CHAPTER 7

LONG PERIOD OSCILLATIONS IN BASINS OF ARBITRARY SHAPES

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ABSTRACT

The equation of continuity and the equation of motion for two-dimensional shallow-water waves are transformed into a set of difference equations which can be solved numerically to obtain the fundamental period and the periods of the higher modes of oscillation of a closed basin of arbitrary shape. From these periods and the difference equations the corresponding amplitude variations are obtained. By modifying the boundary conditions used, with certain restrictions, the modes of oscillation of a fully open harbor can be determined. The data necessary for the solution of both cases are the distribution of the cross section areas and the surface widths along the talweg of the basin (the line of maximum depth).

The exact solutions which describe the lowest mode of oscillation of basins of simple geometry, e.g., circular and eliptical, are compared to the results of this approach. In addition the fundamental and the higher modes of oscillation of three actual harbors are discussed. The agreement obtained by this method compared to others is considered to be reasonably good. Some considerations of the case of three-dimensional forced oscillations are presented.

INTRODUCTION

The study of the free oscillations of closed bodies of water (known as seiche) has aroused the interest of investigators for many years. Considering first a closed rectangular basin of constant depth, the periods of the free oscillations of shallow water waves can be derived from the solution of the Laplace equation for the velocity potential of the fluid, continuity considerations, and the appropriate boundary conditions. Merian's equation developed in this way for the special case of a narrow basin having predominantly longitudinal long-period oscillations, is:

$$T = \frac{2\ell}{n\sqrt{gh}} \qquad (n = 1, 2, 3, \ldots . N) \qquad (1)$$

where T is the period of the oscillation, ℓ is the length of the basin, h is the depth of the basin, and n represents the harmonic order.

Although this expression is sufficient to predict the modes of oscillation of narrow basins which are approximately both rectangular and of constant depth, variations from this shape necessitate a more general approach. Numerous investigators have studied this problem, and only the results of a few will be briefly discussed here. However, the interested reader is referred to Defant (1961) for a detailed summary of various analytical methods.

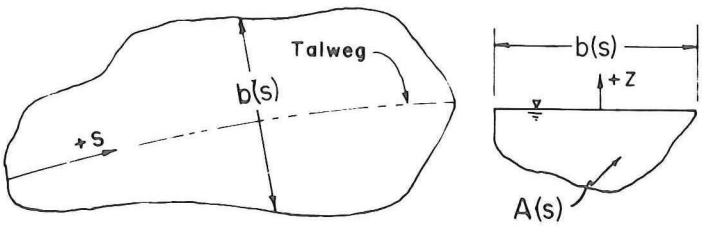


Figure 1. Definition Sketch of Arbitrary Shaped Basin

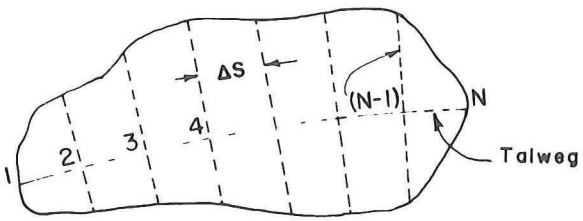


Figure 2. Definition Sketch for Numerical Analysis

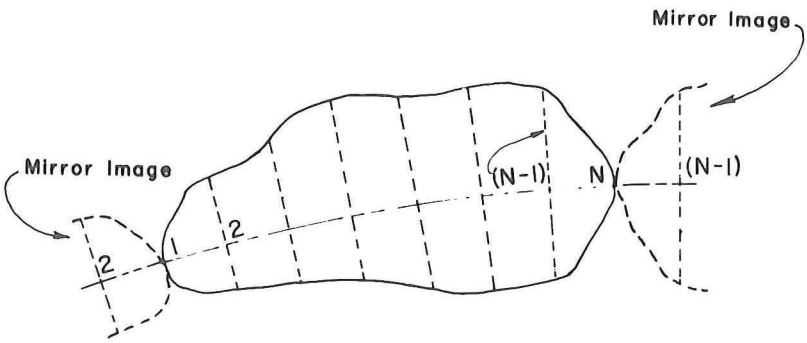


Figure 3. Definition Sketch for Boundary Conditions

$$\frac{\partial}{\partial s}\left[A\frac{\partial \eta}{\partial s}\right] + \frac{b\sigma^2}{g}\eta = 0 \qquad (9)$$

