ or $g(x) = x^8 + ax^6 = 0$ are used to give an approximate solution of the correct root. It must, however, be noted that during the analysis in this report, no instances occurred where more than one root was found. Hence, the method described was never used in practice.

2.2.3. Refining the Solution

In his book, Curtis [10] also describes a way of refining the initial solution from Gauss' method. As explained in subsection 2.2.1, the method of Gauss assumes the time between observations is small to allow the truncation of the Lagrange coefficient series. The proposed refinement aims to remove that constraint by using exact values for the Lagrange coefficients.

An exact equation for the Lagrange coefficients as function of the universal anomaly χ is given in Equation 2.8.

$$\begin{aligned} f(\chi) &= 1 - \frac{\chi^2}{r_2} C(\alpha\chi^2) \\ g(\Delta t, \chi) &= \Delta t - \frac{1}{\sqrt{\mu}} \chi^3 S(\alpha\chi^2) \end{aligned} \quad (2.8)$$

Here, χ is the universal anomaly and α is the reciprocal of the semi-major axis. C and S are the Stumpff functions, defined in Equation 2.9.

$$S(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(2k+3)!} \quad C(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^k}{(2k+2)!} \quad (2.9)$$

The universal anomaly of the object at times t_1 and t_3 can be calculated by solving Equation 2.10 for each Δt .

$$k(\chi) = \frac{r_2 v_{r_2}}{\sqrt{\mu}} \chi^2 C(\alpha\chi^2) + (1 - \alpha r_2) \chi^3 S(\alpha\chi^2) + r_2 \chi - \sqrt{\mu} \Delta t = 0 \quad (2.10)$$

where v_{r_2} is the radial velocity component of $\vec{v}_2$. Equation 2.10 is solved iteratively using the Newton-Raphson method. The radial velocity can be calculated with Equation 2.11.

$$v_r = \frac{\vec{r} \cdot \vec{v}}{r} \quad (2.11)$$

Given an initial solution of $\vec{r}_2$ and $\vec{v}_2$ from the Gauss method, this refinement scheme can be used to calculate exact values for the Lagrange coefficients and update the position and velocity. In every step, the Lagrange coefficients are calculated based on the position and velocity of the previous step. The newly obtained coefficients are averaged with those of the previous iteration and are used to calculate a new position and velocity. For the first iteration, the truncated coefficients are used. This algorithm can be repeated until the update is sufficiently small.

Provided that the first estimate by the Gauss method is sufficiently close to the true solution, the refinement algorithm considered in this section could eliminate the constraint requiring that the time between observations must be small compared to the orbital period. In those cases, the initial result from the Gauss method will be biased due to the truncation error on the Lagrange coefficients, but corrected by the refinement. For very short orbital arcs where the time between observations is very small, the refinement scheme is not expected to considerably improve accuracy, as the correction it provides to the Lagrange coefficients will be relatively smaller.

2.2.4. Circularised Gauss

For very short orbital arcs, methods of minimal orbit determination can be very inconsistent and inaccurate [31]. It will be observed in later chapter that for very short arcs, the IOD methods have a tendency to produce very elliptical and even hyperbolic orbits. In addition, when performing initial orbit determination, it is in general a safe approach to initially assume an object's orbit to be circular, and only take eccentricity into account once observations are available over a sufficiently long orbital arc. Therefore, the approach by Stoker et al. [31] is used, which produces circular orbits from the Gauss method's solution. The solution is coerced into a circular orbit by altering the velocity vector. The approach is detailed below.

Given position and velocity calculated by the Gauss method, the angular momentum can be calculated from Equation 2.12.

$$\vec{h} = \vec{r} \times \vec{v} \quad (2.12)$$

where $\vec{h}$ is the angular momentum vector; and $\vec{r}$ and $\vec{v}$ are the position and velocity calculated by the Gauss method, respectively.

Now, based on the position $\vec{r}$, the circular velocity magnitude equals $v_c = \sqrt{\frac{\mu}{r}}$ with $r = |\vec{r}|$ and the unit vector of the circular velocity is calculated using Equation 2.13. Note that it is ensured the new orbit still lies in the same plane as the original solution by using the angular momentum vector from Equation 2.12 based on the original state vector.

$$\hat{v}_c = \frac{\vec{h} \times \vec{r}}{|\vec{h} \times \vec{r}|} \quad (2.13)$$

The new velocity vector then equals $\vec{v}_c = v_c \cdot \hat{v}_c$. Note that this method, while developed on top of the Gauss method, can be combined with any method of initial orbit determination if so desired.

2.3. Method of Gooding

This section will discuss the method developed by Gooding [21]. It is based on the method developed by Escobar [14] and addresses some of its shortcomings. Gooding's method requires initial estimates for the slant ranges ρ_1 and ρ_3 which are used together with the angle measurements at t_1 and t_3 to obtain two position vectors. Together with the time between the first and third observation, they are used to solve Lambert's problem. Lambert's problem is finding the orbit through two position vectors when the travel time going from one to the other is known. The solution is propagated to the time of the middle observation, where the error with respect to the actual measurement is quantified and used to correct the range estimates.

2.3.1. Calculation

The implementation of the method of Gooding in this report is taken from Gooding [20, 21]. Contrary to Gauss and Laplace, the method of Gooding works iteratively towards a solution. It needs an initial guess of the slant ranges of two observations, which are refined based on the error at the third observation. Slant ranges to the first and third observations are estimated, which allow calculating a position at these times with Equation 2.14 and Equation 2.15.

$$\vec{r}_1 = \vec{R}_1 + \rho_1 \hat{\rho}_1 \quad (2.14)$$

$$\vec{r}_3 = \vec{R}_3 + \rho_3 \hat{\rho}_3 \quad (2.15)$$

Calculating the orbit from the estimate of the positions at times t_1 and t_3 is an example of Lambert's problem, which is solved using the implementation by Gooding [17, 19]. The obtained orbit is propagated from either observation time to t_2 and a computed position P_c is obtained. The error with respect to the true line of sight is expressed in the plane perpendicular to $\hat{\rho}_2$ through position P_c . A perpendicular axis system is constructed with axes x' and y' . The coordinates (x'_p, y'_p) of P_c define the error. The axis system is chosen such that $y'_p = 0$ and $x'_p > 0$. The geometry is shown in Figure 2.3. The estimates for ρ_1 and ρ_3 are now varied to obtain numerical solutions for the partial derivatives of x'_p and y'_p as functions of ρ_1 and ρ_3 , which are used in Newton's root finding method to generate a new estimate for ρ_1 and ρ_3 . The update of the slant ranges is scaled by a constant which is set to 10^{-5} by default. This process is repeated until x'_p falls below a predetermined threshold.

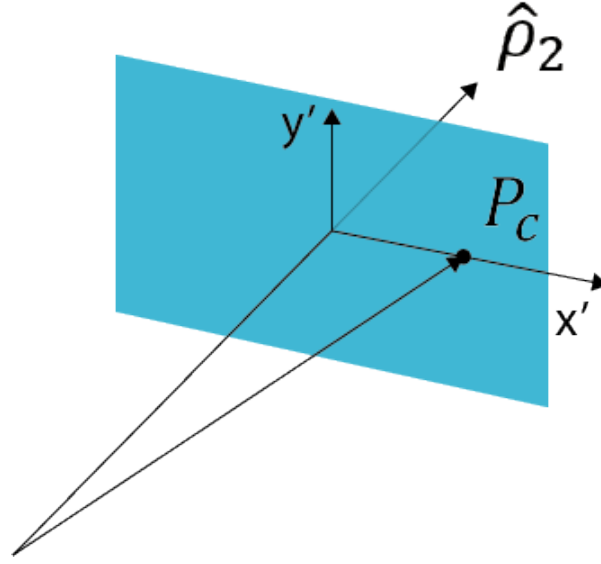


Figure 2.3: Definition of the plane in which the error of P_c is defined

Gooding [21] provides a complete implementation of his method in Fortran in the appendix of his report¹. To ease interaction with the Python scripts used in this study, the Fortran code was compiled as provided by Gooding but equipped with a command line interface to enable calling the method through Python and using its results in subsequent Python routines. Because Gooding's code only returns the final values for the slant ranges ρ_1 , ρ_2 and ρ_3 after a successful iteration, an additional call to a Lambert solver was required to obtain the velocities at the observation times. To avoid having to adapt the Fortran code, an implementation in Python from the pykep [27] library was used, which executes the procedure from Izzo [26].

Gooding's iterative nature differentiates it from the classical methods of Gauss and Laplace, which do not use iterations. This also means that Gooding's method requires an initial guess to start the iteration. This dependency on a good initial guess can be a problem in absence of any a priori information, however, Gooding [21] provides a default starting value which he claims converges for a large range of orbits. Also note that Gooding does not require the time between observations to be small, like Gauss does. In fact, the three observations are allowed to span multiple revolutions, given that the number of revolutions is known.

¹The complete set of all required subroutines can be compiled from Gooding [17, 18, 19, 20, 21], Gooding and Odell [22]

2.3.2. Initial Guess for the Slant Ranges

The iteration of Gooding's method needs to be started with an initial guess of the slant ranges ρ_1 and ρ_3 . While Gooding [21] claims the method has a very large region of convergence, it is nonetheless useful to consider an appropriate starting value, if only to reduce the number of required iterations. In absence of any other a priori information, Gooding [21] proposes a random initial value for the slant ranges as given in Equation 2.16. This condition has also been used to benchmark the method in this paper.

$$\rho_1 = \rho_3 = 2 \cdot (R_1 + R_2 + R_3) \quad (2.16)$$

Alternatively, if some information would be available beforehand, this can be used instead. Depending on the results obtained with the methods of Gauss and the circular orbit method, either or both of them can be used to seed Gooding's method.

It must be noted that, since this report deals with GEO objects, an estimate of the slant range could be produced based on the elevation of the satellite. However, this a priori knowledge is not used in this report.

2.3.3. Selecting the Correct Solution

Gooding's method, like Gauss and Laplace, can generate multiple possible solutions. Unlike Gauss, however, no heuristic method like Wie and Ahn [33] is available to select the correct solution. Gooding [21] states that given the default starting values from Equation 2.16, the first solution that was obtained is statistically the most likely to be correct. Alternatively, if an initial estimate of the slant ranges is available, this could be used to select the solution closest to the initial guess. However, if the accuracy of the initial estimate is unknown, this is not a robust method to determine the correct solution.

The problem of multiple roots is diminished as much as possible by filtering the solutions and reject solutions that are unlikely to be correct for the objects under consideration. More specifically, all parabolic and hyperbolic orbits are rejected as well as all orbits impacting Earth. If multiple solutions still remain, the one closest to the initial guess is selected. If no initial guess is provided, simply the first solution that is not rejected is chosen.

2.4. Circular Orbit Method

The circular orbit method differs from the methods of Gauss and Gooding because it doesn't estimate all six Kepler parameters, but only four. The eccentricity is set to zero and as a consequence, the argument of pericentre is undefined and also set to zero. This means that the method only requires two observations instead of three. Some examples of circular orbit methods were found in literature, but unfortunately, it is unclear how the method used by Choi et al. [9] was implemented and the one described in Kasonsuwan et al. [29] makes some unnecessary approximations. In the latter report, the mean motion of an object is estimated based on its angular velocity as seen by the observer. While this may be a good approximation in some cases and allow for an analytical solution of the orbit, it is entirely possible to obtain a solution without this approximation.

2.4.1. Calculation

The new method estimates a circular trajectory using two requirements on the change in true anomaly $\Delta\theta$. On the one hand, $\Delta\theta$ can be expressed as the product of the time between observations and the angular velocity ω . On the other hand, the change in true anomaly is also the angle between the two observations. Both relations must produce the same change in true anomaly. The resulting equality only depends on the constant orbital radius r and can be solved iteratively. The complete algorithm is discussed below.

Similar to Gauss and Gooding, orbital position can be calculated from observations and observer positions using Equation 2.17 and Equation 2.18.

$$\vec{r}_1 = \vec{R}_1 + \rho_1 \hat{p}_1 \quad (2.17)$$

$$\vec{r}_2 = \vec{R}_2 + \rho_2 \hat{p}_2 \quad (2.18)$$

where $\vec{r}$ is the position of the satellite, $\vec{R}$ is the observer position, $\hat{\rho}$ is the unit vector pointing from the observer to the satellite and the slant range ρ is its length.

First, the square of the length of the position vectors is calculated. This is done by taking the dot products of Equation 2.17 and Equation 2.18 with themselves, which results in Equation 2.19, where $i \in \{1, 2\}$.

$$r_i^2 = \rho_i^2 + 2\rho_i(\vec{R}_i \cdot \hat{\rho}_i) + R_i^2 = r^2 \quad (2.19)$$

Since the orbit is circular, the orbital radius must be constant, hence $r_1 = r_2 = r$. If the observer is stationary on the surface of the Earth, it can also be said that $R_1 = R_2 = R$, however, to allow for observations from different observers, this simplification is not made. Equation 2.19 is quadratic in ρ_i and the solution is given in Equation 2.20. Note that the solution associated with a minus sign in front of the root is disregarded, as it would follow that ρ_i is negative. This is because for the satellite to be in view of an observing station, the dot product of the observer position and line of sight vector must be positive, otherwise the satellite is below the horizon. It follows that the first term on the right hand side is strictly negative. Therefore, the second term must have a positive contribution to obtain a positive value for the slant range. It can be confirmed that the slant range will be strictly positive, since $r > R_i$ for an orbiting object, which means that the second term on the right hand side is strictly greater than the first.

$$\rho_i = -(\vec{R}_i \cdot \hat{\rho}_i) + \sqrt{(\vec{R}_i \cdot \hat{\rho}_i)^2 - R_i^2 + r^2} \quad (2.20)$$

Given an orbital radius r , Equation 2.20 allows the calculation of both ρ_1 and ρ_2 , as well as the positions $\vec{r}_1$ and $\vec{r}_2$ by plugging the obtained slant ranges into Equation 2.17 and Equation 2.18. The change in true anomaly can now be expressed as function of r in Equation 2.21.

$$\Delta\theta = \arccos\left(\frac{\vec{r}_1 \cdot \vec{r}_2}{r^2}\right) \quad (2.21)$$

The second expression for $\Delta\theta$ is obtained from the constant angular velocity of a circular orbit. The angular velocity is given as function of r in Equation 2.22.

$$\omega = \sqrt{\frac{\mu}{r^3}} \quad (2.22)$$

The change in true anomaly is the integral of angular velocity from the time of the first observation to that of the second. This is shown in Equation 2.23.

$$\Delta\theta = \int_{t_1}^{t_2} \omega dt = (t_2 - t_1) \sqrt{\frac{\mu}{r^3}} \quad (2.23)$$

Both Equation 2.21 and Equation 2.23 must be satisfied, and the equality in Equation 2.24 is the result.

$$(t_2 - t_1) \sqrt{\frac{\mu}{r^3}} - \arccos\left(\frac{\vec{r}_1 \cdot \vec{r}_2}{r^2}\right) = 0 \quad (2.24)$$

If it is assumed that the two observations do not span multiple revolutions, Equation 2.24 always has exactly one root. Once a value for r is found with a sufficiently low error tolerance, $\vec{r}_1$ and $\vec{r}_2$ can be calculated. This only leaves the velocity to be computed for a complete state vector. One could obtain the velocity by solving Lambert's problem with the two known positions, but a more straightforward method exists. For a circular orbit, it is known that the magnitude of the velocity $v_c = \sqrt{\frac{\mu}{r}}$. The direction of the angular momentum of the orbit follows from Equation 2.25.

$$\hat{h} = \frac{\vec{r}_1 \times \vec{r}_2}{\|\vec{r}_1 \times \vec{r}_2\|} \quad (2.25)$$

where $\hat{h}$ is the unit vector in the direction of the angular momentum. The direction of the velocity vector at $\vec{r}_i$ for $i \in \{1, 2\}$ then follows from Equation 2.26.

$$\hat{v}_i = \frac{\hat{h} \times \vec{r}_i}{r} \quad (2.26)$$

where $\hat{v}$ is unit vector in the direction of the velocity. The velocity at position $\vec{r}_i$ can now be written as in Equation 2.27.

$$\vec{v}_i = \hat{v}_i \cdot \sqrt{\frac{\mu}{r}} = (\hat{h} \times \vec{r}_i) \cdot \sqrt{\frac{\mu}{r^3}} \quad (2.27)$$

2.4.2. Combining with Gauss Refinement Algorithm

In subsection 2.2.3, an iterative refinement algorithm was discussed to improve the orbit estimate from the Gauss method. The refinement algorithm eliminates the short arc assumption in the Gauss method. No assumptions about the length of the observation arc are made in the circular orbit method, but using the refinement algorithm in combination with the method would relax the assumption that the eccentricity is zero. The Lagrange coefficients are not used in the circular orbit method, but they can easily be computed to start the iterative refinement. The combination of the circular orbit method and the refinement will be tested in this report to determine if the refinement is able to improve the circular orbit solution.

Telescope and Measurement Error

This chapter will detail the measurement error reduction methods that were developed and will be analysed in this report. First, [section 3.1](#) will provide a qualitative discussion on how measurement error affects the quality of the solution. [section 3.2](#) will list the key characteristics of the MeerLICHT telescope and [section 3.3](#) will detail the endpoint determination routine developed during the FOTOS study [34]. The information from these two sections will be then used in [section 3.4](#), which will discuss the two methods of measurement error reduction that will be analysed. The first method analyses the shutter wheel motion to determine a more accurate time stamp for each observation depending on its pixel position, the second will refine the FOTOS endpoint determination routine to resolve a systematic error.

3.1. Effect of Measurement Error

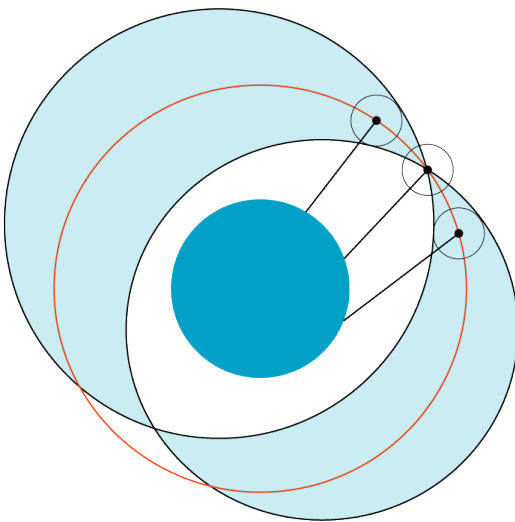


Figure 3.1: Effect of measurement error for a very short observation arc.

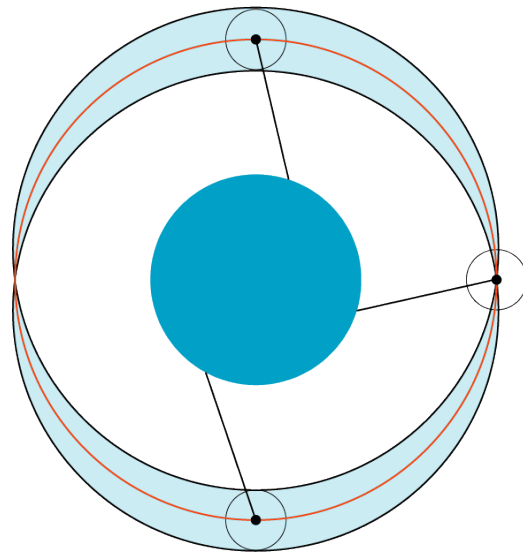


Figure 3.2: Effect of measurement error for a relatively long observation arc.

It is known that initial orbit determination (IOD) methods are very sensitive to measurement error. The cause of this is the short arc of observation data that is usually available. This is illustrated qualitatively in [Figure 3.2](#) and [Figure 3.1](#). Measurement error is represented as a small circle around the observations. The shaded area formed by the two large black circles represents the uncertainty in the resulting orbit solution, while the orange circle denotes the true orbit. Note that the figures are not a correct physical representation, but merely serve to illustrate the problem. It can be seen that in the figure on

the left, the observations are very closely spaced and as a result, the orbit uncertainty is very large. The orbit becomes more degenerate. On the other hand, the observations in the figure on the right cover a relatively long arc. While the orbit error still causes some uncertainty, the shaded area is much smaller and a much better representation of the correct orbit. The figure on the left is a good representation of the observation arcs available for this study, and clearly shows the need for measurement error reduction. Clearly, as the circles representing measurement error shrink, so will the uncertainty of the orbit error.

3.2. The MeerLICHT telescope

This report will use observations from the MeerLICHT telescope [5]. MeerLICHT is a prototype of the BlackGEM telescope array which is being commissioned in La Silla, Chile. MeerLICHT itself is deployed at the Sutherland Observing site in South Africa, where it is to serve as the optical counterpart of the MeerKAT radio array. The telescope has an aperture of 60 [cm] and a focal length of 330 [cm]. At its heart sits an STA1600 CCD chip offering 10560 by 10560 pixels at 9 [μm] per pixel [6]. This gives MeerLICHT a pixel scale of $0.56 \left[\frac{\text{arcsec}}{\text{px}} \right]$ and a field of view of $2.7 \text{ [deg}^2\text{]}$. In front of the CCD, a shutter wheel and a filter wheel with six positions are mounted. MeerLICHT is designed to have seeing-limited image quality. It is expected the telescope can reach a depth of 23 magnitudes with an exposure time of less than 5 [min]. The properties of MeerLICHT are summarised in Table 3.1.

Table 3.1: MeerLICHT characteristics

Property	Value
Aperture	60 [cm]
Focal length	330 [cm]
Resolution	10560×10560 [px^2]
Pixel size	9×9 [μm^2]
Pixel Resolution	$0.56 \left[\frac{\text{arcsec}}{\text{pixel}} \right]$
Field of View	2.7×2.7 [deg^2]

3.3. FOTOS Data Reduction Pipeline

Data reduction on images from the MeerLICHT telescope is performed using the routines from Wijnen et al. [34, 35], with one notable difference. MeerLICHT images are provided as S-corr images, which is an image subtraction technique that also takes into account the signal-to-noise ratio, known point spread functions of stars and uncertainty of the astrometric solution [36]. Generating these images is a computationally intensive process, however, satellite tracks are shown considerably more clearly, hence, S-corr images are used instead of normal difference images generated by the FOTOS routine. Difference images are useful because they immediately eliminate most of the constant light sources in an image. Further data reduction discards the background and creates a binary image. This is shown in Figure 3.3.

In a next step, the Hough transform is applied to find all line segments present in an image. Then, the RANSAC method is used on specific areas of the image containing line segments identified by the Hough transform. The RANSAC method fits a single line through the data points and connects all pieces of the same track. The line found by the RANSAC method is then drawn on top of the subtracted image, shown in the middle panel of Figure 3.3. The endpoints of the streak are determined along the RANSAC line in the S-corr image. The precise algorithm to obtain the endpoints is discussed later.

3.4. Measurement Error Reduction

The observations by the MeerLICHT telescope are subject to measurement error that will corrupt the orbit estimate. This section will address two methods to reduce the error specific for the MeerLICHT system and FOTOS routine. The first aims to reduce the uncertainty on the timestamp given to an observation, the second method is implemented to improve the accuracy of the endpoint determination itself. These sources of error were selected because their magnitude is unknown and specific to the

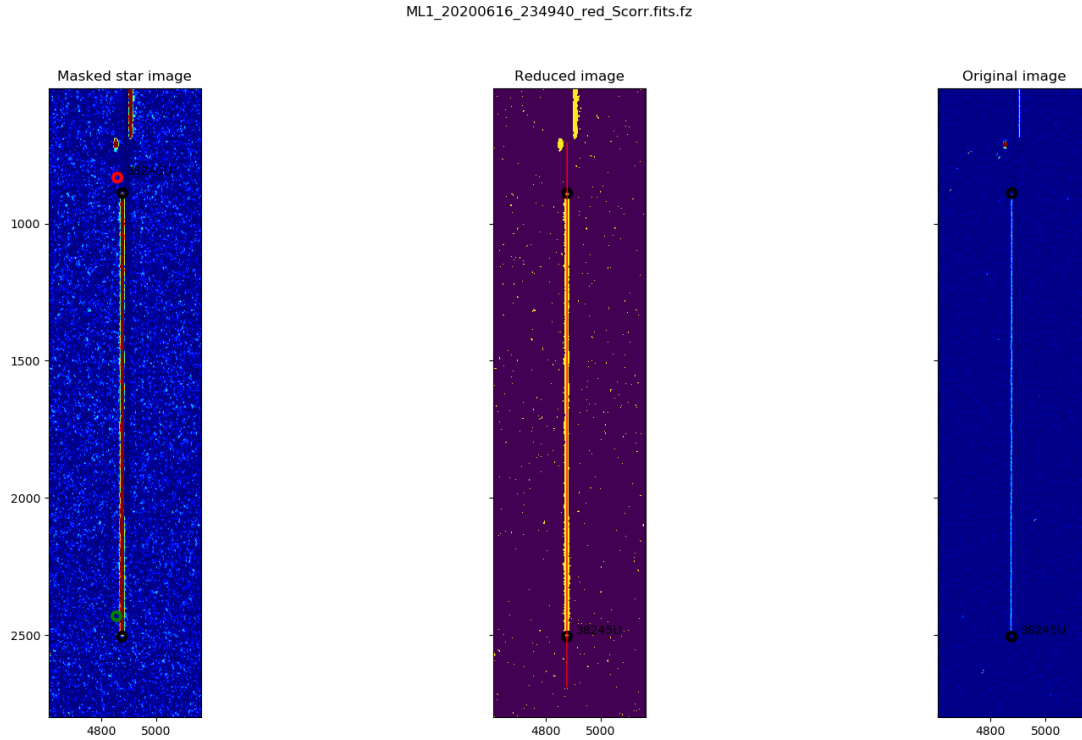


Figure 3.3: Example of a streak in a MeerLICHT image. Left is the unaltered S-corr image, middle is the S-corr image after data reduction and right is the original image.

current observation system. Other errors, such as the light-time effect or aberration due to the velocity of the observer are not taken into account in this study. This is because over the short timespan of observations, the corrections will be approximately equal for each observation. As a result, light time effect would only affect the true anomaly, and aberration would cause a rotation of the orbital plane, but the effect on other orbital parameters will be relatively small and so would the resulting error in the orbit estimate.

