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DOI
10.1109/TAC.2024.3487798

Publication date
2025

Document Version
Final published version

Published in
IEEE Transactions on Automatic Control

Citation (APA)
Kaczmarek, M. B., & Hosseinnia, S. H. (2025). Negative Imaginary Reset Control Systems. *IEEE Transactions on Automatic Control*, *70*(4), 2630-2635. https://doi.org/10.1109/TAC.2024.3487798

Negative Imaginary Reset Control Systems

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***Abstract*—In this note, we present an extension of the nonlinear negative imaginary (NI) systems theory to reset systems. We define the reset NI and reset strictly NI systems and provide a state-space characterization of these systems in terms of linear matrix inequalities. Subsequently, we establish the conditions for the internal stability of a positive feedback interconnection of a (strictly) NI linear time-invariant plant and a reset (strictly) NI controller. The applicability of the proposed method is demonstrated in a numerical example of a reset version of a positive position feedback controller for a plant with resonance.**

***Index Terms*—Negative-imaginary (NI) systems, reset control, stability of hybrid systems.**

I. Introduction

Negative-Imaginary (NI) systems theory, introduced in [1] and [2], offers a framework for stability analysis of energy-dissipating systems that do not fit within the classical dissipativity framework. A prominent application of this theory is found in flexible mechanical structures with collocated force actuators and position sensors. In these systems, energy is associated with the output and its derivative, and they exhibit passivity from the input to the derivative of the output.

For linear time-invariant (LTI) systems, the NI systems theory is well developed. The necessary and sufficient conditions for a system to exhibit the NI and strictly NI (SNI) properties are formulated in terms of both frequency responses and linear matrix inequalities (LMI) for state-space matrices [1], [2], [3], [4]. The class of output NI systems, that unifies the existing subclasses of the NI systems class was introduced in [5]. The relationship between the NI properties of LTI systems and the dissipativity theory has been studied in [6], [7], and [8]. The stability conditions for a positive feedback interconnection of a NI and SNI system involve only the open-loop steady-state gain of the system [1], hence robust stability can be guaranteed for systems with uncertain dynamics and lightly-damped resonances. Thanks to this property, the LTI NI framework found wide adoption in the field of active vibration control of flexible structures [9].

The NI systems theory was extended to Lipschitz continuous nonlinear systems in [10], [11], [12], and [13], using the dissipativity theory. Two stronger notions of the NI property for nonlinear systems were introduced, namely, the weakly strictly nonlinear negative imaginary (WS-NNI) property [11] and the nonlinear output SNI property [12], [13]. The conditions for the closed-loop stability of a positive feedback interconnection of two nonlinear NI systems were derived. The notion of linear time-varying NI systems was introduced in [14].

The existing nonlinear NI systems theory does not accommodate hybrid systems with state jumps. Among such systems, reset systems [15], [16], [17] are being rapidly developed and show potential for wide adoption in the industry for control of LTI plants, which can be explained using two arguments. First, reset control systems have been shown to overcome the inherent limitations of LTI systems, as evidenced by numerous studies, particularly in the field of precision motion control [18], [19], [20], [21], [22]. Second, the design of reset control systems can be conducted in the frequency domain using the describing function approximation [23], [24], [25], [26], following a procedure analogous to the design of commonly employed LTI controllers. This represents a significant advantage compared to other types of nonlinear controllers.

However, the stability analysis of reset control systems often requires parametric models of both the controller and the plant [15], [27], which in practice may be hard to obtain. Although specific frequency-domain methods exist, their applicability is limited to low-order systems [28], [29], [30] or systems with less common reset conditions [31]. Establishing the stability of reset control systems can be achieved through the passivity theory [32], yet the NI systems theory would be better suited to many applications, for example, in the field of active vibration control.

In this note, we extend the nonlinear NI systems theory to reset systems. In this way, we address the challenges related to assessing the stability of reset control systems for LTI plants. We achieve this with the following three contributions.

1) We introduce definitions of reset NI (RNI) and reset strictly NI (RSNI) systems. This is necessary, as the reset systems do not fit existing definitions of NI properties.
2) We provide the necessary and sufficient conditions for a reset system to be RNI (RSNI), analogue to the NI Lemma for LTI systems [3]. In consequence, it can be easily checked if a system possesses the RNI (RSNI) properties.
3) We show that a positive feedback interconnection of an SNI (NI) LTI plant with an RNI (RSNI) reset controller is internally stable under some conditions.

The applicability of the stability result is illustrated with an example of a reset positive position feedback (PPF) [33] controller for a plant with resonance, commonly used to model mechanical systems.

The rest of this article is organized as follows. Section II provides preliminary information on LTI and nonlinear NI systems and reset control. The main contribution of the note is presented in Section III. Section IV illustrates the stability results with an example. Finally, Section V concludes this article.

II. Preliminaries

In this section, we present the studied control system, including the considered class of reset systems. Moreover, we recall the concepts of linear and nonlinear negative imaginary (NNI) systems.