wherein the water surface amplitude η is now only a function of s. Since for the two-dimensional case, variations are in the s-direction alone, this can be expanded to:

$$A\frac{d^2\eta}{ds^2} + \frac{dA}{ds}\frac{d\eta}{ds} + \lambda b\eta = 0 \qquad (10)$$

where λ is a characteristic value (eigenvalue) of the problem and the expression:

$$\lambda_m = \frac{4\pi^2}{gT_m^2} \quad , \; m = 1, 2, \ldots . N \qquad (11)$$

relates a particular characteristic value, λ_m, to the corresponding period T_m where the subscript m = 1, refers to the period of the fundamental mode of oscillation and other values refer to the higher modes of oscillation.

The objective of this investigation is the solution of Eq. 10 with the appropriate boundary conditions for an arbitrary variation in the s-direction of both the area A and the surface width b. This solution will yield the eigenvalues which describe the periods of the modes of oscillation as well as the variation of the amplitude in the s-direction.

For the case of a completely closed basin the obvious boundary condition is a zero water particle velocity normal to the boundaries of the basin. For a vertical boundary in the two-dimensional problem this condition (u = 0) can be replaced by the condition that the slope of the water surface is zero at the boundary, i.e. $\frac{\partial \eta}{\partial s} = 0$. In other words the water surface amplitude is a maximum or a minimum at a solid vertical boundary.

In the investigation of closed basins the criterion of a maximum or minimum amplitude at the basin ends was used as the boundary condition (at s = 0 and s = ℓ in Fig. 1). Admittedly this type of boundary condition becomes approximate when the basin boundaries are inclined at a small angle to the horizontal; however, for the case of a circular basin with a parabolic bottom this approach gave surprisingly good results for the fundamental mode. This will be discussed more fully later.

For a basin which communicates with the open-sea the boundary condition at the entrance used in the analysis was that this section is a nodal region, and the amplitude at that section is zero for all time. Hence, this approach neglects the influence of the entrance and any degree of constriction of the entrance on the induced motion within the harbor, and therefore, it is applicable only to fully open harbors where the basin width is small compared to the length. This also will be discussed later with reference to the applicability of this method to actual harbors.

where n = 1, 2, 3, ...N
m = 1, 2, 3, ...N

Eq. 14a then represents a set of equations which must be solved simultaneously to yield the eigenvalues of the problem Λ_m (or the modal periods T_m).

CLOSED BASIN

A sketch is presented in Fig. 3 showing mirror images attached to the basin at Sections 1 and N (the areas and surface widths at Section 2 and N-1 in the images are equal to the corresponding sections in the basin). These images facilitate the application of the closed basin boundary conditions previously mentioned. By making the amplitude at the corresponding sections in the basin and the images the same, a maximum or minimum amplitude is imposed at Sections 1 and N. Using difference quotients the boundary conditions become:

$$\text{at } s = 0 \ (n = 1) \text{ and at } s = \ell \ (n = N)$$

$$\eta_{n-1} = \eta_{n+1} \tag{15}$$

The set of equations obtained from Eqs. 14 and the appropriate boundary conditions is:

$$\begin{aligned}
a_{11}\eta_1 - a_{11}\eta_2 &= \Lambda_m \eta_1 \\
a_{21}\eta_1 + a_{22}\eta_2 + a_{23}\eta_3 &= \Lambda_m \eta_2 \\
a_{32}\eta_2 + a_{33}\eta_3 + a_{34}\eta_4 &= \Lambda_m \eta_3 \\
&\vdots \\
-a_{NN}\eta_{N-1} + a_{NN}\eta_N &= \Lambda_m \eta_N
\end{aligned} \tag{16}$$