3.4.1. Shutter Rotation Model

In this section, a model will be built to analyse the rotation of the shutter wheel across the charge-coupled device (CCD). The MeerLICHT telescope is designed such that each pixel has exactly the same exposure time, but the timing of each pixel is irrelevant for astronomical purposes and hence unknown. Therefore, this refinement was developed. Due to the finite rotation speed of the shutter wheel, individual pixels in the CCD do not get covered and uncovered at exactly the same time. By determining the motion of the shutter wheel, it is possible to calculate the exact times at which each pixel is covered or uncovered instead of the single timestamp associated with an image. This allows greater precision in the timestamp allocated to an observation.

Figure 3.4 displays the setup of the MeerLICHT telescope. The axis of the shutter wheel is positioned to the east of the CCD. In making a full rotation, the shutter wheel traverses four positions: open (0°), close (90°), open (180°) and close (270°), where θ is the angle with the positive x axis. Since the design is radially symmetric, only the first two positions will be considered, and the behaviour of the other two is assumed to be identical. The shutter wheel is powered by a motor of which a full rotation is divided into 20,000 steps. The shutter wheel rotates counterclockwise, which means the south side of the CCD is uncovered and covered first during opening and closing, respectively.

The recoding of an image is an interaction between different components of the telescope. The process is started by the control computer, which requests a new exposure with certain exposure time T_{exp} from the CCD controller. The CCD controller then starts a timer T and asks the shutter controller to open the

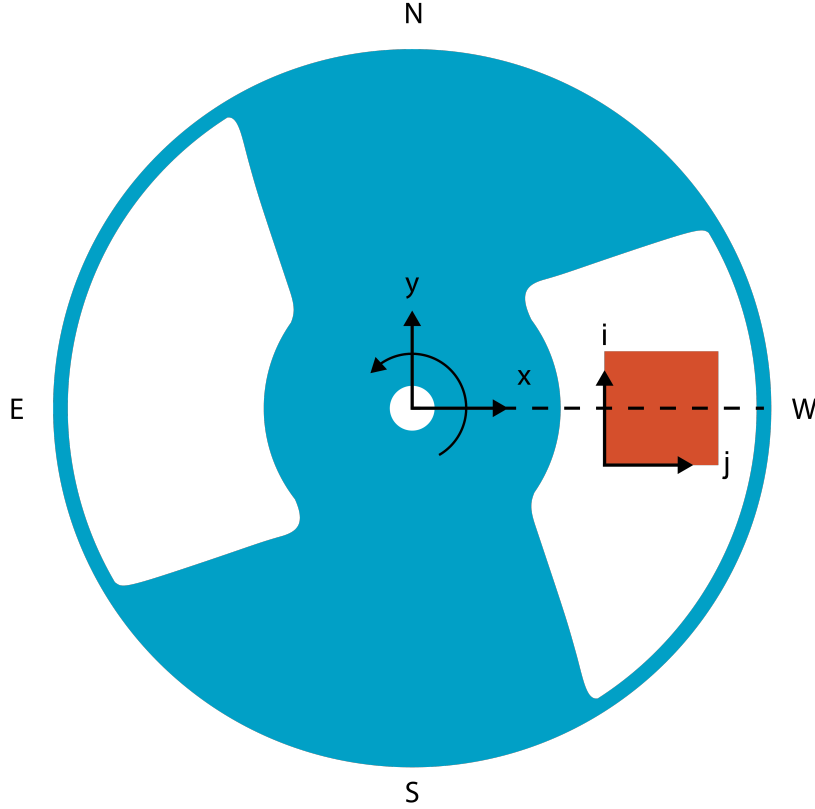


Figure 3.4: MeerLICHT telescope setup as seen when looking at the sky through the telescope.

shutter. This corresponds to $t = 0$ in Figure 3.5. The shutter controller, in turn, controls the GPS unit. When the shutter has moved 10 motor steps, the controller triggers the GPS unit to register GPS_{START} . The time between the shutter starting to open and it reaching 10 steps is denoted dt_1 . Some time later, the shutter has finished opening, turning 5,000 steps (90°). This time is denoted T_m , the moving time of the shutter. The system is now stationary until the CCD controller's timer reaches the requested exposure time T_{exp} . The shutter controller is then asked to close the shutter again. At the moment the shutter has rotated 9,990 steps, the GPS unit registers GPS_{END} . Some time dt_2 later, the shutter has rotated the full 10,000 (180°) steps and is again at rest.



Figure 3.5: Time sequence of the shutter. The steps between events are not to scale.

The headers of the FITS images provide GPS_{START} and GPS_{END} . In the following, GPS_{START} will be used as a basis to calculate other times. The relation between GPS_{START} and the various events mentioned above are shown below the timeline in Figure 3.5. It can be concluded that the exact time at which any

pixel is uncovered can be calculated using Equation 3.1.

$$t_1(px) = GPS_{START} - dt_1 + \tau_1(px) \quad (3.1)$$

where $\tau_1(px)$ is the moving time of the shutter while opening as function of pixel position. Similarly, the time a pixel is covered again is calculated using Equation 3.2.

$$t_2(px) = GPS_{START} - dt_1 + T_{exp} + \tau_2(px). \quad (3.2)$$

Here, $\tau_2(px)$ is the moving time of the shutter while closing as function of pixel position.

Figure 3.6 shows the designed curve of motor steps and θ as functions of time. This profile is interpolated before it is fed to the shutter controller. The accuracy of interpolation is unknown but assumed to be small. The image shows a single opening or closing motion of the shutter wheel. These motions are identical. Over 5,000 motor steps, the shutter rotates 90 [°]. This takes 1.02 [s], with a data point provided every millisecond. In reference to Figure 3.5, dt_1 and dt_2 equal 73 and 74 [ms], respectively. Table 3.2 contains a list of the shutter rotation characteristics.

Table 3.2: Characteristics of shutter rotation

Variable		Value
Total shutter moving time	T_m	1.02 [s]
Number of data points	n	1020 [—]
Shutter moving time before GPS_{START}	dt_1	73 [ms]
Shutter moving time after GPS_{END}	dt_2	74 [ms]

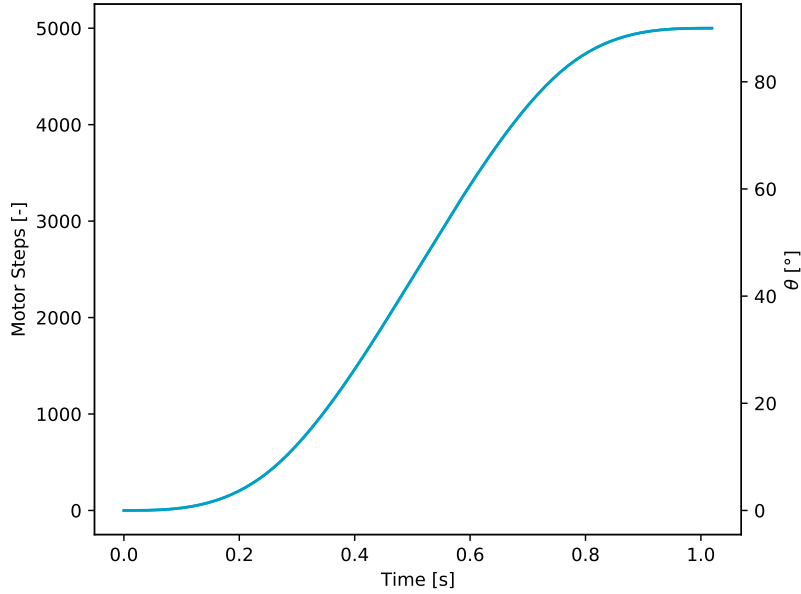


Figure 3.6: Plot of shutter rotation vs time. The profile is identical for opening and closing.

It must be noted that, in addition to GPS_{START} and GPS_{END} , the header of the FITS image also provides a value for the actual moving time T_m^* , defined as $GPS_{END} - GPS_{START} - T_{exp}$. T_m^* relates to T_m as $T_m = T_m^* + dt_1^* + dt_2^*$, where dt_i^* are the unknown actual moving times for the first and last ten motor steps. Although it is observed that T_m^* varies between different images, its exact value will be ignored and instead a fixed value of $T_m = 1.02$ [s] will be used to simplify calculations. This is because the variation is relatively small and its cause is unknown. Figure 3.7 shows the variation of the shutter moving time for all Scorr images in the data set. It can be observed that the variation is small but very irregular. Note that $T_m - dt_1 - dt_2 = 0.873$ [s] based on Table 3.2, which approximately corresponds to

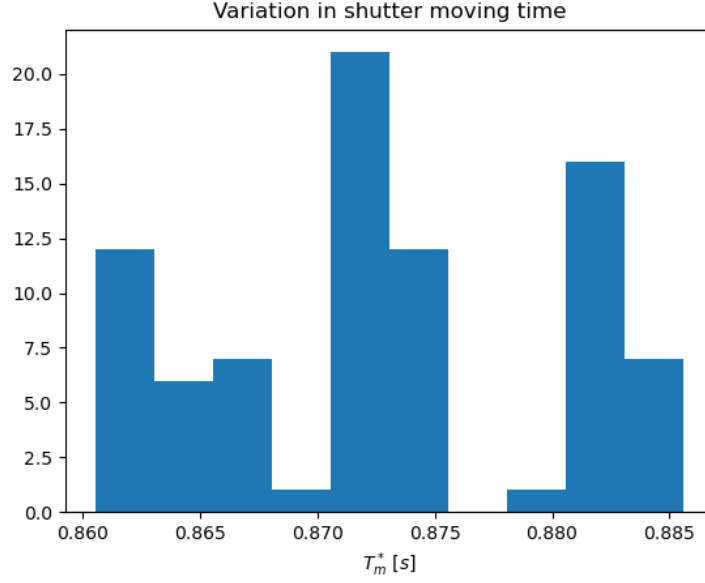


Figure 3.7: Variation of shutter moving time T_m^*

the mean of t_m^* observed in Figure 3.7. If the actual moving time T_m^* would be taken into account, exact values for dt_1^* and dt_2^* would have to be determined as well, and not enough information is available to be able to do this with sufficient accuracy.

The time offset of every pixel with respect to the registered GPS_{START} can be calculated as follows. First, the equation of the lines through the edges of the hole in the shutter blade will be determined and written as a function of the rotation angle of the shutter wheel, θ . Then, requiring a line segment is rotated such that it contains a given pixel, the equation of the line can be solved for θ . The objective of finding the time at which a pixel is covered or uncovered is then mathematically equivalent to finding the time at which the appropriate edge of the hole in the shutter wheel coincides with a pixel position. Finally, knowing the relation between θ and motor steps, which is related to the moving time of the shutter wheel, it is possible to determine the time τ the shutter needs to cover or uncover a pixel when starting from a rest position. For all calculation steps, the reader is referred to Appendix A. Given τ , Equation 3.1 or Equation 3.2 can be solved for a given pixel to get an absolute time pertaining to an observation. Due to symmetry, the time offsets τ_1 for opening and τ_2 for closing of the shutter are identical.

The result of the calculation is shown in Figure 3.8. The image shows the offset time for all pixels in the CCD. It can be seen that covering or uncovering the CCD only takes up a small portion of the shutter's total moving time. The first pixel is crossed after 0.387 [s], the last one 0.273 [s] later at 0.660 [s]. The midpoint of the CCD has a time offset of 0.552 [s].

Without this refinement and the knowledge about the shutter motion, timestamps for the start and endpoint are calculated using Equation 3.3.

$$t_{1,2} = \frac{GPS_{START} + GPS_{END} \mp T_{exp}}{2} \quad (3.3)$$

where t_1 and t_2 are the times of the start and end of exposure, respectively. From Figure 3.5, it is observed that $GPS_{END} - GPS_{START} = T_{exp} + T_m - dt_1 - dt_2$. Using this relation while setting $GPS_{START} = 0$ results in a timestamp at the start of the exposure of 0.4365 [s] for all pixels. Using the new method, this timestamp lies between 0.314 [s] and 0.587 [s] and equals 0.479 [s] for the midpoint. In both cases, the timestamp at the end of exposure is 60 [s] later.

The main source of the time error lies in the difference between time offsets of the begin and end point of the streak. If both pixels containing these points have the same offset, there is a constant difference

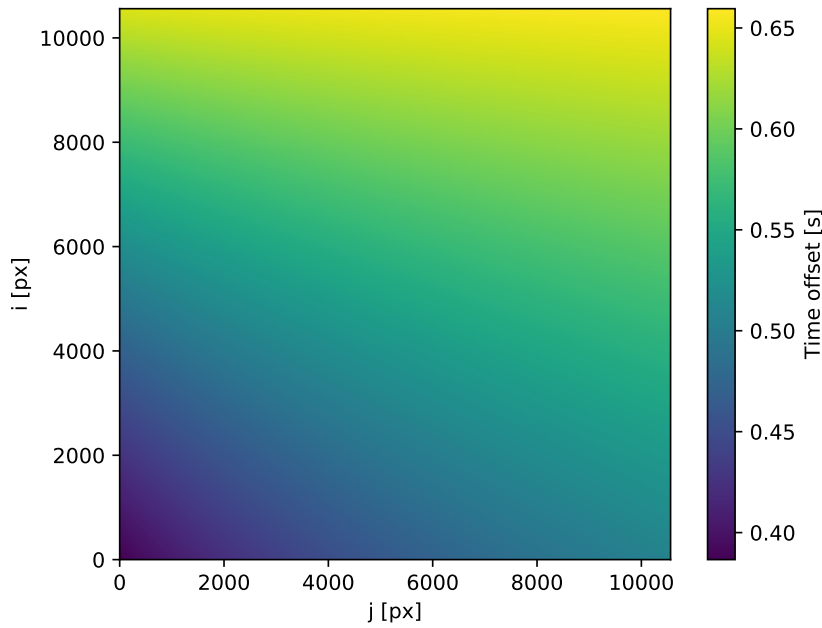


Figure 3.8: Time offset for each pixel in the CCD.

between the observation timestamp and the ‘real’ timestamp. Assuming an otherwise perfect orbit determination, free of any other sources of error, the calculated orbit would be identical to the real one but shifted in time. This means the calculated satellite would lead or chase the real satellite, but both would follow the same orbit. This type of error is less of a concern since the satellite can easily be recovered in follow-up observations. When the time offsets vary between begin and end point, the time between the two observations would be more or less than the expected exposure time of the image. It would seem as if the satellite needed more or less time moving between the two observations, which affects the velocity and hence the orbit solution. This creates a more serious issue. The orbit determination will produce an inaccurate orbit solution, which is only exaggerated when the solution is propagated to a future epoch. It can be observed in Figure 3.8 that, depending on the location of a satellite streak in the frame, the effect of the time offset varies. A streak in the lower left corner along the i axis will have a larger offset difference between begin and end point than a streak of the same length in the upper portion of the image along the j axis. Figure 3.9 shows the deviation of the time between start and endpoint from the nominal exposure time for all observations. It can be observed that the deviation is relatively small.

This concludes the discussion of the time offset calculation. What remains are some recommendations to further refine the calculation should it be desired. A first improvement is the use of the recorded value for the moving time T_m^* . These values are known to vary between images. They were disregarded as it required the determination of dt_1^* and dt_2^* and not enough information is available to perform the calculation. Depending on the results obtained from the current method, it may be worthwhile to take into account these variations. Another improvement can be made by refining the shutter profile from Figure 3.6. Currently, the curve is only known as it was originally designed. The data points are interpolated by the MeerLICHT system and the accuracy of that interpolation is not known. In addition, it is unknown how closely the motor follows the commands it is given. In order to further refine the shutter profile, a series of slew images could be recorded. Instead of moving along with the stars during an exposure, MeerLICHT could keep completely still or move in a different direction, causing the stars to form streaks in the image. Given the known positions of stars from catalogues like GAIA¹

¹<https://sci.esa.int/web/gaia/-/60192-gaia-creates-richest-star-map-of-our-galaxy-and-beyond>, last accessed 05 February 2021

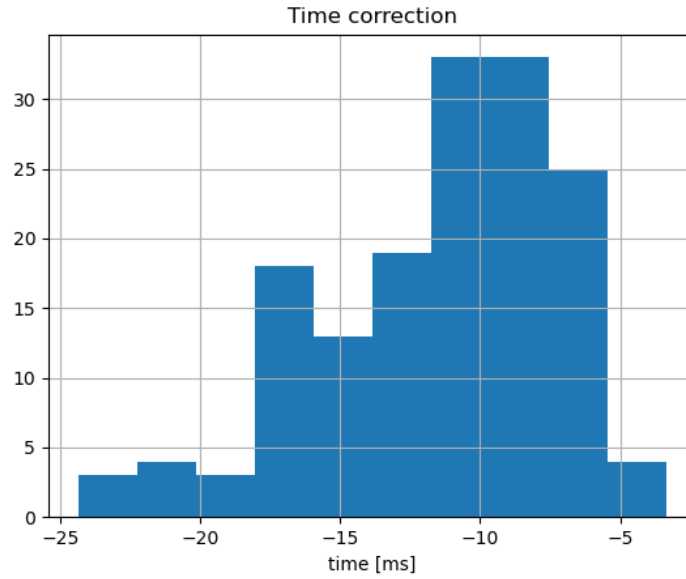


Figure 3.9: Deviation of the time between start and end point of a streak from the nominal exposure time of the image.

or Hipparcos² and the pixels containing the end points of the stars' streaks in the image, it is possible to determine at which time the shutter moved across those pixels, and the curve could be reverse engineered. Finally, a third improvement can be made by abandoning the assumption of symmetry. The current method does not differentiate between the two openings in the shutter blade and assumes both will exhibit identical behaviour. The correctness of this assumption has not been investigated. While the likelihood of manufacturing errors in the shutter blade that could cause irregularities is small, it is possible that for example the motor driving the shutter has some bias and has resting position that differ from the nominal $n * 90$ [°], where $n \in \{0, 1, 2, 3\}$. It has already been shown in this section that the motor exhibits small variations in its moving time. This refinement could also be implemented using the slewing images discussed previously.

3.4.2. End Point Determination

The FOTOS routine uses the RANSAC method to identify line segments [34, 35]. This is explained in section 3.3. To obtain the endpoints of the streak, a cost function is maximised using the values of the pixels on the RANSAC line. A threshold is set as the sum of the local mean pixel value around the track μ_p and one standard deviation of the pixel values σ_p . Pixels with a value above the threshold are taken to be part of the satellite streak. The cost function is given in Equation 3.4.

$$S(n) = \sum_{i=0}^n [p_i - (\mu_p + \sigma_p)] \quad (3.4)$$

In the equation, p_i is the value of pixel i ; μ_p is the mean pixel value of the subset of pixels around the track and σ_p is the standard deviation of pixel values in the subset. The starting point is taken as the brightest pixel along the RANSAC line and the cost function is evaluated once in each direction along the line. It is assumed that the brightest pixel on the RANSAC line is part of the satellite streak, which implies that the first pixel to decrease the cost function must be the first pixel beyond the satellite streak.

In the S-corr images, satellite streaks have a width of approximately 10 to 25 [px]. Assuming the point-spread function (PSF) of the object is homogeneous in every direction, a satellite is expected to have a circular shape in an image with an exposure time short enough for the satellite to stay in one pixel. Its track in the MeerLICHT images would then end in a semicircle with the spacecraft's true location

²<https://sci.esa.int/web/hipparcos/-/52887-the-hipparcos-all-sky-map>, last accessed 05 February 2021

at the centre instead of at the very end of the streak. As a result, the FOTOS method is expected to overestimate the length of streaks by approximately half of the track width at both ends. This is illustrated in Figure 3.10. On the left is an S-corr image, on the right a regular difference image. The S-corr image produces a much clearer track and greatly increases the detectability of fainter tracks. Therefore, these images are used by the FOTOS method. However, the S-corr images also produce a much wider track than the difference image, which increases the error on the endpoint determination using Equation 3.4. It can be observed in Figure 3.10 that the endpoint as determined by the FOTOS method is outside of the streak in the difference image. The expected true position in blue remains in the middle of the semicircle at the end of the track.

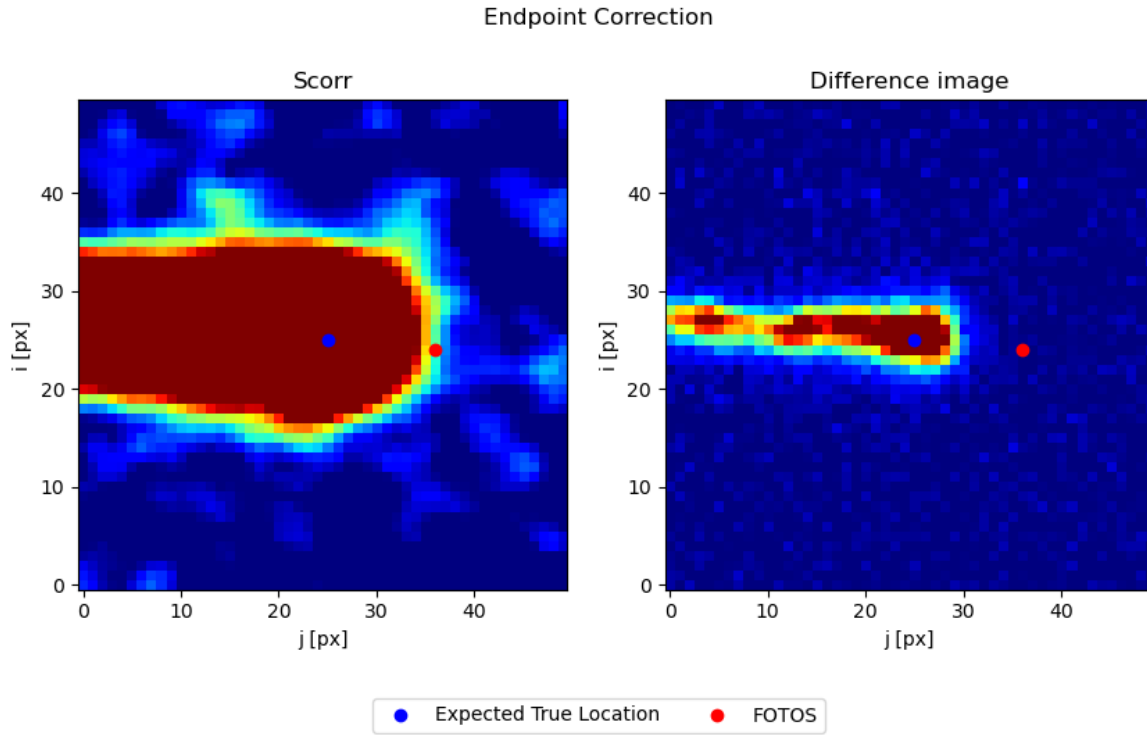


Figure 3.10: Illustration of the satellite location at the end of the streak as determined by the FOTOS method (red) and the expected true position (blue).

The expected true locations of the spacecraft in the images were determined manually for the observations from the night of 16 June 2020. Figure 3.11 shows the difference of the manually determined endpoints with those of the FOTOS routine in pixels. The satellite streaks run through the image approximately parallel to the j axis and in the negative j direction, which results in a clear grouping of begin and endpoints. The group of negative differences in j are the starting points of the streaks, the others are the endpoints. Due to the streaks aligning with the j axis, the difference in i values is relatively small. In theory, since the RANSAC routine draws lines through the brightest pixels randomly, the endpoint could be anywhere on the circumference of the semicircle, but the small variation in i values in Figure 3.11 indicates they are concentrated near the axis of the streak. This is because the RANSAC method attempts to find the longest line, which is near the axis of the streak.

