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On Approximations of First Integrals for a Strongly Nonlinear Forced Oscillator

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Abstract

In this paper a strongly nonlinear forced oscillator will be studied. It will be shown that the recently developed perturbation method based on integrating factors can be used to approximate first integrals. Not only approximations of first integrals will be given, but it will also be shown how, in a rather efficient way, the existence and stability of time-periodic solutions can be obtained from these approximations. In addition phase portraits, Poincaré-return maps, and bifurcation diagrams for a set of values of the parameters will be presented. In particular the strongly nonlinear forced oscillator equation $\ddot{X}+X+\lambda X^3 = \epsilon\left(\delta\dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(2t)\right)$ will be studied in this paper. It will be shown that the presented perturbation method not only can be applied to a weakly nonlinear oscillator problem (that is, when the parameter $\lambda = \mathcal{O}(\epsilon)$) but also to a strongly nonlinear problem (that is, when $\lambda = \mathcal{O}(1)$). The model equation as considered in this paper is related to the phenomenon of galloping of overhead power transmission lines on which ice has accreted.

Keywords: Integrating factor, integrating vector, first integral, perturbation method, asymptotic approximation of first integral, periodic solution, strongly nonlinear forced oscillator.

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1 Introduction

In this paper it will be shown how the perturbation method based on integrating vectors can be applied to the following non-autonomous equation

$$\ddot{X} + \frac{dU(X)}{dX} = \epsilon f(X, \dot{X}, t), \tag{1.1}$$

where $U(X)$ is the potential energy of the unperturbed (that is, $\epsilon = 0$), nonlinear oscillator, where $X = X(t)$, $\dot{X} = \frac{dX}{dt}$, where ϵ is a small parameter satisfying $0 < \epsilon \ll 1$, and where f is a sufficiently smooth function. Many classical perturbation methods have been used to approximate analytically the solution of the weakly nonlinear problem (1.1), that is, when $\frac{dU(X)}{dX}$ in (1.1) is linear in X. However, when $\frac{dU(X)}{dX}$ is non-linear in X most of the perturbation methods can not be applied to construct approximations of the solutions. In this paper it will be shown that the perturbation method based on integrating factors can be applied to a class of non-autonomous nonlinear equations as described by (1.1). In particular equation (1.1) with $\frac{dU(X)}{dX} = X + \lambda X^3$ and $f(X, \dot{X}, t) = \delta\dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(\omega t)$ will be studied in detail in this paper. The existence and stability of time-periodic solutions will be investigated. Bifurcation diagrams will be presented, and the dynamics of the oscillator as described by

$$\ddot{X} + X + \lambda X^3 = \epsilon\left(\delta\dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(\omega t)\right) \tag{1.2}$$

will be studied in this paper. In (1.2) it is assumed that the parameter λ is positive, that is, it is assumed that the oscillator is attached to hard, nonlinear springs. The parameters $\delta > 0$, $\beta > 0$, $\gamma \neq 0$, and ω are assumed to be constants independent of ϵ. The oscillator equation (1.2) originates from the following system of ODEs

$$\begin{cases} \ddot{Y} + \omega_1^2 Y &= \epsilon\left[-a_{1,0}\dot{Y} + a_{2,0}\dot{Y}^2 + a_{0,2}\dot{X}^2\right], \\ \ddot{X} + \omega_2^2 X + \lambda X^3 &= \epsilon\left[b_{0,1}\dot{X} - b_{1,1}\dot{Y}\dot{X} - b_{0,3}\dot{X}^3\right], \end{cases} \tag{1.3}$$

which is a simple model for the flow-induced vibrations of a cable in a windfield. System (1.3) with $\lambda = 0$ or $\lambda = \mathcal{O}(\epsilon)$ has been derived by Van der Beek in [2, 3]. The coefficients $a_{1,0}$, $a_{2,0}$, $a_{0,2}$, $b_{0,1}$, $b_{1,1}$, and $b_{0,3}$ depend on the quasi-static drag and lift forces acting on a conductor line in a uniform windfield. System (1.3) can be considered to be an oscillator with two degrees of freedom which describes flow-induced vibration of cables in a windfield. The displacement of the cable in vertical direction (that is, perpendicular to the windflow) is described by $X(t)$, and the displacement of the cable in horizontal direction (that is, in the direction of the windflow) is given by $Y(t)$. For more (and complete) details the reader is referred to [2, 3, 18]. It is well-known that galloping of conductor lines is an almost purely vertical oscillation of these lines. Based upon the results as obtained in [2, 3, 13] which confirm this vertical oscillation it is now assumed that the oscillator will oscillate in an almost vertical direction and that $Y(t) = A\cos(\omega_1 t)$. In this way system (1.3) can be reduced to

$$\ddot{X} + \omega_2^2 X + \lambda X^3 = \epsilon \left(b_{0,1}\dot{X} - b_{1,1}A\cos(\omega_1 t)\dot{X} - b_{0,3}\dot{X}^3 \right). \tag{1.4}$$

For small values of λ (that is, for $\lambda = \mathcal{O}(\epsilon)$) an interesting internal resonance for system (1.3) is $\omega_1 : \omega_2 = 2 : 1$. This case for instance has been studied in [2, 3]. In this paper it is also assumed that $\omega_1 : \omega_2 = 2 : 1$ when $\lambda = \mathcal{O}(1)$. After some elementary rescalings in (1.4) equation (1.2) is finally obtained with $\omega = 2$. In particular we will be interested in the existence and stability of (order 1) periodic solutions of equation (1.2). Many researchers investigated extensively the behavior of the solutions of equations of the type (1.1). For instance, Nayfeh and Mook [1], Belhaq and Houssni [8] and others investigated the steady-state (periodic) solutions of the weakly nonlinear equation (1.1) with $\frac{dU(X)}{dX} = \omega_0^2(1 + h\cos(\nu t))X$ and $f(X, \dot{X}, t) = -\alpha\dot{X} - \beta X^2 - \xi X^3 + \gamma\cos(\omega t)$ using a generalized averaging method or a multiple time-scales perturbation method. For $\beta = 0$, $h = 0$ Lima and Pettini [10] studied the control of chaos in a periodically forced oscillator. They showed analytically that a small nonlinear parametric perturbation can suppress chaos. Again by using a multiple time-scales perturbation method Burton and Rahman [17] studied the response of a weakly nonlinear oscillator as described by equation (1.1) with $\frac{dU(X)}{dX} = mX$ and $f(X, \dot{X}, t) = -2\eta\dot{X} - g(X) + 2P\cos(\omega t)$, where $g(X)$ is an odd nonlinear function, and where m is an integer which may be either -1, 0, or 1. Roy [11] used an elliptic averaging method to investigate (1.1) with $\frac{dU(X)}{dX} = \alpha X + \gamma X^3$ and $f(X, \dot{X}, t) = -\beta\dot{X} + F\cos(\omega t)$. Brothers and Haberman [5] also studied (1.1) with $f(X, \dot{X}, t) = -h(X, \dot{X}) + \gamma\cos(2\pi\omega t)$, where h is a purely dissipative perturbation (h is odd in $\dot{X}$) by using averaging and matching techniques. Higher-order averaging techniques based on Lie transforms have been used by Yagasaki and Ichikawa [6] to study weakly nonlinear equations like (1.1) with $f(X, \dot{X}, t) = -\delta\dot{X} - \beta X^2 - \alpha X^3 + \gamma\cos(\omega t)$. Van Horssen [20, 21] studied a weakly nonlinear Duffing equation (1.1) with $f(X, \dot{X}, t) = -a\dot{X} - bX^3 + c\cos(t)$ using the perturbation method based on integrating factors and multiple time-scales. In this paper it will be shown that for the weakly non-autonomous and weakly nonlinear equation (1.2) exactly the same results can be obtained as by applying the classical perturbation techniques (such as averaging, multiple time-scales, Poincaré-Lindstedt or others). However, for the strongly nonlinear equation (1.2) with $\lambda = \mathcal{O}(1)$ most of the classical perturbation methods can not be applied. In this paper the recently developed perturbation method based on integrating factors (see [20, 21]) will be used to construct asymptotic approximations of first integrals for (1.2) on long time-scales. In the literature not many analytical results can be found for strongly nonlinear and non-autonomous oscillator equation like (1.2). Only recently Yagasaki [7] studied (1.2) with $\lambda = 1$ and with the perturbation in the right-hand side of (1.2) replaced by $(-\delta + X\cos(\omega t))\dot{X} + \gamma\cos(\omega t)$ using an adapted version of Melnikov's method. This paper is organized as follows. In section 2 of this paper the construction of approximations of first integrals by using the perturbation method based on integrating factors will be discussed briefly for the general oscillator equation (1.1). In section 3 approximations of first integrals will be constructed explicitly for the weakly and the strongly nonlinear, forced oscillator equation (1.2). Using the approximations of the first integrals it will be shown in section 4 how the existence and stability of time-periodic solutions for the oscillator equation (1.2) can be obtained. The bifurcation(s) of periodic solutions will be studied in detail, and a complete set of topological different phase portraits will be presented. Finally in section 5 of this paper some conclusions will be drawn and some remarks will be made.

2 Approximations of First Integrals

In this section we briefly outline how the perturbation method based on integrating vectors can be applied to approximate first integrals (see also [12, 13, 14, 20, 21]). We consider the class of non-linear oscillators described by the equation

$$\ddot{X} + \frac{dU(X)}{dX} = \epsilon f(X, \dot{X}, t), \tag{2.1}$$

where $U(X)$ is a potential, $X = X(t)$, $\dot{X} = \frac{dX}{dt}$, ϵ is a small parameter satisfying $0 < \epsilon \ll 1$, and where f is assumed to be sufficiently smooth. We assume that the unperturbed (that is, $\epsilon = 0$) solutions of (2.1) form a family of periodic orbits in the phase-plane $(X, \dot{X})$. This family may cover the entire "phase plane" $(X, \dot{X})$, or a bounded region $\mathcal{D}$ of the phase plane. Each periodic orbit corresponds to a constant energy level $E = \frac{1}{2}\dot{X}^2 + U(X)$. With each constant energy level E corresponds a phase angle ψ, which is defined to be

$$\psi = \int_0^X \frac{dr}{\sqrt{2E - 2U(r)}}. \tag{2.2}$$

From (2.1)-(2.2) a transformation $(X, \dot{X}) \longmapsto (E, \psi)$ can then be defined with

$$\begin{cases} \dot{E} = \epsilon \dot{X} f & = \quad g_1(E, \psi, t), \\ \\ \dot{\psi} = 1 + \epsilon \left[-\int_0^X \frac{dr}{(2E-2U(r))^{\frac{3}{2}}} \dot{X} f \right] & = \quad g_2(E, \psi, t). \end{cases} \tag{2.3}$$

Multiplying the first and the second equation in (2.3) with $\mu_1(E, \psi, t)$ and $\mu_2(E, \psi, t)$ respectively, it follows that the integrating factors $\mu_1(E, \psi, t)$ and $\mu_2(E, \psi, t)$ have to satisfy (see also [20, 21])

$$\begin{cases} \frac{\partial \mu_1}{\partial \psi} = \frac{\partial \mu_2}{\partial E}, \\ \\ \frac{\partial \mu_1}{\partial t} = -\frac{\partial}{\partial E} \left(\mu_1 g_1 + \mu_2 g_2 \right), \\ \\ \frac{\partial \mu_2}{\partial t} = -\frac{\partial}{\partial \psi} \left(\mu_1 g_1 + \mu_2 g_2 \right). \end{cases} \tag{2.4}$$

Let $g_1 = \epsilon g_{1,1} + \epsilon^2 g_{1,2}$, $g_2 - 1 = \epsilon g_{2,1} + \epsilon^2 g_{2,2}$. Expanding μ_1 and μ_2 in formal power series in ϵ, that is,

$$\mu_i(E, \psi, t; \epsilon) = \mu_{i,0}(E, \psi, t) + \epsilon \mu_{i,1}(E, \psi, t) + \ldots$$

for $i = 1$ and 2, substituting g_1, g_2, and the expansions for μ_1 and μ_2 into (2.4), and by taking together terms of equal powers in ϵ, we finally obtain the following $\mathcal{O}(\epsilon^n)$-problems: for $n = 0$

$$\begin{cases} \frac{\partial \mu_{1,0}}{\partial \psi} = \frac{\partial \mu_{2,0}}{\partial E}, \\ \\ \frac{\partial \mu_{1,0}}{\partial t} = -\frac{\partial \mu_{2,0}}{\partial E}, \\ \\ \frac{\partial \mu_{2,0}}{\partial t} = -\frac{\partial \mu_{2,0}}{\partial \psi}, \end{cases} \tag{2.5}$$

for n=1

$$\begin{cases} \frac{\partial \mu_{1,1}}{\partial \psi} = \frac{\partial \mu_{2,1}}{\partial E}, \\ \\ \frac{\partial \mu_{1,1}}{\partial t} = -\frac{\partial}{\partial E}\left(\mu_{1,0}g_{1,1} + \mu_{2,0}g_{2,1} + \mu_{2,1}\right), \\ \\ \frac{\partial \mu_{2,1}}{\partial t} = -\frac{\partial}{\partial \psi}\left(\mu_{1,0}g_{1,1} + \mu_{2,0}g_{2,1} + \mu_{2,1}\right), \end{cases} \tag{2.6}$$

and for $n \geq 2$

$$\begin{cases} \frac{\partial \mu_{1,n}}{\partial \psi} = \frac{\partial \mu_{2,n}}{\partial E}, \\ \\ \frac{\partial \mu_{1,n}}{\partial t} = -\frac{\partial}{\partial E}\left(\mu_{1,n-2}g_{1,2} + \mu_{1,n-1}g_{1,1} + \mu_{2,n-2}g_{2,2} + \mu_{2,n-1}g_{2,1} + \mu_{2,n}\right), \\ \\ \frac{\partial \mu_{2,n}}{\partial t} = -\frac{\partial}{\partial \psi}\left(\mu_{1,n-2}g_{1,2} + \mu_{1,n-1}g_{1,1} + \mu_{2,n-2}g_{2,2} + \mu_{2,n-1}g_{2,1} + \mu_{2,n}\right). \end{cases} \tag{2.7}$$