The eigenvalues, Λ_m, of this set of equations can be evaluated by finding the eigenvalues of the matrix which is formed from the coefficients of the terms on the left hand side (see Hildebrand, 1958). The matrix of coefficients is:

restrictions it can be applied to the case of a fully open harbor connected to the open-sea. Ippen and Goda (1963) have shown that for a fully open rectangular harbor with an aspect ratio (the ratio of the harbor width to the harbor length) less than 0.1 the basin acts essentially as a quarter wave length resonator. This means that for these dimensions the fully open harbor will have a node at the entrance for all resonant modes of oscillation. Assuming that an irregularly shaped harbor of an aspect ratio less than 0.1 acts in a similar way, this then implies that a simple modification can be made to Eq. 16 still using the same concept of mirror images. To impose a condition of zero amplitude at Section N, the boundary condition:

$$\eta_{N-1} = -\eta_{N+1} \tag{19}$$

is chosen (where Section N + 1 is the first section in the mirror image attached to the entrance). This approach essentially forces the amplitude at Section N to be zero. Incorporating this condition in the solution the last equation in the set Eq. 16 becomes:

$$-a^{*}_{NN}\,\eta_{N-1} + a_{NN}\eta_N = \Lambda_m \eta_N \tag{20}$$

where $$a^{*}_{NN} = \frac{A_{N-1} - A_N}{b_N}$$

Therefore, the matrix presented as Eq. 17 can be modified simply by substituting the coefficient $-a^{*}_{NN}$ for the coefficient $-a_{NN}$ in the Nth row (N - 1)th column keeping all other coefficients the same and then solving this for the eigenvalues.

With the origin of sections at the closed end of the basin the recursion formulas Eq. 18a and 18b used to determine the amplitude variation remain the same. Eq. 18c becomes:

$$\frac{\eta_N}{\eta_1} = \left[\frac{a^{*}_{NN}/a_{NN}}{1 - \frac{\Lambda_m}{a_{NN}}} \right] \frac{\eta_{N-1}}{\eta_1} \tag{21}$$

The confirmation of this approach lies in the results obtained which will be discussed later.

ALTERNATE PROCEDURE FOR OBTAINING EIGENVALUES

For an equation of the form:

$$\frac{d}{dx}\left(p \frac{dy}{dx}\right) + (q + \lambda r) y = 0 \tag{22}$$

the Rayleigh-Ritz method of determining the eigenvalues can be used (see Kopal, 1961). This method is somewhat similar to the method of

Let:
$$F = A\left(\frac{d\eta}{ds}\right)^2 \tag{26a}$$

$$G = b\eta^2 \tag{26b}$$

Then by Simpson's Rule the numerator of Eq. 24 can be expressed:

$$\int_o^\ell F\,ds = \frac{\Delta s}{3}\left\{F_1 + 4F_2 + 2F_3 + \ldots\ldots + 2F_{N-2} + 4F_{N-1} + F_N\right\} \tag{27}$$

A similar expression can be written for the quantity G. Only the solution for the closed basin will be considered, since the only difference between the open and closed basin is the substitution of Eq. 25b instead of Eq. 25a in Eq. 24. Substituting Eq. 25a into Eqs. 26a and 26b one obtains:

$$F_n = \eta_1^2\, A_n \frac{\pi^2}{\ell^2} \sin^2\left[\frac{(n-1)\,\pi\,\Delta s}{\ell}\right] \tag{28a}$$

$$G_n = \eta_1^2\, b_n \cos^2\left[\frac{(n-1)\,\pi\,\Delta s}{\ell}\right] \tag{28b}$$

Therefore, from Eqs. 28a, 28b, 26, and 23, as a first approximation the smallest eigenvalue for a closed basin is:

$$\lambda_1 = \frac{\pi^2}{\ell^2}\left\{4A_2 \sin^2\left(\pi\frac{\Delta s}{\ell}\right) + 2A_3 \sin^2\left(2\pi\frac{\Delta s}{\ell}\right) + \ldots. + 4A_{N-1} \sin^2\left[(N-2)\,\pi\frac{\Delta s}{\ell}\right]\right\} \div \left\{b_1 + 4b_2 \cos^2\left(\frac{\pi\,\Delta s}{\ell}\right) + \ldots + b_N\right\} \tag{29}$$

This expression can be set up quite simply for rapid solution by means of a desk calculator.

A better approximation to λ_1, as well as approximations to the next two eigenvalues, could be obtained by using the following expression for the amplitude:

$$\eta_n = a_o \cos\theta_n + a_1 \cos 2\theta_n + a_2 \cos 3\theta_n$$

$$\text{where } \theta_n = \frac{(n-1)\pi\,\Delta s}{\ell} \tag{30}$$

However, it can be seen from this expression and the previous development that this method rapidly loses its simplicity and ease of computation as the order of approximation increases.