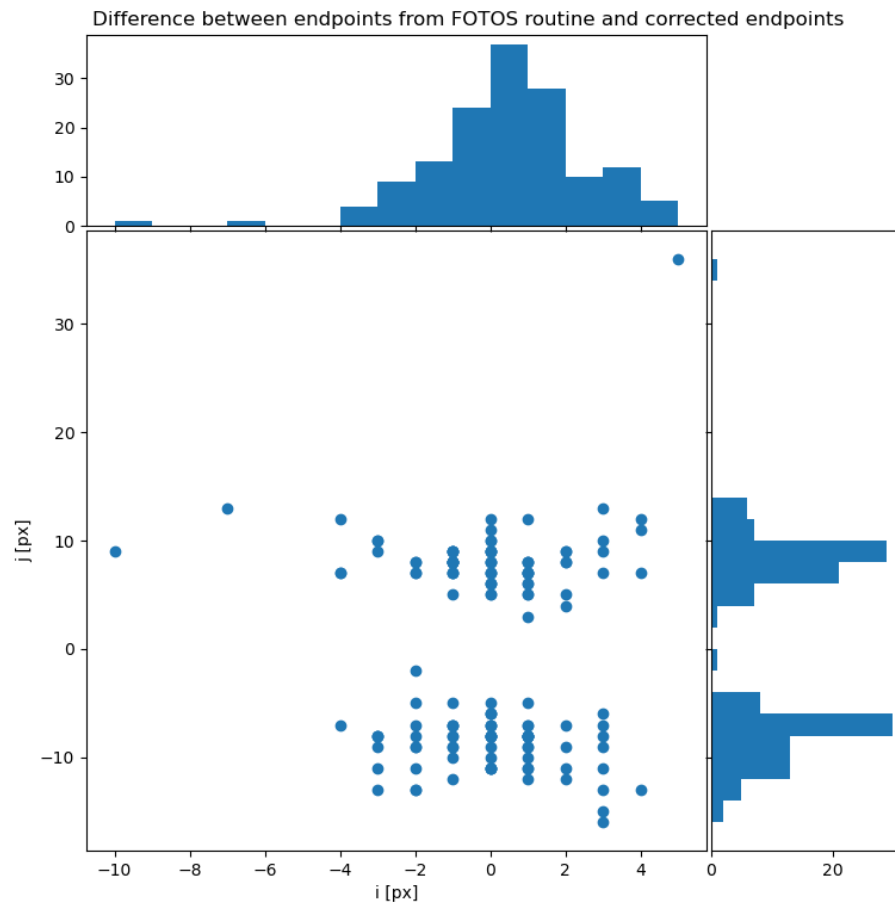


Figure 3.11: Difference of the manually determined endpoints with those of the FOTOS routine

Validation

This chapter will discuss the validation that is performed on the algorithms in this report. Validation is necessary to ensure that the code runs correctly and is free of coding errors.

4.1. Kepler Element Conversion

The conversion between Cartesian and Keplerian elements is validated by defining a set of Kepler elements, converting them to Cartesian and back to Keplerian and confirm that the original elements are successfully recovered. The result is shown in [Table 4.1](#), with the original elements, the results after conversion and the difference between both for each orbital element. The errors are shown to be sufficiently small. It should be noted that it is theoretically possible for errors to still exist in the code if they cancel out against each other to still produce the results in [Table 4.1](#), however, the likelihood of this is deemed negligible.

Table 4.1: Result of Kepler element conversion validation.

Element	Initial	After Conversion	Difference
a [km]	$7.096 \cdot 10^3$	$7.096 \cdot 10^3$	$9.095 \cdot 10^{-13}$
e [—]	$1.122 \cdot 10^{-3}$	$1.122 \cdot 10^{-3}$	$2.496 \cdot 10^{-16}$
i [°]	$9.203 \cdot 10^1$	$9.203 \cdot 10^1$	0.000
Ω [°]	$2.961 \cdot 10^2$	$2.961 \cdot 10^2$	0.000
ω [°]	$1.207 \cdot 10^2$	$1.207 \cdot 10^2$	$1.550 \cdot 10^{-11}$
θ [°]	$2.395 \cdot 10^2$	$2.395 \cdot 10^2$	$1.555 \cdot 10^{-11}$

4.2. Gauss Method

The method of Gauss is validated using the example from Curtis [10]. The book comes with MATLAB scripts that include an example for the method of Gauss. [Table 4.2](#) lists the input parameters for the Gauss method. Based on these inputs, the position and velocity is calculated at the second epoch. The values are listed in [Table 4.3](#). It can be observed that the difference between the values from the MATLAB code from Curtis [10] and those from the implementation for this study is negligible.

Table 4.2: Input for Gauss validation from Curtis [10]. Observer is located at a latitude of 40° and altitude of 1 [km]. $\mu = 398600 \left[\frac{km^3}{s^2} \right]$, $R_e = 6378$ [km] and Earth flattening factor $f = 1/298.26$.

Epoch	Local Sidereal Time [°]	Time Step [s]	Right Ascension [°]	Declination [°]
1	44.5065	0	43.5365	−8.78334
2	45.000	118.104	54.4196	−12.0739
3	45.4992	237.577	64.3178	−15.1054

Table 4.3: Results of Gauss validation. The expected results are from Curtis [10]

Position [km]			Velocity $\frac{km}{s}$		
Expected	Obtained	Difference	Expected	Obtained	Difference
$5.659 \cdot 10^3$	$5.659 \cdot 10^3$	$2.274 \cdot 10^{-11}$	-3.880	-3.880	$5.849 \cdot 10^{-13}$
$6.534 \cdot 10^3$	$6.534 \cdot 10^3$	$3.183 \cdot 10^{-11}$	5.116	5.116	$6.999 \cdot 10^{-13}$
$3.270 \cdot 10^3$	$3.270 \cdot 10^3$	$8.185 \cdot 10^{-12}$	-2.240	-2.240	$1.781 \cdot 10^{-13}$

4.3. Gooding Method

Gooding's method is validated in two steps. First, the slant ranges that are produced by the Fortran code are validated based on an example from Gooding [20]. The input values are given in Table 4.4. All values are normalised such that $\mu = 1$ and the orbital radius of Earth around the sun $r_e = 1$. The example deals with observations of a minor planet in orbit around the Sun, but this does not make the example less valid. The valid solution for the slant ranges is given in Table 4.5 together with the expected result. The difference are deemed to be sufficiently small, given that the results in Gooding [21] are obtained using a computer from the 1990's.

The second step is to validate the position and velocity obtained from the slant ranges. For this, the example from Curtis [10] is again used. The results are given in Table 4.6 and it is concluded that the differences are sufficiently small.

Table 4.4: Input for Gooding validation from Gooding [20]. The times between observations are $\tau_{12} = 0.325593$ and $\tau_{23} = 0.701944$.

Variable	Value		
$\vec{R}_1$	0.7000687	0.6429399	0.2789211
$\vec{R}_2$	0.4306907	0.8143496	0.3532745
$\vec{R}_3$	0.0628371	0.9007098	0.3907417
$\hat{\rho}_1$	0.9028975	0.0606048	0.4255621
$\hat{\rho}_2$	0.9224764	0.0518570	0.3825549
$\hat{\rho}_3$	0.9347684	0.0802269	0.3460802

Table 4.5: Result of Gooding validation of Fortran code.

Expected	Obtained	Difference
2.399	2.399	$7.408 \cdot 10^{-8}$
2.564	2.564	$1.391 \cdot 10^{-7}$
2.824	2.825	$2.514 \cdot 10^{-7}$

4.4. Circular Orbit Method

Since the circular orbit method is newly developed in this report, no validation data from literature is available. The method is validated by defining a set of Kepler elements. Elements are chosen randomly, except for eccentricity and argument of pericentre which are set to zero. The Cartesian position is obtained for $\theta_1 = 0$ and $\theta_2 = 45$ [°]. An observer is defined at 40 [°] latitude and an elevation of 1 [km]. The local sidereal time of the observer at each epoch is $LST_1 = 120$ and $LST_2 = 120 + \omega_e * \sqrt{\frac{a^3}{\mu}}$ [°], where ω_e is the rotational speed of Earth, a is the semi-major axis of the satellite and μ is the gravitational parameter of Earth. From the observer and satellite position, right ascension and declination measurements are calculated. These are used as input to the circular orbit method. The result of the circular orbit method is converted to Kepler elements and listed in Table 4.7 next to the original Kepler elements. It is shown that the difference is negligible.

Table 4.6: Results of Gooding validation. The expected results are from Curtis [10].

Position [km]			Velocity $\frac{km}{s}$		
Expected	Obtained	Difference	Expected	Obtained	Difference
$5.662 \cdot 10^3$	$5.662 \cdot 10^3$	$3.965 \cdot 10^{-9}$	-3.885	-3.885	$4.545 \cdot 10^{-8}$
$6.538 \cdot 10^3$	$6.538 \cdot 10^3$	$5.540 \cdot 10^{-9}$	5.121	5.121	$4.957 \cdot 10^{-8}$
$3.269 \cdot 10^3$	$3.269 \cdot 10^3$	$1.454 \cdot 10^{-9}$	-2.243	-2.243	$3.743 \cdot 10^{-8}$

Table 4.7: Result of circular orbit validation.

Element	Initial	After Conversion	Difference
a [km]	$1.000 \cdot 10^4$	$1.000 \cdot 10^4$	$4.402 \cdot 10^{-10}$
e [-]	0.000	$5.241 \cdot 10^{-16}$	$5.241 \cdot 10^{-16}$
i [°]	$2.000 \cdot 10^1$	$2.000 \cdot 10^1$	$9.415 \cdot 10^{-13}$
Ω [°]	$1.000 \cdot 10^2$	$1.000 \cdot 10^2$	$4.405 \cdot 10^{-12}$
ω [°]	0.000	0.000	0.000
θ [°]	$4.500 \cdot 10^1$	$4.500 \cdot 10^1$	$3.702 \cdot 10^{-12}$

4.5. Gauss Refinement Algorithm

The Gauss refinement algorithm is validated in combination with the Gauss method as well as the circular orbit method. The former can be validated in a straightforward way, identical to the validation of the Gauss method. With the same input values from Table 4.2, Curtis [10] also provides an example of the results with the refinement. The differences are listed in Table 4.8 and are shown to be sufficiently small.

Table 4.8: Results of Gauss refinement validation in combination with Gauss method. The expected results are from Curtis [10]. The algorithm successfully converges after 10 iterations. Minimum update between steps is set to 10^{-8} .

Position [km]			Velocity $\frac{km}{s}$		
Expected	Obtained	Difference	Expected	Obtained	Difference
$5.662 \cdot 10^3$	$5.662 \cdot 10^3$	$5.999 \cdot 10^{-8}$	-3.885	-3.885	$6.902 \cdot 10^{-7}$
$6.538 \cdot 10^3$	$6.538 \cdot 10^3$	$8.386 \cdot 10^{-8}$	5.121	5.121	$7.527 \cdot 10^{-7}$
$3.269 \cdot 10^3$	$3.269 \cdot 10^3$	$2.206 \cdot 10^{-8}$	-2.243	-2.243	$5.685 \cdot 10^{-7}$

For the refinement algorithm in combination with the circular orbit method, the same approach was used as for the validation of the circular orbit method. The eccentricity and argument of pericentre were altered to deviate from a circular orbit and the LST at the first epoch was set to 100° . Instead of two, three epochs were calculated at $\theta_1 = 0$, $\theta_2 = 20$ and $\theta_3 = 40^\circ$. Otherwise, all inputs remained identical. Table 4.9 shows the obtained Kepler elements from the circular orbit method with refinement compared to the original orbital elements. It is shown that the difference between them is negligible.

Table 4.9: Results of Gauss refinement validation in combination with circular orbit method. The algorithm successfully converges after 28 iterations. Minimum update between steps is set to 10^{-12} .

Element	Initial	After Conversion	Difference
a [km]	$1.000 \cdot 10^4$	$1.000 \cdot 10^4$	$3.824 \cdot 10^{-9}$
e [—]	$1.000 \cdot 10^{-1}$	$1.000 \cdot 10^{-1}$	$1.489 \cdot 10^{-12}$
i [°]	$2.000 \cdot 10^1$	$2.000 \cdot 10^1$	$5.951 \cdot 10^{-11}$
Ω [°]	$1.000 \cdot 10^2$	$1.000 \cdot 10^2$	$1.501 \cdot 10^{-11}$
ω [°]	$3.450 \cdot 10^2$	$3.450 \cdot 10^2$	$2.206 \cdot 10^{-9}$
θ [°]	$2.000 \cdot 10^1$	$2.000 \cdot 10^1$	$2.192 \cdot 10^{-9}$

Simulation Results

This chapter will discuss simulations performed using TLE data. They will serve to establish a theoretical performance limit for the IOD methods in absence of any error as well as to analyse their behaviour for various magnitudes of observation error. For the simulations, a set of 42 objects in GEO or a similar high Earth orbit are used. The data is obtained by propagating TLE data of the GEO satellites in the data set. The TLE values are regarded as the truth, since any deviations with respect to the objects they pertain to are of no concern in this analysis. The middle observation is on 16 June 2020 at 22:00, two additional observations are generated at time steps equal to half of the observation arc before and after the middle epoch. Hence, when looking at an observation arc of 30% of the orbital period, the time between observations equals 15% of the orbital period. The propagation of the satellite position to the observation epoch is performed with the SDP4 propagator, a fast analytical integrator designed to work with TLE data [32]. It takes into account the most important harmonics of Earth's gravity field, third body gravitation from the Moon and the Sun as well as solar radiation pressure. The satellites and specific TLE epochs and orbital elements used for the simulation are listed in [Table C.1](#). For the observer, the MeerLICHT telescope's position is used, which is given in [Table 2.1](#).

In this report, the orbital elements that are analysed are the semi-major axis, eccentricity, inclination and the sum of the right ascension of the ascending node, the argument of pericentre and the true anomaly. None of the latter three quantities is well-defined for equatorial circular orbits, but their sum is. In that case, the sum of these elements denotes the right ascension of the satellite, and it will be referred to as such in this report.

5.1. Inherent method error

The orbit determination methods discussed in [chapter 2](#) have inherent shortcomings. This is unavoidable with the number of data points they use. In this section, the magnitude of the error inherent to the methods will be analysed and discussed. This will then be used in following sections to determine when the inherent error starts to dominate errors coming from input errors.

[Figure 5.1](#) shows the error of the methods of Gauss, refined Gauss, Gooding and the circular orbit method as function of arc length expressed as percentage of one complete revolution. The refinement algorithm is discussed in [subsection 2.2.3](#) and is specific to the Gauss method. However, given an additional observation, it can also prove valuable in combination with the circular orbit method, where it can relax the eccentricity constraint after an initial circular orbit estimate. The graph displays the average of the absolute errors of the results of all satellites in the dataset. The circular orbit method only uses the first and third observation to determine the orbit, but the error is calculated for the middle point for consistency with the other methods. The solution at the middle observations is also needed when applying the refinement algorithm to the circular orbit method, since it requires three positions. It can be observed that Gooding's method has the lowest error and that the error is constant over the range of observation arcs. The circular orbit method has a significantly larger error that also remains constant with time. The cause of this inherent error is the constraint on the eccentricity, which only allows four orbital element to be fit to the observations. Gauss' method as well as its refinement algorithm initially

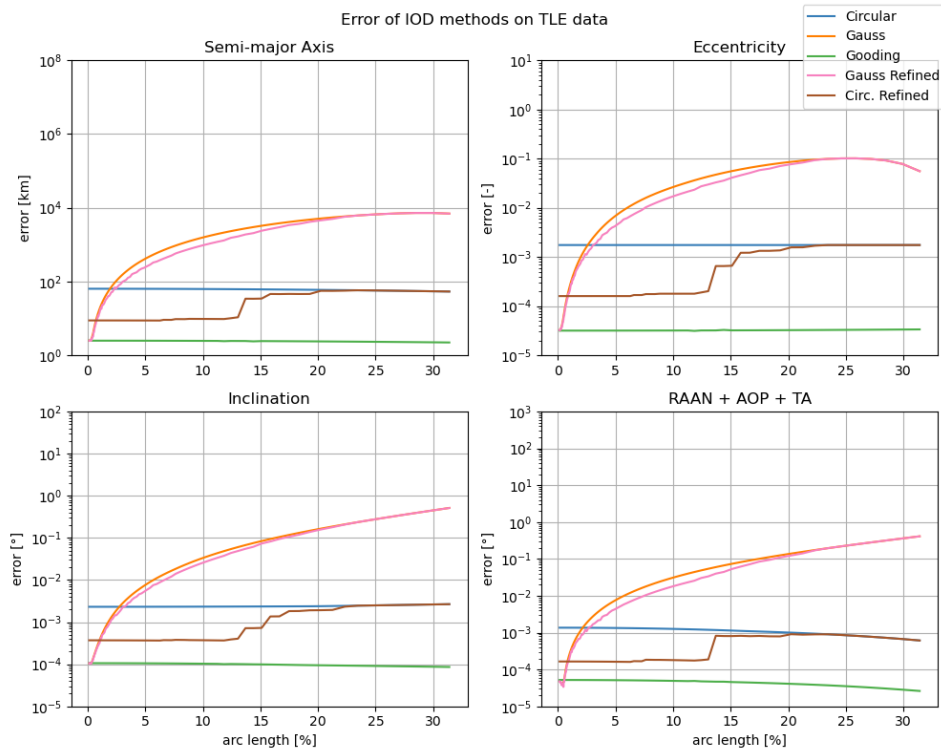


Figure 5.1: Inherent error of the methods of Gauss, Gooding and circular orbit method as function of arc length expressed as percentage of one revolution. The plots show the average error of 42 satellites in GEO and similar orbits.

show very small errors; however they quickly increase with increasing observation arc. This is due to the assumption in the Gauss method that the time between observations is short relative to the orbital period. Clearly, this assumption breaks down very quickly with the error in the semi-major axis already exceeding 1000 [km] at approximately 8% of the orbital period. As seen in Figure 2.2, The error in the truncated Lagrange coefficients increases with time and results in a large error on the orbit estimate.

The refinement technique for the Gauss method laid out in subsection 2.2.1 seems to have very little effect on the inherent error of the Gauss method when looking at Figure 5.1. However, The averaging over all satellites is misleading. In Figure 5.3, each satellite is plotted separately and the effect of the refinement is clearly shown. When the algorithm is able to converge, an approximately constant error is the result. When the algorithm can no longer converge for a satellite, it falls back to the original Gauss solution. This is confirmed when comparing the magnitudes of the error in Figure 5.3 to the average Gauss error in Figure 5.1.

The reason for the algorithm failing to converge at some point is unclear. Large differences can be observed between one satellite and another in terms of the moment at which the algorithm fails to converge. Given the similarity of the orbits of all satellites in the dataset, it is unclear on which quantities the convergence depends. The fact that once the algorithm stops converging, it can never recover for a longer arc indicates some dependency on the observation arc. Possibly, the initial error of the Lagrange coefficients shown in Figure 2.2 is one of the causes. However, there must be other factors at play to explain the difference in behaviour between individual satellites.

Finally, it must be noted that the remaining error with successful refinement is identical to that observed for Gooding's method in Figure 5.2. It is likely that the remaining error is a reflection of the orbit's deviation from a Kepler orbit and represents a theoretical limit of the precision that can be obtained using the assumption of a Kepler orbit.

Since the circular orbit method only requires two observations, the middle observation that is generated is not used for the orbit determination, but the position and velocity at the middle observation are calculated and the error with respect to the TLE at the middle observation is plotted. Figure 5.4 shows the inherent model error as function of arc length for the circular orbit method. The main assumption in

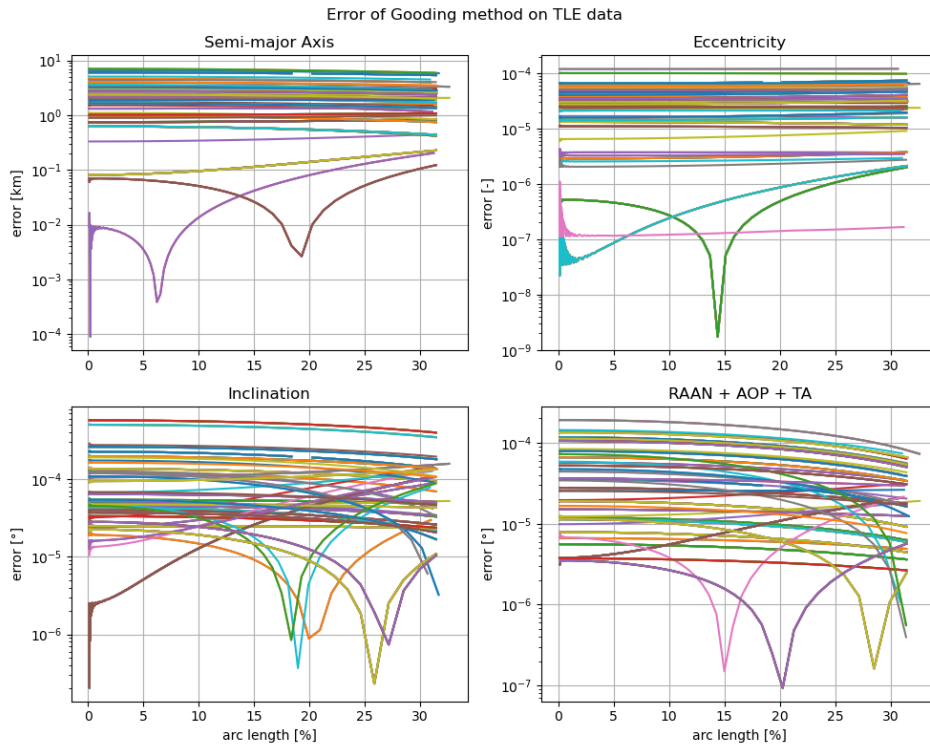


Figure 5.2: Inherent error of method of Gooding, refined, as function of arc length expressed as percentage of one revolution. The error for each satellite is plotted separately.

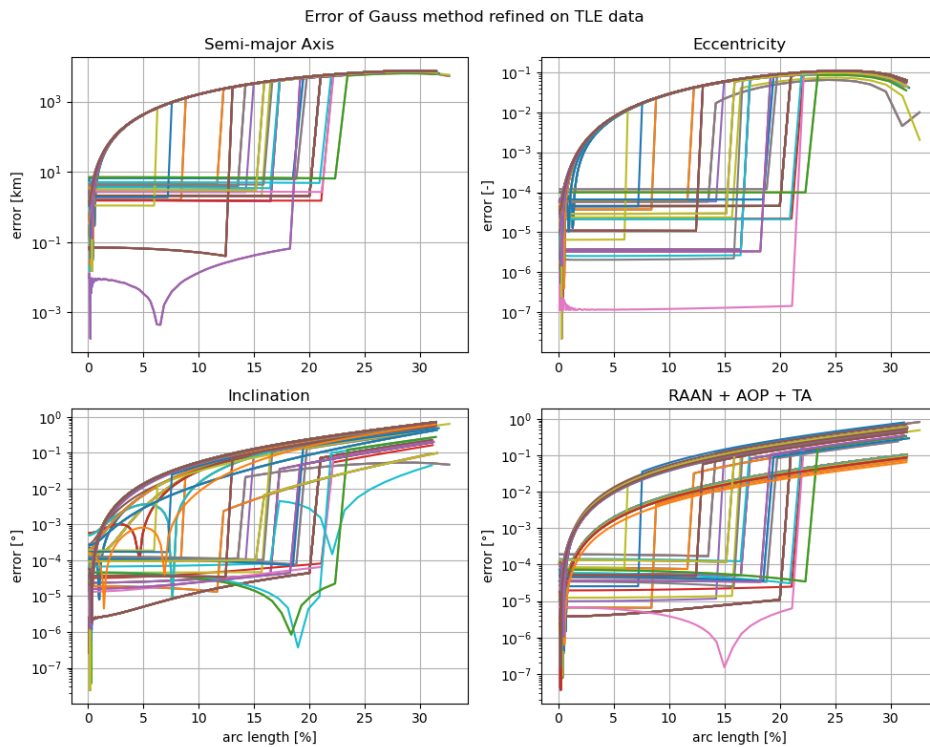


Figure 5.3: Inherent error of method of Gauss, refined, as function of arc length expressed as percentage of one revolution. The error for each satellite is plotted separately.

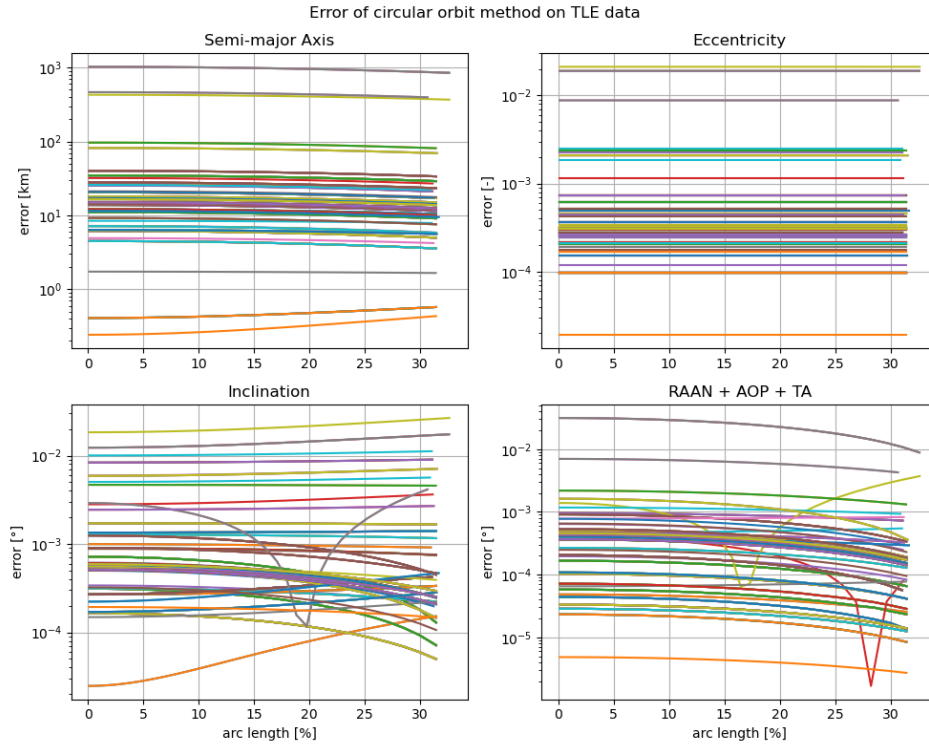


Figure 5.4: Inherent error of the circular orbit method as function of arc length expressed as percentage of one revolution. The error for each satellite is plotted separately.