Received 19 August 2024; accepted 26 October 2024. Date of publication 29 October 2024; date of current version 31 March 2025. This work was supported by the NWO HTSM Applied and Technical Science Program through project MetaMech under Grant 17976. Recommended by Associate Editor C. Mahulea. *(Corresponding author: Marcin B. Kaczmarek.)*

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Digital Object Identifier 10.1109/TAC.2024.3487798

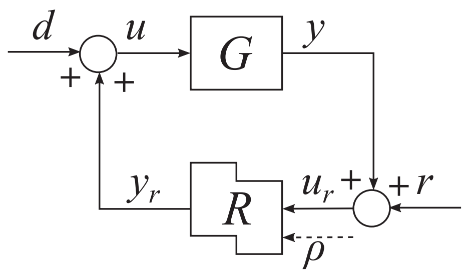

Fig. 1. Positive feedback interconnection.

A. Notations

A^*	Complex conjugate transpose of the complex matrix A.
A^T	Transpose of the matrix A.
$A > 0$	The matrix A is positive definite.
$A \geq 0$	The matrix A is positive semidefinite.
$\Re\{s\}$	Real part of the complex number s.
$\lambda_{\max}(A)$	Maximum eigenvalue of the matrix A.

B. System Description

We focus on the control architecture presented in Fig. 1. A positive feedback interconnection is used to follow the conventions of the NI systems theory. The closed-loop system consists of a LTI plant and a reset controller. The plant is described by

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, and A, B, C, D are constant matrices of appropriate dimensions. The plant (1) has the $m \times m$ real-rational proper transfer function $G(s) := C(sI - A)^{-1}B + D$, which is said to be strictly proper if $G(\infty) = D = 0$.

The state-space representation of the reset element is

$$R : \begin{cases} \dot{x}_r(t) = A_r x_r(t) + B_r u_r(t), & \rho(t) \neq 0 \\ x_r(t^+) = A_\rho x_r(t), & \rho(t) = 0 \\ y_r(t) = C_r x_r(t) + D_r u_r(t) \end{cases} \tag{2}$$

where $x_r \in \mathbb{R}^{n_r}$ is the state of R, $x_r(t^+) = x_r^+ = \lim_{\epsilon \to 0^+} x(t + \epsilon)$ is the after reset state value, $u_r \in \mathbb{R}^m$ is the input of R, $y_r \in \mathbb{R}^m$ is the output of R and A_r, B_r, A_ρ, C_r, and D_r are constant matrices of appropriate dimensions.

The base linear system (BLS) R_{bls} is an LTI system with a state-space realization (A_r, B_r, C_r, D_r) and describes dynamics of R in the absence of reset.

The reset is triggered by a signal $\rho(t)$. The linear reset law $x_r^+ = A_\rho x_r(t)$ describes the change of state that occurs at reset instants $t_k, k = 1, 2, \ldots$, that is when the reset condition $\rho = 0$ is satisfied.

The closed-loop system, in the absence of external inputs d, r and assuming that $DD_r = 0$, is given by

$$\begin{cases} \dot{x}_{\text{CL}}(t) = A_{\text{CL}} x_{\text{CL}}(t), & x_{\text{CL}}(t) \notin \mathcal{M}(t) \\ x_{\text{CL}}(t^+) = A_R x_{\text{CL}}(t), & x_{\text{CL}}(t) \in \mathcal{M}(t) \\ y_{\text{CL}}(t) = C_{\text{CL}} x_{\text{CL}}(t) \end{cases} \tag{3}$$

where $x_{\text{CL}} = [x, \quad x_r]^T$, the reset surface $\mathcal{M}(t)$ defines states triggering reset and is defined

$$\mathcal{M}(t) = \{\xi \in \mathbb{R}^{n+n_r} : \rho(t) = 0, (I - A_R)\xi \neq 0\} \tag{4}$$

and

$$A_{CL} = \begin{bmatrix} A + BD_r C & BC_r \\ B_r C & A_r + B_r DC_r \end{bmatrix}$$

$$A_R = \begin{bmatrix} I & 0 \\ 0 & A_\rho \end{bmatrix}, C_{\text{CL}} = \begin{bmatrix} C & DC_r \end{bmatrix}.$$

The ordered set of reset time instants is

$$\mathcal{T}(x_0) \triangleq \{t_i \in \mathbb{R} : t_i < t_{i+1}; x_{CL} \in \mathcal{M}(t_i), i \in \mathbb{N}\}. \tag{5}$$

Note that, different from most available results, the subsystems may have multiple inputs and outputs and are not assumed to be strictly proper.

The stability of the unforced system can be concluded using the Lyapunov-like condition.