The $\mathcal{O}(\epsilon^0)$-problem (2.5) can readily be solved, yielding $\mu_{1,0} = h_{1,0}(E, \psi - t)$ and $\mu_{2,0} = h_{2,0}(E, \psi - t)$ with $\frac{\partial h_{1,0}}{\partial \psi} = \frac{\partial h_{2,0}}{\partial E}$. The functions $h_{1,0}$ and $h_{2,0}$ are still arbitrary and will now be chosen as simple as possible. We choose $h_{1,0} \equiv 1$ and $h_{2,0} \equiv 0$, and so (see also [12, 20])

$$\mu_{1,0} = 1,\ \mu_{2,0} = 0. \tag{2.8}$$

It follows from the order ϵ-problem (2.6) that $\mu_{1,1}$ and $\mu_{2,1}$ have to satisfy

$$\begin{cases} \frac{\partial \mu_{1,1}}{\partial t} + \frac{\partial \mu_{1,1}}{\partial \psi} = -\frac{\partial}{\partial E}\left(g_{1,1}\right), \\ \\ \frac{\partial \mu_{2,1}}{\partial t} + \frac{\partial \mu_{2,1}}{\partial \psi} = -\frac{\partial}{\partial \psi}\left(g_{1,1}\right). \end{cases} \tag{2.9}$$

By using the method of characteristics for first order PDEs we then obtain

$$\begin{cases} \mu_{1,1} = h_{1,1}(E, \psi - t) - \int^t \left(\frac{\partial}{\partial E}(g_{1,1})\right) d\bar{t}, \\ \\ \mu_{2,1} = h_{2,1}(E, \psi - t) - \int^t \left(\frac{\partial}{\partial \psi}(g_{1,1})\right) d\bar{t}, \end{cases} \tag{2.10}$$

where $h_{1,1}$, $h_{2,1}$ are arbitrary functions which have to satisfy

$$\frac{\partial h_{1,1}}{\partial \psi} - \frac{\partial}{\partial \psi}\int^t \left(\frac{\partial}{\partial E}(g_{1,1})\right) d\bar{t} = \frac{\partial h_{2,1}}{\partial E} - \frac{\partial}{\partial E}\int^t \left(\frac{\partial}{\partial \psi}(g_{1,1})\right) d\bar{t}. \tag{2.11}$$

We choose $h_{1,1}$ and $h_{2,1}$ as simple as possible, that is, we take $h_{1,1} = 0$, $h_{2,1} = 0$. We then obtain for $\mu_{1,1}$ and $\mu_{2,1}$

$$\begin{cases} \mu_{1,1} = -\frac{\partial}{\partial E}\left(\int^t g_{1,1} d\bar{t}\right), \\ \\ \mu_{2,1} = -\frac{\partial}{\partial \psi}\left(\int^t g_{1,1} d\bar{t}\right). \end{cases} \tag{2.12}$$

The $\mathcal{O}(\epsilon^2)$-problem (2.7) can be solved , yielding

$$\begin{cases} \mu_{1,2} = -\frac{\partial}{\partial E}\left(\int^t (g_{1,2} + \mu_{1,1} g_{1,1} + \mu_{2,1} g_{2,1})\, d\bar{t}\right), \\ \\ \mu_{2,2} = -\frac{\partial}{\partial \psi}\left(\int^t (g_{1,2} + \mu_{1,1} g_{1,1} + \mu_{2,1} g_{2,1})\, d\bar{t}\right). \end{cases} \tag{2.13}$$

The $\mathcal{O}(\epsilon^n)$-problems (2.7) with $n > 2$ can be solved in a similar way. An approximation F_1 of a first integral $F = constant$ of system (2.3) can now be obtained from (2.8), (2.12) and (2.13) yielding (see also [20, 21])

$$F_1(E, \psi, t) = E - \epsilon \left[\int^t g_{1,1} d\bar{t}\right] - \epsilon^2 \left[\int^t (g_{1,2} + \mu_{1,1} g_{1,1} + \mu_{2,1} g_{2,1})\, d\bar{t}\right]. \tag{2.14}$$

How well F_1 approximates a first integral $F = constant$ can be deduced from (see also [20, 21])

$$\begin{aligned} \frac{dF_1}{dt} &= \left[g_1 + \epsilon\mu_{1,1} g_1 + \epsilon^2 \mu_{1,2} g_1 + \epsilon\mu_{2,1} g_2 + \epsilon^2 \mu_{2,2} g_2\right]_{**} \\ &= \epsilon^3 \mathcal{R}_1(E, \psi, t), \end{aligned} \tag{2.15}$$

where $g_1, g_2, \mu_{1,1}, \mu_{2,1}, \mu_{1,2}, \mu_{2,2}$ are given by (2.3), (2.12) and (2.13), and where the $**$ indicates that only terms of $\mathcal{O}(\epsilon^m)$ with $m \geq 3$ are included. From the existence and uniqueness theorems for ODEs we know that initial value problems for (2.1) (with sufficiently smooth potential $U(X)$ and nonlinearity $f(X, \dot{X}, t)$) are well-posed on a time-scale of order $\frac{1}{\epsilon}$. This implies that also an initial-value problem for system (2.3) is well-posed on this time-scale. From (2.3) it then follows on this time-scale that if $E(0)$ is bounded then $E(t)$ is bounded and $\psi(t)$ is bounded by a constant plus t. Since $|\mathcal{R}_1| \leq c_0 + c_1 t$ on a time scale of order $\frac{1}{\epsilon}$, where c_0, c_1 are constants, it follows from (2.15) that

$$F_1(E(t), \psi(t), t; \epsilon) = constant + \epsilon^3 \int_0^t \mathcal{R}_1(E(s), \psi(s), s; \epsilon) ds,$$

and so,

$$\begin{aligned} F_1(E(t), \psi(t), t; \epsilon) &= constant + \mathcal{O}(\epsilon^3),\ 0 \leq t \leq T_0 < \infty, \\ F_1(E(t), \psi(t), t; \epsilon) &= constant + \mathcal{O}(\epsilon),\ 0 \leq t \leq \frac{L}{\epsilon}, \end{aligned} \tag{2.16}$$

where T_0 and L are ϵ-independent constants. Another (functionally independent) approximation of a first integral can be obtained by putting in (2.5)

$$\mu_{2,0} = 1,\ \mu_{1,0} = 0, \tag{2.17}$$

instead of (2.8). The $\mathcal{O}(\epsilon)$-problem (2.6) can now be solved, yielding

$$\begin{cases} \mu_{1,1} = k_{1,1}(E, \psi - t) - \int^t \left(\frac{\partial}{\partial E}(g_{2,1})\right) d\bar{t}, \\ \\ \mu_{2,1} = k_{2,1}(E, \psi - t) - \int^t \left(\frac{\partial}{\partial \psi}(g_{2,1})\right) d\bar{t}, \end{cases} \tag{2.18}$$

where the functions $k_{1,1}$ and $k_{2,1}$ are arbitrary functions which have to satisfy

$$\frac{\partial k_{1,1}}{\partial \psi} - \frac{\partial}{\partial \psi}\int^t \left(\frac{\partial}{\partial E}(g_{2,1})\right) d\bar{t} = \frac{\partial k_{2,1}}{\partial E} - \frac{\partial}{\partial E}\int^t \left(\frac{\partial}{\partial \psi}(g_{2,1})\right) d\bar{t}. \tag{2.19}$$

We choose these functions as simple as possible, that is, $k_{1,1} = 0$ and $k_{2,1} = 0$. Then we obtain

$$\begin{cases} \mu_{1,1} = -\frac{\partial}{\partial E}\left(\int^t g_{2,1} d\bar{t}\right), \\ \\ \mu_{2,1} = -\frac{\partial}{\partial \psi}\left(\int^t g_{2,1} d\bar{t}\right). \end{cases} \tag{2.20}$$

The $\mathcal{O}(\epsilon^2)$-problem (2.7) can be solved , yielding

$$\begin{cases} \mu_{1,2} = -\frac{\partial}{\partial E}\left(\int^t (g_{2,2} + \mu_{1,1}g_{1,1} + \mu_{2,1}g_{2,1})\, d\bar{t}\right), \\ \\ \mu_{2,2} = -\frac{\partial}{\partial \psi}\left(\int^t (g_{2,2} + \mu_{1,1}g_{1,1} + \mu_{2,1}g_{2,1})\, d\bar{t}\right). \end{cases} \tag{2.21}$$

An approximation F_2 of a first integral $F = constant$ of system (2.3) can now be obtained from (2.17), (2.20) and (2.21) yielding (see also [20, 21])

$$F_2(E, \psi, t) = (\psi - t) - \epsilon\left[\int^t g_{2,1} d\bar{t}\right] - \epsilon^2\left[\int^t (g_{2,2} + \mu_{1,1}g_{1,1} + \mu_{2,1}g_{2,1})\, d\bar{t}\right]. \tag{2.22}$$

How well F_2 approximates a first integral $F = constant$ can be deduced from (see also [20, 21])

$$\begin{aligned} \frac{dF_2}{dt} &= \left[g_1 + \epsilon\mu_{1,1}g_1 + \epsilon^2\mu_{1,2}g_1 + \epsilon\mu_{2,1}g_2 + \epsilon^2\mu_{2,2}g_2\right]_{**} \\ &= \epsilon^3\mathcal{R}_1(E, \psi, t), \end{aligned} \tag{2.23}$$

where $g_1, g_2, \mu_{1,1}, \mu_{2,1}, \mu_{1,2}, \mu_{2,2}$ are given by (2.3), (2.20) and (2.21), and where the $**$ indicates that only terms of $\mathcal{O}(\epsilon^m)$ with $m \geq 3$ are included. In the following section we will apply this perturbation method to the oscillator equation (1.2).

3 Approximations of First Integrals for a Nonlinear, Forced Oscillator

In this section we will consider the following nonlinear, forced oscillator equation

$$\ddot{X} + \frac{dU(X)}{dX} = \epsilon f(X, \dot{X}, t), \tag{3.1}$$

where $\frac{dU(X)}{dX} = X + \lambda X^3$ in which $\lambda > 0$ is a parameter, and where $f(X, \dot{X}, t) = \delta \dot{X} - \beta \dot{X}^3 + \gamma \dot{X} \cos(2t)$ in which $\delta > 0$, $\beta > 0$, and $\gamma \neq 0$ are parameters, and where ϵ is a small parameter with $0 < \epsilon \ll 1$. As explained in the introduction the oscillator equation (3.1) can be considered to be a simple model describing the vertical displacement of an overhead power transmission line (on which ice has accreted) in a windfield. The function $X(t)$ describes the vertical displacement. In this section asymptotic approximations of first integrals for (3.1) will be constructed explicitly. To give a rather complete analysis of (3.1) and to understand the bifurcation(s) of the periodic solutions in section 4 we will now consider three cases: (i) $\lambda = \mathcal{O}(\epsilon)$, (ii) $\lambda = \mathcal{O}(\sqrt{\epsilon})$ and (iii) $\lambda = \mathcal{O}(1)$.

