A brief introduction of curvilinear coordinates with help of tensor calculus

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1 Introduction.

For solving of field theoretical or numerical problems, it is advantageous to make use of curvilinear coordinates, because the bodies and surfaces are curved generally. By making a proper choice of the coordinates, the geometry of these bodies can be treated more easily. Especially for flow calculations of cavitating profiles curvilinear coordinates can be used. The wing and cavity are assumed to be (doubly) curved. With the panel method the potential is solved at the collocation points. The collocation points are defined at the stations ip, and jp. These variables ip and jp can be used as the curvilinear parameters. By making use of the curvilinear coordinates, with help of tensor calculus, the velocities at the collocation points can be determined. Also it is possible to make an expansion of several variables, like potential and velocity, into the field, with help of the divergence and rotation theorem of potentials. This report is written to give a summary of the curvilinear equations. So, in other reports the equations in this report can be referred to. The text around the equations will give briefly the meaning of the equations. The derivation of the equations is not explained. The derivations can be found in the literature of tensor calculus (see[1]).

2 Curvilinear coordinates.

The cartesian coordinates of a point P with the components (x,y,z) are functions of the curvilinear coordinates $\xi^1 = \xi$,$\xi^2 = \eta$ and $\xi^3 = \zeta$:

$$\begin{aligned} x &= x(\xi^1, \xi^2, \xi^3) \\ y &= y(\xi^1, \xi^2, \xi^3) \\ z &= z(\xi^1, \xi^2, \xi^3) \end{aligned} \tag{1}$$

These coordinates ξ^1, ξ^2 and ξ^3 will be shortly written as ξ^i. From now on the general theory of the curvilinear coordinates calculus in a R^n-space is treated (, where the index range of the curvilinear parameters is from 1 to n).

3 Base_vectors.

The base_vectors can be determined by taking the partial derivative of each curvilinear variable:

$$\vec{e}_i = \frac{\partial \vec{x}}{\partial \xi^i} \tag{2}$$

In fact these base_vectors are the tangent vectors of each curvilinear variable. The components of the base_vector $\vec{e}_i$, calculated in this way, are given in cartesian coordinates. These base_vectors are called the natural base and are used for contra_variant components. These base_vectors with their components describe a vector_space V. The curvilinear components, which belong to these base_vectors are v^i. They are called the contra_variant components. To get the global vector $\vec{v}_{glob}$ in cartesian coordinates, the components of the contra_variant curvilinear coordinates have to be multiplied by each base_vector:

$$\vec{v}_{glob} = v^i \vec{e}_i \tag{3}$$

4 Dual space.

There also exists a dual space. The base_vectors of this space are perpendicular to the base_vectors as described as before. The base_vector in this space are called dual base_vectors and they are noted as a vector with a superscript: $\vec{\varepsilon}^i$. The relation between the base_vectors and dual base_vectors is given as:

$$< \vec{\varepsilon}^i \cdot \vec{e}_j > = \delta^i_j \tag{4}$$

in which: δ^i_j the kronecker delta symbol.

The components of the dual base_vector are also given in cartesian components. The dual base_vectors are used for co_variant description of a vector.

The curvilinear components, which belong to these dual base_vectors are v_i. They are called the co_variant components. To get the global vector $\vec{v}_{glob}$ in cartesian coordinates, the components of the co_variant curvilinear coordinates have to be multiplied by each dual_base_vector:

$$\vec{v}_{glob} = v_i \vec{\varepsilon}^i \tag{5}$$

5 Fundamental tensor of the first kind or metric tensor.

The fundamental or metric tensor G is defined as the dot_product of the base_vectors:

$$g_{ij} = < \vec{e}_i, \vec{e}_j > \tag{6}$$

The metric tensor gives an indication of the length of the base_vectors and their direction cosines. The metric tensor can also be used to transform the contra_variant components into the co_variant components:

$$v_i = g_{ij} v^j \tag{7}$$

The inverse of the metric tensor G^{-1} is the metric tensor of the dual space. This tensor can be determined by taking the inverse of g_{ij}:

$$g^{ij} = (g_{ij})^{-1} \tag{8}$$

or in the same way as the natural base, by taking the dot_product of the dual base_vectors:

$$g^{ij} =< \vec{\varepsilon}^{i} \cdot \vec{\varepsilon}^{j} > \tag{9}$$

Because g^{ij} is the inverse of g_{ij} the following equation is valid:

$$g^{ij} g_{jk} = g^{i}_{k} = \delta^{i}_{k} \tag{10}$$

In which:

δ^i_k the kronecker delta symbol.

g^i_k the mix metric tensor.