Theorem 1 ([27]): Let $V(x) : \mathbb{R}^{n+n_r} \to \mathbb{R}$ be a continuously differentiable, positive-definite unbounded function such that

$$\dot{V}(x) \triangleq \left[\frac{\partial V}{\partial x}\right] A_{\text{CL}} x_{\text{CL}} < 0, x_{\text{CL}} \neq 0 \tag{6}$$

$$\Delta V \triangleq V(A_R x_{\text{CL}}) - V(x_{\text{CL}}) \leq 0, x_{\text{CL}} \in \mathcal{M}. \tag{7}$$

Then

1) there is a left-continuous function $x_{\text{CL}}(t)$ satisfying (3) for all $t \geq 0$ and
2) the equilibrium point $x_{\text{CL}} = 0$ is globally uniformly asymptotically stable.

In practice, the existence and uniqueness of the solution of reset systems are assured by time-regularization [34], [35]. Time-regularization is a modification of reset system, such that reset instants happen only if a minimum time between resets $\Delta_m > 0$ has lapsed. Any discrete-time implementation inherently features time regularization with Δ_m equal to the sampling time [36]. Therefore, in the remainder of this note, it is assumed that solutions of R are well defined [15].

C. NI Systems

Below, to make this note self-contained, we summarize key results from the literature on the NI systems theory.

1) LTI Systems: *Definition 2 ([1], [3]).* A square transfer matrix $G(s)$ is NI if the following holds:

1) $G(s)$ has no pole at the origin and in $\Re\{s\} > 0$;
2) $j(G(j\omega) - G^*(j\omega)) \geq 0$ for all $\omega \in (0, \infty)$ such that $j\omega$ is not a pole of $G(s)$;
3) If $j\omega_0, \omega_0 \in (0, \infty)$ is a pole of $G(s)$, it is at most a simple pole and the residue matrix $K_0 = \lim_{s \to j\omega_0}(s - j\omega_0)jG(s)$ is positive semi-definite Hermitian.

Definition 3 ([1]): A square real-rational proper transfer function matrix $G(s)$ is SNI if

1) $G(s)$ has no poles in $\Re\{s\} \geq 0$ and
2) $j(G(j\omega) - G^*(j\omega)) > 0$ for all $\omega \in (0, \infty)$.

In the single-input single-output (SISO) case, a transfer function is NI if and only if it has no poles in the open right half plane or the origin and its phase is in $[-180^\circ, 0^\circ]$ at all frequencies.

Lemma 4 ([4]): Let (A, B, C, D) be a minimal state-space realization of transfer function matrix $G(s)$. Then, $G(s)$ is NI if and only if $\det(A) \neq 0, D = D^T$ and there exist matrices $P = P^T > 0, W \in \mathbb{R}^{m \times m}$ and $L \in \mathbb{R}^{m \times n}$ such that the following LMI is satisfied:

$$\begin{bmatrix} PA + A^T P & PB - A^T C^T \\ B^T P - CA & -(CB + B^T C^T) \end{bmatrix} = \begin{bmatrix} -L^T L & -L^T W \\ -W^T L & -W^T W \end{bmatrix} \leq 0. \tag{8}$$

Remark 5 ([3]): The LMI (8) can be simplified to $AP + PA^T \leq 0$ and $B + APC^T = 0$.

Lemma 6 ([3]): Let (A, B, C, D) be a minimal state-space realization of transfer function matrix $G(s)$. Then $G(s)$ is SNI if and only if:

1) $\det(A) \neq 0, D = D^T$,
2) there exists a matrix $P = P^T > 0, P \in \mathbb{R}^{n\times n}$, such that $AP + PA^T \leq 0$ and $B + APC^T = 0$, and
3) the transfer function matrix $M(s) \sim LP^{-1}A^{-1}(sI - A)^{-1}B$ has full column rank at $s = j\omega$ for any $\omega \in (0, \infty)$, where $L^T L = -AP - PA^T$.

2) Nonlinear Systems: Consider now the MIMO nonlinear system of the form

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{9}$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a Lipschitz continuous function and $h : \mathbb{R}^n \to \mathbb{R}^m$ is continuously differentiable function such that $h(0) = 0$. Note that the reset system (2) does not fit in this definition.

Definition 7 ([11]): The system (9) is NNI if there exists a positive definite continuously differentiable storage function $V : \mathbb{R}^n \to \mathbb{R}$ such that

$$\dot{V}(x(t)) \leq \dot{y}(t)^T u(t) \quad \forall t \geq 0. \tag{10}$$

The dissipative inequality of the Definition 7 can also be given in the integral equivalent form

$$V(x(t)) \leq V(x(t_0)) + \int_0^t \dot{y}^T(\tau)u(\tau)\mathrm{d}\tau \quad \forall t \geq 0. \tag{11}$$

Note, that the supply rate used in the definition of an NNI system involves a derivative of the output of the system. This is a major difference when compared with the definition of passive systems. There exist also stronger notions of the NNI properties that are used in the stability analysis of feedback systems.

Definition 8 ([11]): The system (9) is marginally strictly nonlinear negative imaginary (MS-NNI) if the dissipativity inequality (11) is satisfied, and in addition, if u, x are such that

$$\dot{V}(x) = \dot{y}^T(t)u(t) \quad \forall t > 0 \tag{12}$$

then $\lim_{t\to\infty} u(t) = 0$.

