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**DOI**

[10.1109/YAC57282.2022.10023826](https://doi.org/10.1109/YAC57282.2022.10023826)

**Publication date**

2022

**Document Version**

Final published version

**Published in**

Proceedings - 2022 37th Youth Academic Annual Conference of Chinese Association of Automation, YAC 2022

**Citation (APA)**

Zhang, H. W., Wang, J. M., & Wang, J. (2022). Input-to-state stabilization for a  $2 \times 2$  hyperbolic system cascaded with an ODE. In *Proceedings - 2022 37th Youth Academic Annual Conference of Chinese Association of Automation, YAC 2022* (pp. 296-301). (Proceedings - 2022 37th Youth Academic Annual Conference of Chinese Association of Automation, YAC 2022). IEEE.  
<https://doi.org/10.1109/YAC57282.2022.10023826>

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# Input-to-state stabilization for a $2 \times 2$ hyperbolic system cascaded with an ODE

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**Abstract**—The paper deals with the input-to-state stabilization for the  $2 \times 2$  system of first-order hyperbolic equations, which convect in opposite directions cascaded with an unstable ODE equation. First, an inverse backstepping transformation is introduced to obtain a target system. Then, by active disturbance rejection control (ADRC) method, the disturbance is estimated via a disturbance estimator with time-varying gain. When the unmatched disturbances are absent, the disturbance estimator is exponentially convergent to the matched disturbance. Furthermore, in order to reject the matched disturbance and obtain the input-to-state stability of the system, the controller is proposed by using the disturbance estimator. Finally, numerical simulations are presented to validate theoretical results.

**Index Terms**—First-order transport equations, backstepping transformation, active disturbance rejection control, input-to-state stabilization

## I. INTRODUCTION

As mathematical models, coupled hyperbolic systems appear in many application fields, for example: aerospace, road transports, chemical reactors, civil structures, etc.

For coupled hyperbolic systems, there are many papers devoted to considering the problem of stability. By the backstepping transformation, the boundary feedback controller of coupled hyperbolic systems is studied in [1]. Recently, the first-order hyperbolic system cascaded with an ODE is considered, and the feedback controller has been given in [2]. When coupled hyperbolic systems undergo the boundary disturbance, some authors have contributed to the stability analysis. For coupled linear hyperbolic equations, the boundary stabilization has been obtained by the sliding mode control (SMC) method [3]. In [4], by the SMC method and the active disturbance rejection control (ADRC) method, the feedback controllers of the hyperbolic system cascaded with an ODE subject to boundary disturbance have been proposed, respectively. Subsequently, the ADRC approach is extended (see [5]), this extended ADRC method such that the extended state observer (ESO) is exponentially convergent to the matched disturbance.

When the equation undergoes the external disturbances, the rejection of the unmatched disturbance is usually difficult.

This work was supported by the National Natural Science Foundation of China (No. 62073037, 12131008).

Hence, Sontag proposed the ISS notion for ODEs (see [6]) to characterize the influence of the disturbances. In recent years, this concept has been extended to infinite dimensional systems [7]. For infinite-dimensional systems, Mironchenko and Prieur study the ISS properties in [8, 9]. The stability for coupled hyperbolic systems is studied in [7] by small-gain theorem. The state and output feedback controllers for an ODE-wave cascade system have been proposed to obtain the ISS properties in [10]. In [11], some sufficient conditions for local input-to-state stability to quasilinear hyperbolic systems are given. However, few studies have dealt with the ISS problem of coupled first-order hyperbolic equations.

This paper considers the following system:

$$\begin{aligned} \bar{\vartheta}_{\bar{\omega}}(0, \bar{\omega}) &= a\bar{\vartheta}(0, \bar{\omega}) + \bar{\lambda}(0, \bar{\omega}), \\ \bar{\kappa}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= -\bar{\kappa}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) + d_1(\bar{\varrho}, \bar{\omega}) \\ &\quad + \int_0^{\bar{\varrho}} \varsigma_1(\bar{\varrho}, \iota) (\bar{\lambda}(\iota, \bar{\omega}) - \bar{\kappa}(\iota, \bar{\omega})) d\iota, \\ \bar{\kappa}(0, \bar{\omega}) &= \bar{\lambda}(0, \bar{\omega}) + 2a\bar{\vartheta}(0, \bar{\omega}), \\ \bar{\lambda}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= \bar{\lambda}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) \\ &\quad + \int_0^{\bar{\varrho}} \varsigma_1(\bar{\varrho}, \iota) (\bar{\lambda}(\iota, \bar{\omega}) - \bar{\kappa}(\iota, \bar{\omega})) d\iota, \\ \bar{\lambda}(1, \bar{\omega}) &= \bar{U}(\bar{\omega}) + d_2(\bar{\omega}), \end{aligned} \quad (1)$$

where  $\bar{\varrho} \in (0, 1)$ ,  $\bar{\omega} > 0$  are the space and time variables,  $\bar{\vartheta}(0, \bar{\omega})$  is the state of ODE,  $\bar{\kappa}(\bar{\varrho}, \bar{\omega})$  and  $\bar{\lambda}(\bar{\varrho}, \bar{\omega})$  are states of first-order transport equations, respectively.  $a, b > 0$  are two constants,  $\bar{U}(\bar{\omega})$  is a control. Disturbances  $d_1 \in L_{loc}^{\infty}(0, \infty; L^2(0, 1))$ , and  $d_2 \in L_{loc}^{\infty}(0, \infty)$ .

The paper is structured as follows. Section II introduces the backstepping transformation to transform the coupled first-order transport equations into the target system. Then, in order to estimate the disturbance, we design a disturbance estimator by ADRC method. Furthermore, combining the backstepping method and disturbance estimator, the controller is presented in this section. Section III focuses on proving the well-posedness of the closed-loop system. The ISS of the closed-

loop system is shown in Section IV. The main results are verified in Section V by numerical simulations.