This equation can be used for the calculus of derivations from the contra_variant components to co_variant components (or vice versa). The contra_variant metric tensor can also be used to transform the co_variant components of a vector into the contra_variant components of a vector:

$$v^{i} = g^{ij} v_{j} \tag{11}$$

The base vectors of a natural coordinate system form the ribs of a parallelepiped. It's well known that the determinant of the vectors delivers the volume of this parallelepiped:

$$V = det(\vec{e}_1, \cdots, \vec{e}_n) \tag{12}$$

It is also known that the determinant of a transposed 2-tensor (= matrix) is equal to the determinant of the original 2-tensor and that the determinant of a product of two 2-tensors is equal to the product of the determinants of the two 2-tensors. If a 2-tensor is made of the base_vectors:

$$E = (\vec{e}_1, \cdots, \vec{e}_n) \tag{13}$$

then the co_variant metric tensor G (with components g_{ij}) can be calculated as:

$$G = E^{T} E \tag{14}$$

The determinant of G will be:

$$det(G) = det(E^{T}) det(E) = (det(E))^2 = (det(\vec{e}_1, \cdots, \vec{e}_n))^2 \tag{15}$$

From this the square root can be taken and this gives:

$$V = \sqrt{det(G)} = \sqrt{g} = det(\vec{e}_1, \cdots, \vec{e}_n) \tag{16}$$

So the square root of the determinant of the co_variant metric tensor is the volume of the base_vectors. This expression can be used for the Jacobian of integral expressions when the integral is calculated in the transformed curvilinear coordinates.

6 Christoffel symbols.

Using curvilinear coordinates and their base_vector in several points in a space V doesn't have the same length and direction, so the base vectors are a function of the curvilinear coordinates (contra_variant notation):

$$\vec{e}_i = \vec{e}_i(\xi^1, \cdots, \xi^n) \tag{17}$$

When this function is differentiated to one of these curvilinear coordinates, the newly function can be expressed as a linear combination of the origin base_vectors. The components of these base_vectors are the Christoffel symbols of the second kind Γ^k_{ij}:

$$\frac{\partial \vec{e}_i}{\partial \xi^j} = \Gamma^k_{ij}\vec{e}_k \tag{18}$$

The calculation of the Christoffel symbols will be discussed later.

It is also possible that the curvilinear coordinates are dependent of a parameter t (such as streamlines). The base_vectors then are:

$$\vec{e}_i = \vec{e}_i(\xi^1(t), \cdots, \xi^n(t)) \tag{19}$$

Differentiating to t gives, using the chain rule:

$$\frac{d\vec{e}_i}{dt} = \frac{\partial \vec{e}_i}{\partial \xi^j}\frac{d\xi^j}{dt} = \Gamma^k_{ij}\frac{d\xi^j}{dt}\vec{e}_k \tag{20}$$

So the rules of the calculus of differentiation of base_vectors are defined.

Now a vector field $\vec{v}$ is defined with a natural base $(e_1, \cdots, e_n)$ and components v^i. These components are functions of t along a curve. Differentiation to t gives:

$$\begin{aligned}\frac{d\vec{v}}{dt} &= \frac{d(v^i\vec{e}_i)}{dt} = \frac{dv^i}{dt}\vec{e}_i + v^i\frac{d\vec{e}_i}{dt},\\ \frac{d\vec{v}}{dt} &= \frac{dv^i}{dt}\vec{e}_i + v^i\Gamma^k_{ij}\frac{d\xi^j}{dt}\vec{e}_k\end{aligned} \tag{21}$$

Changing the summation indices i and k gives:

$$\frac{d\vec{v}}{dt} = \frac{dv^i}{dt}\vec{e}_i + v^k\Gamma^i_{jk}\frac{d\xi^j}{dt}\vec{e}_i \tag{22}$$