Each of the basins described in Table 1 was divided into 21 sections with a spacing between each section (excluding the first and last sections) of $\Delta s = 10.0$ ft. Due to the truncation shown in Table 1 the distance between the first and the second sections and the next to last and the last sections was somewhat less than 10 ft. In the three cases where the surface width of the first section was 1 ft. this distance was 9.999 ft. and for the case where the surface width of the first section was 5 ft. the first incremental distance Δs, was 9.969 ft.

The periods of the lowest mode of oscillation, the "sloshing mode", for the three different shapes described in Table 1 obtained by the matrix method (Eq. 17) are presented in Table 2 column one. The exact solutions, where available, are presented in the second column, and the periods obtained from Rayleigh-Ritz approach (Eq. 29) are shown in column three of this table.

Table 2. Fundamental Period of Test Basins

Basin Configuration	T_1 (sec.) Matrix	T_1 (sec.) Exact	T_1 (sec.) Rayleigh-Ritz
Circular Const. Depth	18.40	19.00	18.45
Circular Parabolic Bottom	24.45* 24.50**	24.75	24.0
Elliptic Const. Depth	18.38	——	18.58
Rectangular Const. Depth	——	22.30	22.30

The fundamental period of a rectangular basin 200 ft. long and 10 ft. deep is included in Table 2 only for comparison with the periods of the other basins which have the same length.

In the cases where exact solutions are available the agreement between these fundamental periods and the solutions from the matrix of coefficients is within 3%. The agreement between the periods obtained by the Rayleigh-Ritz method is better, which would be expected since both are approximate methods which incorporate the same assumptions.

The case of the circular basin with a parabolic bottom is of particular interest because of the shape of the basin, the two-dimensional approach, and the boundary conditions employed. The cross-section

* Basin truncated at 1 ft. surface width.
** Basin truncated at 5 ft. surface width.

area and the surface width fall off quite rapidly with distance from the center of the basin. However, it is seen that the fundamental period obtained by this numerical method is within 2% of the exact value. Probably even better agreement could be realized between the exact and the approximate approach by taking a larger number of sections. It is also seen that the truncation of the basin does not appear to have a serious effect upon the natural period for this case. This is seen by the agreement between fundamental periods of the two basins which are identical except for the degree of truncation.

For the three cases for which the matrix method was employed 21 eigenvalues were obtained. However, it is felt that for the circular basins and for an elliptical basin with a ratio of the length to the width of two it is not realistic to expect a two-dimensional approach to yield the correct eigenvalues for amplitude variations which are highly three-dimensional, as are the higher modes of oscillation for these basins.

Amplitude distribution - The variation of the amplitude in the circular basins with a horizontal bottom and with a parabolic bottom for the fundamental mode of oscillation was evaluated from Eqs. 18a, 18b, and 18c. These approximate solutions are compared to the following exact solutions for these cases (see Lamb, 1945):

1. Circular basin, constant depth

$$\frac{\eta}{\eta_1} = \frac{J_1(kr)}{J_1(kR)} \tag{31}$$

where J_1 is the Bessel function of the first order, k is the wave number, R is the radius of the basin, and kR = 1.84 corresponds to the period of the "sloshing mode".

2. Circular basin, parabolic bottom

$$\frac{\eta}{\eta_1} = \frac{r}{R} \tag{32}$$

The results of the approximate method and the exact solution are presented in Fig. 4 where the relative amplitude η/η_R is plotted as a function of the relative radius r/R. The numerical computation proceeds from r/R = + 1.0 to r/R = - 1.0. The results of the approximate method for the circular basin with constant depth agrees quite well with the exact solution throughout the full range of relative radius. However, for the circular basin with a parabolic bottom the agreement is not nearly as good in the region from r/R = - 0.6 to r/R = - 1.0 as in other regions. The major problem in a difference equation approach such as this is that errors are cumulative (see Eqs. 18a, 18b, and 18c), hence any error in the eigenvalue Λ_m is of increasing importance as the distance from the first section increases. In addition, the boundary condition which imposes a maximum or minimum amplitude at the ends of the basin is an approximation which does not apply as well to this basin as it does to the one with a vertical boundary.

