CONTINUOUS REPRESENTATION FOR SHELL MODELS OF TURBULENCE

Alexei A. Mailybaev
Instituto Nacional de Matemática Pura e Aplicada – IMPA, Rio de Janeiro, Brazil
Email: alexei@impa.br

Abstract In this work we construct and analyze continuous hydrodynamic models in one space dimension, which are induced by shell models of turbulence. After Fourier transformation, such continuous models split into an infinite number of uncoupled subsystems, which are all identical to the same shell model. The two shell models, which allow such a construction, are considered: the dyadic (Desnyansky–Novikov) model with the intershell ratio $\lambda = 2^{3/2}$ and the Sabra model of turbulence with $\lambda = \sqrt{2+\sqrt{5}} \approx 2.058$. The continuous models allow understanding various properties of shell model solutions and provide their interpretation in physical space. We show that the asymptotic solutions of the dyadic model with Kolmogorov scaling correspond to the shocks (discontinuities) for the induced continuous solutions in physical space, and the finite-time blowup together with its viscous regularization follow the scenario similar to the Burgers equation. For the Sabra model, we provide the physical space representation for blowup solutions and intermittent turbulent dynamics.

INTRODUCTION

Shell models represent simplified "toy" models demonstrating various nontrivial phenomena of developed hydrodynamic turbulence, many of which are still not fully understood [4, 2]. These models consider a geometric series of wavenumbers, $k_n = k_0\lambda^n$ with $n \in \mathbb{Z}$ and fixed $\lambda > 1$. Each shell is represented by one or several (real or complex) numbers u_n called shell speeds, which mimic the velocity field at a given scale $\delta r \sim 2\pi/k_n$. The evolution of shell speeds is governed by infinite-dimensional systems of ordinary differential equations, which allow for local interaction among the shells and must share several properties like scaling invariance, quadratic nonlinearity, energy conservation etc. with the Navier-Stokes equations or other hydrodynamic models. Originating in early 70s, see e.g. [6, 5, 3], shell models became especially popular with the construction of the Gledzer–Ohkitani–Yamada (GOY) model [5, 9], which has the chaotic intermittent dynamics in the inertial interval with the statistical properties close to the Navier-Stokes developed turbulence. In this paper, we consider the following shell models: the dyadic (Desnyansky–Novikov) model [3] and the Sabra model of turbulence [7] representing a modified version of the GOY model.
The derivation of shell models is usually based of restricting the Fourier transformed hydrodynamic equations to a very limited number of modes (shells). Such simplification leads to a strongly reduced system, so that the same shell model can be derived from different original systems, e.g., Burgers or Navier-Stockes equations [10, 8]. As a result, shell model solutions generally lose their quantitative relation with the original systems, though they may retain some important qualitative properties. This leads to the problem of interpreting the results obtained for shell models in their relation with the continuous solutions in physical space.

CONTINUOUS REPRESENTATION OF SHELL MODELS

In this paper, we construct one-dimensional continuous hydrodynamic models, from which the shell models can be derived without any simplifications, i.e., in a rigorous way. This means that the Fourier transformed continuous models split into an infinite set of uncoupled subsystem, where each subsystem is equivalent to the same shell model under consideration. Such continuous models are characterized by nonlocal quadratic nonlinearity (similarly to the nonlocal term induced by pressure in incompressible flows), conserve energy and may have some other properties like the Hamiltonian structure etc. This construction is carried out for the two cases. The first case corresponds to the dyadic (Desnyansky–Novikov) model

$$\frac{\partial u_n}{\partial t} = k_n u_{n-1}^2 - k_{n+1} u_n u_{n+1} - \nu k_n^2 u_n + f_n \tag{1}$$

with the intershell ratio $\lambda = 2^{3/2}$. The corresponding coniunous model takes the form

$$\frac{\partial u}{\partial t} + \frac{\partial g}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} + f, \quad x, t \in \mathbb{R}, \tag{2}$$

where ν is a viscous coefficient, $f(x,t)$ is the forcing term and

$$g(x,t) = \frac{1}{2\pi} \iint K(y-x, z-x) u(y,t) u(z,t) dy dz, \quad K(y,z) = -\frac{4}{(y+z)^2} - \frac{4}{(y-2z)^2} - \frac{4}{(z-2y)^2}. \tag{3}$$