3.1 The case $\lambda = \mathcal{O}(\epsilon)$

Let $\lambda = \tilde{\lambda}\epsilon$ with $\tilde{\lambda} > 0$. To study (3.1) with $\lambda = \tilde{\lambda}\epsilon$ in detail we will use straightforward calculations as presented in section 2 to obtain approximations of the first integrals. By introducing the rescalings $\epsilon\delta = \tilde{\epsilon}$, $X = \sqrt{\frac{\delta}{\tilde{\lambda}}}\tilde{X}$, $\tilde{\beta}\tilde{\lambda} = \beta$, and $\tilde{\gamma}\delta = \gamma$ it follows that (3.1) becomes

$$\ddot{\tilde{X}} + \tilde{X} = \tilde{\epsilon}(\dot{\tilde{X}} - \tilde{\beta}\dot{\tilde{X}}^3 - \tilde{X}^3 + \tilde{\gamma}\dot{\tilde{X}} \cos(2t)). \tag{3.2}$$

In the further analysis the tildes will be dropped for convenience. By introducing the transformation $(X, \dot{X}) \longmapsto (E, \psi)$ as defined by

$$\begin{cases} E &= \frac{1}{2}\dot{X}^2 + \frac{1}{2}X^2, \\ \psi &= \int_0^X \frac{dr}{\sqrt{2E - r^2}} = \sin^{-1}\left(\frac{X}{\sqrt{2E}}\right), \end{cases} \tag{3.3}$$

(where E and ψ are the energy and the phase angle of the unperturbed (that is, $\epsilon = 0$) oscillator respectively) it follows from (3.2) that

$$\begin{cases} \dot{E} = \epsilon \dot{X} g & = \xi_1(E, \psi, t) = \epsilon \xi_{1,1}(E, \psi, t), \\ \dot{\psi} = 1 + \epsilon \left[-\int_0^X \frac{dr}{(2E - r^2)^{\frac{3}{2}}} \dot{X} g \right] & = \xi_2(E, \psi, t) = 1 + \epsilon \xi_{2,1}(E, \psi, t), \end{cases} \tag{3.4}$$

where $g = \dot{X} - \beta \dot{X}^3 - X^3 + \gamma \dot{X} \cos(2t)$. From the calculations as presented in section 2 of this paper it follows that two functionally independent approximations of the first integrals for (3.2) are given by

$$\begin{aligned}
F_1(E,\psi,t) &= E-\epsilon\int^t \xi_{1,1}d\bar{t} \\
&= E-\epsilon\int^t \big(2E\cos(\psi)^2-4\beta E^2\cos(\psi)^4-4E^2\sin(\psi)^3\cos(\psi) \\
&\quad +2E\gamma\cos(\psi)^2\cos(2t)\big)\,d\psi \\
&= E-\epsilon\left(\left(E-\frac{3}{2}E^2\beta\right)\psi+\left(\frac{1}{2}E-E^2\beta\right)\sin(2\psi)-\frac{1}{8}E^2\beta\sin(4\psi)\right. \\
&\quad \left.+\frac{1}{2}E\gamma\sin(2t)+\frac{1}{2}E^2+\frac{1}{8}E\gamma\sin(2\psi+2t)+\frac{1}{2}E\gamma\psi\cos(2\psi-2t)\right) \qquad (3.5)
\end{aligned}$$

and

$$\begin{aligned}
F_2(E,\psi,t) &= (\psi-t)-\epsilon\int^t \xi_{2,1}d\bar{t} \\
&= (\psi-t)+\frac{\epsilon}{2E}\int^t \big(2E\sin(\psi)\cos(\psi)-2E^2\beta\sin(\psi)\cos(\psi)^3 \\
&\quad -4E^2\sin(\psi)^4+2E\gamma\sin(\psi)\cos(\psi)\cos(2t)\big)\,d\psi \\
&= (\psi-t)+\epsilon\left(\left(-\frac{1}{4}+\frac{1}{4}E\beta\right)\cos(2\psi)+\frac{1}{16}E\beta\cos(4\psi)+\frac{1}{2}E\sin(2\psi)\right. \\
&\quad \left.-\frac{1}{16}E\sin(4\psi)-\frac{3}{4}E\psi+\frac{1}{4}\gamma\psi\sin(2\psi-2t)-\frac{1}{16}\gamma\cos(2\psi+2t)\right). \qquad (3.6)
\end{aligned}$$

How well F_1 and F_2 approximate a first integral $F = constant$ can be deduced from

$$\frac{dF_j}{dt} = \epsilon\mu_{1,1}\xi_1+\epsilon\mu_{2,1}(\xi_2-1)=\epsilon^2\mathcal{R}_j(E,\psi,t), \qquad (3.7)$$

where ξ_1 and ξ_2 are given by (3.4). It follows from (3.7) that for $j=1,2$ (see also (2.15)-(2.16))

$$F_j(E(t),\psi(t),t;\epsilon)=constant+\epsilon^2\int_0^t \mathcal{R}_j(E(s),\psi(s),s;\epsilon)ds, \qquad (3.8)$$

and so,

$$\begin{aligned}
F_j(E(t),\psi(t),t;\epsilon) &= constant+\mathcal{O}(\epsilon^2),\ 0\le t\le T_0<\infty, \\
F_j(E(t),\psi(t),t;\epsilon) &= constant+\mathcal{O}(\epsilon),\ 0\le t\le \frac{L}{\sqrt{\epsilon}}, \qquad (3.9)
\end{aligned}$$

where T_0 and L are ϵ-independent constants.

3.2 The case $\lambda = \mathcal{O}(\sqrt{\epsilon})$

Let $\lambda = \sqrt{\epsilon}\bar{\lambda}$ with $\bar{\lambda} > 0$. By introducing the rescalings $\epsilon\delta = \bar{\epsilon}$, $X = \sqrt{\frac{\sqrt{\delta}}{\bar{\lambda}}}\bar{X}$, $\bar{\beta}\bar{\lambda}\sqrt{\delta} = \beta$, and $\bar{\gamma}\delta = \gamma$ it follows that (3.1) becomes

$$\ddot{\bar{X}} + \bar{X} + \sqrt{\bar{\epsilon}}\bar{X}^3 = \bar{\epsilon}(\dot{\bar{X}} - \bar{\beta}\dot{\bar{X}}^3 + \bar{\gamma}\dot{\bar{X}}\cos(2t)). \tag{3.10}$$

In the further analysis the bars will be dropped for convenience. By introducing the transformation $(X, \dot{X}) \longmapsto (E, \psi)$ as defined by

$$\begin{cases} E &= \frac{1}{2}\dot{X}^2 + \frac{1}{2}X^2, \\ \psi &= \int_0^X \frac{dr}{\sqrt{2E - r^2}} = \sin^{-1}\left(\frac{X}{\sqrt{2E}}\right), \end{cases} \tag{3.11}$$

(where E and ψ are the energy and the phase angle of the unperturbed (that is, $\epsilon = 0$) oscillator respectively) it follows from (3.10) that

$$\begin{cases} \dot{E} = \sqrt{\epsilon}\dot{X}g = \xi_3(E,\psi,t) = \sqrt{\epsilon}\xi_{3,1}(E,\psi,t) + \epsilon\xi_{3,2}(E,\psi,t), \\ \dot{\psi} = 1 + \sqrt{\epsilon}\left[-\int_0^X \frac{dr}{(2E-r^2)^{\frac{3}{2}}}\dot{X}g\right] = \xi_4(E,\psi,t) = 1 + \sqrt{\epsilon}\xi_{4,1}(E,\psi,t) + \epsilon\xi_{4,2}(E,\psi,t), \end{cases} \tag{3.12}$$

where $g = -X^3 + \sqrt{\epsilon}\left(\dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(2t)\right)$. From the calculations as presented in section 2 of this paper it follows that two functionally independent approximations of the first integrals for system (3.10) are given by

$$\begin{aligned} F_3(E,\psi,t) &= E + \sqrt{\epsilon}\int^t -\xi_{3,1}d\bar{t} + \epsilon\int^t -\left(\xi_{3,2} + \mu_{3,1}\xi_{3,1} + \mu_{4,1}\xi_{4,1}\right)d\bar{t} \\ &= E + \sqrt{\epsilon}\int^t \left(E^2\sin(2\psi) - \frac{1}{2}E^2\sin(4\psi)\right)d\psi \\ &\quad +\epsilon\int^t \Big(-E\cos(2\psi) - E + \frac{1}{2}E^2\beta\cos(4\psi) + 2E^2\beta\cos(2\psi) + \frac{3}{2}E^2\beta \\ &\quad -\frac{1}{2}E\gamma\cos(2\psi - 2t) - \frac{1}{2}E\gamma\cos(2\psi + 2t) - E\gamma\cos(2t) \\ &\quad +\frac{3}{8}E^3\sin(4\psi) - \frac{3}{4}E^3\sin(2\psi)\Big)d\psi \\ &= E + \sqrt{\epsilon}\left(-\frac{1}{2}E^2\cos(2\psi) + \frac{1}{8}E^2\cos(4\psi)\right) \\ &\quad +\epsilon\Bigg(\left(E^2\beta - \frac{1}{2}E\right)\sin(2\psi) + \left(\frac{3}{2}E^2 - E\right)\psi + \frac{1}{8}E^2\beta\sin(4\psi) \\ &\quad -\frac{1}{2}\gamma\psi\cos(2\psi - 2t) - \frac{1}{8}E\gamma\sin(2\psi + 2t) - \frac{1}{2}E\gamma\sin(2t) \\ &\quad -\frac{3}{32}E^3\cos(4\psi) + \frac{3}{8}E^3\cos(2\psi)\Bigg), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
F_4(E,\psi,t) &= (\psi - t) + \sqrt{\epsilon}\int^t -\xi_{4,1} d\bar{t} + \epsilon \int^t -\left(\xi_{4,2} + \mu_{3,1}\xi_{3,1} + \mu_{4,1}\xi_{4,1}\right) d\bar{t} \\
&= (\psi - t) + \sqrt{\epsilon}\int^t \left(-\frac{3}{4}E + E\cos(2\psi) - \frac{1}{4}E\cos(4\psi)\right) d\psi \\
&\quad + \epsilon \int^t \left(\frac{1}{2}\sin(2\psi) - \frac{1}{4}\beta E \sin(4\psi) - \frac{1}{2}\beta E\sin(2\psi) + \frac{1}{4}\gamma\sin(2\psi + 2t)\right. \\
&\quad - \frac{1}{4}\gamma\sin(2\psi - 2t) - \frac{3}{4}E^2\psi\sin(2\psi) + \frac{3}{8}E^2\psi\sin(4\psi) + \frac{5}{8}E^2\cos(4\psi) \\
&\quad \left. + \frac{87}{64}E^2 - \frac{61}{32}E^2\cos(2\psi) - \frac{3}{32}E^2\cos(6\psi) + \frac{1}{64}E^2\cos(8\psi)\right) d\psi \\
&= (\psi - t) + \sqrt{\epsilon}\left(-\frac{3}{4}E\psi + \frac{1}{2}E\sin(2\psi) - \frac{1}{16}E\sin(4\psi)\right) \\
&\quad + \epsilon\left(\left(-\frac{1}{4} + \frac{1}{4}\beta + \frac{3}{8}E^2\psi\right)\cos(2\psi) - \left(\frac{3}{16}E^2 + \frac{61}{64}E^2\right)\sin(2\psi)\right. \\
&\quad + \left(\frac{1}{16}E\beta - \frac{3}{32}E^2\psi\right)\cos(4\psi) + \left(\frac{3}{128}E^2 + \frac{5}{32}E^2\right)\sin(4\psi) \\
&\quad - \frac{1}{16}\gamma\cos(2\psi + 2t) + \frac{1}{4}\gamma\psi\sin(2\psi - 2t) - \frac{1}{64}E^2\sin(6\psi) \\
&\quad \left. + \frac{1}{512}E^2\sin(8\psi)\right). \qquad (3.14)
\end{aligned}$$

How well F_3 and F_4 approximate a first integral $F = \mathit{constant}$ can deduced from

$$\begin{aligned}
\frac{dF_j}{dt} &= \left[\xi_3 + \sqrt{\epsilon}\mu_{3,1}\xi_3 + \epsilon\mu_{3,2}\xi_3 + \sqrt{\epsilon}\mu_{4,1}\xi_4 + \epsilon\mu_{4,2}\xi_4\right]_{**} \\
&= \epsilon\sqrt{\epsilon}\mathcal{R}_j(E,\psi,t), \qquad (3.15)
\end{aligned}$$

where ξ_3 and ξ_4 are given by (3.12). It follows from (3.15) that for $j = 3, 4$ (see also (2.15)-(2.16))

$$F_j(E(t),\psi(t),t;\epsilon) = \mathit{constant} + \epsilon\sqrt{\epsilon}\int_0^t \mathcal{R}_j(E(s),\psi(s),s;\epsilon)ds, \qquad (3.16)$$

and so,

$$\begin{aligned}
F_j(E(t),\psi(t),t;\epsilon) &= \mathit{constant} + \mathcal{O}(\epsilon\sqrt{\epsilon}),\ 0 \le t \le T_0 < \infty, \\
F_j(E(t),\psi(t),t;\epsilon) &= \mathit{constant} + \mathcal{O}(\sqrt{\epsilon}),\ 0 \le t \le \frac{L}{\epsilon}, \qquad (3.17)
\end{aligned}$$

where T_0 and L are ϵ-independent constants.