Definition 9 ([11]): The system (9) is said to be WS-NNI if it is MS-NNI and globally asymptotically stable when $u \equiv 0$.

For an LTI system (1) the NNI property reduces to the NI property and the WS-NNI property reduces to the SNI property. This equivalence has been shown for systems with the feedthrough term $D = 0$ in [10], [11]. The relationship between dissipativity and the NI property is also explored in [6] and [7]. Note that the notions MS-NNI and WS-NNI are restrictive due to the constraint on the input signal $u(t)$ included in their definitions.

III. NNI Reset Systems

In this section, we present the contribution of this note. The reset system (2) does not fit in the definition (9). Therefore, we propose a new suitable definition of the NI property, enforcing that the storage function does not increase due to the reset actions. Subsequently, we provide two lemmas to characterize NI reset systems. Finally, we provide the internal stability theorem for closed-loop reset systems based on the introduced property.

Definition 10: The system (2) is reset negative imaginary (RNI) if there exists a positive definite continuously differentiable storage function $V : \mathbb{R}^{n_r} \to \mathbb{R}$ such that

$$\dot{V}(x_r(t)) \leq \dot{y}_r(t)u_r(t), \quad t_k < t \leq t_{k+1} \tag{13}$$

$$\Delta V(x_r) = V(x_r(t_k^+)) - V(x_r(t_k)) \leq 0, \quad t_k \in \mathcal{T}. \tag{14}$$

Definition 11: The system (2) is reset strictly negative imaginary (RSNI) if it is RNI, and in addition, if u_r, x_r are such that

$$\dot{V}(x_r) = \dot{y}_r^T(t)u_r(t) \quad \forall t > 0 \tag{15}$$

then $\lim_{t\to\infty} u_r(t) = 0$.

A reset system R is characterized by base linear dynamics R_{bls} and the reset law. Moreover, NNI properties are reduced to NI properties for LTI systems. Therefore, to conclude that a reset system R is RNI (RSNI) it is sufficient to show that R_{bls} is NI (SNI), which can be done using Lemmas 4 and 6, and assure that the storage function does not increase after reset. This is formally expressed in the following two lemmas:

Lemma 12: Consider a reset system R defined by (2) with the BLS R_{bls} being a minimal realization of a transfer function. Then, R is RNI if and only if there exists matrix $P = P^T > 0$ such that the conditions of Lemma 4 are satisfied by R_{bls} and $A_\rho^T P A_\rho - P \leq 0$.

Proof: The conditions related to the properties of the BLS R_{bls} follow from the equivalence of the NI and the NNI properties for LTI systems and the Lemma 4 and its proof in [3]. Consider now the change of the quadratic storage function $V(x_r) = \frac{1}{2}x_r^T P x_r$ due to a reset

$$\Delta V(x_r) = \frac{1}{2}x_r^T(A_\rho^T P A_\rho - P)x_r$$

which is nonpositive for arbitrary x_r if and only if $A_\rho^T P A_\rho - P \leq 0$, which is a condition of the lemma. ■

Lemma 13: Consider a reset system R defined by (2) with the BLS R_{bls} being a minimal realization of a transfer function. Then, R is RSNI if and only if there exists matrix $P = P^T > 0$ such that the conditions of Lemma 6 are satisfied by R_{bls} and $A_\rho^T P A_\rho - P \leq 0$.

Proof: The conditions related to the properties of the BLS R_{bls} follow from the equivalence of the SNI and the WS-NNI properties for LTI systems and the Lemma 6 and its proof in [3]. The rest of the proof follows as in the previous lemma. ■

The feedback system presented in Fig. 1 consists of an LTI plant (1) and a reset controller (2). We have two possible cases: the interconnection of SNI plant and RNI controller or NI plant and RSNI controller. In addition, it is possible to obtain results independent on the reset condition or to assume specific reset conditions. Stability results for some of the possible combinations are presented in separate theorems.

To be able to prove the stability, we introduce restrictions either on the structure of the reset controller or on the reset condition. This is necessary due to the structure of the Lyapunov functions used in the available literature for NI systems [37]. Nevertheless, the results we obtain are sufficiently general to design practical controllers. In Theorem 14, the structure of the reset controller is restricted to allow for the use of any reset condition. For example, the reset can be triggered by a signal from an additional shaping filter [30].

Theorem 14: Consider an LTI SNI $G(s)$ with the minimal realization (1) and an RNI systems R defined by (2) with a BLS $R_{\text{bls}}(s)$, such that $G(\infty)R_{\text{bls}}(\infty) = 0$ and $G(\infty) \geq 0$. Assume that the output of the reset system does not depend directly on the reset state $y_r(t_k^+) = y_r(t_k)$ (which means $C_r A_\rho = C_r$) and that $\lambda_{\max}(G(0)R_{\text{bls}}(0)) < 1$. Then, the positive feedback interconnection of $G(s)$ and R is internally stable for any reset condition.