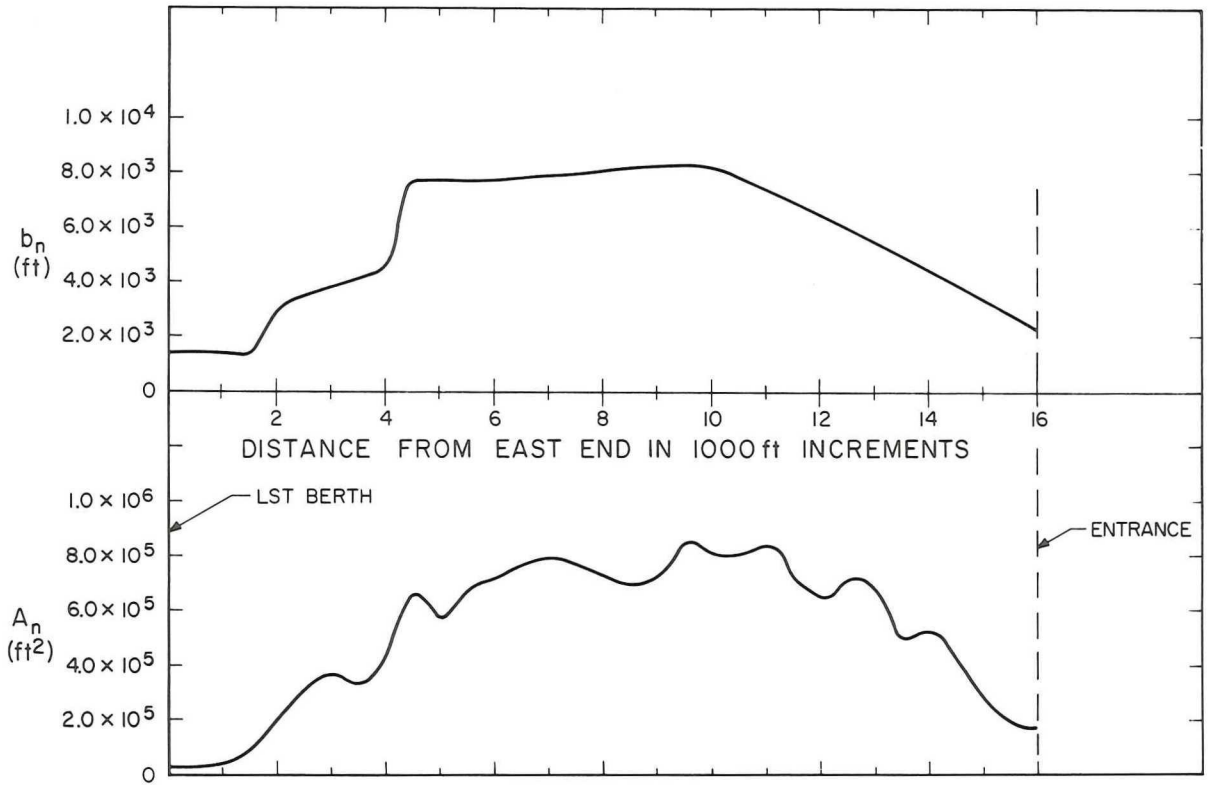


Figure 6. Variation of Surface Width and Cross-Section Area, Apra Harbor, Guam, M.I.