3.3 The case $\lambda = \mathcal{O}(1)$

In this case the rescalings $\epsilon\delta = \hat{\epsilon}$, $X = \frac{1}{\sqrt{\lambda}}\hat{X}$, $\hat{\beta}\delta\lambda = \beta$, and $\hat{\gamma}\delta = \gamma$ are introduced, and (3.1) then becomes

$$\ddot{\hat{X}} + \hat{X} + \hat{X}^3 = \hat{\epsilon}(\dot{\hat{X}} - \hat{\beta}\dot{\hat{X}}^3 + \hat{\gamma}\dot{\hat{X}}\cos(2t)). \tag{3.18}$$

In the further analysis the heads will be dropped for convenience. By introducing the transformation $(X, \dot{X}) \longmapsto (E, \psi)$ as defined by

$$\begin{cases} E &= \frac{1}{2}\dot{X}^2 + \frac{1}{2}X^2 + \frac{1}{4}X^4, \\ \psi &= \int_0^X \frac{dr}{\sqrt{2E - r^2 - \frac{1}{2}r^4}}, \end{cases} \tag{3.19}$$

(where E and ψ are the energy and the phase angle of the unperturbed (that is, $\epsilon = 0$) oscillator) the following system of ODEs is obtained from (3.18)

$$\begin{cases} \dot{E} = \epsilon\dot{X}g &= \xi_5(E,\psi,t) = \epsilon\xi_{5,1}(E,\psi,t), \\ \dot{\psi} = 1 + \epsilon\left[-\int_0^X \frac{dr}{(2E - r^2 - \frac{1}{2}r^4)^{\frac{3}{2}}}\dot{X}g\right] &= \xi_6(E,\psi,t) = 1 + \epsilon\xi_{6,1}(E,\psi,t), \end{cases} \tag{3.20}$$

where $g = \dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(2t)$. The solution of the unperturbed (that is, $\epsilon = 0$) equation (3.18) is $X = A_0 cn(\vartheta, k)$ with $\vartheta = \omega_0\psi$, where $\psi = t + constant$, k is a modulus given by $k^2 = \frac{A_0^2}{2\omega_0^2}$, and $\omega_0^2 = 1 + A_0^2$ (see also [4, 9, 11, 15, 16, 19]). The relationship between the energy E and the "amplitude" A_0 is given by $E = \frac{1}{2}A_0^2 + \frac{1}{4}A_0^4$. The function $cn(\vartheta, k)$ is a Jacobian elliptic function with argument ϑ and modulus k. From the calculations as presented in section 2 of this paper it follows that two functionally independent approximations of the first integrals for system (3.18) are given by

$$\begin{aligned} F_5(E,\psi,t) &= E - \epsilon\int^t \xi_{5,1}d\bar{t} \\ &= E - \epsilon\Bigg[\int^t (\omega_0^2 A_0^2 sn(\vartheta,k)^2 dn(\vartheta,k)^2 - \beta\omega_0^4 A_0^4 sn(\vartheta,k)^4 dn(\vartheta,k)^4) \\ &\quad + \gamma\omega_0^2 A_0^2 sn(\vartheta,k)^2 dn(\vartheta,k)^2 \cos(\frac{\vartheta}{\omega_0}\omega)\frac{d\vartheta}{\omega_0}\Bigg], \end{aligned} \tag{3.21}$$

and

$$\begin{aligned} F_6(E,\psi,t) &= (\psi - t) - \epsilon\int^t \xi_{6,1}d\bar{t} \\ &= (\psi - t) + \epsilon\Bigg[\int^t P_1(\vartheta,k)\,(\omega_0 A_0 sn(\vartheta,k)dn(\vartheta,k) \\ &\quad - \beta\omega_0^3 A_0^3 sn(\vartheta,k)^3 dn(\vartheta,k)^3) + \gamma\omega_0 A_0 sn(\vartheta,k)dn(\vartheta,k)\cos(\frac{\vartheta}{\omega_0}\omega)\frac{d\vartheta}{\omega_0}\Bigg], \end{aligned} \tag{3.22}$$

where $P_1(\vartheta,k)=\frac{dA_0}{dE}cn(\vartheta,k)-A_0\psi sn(\vartheta,k)dn(\vartheta,k)\frac{d\omega_0}{dE}+A_0\frac{\partial}{\partial k}cn(\vartheta,k)\frac{dk}{dE}$, in which $sn(\vartheta,k)$, and $dn(\vartheta,k)$ are elliptic functions , and where $\frac{dA_0}{dE}$, $\frac{d\omega_0}{dE}$, and $\frac{dk}{dE}$ are given by

$$\frac{dA_0}{dE}=\frac{1}{A_0+A_0^3},\quad \frac{d\omega_0}{dE}=\frac{A_0}{\omega_0\left(A_0+A_0^3\right)},\quad \frac{dk}{dE}=\frac{A_0\left(1-2k^2\right)}{2k\omega_0^2\left(A_0+A_0^3\right)}.$$

How well F_5 and F_6 approximate a first integral $F=constant$ can be deduced from

$$\frac{dF_j}{dt} = \epsilon\mu_{5,1}\xi_5+\epsilon\mu_{6,1}(\xi_6-1)=\epsilon^2\mathcal{R}_j(E,\psi,t), \tag{3.23}$$

where ξ_5 and ξ_6 are given by (3.20). It follows from (3.23) that for $j=5,6$ (see also (2.15)-(2.16))

$$F_j(E(t),\psi(t),t;\epsilon)=constant+\epsilon^2\int_0^t\mathcal{R}_j(E(s),\psi(s),s;\epsilon)ds, \tag{3.24}$$

and so,

$$\begin{aligned}F_j(E(t),\psi(t),t;\epsilon) &= constant+\mathcal{O}(\epsilon^2),\ 0\le t\le T_0<\infty,\\ F_j(E(t),\psi(t),t;\epsilon) &= constant+\mathcal{O}(\epsilon),\ 0\le t\le \frac{L}{\sqrt{\epsilon}},\end{aligned} \tag{3.25}$$

where T_0 and L are ϵ-independent constants.

4 Time-periodic solutions and a bifurcation analysis

In the previous section it has been shown explicitly how asymptotic approximations of first integrals can be obtained. In this section we will show how the existence of non-trivial, time-periodic solutions can be determined from the asymptotic approximations of the first integrals. Bifurcation diagrams will be presented, and the analytical obtained approximations for the periodic solutions will be compared with numerical results such as obtained by Poincaré map techniques and obtained by numerical integration of the ODEs (phase portraits).

4.1 The case $\lambda=\mathcal{O}(\epsilon)$

The two functionally independent, asymptotic approximations (3.5) and (3.6) for the first integrals of equation (3.2) can be used to determine the existence and stability of the time-periodic solutions. Moreover, from (3.5) and (3.6) an approximation of a periodic solution can easily be constructed. Let $T<\infty$ be the period of a periodic solution (obviously T should be a multiple of π for $\gamma\neq 0$). Let $G_1(E,\psi,t;\epsilon)=constant$ and $G_2(E,\psi,t;\epsilon)=$ *constant* be two independent first integrals, where G_1 and G_2 are approximated by F_1 and F_2, respectively, and where F_1 and F_2 are given by (3.5) and (3.6), respectively. Let c_1 and

c_2 be constants in the two independent first integrals G_1 and G_2 respectively for which a periodic solution exists. Now consider $G_1 = c_1$ and $G_2 = c_2$ for $t = nT$ and $t = (n-1)T$ with $n \in \mathbb{N}^+$:

$$\begin{cases} G_1\left(E(nT), \psi(nT), nT; \epsilon\right) = c_1, \\ G_1\left(E\left((n-1)T\right), \psi((n-1)T), (n-1)T; \epsilon\right) = c_1, \\ G_2\left(E(nT), \psi(nT), nT; \epsilon\right) = c_2, \\ G_2\left(E\left((n-1)T\right), \psi((n-1)T), (n-1)T; \epsilon\right) = c_2. \end{cases} \tag{4.1}$$

Approximating G_1 by F_1 and G_2 by F_2, eliminating c_1 and c_2 from (4.1) by subtractions, we then obtain

$$\begin{aligned} E(nT) &= E\left((n-1)T\right) + \epsilon T \Big(E\left((n-1)T\right) - \frac{3}{2} E\left((n-1)T\right)^2 \beta \\ &\quad + \frac{1}{2}\gamma E((n-1)T)\cos(2\psi\left((n-1)T\right)) \Big) + \mathcal{O}(\epsilon^2 t), \\ \psi(nT) &= \psi\left((n-1)T\right) - T + \epsilon T \left(\frac{3}{4} E((n-1)T) - \frac{1}{4}\gamma \sin(2\psi\left((n-1)T\right)) \right) + \mathcal{O}(\epsilon^2 t), \end{aligned} \tag{4.2}$$

on a time scale of order $\frac{1}{\epsilon}$. In fact (4.2) defines a map $Q : E \to Q(E) \Leftrightarrow E_n = Q(E_{n-1})$ which we will use to determine the nontrivial periodic solution(s) of (3.2). By neglecting the $\mathcal{O}(\epsilon^2 t)$ terms in (4.2) we can define a new map $P : \tilde{E} \to P(\tilde{E}) \Leftrightarrow \tilde{E}_n = P(\tilde{E}_{n-1})$. It should be remarked that the second equation in the map Q (and in the map P) will always be considered modulo T. From the well-known theorem of Hartman-Grobman it follows that when the map P has a hyperbolic fixed point then the map Q also has a fixed point which is ϵ-close to the one of the map P. Moreover, the fixed point of the map Q has the same stability properties as the corresponding fixed point of the map P. It is also well-known that a fixed point corresponds to a periodic solution of the original ODE, that is, (3.2). In this case it follows from (4.2) with $\gamma \neq 0$ that the map P has as nontrivial fixed points (E_0, ψ_0), where

$$E_0 = \frac{2\beta \pm \sqrt{\gamma^2(\beta^2+1) - 4}}{3(\beta^2+1)}, \tag{4.3}$$

and where ψ_0 is given by

$$\begin{cases} \gamma\cos(2\psi_0) = 3E_0\beta - 2 \ \text{ and} \\ \gamma\sin(2\psi_0) = 3E_0. \end{cases} \tag{4.4}$$

Since we are interested in nontrivial periodic solutions of (3.2) (that is, $E_0 > 0$) it follows from (4.3) that we have to consider the following three cases

(a) for $\gamma^2(\beta^2+1) > 4$ and $-2 < \gamma < 2$ there are two nontrivial solutions for E_0,

(b) for $\gamma^2(\beta^2+1) = 4$ or $\gamma \geq 2$ or $\gamma \leq -2$ there is one nontrivial solution for E_0, and

(c) for $\gamma^2(\beta^2+1) < 4$ and $\gamma \neq 0$ there is no nontrivial solution for E_0.

The linearized map of map P around a fixed point of map P, is given by

$$DP = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \epsilon T \begin{pmatrix} 1 - 3E_0\beta + \frac{1}{2}\gamma\cos(2\psi_0) & -E_0\gamma\sin(2\psi_0) \\ \frac{3}{4} & -\frac{1}{2}\gamma\cos(2\psi_0) \end{pmatrix}. \tag{4.5}$$

By using (4.4) it follows from (4.5) that the eigenvalues of DP are

$$\bar{\lambda}_{1,2} = 1 + \epsilon T\left(\frac{1}{2} - \frac{3}{2}\beta E_0 \pm \frac{1}{2}\sqrt{1 - 9E_0^2}\right). \tag{4.6}$$

If the eigenvalues as given by (4.6) are not equal to one in modulus, then the fixed point (E_0, ψ_0) is hyperbolic. The results as given by (4.3) and (4.4) are exactly the same results as the ones which can be obtained by using the averaging method or the two time-scales perturbation method. The bifurcation diagram in the (β, γ)-plane is given in Figure 1. For

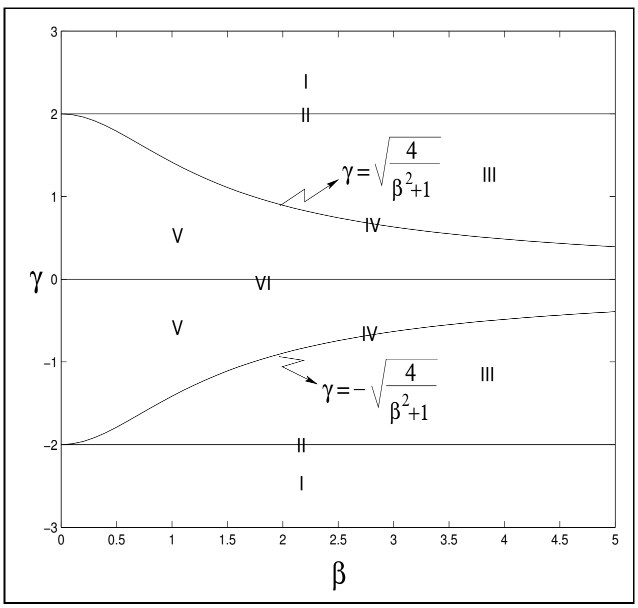


Figure 1: The bifurcation diagram in the (β, γ)-plane for the weakly nonlinear forced oscillator equation (3.2).