Expliciting the base_vector results in:

$$\frac{d\vec{v}}{dt} = (\frac{dv^i}{dt} + v^k\Gamma^i_{jk}\frac{d\xi^j}{dt})\vec{e}_i \tag{23}$$

The components of the derivative of $\frac{d\vec{v}}{dt}$ are called $\frac{Dv^i}{dt}$. For $\frac{Dv^i}{dt}$ remains:

$$\frac{Dv^i}{dt} = \frac{dv^i}{dt} + v^k \Gamma^i_{jk} \frac{d\xi^j}{dt} \tag{24}$$

This method of differentiation of the components of $\vec{v}$ is called absolute, intrinsic or contra_variant differentiation. This gives the components of the derivative vector of a vector field $\vec{v}$, with a curvature, expressed in the natural base of each point.

If a vector field is defined only with the curvilinear coordinates ξ^i:

$$v^i = v^i(\xi^1, \cdots, \xi^n) \tag{25}$$

Then the derivative of the h-th coordinate line ξ^h of v^i gives:

$$\frac{Dv^i}{d\xi^h} = \frac{\partial v^i}{\partial \xi^h} + v^k \Gamma^i_{jk} \frac{\partial \xi^j}{\partial \xi^h} \tag{26}$$

(This equation is found by substituting t by ξ^h in equation (24).) The curvilinear coordinates ξ^i are independent of each other. So the partial derivative of the curvilinear coordinates gives the kronecker delta function:

$$\frac{\partial \xi^j}{\partial \xi^h} = \delta^j_h \tag{27}$$

This expression can be replaced in equation (26). By making use of a new notation for this kind of derivating; $v^i{}_{,h}$, it yields:

$$v^i{}_{,h} = \frac{Dv^i}{d\xi^h} = \frac{\partial v^i}{\partial \xi^h} + \Gamma^i_{jk} v^k \tag{28}$$

This equation can also be used for the absolute differentiation. It can then be written as:

$$\frac{Dv^i}{dt} = v^i{}_{,j} \frac{d\xi^j}{dt} \tag{29}$$

In the same way it is possible to derive a simular expression for the co_variant derivatives. The results of the co_variant derivatives are:

$$\frac{Dv_i}{dt} = \frac{dv_i}{dt} - v_k \Gamma^k_{ji} \frac{d\xi^j}{dt} \tag{30}$$

(This is called the co_variant differentiation)

$$v_{i,h} = \frac{Dv_i}{d\xi^h} = \frac{\partial v_i}{\partial \xi^h} - \Gamma^k_{hi} v_k \tag{31}$$

and

$$\frac{Dv_i}{dt} = v_{i,j} \frac{d\xi^j}{dt} \tag{32}$$

7 Calculation of the Christtoffels symbols.

In section (3. Base_vectors.) it has been derived that the partial derivative of each curvilinear parameter gives the base_vector:

$$\vec{e}_i = \frac{\partial \vec{x}}{\partial \xi^i} \tag{33}$$

Repeated differentiation of this newly obtained equation to ξ^i, and remembering the definition of the Christoffel symbol of the second kind, it results in:

$$\frac{\partial^2 \vec{x}}{\partial \xi^j \partial \xi^i} = \frac{\partial \vec{e}_i}{\partial \xi^j} = \Gamma^k_{ji} \vec{e}_k = \Gamma^k_{ji} \frac{\partial \vec{x}}{\partial \xi^k} \tag{34}$$

This equation is multiplied (inner product) by the vector $\vec{e}_l$. Keeping in mind the definition of the metric tensor equation (34) can be rewritten as:

$$< \frac{\partial \vec{x}}{\partial \xi^l}, \frac{\partial^2 \vec{x}}{\partial \xi^j \partial \xi^i} > = < \vec{e}_l, \frac{\partial \vec{e}_i}{\partial \xi^j} > = \Gamma^k_{ji} g_{kl} \tag{35}$$

The right hand side is the definition of the Christoffel symbol of the first kind and is notated as:

$$\Gamma_{ji|l} = \Gamma^k_{ji} g_{kl} \tag{36}$$

So the inner product of the base_vectors and the derivative of the base _vectors gives the Christoffel symbols of the first kind:

$$\Gamma_{ji|l} = < \vec{e}_l \cdot \frac{\partial \vec{e}_i}{\partial \xi^j} > \tag{37}$$