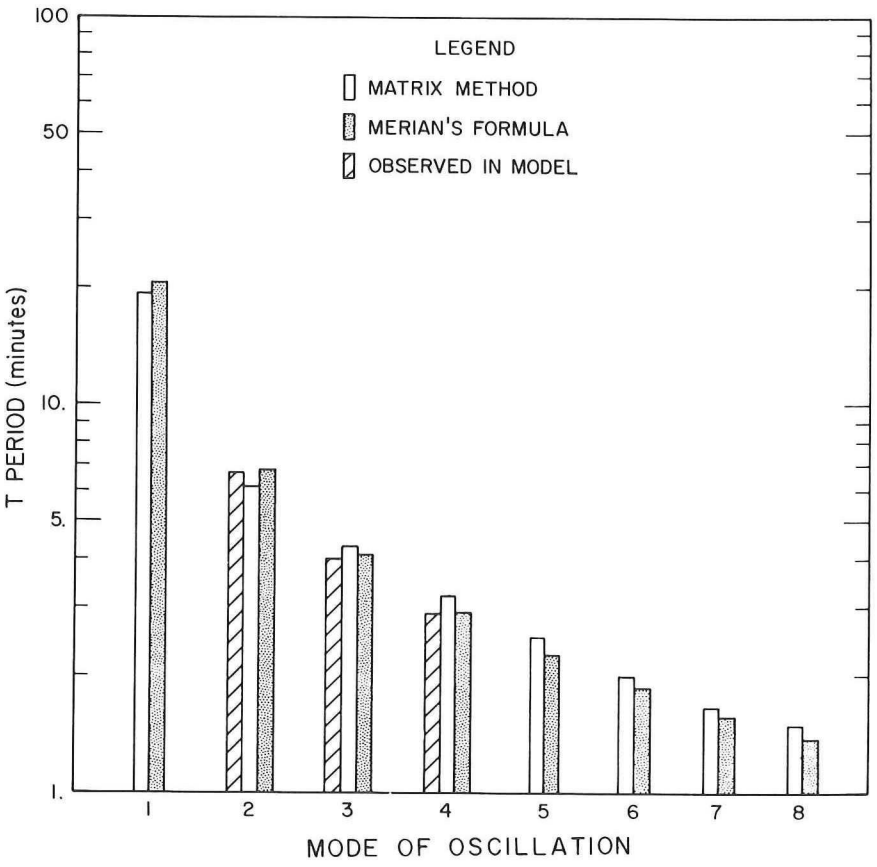


Figure 7. Periods of Oscillation of "Mode 2", Apra Harbor, Guam, M.I.