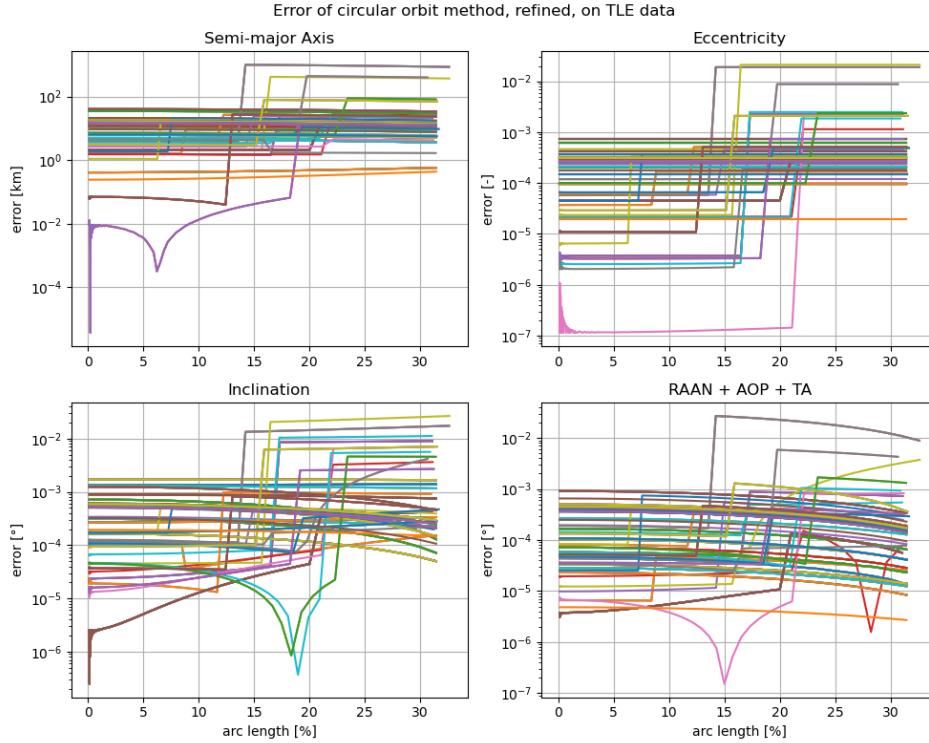


Figure 5.5: Inherent error of the circular orbit method, refined, as function of arc length expressed as percentage of one revolution. The error for each satellite is plotted separately.

the circular orbit method that distinguishes it from the methods of Gauss and Gooding is that the orbit is circular. It can be seen in [Figure 5.4](#) that the satellites with the highest eccentricity also have the largest error in semi-major axis and show varying error for inclination and satellite right ascension. The objects with very low eccentricity show no error variation. The relatively large errors in semi-major axis for the most eccentric orbits indicate a high sensitivity to the object's eccentricity. This is not unexpected, but might prove problematic when using the method to obtain an initial estimate for objects with higher eccentricity. This is a subject of further research. The object with the highest eccentricity in the current dataset is object 41585U with an eccentricity of $2.115 \cdot 10^{-2}$.

The Gauss refinement algorithm is also applied to the circular orbit method. For this, all three observations are used. The results are shown in [Figure 5.5](#) for individual satellites. Comparing this figure to [Figure 5.3](#), it should be noted that the point at which the refinement algorithm stops converging is the same for each satellite in both graphs. This implies that the convergence of the refinement algorithm is independent of the magnitude of the orbit error, since this is very different for both methods. As long as the algorithm can converge, it greatly reduces the inherent error of the circular orbit method, as it removes the constraint on the eccentricity.

5.2. Effect of input error

In the previous section, it was observed that the methods of initial orbit determination show small errors for short observation arcs. The circular orbit method and the Gooding method have a constant error over varying observation arc length, while Gauss shows an increasing error. However, those results were gathered in absence of error on the input values; right ascension, declination and time stamps. For an accurate prediction of each method's performance, these must also be taken into account. The data from the MeerLICHT telescope contains very short observations arcs of no more than 5 minutes of observations spaced 60 [s] apart. For GEO satellites, this corresponds to an observations arc of less than 0.5%. In this regime, IOD methods are known to be extremely sensitive to input errors. In this section, input errors will be modelled as normal distribution with zero mean and varying standard deviation σ . Errors in right ascension and declination are analysed independently from errors on the timestamps of the observations. Both right ascension and declination are given a random error with no correlation between the two. Errors in timestamp cause an along track error in the orbit determination. If the error is equal for all observations, the error mainly affects the true anomaly. Errors in right ascension and declination generally act both along and cross track. In the case objects orbiting in the equatorial plane, a decoupling occurs, where right ascension errors mainly act along track and declination errors mainly cross track.

5.2.1. Error in Right Ascension and Declination

Four different magnitudes of standard deviation are plotted in [Figure 5.6](#): 0.1, 1, 3 and 5 [*arcsec*]. The first is lower than the angular resolution of the MeerLICHT telescope, which is 0.28 [*arcsec*], half of the pixel resolution. This will give an idea of the the performance limits of the MeerLICHT telescope for GEO observation with very short arcs. In practice, the resolution of the MeerLICHT telescope is likely lower, since eliminating all sources of error is no trivial task and the telescope is seeing limited. The largest standard deviation approximately matches the estimated error in the endpoint determination of the FOTOS routine based on [Figure 3.11](#), which will give an indication of the effect of this specific error. The two intermediate values serve to establish a sense of the rate at which the orbit error increases.

[Figure 5.6](#) shows the orbit estimate errors on the IOD methods for various random errors in right ascension and declination of the observations. It can be observed that for very short arcs, all methods are extremely sensitive to input error of any magnitude, with a notable exception for the circular orbit method, whose constraint on the eccentricity seems to render the method relatively robust against measurement error. In addition, the methods of Gauss, including refined Gauss, and Gooding show a similar trend of the error until the short arc assumption of Gauss starts to dominate the input error. The behaviour of Gauss' method for the shortest arcs is consistent to findings from Dichter and Wojcik [13], who investigated the minimum time interval at which the Gauss method will convergence given the magnitude of the measurement error and concluded that for $\sigma = 1$ [*arcsec*], the Gauss method starts converging with an observation arc of approximately 300 [s].

The error in the method of Gauss shows a clear minimum that varies with the magnitude of the input error. The minimum is located at approximately 2%, 4%, 5% and 6% for input errors of 0.1, 1, 3 and

5 [arcsec], respectively. It follows that when the approximate magnitude of the input error is known, a theoretical limit to the accuracy of the Gauss method can be derived as well as an optimal length of the observation arc. While for the range of random errors applied, this minimum error can be considered to be acceptable for an initial orbit estimate, it is still higher than what can be achieved with the other IOD methods.

The refinement algorithm for the Gauss method is unable to improve the orbit estimate for short arcs. Only when the orbit error starts increasing again with the TLE data, an effect can be observed which is very similar to the refinement algorithm's behaviour in Figure 5.1. In practice, this means that the refinement can decrease the minimum error achievable by the Gauss method, allowing it to match Gooding's performance for longer arcs, until it is no longer able to converge and falls back to the Gauss solution, similar to Figure 5.3.

Where Gauss' method starts suffering from its inherent error in Figure 5.6, the orbit error of Gooding's method keeps decreasing with increasing arc length. It can clearly be seen how a larger input error requires a longer orbital arc for the same orbit error. In addition, the larger errors prevent Gooding's method from achieving the same accuracy as was observed in Figure 5.1 for the longest arc length in the figure. Nevertheless, given a sufficiently long arc of observations, Gooding's method can provide relatively accurate information regarding an object's orbit and should prove more practical to use than the Gauss method, mainly because Gooding's method can achieve smaller errors.

As indicated before, the circular orbit method shows the most stable behaviour in the presence of input error in Figure 5.6. The orbit error increase for the shortest arc is very small compared to the other methods and quickly decreases to values close to those in Figure 5.1. This changes, however, when applying Gauss' refinement to the circular orbit method. While the refinement improves the orbit estimate in Figure 5.1, this is no longer true when considering observation errors. Previously, the relaxation of the eccentricity constraint allowed increased accuracy of the orbit estimate, but now, this relaxation is the cause of a large increase in eccentricity due to the input errors, which makes the solution more degenerate. However, the refinement algorithm can still reduce the error on the circular orbit method on a small range of observation arcs between 2 and 15%, depending on error magnitude; however, not for the arc lengths available in the observation data from MeerLICHT.

5.2.2. Error in Observation time

Figure 5.7 shows the errors for four different values of the standard deviation of the time error. The first is the estimated highest accuracy that can be achieved with the MeerLICHT telescope, while the other three values serve to illustrate the effect of different magnitudes of measurement error.

Figure 5.7 shows the results of adding a random error to the time stamp of the observations. Qualitatively, the behaviour is very similar to Figure 5.6 which shows the results for errors in right ascension and declination. A striking difference is the apparent insensitivity of orbit solution for the semi-major axis to the magnitude of the error. The existence of an error causes very large errors, but no difference between different error magnitudes can be observed for the methods of Gauss and Gooding. The cause of this is unknown, however, it implies that in order to reduce the error on the orbital elements caused by time errors, the time error must be eliminated completely, or at least below some unknown threshold, for the effort to have any effect on the quality of the orbit solution.

While it is unknown why the error magnitude does not affect Gauss' and Gooding's method, its effect on the circular orbit method is according to expectations, as the time difference between observations directly leads to the expected angle between them. In fact, when comparing the errors in Figure 5.7 to those in Figure 5.6, it can be concluded that the circular orbit method is relatively more sensitive to time errors. This is most apparent for the satellite right ascension, where the errors quickly stop decreasing and show a larger residual orbit error for a larger time error.

The behaviour of the refinement algorithm is nearly identical to Figure 5.6 for both the Gauss and circular orbit method. It severely deteriorates the estimate for the circular orbit method and only shows a small average improvement for the Gauss method after its minimum. For the circular orbit method, the refinement algorithm can only improve the eccentricity for observation arcs between 10% and 15%, which is a much smaller improvement than for right ascension and declination errors.

Finally, a time error causes a slower decrease of the orbit error with increasing orbital arc as compared to an error in right ascension and declination. This holds equally for all methods considered. GEO

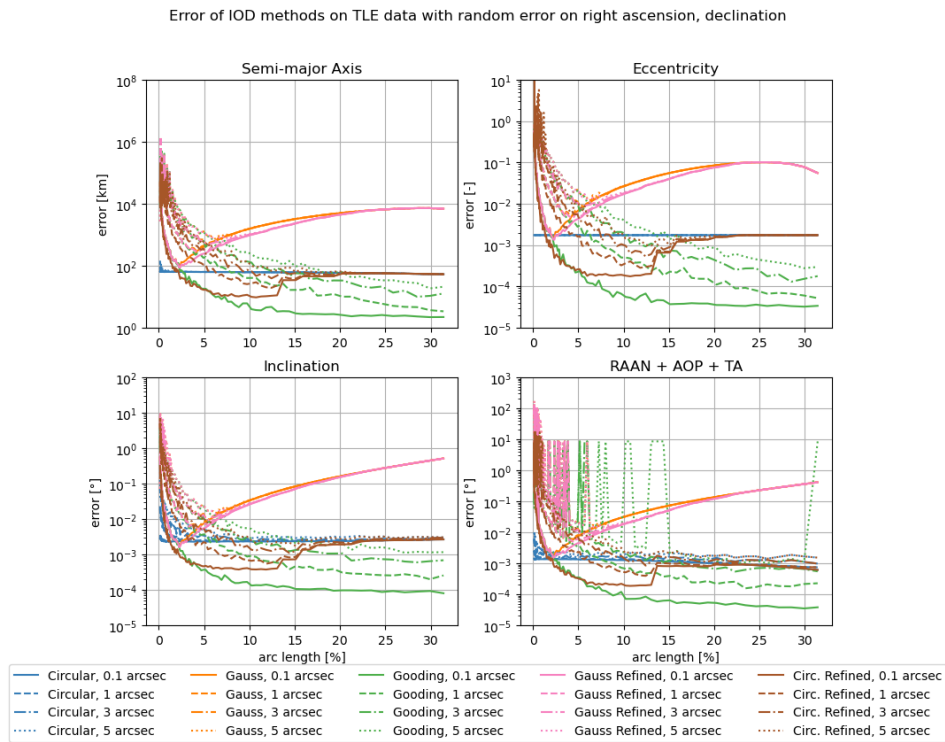


Figure 5.6: Error of initial orbit determination methods with observation errors as function of arc length expressed as percentage of one revolution.

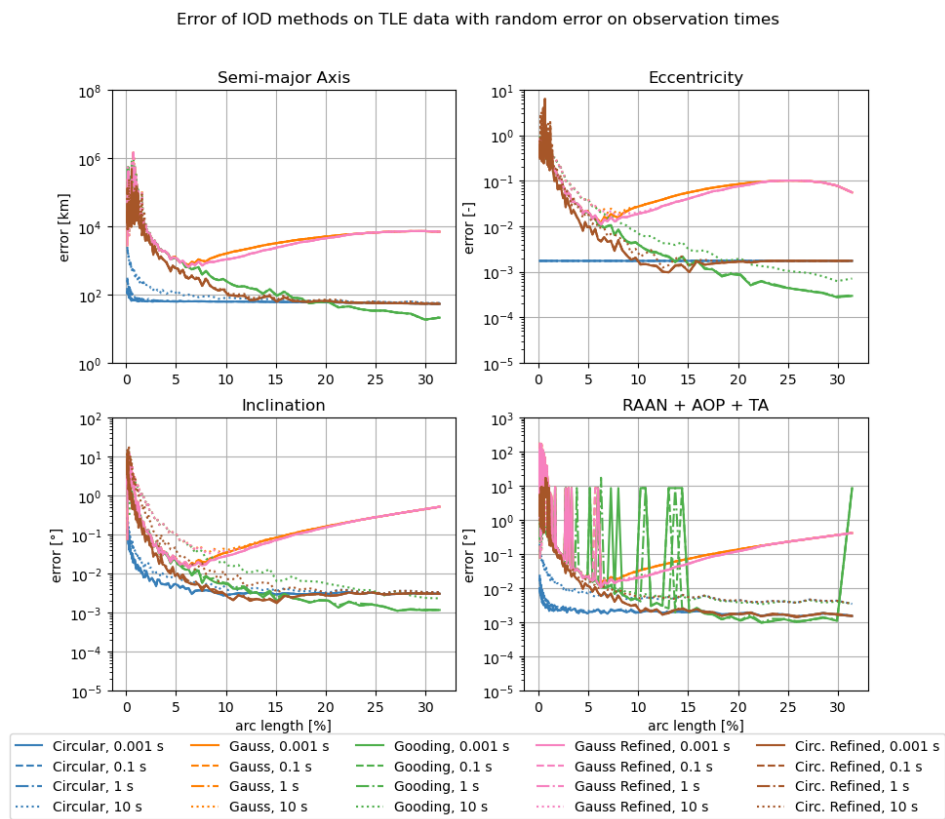


Figure 5.7: Error of initial orbit determination methods with observation errors as function of arc length expressed as percentage of one revolution.

satellites move through the image at a speed of approximately $30 \left[\frac{px}{s} \right]$, depending on elevation. A time error of 10 [s] is then equivalent to an along track error of approximately 300 [px] . Similarly, an error of $1 \cdot 10^{-3} \text{ [s]}$ is equivalent to an along track error of 0.03 [px] . This calculation does not take into account the error in the observer position, however, the latter is relatively small. Due to Earth rotation, the MeerLICHT telescope, located at $-32.3799 \text{ [}^\circ\text{]}$ latitude, has a velocity of approximately $400 \left[\frac{m}{s} \right]$ in the inertial frame. An error in the observer position causes a translation of the line of sight vector inversely proportional to the angle between the observer position vector and the line of sight vector: $400 \cdot \epsilon_t \cdot \frac{\vec{R} \cdot \hat{\rho}}{R}$, where ϵ_t is the time error and $\frac{\vec{R} \cdot \hat{\rho}}{R}$ is the cosine of the angle between observer position and line of sight vector. The magnitude of the translation varies between $400 \cdot \epsilon_t \text{ [m]}$ for observations in zenith and 0 [m] for observation on the horizon. This is equivalent to $40 \cdot 4 \cdot 10^{-3}$ for observations in zenith for $\sigma = 10$ and $\sigma = 10^{-3} \text{ [s]}$, respectively. While the along track error explains the cause of the relatively large orbit error for $\sigma = 10 \text{ [s]}$, it does not provide a rationale for why there is little difference in orbit error due to varying σ . This is a topic of further investigation.

5.3. Conclusions

From the simulations performed in this chapter, it can be concluded that none of the classical orbit determination methods are expected to perform well for the short arcs present in the observation data: given the magnitude of the errors, it is expected to be difficult to obtain follow up observations based on the initial estimates. The method of Gauss only has a small region where the error is relatively low, with a minimum error at approximately 6% of the orbital period, depending on the input error magnitude. Before the minimum, it is on par with Gooding's method, however, the latter shows a further decreasing orbit error with increasing observation arc whereas Gauss does not. Hence, overall, Gooding's method performs better than Gauss'. The circular orbit method proves to be much more robust against input errors and provides relatively stable and accurate results for GEO objects. The results of the circular orbit method are very promising for obtaining follow up observations. Looking at the orbit error behaviour for the smallest measurement errors in the figures, it can be concluded that, given the resolution of the MeerLICHT telescope both in right ascension and declination as well as time, even a theoretically ideal set of observations where no other sources of error are present has insufficient accuracy to allow a good orbit estimate from the methods of Gooding and Gauss for observation arcs of only a few minutes.

The refinement algorithm is of limited practical use. For observations without error, it can significantly reduce the error on both the Gauss and circular methods. However, the method shows large variations in performance between individual satellites and the cause of this is unknown. When adding random errors to the right ascension and declination, the refinement actually deteriorates the orbit estimate of the circular orbit method for short arcs up to 2-10%, depending on the input error magnitude, and beyond this only improves the estimate for a small portion of arc lengths up to 15%. For errors in time, the refinement algorithm generates more degenerate orbits until approximately 15%, after which it is equal to the circular orbit method. For the Gauss method, the algorithm delays the point at which the inherent error starts to dominate the solution for both types of input error.

Results of Measurement Error Reduction

This chapter will analyse the effect of the measurement error reduction techniques that were discussed in [section 3.4](#). The analysis is performed on observations solely from the night of 16 June 2020. The reason for this is that correcting the endpoints is a manual and time consuming effort. In this chapter, only the results on 120 [s] arcs are shown and analysed. The same analysis was performed for 140 [s] arc as well, however, the results were nearly identical and therefore not discussed in detail.

The orbit estimates are filtered: only the bound orbits on a non-impacting trajectory with Earth are considered to be valid estimates. Parabolic and hyperbolic orbits as well as elliptical orbit intersecting Earth are considered invalid. For each error reduction method, the number of valid solutions found by each IOD method out of the 64 observations sets is listed in [Table 6.1](#). This is an important measure, since it is a direct indication of the effect of the input error reduction. The results in the table are discussed in [section 6.1](#), [section 6.2](#) and [section 6.3](#) for endpoint correction, timing correction and both corrections together, respectively. Next to that, [Figure 6.1](#), [Figure 6.2](#) and [Figure 6.3](#) show the error on the orbit estimate for data without corrections, with the endpoint correction, the time correction and both corrections applied for the valid solutions for the circular orbit method, the method of Gauss and of Gooding, respectively. The numbers shown in the top left corner of each plot are the average error for each data set. The discussion of the results for the error corrections is referred to their respective sections; the endpoint correction in [section 6.1](#), the timing correction in [section 6.2](#) and both applied together in [section 6.3](#). This chapter is concluded in [section 6.4](#) with a decision on which corrections to implement for the analysis in [chapter 7](#).

Table 6.1: Number of valid orbits found with measurement error corrections out of 64.

IOD Method	Valid Orbits			
	No Corr.	Endpoint	Time	All
Circular	64	64	64	64
Gauss	16	15	16	15
Gooding	19	16	15	18

6.1. Endpoint Correction

The endpoint correction method is discussed in [subsection 3.4.2](#). [Appendix E](#) lists the right ascension, declination and timestamp for all manually determined endpoints. The number of valid orbit estimates in [Table 6.1](#) for endpoint correction are slightly below those for uncorrected input data. While the limited size of the dataset does not allow a strong conclusion, it does not appear that the endpoint correction generates a higher number of valid orbits. This is discussed below.

The endpoint correction causes the error on the semi-major axis to increase by almost a factor two for the method of Gauss. For the eccentricity, inclination and satellite right ascension, a decrease in error is observed. Gooding's method shows an increase in error on the semi-major axis by almost an order

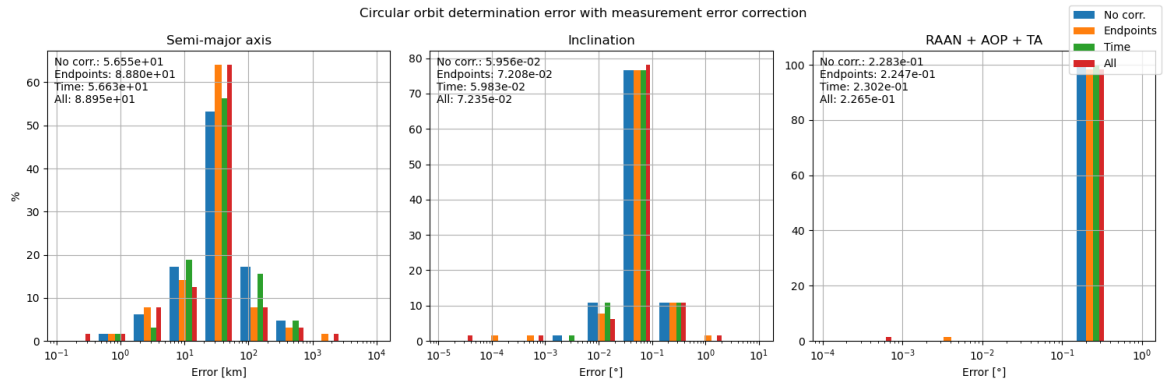


Figure 6.1: Orbit estimate errors for different measurement error corrections for circular orbit method

of magnitude and a decrease in error on the inclination and satellite right ascension. The error on the eccentricity remains approximately the same. For the circular orbit method, the endpoint correction causes the error on the semi-major axis and inclination to increase, while the error on satellite right ascension is approximately the same.

It appears the endpoint correction does not improve the orbit estimates. The errors on semi-major axis and eccentricity, the orbital elements with the relatively largest error, strongly increase and remain approximately the same, respectively. An explanation for this can be found in Figure 6.4. The figure shows the relative improvement of the error of individual observation sets with respect to the unaltered data for the circular orbit method. The data is generated using observation arcs of 60 [s]. With two observations spanning a 60 [s] arc, most observation sets in this dataset are simply one streak in an image. In the figure, values between -1 and 1 indicate an improvement. Note that the figure is zoomed in on the region around the origin and does not show all 94 available data points. Figure 6.4 shows that the circular orbit solutions for the semi-major axis with endpoint correction are concentrated on the negative part of the axis. This means the error has changed sign and indicates that the new endpoints are likely an overcorrection of the error. This is confirmed by examining Equation 2.24, which provides a straightforward relation between semi-major axis and angular separation of the observations for the circular orbit method. An overestimation, respectively underestimation of the streak length would result in an underestimate, respectively overestimate of the semi-major axis. Hence, while it was concluded in subsection 3.4.2 that the FOTOS endpoint determination method consistently makes streaks too long, the fact that the error on the semi-major axis switches sign in Figure 6.4 leads to the conclusion that the new endpoint determination makes streaks too short. Too short streaks would not affect the inclination or the satellite right ascension, which is also confirmed in the figure and supports this theory.

6.2. Timing Correction

The number of successful orbit estimates in Table 6.1 shows no difference for the Gauss method, yet a decrease for the Gooding method. This difference in results between the methods of Gauss and Gooding will be discussed in the next paragraph. Based on the values in Table 6.1, it can be concluded that the timing correction does not generate more valid orbit estimates.

From the plots and values in Figure 6.1, Figure 6.2 and Figure 6.3, it is clear that the timing correction has a smaller effect on the orbit estimate error than the endpoint corrections. The difference between the time correction and the original solution is negligible for the circular orbit method and the method of Gauss. For the method of Gooding, the timing error produces worse results for all elements except the satellite right ascension, for which the error decreases. From Figure 5.7, it was concluded that the existence of a time error, regardless of its magnitude, can generate very large orbit errors. Based on the results, this could mean that another (small) source time error still remains, which would explain the lack of improvement observed with the circular orbit method and the method of Gauss. However, this is not consistent with the behaviour of the method of Gooding. In Figure 5.7, similar trends were observed for the methods of Gooding and Gauss up to an observation arc of 6%, but Figure 6.2 and Figure 6.3 show very different results from both methods. It is unclear by what this inconsistent behaviour is caused.