## II. CONTROL DESIGN

### A. Backstepping transformation

Now we introduce the following backstepping transformation:

$$\begin{aligned} \bar{\sigma}(\bar{\varrho}, \bar{\omega}) &= \bar{\lambda}(\bar{\varrho}, \bar{\omega}) - \int_0^{\bar{\varrho}} k(\bar{\varrho}, \iota) \bar{\lambda}(\iota, \bar{\omega}) d\iota \\ &\quad - \int_0^{\bar{\varrho}} l(\bar{\varrho}, \iota) \bar{\kappa}(\iota, \bar{\omega}) d\iota + m(\bar{\varrho}) \bar{\vartheta}(0, \bar{\omega}), \end{aligned} \quad (2)$$

and its inverse transformation:

$$\begin{aligned} \bar{\lambda}(\bar{\varrho}, \bar{\omega}) &= \bar{\sigma}(\bar{\varrho}, \bar{\omega}) + \int_0^{\bar{\varrho}} p(\bar{\varrho}, \iota) \bar{\sigma}(\iota, \bar{\omega}) d\iota \\ &\quad + \int_0^{\bar{\varrho}} q(\bar{\varrho}, \iota) \bar{\kappa}(\iota, \bar{\omega}) d\iota - n(\bar{\varrho}) \bar{\vartheta}(0, \bar{\omega}), \end{aligned}$$

where kernel functions  $k(\bar{\varrho}, \iota)$ ,  $l(\bar{\varrho}, \iota)$ ,  $m(\bar{\varrho})$ ,  $p(\bar{\varrho}, \iota)$ ,  $q(\bar{\varrho}, \iota)$ ,  $n(\bar{\varrho})$  satisfy following equations on  $\Omega = \{(\bar{\varrho}, \iota) \in \mathbb{R}^2 \mid 0 \leq \iota \leq \bar{\varrho} \leq 1\}$ :

$$\left\{ \begin{aligned} k_{\bar{\varrho}}(\bar{\varrho}, \iota) &= -k_{\iota}(\bar{\varrho}, \iota) + \int_{\iota}^{\bar{\varrho}} k(\bar{\varrho}, \varphi) s_1(\varphi, \iota) d\varphi \\ &\quad - s_1(\bar{\varrho}, \iota) + \int_{\iota}^{\bar{\varrho}} l(\bar{\varrho}, \varphi) s_1(\varphi, \iota) d\varphi, \\ l_{\bar{\varrho}}(\bar{\varrho}, \iota) &= l_{\iota}(\bar{\varrho}, \iota) - \int_{\iota}^{\bar{\varrho}} l(\bar{\varrho}, \varphi) s_1(\varphi, \iota) d\varphi \\ &\quad + s_1(\bar{\varrho}, \iota) - \int_{\iota}^{\bar{\varrho}} k(\bar{\varrho}, \varphi) s_1(\varphi, \iota) d\varphi, \\ l(\bar{\varrho}, \bar{\varrho}) &= 0, \\ k(\bar{\varrho}, 0) &= l(\bar{\varrho}, 0) - m(\bar{\varrho}), \\ m'(\bar{\varrho}) &= am(\bar{\varrho}) - 2al(\bar{\varrho}, 0), \\ m(0) &= a + c, \end{aligned} \right. \quad (3)$$

and

$$\left\{ \begin{aligned} p_{\bar{\varrho}}(\bar{\varrho}, \iota) &= -p_{\iota}(\bar{\varrho}, \iota) + \int_{\iota}^{\bar{\varrho}} q(\bar{\varrho}, \varphi) s_1(\varphi, \iota) d\varphi \\ &\quad - s_1(\bar{\varrho}, \iota) - \int_{\iota}^{\bar{\varrho}} s_1(\bar{\varrho}, \varphi) p(\varphi, \iota) d\varphi \\ &\quad + \int_{\iota}^{\bar{\varrho}} \int_{\iota}^{\varphi} q(\bar{\varrho}, \varphi) s_1(\varphi, \vartheta) p(\vartheta, \iota) d\vartheta d\varphi, \\ q_{\bar{\varrho}}(\bar{\varrho}, \iota) &= q_{\bar{\varrho}}(\bar{\varrho}, \iota) + \int_{\iota}^{\bar{\varrho}} s_1(\bar{\varrho}, \varphi) q(\varphi, \iota) d\varphi \\ &\quad - s_1(\bar{\varrho}, \iota) + \int_{\iota}^{\bar{\varrho}} q(\bar{\varrho}, \varphi) s_1(\varphi, \iota) d\varphi \\ &\quad - \int_{\iota}^{\bar{\varrho}} \int_{\iota}^{\varphi} q(\bar{\varrho}, \varphi) s_1(\varphi, \vartheta) q(\vartheta, \iota) d\vartheta d\varphi, \\ n'(\bar{\varrho}) &= -cn(\bar{\varrho}) + \int_0^{\bar{\varrho}} \int_{\iota}^{\bar{\varrho}} q(\bar{\varrho}, \varphi) s_1(\varphi, \iota) n(\iota) d\varphi d\iota \\ &\quad - (a - c)q(\bar{\varrho}, 0) - \int_0^{\bar{\varrho}} s_1(\bar{\varrho}, \iota) n(\iota) d\iota, \\ p(\bar{\varrho}, 0) &= q(\bar{\varrho}, 0) - n(\bar{\varrho}), \\ q(\bar{\varrho}, \bar{\varrho}) &= 0, \\ n(0) &= a + c, \end{aligned} \right. \quad (4)$$