Proof: Let $V_g(x) = x^T P_g x$ and $V_r(x_r) = x_r^T P_r x_r$, where $P_g = P_g^T > 0$, $P_r = P_r^T > 0$ are matrices of appropriate dimensions, and consider the Lyapunov candidate function for the feedback system

$$V(x, x_r) = x_{\text{CL}}^T \begin{bmatrix} P_g - C^T D_r C & -C^T C_r \\ -C_r^T C & P_r - C_r^T D C_r \end{bmatrix} x_{\text{CL}}.$$

From [37, Lemma 4] we have that $V(x, x_r)$ is positive definite if and only if $\lambda_{\max}(G(0)R_{\text{bls}}(0)) < 1$. In the proof of [37, Thm. 1], we find that if the conditions on the subsystems stated here are satisfied, we have $\dot{V}(x, x_r) \leq 0$. This implies that the BLS of the closed loop (3) is at least Lyapunov stable. Moreover, Ghallab et al. [37] showed that the A_{CL} matrix does not have eigenvalues on the imaginary axis, which implies the asymptotic stability of the BLS of (3).

To show the stability of the complete reset system, we consider the change of $V(x, x_r)$ due to the reset $\Delta V(x, x_r) = V(x, x_r^+) - V(x, x_r)$ given in equation (16), shown at the bottom of this page. Using $C_r A_\rho = C_r$, we obtain

$$\Delta V(x, x_r) = x_r^T (A_\rho^T P_r A_\rho - P_r) x_r \tag{17}$$

which as can be seen in the Lemma 12, is smaller or equal to 0 for any x_r. ∎

Remark 15: The stability of a feedback connection of an LTI NI $G(s)$ and an RSNI R, assuming that the output of the reset system does not depend directly on the reset state, can be proven in analogue to the Theorem 14.

Remark 16: The requirement that the output of the reset system does not depend directly on the reset state $y_r(t_k^+) = y_r(t_k)$, which is equivalent to $C_r A_\rho = C_r$, is satisfied by any reset control system connected in series with an LTI low-pass filter. The complete series interconnection should then satisfy the conditions of the Lemma 12 in the case of the Theorem 14, or Lemma 13 if an LTI NI plant is considered.

In Theorem 17, the classical zero-crossing reset condition is assumed to remove the restrictions on the structure of the controller.

Theorem 17: Consider an LTI SNI $G(s)$ with the minimal realization (1) and an RNI systems R defined by (2) with a BLS $R_{\text{bls}}(s)$, such that $G(\infty)R_{\text{bls}}(\infty) = 0$ and $G(\infty) \geq 0$. Assume the reset condition $\rho(t) = u_r(t)$ is set and that $\lambda_{\max}(G(0)R_{\text{bls}}(0)) < 1$. Then, the positive feedback interconnection of $G(s)$ and R is internally stable.

Proof: The first part of the proof, related to the BLS, is the same as in the proof of Theorem 14. Consider $\Delta V(x, x_r)$ given by (16). Using the knowledge of the reset condition at the reset instant we have $u_r(t_k) = y(t_k) = 0$, that is

$$y(t_k) = Cx(t_k) + D(C_r x_r(t_k) + D_r y(t_k)). \tag{18}$$

Using the assumption $G(\infty)R_{\text{bls}}(\infty) = DD_r = 0$ we have $Cx(t_k) = -DC_r x_r(t_k)$. Substituting to (16) we obtain

$$\begin{aligned} \Delta V(x, x_r) = & -x_r^T \left((I - A_\rho^T) C_r^T D C_r (I - A_\rho)\right) x_r \\ & - x_r^T \left(A_\rho^T P A_\rho - P_r\right) x_r \end{aligned} \tag{19}$$

which is smaller of equal to 0 for any x_r if R is RNI (see Lemma 12). ∎

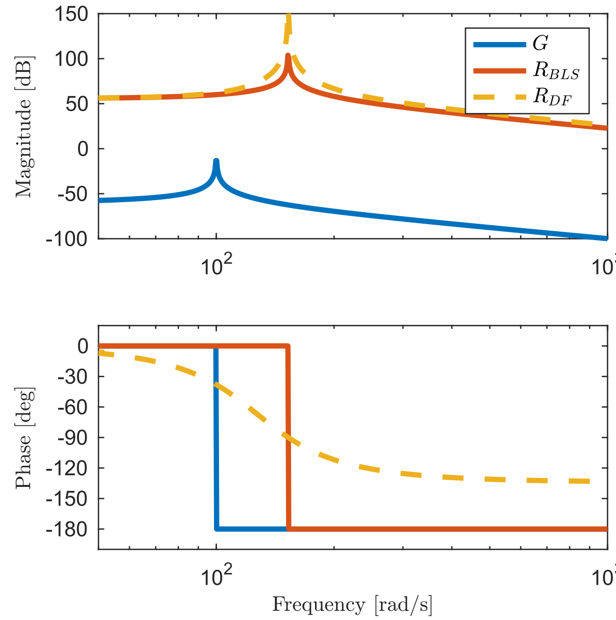

Fig. 2. Frequency response of the plant and the baseline system of the controller and describing function prescription of the reset controller.