As the second case, we considered the Sabra model

$$\frac{\partial u_n}{\partial t} = i \left[k_{n+1} u_{n+2} u_{n+1}^* - (1+c) k_n u_{n+1} u_{n-1}^* - c k_{n-1} u_{n-1} u_{n-2} \right] - \nu k_n^2 u_n + f_n \tag{4}$$

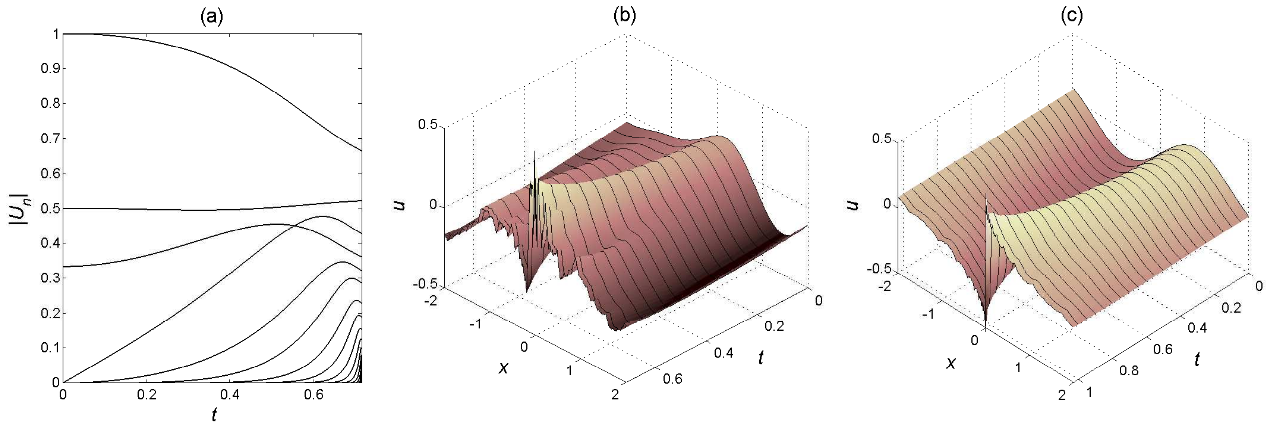

Figure 1. Finite-time blowup in inviscid Sabra model with no forcing in the 3D regime ($c = -\lambda^{-1}$). (a) Evolution of shell speed amplitudes $|U_n(t)|$ for complex initial conditions. (b) Physical space representation $u(x,t)$ of the shell model solution $U_n(t)$. (c) Physical space representation $u(x,t)$ of the purely imaginary solution $U_n(t)$. The final time in all the figures corresponds to the blowup point.

with the intershell ratio $\lambda = \sqrt{2+\sqrt{5}} \approx 2.058$. The continuous representation takes the form of Eqs. (2) and (3) with the different kernel

$$K(y,z) = K_\psi(y,z) + K_\psi(z,y), \quad K_\psi(y,z) = \frac{\sigma}{(\sigma y - z)^2} - \frac{(1+c)\sigma^2}{(\sigma^2 y - z)^2} - \frac{c\sigma}{(\sigma y + z)^2}, \quad \sigma = \frac{1+\sqrt{5}}{2} \approx 1.618. \quad (5)$$

Note that our approach uses the fixed values of λ, as opposed to the limiting models obtained as $\lambda \to 1$ [1].

BLOWUP AND INTERMITTENCY IN CONTINUOUS MODELS

The proposed continuous models resolve the problem of representing the shell model solutions in physical space. This allows interpreting various results for shell models by relating them with the known properties of continuous flows or, more generally, of evolutionary partial differential equations. We demonstrate this by showing that the asymptotic solution with the Kolmogorov scaling, $u_n \propto k_n^{-1/3}$, in the dyadic shell model (1) corresponds to a shock (discontinuity) for the induced continuous solution in physical space. Furthermore, the finite-time blowup together with its viscous regularization in the dyadic model follow the scenario similar to the Burgers equation for the continuous representation. Another implication of this approach is the physical space representation of finite-time blowup and intermittent turbulent dynamics in the Sabra model, Fig. 1.

Acknowledgments The work was supported by the CNPq (grant 305519/2012-3) and by the FAPERJ (Pensa Rio 2014).

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