where  $c > 2a$ . Then, we get

$$\begin{aligned} \bar{\vartheta}_{\bar{\omega}}(0, \bar{\omega}) &= -c\bar{\vartheta}(0, \bar{\omega}) + \bar{\sigma}(0, \bar{\omega}), \\ \bar{\kappa}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= -\bar{\kappa}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) + \int_0^{\bar{\varrho}} s_2(\bar{\varrho}, \iota) \bar{\sigma}(\iota, \bar{\omega}) d\iota \\ &\quad + \int_0^{\bar{\varrho}} s_3(\bar{\varrho}, \iota) \bar{\kappa}(\iota, \bar{\omega}) d\iota + d_1(\bar{\varrho}, \bar{\omega}) \\ &\quad - \int_0^{\bar{\varrho}} s_1(\bar{\varrho}, \iota) n(\iota) \bar{\vartheta}(0, \bar{\omega}) d\iota, \\ \bar{\kappa}(0, \bar{\omega}) &= \bar{\sigma}(0, \bar{\omega}) + (a - c)\bar{\vartheta}(0, \bar{\omega}), \\ \bar{\sigma}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= \bar{\sigma}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) + D(\bar{\varrho}, \bar{\omega}), \\ \bar{\sigma}(1, \bar{\omega}) &= \bar{U}(\bar{\omega}) + d_2(\bar{\omega}) - \int_0^1 k(1, \iota) \bar{\lambda}(\iota, \bar{\omega}) d\iota \\ &\quad - \int_0^1 l(1, \iota) \bar{\kappa}(\iota, \bar{\omega}) d\iota + m(1) \bar{\vartheta}(0, \bar{\omega}), \end{aligned} \quad (5)$$

where  $\bar{\varrho} \in (0, 1)$ ,  $\bar{\omega} > 0$ , and

$$\begin{aligned} s_2(\bar{\varrho}, \iota) &= s_1(\bar{\varrho}, \iota) + \int_{\iota}^{\bar{\varrho}} s_1(\bar{\varrho}, \varphi) p(\varphi, \iota) d\varphi, \\ s_3(\bar{\varrho}, \iota) &= \int_{\iota}^{\bar{\varrho}} s_1(\bar{\varrho}, \varphi) q(\varphi, \iota) d\varphi - s_1(\bar{\varrho}, \iota), \\ D(\bar{\varrho}, \bar{\omega}) &= - \int_0^{\bar{\varrho}} l(\bar{\varrho}, \iota) d_1(\iota, \bar{\omega}) d\iota. \end{aligned}$$

**Proposition 1.** Equations (3) and (4) have smooth solutions  $(k(\bar{\varrho}, \iota), l(\bar{\varrho}, \iota), m(\bar{\varrho}))$ ,  $(p(\bar{\varrho}, \iota), q(\bar{\varrho}, \iota), n(\bar{\varrho})) \in C^1(\Omega) \times C^1(\Omega) \times C^1([0, 1])$ , respectively.

*Proof.* The proof is omitted. The interested reader may refer to [2].  $\square$

Next, we set

$$\begin{aligned}\bar{U}(\bar{\omega}) &= \bar{U}_1(\bar{\omega}) + \bar{U}_2(\bar{\omega}) \\ &= \int_0^1 k(1, \iota) \bar{\lambda}(\iota, \bar{\omega}) d\iota - m(1) \bar{\vartheta}(0, \bar{\omega}) \\ &\quad + \int_0^1 l(1, \iota) \bar{\kappa}(\iota, \bar{\omega}) d\iota + \bar{U}_2(\bar{\omega}),\end{aligned}\quad (6)$$

where  $\bar{U}_2$  is to be designed later. Substituting (6) into (5), we obtain

$$\begin{aligned}\bar{\vartheta}_{\bar{\omega}}(0, \bar{\omega}) &= -c\bar{\vartheta}(0, \bar{\omega}) + \bar{\sigma}(0, \bar{\omega}), \\ \bar{\kappa}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= -\bar{\kappa}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) + \int_0^{\bar{\varrho}} s_2(\bar{\varrho}, \iota) \bar{\sigma}(\iota, \bar{\omega}) d\iota \\ &\quad + \int_0^{\bar{\varrho}} s_3(\bar{\varrho}, \iota) \bar{\kappa}(\iota, \bar{\omega}) d\iota + d_1(\bar{\varrho}, \bar{\omega}) \\ &\quad - \int_0^{\bar{\varrho}} s_1(\bar{\varrho}, \iota) n(\iota) \bar{\vartheta}(0, \bar{\omega}) d\iota,\end{aligned}\quad (7)$$

$$\bar{\kappa}(0, \bar{\omega}) = \bar{\sigma}(0, \bar{\omega}) + (a - c) \bar{\vartheta}(0, \bar{\omega}), \quad (9)$$

$$\bar{\sigma}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) = \bar{\sigma}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) + D(\bar{\varrho}, \bar{\omega}), \quad (10)$$

$$\bar{\sigma}(1, \bar{\omega}) = \bar{U}_2(\bar{\omega}) + d_2(\bar{\omega}). \quad (11)$$

### B. Active disturbance rejection control

Now we use ADRC method (see [5, 12]) to estimate the disturbance.

Define a system operator  $\mathcal{A}_{\bar{\sigma}} : D(\mathcal{A}_{\bar{\sigma}}) \rightarrow L^2(0, 1)$  for (10)-(11) as follows:

$$\begin{cases} \mathcal{A}_{\bar{\sigma}} \bar{g} = \bar{g}', \\ D(\mathcal{A}_{\bar{\sigma}}) = \{\bar{g} \in H^1(0, 1) \mid \bar{g}(1) = 0\}. \end{cases}$$

System (10)-(11) can be represented as:

$$\frac{d}{d\bar{\omega}} \bar{\sigma}(\cdot, \bar{\omega}) = \mathcal{A}_{\bar{\sigma}} \bar{\sigma}(\cdot, \bar{\omega}) + \mathcal{I}D(\cdot, \bar{\omega}) + \mathcal{B}_{\bar{\sigma}} [\bar{U}_2(\bar{\omega}) + d_2(\bar{\omega})], \quad (12)$$

where  $\mathcal{I}$  is an identity operator and  $\mathcal{B}_{\bar{\sigma}} = \delta(\bar{\varrho} - 1)$ .