$E_0 > 0$ and $0 \leq \psi_0 < \pi$ the following conclusions can be drawn from (4.3)-(4.6) and from Figure 1. In region I in Figure 1 we will have one stable fixed point (E_0, ψ_0). Crossing the line II a second unstable fixed point is bifurcated. In region III we will have one stable and one unstable fixed point. These two critical points will coincide on the line IV, and a saddle node occurs on this line, and in region V no fixed points occur. Finally on the line VI, that is, for $\gamma = 0$ we have infinitely many fixed points (E_0, ψ_0) with $E_0 = \frac{2}{3\beta}$ and ψ_0 arbitrary. It should be remarked that for $\gamma = 0$ equation (3.2) reduces to well-known autonomous Rayleigh equation. The existence of stable and unstable nontrivial periodic solutions for $\beta = 2$ is given in Figure 2 in the (γ, E_0)-plane. In Figure 3 phase portraits

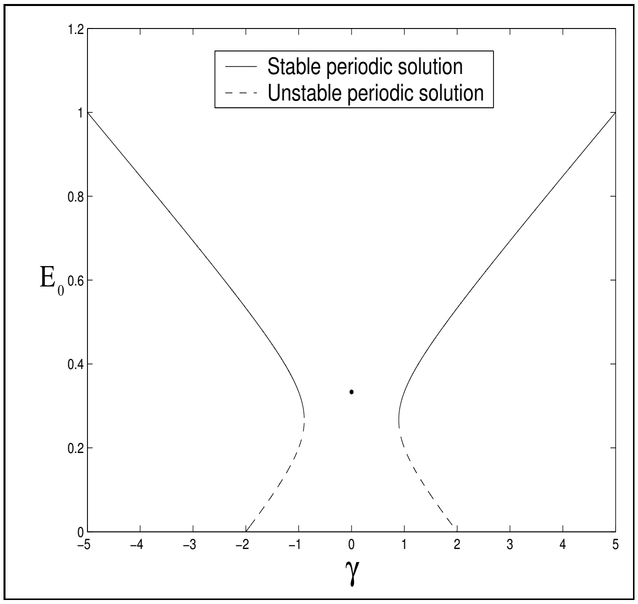
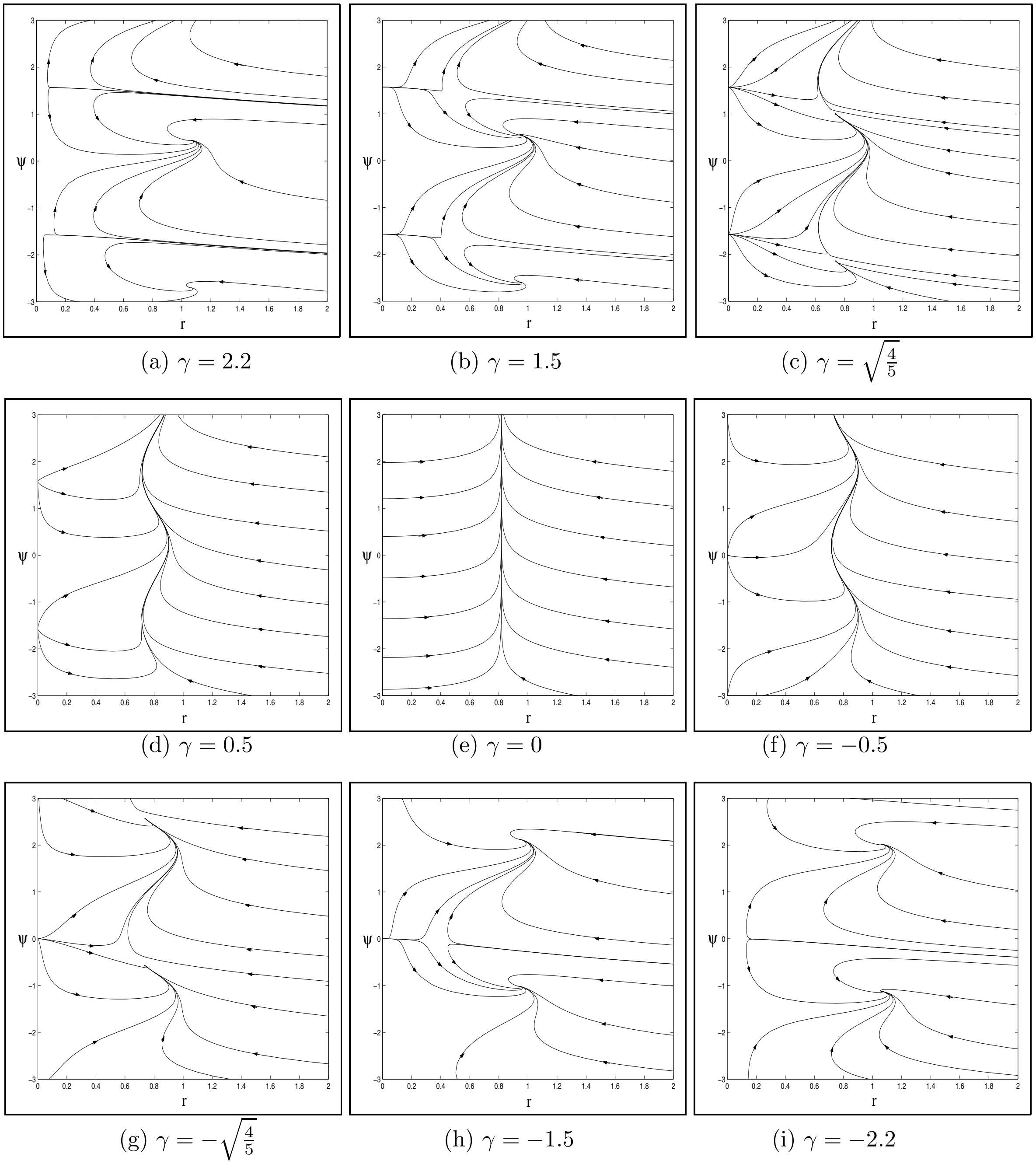


Figure 2: The bifurcation diagram in the (γ, E_0)-plane for the weakly nonlinear forced oscillator equation (3.2) with $\beta = 2$.

in the (r, ψ)-plane (with $E = \frac{1}{2}r^2$) are given for the first order averaged weakly nonlinear forced oscillator equation (3.2) with $\beta = 2$ and for several values of γ. From Figure 2 and from Figure 3 it can readily be seen that we have only one stable periodic solution of (3.2) for $\gamma^2 > 4$ and $0 \leq \psi_0 < \pi$. For $\gamma^2 = 4$ a second, unstable periodic solution is bifurcated, and for $\frac{4}{5} < \gamma^2 < 4$ we have two periodic solutions. A saddle node bifurcation occurs for $\gamma^2 = \frac{4}{5}$, and for $0 < \gamma^2 < \frac{4}{5}$ we have no periodic solutions. In Figure 4 the Poincaré map technique is used, and $X(t)$ and $\dot{X}(t)$ are depicted in the $(X, \dot{X})$-plane at times t equal to 2π, 4π, 6π, 8π, ... To compare the analytical results (as given in Figure 1 and Figure 2) with the numerical results (as given in Figure 3 and Figure 4) it should be noted that $X = r\sin(\psi)$, $E = \frac{1}{2}r^2 = \frac{1}{2}(X^2 + \dot{X}^2)$. Then, it can readily be seen that the analytical results and the numerical results are in good agreement. Finally it should be remarked that an order ϵ approximation of an order 1, 2π-periodic solution is given by $X(t) = \sqrt{2E}\sin(\psi(t))$, where $E(t) = E_0 + \mathcal{O}(\epsilon)$ and $\psi(t) = t + \psi_0 + \mathcal{O}(\epsilon)$, and where E_0 and ψ_0 are solutions of (4.3) and (4.4).

4.2 The case $\lambda = \mathcal{O}(\sqrt{\epsilon})$

The two functionally independent, asymptotic approximations (3.13) and (3.14) for the first integrals of equation (3.10) can be used to approximate the solutions. Moreover,

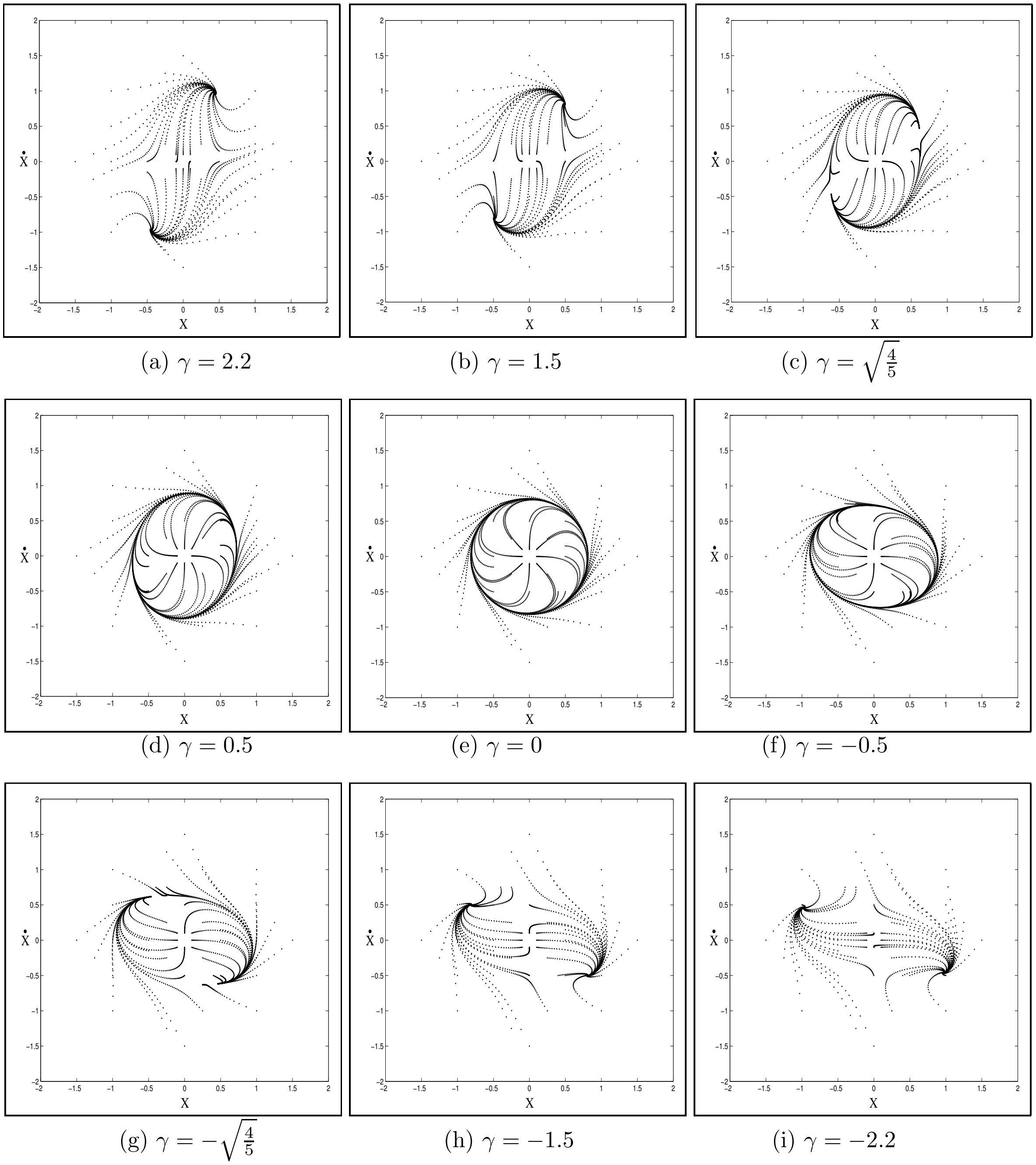


(a) $\gamma = 2.2$ (b) $\gamma = 1.5$ (c) $\gamma = \sqrt{\frac{4}{5}}$

(d) $\gamma = 0.5$ (e) $\gamma = 0$ (f) $\gamma = -0.5$

(g) $\gamma = -\sqrt{\frac{4}{5}}$ (h) $\gamma = -1.5$ (i) $\gamma = -2.2$

Figure 3: Phase Portraits in the (r, ψ)-plane for the weakly nonlinear forced oscillator equation (3.2) with $\beta = 2$ and for several values of γ.