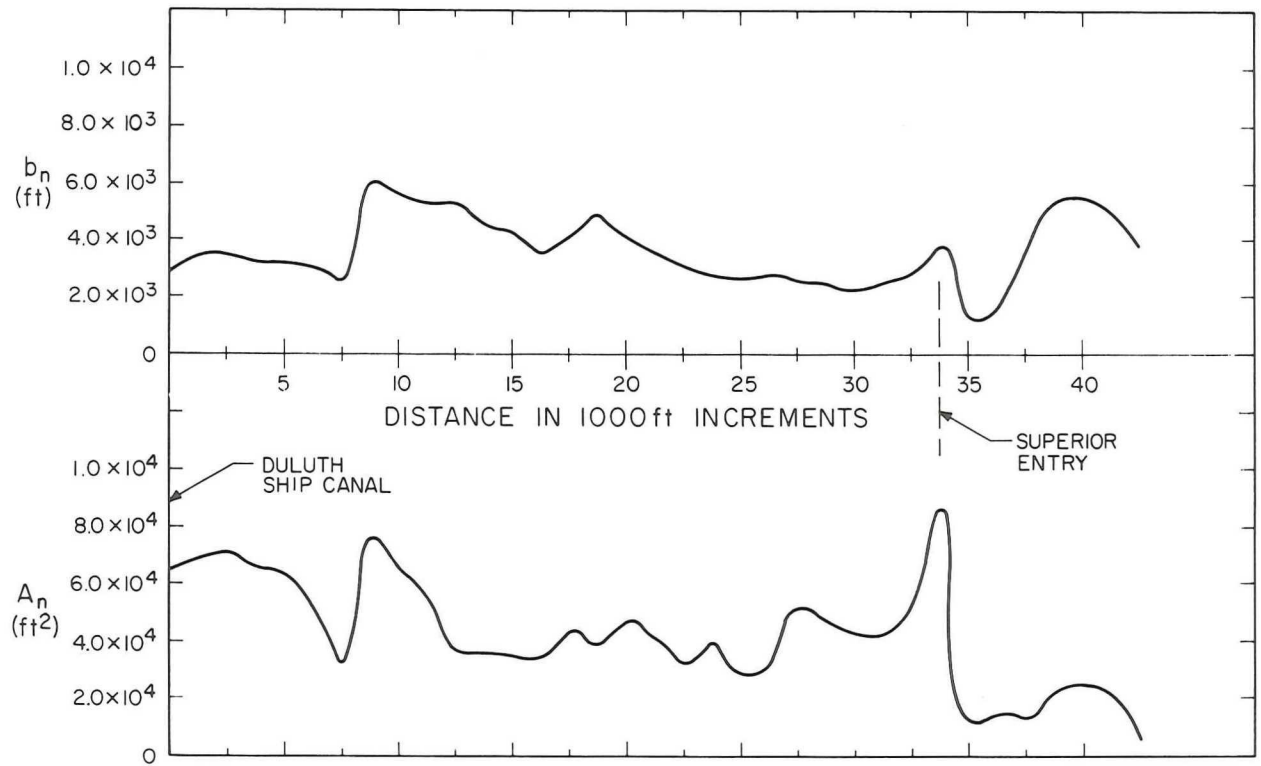


Figure 9. Variation of Surface Width and Cross-Section Area, Duluth-Superior Harbor, Minnesota-Wisconsin

In Fig. 7 the periods observed in the model are indicated for the 2nd, 3rd, and 4th mode. In addition the periods obtained from Merian's formula:

$$T = \frac{4}{n\sqrt{gh}} \quad (n = 1, 3, 5, \ldots) \tag{33}$$

are shown. This expression assumes, as does the matrix method, that a node exists at the harbor entrance. It is seen that the observed values agree relatively well with the computed periods and that the matrix method is in fairly good agreement with the periods obtained from Eq. 33.

The periods shown in Fig. 7 are probably too small for the case of a true forced oscillation because of the effect of the aspect ratio of the harbor (ratio of width to length) and the effect of the restricted entrance. Both of these effects tend to cause the amplitude at the entrance to be different from zero, and the harbor will resonate at a larger period than that indicated by the analysis. In fact, using the results of Ippen and Goda (1963) and considering Apra Harbor as a rectangular basin of an aspect ratio of 0.41 and a ratio of entrance width to maximum width of 0.19 the period of the first harmonic should be 74% greater than shown and that of the second harmonic should have a period 32% greater. Although this approach does not apply directly since the harbor is not rectangular and the entrance conditions are not exactly the same in the two cases it does indicate some of the problems inherent in the analysis where the entrance effects and the accompanying radiation of energy to the open-sea are not considered.

The amplitude variation along the talweg obtained for the fundamental and the second harmonic are shown in Fig. 8. It is seen that the imposed entrance boundary condition is met in the two cases.

Duluth-Superior Harbor - The second case chosen for analytical study was Duluth-Superior Harbor. The problem of seiche in this basin was treated previously by Housley (1962) using the method of Defant to compare predicted and observed periods of oscillation. Since Housley treated the problem as the free oscillations of a closed basin, assuming that the Duluth and the Superior entries, and St. Louis Bay do not significantly affect the oscillations of the harbor, this was a useful case to check. Duluth-Superior Harbor is approximately 43,000 ft. long and 4000 ft. wide with an average depth of all sections of 12.4 ft.

For this analysis 35 sections were taken spaced 1250 ft. apart and the variation of the surface width and cross-section area with distance obtained at these sections are shown in Fig. 9 where the origin is at the Superior end of the Harbor. The periods of free oscillation of the basin obtained by the matrix method are presented in Fig. 10 for the first through the eighth modes of oscillation. In addition, the periods of the five modes obtained by Defant's method and those obtained from Merian's formula, Eq. 1 are also shown in Fig. 10. It is seen that the periods obtained from the three different methods agree within 10% except for the 2nd mode where the difference is somewhat greater. Some of the differences between the results of this method and Defant's method can