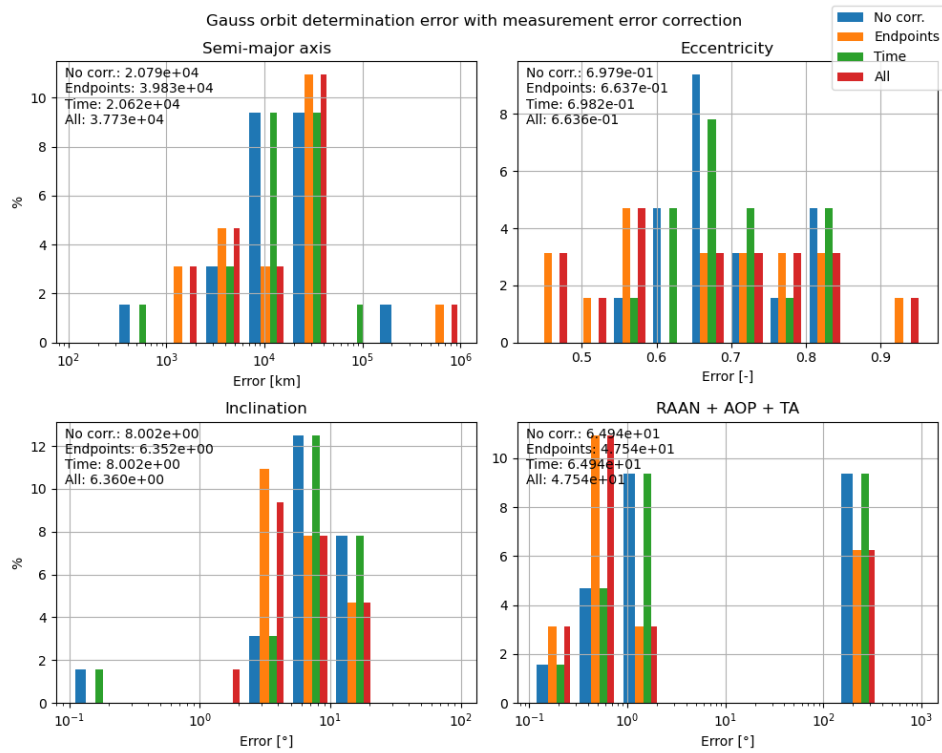


Figure 6.2: Orbit estimate errors for different measurement error corrections for Gauss method

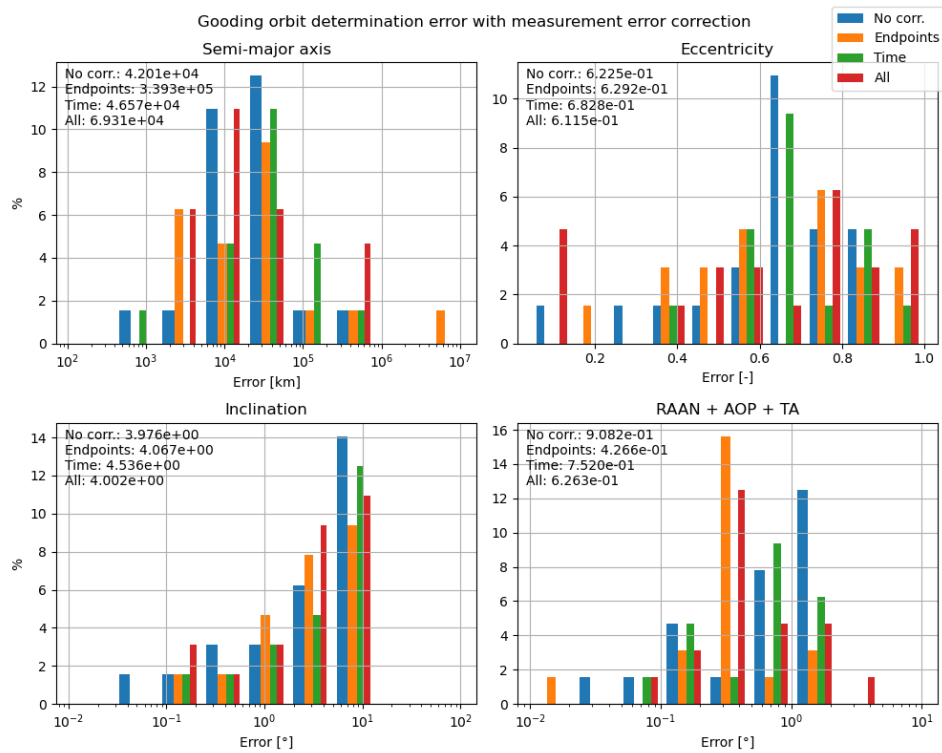


Figure 6.3: Orbit estimate errors for different measurement error corrections for Gooding method

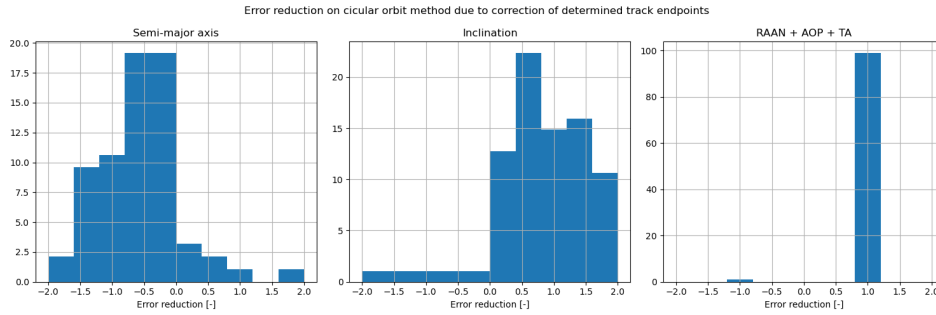


Figure 6.4: Orbit estimate errors for endpoints correction on circular orbit method.

Gooding's method is configured the same way in both chapters.

It must be noted that the magnitude of the timing correction is relatively small compared to the endpoint correction. The latter changes the observed endpoints by approximately 10 [px] as seen in Figure 3.11, while the former produces a correction on the order of 10 [ms], which is equivalent to approximately 0.3 [px]. Hence, the presence of a relatively large error in right ascension and declination might also prevent the timing correction to improve the orbit estimate.

6.3. Combining all corrections

The previous sections showed that the input error corrections considered cannot improve the orbit estimate from any of the IOD methods. Looking back to chapter 5, which displayed an extreme sensitivity of all methods to input errors, it is likely one error correction measure cannot produce better results because other sources of input error remain. Hence, all errors should be corrected simultaneously to obtain the best results. Unfortunately, it can immediately be observed from the Figure 6.1, Figure 6.2 and Figure 6.3 that implementing both corrections does not produce better results. Since the endpoint correction is relatively larger than the timing correction, the solution with all corrections is almost identical to the one with only the endpoint correction. This is also observed in Table 6.1.

It must be noted the timing and endpoint correction applied in this chapter are not the only sources of error. Several other sources of error are still present in the data, however, their impact is expected to be relatively small. This is briefly discussed in chapter 3. In chapter 5, it was observed that for short arcs of less than 1% of the orbital period, the magnitude of the measurement error does not affect the magnitude of the orbit error. The lack of improvement from the measurement error reduction techniques in this chapter is consistent with those findings and leads to the conclusion that measurement error should be reduced to negligible magnitudes in order to obtain accurate orbit estimates on arcs of less than 1%.

6.4. Conclusion

The measures taken to reduce the input error in the hopes of improving the orbit estimates have proven ineffective. The most likely cause is the presence of other sources error on the input values. In addition, it must be considered that the manual determination of endpoints is not accurate enough. It should be investigated whether the assumption in the method were invalid or the manual endpoint determination was carried out with insufficient accuracy. Due to the lack of improvement on the orbit estimate, the error reductions will not be applied for the analysis in the next chapter. It should be noted that the findings from this chapter largely agree with those of chapter 5 and that it can be concluded that for very short arcs, the magnitude of the error is of little concern, but instead the mere presence of an error source is the cause of the orbit error.

This same analysis was performed for observations arcs of 240 [s] for the methods of Gooding and Gauss and the circular orbit method. The results were nearly identical to those shown for the 120 [s] arcs discussed here, with the single exception of a small decrease in error on the semi-major axis for the circular orbit method.

Observation Results

In this chapter, the results of the orbit determination on actual observations will be discussed. The different IOD methods and their variations will be used to produce orbit estimates for a set of GEO satellites. The findings will be compared to those from [chapter 5](#) regarding the error behaviour of the IOD methods on simulated observations for varying orbital arc and measurement error magnitude. The data set used in this chapter has no corrections on the inputs, since the analysis in [chapter 6](#) did not indicate any improvement in the orbit estimates. Hence, the endpoints as determined by the FOTOS routine are used without timing correction for the shutter wheel motion. The data used in this chapter spans all four nights of observations. The observation data is available in [Appendix D](#).

7.1. Observation Strategy

Observations were made on the nights of 16, 17, 19 and 24 June 2020. The observation set was created taking into account the time of year and visibility of satellites as well as the scientific objectives of the MeerLICHT telescope. The latter was accomplished by observing satellites while they were in fields of scientific interest. The goal was to obtain observations of all satellites multiple times during each night, however, due to unfortunate weather and technical issues, this number of observations was only obtained for IntelSat 602 (20135U).

As mentioned before, the MeerLICHT telescope has an exposure time of 60 [s]. In addition, there is approximately 60 [s] overhead between the end of one observation and the start of the next. The result is an approximately equal spacing of streak endpoints in time. Observation sets are considered with varying time between individual observations. The circular orbit method has observation sets spanning 1, 2, 3, 4 and 5 minutes, as well as several hours for those satellites that were observed twice in a single night. For the methods using three observations, sets are available with a total timespan of two or four minutes. The reason that no sets of 3 or 5 [min] were considered for the methods of Gauss and Gooding is the dependency of Gauss' method on the symmetry of the observations. Celletti and Pinzari [8] show that the method of Gauss produces better results for shorter arcs if the observations are equally spaced. For the same reason, an arc of several hours is not used, since only one approximately equally spaced observation set for that arc length is available, for IntelSat 602. For all other objects, at most two observations with a spacing of several hours are available, which would give extremely asymmetrically spaced observations. [Table 7.1](#) lists the amount of observation sets that are available for each time span. It can be seen that the number of observations quickly decreases with increasing arc.

Table 7.1: Observations sets available for each time span

	60 [s]	120 [s]	180 [s]	240 [s]	300 [s]	4 – 6 [hrs]
Observations sets	209	136	88	40	20	10

7.2. Error Definition and Filtering of Results

The error shown in the graphs in this report are defined with respect to the two line element set (TLE) of the object propagated to the time of the observation. According to Fröh and Schildknecht [16], the TLE of a GEO object is accurate to 25 [km] along track and 10 [km] cross track on average, but when the error from the IOD method is much larger, they can be regarded as truth. The TLE errors correspond to an angular error of approximately $3.9 \cdot 10^{-2}$ and $1.6 \cdot 10^{-2}$ [°] for along track and cross track directions, respectively. For the MeerLICHT telescope, which has a pixel resolution of approximately 100 [m] at GEO altitudes, this means an average pixel difference between TLE and observations on the order of several hundred pixels. The difference between the observations and the TLE data is plotted in Figure 7.1, which confirms the error of the TLE data. The colour indicates the difference between the TLE epoch and the observation epoch in days. It can be seen that the age of the TLE does not affect the error of the available observations. The outlier with a pixel error of approximately 1600 is object 41744U, USA 270.

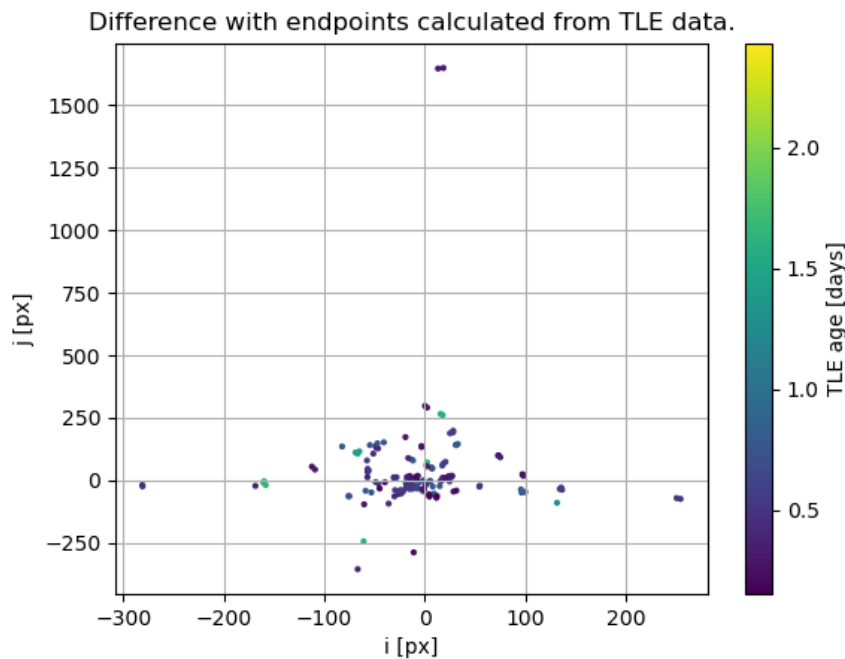


Figure 7.1: Difference between FOTOS endpoints and calculated TLE positions

The error is shown for the semi-major axis, the eccentricity, the inclination and the sum of right ascension of the ascending node (RAAN), argument of periapsis (AOP) and true anomaly (TA). For those object that do not orbit in the equatorial plane, the error on RAAN is also analysed. For these objects, RAAN is defined and should be estimated to be able to determine the orbital plane of the object, which is important for obtaining subsequent observations.

For every IOD method, the final result is filtered and only accepted if the result is a closed orbit around Earth that does not impact the surface. This is done because parabolic or hyperbolic orbits and orbits that impact Earth are not a valid solution when calculating Earth orbiting satellite trajectories. The percentage of valid orbits found for each IOD method then serves as an additional performance parameter.

7.3. Gauss Method

Figure 7.2 shows the results of the Gauss method on arcs of 120 and 240 [s]. The errors of the unaltered Gauss method from Curtis [10] are shown in blue and green for 120 [s] and 240 [s] arcs, respectively. Following Stoker et al. [31], the solutions of the Gauss method are circularised and plotted in orange and red for both arc lengths. Table 7.2 lists the percentages of observation sets for which

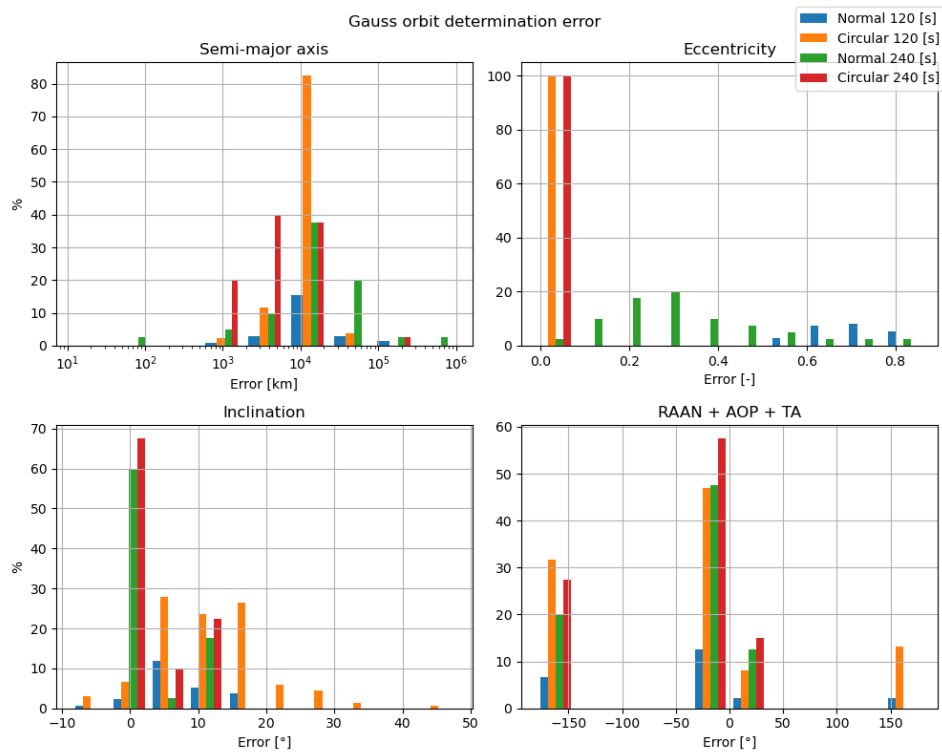


Figure 7.2: Error with respect to TLE data of orbit estimates using Gauss method for various arc lengths.

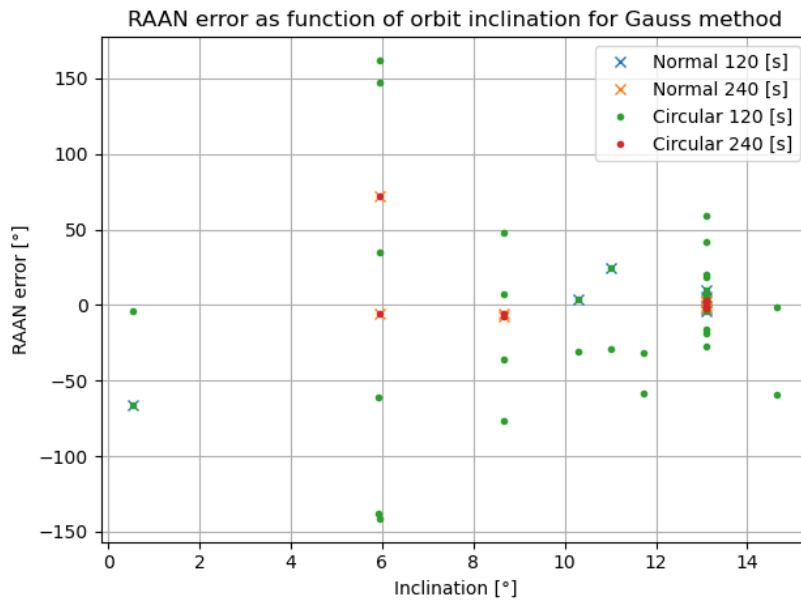


Figure 7.3: Error with respect to TLE data of orbit estimate RAAN values as function of inclination using Gauss method for various arc lengths.

unaltered Gauss could generate an accepted solution. It can be observed that for the shortest arc, only approximately a quarter of the observation sets generates an elliptic solution. This number increases sharply for the longer arc. The circularised Gauss method has 100% accepted solutions for both arc lengths, since none have an orbital radius of less than the equatorial radius of the Earth.

Table 7.2: Number of accepted solutions for each arc length. Accepted solutions are bound orbits around Earth. The total number of observations sets is 136 and 40 for 120 and 240 [s] arcs, respectively.

Arc Length [s]	Obs. Sets [%]
120	32
240	32

The orbit error in Figure 7.2 is large for all Kepler elements. For all arc lengths and both the unaltered and circularised Gauss methods, the satellite right ascension error is concentrated around 0 and ± 150 [deg]. The latter error is due to inversion of the $\rho_i \hat{\rho}_i$ term in Equation 2.3 caused by a calculated negative slant range. The root cause is a sign switch in the determinant D_0 of the line of sight matrix $[\hat{\rho}_1 \ \hat{\rho}_2 \ \hat{\rho}_3]^T$ (Equation 5.108 [10]), which propagates through the solution procedure and also causes the sign change of the slant ranges (Equations 5.112, 5.113 and 5.114 [10]). The result is a position on the far side of Earth relative to the observer, which explains the large error in satellite right ascension.

For the unaltered orbit solutions, the error on the semi-major axis is approximately of the same order as the expected semi-major axis for a geosynchronous orbit. The main advantage of the longer observation arc is that it generates more accepted solutions, however, their accuracy does not improve for the semi-major axis. Looking at eccentricity and inclination, the longer arc shows a clear decrease in error magnitude. This is especially obvious for the eccentricity, where two distributions can be observed, with distinct averages at approximately 0.3 and 0.7 for 120 [s] and 240 [s] arcs, respectively. For the inclination, the distribution for the short arc already has a mean close to zero, but the longer arc shows a much higher concentration of errors near zero.

The circularised solutions show the most improvement on the semi-major axis. Here, longer arc shows a clear decrease in the error compared to the unaltered solution. The error in eccentricity is very small by design. The differences observed for the inclination and satellite right ascension are the result of a difference in filtering. The hyperbolic orbits produced by the unaltered Gauss method are filtered out, but these same solutions are circularised in the circularisation procedure and hence not filtered afterwards. This explains the differences in the inclination and satellite right ascension, as these parameters are not affected by circularisation.

Figure 7.3 shows the error on RAAN as a function of inclination. The solutions of unaltered Gauss are indicated by an 'x' symbol to differentiate them from the circularised solutions which are identical for a specific observation set. It can be observed that the circularised Gauss method has extremely large errors on RAAN for the shortest observation arc. The solutions of unaltered Gauss are much smaller and within ± 20 [°] for inclination above 2 [°]. The longer observation arc produces better results for both versions of the Gauss method.

It can be concluded that for the 240 [s] observation arc, the circularised Gauss method performs better than the unaltered method. However, for the 120 arc, the fact that the hyperbolic solutions are not filtered but instead circularised results in worse results, with especially large errors on the RAAN.

7.4. Circular Orbit Method

The circular orbit method performs very well for geostationary orbits. Figure 7.4 shows the error of the semi-major axis, inclination and satellite right ascension with respect to TLE data for various orbital arcs. The eccentricity is not plotted as it is zero by definition for the circular orbit method. It can be seen how the error shows decreasing spread for increasing orbital arc lengths. However, a bias in the results can be observed. For increasing orbital arc, the inclination converges to an error of approximately 0.05 [deg] and satellite right ascension to -0.23 [deg]. The cause of this is not known, but is likely related to the error on the TLE. The errors in Figure 7.4 approach the order of magnitude of the TLE error,

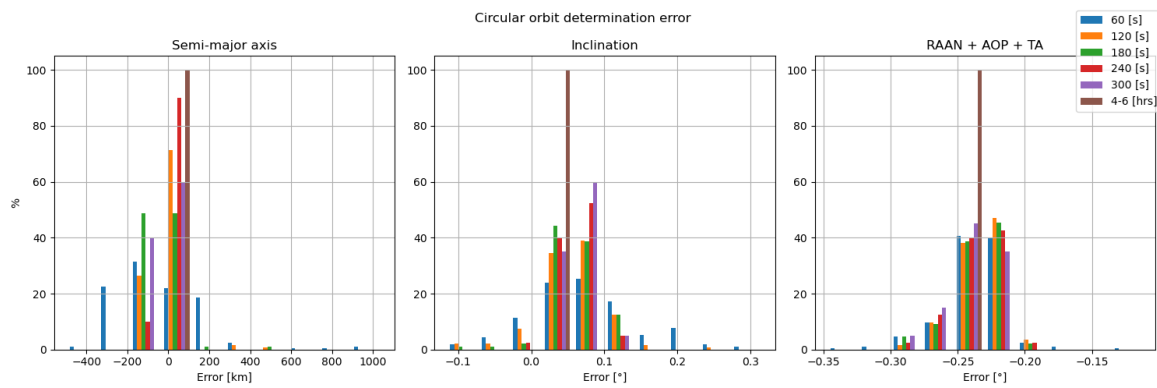


Figure 7.4: Error with respect to TLE data of orbit estimates using circular orbit method for various arc lengths.

which means the TLE error is no longer negligible compared to the orbit estimates from the circular orbit method.

Figure 7.5 plots the error of RAAN as function of inclination for inclinations higher than 0.5° . It can be observed that the RAAN is relatively accurately defined, with an error on the order of 1° . The error on the inclination is also very small, which gives a relatively accurate definition of the orbital plane of the object. This greatly simplifies obtaining additional observations, since a simple plane scan will allow the recovery of the object.

7.5. Gooding Method

Table 7.3: Number of accepted solutions for each initial value for Gooding's method. Accepted solutions are bound orbits around Earth. The total number of observation sets is 136 and 40 for 120 and 240 [s], respectively.

Arc Length Update step	120 [s]		240 [s]	
	10^{-5}	10^{-8}	10^{-5}	10^{-8}
Default	37	67	24	37
Circular	30	55	26	36
TLE	35	48	26	37

Table 7.4: Number of accepted solutions shared between different initial values for Gooding's method.

Arc Length Update step	120 [s]		240 [s]	
	10^{-5}	10^{-8}	10^{-5}	10^{-8}
Default & Circular	22	42	24	36
Default & TLE	22	37	24	37
Circular & TLE	22	33	25	36
All	20	29	24	37

To assess the magnitude of the convergence region of the method, three different configurations of the Gooding method were considered. All use the same sets of observations, but differ in the initial guess for the slant ranges. The first (called 'Default') uses Gooding's proposal for an initial estimate of ρ_1 and ρ_3 , which is given in Equation 2.16. By chance, this initial value is relatively close to the expected values for geostationary orbits. The second initial value considered (called 'Circ seed') is the one obtained from the circular solution of the circular orbit method for the first and last observation used in the Gooding method. Finally, the third initial value (called 'TLE seed') is the result of the TLE data of the object propagated to the time of the second observation. Of course, this last initial value is of little practical use when the orbit of the satellite is unknown, but might provide theoretical insights into the method's convergence behaviour.