A direct computation shows that:

$$\begin{cases} \mathcal{A}_{\bar{\sigma}}^* \hat{k} = -\hat{k}', \\ D(\mathcal{A}_{\bar{\sigma}}^*) = \{\hat{k} \in H^1(0, 1) \mid \hat{k}(0) = 0\}. \end{cases}$$

Then, we have

$$\begin{aligned}\frac{d}{d\bar{\omega}} \langle \bar{\sigma}(\cdot, \bar{\omega}), \hat{k} \rangle &= \langle \bar{\sigma}(\cdot, \bar{\omega}), \mathcal{A}_{\bar{\sigma}}^* \hat{k} \rangle + \langle \mathcal{I}D(\cdot, \bar{\omega}), \hat{k} \rangle \\ &\quad + [\bar{U}_2(\bar{\omega}) + d_2(\bar{\omega})] \mathcal{B}_{\bar{\sigma}}^* \hat{k},\end{aligned}\quad (13)$$

where  $\hat{k}(\bar{\varrho})$  is a test function. System (13) is equivalent to

$$\dot{\ell}_1(\bar{\omega}) = \ell_2(\bar{\omega}) + \ell_3(\bar{\omega}) + [\bar{U}_2(\bar{\omega}) + d_2(\bar{\omega})] \hat{k}(1), \quad (14)$$

where

$$\begin{aligned}\ell_1(\bar{\omega}) &= \int_0^1 \bar{\sigma}(\bar{\varrho}, \bar{\omega}) \hat{k}(\bar{\varrho}) d\bar{\varrho}, \\ \ell_2(\bar{\omega}) &= - \int_0^1 \bar{\sigma}(\bar{\varrho}, \bar{\omega}) \hat{k}'(\bar{\varrho}) d\bar{\varrho},\end{aligned}$$

$$\ell_3(\bar{\omega}) = \int_0^1 D(\bar{\varrho}, \bar{\omega}) \hat{k}(\bar{\varrho}) d\bar{\varrho}.$$

Taking specially  $\hat{k}(\bar{\varrho}) = \bar{\varrho} \in D(\mathcal{A}_{\bar{\sigma}}^*)$  in (14), we obtain

$$\dot{\ell}_1(\bar{\omega}) = \ell_2(\bar{\omega}) + \bar{U}_2(\bar{\omega}) + D_2(\bar{\omega}), \quad (15)$$

where

$$\ell_1(\bar{\omega}) = \int_0^1 \bar{\varrho} \bar{\sigma}(\bar{\varrho}, \bar{\omega}) d\bar{\varrho}, \quad \ell_2(\bar{\omega}) = - \int_0^1 \bar{\sigma}(\bar{\varrho}, \bar{\omega}) d\bar{\varrho},$$

$$D_2(\bar{\omega}) = \int_0^1 \bar{\varrho} D(\bar{\varrho}, \bar{\omega}) d\bar{\varrho} + d_2(\bar{\omega}).$$

Next, by using the time-varying gain function  $\bar{h}(\bar{\omega})$ , ESO is designed as:

$$\begin{aligned}\dot{\hat{\ell}}_1(\bar{\omega}) &= \ell_2(\bar{\omega}) + \bar{U}_2(\bar{\omega}) + \hat{D}_2(\bar{\omega}) \\ &\quad - \bar{h}(\bar{\omega}) [\hat{\ell}_1(\bar{\omega}) - \ell_1(\bar{\omega})], \\ \dot{\hat{D}}_2(\bar{\omega}) &= -\bar{h}^2(\bar{\omega}) [\hat{\ell}_1(\bar{\omega}) - \ell_1(\bar{\omega})],\end{aligned}\quad (16)$$

where  $\bar{h}(\bar{\omega})$  satisfies:

$$\begin{cases} \bar{h}(\bar{\omega}) > 0, \quad \dot{\bar{h}}(\bar{\omega}) > 0, \quad \forall \bar{\omega} \geq 0, \\ \bar{h}(\bar{\omega}) \rightarrow \infty \text{ as } \bar{\omega} \rightarrow \infty, \\ \sup_{\bar{\omega} \in [0, \infty)} \left| \frac{\dot{\bar{h}}(\bar{\omega})}{\bar{h}(\bar{\omega})} \right| = \bar{\delta} < \infty \text{ for some } \bar{\delta} > 0, \\ \text{and } \lim_{\bar{\omega} \rightarrow \infty} \frac{e^{\mu \bar{\omega}}}{\bar{h}(\bar{\omega})} = 0. \end{cases}\quad (17)$$

Then, we obtain

**Lemma 1.** Assume  $\hat{\ell}_1, \hat{D}_2$  are solutions of (16), and  $D_2(\bar{\omega})$  satisfies  $|\dot{D}_2(\bar{\omega})| \leq C e^{\bar{\nu} \bar{\omega}}$ ,  $\bar{\omega} \geq 0$ , where  $C, \bar{\nu} > 0$  are constants. If  $\bar{h}(\bar{\omega}) = k e^{\nu \bar{\omega}}$  with  $k > 0, \nu > \bar{\nu}$ , we can find constants  $k_0, \epsilon > 0$  so that

$$\bar{h}^2(\bar{\omega}) |\hat{\ell}_1(\bar{\omega}) - \ell_1(\bar{\omega})|^2 + |\hat{D}_2(\bar{\omega}) - D_2(\bar{\omega})|^2 \leq k_0 e^{-\epsilon \bar{\omega}}. \quad (18)$$

*Proof.* Due to space limits, the proof is omitted. The interested reader may refer to [5].  $\square$