(a) $\gamma = 2.2$ (b) $\gamma = 1.5$ (c) $\gamma = \sqrt{\frac{4}{5}}$

(d) $\gamma = 0.5$ (e) $\gamma = 0$ (f) $\gamma = -0.5$

(g) $\gamma = -\sqrt{\frac{4}{5}}$ (h) $\gamma = -1.5$ (i) $\gamma = -2.2$

Figure 4: Poincaré-map results for the weakly nonlinear forced oscillator equation (3.2) in the $(X, \dot{X})$-plane for several values of γ with $\beta = 2$ and $\epsilon = \frac{2}{100}$, and with sample-times t equal to 2π, 4π, 6π, 8π, ...

from (3.13) and (3.14) an approximation of a periodic solution (if it exists) can easily be constructed. Let $T < \infty$ be the period of a periodic solution (obviously T should be a multiple of π for $\gamma \neq 0$). Let $G_3(E, \psi, t; \epsilon) = constant$ and $G_4(E, \psi, t; \epsilon) = constant$ be two independent first integrals, where G_3 and G_4 are approximated by F_3 and F_4, respectively, and where F_3 and F_4 are given by (3.13) and (3.14), respectively. Let c_3 and c_4 be constants in the two independent first integrals G_3 and G_4 respectively for which a periodic solution exists. Now consider $G_3 = c_3$ and $G_4 = c_4$ for $t = nT$ and $t = (n-1)T$ with $n \in \mathbb{N}^+$:

$$\begin{cases} G_3\left(E(nT), \psi(nT), nT; \epsilon\right) = c_3, \\ G_3\left(E\left((n-1)T\right), \psi((n-1)T), (n-1)T; \epsilon\right) = c_3, \\ G_4\left(E(nT), \psi(nT), nT; \epsilon\right) = c_4, \\ G_4\left(E\left((n-1)T\right), \psi((n-1)T), (n-1)T; \epsilon\right) = c_4. \end{cases} \tag{4.7}$$

Approximating G_3 by F_3 and G_4 by F_4 respectively, eliminating c_3 and c_4 from (4.7) by subtractions, and using the transformation $\psi(t) = \theta(t) + \sqrt{\epsilon}\frac{3}{4}tE(t)$, we then obtain

$$\begin{aligned} E(nT) &= E\left((n-1)T\right) + \epsilon T \Bigg(E\left((n-1)T\right) - \frac{3}{2} E\left((n-1)T\right)^2 \beta \\ &\quad + \frac{1}{2}\gamma E((n-1)T)\cos(2\theta\left((n-1)T\right))\Bigg) + \mathcal{O}(\epsilon\sqrt{\epsilon}t), \\ \theta(nT) &= \theta\left((n-1)T\right) - T + \epsilon T \left(-\frac{105}{64} E((n-1)T)^2 - \frac{1}{4}\gamma \sin(2\theta\left((n-1)T\right)) \right) \\ &\quad + \mathcal{O}(\epsilon\sqrt{\epsilon}t), \end{aligned} \tag{4.8}$$

on a time scale of order $\frac{1}{\sqrt{\epsilon}}$. In fact (4.8) defines a map $R : E \to R(E) \Leftrightarrow E_n = R(E_{n-1})$ which we will use to determine the nontrivial periodic solution(s) of equation (3.10). By neglecting terms of $\mathcal{O}(\epsilon\sqrt{\epsilon}t)$ in (4.8) we can define a new map $S : \tilde{E} \to S(\tilde{E}) \Leftrightarrow \tilde{E}_n = S(\tilde{E}_{n-1})$. It should be remarked that the second equation in the map S (and in the map R) will always be considered modulo T. From the well-known theorem of Hartman-Grobman it follows that when the map S has a hyperbolic fixed point then the map R also has a fixed point which is ϵ-close to the one of the map S. Moreover, the fixed point of the map R has the same stability properties as the corresponding fixed point of the map S. In this case it follows from (4.8) with $\gamma \neq 0$ that the map S has as nontrivial fixed points (E_0, θ_0), where E_0 is given by

$$(3\beta E_0 - 2)^2 + \left(-\frac{105}{16} E_0^2 \right)^2 = \gamma^2, \tag{4.9}$$

and where θ_0 is given by

$$\begin{cases} \gamma\cos(2\theta_0) &= 3E_0\beta - 2 \text{ and} \\ \gamma\sin(2\theta_0) &= \left(-\frac{105}{16}E_0^2\right). \end{cases} \tag{4.10}$$

The linearized map of map S around a fixed point of map S, is given by

$$DP = \begin{pmatrix} 1 & 0 \\ & \\ 0 & 1 \end{pmatrix} + \epsilon T \begin{pmatrix} 1 - 3E_0\beta + \frac{1}{2}\gamma\cos(2\theta_0) & -E_0\gamma\sin(2\theta_0) \\ & \\ -\frac{105}{32}E_0 & -\frac{1}{2}\gamma\cos(2\theta_0) \end{pmatrix}. \qquad (4.11)$$

By using (4.9) it follows from (4.11) that the eigenvalues of DP are

$$\bar{\lambda}_{1,2} = 1 + \epsilon T\left(\frac{1}{2} - \frac{3}{2}\beta E_0 \pm \frac{1}{32}\sqrt{256 - 22050E_0^4}\right). \qquad (4.12)$$

Again if the eigenvalues (4.12) are not equal to one in modules, then the fixed point (E_0, θ_0) is hyperbolic. The results as given by (4.9) and (4.10) are exactly the same results as the ones which can be obtained by using the second order averaging method or the multiple time-scales method or other perturbation techniques. Using the formulas of Cardano the bifurcation diagram in the (β, γ)-plane can be derived from (4.9) and is given in Figure 5. The regions I-V in Figure 5 are as defined in section 4.1. The existence of stable and

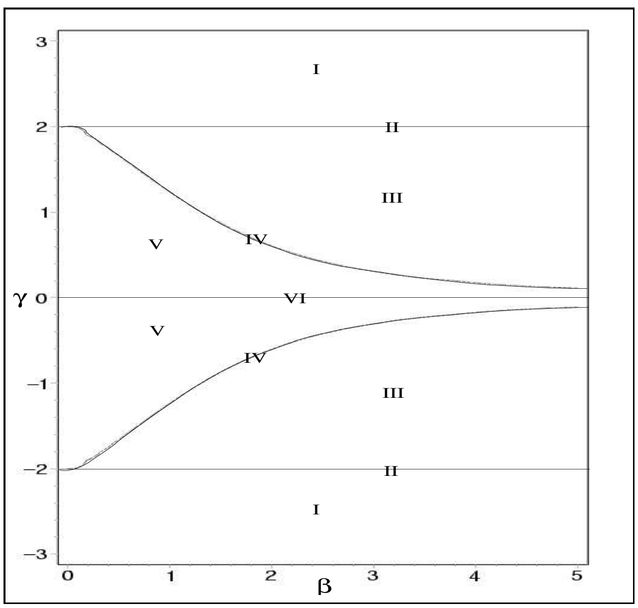


Figure 5: The bifurcation diagram in the (β, γ)-plane for the nonlinear map (4.8).

unstable nontrivial equilibrium solutions for the nonlinear map (4.8) with $\beta = 2$ can be determined from Figure 6 in the (γ, E_0)-plane. In Figure 7 the phase portraits in the (r, θ)-plane (with $E = \frac{1}{2}r^2$) are given for the second order averaged nonlinear forced oscillator equation (3.10) with $\beta = 2$ and for several values of γ. It should be remarked that the fixed points as given by (4.9) and (4.10) are not corresponding with the 2π-periodic solutions of the equation (3.10) due to the transformation $\theta(t) = \psi(t) - \sqrt{\epsilon}\frac{3}{4}E(t)t$. So, the periodic solutions as given in Figure 5 and 6 for $\gamma \neq 0$ are not the 2π-periodic solutions for the

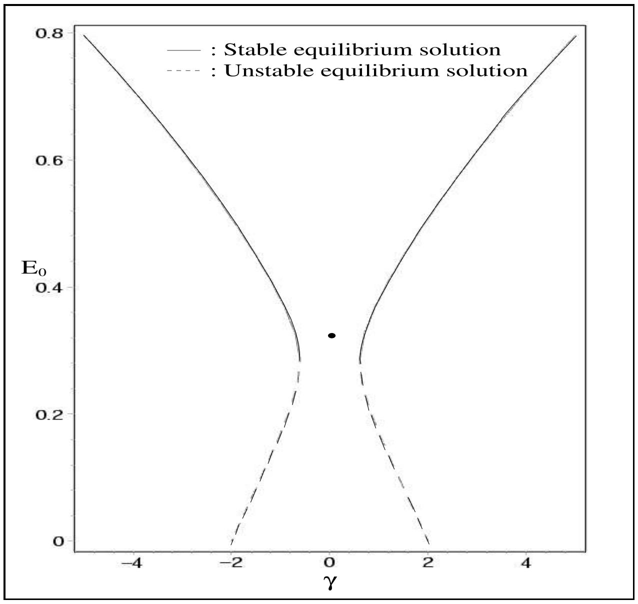


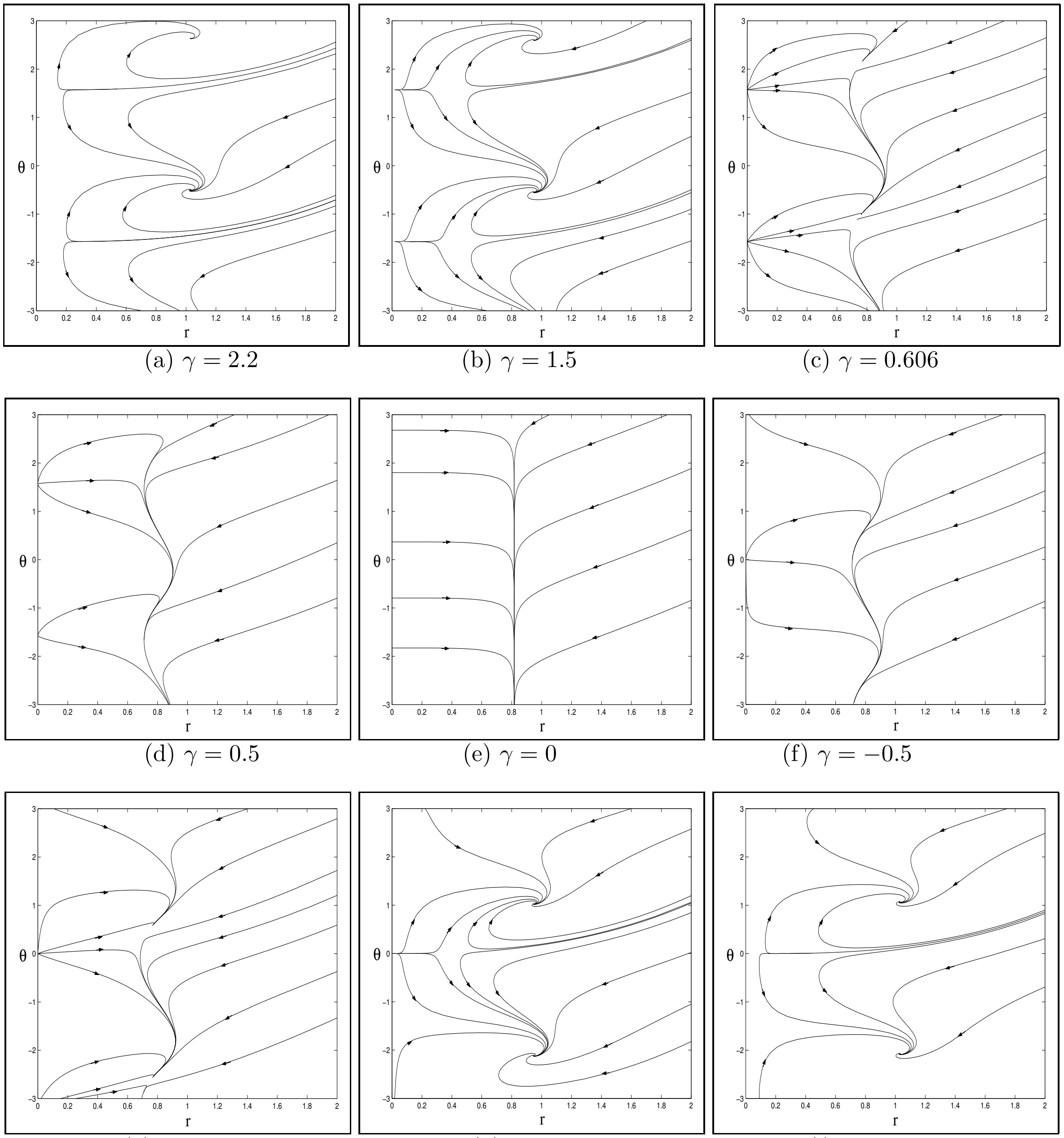
Figure 6: The bifurcation diagram in the (γ, E_0)-plane for the nonlinear map (4.8) with $\beta = 2$.

original equation (3.10). Finally it should be remarked that an approximation of a solution for the nonlinear forced oscillator equation (3.10) (in a neighborhood of the equilibrium points of the nonlinear map (4.8)) is given by $X(t) = \sqrt{2E_0}\sin\left(\psi(0) + t + \sqrt{\epsilon}\frac{3}{4}E_0 t\right) + \mathcal{O}(\sqrt{\epsilon})$ on a time scale of order $\frac{1}{\sqrt{\epsilon}}$, where E_0 and $\psi(0) = \theta(0) = \theta_0$ are the solutions of (4.9) and (4.10). We can see from these approximations that the "periods" of the solutions of (3.10) (which are $\mathcal{O}(1)$, and not $o(1)$) are less than 2π. This implies that there are no 2π-periodic solutions which are strict $\mathcal{O}(1)$ (that is, are $\mathcal{O}(1)$ but not $o(1)$). These results are confirmed in Figure 8, in which Poincaré-return map results are given for the nonlinear forced oscillator equation (3.10) in the $(X, \dot{X})$-plane for several values of γ. It is still possible that equation (3.10) has small amplitude, 2π-periodic solutions. From the applicational point of view these small amplitude oscillations in vertical direction are not so interesting, but from a mathematical point of view these solutions might be of interest to understand the bifurcations that are occurring. To study these small amplitude solutions the following rescaling is usually introduced in (3.10): $X(t) = \epsilon^{\alpha} Z(t)$ with $\alpha > 0$, yielding

$$\ddot{Z} + Z + \epsilon^{\frac{1}{2}+2\alpha} Z^3 = \epsilon\left(\dot{Z} - \beta\epsilon^{2\alpha}\dot{Z}^3 + \gamma\dot{Z}\cos(2t)\right). \tag{4.13}$$