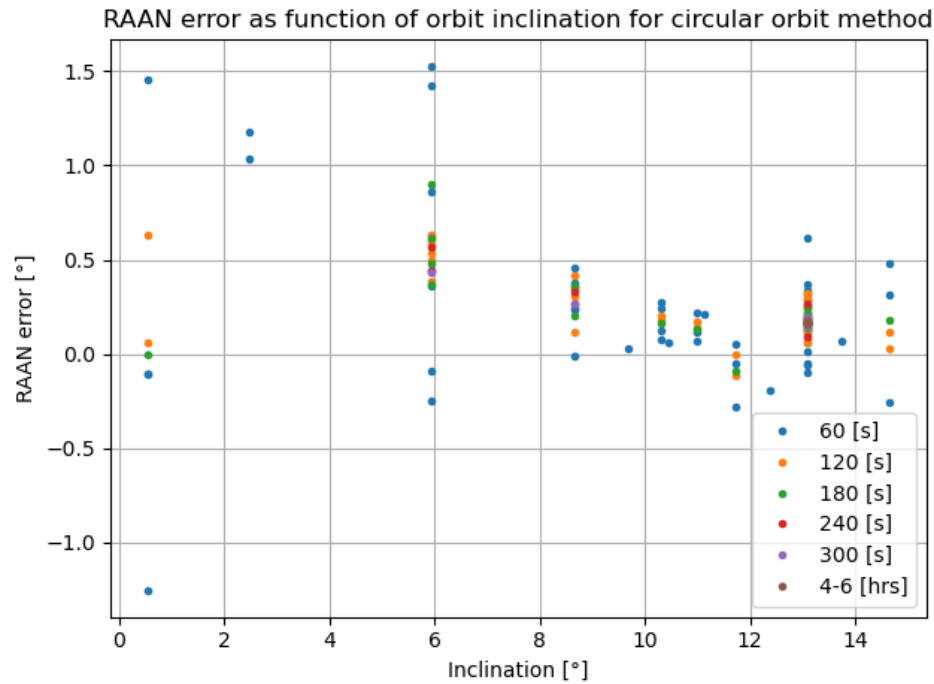


Figure 7.5: Error with respect to TLE data of orbit estimate RAAN values as function of inclination using circular orbit method for various arc lengths.

Table 7.3 lists the numbers of bound orbits each initial value type produced and Table 7.4 how many observation sets produced results for multiple initial value types. The tables list results for different update steps. The update step is a scaling factor applied to the update for the slant ranges, as discussed in subsection 2.3.1. The effect of the update step on the number of solutions was analysed, and it was found that a step of 10^{-8} gave the highest number of valid orbits for both arc lengths. The other update step of 10^{-5} is the default setting which was used in chapter 5 and chapter 6. It is observed in Table 7.3 that for the 120 [s] arc, the default initial values generates more acceptable solutions than the other two for both update steps. For the longer arc, approximately equal numbers of valid solutions are produced. It can also be seen that the increase to an arc of 240 [s] greatly increases the relative number of solutions found. Looking at Table 7.4 for the default update step, it is observed that there is very little difference in the percentage of observation sets that produce results for two out of the three methods and all methods. For both arc lengths, there exists a subset of the observations that generates solutions with any initial value type, and the difference with the values in Table 7.3 is made up of observation sets that only give a solution with the one initial value type, but not for the others. For the 240 [s] arc, almost all solutions found are obtained using each initial value type. For the 120 [s] arc, it is likely that the subset giving solutions for all types is the set of observations with the smallest observation error. For the smaller update step, the majority of solutions is common to two or three initial value types, however, a larger difference between combinations can be observed. This is likely caused by the larger difference in the number of valid solutions found for each initial value type. It is unclear why the default seed generates more valid solutions than the others, which are expected to be closer to the solution. Gooding [21] mentions that his default initial value performs well compared to others he tested, but it is still unclear how it can perform better than initial estimates leveraging a priori knowledge. A cause could be the handling of the initial values for finding more than one solution. When repeating the algorithm to find more solutions on the same observation data, the initial values are changed to steer away from the solution(s) already found. The manner in which this is done is not clear and could cause the observed behaviour. This is a topic for further research.

In terms of the convergence behaviour of Gooding's method, it can be concluded that the seeding of the slant ranges has more effect on the shorter arc. For the longer arc, the difference in results with different seeds is relatively smaller. This leads to the conclusion that the performance of the seed is

likely dependent on the arc length, as the default seed performs best for the shorter arc, but worst for the longer arc with the larger step size, while no difference can be observed for the smaller step size. The circular seed performs worst for the short arc but is tied with the others on the long arc.

It is clear from Table 7.3 and Table 7.4 that the smaller update step of 10^{-8} produces more results, hence, the results in Figure 7.6 and Figure 7.7 are produced using only the update step of 10^{-8} .

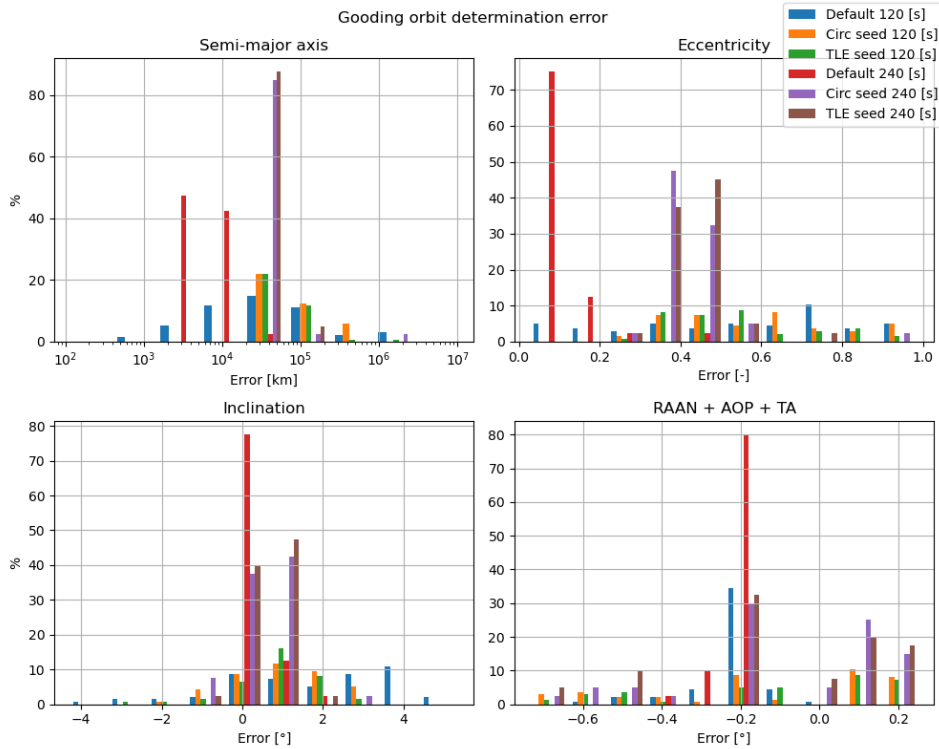


Figure 7.6: Error with respect to TLE data of orbit estimates using Gooding method for various arc lengths.

Overall, the default initial value outperforms the other two. For the short observation arc, it shows a lower error in semi-major axis, eccentricity and satellite right ascension. For the longer arc, the default initial value also performs better on the inclination. The circular and TLE seed show very similar performance. Compared to the results from the method of Gauss, the error on the semi-major axis is similar for the default seed, but larger for the other seeds. Gooding shows a larger spread of errors on eccentricity for the short arc; the default seed clearly shows better results than Gauss, the circular and TLE seeds perform slightly worse. All seeds of Gooding's method are able to relatively accurately determine the inclination and satellite right ascension compared to the Gauss method. Gooding's method also allows negative values for the slant ranges, but if those were found, they were rejected before a complete state was calculated.

The method of Gooding shows decreasing error in its estimate for RAAN with increasing satellite inclination. This is plotted in Figure 7.7. For larger values of inclination, however, the error on the right ascension of the ascending node is still relatively large, which results in a relatively larger uncertainty in the orientation of the satellite's orbital plane. The error is comparable to the results from Figure 7.3 for the unaltered Gauss method. The different seeds give different errors for an object at a specific inclination and it seems the default seed in this case tends to have larger errors for the short arc.

The default seed should be preferred for Gooding's method. For both observations arcs, it produces better results than the other seed on most orbital elements, with the exception of RAAN on the short arc. However, the reason for this better performance should be investigated.

7.6. Error on Propagated Results

While the graphs in the previous sections provide a lot of information about the orbit error for the different IOD methods, the main measure that is of interest is the ability to predict an object's position at a future epoch for additional observations. This is plotted in [Figure 7.8](#), [Figure 7.9](#) and [Figure 7.10](#) for the circular orbit, Gauss and Gooding methods, respectively. The graphs were obtained by propagating orbit estimates from observations sets with varying arc length to epochs of other observations of the same object. Propagation is performed using the SDP4 propagator, which was also used in [chapter 5](#) to perform the simulations [32]. The error is the angle difference between the propagated orbit and the observations at the new epoch as seen by the observer. To maximise the number of data points, propagation was performed forward and backward in time. The figures show the minimum ($\blacktriangledown$), maximum ($\blacktriangle$) and median value ($\bullet$) of the error. The x axis shows the number of nights the orbit is propagated, with zero being within the same night. For some points, no minimum or maximum is given, only a median. In these cases, there is only one data point available. The red line in the figures defines half of the field of view of the MeerLICHT telescope. If the error is below this line and the MeerLICHT telescope is centred on the expected position, the satellite will be in the image. In [Appendix F](#), the number of data points for every observation arc and propagation time is provided for each IOD method.

The method of Gauss shows very large errors. While the minimum error is relatively low for all propagation times, the median and maximum errors are very high. Given how the angle of the error can never exceed 180° , the maximum error is already close to its highest possible value for the shortest propagation time. There is no trend of the median error as function of propagation time for the shortest observations arc, while an increasing median error would be expected. This could indicate a relatively large spread of the error. In addition, the shorter arcs show little difference between unaltered and circularised Gauss. It can be observed that when propagating the longer arc solution by less than two nights, the circularised Gauss method performs considerably better. For propagation within one night, the maximum and median error are lower and for one night propagation, the median and minimum error are lower. For two and three nights propagation, the results of unaltered and circularised Gauss grow closer to each other.

For Gooding's method, it is again observed that on average, the default seed performs better than the others, although in most cases the error is still very large. However, it can be observed that for propagation within the same night and to the next night, the longer observation arc with the default seed shows a relatively small median error. While no predictions are accurate enough to immediately be within the field of view of MeerLICHT, recovery of some objects is likely possible within the same night using 240 [s] observation arcs. For propagation within the same night, Gooding's method performs considerably better than Gauss', with the median error on the shortest arc approximately one third of the error from the Gauss method. A large decrease in error can also be observed for the longer observation arc. For propagation to the next night, Gooding with the default seed still outperforms the method of Gauss, but the results of the other seeds are larger for the short arc and approximately equal for the long arc. For even longer propagation times, the median error of the method of Gooding is actually worse than that of Gauss.

The circular orbit method shows much lower errors over the whole range of propagation times considered. It can be observed that the orbit estimates from 120 [s] arcs on average are able to provide an estimate accurate enough to stay in MeerLICHT's field of view after a propagation time of three days. The shortest arc shows a much larger maximum error and an increasing median for increasing propagation time. The observation arcs of 180 [s] show very similar performance to the [120] [s] arcs. For longer observation arcs, the median error does not improve, however, for one night propagation and less, the spread of the data becomes much smaller and the maximum error is lower as a result. The results in this graph in combination with the accurate estimates of the orbital plane of the objects make the circular orbit method a very suitable candidate to perform initial orbit determination on GEO objects. Even when the true position of an object is outside the field of view of MeerLICHT, the fact that the orbital plane is well defined allows MeerLICHT to make multiple observations in the plane to recover the object.

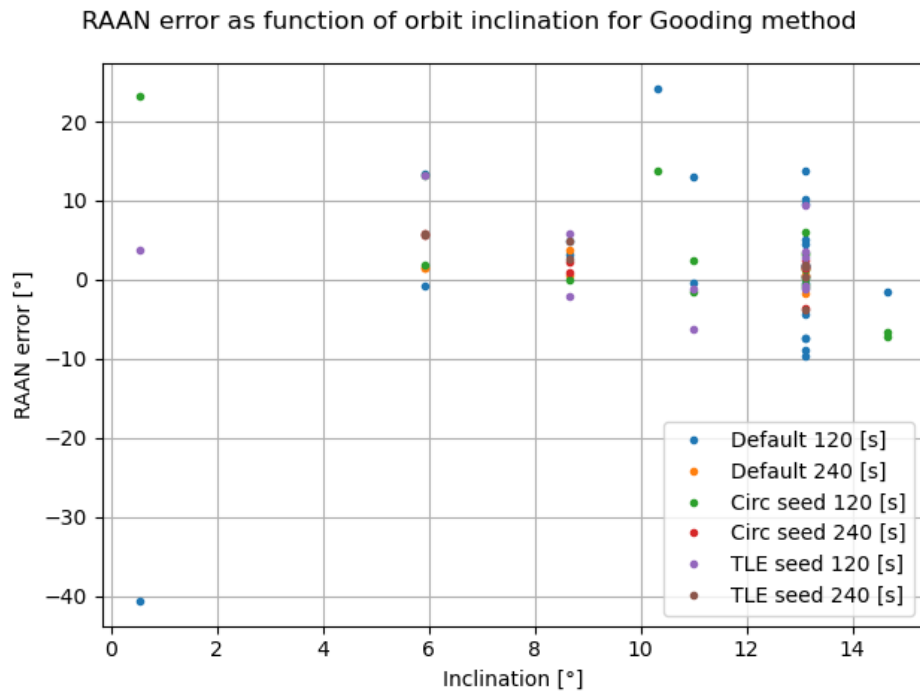


Figure 7.7: Error with respect to TLE data of orbit estimate RAAN values as function of inclination using Gooding method for various arc lengths.

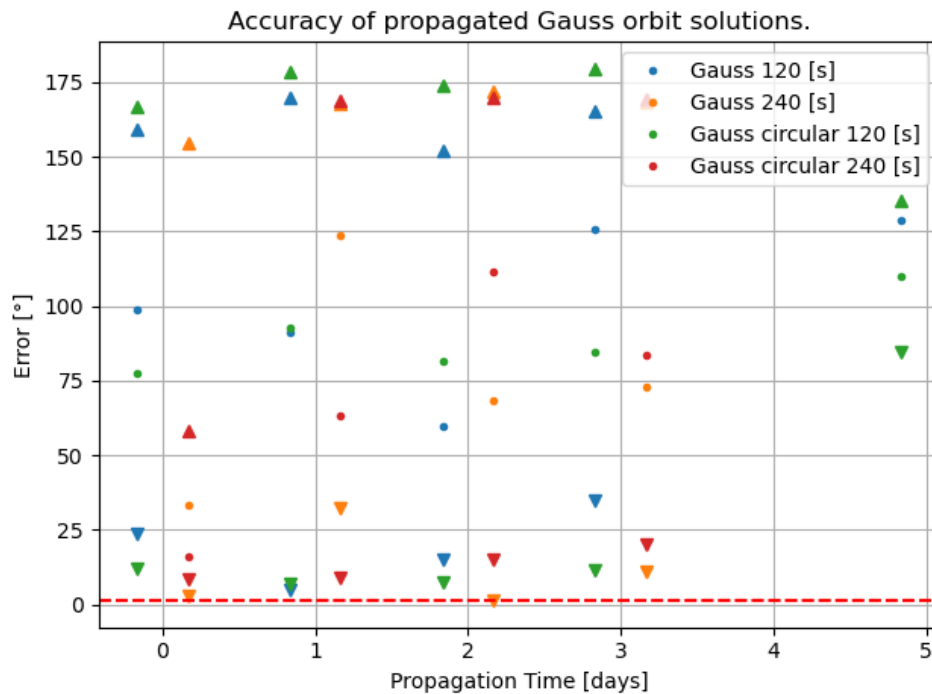


Figure 7.8: Error on predicted position for Gauss method

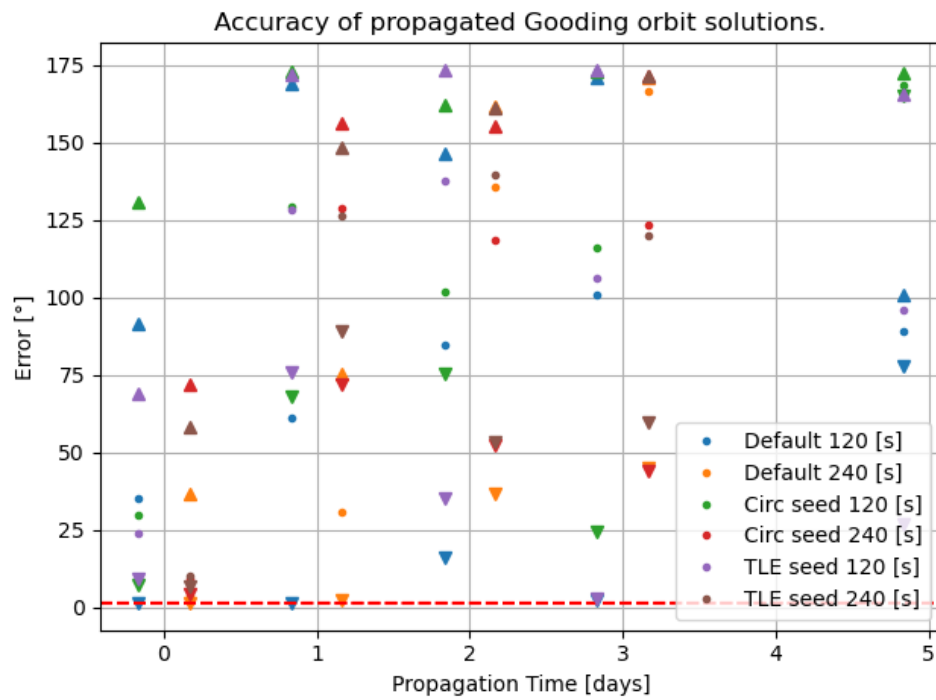


Figure 7.9: Error on predicted position for Gooding method

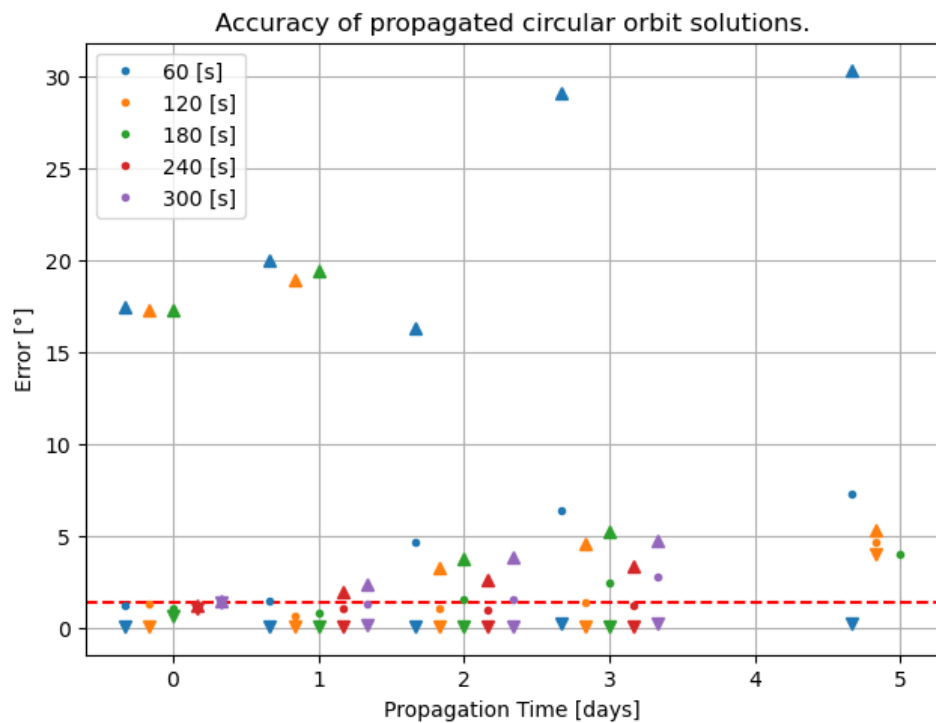


Figure 7.10: Error on predicted position for circular orbit method

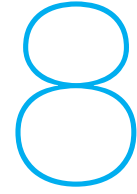
7.7. Conclusion

It can be concluded that the circular orbit method performs best for the short observation arcs in the dataset. The method is robust against the measurement error that is known to be present in the data and is able to provide accurate predictions of an object's position several nights into the future if necessary.

The methods of Gooding and Gauss show very large errors, mainly on the semi-major axis and the eccentricity. For the method of Gauss, the circularisation improves the orbit estimates for longer arcs, however, the circularisation of invalid orbit estimates from the short observation arc causes large errors in inclination and RAAN without notable improvements on the other orbital elements. The method of Gauss gives the worst results on inclination, and RAAN for non-equatorial orbits, which results in a poor estimate of the orbital plane. This severely complicates recovery of an object at a future epoch.

Gooding's method shows large differences in results depending on the initial estimate of the slant ranges. Counter intuitively, the default initial estimate performs better than estimates based on the circular orbit method and the TLE of the object, which likely are more accurate. The default seed both increases the number of valid orbits found and decreases the error on most orbital elements for both observation arc lengths. Gooding's method produces smaller errors on inclination and RAAN compared to the Gauss method and hence produces a better estimate of the orbital plane. In addition, the angular error of the prediction after propagation of the longer observation arc estimate within the same night are relatively low and likely allow the recovery of some objects.

The quality of the results obtained for the observation arcs that are available also leads to the conclusion that longer observation arcs are desired for the methods of Gauss and Gooding. Considering the improvement of the errors going from 120 to 240 [s] arcs, an increase in the observation arc to 360 or 480 [s] might already be sufficient to produce relatively accurate orbit estimate, also considering that measurement error reduction can likely reduce the error far enough for the Gauss method to converge consistently on these longer arcs. A similar improvement should also be expected from Gooding's method.



Conclusions

In this report, an analysis was performed on different initial orbit determination (IOD) methods and their suitability to perform initial orbit determination on very short arc data from the MeerLICHT telescope. Using simulated observations, the error behaviour of the IOD methods was investigated for different arcs. The effect of measurement error reduction techniques was analysed and finally, the results of the IOD methods on real observations from the MeerLICHT telescope were discussed, including their ability to predict an object's position for future observations.

From simulations, it can be concluded that the methods of Gooding and Gauss are very sensitive to measurement error for short observation arcs. The sensitivity rapidly decreases with increasing orbit arcs, and relatively accurate orbit estimates are possible from an orbital arc of 6% onward, however, the effect of measurement error remains noticeable up to the maximum orbital arc of 30%. The magnitude of the orbit error for orbital arcs of less than 2% is approximately constant for increasing measurement error. The method of Gauss shows a large increase in error when the observations arc becomes too long, as then the short arc assumption inherent to the method breaks down. The circular orbit method is much less sensitive to measurement error. It is observed that the methods are relatively more sensitive to errors in the timestamp and that the magnitude of the error has less influence on the magnitude of the orbit error. The reason for this is not yet understood.

The analysis of the effect of measurement error reduction shows that the methods considered were ineffective. Two methods were tested. The first corrects a systematic error in the FOTOS endpoint determination algorithm; the second refines the time stamp given to observations by analysing the shutter wheel motion. Neither method could increase the number of elliptic solutions found or reduce the error on the orbit estimates when compared to TLE data. Furthermore, indications were found that the endpoint correction model itself still contains a systematic error and has a tendency to overcompensate the original error. It was therefore decided that they would not be used on the observation data in the remainder of the report. It was concluded that the error reduction measures work in theory and would most likely reduce the observation error, but that as long as the total measurement error remains higher than some critical value for a given observation arc length, no reduction in the orbit estimate error would be observed. This hypothesis is supported by the findings from the simulations, where it was observed that for very short arcs of less than 2%, the magnitude of the measurement error does not affect the magnitude of the orbit estimate error. This was equally true for errors in right and declination as well as time stamp.

From observation data, it was concluded that the circular orbit method performed best and is suitable for IOD of objects in geostationary Earth orbit (GEO) using the same telescope and observation strategy. The errors on the orbital elements were very small, the orbital plane was defined relatively accurately and propagating one solution to other observation times revealed that in most cases, the MeerLICHT telescope would easily be able to recover the object, even after a propagation time of several days.

The method of Gooding, in combination with the default initial value for the slant ranges and an observation arc of 240 [s], is able to give moderately accurate prediction of an object's position for propagation within the same night. The prediction error is larger than for the circular orbit method,

but likely allows the recovery of some objects. Of the three initial value types used with the Gooding method, the default one performed best. This is unexpected since the other two provided estimates which are closer to the correct values. This could be caused by how Gooding's method changes the initial values when calculating a second and third solution.

The method of Gauss is not suitable for initial orbit determination with the MeerLICHT telescope and the short observation arcs that were available. The errors on the orbital elements compared to TLE data were very large and the orbital plane could not be accurately determined in most cases. Propagating orbit estimates to other observation times of the same object showed very large prediction errors as well. The circularisation of the solution provided improved orbit estimates for the 240 [s], but the improvement was insufficient to allow follow up observations.

It was also concluded that, while the circular orbit method performs very well for the current observation strategy, a small increase in the observation arc to six or eight minutes could result in relatively accurate orbit estimates from the methods of Gauss and Gooding as well.

Recommendations

The final chapter of this thesis will formulate some recommendations. First, [section 9.1](#) will provide recommendations on an observation strategy for the MeerLICHT and BlackGEM telescopes based on the results of this report. Then, [section 9.2](#) will discuss some points of further research.