By Lemma 1, let

$$\bar{U}_2(\bar{\omega}) = -\hat{D}_2(\bar{\omega}), \quad (19)$$

then, we get

$$\begin{cases}
\bar{\vartheta}_{\bar{\omega}}(0, \bar{\omega}) &= a\bar{\vartheta}(0, \bar{\omega}) + \bar{\lambda}(0, \bar{\omega}), \\
\bar{\kappa}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= -\bar{\kappa}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) + d_1(\bar{\varrho}, \bar{\omega}) \\
&\quad + \int_0^{\bar{\varrho}} \varsigma_1(\bar{\varrho}, \iota)(\bar{\lambda}(\iota, \bar{\omega}) - \bar{\kappa}(\iota, \bar{\omega}))d\iota, \\
\bar{\kappa}(0, \bar{\omega}) &= \bar{\lambda}(0, \bar{\omega}) + 2a\bar{\vartheta}(0, \bar{\omega}), \\
\bar{\lambda}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= \bar{\lambda}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) \\
&\quad + \int_0^{\bar{\varrho}} \varsigma_1(\bar{\varrho}, \iota)(\bar{\lambda}(\iota, \bar{\omega}) - \bar{\kappa}(\iota, \bar{\omega}))d\iota, \\
\bar{\lambda}(1, \bar{\omega}) &= \int_0^1 k(1, \iota)\bar{\lambda}(\iota, \bar{\omega})d\iota - m(1)\bar{\vartheta}(0, \bar{\omega}) \\
&\quad + \int_0^1 l(1, \iota)\bar{\kappa}(\iota, \bar{\omega})d\iota - \hat{D}_2(\bar{\omega}) + d_2(\bar{\omega}), \\
\dot{\hat{\ell}}_1(\bar{\omega}) &= \ell_2(\bar{\omega}) - \hbar(\bar{\omega})[\hat{\ell}_1(\bar{\omega}) - \ell_1(\bar{\omega})], \\
\dot{\hat{D}}_2(\bar{\omega}) &= -\hbar^2(\bar{\omega})[\hat{\ell}_1(\bar{\omega}) - \ell_1(\bar{\omega})], \\
\bar{\lambda}(\bar{\varrho}, 0) &= \bar{\lambda}_0(\bar{\varrho}), \bar{\kappa}(\bar{\varrho}, 0) = \bar{\kappa}_0(\bar{\varrho}), \bar{\vartheta}(0, 0) = \bar{\vartheta}_0, \\
\hat{\ell}_1(0) &= \hat{\ell}_1(0), \hat{D}_2(0) = \hat{D}_2(0),
\end{cases} \quad (20)$$

where

$$\ell_1(\bar{\omega}) = \int_0^1 \bar{\varrho}\bar{\sigma}(\bar{\varrho}, \bar{\omega})d\bar{\varrho}, \quad \ell_2(\bar{\omega}) = -\int_0^1 \bar{\sigma}(\bar{\varrho}, \bar{\omega})d\bar{\varrho}.$$

Consider system (20) in  $\mathcal{H}_1 = \mathbb{R} \times L^2(0, 1) \times L^2(0, 1) \times \mathbb{R}^2$ .

### III. WELL-POSEDNESS OF THE SYSTEM (20)

Using the error variables

$$\tilde{\ell}_1(\bar{\omega}) = \hbar(\bar{\omega})[\hat{\ell}_1(\bar{\omega}) - \ell_1(\bar{\omega})], \quad \tilde{D}_2(\bar{\omega}) = \hat{D}_2(\bar{\omega}) - D_2(\bar{\omega}) \quad (21)$$

and the invertible transformation (2), we get the following equivalent system:

$$\bar{\vartheta}_{\bar{\omega}}(0, \bar{\omega}) = -c\bar{\vartheta}(0, \bar{\omega}) + \bar{\sigma}(0, \bar{\omega}), \quad (22)$$

$$\begin{aligned}
\bar{\kappa}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) &= -\bar{\kappa}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) + \int_0^{\bar{\varrho}} \varsigma_2(\bar{\varrho}, \iota)\bar{\sigma}(\iota, \bar{\omega})d\iota \\
&\quad + \int_0^{\bar{\varrho}} \varsigma_3(\bar{\varrho}, \iota)\bar{\kappa}(\iota, \bar{\omega})d\iota + d_1(\bar{\varrho}, \bar{\omega}) \\
&\quad - \int_0^{\bar{\varrho}} \varsigma_1(\bar{\varrho}, \iota)n(\iota)\bar{\vartheta}(0, \bar{\omega})d\iota, \quad (23)
\end{aligned}$$

$$\bar{\kappa}(0, \bar{\omega}) = \bar{\sigma}(0, \bar{\omega}) + (a-c)\bar{\vartheta}(0, \bar{\omega}), \quad (24)$$

$$\bar{\sigma}_{\bar{\omega}}(\bar{\varrho}, \bar{\omega}) = \bar{\sigma}_{\bar{\varrho}}(\bar{\varrho}, \bar{\omega}) + D(\bar{\varrho}, \bar{\omega}), \quad (25)$$

$$\bar{\sigma}(1, \bar{\omega}) = -\hat{D}_2(\bar{\omega}) + d_2(\bar{\omega}), \quad (26)$$

$$\begin{aligned}
\dot{\tilde{\ell}}_1(\bar{\omega}) &= -\hbar(\bar{\omega})[\tilde{\ell}_1(\bar{\omega}) - \tilde{D}_2(\bar{\omega})] \\
&\quad + \frac{\dot{\hbar}(\bar{\omega})}{\hbar(\bar{\omega})}\tilde{\ell}_1(\bar{\omega}), \quad (27)
\end{aligned}$$

$$\dot{\tilde{D}}_2(\bar{\omega}) = -\hbar(\bar{\omega})\tilde{\ell}_1(\bar{\omega}) - \dot{\tilde{D}}_2(\bar{\omega}). \quad (28)$$

The ‘ $(\tilde{\ell}_1, \tilde{D}_2)$  part’ of (22)-(28) is ODEs, so we only need to show the well-posedness of system (22)-(26).