The most interesting cases occur for $\alpha = \frac{1}{4}$ and $\alpha > \frac{1}{4}$. For $\alpha = \frac{1}{4}$ equation (4.13) becomes equation (3.2) with β near zero, and this equation has 2π-periodic solutions for special values of the parameters (see section 4.1). For $\alpha > \frac{1}{4}$ equation (4.13) becomes (up to $\mathcal{O}(\epsilon^{\frac{1}{2}+2\alpha})$)

$$\ddot{Z} + Z = \epsilon\left(\dot{Z} + \gamma\dot{Z}\cos(2t)\right). \tag{4.14}$$

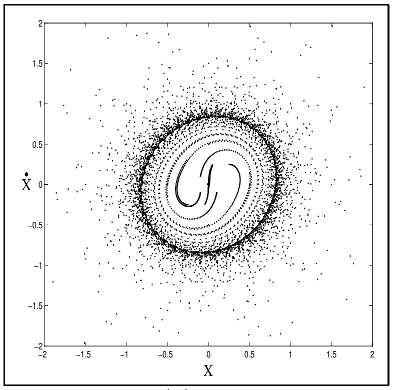


(a) $\gamma = 2.2$ (b) $\gamma = 1.5$ (c) $\gamma = 0.606$

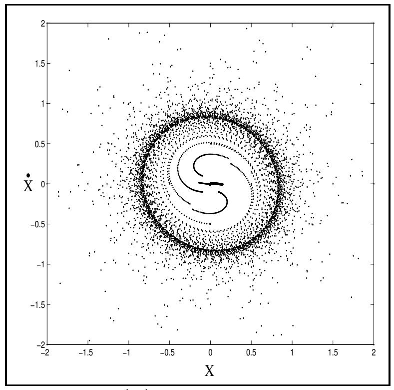
(d) $\gamma = 0.5$ (e) $\gamma = 0$ (f) $\gamma = -0.5$

(g) $\gamma = -0.606$ (h) $\gamma = -1.5$ (i) $\gamma = -2.2$

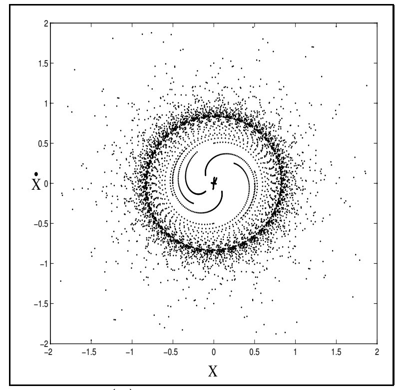
Figure 7: Phase Portraits in the (r, θ)-plane for the nonlinear forced oscillator equation (3.10) with $\beta = 2$ and for several values of γ.



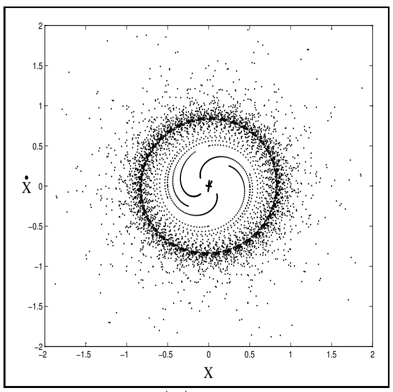
(a) $\gamma = 2.2$



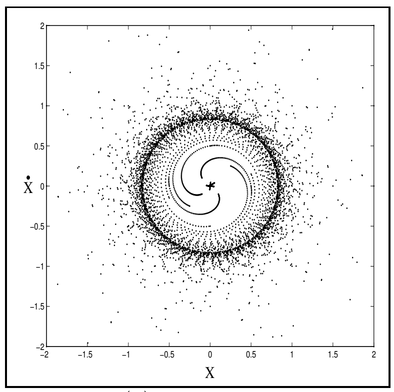
(b) $\gamma = 1.5$



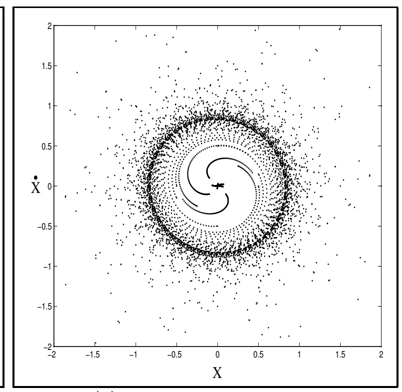
(c) $\gamma = 0.606$



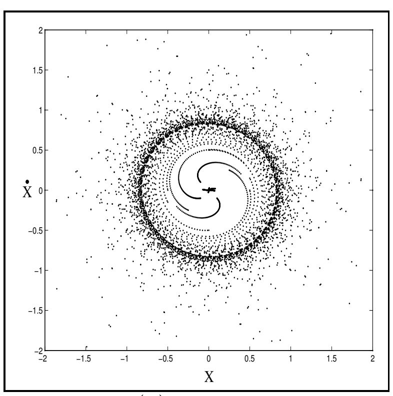
(d) $\gamma = 0.5$



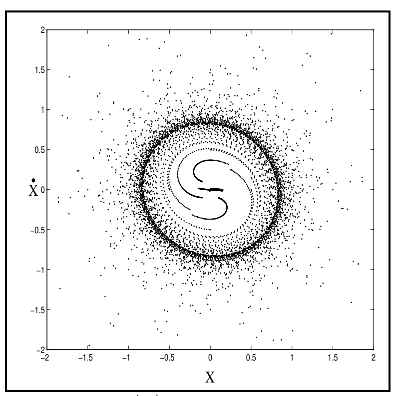
(e) $\gamma = 0$



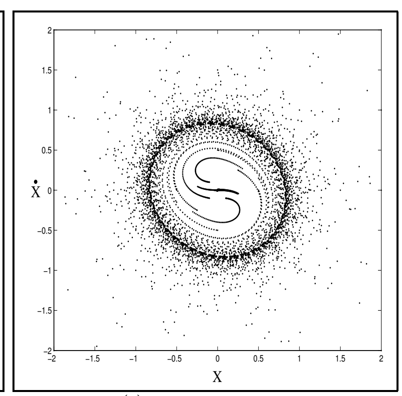
(f) $\gamma = -0.5$



(g) $\gamma = -0.606$



(h) $\gamma = -1.5$



(i) $\gamma = -2.2$

Figure 8: Poincaré-map results for the nonlinear forced oscillator equation (3.10) in the $(X, \dot{X})$-plane for $\beta = 2$ and for several values of γ, and $\sqrt{\epsilon} = \frac{5}{100}$, and with sample-times $t = 2\pi + 4\pi n$, where $n \in \mathbb{N}$.

In the Appendix 1 (4.14) is studied briefly. From the Poincaré expansion theorem it follows that all solutions of (3.10) can be expand in $X_0(t)+\sqrt{\epsilon}X_1(t)+\epsilon X_2(t)+\ldots$ on a time-scale of order 1. Obviously $\ddot{X}_0 + X_0 = 0$. So, from the Poincaré expansion theorem and the results obtained in this section it follows that equation (3.10) can only have small amplitude, 2π-periodic solutions as periodic solutions.

4.3 The case $\lambda = \mathcal{O}(1)$

The two functionally independent, asymptotic approximation (3.21) and (3.22) for the first integrals of equation (3.18) can be used to determine the existence of the time-periodic solutions. Moreover, from (3.21) and (3.22) an approximation of a periodic solution can easily be constructed. Let $T < \infty$ be the period of a periodic solution (obviously T should be πl, with $l \in \mathbb{N}^+$ for $\gamma \neq 0$). Let $G_5(E, \psi, t; \epsilon) = constant$ and $G_6(E, \psi, t; \epsilon) = constant$ be two independent first integrals, where G_5 and G_6 are approximated by F_5 and F_6, respectively, and where F_5 and F_6 are given by (3.21) and (3.22), respectively. Let c_5 and c_6 be constants in the two independent first integrals for which a periodic solution exists. Now consider $G_5 = c_5$ and $G_6 = c_6$ for $t = 0$ and $t = T$. Approximating G_5 by F_5 and G_6 by F_6 (as given by (3.21) and (3.22)), eliminating c_5 and c_6 by subtractions, we then obtain (using the fact that $E(0) = E(T)$ for a periodic solution)

$$\left\{\begin{array}{lcl} \epsilon \int_0^T \left(\dot{X}^2 - \beta\dot{X}^4 + \gamma\dot{X}^2\cos(2s)\right) ds & = & \mathcal{O}(\epsilon^2), \\ \\ \epsilon \int_0^T P_1(\vartheta, k)\left(\dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(2s)\right) ds & = & \mathcal{O}(\epsilon^2), \end{array}\right. \tag{4.15}$$

where $\dot{X} = -\omega_0 A_0 sn(\vartheta, k) dn(\vartheta, k)$. We can rewrite equation (4.15) as

$$\left\{\begin{array}{lcl} \epsilon I(E, \psi, \beta, \gamma) & = & \mathcal{O}(\epsilon^2), \\ \\ \epsilon J(E, \psi, \beta, \gamma) & = & \mathcal{O}(\epsilon^2). \end{array}\right. \tag{4.16}$$

To have a periodic solution for (3.18) we have to find an energy E and a phase angle ψ such that $I(E, \psi, \beta, \gamma)$ and $J(E, \psi, \beta, \gamma)$ are equal to zero (see also [12, 14]). To find this energy and phase angle we rewrite $I(E, \psi, \beta, \gamma)$ and $J(E, \psi, \beta, \gamma)$ in

$$\left\{\begin{array}{lcl} I & \equiv & I_1 - \beta I_2 + \gamma I_3 = 0, \\ \\ J & \equiv & J_1 - \beta J_2 + \gamma J_3 = 0, \end{array}\right. \tag{4.17}$$

where

$$\left\{\begin{array}{lcl} I_1 & = & \int_0^T \dot{X}^2 ds,\ I_2 = \int_0^T \dot{X}^4 ds,\ I_3 = \int_0^T \left(\dot{X}^2\cos(2s)\right) ds, \\ \\ J_1 & = & \int_0^T P_1(\vartheta, k)\dot{X} ds,\ J_2 = \int_0^T P_1(\vartheta, k)\dot{X}^3 ds,\ J_3 = \int_0^T P_1(\vartheta, k)\left(\dot{X}\cos(2s)\right) ds. \end{array}\right. \tag{4.18}$$

Let $D = I_3J_2 - I_2J_3$. It follows from (4.17) that for a periodic solution to exist β and γ, can be considered to be functions of the energy E and the phase angle ψ, that is,

$$\begin{cases} \beta = \frac{1}{D}\left(I_3J_1 - I_1J_3\right), \\ \gamma = \frac{1}{D}\left(I_2J_1 - I_1J_2\right), \end{cases} \tag{4.19}$$

for $D \neq 0$. By using an adaptive recursive Simpson rule the values of the parameters β and γ can be calculated from (4.17)-(4.19) for which a periodic solution exists. From (3.18) it is obvious that the period T should be a multiple of π. The expansion theorem of Poincaré implies that the solution(s) of (3.18) can be expanded in $X_0(t)+\epsilon X_1(t)+\epsilon^2X_2(t)+\ldots$ on a time-scale of order 1, where X_0 satisfies $\ddot{X}_0+X_0+X_0^3 = 0$. Now $X_0(t)$ is a periodic function with period $T_0(E_0) = 4\int_0^{A_0} \frac{1}{\sqrt{2E_0-X_0^2-\frac{1}{2}X_0^4}}dX_0$, where $A_0 > 0$ satisfies $2E_0 - A_0^2 - \frac{1}{2}A_0^4 = 0$. In Figure 9 $T_0(E_0)$ is plotted. From Figure 9 and from the fact that T should be a multiple

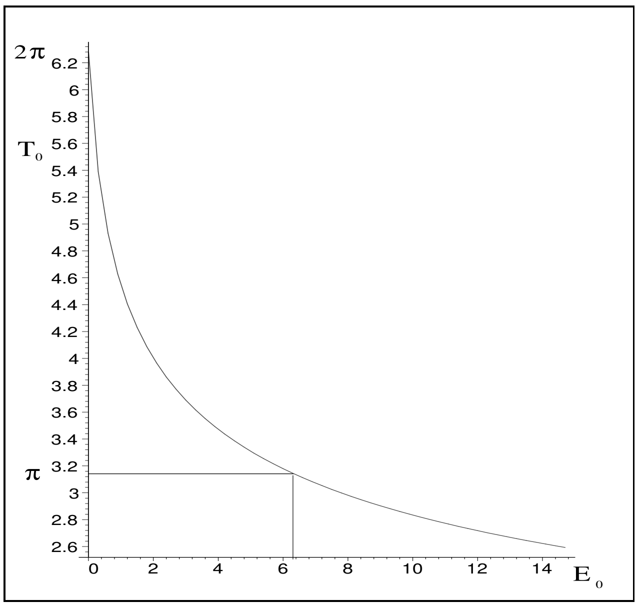


Figure 9: The period T_0 of the unperturbed equation (3.18) (that is, (3.18) with $\epsilon = 0$) as function of the energy $E_0 = \frac{1}{2}\dot{X}_0^2 + \frac{1}{2}X_0^2 + \frac{1}{4}X_0^4$.