9.1. Observation Strategy

To enable initial orbit determination of new objects using the same type of very short arcs that were analysed in this report, the following observation strategy using the circular orbit method in combination with Gooding's method is recommended. First, a circular orbit should be estimated based on the first observations of a new object. An initial observation arc of at least two minutes is desirable, but not strictly necessary. Using this circular orbit estimate, a new epoch should be selected to obtain follow up observations of the object. For objects in geostationary Earth orbit, results have shown that for observation arcs of at least two minutes, follow up observation can be made several nights in the future. However, since the true orbit of the object is unknown and may not be circular, the new epoch should be close to the first observation. More focused analysis is needed to determine how close the follow up observations should be. As a first estimate, follow up observations approximately one hour after the initial set would likely give a high chance at recovering the object while also providing a long enough arc for the method of Gooding to estimate a relatively accurate orbit. With the longer arc, it would be worthwhile to implement measurement error reduction techniques, as it is shown in the simulation that lower observation error will give better orbit estimates for longer arcs. The reason Gooding's method is selected instead of Gauss is because it showed better performance on the observation data, and the method of Gauss has an inherent error for longer arcs.

With a BlackGEM telescope as a dedicated observation platform, a different observation strategy would be preferred. In this case, it is recommended to follow a new object as long as possible after discovery to obtain an observation arc of at least 10 to 15% of the orbital period. This would likely provide enough observations to make use of other, more accurate orbit determination methods.

9.2. Further Research

Before the recommended IOD strategy can be implemented reliably, more research is required. Below follow some recommendation for further work to be carried out.

It was established in [chapter 6](#) that the correction on the endpoint determination is still biased. It is unclear whether the assumptions made are wrong or incomplete or the manual determination of the endpoint was carried out incorrectly. This should be investigated as an error on the order of 10 pixels has a severe effect, even for longer observation arcs.

The analysis of the observations showed that the default initial value of Gooding's method provided better results than initial values based on the circular orbit method and TLE data. This was an unexpected result since the latter two are closer to the correct solution. A possible cause of this could be the method used by Gooding to alter the initial values in order to find additional solutions after the first

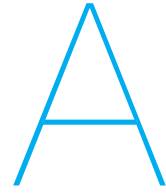
is computed. It is currently unknown how the initial values are changed and it is possible they favour the default starting values.

The error of the circular orbit method should be investigated on other orbits. The method is expected to work similarly for any circular orbit, regardless of altitude, but the error behaviour is unknown for eccentric orbits both in low and higher orbits. The proposed IOD strategy hinges on the fact that the circular orbit method is able to accurately define the orbital plane of the object, which is a crucial factor in the ability to obtain follow up observations. If that would no longer be the case for eccentric orbits, the effectiveness of the IOD strategy is greatly diminished.

The circular orbit method could be expanded in several ways. First, it could use more observations instead of two to fit a circular orbit and the orbital radius could be computed by finding the minimum error over all observations. In addition, the method could be extended so it also provides an estimate for the eccentricity in a second stage of the method. This would be interesting considering the MeerLICHT telescope also detects GTO and HEO objects in highly eccentric orbits. One possible method is explained here as a recommendation for future research.

Using at least two more observations for a total of four, the eccentricity could be estimated as follows. The orbital plane by definition contains the two initial observations used to construct it. The two additional observations, in combination with a slant range, will have some distance to the orbital plane, depending on the magnitude of the slant range. Since four observations is one more than strictly necessary, the system is overdetermined and in the presence of measurement error, no ellipse can be drawn through all four. If the orbital plane is kept constant, but all four slant ranges are allowed to vary, they can be chosen such that the normal projections of the observations onto the orbital plane allow the construction of an ellipse through them with the origin of the frame in one of the focal points. Now, the total distance of the observations to the orbital plane can be minimised to obtain a complete orbit definition.

The Gauss refinement algorithm holds promise for both the methods of Gauss and the circular orbit method. If it can be made to behave more consistently across different objects, it could eliminate the short arc assumption in Gauss' method, putting it on par with Gooding's method, as well as allow the circular orbit method to relax the zero eccentricity constraint to produce a more realistic orbit. Currently, the refinement algorithm is very inconsistent for a set of GEO objects with very similar characteristics, which results in the algorithm being unreliable and unusable. If the cause of the inconsistent behaviour could be identified and resolved, the refinement could prove to be very helpful.



Shutter Offset Time Calculation

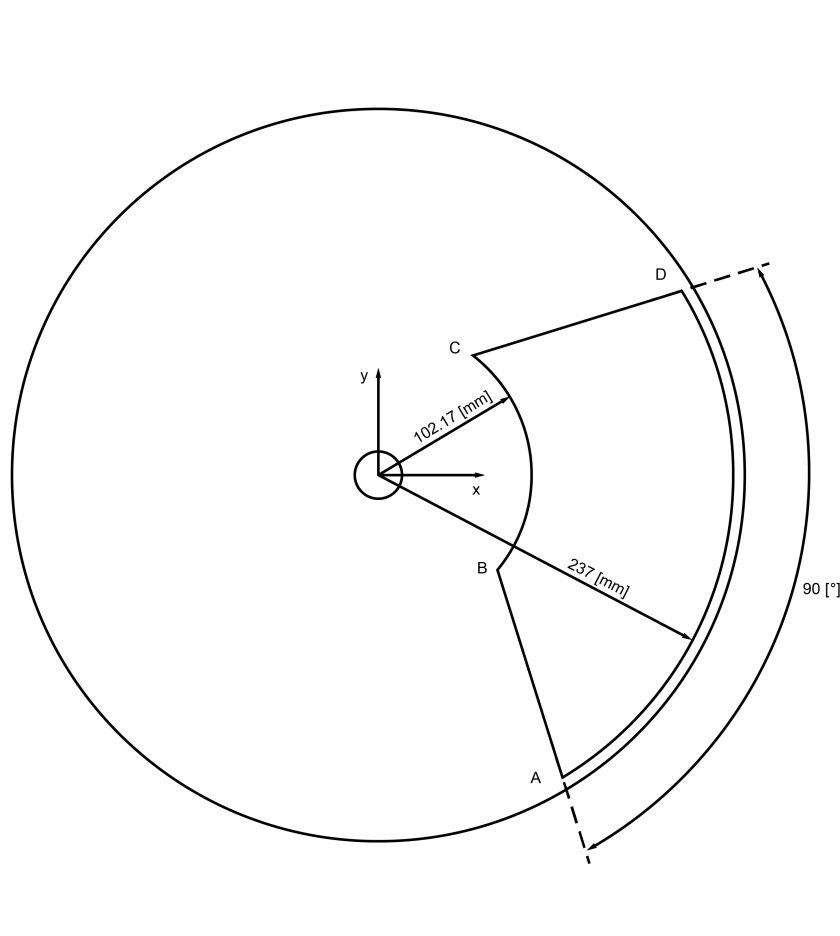


Figure A.1: Shutter wheel geometry. Points A through D are the intersections of the circular and straight sections of the hole in the wheel. The figure is constructed from the technical drawing and the CAD model, since the former is not fully constrained. The wheel rotates counterclockwise.

The shutter wheel is shown in [Figure A.1](#). The figure is based on the CAD drawing of the wheel, provided in [Appendix B](#). Points A through D denote the intersections of the circle arcs and the straight line segments forming the shutter opening. Here, the fillets in the corners were ignored. This simplification can be made since the filleted parts never intersect with any part of the CCD. Since the information in the technical drawing is insufficient to calculate the coordinates of points A through D, they were

determined using the computer model of the shutter blade. The coordinates are given for the case where the wheel is in an open position ($\theta = 0$ [°]). The values of the four corner points are listed in [Table A.1](#).

Table A.1: Coordinates of corner points of the hole in the shutter wheel.

Point	Coordinates (x, y) [mm]
A	(123.159, -202.486)
B	(79.74, -63.876)
C	(63.876, 79.741)
D	(202.486, 123.16)

It can be observed that points *C* and *D* exactly coincide with points *A* and *B* when the latter are rotated counterclockwise by 90 [°] around the origin. It follows that the position of line *AB* when the shutter is open is identical to that of line *CD* when the shutter is closed, which, given the fact that the rotation profile is identical for opening and closing, means the time offset of a pixel is the same for opening and closing of the shutter. Hence, it is sufficient to only consider line *AB* in this analysis.

When the wheel rotates, the coordinates of the points *A* (x_A, y_A) and *B* (x_B, y_B) will change. The new coordinates of these points when $\theta \neq 0$ [°] are denoted as *A'* (x'_A, y'_A) and *B'* (x'_B, y'_B). Their values can be calculated using [Equation A.1](#).

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} \quad (\text{A.1})$$

An equation of the form $ax + by + c = 0$ of the line through points *A* and *B* is constructed knowing $a = y'_B - y'_A$, $b = -(x'_B - x'_A)$ and $c = x'_B y'_A - x'_A y'_B$. This gives

$$(y'_B - y'_A) \cdot x - (x'_B - x'_A) \cdot y + x'_B y'_A - x'_A y'_B = 0$$

Combined with [Equation A.1](#), the result is given in [Equation A.2](#). The equation of the line only depends on the coordinates of points *A* and *B* at $\theta = 0$ [°] and the angle θ .

$$[(x_B - x_A) \sin \theta + (y_B - y_A) \cos \theta] \cdot x + [(y_B - y_A) \sin \theta - (x_B - x_A) \cos \theta] \cdot y + (x_B y_A - x_A y_B) = 0 \quad (\text{A.2})$$

In order to find the angle θ for which line *AB* coincides with a pixel *P* with coordinates (x_P, y_P), it suffices to fill x_P and y_P into [Equation A.2](#) and solve for θ . A solution for the resulting equation, [Equation A.3](#), can be obtained by using the known relation between sines and cosines $\sin^2 \theta + \cos^2 \theta = 1$.

$$[(y_B - y_A) x_P - (x_B - x_A) y_P] \cdot \cos \theta + [(x_B - x_A) x_P + (y_B - y_A) y_P] \cdot \sin \theta + (x_B y_A - x_A y_B) = 0 \quad (\text{A.3})$$

For simplicity's sake, [Equation A.3](#) is rewritten as

$$u \cos \theta + v \sin \theta + w = 0$$

where

$$u = (y_B - y_A) x_P - (x_B - x_A) y_P$$

$$v = (x_B - x_A) x_P + (y_B - y_A) y_P$$

$$w = x_B y_A - x_A y_B$$

The system to solve is given in [Equation A.4](#).

$$\begin{cases} u \cos \theta + v \sin \theta + w = 0 \\ \sin^2 \theta + \cos^2 \theta = 1 \end{cases} \quad (\text{A.4})$$

Rewriting the first equation to $\cos \theta$ and substituting the result for $\cos \theta$ into the second equation, a second-degree polynomial in $\sin \theta$ appears:

$$(u^2 + v^2) \sin^2 \theta + 2vw \sin \theta + (w^2 - u^2) = 0$$

The solution for this equation is given in [Equation A.5](#).

$$\sin \theta = \frac{-vw \pm u\sqrt{u^2 + v^2 - w^2}}{u^2 + v^2} \quad (\text{A.5})$$

Taking the inverse sine of both sides gives θ in [Equation A.6](#). The equation gives two mathematically valid solutions, however, only one is of interest here. The two solutions are shown in [Figure A.2](#) for the case where P is the midpoint of the CCD.

$$\theta = \sin^{-1} \left(\frac{-vw \pm u\sqrt{u^2 + v^2 - w^2}}{u^2 + v^2} \right) + k2\pi, \quad k \in \mathbb{Z} \quad (\text{A.6})$$

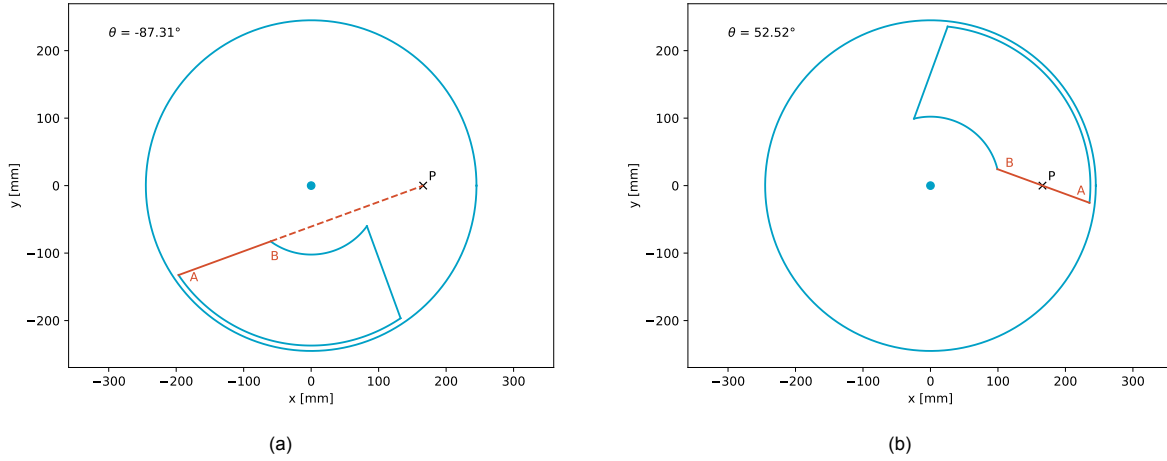


Figure A.2: Graphical depiction of the solutions of [Equation A.6](#).

Clearly, the sought after solution is depicted in [Figure A.2b](#). This solution can easily be distinguished from the other one by requiring the condition given in [Equation A.7](#).

$$(x_P - x'_A)(x_P - x'_B) + (y_P - y'_A)(y_P - y'_B) < 0 \quad (\text{A.7})$$

This is equivalent to taking the dot product of vectors $(\vec{P} - \vec{A})$ and $(\vec{P} - \vec{B})$, where $\vec{A}$, $\vec{B}$ and $\vec{P}$ are the vectors from the origin to A , B and P , respectively. In [Figure A.2b](#), vectors $(\vec{P} - \vec{A})$ and $(\vec{P} - \vec{B})$ point in opposite directions, hence, their dot product must be strictly smaller than zero. The configuration in [Figure A.2a](#) has these vectors pointing in the same direction, resulting in a strictly positive result when taking the dot product.

A relation has now been established between the coordinates of a pixel in Cartesian space and the required angle θ by which the shutter wheel needs to rotate to cover or uncover said pixel. The Cartesian coordinates of a pixel can be obtained as follows.

MeerLICHT's properties are listed in [Table 3.1](#). It is known that the CCD has dimensions of 95×95 [mm], which in combination with the resolution and pixel size leads to the conclusion that the distance between pixels δ_p is equal to the pixel size, listed in [Table A.2](#) as s . From the CAD model, it is determined that

Table A.2: Characteristics of STA1600LN CCD in MeerLICHT telescope

Variable		Value
Side length	S	95 [mm]
Pixel pitch	δ_p	9 [μm]
CCD midpoint coordinates	(x_{CCD}, y_{CCD})	(166, 0) [mm]

the midpoint of the CCD is located at $(x_{CCD}, y_{CCD}) = (166, 0)$ [mm]. The values are summarised in Table A.2.

Pixel indices are given in the ij coordinate system shown in Figure 3.4. The transformation from pixel indices to Cartesian coordinates can now be written as in Equation A.8. The pixel in the first row and column has indices $(i, j) = (0, 0)$.

$$\begin{bmatrix} x_{px} \\ y_{px} \end{bmatrix} = \begin{bmatrix} j_{px} \\ i_{px} \end{bmatrix} \cdot \delta_p + \begin{bmatrix} x_{CCD} \\ y_{CCD} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} S \\ S \end{bmatrix} \quad (\text{A.8})$$

The final step is to relate the angle θ to moving time. The inverse relation is already given in Figure 3.6. To be able to generate a unique solution for every pixel in the CCD, cubic spline interpolation is used on the data points.

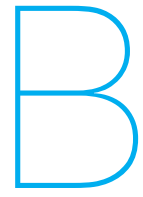
Given a set of data points $(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))$, cubic spline interpolation constructs a spline S to approximate f by generating polynomials S_j of the third degree between each two consecutive data points x_j, x_{j+1} , ensuring

$$\begin{aligned} S_j(x_j) &= f(x_j) & \forall j \in \{0, 1, \dots, n-1\}; \\ S_j(x_{j+1}) &= f(x_{j+1}) & \forall j \in \{0, 1, \dots, n-1\}; \\ S_{j+1}(x_{j+1}) &= S_j(x_{j+1}) & \forall j \in \{0, 1, \dots, n-2\}; \\ \dot{S}_{j+1}(x_{j+1}) &= \dot{S}(x_{j+1}) & \forall j \in \{0, 1, \dots, n-2\}; \end{aligned}$$

where $\dot{S}$ is the first derivate of S with respect to x . These constraints, together with two additional boundary conditions at x_0 and x_n , allow the construction of S with continuous first and second derivatives in the domain $[x_0, x_n]$ [7].

Figure 3.6 contains $n_0 = 1020$ data points, uniformly spaced in time. However, the first and last few time step values have identical values of θ and hence cannot be used to construct a relation for θ as function of time. The first and last 10 data points are therefore omitted, resulting in $n = 1000$. So called natural boundary conditions were chosen for θ_0 and θ_n , which means $\dot{S}(\theta_0) = \dot{S}(\theta_n) = 0$. Multiple other types of boundary condition options were explored, including defining derivatives $\dot{f}(\theta_0)$ and $\dot{f}(\theta_n)$, and having one polynomial for the first and last two segments. The difference in results was negligible over the range of θ covering the CCD and natural boundary conditions were selected for simplicity. The cubic spline is generated using Scipy's CubicSpline¹ class.

¹<https://docs.scipy.org/doc/scipy/reference/generated/scipy.interpolate.CubicSpline.html>, last accessed 01 February 2021



Shutter Wheel Technical Drawing

C

Simulation Data

Table C.1: List of satellites used in simulations. The table lists all satellites used in the simulations, the epoch of the TLE used, and the average orbital elements at the epoch.

ID	Epoch of TLE	a [km]	e [–]	i [°]	Ω [°]	ω [°]	M [°]
11353U	16-06-2020 01:22:46.392	42656.5	1.8424e-03	12.369	321.87	34.84	337.98
12851U	16-06-2020 02:52:58.089	41960.5	4.8480e-04	10.983	313.93	39.44	317.79
13907U	16-06-2020 02:55:21.297	42871.6	8.7898e-03	11.725	307.59	89.37	284.68
20315U	16-06-2020 08:04:26.559	42522.6	2.7240e-04	13.084	23.16	61.55	309.75
21227U	15-06-2020 02:36:19.510	42362.1	7.7340e-04	11.896	29.06	320.15	323.34
23199U	16-06-2020 05:57:12.597	42454.7	4.5800e-04	9.670	40.77	52.64	314.97
23656U	15-06-2020 18:26:24.666	42191.4	2.3993e-03	14.648	10.42	4.20	172.23
23779U	15-06-2020 13:31:05.554	42357.7	3.8770e-04	6.356	61.58	50.93	67.64
23943U	16-06-2020 08:04:34.097	42469.7	2.3027e-03	10.294	32.97	308.78	52.75
24740U	15-06-2020 07:25:02.827	42381.6	2.0757e-03	13.732	20.54	127.05	233.13
24748U	15-06-2020 07:55:38.391	42515.6	2.5196e-03	10.436	37.02	278.08	69.60
24957U	16-06-2020 03:12:42.318	42164.0	3.6420e-04	5.966	63.79	19.37	280.35
25913U	16-06-2020 08:05:41.931	42597.5	5.4980e-04	8.134	50.18	347.47	0.91
26863U	16-06-2020 08:02:10.890	42440.7	1.1207e-03	13.597	23.04	88.62	274.08
27712U	15-06-2020 09:10:12.585	42080.0	2.0768e-03	8.643	33.67	239.47	120.26
27825U	15-06-2020 06:24:16.752	42164.3	5.3470e-04	5.914	32.94	48.50	322.63
28187U	16-06-2020 10:17:42.438	42163.9	4.5420e-04	0.330	86.47	3.19	29.64
28194U	16-06-2020 04:50:25.357	42191.5	2.0200e-04	11.114	37.02	331.43	22.21
28885U	16-06-2020 09:59:04.678	42165.1	3.4510e-04	0.017	68.41	18.30	15.17
29526U	16-06-2020 04:42:53.126	42165.0	2.3820e-04	0.049	339.40	81.13	301.10
32294U	16-06-2020 08:08:57.898	42165.2	3.3220e-04	0.071	351.60	98.55	322.26
32479U	16-06-2020 00:46:42.733	41150.2	1.8907e-02	8.365	42.04	185.53	24.40
33154U	16-06-2020 04:42:53.126	42163.8	6.4500e-04	0.051	27.75	61.55	272.36
33460U	16-06-2020 11:42:42.856	42163.2	2.7930e-04	0.021	25.81	93.50	9.66
36592U	16-06-2020 06:26:12.481	42164.0	4.9040e-04	0.051	117.37	2.14	268.02
37393U	16-06-2020 04:50:25.357	42165.4	3.3910e-04	0.028	87.56	3.11	299.36
38245U	16-06-2020 09:59:12.215	42164.4	3.0270e-04	0.029	93.38	356.22	12.94
39034U	16-06-2020 04:50:25.357	42164.1	3.5690e-04	0.072	7.24	88.78	294.28
39079U	16-06-2020 03:09:54.369	42164.1	1.5670e-04	0.015	328.37	145.51	244.45
39215U	16-06-2020 11:36:03.385	42164.0	1.9450e-04	2.450	20.28	80.81	2.99
39233U	16-06-2020 06:26:04.943	42164.8	1.3540e-04	0.048	17.78	62.80	306.41
40277U	16-06-2020 11:44:05.766	42164.5	2.3330e-04	0.045	234.19	285.33	334.70
40617U	16-06-2020 10:00:27.587	42164.0	2.9540e-04	0.013	353.63	112.86	0.79
41029U	16-06-2020 11:36:25.998	42164.8	3.4180e-04	0.044	263.69	301.72	259.87
41238U	16-06-2020 04:50:10.284	42164.6	2.5100e-04	0.035	224.65	260.46	263.86
41585U	16-06-2020 03:34:34.235	41129.9	2.1166e-02	6.644	349.07	220.51	137.87
41744U	04-06-2020 19:44:14.973	42136.3	1.2440e-04	0.528	96.56	233.87	262.66
42747U	16-06-2020 03:05:46.882	42164.0	7.7190e-04	0.019	34.10	53.87	271.21
43700U	16-06-2020 06:26:12.481	42164.1	1.2690e-04	0.014	315.78	159.61	271.96
44035U	16-06-2020 09:59:19.752	42164.0	9.8000e-05	0.053	121.14	265.15	76.55
44071U	16-06-2020 07:35:42.715	42164.3	2.4200e-05	0.105	100.69	4.24	298.07
44307U	16-06-2020 04:49:32.598	42164.6	2.0910e-04	0.010	35.77	45.87	304.67



Observation Data

Table D.1: Observation data

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
20315U	59016.719145	59016.719151	169.365783	13.103087
23943U	59016.718451	59016.718456	169.091580	13.035218
23943U	59016.719145	59016.719151	169.336279	12.998280
20315U	59016.719834	59016.719839	169.609507	13.050439
20315U	59016.720529	59016.720534	169.853630	12.996926
23943U	59016.719834	59016.719839	169.582808	12.961503
23943U	59016.720529	59016.720533	169.829634	12.924193
21227U	59016.721229	59016.721236	168.865093	13.582167
21227U	59016.721924	59016.721930	169.054081	13.547612
21227U	59016.722617	59016.722623	169.397315	13.484834
21227U	59016.723311	59016.723317	169.585423	13.449927
39215U	59016.745702	59016.745708	199.141937	5.436209
39215U	59016.746396	59016.746402	199.394937	5.423512
44071U	59016.745702	59016.745708	198.158180	5.482045
44071U	59016.746396	59016.746402	198.410477	5.481393
24748U	59016.925977	59016.925982	229.101372	1.936154
24748U	59016.926672	59016.926676	229.346752	1.887128
27712U	59016.925977	59016.925983	227.940207	2.185528
27712U	59016.926672	59016.926677	228.192300	2.144231
27712U	59016.927366	59016.927372	228.440544	2.103674
27712U	59016.928061	59016.928066	228.690869	2.062721
27712U	59016.928776	59016.928782	228.946853	2.021517
27712U	59016.929471	59016.929476	229.199273	1.980983
12851U	59016.936451	59016.936456	245.799914	-6.136303
12851U	59016.937145	59016.937151	246.056302	-6.113771
13907U	59016.936451	59016.936456	246.061320	-6.349372
13907U	59016.937145	59016.937150	246.297974	-6.321287
12851U	59016.937843	59016.937848	246.313233	-6.091078
12851U	59016.938537	59016.938542	246.571137	-6.067877
13907U	59016.937843	59016.937848	246.546182	-6.291766
13907U	59016.938537	59016.938542	246.785948	-6.263090
20315U	59016.937843	59016.937849	245.505159	-5.386106
20315U	59016.938537	59016.938544	245.753382	-5.431874
20315U	59016.939264	59016.939270	246.010395	-5.479653
20315U	59016.939959	59016.939965	246.257906	-5.525269

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Table D.1 continued from previous page