Consider system (22)-(26) in  $\mathcal{H}_2 = \mathbb{R} \times L^2(0, 1) \times L^2(0, 1)$ , and the inner product of this space is

$$\begin{aligned}
&\langle (X_1, \bar{y}_1, \bar{g}_1)^\top, (X_2, \bar{y}_2, \bar{g}_2) \rangle_{\mathcal{H}_2} \\
&= cX_1X_2 + \int_0^1 \bar{y}_1(\bar{\varrho})\bar{y}_2(\bar{\varrho})d\bar{\varrho} + c_1 \int_0^1 \bar{g}_1(\bar{\varrho})\bar{g}_2(\bar{\varrho})d\bar{\varrho},
\end{aligned}$$

where  $((X_1, \bar{y}_1, \bar{g}_1), (X_2, \bar{y}_2, \bar{g}_2)) \in \mathcal{H}_2$  and  $c_1 \geq 2$ .

The operator of system (22)-(26) is:

$$\begin{cases}
\mathcal{A}(X, \bar{y}, \bar{b}) = (-cX + \bar{b}(0), -\bar{y}', -\bar{b}'), \\
D(\mathcal{A}) = \left\{ (X, \bar{y}, \bar{b}) \in \mathbb{R} \times H^1(0, 1) \times H^1(0, 1) \mid \right. \\
\left. \bar{y}(0) = \bar{b}(0) + (a-c)X, \bar{b}(1) = 0 \right\}. \quad (29)
\end{cases}$$

Then,

$$\begin{cases}
\mathcal{A}^*(Y, f, k) = (-cY + \frac{a-c}{c}f(0), f', -k'), \\
D(\mathcal{A}^*) = \left\{ (Y, f, k) \in \mathbb{R} \times H^1(0, 1) \times H^1(0, 1) \mid \right. \\
\left. f(1) = 0, k(0) = cY + f(0) \right\}.
\end{cases}$$

Hence, system (22)-(26) can be expressed as:

$$\begin{aligned}
\frac{d}{d\bar{\omega}}\Theta(\cdot, \bar{\omega}) &= \mathcal{A}\Theta(\cdot, \bar{\omega}) + \mathcal{B}_1\Theta(\cdot, \bar{\omega}) + \mathcal{I}(0, d_1(\cdot, \bar{\omega}), \\
&\quad D(\cdot, \bar{\omega}))^\top + \mathcal{B}_2(-\hat{D}_2(\bar{\omega}) + d_2(\bar{\omega})), \quad (30)
\end{aligned}$$

where  $\Theta(\cdot, \bar{\omega}) = [\bar{\vartheta}(0, \bar{\omega}), \bar{\kappa}(\cdot, \bar{\omega}), \bar{\sigma}(\cdot, \bar{\omega})]^\top$ ,  $\mathcal{I}$  is an identity operator and

$$\begin{aligned}
\mathcal{B}_1\Theta(\cdot, \bar{\omega}) &= \left( 0, \int_0^{\bar{\varrho}} \varsigma_3(\bar{\varrho}, \iota)\bar{\kappa}(\iota, \bar{\omega})d\iota \right. \\
&\quad + \int_0^{\bar{\varrho}} \varsigma_2(\bar{\varrho}, \iota)\bar{\sigma}(\iota, \bar{\omega})d\iota \\
&\quad \left. - \int_0^{\bar{\varrho}} \varsigma_1(\bar{\varrho}, \iota)n(\iota)\bar{\vartheta}(0, \bar{\omega})d\iota, 0 \right), \\
\mathcal{B}_2 &= (0, 0, \delta(\bar{\varrho} - 1)). \quad (31)
\end{aligned}$$

**Theorem 1.** Given  $T > 0$  and  $(\bar{\vartheta}_0, \bar{\kappa}_0(\cdot), \bar{\sigma}(\cdot, 0)) \in \mathcal{H}_2$ , system (22)-(26) exists the unique mild solution  $(\bar{\vartheta}(0, \cdot), \bar{\kappa}, \bar{\sigma}) \in C(0, T; \mathcal{H}_2)$ .

*Proof.* The well-posedness of system (22)-(26) can be shown through the method of  $C_0$ -semigroup, and the details are omitted.  $\square$

Then, owing to the equivalence of systems (20) and (22)-(28), we obtain:

**Theorem 2.** Given  $T > 0$  and  $(\bar{\vartheta}_0, \bar{\lambda}_0, \bar{\kappa}_0, \hat{\ell}_1(0), \hat{D}_2(0)) \in \mathcal{H}_1$ , system (20) has the unique mild solution  $(\bar{\vartheta}(0, \cdot), \bar{\lambda}, \bar{\kappa}, \hat{\ell}_1, \hat{D}_2) \in C(0, T; \mathcal{H}_1)$ .

#### IV. INPUT-TO-STATE STABILITY

**Theorem 3.** Assume the disturbance  $D_2$  satisfies  $|\dot{D}_2(\bar{\omega})| \leq C e^{\bar{\omega}}$  and  $h(\bar{\omega})$  satisfies (17). Then, for any initial value  $(\bar{\vartheta}_0, \bar{\lambda}_0, \bar{\kappa}_0) \in \mathcal{H}_2$ , system (20) is ISS, that is, there are positive constants  $E_1, E_2$  and  $E_3$  independent of  $d_1, d_2$  and  $(\bar{\vartheta}_0, \bar{\lambda}_0, \bar{\kappa}_0)$ , so that

$$\begin{aligned} & \|(\bar{\vartheta}(0, \bar{\omega}), \bar{\lambda}(\cdot, \bar{\omega}), \bar{\kappa}(\cdot, \bar{\omega}))\|_{\mathcal{H}_2}^2 \\ & \leq E_1 e^{-E_2 \bar{\omega}} \|(\bar{\vartheta}_0, \bar{\lambda}_0, \bar{\kappa}_0)\|_{\mathcal{H}_2}^2 \\ & \quad + E_3 \sup_{0 \leq s \leq \bar{\omega}} (\|d_1(\cdot, s)\|_{L^2(0,1)}^2). \end{aligned}$$