IV. Illustrative Examples

In this section, we demonstrate the applicability of the stability results by presenting an example of a reset controller for damping an LTI plant with a resonance that achieves a finite-time convergence.

The plant represented by a transfer function

$$G(s) = \frac{1/k}{s^2/\omega_0^2 + 2\zeta s/\omega_0 + 1} \tag{20}$$

can be seen as an approximation of a flexible mechanical structure with stiffness k, natural frequency ω_0 and damping ratio ζ. As a controller, a second-order reset element given by the (2) with

$$A_r = \begin{bmatrix} 0 & 1 \\ -\frac{7}{3}\omega_0^2 & -2\zeta\omega_0 \end{bmatrix}, \qquad B_r = \begin{bmatrix} 0 \\ \omega_0^2 \end{bmatrix}, A_\rho = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$
$$C_r = \begin{bmatrix} \frac{4}{3}k & 0 \end{bmatrix}, \qquad D_r = 0$$

and with reset triggered by the derivative of the input signal $\rho = \dot{u}_r$. The controller is a reset version of the PPF [33], commonly used in active vibration control, as the transfer function of the BLS is

$$R_{\text{bls}}(s) = \frac{4k}{3s^2/\omega_0^2 + 6\zeta s/\omega_0 + 7}.$$

In the rest of this article, values $k = 10^3$, $\omega_0 = 10^2$rad/s and $\zeta = 10^{-6}$ are used for demonstration.

Fig. 2 presents the frequency responses of the plant, the BLS of the controller and its describing function description [23]. Since the

$$\begin{aligned} \Delta V(x, x_r) &= x_{\text{CL}}^T \begin{bmatrix} 0 & -(C^T C_r A_\rho - C^T C_r) \\ -(A_\rho^T C_r^T C - C_r^T C) & A_\rho^T (P_r - C_r^T D C_r) A_\rho - (P_r - C_r^T D C_r) \end{bmatrix} x_{\text{CL}} \\ &= -x_r^T (A_\rho^T C_r^T C - C_r^T C) x - x^T (C^T C_r A_\rho - C^T C_r) x_r \\ &\quad + x_r^T A_\rho^T (P_r - C_r^T D C_r) A_\rho x_r - x_r^T (P_r - C_r^T D C_r) x_r. \end{aligned} \tag{16}$$

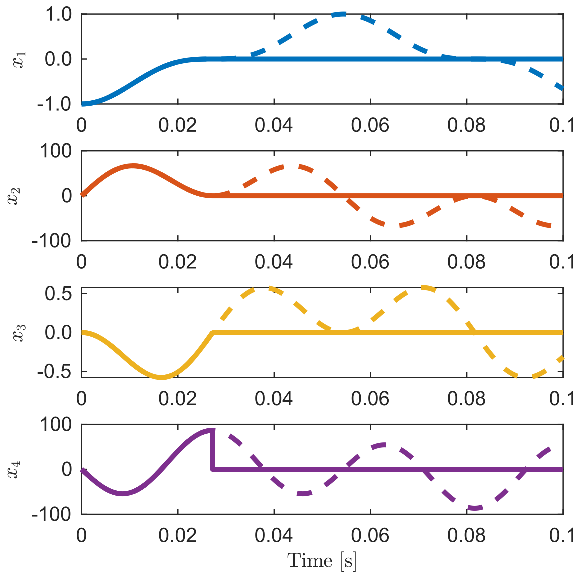

Fig. 3. State trajectories for the closed-loop interconnection. Dashed lines correspond to the BLS and solid lines to the system with reset.

considered subsystems are SISO, we can conclude that the plant (20) is SNI. The same can be concluded for the BLS of the controller. To show that the complete reset element is RNI, we use the LMI that follow from Lemma 12, which are satisfied by

$$P = \begin{bmatrix} 0.0003 & 0 \\ 0 & 7.5000 \end{bmatrix}.$$

For the considered reset controller we have $C_r A_\rho = C_r$, $|G(\infty)R_{\text{bls}}(\infty)| = 0$ and $|G(0)R_{\text{bls}}(0)| = 4/7$, so the stability can be concluded using the Theorem 14.

Fig. 3 shows the state trajectories in response to initial condition $x_{CL}(0) = [-1 \quad 0 \quad 0 \quad 0]^T$ for the closed-loop interconnection for both the reset control system (solid lines) and its base linear dynamics (dashed lines). In the reset case, the state x_1, which corresponds to the position of the plant, converges 0 in finite time. This behavior can be understood by observing that at the reset instant, the only nonzero state is the state to be reset. It can be said, all the energy of the system is associated with that state and is dissipated by the reset action. Similar effect can be seen in [38].