The Christoffel symbols of the first kind can also be calculated on an other way. Using the definition of the metric tensor (see equation (6))

$$g_{ij} = < \vec{e}_i \cdot \vec{e}_j >$$

and differentiating this equation partial to ξ^m gives:

$$\frac{\partial g_{ij}}{\partial \xi^m} = < \frac{\partial \vec{e}_i}{\partial \xi^m} \cdot \vec{e}_j > + < \vec{e}_i \cdot \frac{\partial \vec{e}_j}{\partial \xi^m} > \tag{38}$$

In the right hand side the definitions of the Christoffel symbols of the first kind appear (see equation(37)). So equation (38) can be rewritten as:

$$\frac{\partial g_{ij}}{\partial \xi^m} = \Gamma_{mj|i} + \Gamma_{im|j} \tag{39}$$

Cyclic changes of the indices i,j,m gives respectively:

$$\frac{\partial g_{mi}}{\partial \xi^j} = \Gamma_{ji|m} + \Gamma_{mj|i} \quad (40)$$

$$\frac{\partial g_{jm}}{\partial \xi^i} = \Gamma_{im|j} + \Gamma_{ji|m} \quad (41)$$

Adding equation (40) and (41) and substracting equation (39) gives:

$$2\Gamma_{ji|m} = \frac{\partial g_{jm}}{\partial \xi^i} + \frac{\partial g_{mi}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^m} \quad (42)$$

The Christoffel symbol of the first kind can be determined by dividing this equation by 2:

$$\Gamma_{ji|m} = \frac{1}{2}\left(\frac{\partial g_{jm}}{\partial \xi^i} + \frac{\partial g_{mi}}{\partial \xi^j} - \frac{\partial g_{ij}}{\partial \xi^m}\right) \quad (43)$$

Equation (37) can be multiplied with the contra_variant metric tensor G^{-1} with components g^{hl}. Making use of that the product of the co_variant metric tensor and contra_variant metric tensor gives the Kronecker function (see equation (10)). The Christoffel symbol of the second kind can be written explicitly:

$$g^{hl}\Gamma_{ji|l} = g^{hl} g_{lk} \Gamma^k_{ji} = \delta^k_h \Gamma^k_{ji} = \Gamma^k_{ji} \quad (44)$$

or

$$\Gamma^h_{ji} = g^{hl}\Gamma_{ji|l} \quad (45)$$

8 Gradient, divergence and rotation.

In this section the definition of gradient, divergence and rotation is given in a curvilinear coordinate system, without further derivation.
The gradient of a scalar F is defined as:

$$grad(F)^i = \lim_{\Delta\xi^i \to 0} \frac{\Delta F}{e_i \Delta \xi^i} = g^{ij}\frac{\partial F}{\partial \xi^j} \quad (46)$$

The divergence of a vector $\vec{v}$ is defined as:

$$div(\vec{v}) = \lim_{V \to 0} \frac{\int\int <\vec{v}\cdot\vec{n}> dO}{V} = v^i{}_{,i} = \frac{\partial v^i}{\partial \xi^i} + \Gamma^i_{ih} v^h \quad (47)$$

The rotation of a vector $\vec{v}$ is calculated as:

$$< \vec{n}.rot\vec{v} >= \lim_{\Delta O \to 0} \frac{\oint \vec{v} \cdot d\vec{s}}{\Delta O} \quad (48)$$

Then the definition of rotation is found by eliminating the normal vector:

$$rot(\vec{v})_{ij} = \vec{v}_{j,i} - \vec{v}_{i,j} = \frac{\partial v_j}{\partial \xi^i} - \frac{\partial v_i}{\partial \xi^j} \quad (49)$$

9 The geometry of a scale and of a surface.

The surface of the body is described with two parameters $\xi^1 = \xi$ and $\xi^2 = \eta$ in R^3.