*Proof.* Using variables  $\tilde{\ell}_1, \tilde{D}_2$  given by (21) and the invertible transformation (2), we can obtain system (22)-(28), which is the equivalent system of (20). The energy of system (22)-(26) is

$$E(\bar{\omega}) = c_0 \int_0^1 \bar{\kappa}^2(\bar{\varrho}, \bar{\omega}) d\bar{\varrho} + \int_0^1 \bar{\sigma}^2(\bar{\varrho}, \bar{\omega}) d\bar{\varrho} + c \bar{\vartheta}^2(0, \bar{\omega}).$$

Let

$$\begin{aligned} \hat{V}(\bar{\omega}) &= \frac{1}{2} c \bar{\vartheta}^2(0, \bar{\omega}) + \frac{1}{2} \int_0^1 \lambda e^{-\rho \bar{\varrho}} \bar{\kappa}^2(\bar{\varrho}, \bar{\omega}) d\bar{\varrho} \\ & \quad + \frac{1}{2} \int_0^1 \sigma e^{\mu \bar{\varrho}} \bar{\sigma}^2(\bar{\varrho}, \bar{\omega}) d\bar{\varrho}, \end{aligned} \quad (32)$$

where parameters  $\lambda, \rho, \mu, \sigma > 0$ . Applying Young's and the Cauchy-Schwarz inequalities[13], we find

$$\delta_1 E(\bar{\omega}) \leq \hat{V}(\bar{\omega}) \leq \delta_2 E(\bar{\omega}),$$

where  $\delta_1 = \frac{\min\{\frac{1}{2}, \frac{\sigma}{2}, \frac{\lambda e^{-\rho}}{2}\}}{\max\{c, c_0 + 1\}}$  and  $\delta_2 = \frac{\max\{\frac{1}{2}, \frac{\lambda}{2}, \frac{\sigma e^{\mu}}{2}\}}{\min\{1, c, c_0\}}$ . Let

$$\begin{aligned} M_1 &> \max \left\{ \max_{0 \leq \iota \leq \bar{\varrho} \leq 1} \{|\varsigma_2(\bar{\varrho}, \iota)|\}, \max_{0 \leq \iota \leq \bar{\varrho} \leq 1} \{|\varsigma_3(\bar{\varrho}, \iota)|\}, \right. \\ & \quad \left. \max_{0 \leq \iota \leq \bar{\varrho} \leq 1} \{|\varsigma_1(\bar{\varrho}, \iota)n(\iota)|\} \right\}. \end{aligned}$$

Then,

$$\begin{aligned} \dot{\hat{V}}(\bar{\omega}) &\leq -\eta \hat{V}(\bar{\omega}) + 4\lambda \int_0^1 d_1^2(\bar{\varrho}, \bar{\omega}) d\bar{\varrho} \\ & \quad + \frac{(2+\mu)\sigma e^{\mu}}{\mu} \int_0^1 D^2(\bar{\varrho}, \bar{\omega}) d\bar{\varrho} + \sigma e^{\mu} k_0 e^{-\epsilon \bar{\omega}}, \end{aligned}$$

where

$$\begin{aligned} \eta &= \min \left\{ \frac{3}{2}c - \frac{3\lambda(a-c)^2}{c} - \frac{2b^2\lambda M_1}{c\rho}, \frac{3\mu}{4} - \frac{2b^2\lambda M_1}{\sigma\rho}, \right. \\ & \quad \left. \rho\lambda - \frac{3b^2\lambda M_1}{8} - \frac{2b^2\lambda M_1}{\rho} - \frac{\lambda}{8} \right\} > 0, \end{aligned}$$

and  $\sigma$  and  $\lambda$  satisfy

$$\frac{\sigma}{2} - \frac{3\lambda}{2} - 2 > 0.$$

Therefore, due to the fact that  $D$  is a disturbance dependent on  $d_1$ , we find a positive constant  $\hat{\iota}$  so that

$$\dot{\hat{V}}(\bar{\omega}) \leq -\eta \hat{V}(\bar{\omega}) + \hat{\iota} \|d_1(\cdot, \bar{\omega})\|_{L^2(0,1)}^2 + \sigma e^{\mu} k_0 e^{-\epsilon \bar{\omega}}.$$

Then,

$$\begin{aligned} & \|\bar{\kappa}(\cdot, \bar{\omega})\|^2 + \|\bar{\sigma}(\cdot, \bar{\omega})\|^2 + c|\bar{\vartheta}(0, \bar{\omega})|^2 \\ & \leq \frac{\delta_2}{\delta_1} e^{-\eta \bar{\omega}} \left( \|\bar{\kappa}(\cdot, 0)\|^2 + \|\bar{\sigma}(\cdot, 0)\|^2 + c|\bar{\vartheta}(0, 0)|^2 \right) \\ & \quad + \frac{\hat{\iota}}{\delta_1 \eta} \sup_{0 \leq s \leq \bar{\omega}} (\|d_1(\cdot, s)\|_{L^2(0,1)}^2) \\ & \quad + \frac{\sigma e^{\mu} k_0}{\delta_1 \eta} \frac{1}{\eta - \epsilon} e^{-\epsilon \bar{\omega}}. \end{aligned}$$