V. Conclusion

In this article, the NI systems theory has been extended to a class of reset control systems. We introduced definitions for RNI and RSNI systems. Moreover, we established necessary and sufficient conditions for a system to exhibit RNI and RSNI behaviors in the form of LMI. We considered positive feedback interconnections of an LTI plant and a reset controller and proven internal stability in the absence of external inputs in three different cases. Due to the structure of the Lyapunov functions used currently for NI systems, it was necessary to introduce restrictions either on the structure of the reset controller or on the reset condition. Relaxing these conditions is a remaining challenge. What is more, the definition of the RSNI system is based on the MS-NNI and, in consequence, also restrictive due to the conditions on the input signal. To increase the applicability of this result, finding an alterative definition for the RSNI would be necessary.

To exemplify the applicability of the derived stability results, we provide an illustration involving a flexible plant controlled by a second-order reset element. The obtained results can be used to show the stability of reset controllers providing finite-time convergence. The developed theory is important from a practical point of view since it allows us to conclude the stability of feedback systems consisting of a known nonlinear reset controller and an LTI plant without the need for a parametric model of the plant, as the NI properties may be concluded base on measured frequency response functions.

References

[1] A. Lanzon and I. R. Petersen, "Stability robustness of a feedback interconnection of systems with negative imaginary frequency response," *IEEE Trans. Autom. Control*, vol. 53, no. 4, pp. 1042–1046, May 2008.

[2] I. R. Petersen and A. Lanzon, "Feedback control of negative-imaginary systems," *IEEE Control Syst.*, vol. 30, no. 5, pp. 54–72, Oct. 2010.

[3] J. Xiong, I. R. Petersen, and A. Lanzon, "A negative imaginary lemma and the stability of interconnections of linear negative imaginary systems," *IEEE Trans. Autom. Control*, vol. 55, no. 10, pp. 2342–2347, Oct. 2010.

[4] M. A. Mabrok, A. G. Kallapur, I. R. Petersen, and A. Lanzon, "Stability analysis for a class of negative imaginary feedback systems including an integrator," in *Proc. 8th Asian Control Conf.-Final Prog. Proc.*, 2011, pp. 1481–1486.

[5] P. Bhowmick and A. Lanzon, "Time-domain output negative imaginary systems and its connection to dynamic dissipativity," in *Proc. IEEE Conf. Decis. Control*, 2020, pp. 5167–5172.

[6] P. Bhowmick and S. Patra, "On LTI output strictly negative-imaginary systems," *Syst. Control Lett.*, vol. 100, pp. 32–42, Feb. 2017.

[7] P. Bhowmick and A. Lanzon, "Output strictly negative imaginary systems and its connections to dissipativity theory," in *Proc. IEEE Conf. Decis. Control*, Dec. 2019, pp. 6754–6759.

[8] A. Lanzon and P. Bhowmick, "Characterization of input-output negative imaginary systems in a dissipative framework," *IEEE Trans. Autom. Control*, vol. 68, no. 2, pp. 959–974, Feb. 2023.

[9] I. R. Petersen, "Negative imaginary systems theory and applications," *Annu. Rev. Control*, vol. 42, pp. 309–318, Jan. 2016.

[10] A. G. Ghallab, M. A. Mabrok, and I. R. Petersen, "Extending negative imaginary systems theory to nonlinear systems," in *Proc. IEEE Conf. Decis. Control*, Institute of Electrical and Electronics Engineers Inc., Jul. 2018, vol. 2018-Decem, pp. 2348–2353.

[11] A. G. Ghallab and I. R. Petersen, "Negative imaginary systems theory for nonlinear systems: A. dissipativity approach," 2022, *arXiv:2201.00144*.

[12] K. Shi, I. G. Vladimirov, and I. R. Petersen, "Robust output feedback consensus for networked identical nonlinear negative-imaginary systems," *IFAC-PapersOnline*, vol. 54, Jan. 2021, pp. 239–244.

[13] K. Shi, I. R. Petersen, and I. G. Vladimirov, "Nonlinear negative imaginary systems with switching," in *IFAC-PapersOnLine*, vol. 56, pp. 3936–3941, 2023.

[14] S. Kurawa, P. Bhowmick, and A. Lanzon, "Negative imaginary theory for a class of linear time-varying systems," *IEEE Control Syst. Lett.*, vol. 5, no. 3, pp. 1001–1006, Jul. 2021.

[15] A. Barreiro and A. Bãnos, *Reset Control Systems*. London, U.K.: Springer, 2012.

[16] Y. Guo, L. Xie, and Y. Wang, *Analysis and Design of Reset Control Systems*. Institution of Engineering and Technology, Jan. 2016.

[17] C. Prieur, I. Queinnec, S. Tarbouriech, and L. Zaccarian, "Analysis and synthesis of reset control systems," *Found. Trends Syst. Control*, vol. 6, no. 2-3, pp. 119–338, 2018.