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
20315U	59016.940645	59016.940651	246.500811	-5.569421
20315U	59016.941339	59016.941345	246.748679	-5.615403
27825U	59016.949454	59016.949460	294.373981	-1.398005
27825U	59016.950149	59016.950154	294.628624	-1.404800
11353U	59016.950862	59016.950866	294.995377	-1.928337
11353U	59016.951556	59016.951560	295.236267	-1.876485
27825U	59016.950862	59016.950867	294.886032	-1.410577
27825U	59016.951556	59016.951561	295.138186	-1.417067
33154U	59016.957986	59016.957992	277.056654	5.182501
33154U	59016.958680	59016.958686	277.307915	5.182172
36592U	59016.957986	59016.957992	277.081017	5.245735
36592U	59016.958680	59016.958686	277.332490	5.245738
41029U	59016.957986	59016.957992	277.059824	5.252830
41029U	59016.958680	59016.958686	277.311582	5.252534
43700U	59016.957986	59016.957992	276.829564	5.232758
43700U	59016.958680	59016.958686	277.081924	5.232574
29526U	59016.959379	59016.959384	277.513721	5.187817
29526U	59016.960073	59016.960079	277.766020	5.186502
33154U	59016.959379	59016.959384	277.558549	5.182189
33154U	59016.960073	59016.960079	277.810950	5.180700
36592U	59016.959379	59016.959384	277.583453	5.244489
36592U	59016.960073	59016.960079	277.835528	5.244370
39233U	59016.959379	59016.959385	276.951903	5.187262
39233U	59016.960073	59016.960079	277.204228	5.185881
41029U	59016.959379	59016.959384	277.562393	5.252551
41029U	59016.960073	59016.960079	277.815286	5.252750
43700U	59016.959379	59016.959385	277.332411	5.231928
43700U	59016.960073	59016.960079	277.585457	5.231630
32294U	59016.960782	59016.960788	277.027679	5.159521
32294U	59016.961477	59016.961482	277.281242	5.159814
39233U	59016.960782	59016.960788	277.457862	5.185168
39233U	59016.961477	59016.961482	277.710843	5.185287
43700U	59016.960782	59016.960788	277.838321	5.231045
43700U	59016.961477	59016.961482	278.092027	5.230947
32294U	59016.962187	59016.962193	277.535893	5.159058
32294U	59016.962882	59016.962887	277.787065	5.159004
39233U	59016.962187	59016.962192	277.965721	5.185127
39233U	59016.962882	59016.962887	278.217277	5.184404
37393U	59016.970061	59016.970067	311.422812	5.053106
37393U	59016.970755	59016.970761	311.674308	5.052416
39034U	59016.970061	59016.970066	311.738008	5.006520
39034U	59016.970755	59016.970761	311.989166	5.006370
40277U	59016.970061	59016.970066	312.022946	5.099238
40277U	59016.970755	59016.970761	312.274243	5.099034
37393U	59016.971453	59016.971458	311.925005	5.052388
37393U	59016.972147	59016.972152	312.176616	5.052057
39034U	59016.971453	59016.971458	312.239709	5.006907
39034U	59016.972147	59016.972152	312.491658	5.006530
40277U	59016.971453	59016.971458	312.524679	5.099003
40277U	59016.972147	59016.972152	312.776738	5.098637
40617U	59016.971453	59016.971458	311.419445	5.062596
40617U	59016.972147	59016.972153	311.671093	5.061912
37393U	59016.972839	59016.972844	312.424528	5.051923

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Table D.1 continued from previous page

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
37393U	59016.973534	59016.973539	312.677186	5.051009
40617U	59016.972839	59016.972845	311.918812	5.062043
40617U	59016.973534	59016.973539	312.171847	5.061713
41238U	59016.972839	59016.972845	311.347087	5.106623
41238U	59016.973534	59016.973539	311.600110	5.106784
40617U	59016.974246	59016.974251	312.427647	5.061347
40617U	59016.974940	59016.974945	312.678744	5.061210
41238U	59016.974246	59016.974251	311.855631	5.106743
41238U	59016.974940	59016.974946	312.106777	5.105812
33460U	59016.989646	59016.989651	313.583968	5.068406
33460U	59016.990340	59016.990345	313.835855	5.068049
38245U	59016.989646	59016.989651	313.001905	5.085841
38245U	59016.990340	59016.990346	313.252591	5.085801
44035U	59016.989646	59016.989651	313.333538	5.079510
44035U	59016.990340	59016.990345	313.586202	5.079216
28885U	59016.991053	59016.991059	312.818303	5.087318
28885U	59016.991747	59016.991753	313.068713	5.086543
33460U	59016.991053	59016.991058	314.092279	5.067973
33460U	59016.991747	59016.991752	314.343423	5.067652
38245U	59016.991053	59016.991058	313.508484	5.085253
38245U	59016.991747	59016.991753	313.759473	5.085072
42747U	59016.991053	59016.991058	313.792681	5.089683
42747U	59016.991747	59016.991753	314.043224	5.088484
44035U	59016.991053	59016.991058	313.842505	5.078886
44035U	59016.991747	59016.991753	314.093730	5.077992
28885U	59016.992482	59016.992488	313.333214	5.086227
28885U	59016.993176	59016.993182	313.585567	5.085623
38245U	59016.992482	59016.992487	314.022926	5.084879
38245U	59016.993176	59016.993181	314.276192	5.084392
20315U	59017.115515	59017.115521	310.049244	-9.000285
20315U	59017.116210	59017.116215	310.302585	-8.978796
23656U	59017.115515	59017.115521	310.838770	-8.982011
23656U	59017.116210	59017.116215	311.094664	-8.942596
20315U	59017.116887	59017.116893	310.548231	-8.958001
20315U	59017.117582	59017.117587	310.803510	-8.936151
23656U	59017.116887	59017.116893	311.342368	-8.903365
23656U	59017.117582	59017.117587	311.599933	-8.863346
20315U	59017.118300	59017.118305	311.064064	-8.913277
20315U	59017.118994	59017.119000	311.316830	-8.890862
28194U	59017.724070	59017.724075	223.702132	4.642496
28194U	59017.724764	59017.724769	223.946643	4.587879
37393U	59017.724070	59017.724076	223.666284	5.226478
37393U	59017.724764	59017.724770	223.916699	5.225818
39034U	59017.724070	59017.724075	223.988851	5.138970
39034U	59017.724764	59017.724770	224.239344	5.137532
40277U	59017.724070	59017.724075	224.227099	5.194298
40277U	59017.724764	59017.724770	224.477792	5.194151
23199U	59017.732866	59017.732872	222.544850	5.547936
23199U	59017.733560	59017.733566	222.789774	5.500155
33460U	59017.732866	59017.732872	221.904963	5.246906
33460U	59017.733560	59017.733566	222.156621	5.246201
42747U	59017.732866	59017.732872	221.717729	5.256962
42747U	59017.733560	59017.733566	221.969492	5.256371

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Table D.1 continued from previous page

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
44307U	59017.732866	59017.732871	222.916728	5.232123
44307U	59017.733560	59017.733565	223.168585	5.231344
33460U	59017.734256	59017.734261	222.405604	5.245965
33460U	59017.734950	59017.734956	222.658791	5.245132
38245U	59017.734256	59017.734261	221.842819	5.267466
38245U	59017.734950	59017.734956	222.096062	5.266652
41744U	59017.734256	59017.734262	222.211110	5.777242
41744U	59017.734950	59017.734956	222.461332	5.775225
42747U	59017.734256	59017.734261	222.219006	5.255932
42747U	59017.734950	59017.734956	222.471102	5.255678
44035U	59017.734256	59017.734261	222.135441	5.296665
44035U	59017.734950	59017.734956	222.388742	5.296312
28885U	59017.735645	59017.735651	221.662368	5.275279
28885U	59017.736340	59017.736345	221.913305	5.274895
33460U	59017.735645	59017.735651	222.908912	5.244498
33460U	59017.736340	59017.736345	223.160034	5.244026
38245U	59017.735645	59017.735651	222.345744	5.266947
38245U	59017.736340	59017.736345	222.596926	5.265846
41744U	59017.735645	59017.735651	222.714113	5.773456
41744U	59017.736340	59017.736346	222.962989	5.771454
42747U	59017.735645	59017.735651	222.721167	5.255295
42747U	59017.736340	59017.736345	222.971725	5.254766
44035U	59017.735645	59017.735651	222.638079	5.295971
44035U	59017.736340	59017.736345	222.889287	5.295306
28885U	59017.737055	59017.737061	222.170734	5.274472
28885U	59017.737750	59017.737755	222.422797	5.273780
38245U	59017.737055	59017.737061	222.854214	5.265776
38245U	59017.737750	59017.737755	223.106435	5.265148
43700U	59017.741409	59017.741415	199.644484	5.351920
43700U	59017.742104	59017.742110	199.896967	5.351162
33154U	59017.742810	59017.742815	200.461227	5.376254
33154U	59017.743504	59017.743510	200.712474	5.375840
36592U	59017.742810	59017.742815	200.425980	5.379765
36592U	59017.743504	59017.743510	200.677886	5.379533
39233U	59017.742810	59017.742816	199.771341	5.368535
39233U	59017.743504	59017.743510	200.022356	5.368332
43700U	59017.742810	59017.742815	200.150271	5.351006
43700U	59017.743504	59017.743510	200.402321	5.350823
32294U	59017.744203	59017.744208	199.864997	5.329426
32294U	59017.744897	59017.744903	200.117673	5.328960
36592U	59017.744203	59017.744208	200.928149	5.379273
36592U	59017.744897	59017.744902	201.181486	5.379423
39233U	59017.744203	59017.744208	200.273448	5.367006
39233U	59017.744897	59017.744903	200.526182	5.367360
43700U	59017.744203	59017.744208	200.653115	5.350311
43700U	59017.744897	59017.744903	200.906435	5.350229
32294U	59017.745599	59017.745605	200.369320	5.328211
32294U	59017.746293	59017.746299	200.621087	5.327405
39215U	59017.745599	59017.745605	200.088181	5.391044
39215U	59017.746293	59017.746299	200.340610	5.377764
39233U	59017.745599	59017.745605	200.777927	5.366476
39233U	59017.746293	59017.746299	201.029538	5.365984
20315U	59017.755285	59017.755290	177.886245	10.964278

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Table D.1 continued from previous page

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
20315U	59017.755979	59017.755984	178.129752	10.906341
23943U	59017.755285	59017.755290	178.698036	11.366174
23943U	59017.755979	59017.755984	178.943678	11.323988
20315U	59017.756672	59017.756676	178.370546	10.848599
20315U	59017.757366	59017.757371	178.612677	10.790351
20315U	59019.751470	59019.751475	168.361852	12.750615
20315U	59019.752164	59019.752169	168.605703	12.697609
24740U	59019.751470	59019.751474	168.842901	12.457581
24740U	59019.752164	59019.752169	169.090197	12.399220
20315U	59019.752857	59019.752862	168.848253	12.644971
20315U	59019.753551	59019.753556	169.092257	12.591094
27825U	59019.759433	59019.759439	228.783391	3.779431
27825U	59019.760128	59019.760134	229.035788	3.750849
27825U	59019.760815	59019.760821	229.279123	3.722923
27825U	59019.761509	59019.761515	229.531028	3.693522
27825U	59019.762205	59019.762211	229.778595	3.665354
27825U	59019.762899	59019.762905	230.029537	3.636213
37393U	59019.770402	59019.770408	242.353403	5.185470
37393U	59019.771096	59019.771102	242.605573	5.184601
39034U	59019.770402	59019.770407	242.677351	5.094148
39034U	59019.771096	59019.771102	242.928786	5.092968
40277U	59019.770402	59019.770407	242.917884	5.173881
40277U	59019.771096	59019.771102	243.170064	5.174643
37393U	59019.771779	59019.771784	242.850291	5.184353
37393U	59019.772473	59019.772479	243.103307	5.183883
39034U	59019.771779	59019.771784	243.173571	5.092922
39034U	59019.772473	59019.772478	243.426670	5.091913
40617U	59019.771779	59019.771785	242.328820	5.179345
40617U	59019.772473	59019.772479	242.582181	5.178477
37393U	59019.773173	59019.773178	243.352420	5.182617
37393U	59019.773867	59019.773872	243.606438	5.183176
40617U	59019.773173	59019.773178	242.831276	5.178212
40617U	59019.773867	59019.773873	243.085089	5.176970
41238U	59019.773173	59019.773179	242.257094	5.203648
41238U	59019.773867	59019.773873	242.512807	5.203776
40617U	59019.774549	59019.774554	243.328769	5.177312
40617U	59019.775243	59019.775248	243.580986	5.176489
41238U	59019.774549	59019.774555	242.755570	5.203431
41238U	59019.775243	59019.775249	243.007374	5.203011
33460U	59019.788940	59019.788945	244.095925	5.199962
33460U	59019.789634	59019.789640	244.349923	5.198711
41744U	59019.788940	59019.788946	244.576930	5.565812
41744U	59019.789634	59019.789640	244.826819	5.562930
42747U	59019.788940	59019.788945	243.902064	5.216325
42747U	59019.789634	59019.789640	244.155861	5.216278
44035U	59019.788940	59019.788946	243.842265	5.251642
44035U	59019.789634	59019.789640	244.096069	5.250876
28885U	59019.793407	59019.793412	244.473663	5.229588
28885U	59019.794101	59019.794107	244.727283	5.228142
28885U	59019.794789	59019.794795	244.972966	5.227989
28885U	59019.795484	59019.795489	245.224959	5.227227
39079U	59019.794789	59019.794795	243.882998	5.215939
39079U	59019.795484	59019.795490	244.136463	5.214637

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Table D.1 continued from previous page

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
29526U	59019.796570	59019.796576	221.719324	5.291071
29526U	59019.797264	59019.797270	221.972361	5.289885
33154U	59019.796570	59019.796576	221.798956	5.329658
33154U	59019.797264	59019.797270	222.052086	5.328772
36592U	59019.796570	59019.796576	221.810456	5.356560
36592U	59019.797264	59019.797270	222.064561	5.354752
41029U	59019.796570	59019.796576	221.715461	5.315629
41029U	59019.797264	59019.797270	221.968490	5.314786
29526U	59019.797949	59019.797955	222.218387	5.289800
29526U	59019.798644	59019.798649	222.469847	5.289564
36592U	59019.797949	59019.797955	222.309952	5.353546
36592U	59019.798644	59019.798649	222.561601	5.355289
39233U	59019.797949	59019.797955	221.650299	5.323320
39233U	59019.798644	59019.798649	221.901370	5.322838
41029U	59019.797949	59019.797955	222.214985	5.315189
41029U	59019.798644	59019.798649	222.466448	5.314645
43700U	59019.797949	59019.797955	222.009701	5.325533
43700U	59019.798644	59019.798649	222.260947	5.325129
29526U	59019.799337	59019.799342	222.718481	5.288548
29526U	59019.800031	59019.800036	222.971505	5.288469
32294U	59019.799337	59019.799343	221.731790	5.280245
32294U	59019.800031	59019.800037	221.984548	5.278882
36592U	59019.799337	59019.799342	222.809817	5.353876
36592U	59019.800031	59019.800036	223.063234	5.353673
39233U	59019.799337	59019.799342	222.150595	5.322492
39233U	59019.800031	59019.800037	222.402797	5.321681
41029U	59019.799337	59019.799342	222.715719	5.314092
41029U	59019.800031	59019.800036	222.968752	5.313682
43700U	59019.799337	59019.799342	222.509507	5.324397
43700U	59019.800031	59019.800037	222.763066	5.324591
32294U	59019.800711	59019.800717	222.227703	5.278691
32294U	59019.801406	59019.801411	222.480263	5.278290
39233U	59019.800711	59019.800717	222.646211	5.321398
39233U	59019.801406	59019.801411	222.898920	5.319935
28187U	59024.762301	59024.762307	242.265279	5.368258
28187U	59024.762995	59024.763001	242.507085	5.367180
20315U	59024.807066	59024.807072	168.019976	11.861552
20315U	59024.807761	59024.807767	168.265290	11.808993



Manually Determined Endpoints

Table E.1: Right ascension and declination of manually determined endpoints.

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
20315U	59016.719145	59016.719151	169.364979	13.103401
23943U	59016.718451	59016.718456	169.092705	13.035061
23943U	59016.719145	59016.719151	169.334993	12.998124
20315U	59016.719834	59016.719839	169.610953	13.050123
20315U	59016.720529	59016.720534	169.851547	12.997400
23943U	59016.719834	59016.719839	169.584736	12.961344
23943U	59016.720529	59016.720533	169.827872	12.924822
39215U	59016.745702	59016.745708	199.143667	5.436204
39215U	59016.746396	59016.746402	199.393055	5.423676
44071U	59016.745702	59016.745708	198.159119	5.482046
44071U	59016.746396	59016.746402	198.408592	5.481393
24748U	59016.925977	59016.925982	229.102466	1.935839
24748U	59016.926672	59016.926676	229.345348	1.886974
27712U	59016.925977	59016.925983	227.942232	2.185057
27712U	59016.926672	59016.926677	228.190893	2.144546
27712U	59016.927366	59016.927372	228.442583	2.104140
27712U	59016.928061	59016.928066	228.688988	2.063352
27712U	59016.928776	59016.928782	228.948889	2.021198
27712U	59016.929471	59016.929476	229.197866	1.980830
12851U	59016.936451	59016.936456	245.801485	-6.136466
12851U	59016.937145	59016.937151	246.055199	-6.113924
13907U	59016.936451	59016.936456	246.062422	-6.349532
13907U	59016.937145	59016.937150	246.299865	-6.320979
12851U	59016.937843	59016.937848	246.314494	-6.090924
12851U	59016.938537	59016.938542	246.570035	-6.068032
13907U	59016.937843	59016.937848	246.546496	-6.292079
13907U	59016.938537	59016.938542	246.788463	-6.262621
20315U	59016.937843	59016.937849	245.506409	-5.386107
20315U	59016.938537	59016.938544	245.752442	-5.431716
20315U	59016.939264	59016.939270	246.011810	-5.479813
20315U	59016.939959	59016.939965	246.256963	-5.525111
20315U	59016.940645	59016.940651	246.502541	-5.569895
20315U	59016.941339	59016.941345	246.747580	-5.614774
27825U	59016.949454	59016.949460	294.375389	-1.398323
27825U	59016.950149	59016.950154	294.627057	-1.404795

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Table E.1 continued from previous page

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
11353U	59016.950862	59016.950866	294.996317	-1.927872
11353U	59016.951556	59016.951560	295.235018	-1.876485
27825U	59016.950862	59016.950867	294.887285	-1.410737
27825U	59016.951556	59016.951561	295.136934	-1.417065
33154U	59016.957986	59016.957992	277.058069	5.182811
33154U	59016.958680	59016.958686	277.306814	5.182175
36592U	59016.957986	59016.957992	277.082275	5.245889
36592U	59016.958680	59016.958686	277.331072	5.245272
41029U	59016.957986	59016.957992	277.061238	5.252984
41029U	59016.958680	59016.958686	277.310638	5.252537
43700U	59016.957986	59016.957992	276.831445	5.232911
43700U	59016.958680	59016.958686	277.080510	5.232420
29526U	59016.959379	59016.959384	277.514979	5.187343
29526U	59016.960073	59016.960079	277.764922	5.186976
33154U	59016.959379	59016.959384	277.559964	5.181714
33154U	59016.960073	59016.960079	277.809537	5.181175
36592U	59016.959379	59016.959384	277.584556	5.244955
36592U	59016.960073	59016.960079	277.834114	5.244375
39233U	59016.959379	59016.959385	276.953157	5.186789
39233U	59016.960073	59016.960079	277.202813	5.186198
41029U	59016.959379	59016.959384	277.564124	5.252545
41029U	59016.960073	59016.960079	277.814184	5.252128
43700U	59016.959379	59016.959385	277.333670	5.232082
43700U	59016.960073	59016.960079	277.583884	5.231636
32294U	59016.960782	59016.960788	277.029407	5.159673
32294U	59016.961477	59016.961482	277.279667	5.159348
39233U	59016.960782	59016.960788	277.459280	5.185634
39233U	59016.961477	59016.961482	277.709742	5.184978
43700U	59016.960782	59016.960788	277.840050	5.231352
43700U	59016.961477	59016.961482	278.090773	5.230796
32294U	59016.962187	59016.962193	277.536994	5.159055
32294U	59016.962882	59016.962887	277.785962	5.158381
39233U	59016.962187	59016.962192	277.966974	5.184653
39233U	59016.962882	59016.962887	278.215712	5.183942
37393U	59016.970061	59016.970067	311.423908	5.053104
37393U	59016.970755	59016.970761	311.673052	5.052576
39034U	59016.970061	59016.970066	311.739265	5.006674
39034U	59016.970755	59016.970761	311.988065	5.006529
40277U	59016.970061	59016.970066	312.024049	5.099549
40277U	59016.970755	59016.970761	312.272985	5.099194
37393U	59016.971453	59016.971458	311.926264	5.052385
37393U	59016.972147	59016.972152	312.175356	5.051747
39034U	59016.971453	59016.971458	312.241438	5.006588
39034U	59016.972147	59016.972152	312.490559	5.006533
40277U	59016.971453	59016.971458	312.526249	5.099154
40277U	59016.972147	59016.972152	312.775643	5.098797
40617U	59016.971453	59016.971458	311.420854	5.062437
40617U	59016.972147	59016.972153	311.670307	5.061914
37393U	59016.972839	59016.972844	312.425784	5.051606
37393U	59016.973534	59016.973539	312.675621	5.051484
40617U	59016.972839	59016.972845	311.920228	5.062040
40617U	59016.973534	59016.973539	312.170744	5.061403
41238U	59016.972839	59016.972845	311.348340	5.107090

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Table E.1 continued from previous page

Satellite ID	Modified Julian Date	MJD w. shutter corr.	Right Ascension [°]	Declination [°]
41238U	59016.973534	59016.973539	311.598853	5.106630
40617U	59016.974246	59016.974251	312.428904	5.061500
40617U	59016.974940	59016.974945	312.677489	5.061058
41238U	59016.974246	59016.974251	311.856889	5.106583
41238U	59016.974940	59016.974946	312.105519	5.106128
33460U	59016.989646	59016.989651	313.585070	5.068560
33460U	59016.990340	59016.990345	313.834754	5.068052
38245U	59016.989646	59016.989651	313.001278	5.086155
38245U	59016.990340	59016.990346	313.251177	5.085648
44035U	59016.989646	59016.989651	313.335111	5.079507
44035U	59016.990340	59016.990345	313.585100	5.079062
28885U	59016.991053	59016.991059	312.819397	5.087159
28885U	59016.991747	59016.991753	313.067771	5.086545
33460U	59016.991053	59016.991058	314.093534	5.067969
33460U	59016.991747	59016.991752	314.342016	5.067657
38245U	59016.991053	59016.991058	313.510216	5.085406
38245U	59016.991747	59016.991753	313.758214	5.084919
42747U	59016.991053	59016.991058	313.793780	5.089053
42747U	59016.991747	59016.991753	314.041969	5.088801
44035U	59016.991053	59016.991058	313.843919	5.078569
44035U	59016.991747	59016.991753	314.041812	5.088802
28885U	59016.992482	59016.992488	313.334315	5.086225
28885U	59016.993176	59016.993182	313.584466	5.085782
38245U	59016.992482	59016.992487	314.024653	5.084874
38245U	59016.993176	59016.993181	314.274939	5.084083
20315U	59017.115515	59017.115521	310.050190	-9.000289
20315U	59017.116210	59017.116215	310.301793	-8.978950
23656U	59017.115515	59017.115521	310.839564	-8.982012
23656U	59017.116210	59017.116215	311.093713	-8.942595
20315U	59017.116887	59017.116893	310.549500	-8.957847
20315U	59017.117582	59017.117587	310.802399	-8.936305
23656U	59017.116887	59017.116893	311.343474	-8.903209
23656U	59017.117582	59017.117587	311.598831	-8.863189
20315U	59017.118300	59017.118305	311.064857	-8.913121
20315U	59017.118994	59017.119000	311.316039	-8.890861

Number of Data Points for Orbit Propagation

This appendix lists the number of data points that is available for each observation arc and propagation time that are used to build the plots in [section 7.6](#). For example, 8 data points for an observation arc of 120 [s] and propagation within the same night in [Table F.1](#) means that there are 8 orbit predictions based on an observation arc of 120 [s] that are propagated within the same night.

Table F.1: Number of data points for every propagation time and observation arc length for unaltered Gauss method.

Observation Arc [s]	Propagation Time [Days]				
	0	1	2	3	5
120	8	98	52	66	1
240	8	70	56	71	0

Table F.2: Number of data points for every propagation time and observation arc length for circularised Gauss method.

Observation Arc [s]	Propagation Time [Days]				
	0	1	2	3	5
120	32	283	192	287	2
240	8	86	74	96	0

Table F.3: Number of data points for every propagation time and observation arc length for Gooding method with default seed.

Observation Arc [s]	Propagation Time [Days]				
	0	1	2	3	5
120	24	163	85	110	1
240	8	83	68	83	0

Table F.4: Number of data points for every propagation time and observation arc length for Gooding method with circular orbit method seed.

Observation Arc [s]	Propagation Time [Days]				
	0	1	2	3	5
120	20	129	55	107	2
240	8	79	64	83	0

Table F.5: Number of data points for every propagation time and observation arc length for Gooding method with TLE seed.

Observation Arc [s]	Propagation Time [Days]				
	0	1	2	3	5
120	10	163	65	101	2
240	8	83	68	83	0

Table F.6: Number of data points for every propagation time and observation arc length for circular orbit method.

Observation Arc [s]	Propagation Time [Days]				
	0	1	2	3	5
60	46	417	294	415	5
120	36	282	190	284	2
180	22	173	127	179	1
240	8	64	64	74	0
300	4	32	32	37	0

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