Next, we find a positive constant  $A_1$  so that

$$\begin{aligned} & \|\bar{\kappa}(\cdot, \bar{\omega})\|^2 + \|\bar{\sigma}(\cdot, \bar{\omega})\|^2 + c|\bar{\vartheta}(0, \bar{\omega})|^2 \\ & \leq A_1 e^{-\omega \bar{\omega}} \left( \|\bar{\kappa}(\cdot, 0)\|^2 + \|\bar{\sigma}(\cdot, 0)\|^2 + c|\bar{\vartheta}(0, 0)|^2 \right) \\ & \quad + \frac{\hat{\iota}}{\delta_1 \eta} \sup_{0 \leq s \leq \bar{\omega}} (\|d_1(\cdot, s)\|_{L^2(0,1)}^2), \end{aligned} \quad (33)$$

where

$$\omega = \min\{\eta, \epsilon\}.$$

The invertibility and boundedness of transformation (2) imply that we can find constants  $\nu_i > 0 (i = 1 - 6)$  so that

$$\begin{aligned} \|\bar{\sigma}(\cdot, \bar{\omega})\|^2 &\leq \nu_1 \|\bar{\lambda}(\cdot, \bar{\omega})\|^2 + \nu_2 \|\bar{\kappa}(\cdot, \bar{\omega})\|^2 + \nu_3 |\bar{\vartheta}(0, \bar{\omega})|^2, \\ \|\bar{\lambda}(\cdot, \bar{\omega})\|^2 &\leq \nu_4 \|\bar{\sigma}(\cdot, \bar{\omega})\|^2 + \nu_5 \|\bar{\kappa}(\cdot, \bar{\omega})\|^2 + \nu_6 |\bar{\vartheta}(0, \bar{\omega})|^2. \end{aligned}$$

Hence,

$$\begin{aligned} & \nu_7 \left( \|\bar{\sigma}(\cdot, \bar{\omega})\|^2 + \|\bar{\kappa}(\cdot, \bar{\omega})\|^2 + c|\bar{\vartheta}(0, \bar{\omega})|^2 \right) \\ & \leq \|\bar{\kappa}(\cdot, \bar{\omega})\|^2 + \|\bar{\lambda}(\cdot, \bar{\omega})\|^2 + c|\bar{\vartheta}(0, \bar{\omega})|^2 \\ & \leq \nu_8 \left( \|\bar{\sigma}(\cdot, \bar{\omega})\|^2 + \|\bar{\kappa}(\cdot, \bar{\omega})\|^2 + c|\bar{\vartheta}(0, \bar{\omega})|^2 \right), \end{aligned} \quad (34)$$

where

$$\begin{aligned} \nu_7 &= \frac{1}{\max\left\{\nu_1, \nu_2 + 1, \frac{\nu_3 + c}{c}\right\}}, \\ \nu_8 &= \max\left\{\nu_4, \nu_5 + 1, \frac{\nu_6 + c}{c}\right\}. \end{aligned}$$

Then,

$$\begin{aligned} & \|(\bar{\vartheta}(0, \bar{\omega}), \bar{\lambda}(\cdot, \bar{\omega}), \bar{\kappa}(\cdot, \bar{\omega}))\|_{\mathcal{H}_2}^2 \\ & \leq E_1 e^{-E_2 \bar{\omega}} \|(\bar{\vartheta}_0, \bar{\lambda}_0, \bar{\kappa}_0)\|_{\mathcal{H}_2}^2 \\ & \quad + E_3 \sup_{0 \leq s \leq \bar{\omega}} (\|d_1(\cdot, s)\|^2), \end{aligned}$$

where

$$E_1 = \frac{\nu_8 A_1}{\nu_7}, \quad E_2 = \omega, \quad E_3 = \frac{\nu_8 \hat{\iota}}{\delta_1 \eta}.$$

The proof is complete.  $\square$

#### V. NUMERICAL EXAMPLES

In this section, we apply the finite difference method to discretize equations. The time-space step variation  $(\Delta \bar{\omega}, \Delta \bar{\varrho}) = (0.002, 0.02)$ . Let  $a = 0.9, b = 1, c = 4, k_1 = 3$  and  $v = 0.1$ . Set  $d_1(\bar{\varrho}, \bar{\omega}) = 3 \sin(2\bar{\varrho}) + 2 \cos(\bar{\omega})$  and  $d_2(\bar{\omega}) = 2 \cos(\bar{\omega})$ . We choose  $\bar{\vartheta}_0 = 0.5, \bar{\lambda}_0(\bar{\varrho}) = \bar{\varrho}^2$  and  $\bar{\kappa}_0(\bar{\varrho}) = 2\bar{\varrho}^3 + 5\bar{\varrho}$ . Fig. 1 shows that system (20) is ISS. Fig. 2 shows that estimation  $\hat{D}_2$  tracks well the true value of disturbance  $D_2$ .

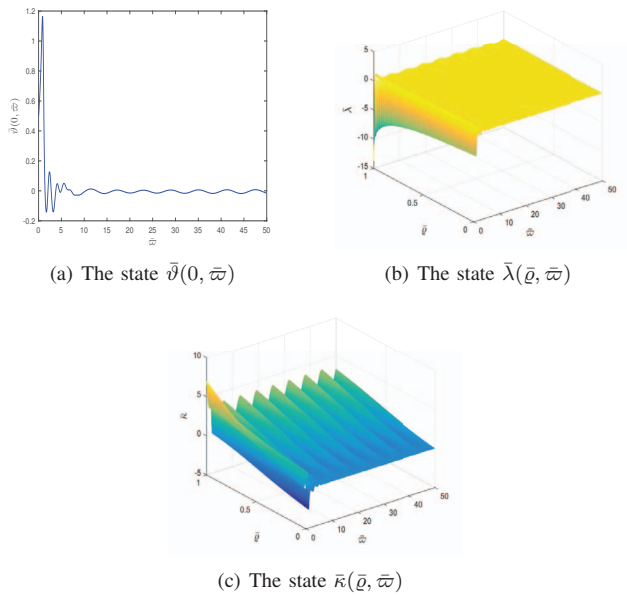


Fig. 1: The states of close-loop system (20)

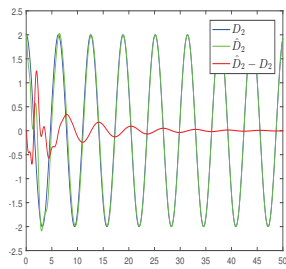


Fig. 2: Disturbance  $D_2(\bar{\omega})$ , its estimation  $\hat{D}_2(\bar{\omega})$  and the error of  $\hat{D}_2(\bar{\omega})$  and  $D_2(\bar{\omega})$

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