[18] O. Beker, C. V. Hollot, and Y. Chait, "Plant with integrator: An example of reset control overcoming limitations of linear feedback," *IEEE Trans. Autom. Control*, vol. 46, no. 11, pp. 1797–1799, Nov. 2001.

[19] D. Wu, G. Guo, and Y. Wang, "Reset integral-derivative control for HDD servo systems," *IEEE Trans. Control Syst. Technol.*, vol. 15, no. 1, pp. 161–167, Jan. 2007.

[20] G. Zhao, D. Nešić, Y. Tan, and C. Hua, "Overcoming overshoot performance limitations of linear systems with reset control," *Automatica*, vol. 101, pp. 27–35, 2019.

[21] N. Karbasizadeh, A. A. Dastjerdi, N. Saikumar, D. Valerio, and S. H. H. Nia, "Benefiting from linear behaviour of a nonlinear reset-based element at certain frequencies," in *Proc. 2020 Australian New Zealand Control Conf.*, Institute of Electrical and Electronics Engineers Inc., Nov. 2020, pp. 226–231.

[22] N. Karbasizadeh, N. Saikumar, and S. H. HosseinNia, “Fractional-order single state reset element,” *Nonlinear Dyn.*, vol. 104, pp. 413–427, Mar. 2021.

[23] Y. Guo, Y. Wang, and L. Xie, “Frequency-domain properties of reset systems with application in hard-disk-drive systems,” *IEEE Trans. Control Syst. Technol.*, vol. 17, no. 6, pp. 1446–1453, Nov. 2009.

[24] N. Saikumar, K. Heinen, and S. H. HosseinNia, “Loop-shaping for reset control systems: A higher-order sinusoidal-input describing functions approach,” *Control Eng. Pract.*, vol. 111, 2021, Art. no. 104808.

[25] A. A. Dastjerdi and S. H. Hosseinnia, “A frequency-domain tuning method for a class of reset control systems,” *IEEE Access*, vol. 9, pp. 40950–40962, 2021.

[26] X. Zhang, M. B. Kaczmarek, and S. H. HosseinNia, “Frequency response analysis for reset control systems: Application to predict precision of motion systems,” *Control Eng. Pract.*, vol. 152, 2024, Art. no. 106063.

[27] O. Beker, C. V. Hollot, Y. Chait, and H. Han, “Fundamental properties of reset control systems,” *Automatica*, vol. 40, pp. 905–915, Jun. 2004.

[28] C. V. Hollot, Y. Zheng, and Y. Chait, “Stability analysis for control systems with reset integrators,” in *Proc. IEEE Conf. Decis. Control*, 1997, pp. 1717–1719.

[29] A. A. Dastjerdi, A. Astolfi, and S. H. Hosseinnia, “A frequency-domain stability method for reset systems,” in *Proc. IEEE Conf. Decis. Control*, Dec. 2020, pp. 5785–5791.

[30] A. A. Dastjerdi, A. Astolfi, and S. H. HosseinNia, “Frequency-domain stability methods for reset control systems,” *Automatica*, vol. 148, Feb. 2023, Art. no. 110737.

[31] S. J. L. M. van Loon, K. G. J. Gruntjens, M. F. Heertjes, N. van de Wouw, and W. P. M. H. Heemels, “Frequency-domain tools for stability analysis of reset control systems,” *Automatica*, vol. 82, pp. 101–108, Aug. 2017.

[32] J. Carrasco, A. Baños, and A. van der Schaft, “A passivity-based approach to reset control systems stability,” *Syst. Control Lett.*, vol. 59, pp. 18–24, Jan. 2010.

[33] J. L. Fanson and T. K. Caughey, “Positive position feedback control for large space structures,” *AIAA J.*, vol. 28, pp. 717–724, May 1990.

[34] D. Nesic, L. Zaccarian, and A. R. Teel, “Stability properties of reset systems,” *Automatica*, vol. 44, no. 8, pp. 2019–2026, 2008.

[35] L. Zaccarian, D. Nešić, and A. R. Teel, “First order reset elements and the clegg integrator revisited,” in *Proc. 2005, Amer. Control Conf.*, 2005, pp. 563–568.

[36] M. F. Heertjes, K. G. J. Gruntjens, S. J. L. M. van Loon, N. van de Wouw, and W. P. M. H. Heemels, “Experimental evaluation of reset control for improved stage performance,” *IFAC-PapersOnLine*, vol. 49, no. 13, pp. 93–98, 2016.

[37] A. G. Ghallab, M. A. Mabrok, and I. R. Petersen, “Lyapunov-based stability of feedback interconnections of negative imaginary systems,” *IFAC-PapersOnline*, vol. 50, Jul. 2017, pp. 3424–3428.

[38] R. T. Bupp, D. S. Bernstein, V. S. Chellaboina, and W. M. Haddad, “Finite settling time control of the double integrator using a virtual trap-door absorber,” *IEEE Trans. Autom. Control*, vol. 45, no. 4, pp. 776–780, Apr. 2000.