It's assumed that the surface S varies smoothly and that the functions, which described this surface are sufficiently differentiable. In each point P on S there are two base_vectors $\vec{e}_1 = \vec{X}_\xi$ and $\vec{e}_2 = \vec{X}_\eta$. These are the tangent vectors. Perpendicular to these base_vectors a normal can be defined:

$$\vec{N} = \frac{\vec{X}_\xi \otimes \vec{X}_\eta}{\| \vec{X}_\xi \otimes \vec{X}_\eta \|} \quad (50)$$

9.1 Definition of a scale.

A scale is defined as a parameterisation of the surface coordinates and a linear normal-coordinate:

$$\vec{x}(\xi, \eta, \zeta) = \vec{X}(\xi, \eta) + \zeta \vec{N}(\xi, \eta) \quad (51)$$

The base_vectors, which belong to these parameters are:

$$\begin{aligned} \vec{e}_1 &= \vec{X}_\xi + \zeta \vec{N}_\xi \\ \vec{e}_2 &= \vec{X}_\eta + \zeta \vec{N}_\eta \\ \vec{e}_3 &= \vec{N} \end{aligned} \quad (52)$$

If $\zeta = 0$, then the metric of a surface appears.

9.2 The metric tensor of the scale and surface.

According to the definition of the metric tensor (see equation (6)) the metric tensor in a scale with indices i,j = 1,2 can be determined by:

$$\begin{aligned} g_{ij} &= < \vec{e}_i \cdot \vec{e}_j >= < \vec{X}_\xi + \zeta \vec{N}_\xi \cdot \vec{X}_\eta + \zeta \vec{N}_\eta > \\ &= < \vec{X}_\xi \cdot \vec{X}_\eta > + (< \vec{X}_\xi \cdot \vec{N}_\eta > + < \vec{X}_\eta, \vec{N}_\xi >)\zeta + < \vec{N}_\xi \cdot \vec{N}_\eta > \zeta^2 \end{aligned} \quad (53)$$

If sub_tensors are defined as:

$$
\begin{aligned}
a_{ij} &= <\vec{X}_\xi \cdot \vec{X}_\eta> \\
b_{ij} &= <\vec{X}_\xi \cdot \vec{N}_\eta> = <\vec{X}_\eta \cdot \vec{N}_\xi> = -<\vec{N} \cdot \vec{X}_{\xi\eta}> \\
c_{ij} &= <\vec{N}_\xi \cdot \vec{N}_\eta>
\end{aligned} \tag{54}
$$

then the metric tensor in the scale is composed of:

$$
g_{ij} = a_{ij} + 2b_{ij}\zeta + c_{ij}\zeta^2 \tag{55}
$$

The components with index 3 in this equation are:

$$
\begin{aligned}
g_{i3} &= g_{3i} = <\vec{e}_i \cdot \vec{e}_3> = <\vec{X}_\xi + \zeta\vec{N}_\xi \cdot \vec{N}> = 0 \\
g_{33} &= <\vec{N} \cdot \vec{N}> = 1
\end{aligned} \tag{56}
$$

So the metric tensors in the scale have the following forms:

$$
G = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{12} & g_{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{57}
$$

and:

$$
G^{-1} = \begin{pmatrix} g^{11} & g^{12} & 0 \\ g^{12} & g^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{58}
$$

From this equation it is easy to see that the third co_variant component is equal to the third contra_variant component:

$$
v_3 = v^3 \tag{59}
$$

When $\zeta = 0$ the metric tensor of a surface appears and the components of this metric tensor are a_{ij}.

9.3 Christoffel symbols of the scale with an index 3.

Here a summary of the Christoffel symbols with the index 3 is given, without derivation. The Christoffel symbols of the first kind are:

$$
\begin{aligned}
\Gamma_{i3|3} &= \Gamma_{3i|3} = \Gamma_{33|i} = \Gamma_{33|3} = 0 \\
\Gamma_{ij|3} &= -(b_{ij} + c_{ij}\zeta) \\
\Gamma_{3i|j} &= b_{ij} + c_{ij}\zeta
\end{aligned} \tag{60}
$$

The Christoffel symbols of the second kind are:

$$
\begin{aligned}
\Gamma^3_{i3} &= \Gamma^3_{3i} = \Gamma^i_{33} = \Gamma^3_{33} = 0 \\
\Gamma^3_{ij} &= -(b_{ij} + c_{ij}\zeta) \\
\Gamma^j_{3i} &= g^{jk}\Gamma_{3i|k} = g^{jk}(b_{ik} + c_{ik}\zeta)
\end{aligned} \tag{61}
$$

10 references.

[1] Kallenberg, G.W.M.

"Tensoranalyse",T.U. Delft, september 1978