of π it immediately follows that T should be equal to π or 2π. For 2π-periodic solutions it follows from Figure 9 that E_0 should be zero, and so a 2π-periodic solution (if it exists) should have a small amplitude. To study these small amplitude solutions the following rescaling is introduced in (3.18): $X(t) = \epsilon^\alpha Z(t)$ with $\alpha > 0$, yielding

$$\ddot{Z} + Z = -\epsilon^{2\alpha}Z^3 + \epsilon\dot{Z} - \epsilon^{1+2\alpha}\beta\dot{Z}^3 + \epsilon\gamma\dot{Z}\cos(2t). \tag{4.20}$$

For 2π-periodic solutions $Z(t)$ only the case $\alpha = \frac{1}{2}$ and the case $\alpha > \frac{1}{2}$ have to be considered. For $\alpha = \frac{1}{2}$ equation (4.20) becomes equation (3.2) with β near zero, and this equation has

2π-periodic solutions for special values of the parameters (see section 4.1). For $\alpha > \frac{1}{2}$ equation (4.14) up to $\mathcal{O}(\epsilon^{2\alpha})$ is again obtained, and this equation has been studied briefly in Appendix 1. For π-periodic solutions it follows from Figure 9 that E_0 should be near 6.33552259..., and so a π-periodic solution (if it exists) should have an amplitude of (strict) $\mathcal{O}(1)$. To determine the values of β and γ for which a π-periodic solution exists it should be observed that $\beta = \beta(E_0, \psi_0)$ and $\gamma = \gamma(E_0, \psi_0)$, where $E_0 = 6.33552259...$ and $0 \leq \psi_0 \leq 4K(k)$ (in which $K(k)$ is the complete elliptic integral of the first kind). For different values of ψ_0 (with $E_0 = 6.33552259...$) the integrals in (4.18) and (4.19) have been calculated by using an adaptive recursive Simpson rule. It should be observed that in (4.18) $X(t)$ (that is, the solution of the unperturbed equation (3.18) with $\epsilon = 0$) depends on the initial energy E_0 and on the initial phase angle ψ_0. For $E_0 = 6.33552259...$ $\beta = \beta(E_0, \psi_0)$ and $\gamma = \gamma(E_0, \psi_0)$ will give a curve in the (β, γ)-plane. This curve has been determined numerically, and is given in Figure 10. From a practical point of view it is obvious that the chance that the

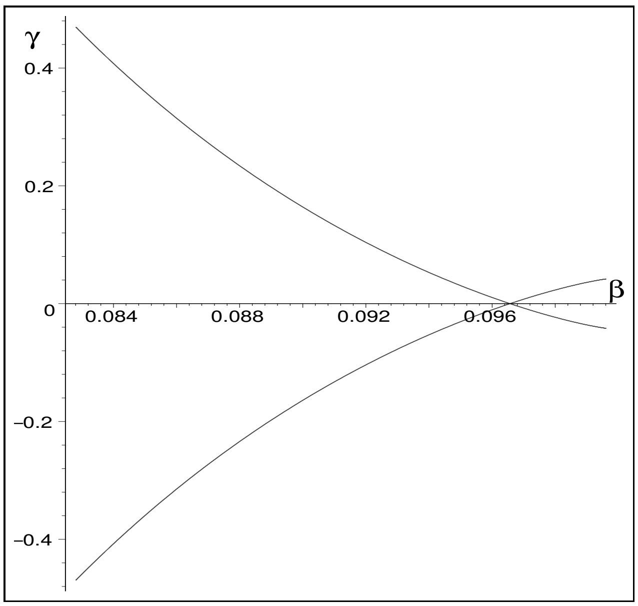
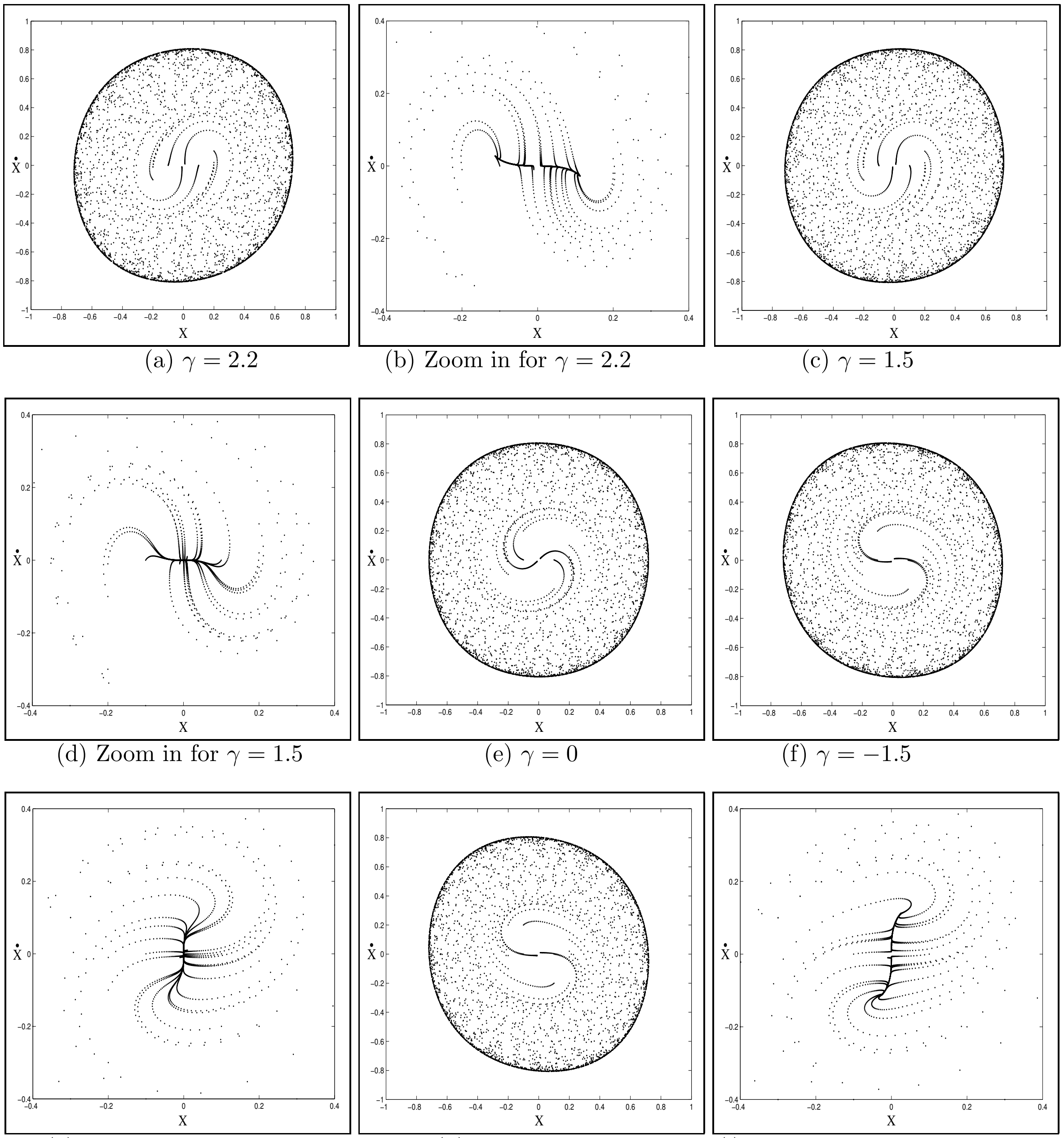


Figure 10: The curve in the (β, γ)-plane for which the strongly nonlinear forced equation (3.18) has π-periodic solutions of order 1.

parameters β and γ are on this curve is of course zero. For that reason also Poincaré-map results are given in Figure 11 for different values of β and γ.

5 Conclusions and remarks

In this paper it has been shown that the perturbation method based on integrating factors can be used efficiently to approximate first integrals for strongly nonlinear forced oscillators. In section 2 (and 3) of this paper a justification of the presented perturbation method has been given. It has also been shown how the existence and stability of time-periodic solutions



(a) $\gamma = 2.2$ (b) Zoom in for $\gamma = 2.2$ (c) $\gamma = 1.5$

(d) Zoom in for $\gamma = 1.5$ (e) $\gamma = 0$ (f) $\gamma = -1.5$

(g) Zoom in for $\gamma = -1.5$ (h) $\gamma = -2.2$ (i) Zoom in for $\gamma = -2.2$

Figure 11: Poincaré-map results for the nonlinear forced oscillator equation (3.18) in the $(X, \dot{X})$-plane for several values of γ with $\beta = 2$ and $\epsilon = \frac{2}{100}$, and with sample-times $t = \pi n$ with $n \in \mathbb{Z}^+$ for the figures (a), (c), (e), (f), and (h), and $t = -2\pi n$ with $n \in \mathbb{Z}^+$ for the figures (b), (d), (g), and (i).

can be deduced from the approximations of the first integrals for the strongly nonlinear forced oscillators.

In this paper the following three oscillator equations have been studied in detail:

$$\ddot{X} + X = \epsilon\left(\dot{X} - \beta\dot{X}^3 - X^3 + \gamma\dot{X}\cos(2t)\right), \tag{5.1}$$

$$\ddot{X} + X + \sqrt{\epsilon}X^3 = \epsilon\left(\dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(2t)\right), \text{ and} \tag{5.2}$$

$$\ddot{X} + X + X^3 = \epsilon\left(\dot{X} - \beta\dot{X}^3 + \gamma\dot{X}\cos(2t)\right), \tag{5.3}$$

where ϵ is a small parameter with $0 < \epsilon \ll 1$, and where $\beta > 0$ and $\gamma \neq 0$ are constants (of order 1). In particular the $\mathcal{O}(1)$ behavior of the solutions has been studied. From the applicational point of view this $\mathcal{O}(1)$-behavior is the most interesting behavior when galloping is studied. For equation (5.1) it has been shown for what values of the parameters the solutions will tend to a 2π-periodic solution of order 1, and for what values of the parameters the solutions will tend to a (non-periodic) bounded attractor. The results obtained for (5.1) are in agreement with the results as obtained in [2, 3]. For equation (5.2) it has been shown that there are no periodic solutions of order 1. Small amplitude, 2π-periodic solutions, however, exist for certain values of the parameters. In general the solutions will tend to a bounded, non-periodic attractor of order 1. For equation (5.3) it has been shown that there are π-periodic solutions of order 1 for special values of parameters. These π-periodic solutions are, however, structurally unstable. Also small amplitude, 2π-periodic solutions exist for certain values of parameters. In general the solutions will tend to a bounded, non-periodic attractor of order 1.

A Appendix 1

In section 4.2 and in section 4.3 the following ODEs have been derived to describe the small amplitude solutions of the oscillator equations

$$\ddot{Z} + Z = \epsilon\left(\dot{Z} + \gamma\dot{Z}\cos(2t)\right) - \epsilon^{\frac{1}{2}+2\alpha}Z^3 - \epsilon^{1+2\alpha}\beta\dot{Z}^3, \text{ and} \tag{1.4}$$

$$\ddot{Z} + Z = \epsilon\left(\dot{Z} + \gamma\dot{Z}\cos(2t)\right) - \epsilon^{2\alpha}Z^3 - \epsilon^{1+2\alpha}\beta\dot{Z}^3, \tag{1.5}$$

with $\alpha > \frac{1}{4}$ and with $\alpha > \frac{1}{2}$ respectively. In this appendix (1.4) and (1.5) will be studied briefly. By introducing the transformation

$$\begin{cases} Y(t) = Y_1(t)\cos(t) + Y_2(t)\sin(t), \\ \dot{Y}(t) = -Y_1(t)\sin(t) + Y_2(t)\cos(t), \end{cases} \tag{1.6}$$

the first order averaged system of equation (1.4) or of equation (1.5) becomes

$$\begin{cases} \dot{Y}_1 = \epsilon\left(\frac{1}{2}Y_1 - \frac{1}{4}\gamma Y_1\right), \\ \dot{Y}_2 = \epsilon\left(\frac{1}{4}\gamma Y_2 + \frac{1}{2}Y_2\right). \end{cases} \tag{1.7}$$

For γ^2 not in an $o(1)$ neighborhood of 4 system (1.7) has only as fixed point(s) the trivial fixed point $(0, 0)$. This fixed point turns out to be unstable. So, for γ^2 not in an $o(1)$ neighborhood of 4 it can be conclude that (1.4) and (1.5) do not have nontrivial, 2π-periodic solutions. For γ^2 in an $o(1)$ neighborhood of 4 second order or higher order averaging has to be applied to (1.4) or (1.5). Again it can be shown that (1.4) and (1.5) do not have nontrivial, 2π-periodic solutions. The elementary calculations to prove this will be omitted.

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