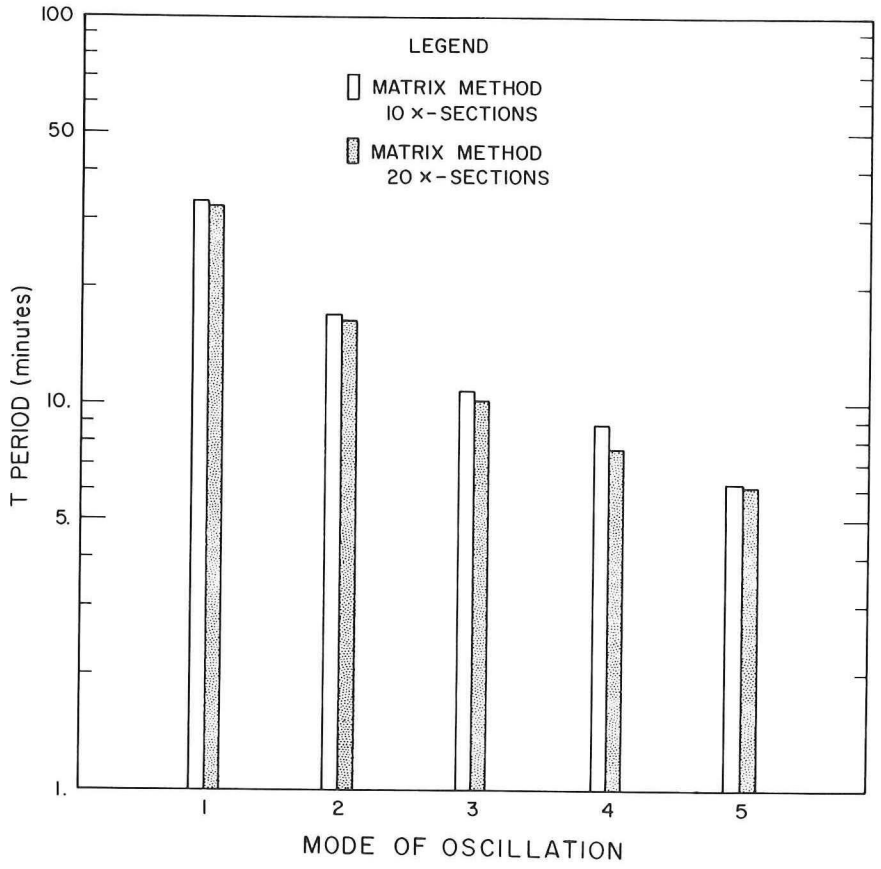


Figure 12. Periods of Oscillation of Monterey Bay, California

perhaps be attributed to the different number of cross-sections used. In the matrix approach 35 sections were used to define the matrix of coefficients compared to 154 sections used by Housley.

Monterey Bay - The third basin investigated was Monterey Bay, California. This is a large bay with a length of approximately 56,000 ft. from the shore to the entrance, a maximum width of 138,000 ft. and an average depth which decreases from 581 ft. at the entrance to essentially zero at the shore. The longitudinal variation of the surface widths and the cross-section areas are presented in Fig. 11.

Twenty sections were used (2960 ft. apart) in this analysis and the bay was treated as a fully open basin with a node located at the entrance to the bay. (For comparison, the basin was also divided into 10 sections.) This assumption is probably considerably in error due to the large aspect ratio of the bay and the resultant effect on the water particle velocity at the entrance of the energy radiated to the open sea. Nevertheless this treatment does indicate some interesting trends.

The periods of the first five modes of oscillation obtained from the eigenvalues of the matrix of coefficients are presented in Fig. 12. In addition the periods of these modes obtained by using 10 cross sections (a 10 x 10 matrix as opposed to a 20 x 20) are shown. The agreement between the results obtained from the two cases is fairly good indicating that as a first approximation a coarse spacing of the cross sections is probably sufficient for determining the eigenvalues.

The amplitude variation obtained for the first four modes using the finer cross-section spacing (20 sections) is presented in Fig. 13. The shape of the fundamental looks reasonable considering the configuration of the basin and the amplitude variations shown in Fig. 4 for the sloshing modes of circular basins with a horizontal bottom and with a parabolic bottom. However, due to the planform and the depth variation of Monterey Bay, it is felt that the oscillations of the higher modes would be of a more three-dimensional nature. In addition, as mentioned previously, since the bay communicates directly with the open-sea in an unrestricted sense, the effect of energy radiation from the bay on the modes of oscillation would no doubt be more important than the case of Apra Harbor discussed previously. Therefore, the shape of the amplitude distribution for the higher harmonics should be considered to be more qualitative than quantitative.

SOME CONSIDERATIONS FOR THREE-DIMENSIONAL MOTION

The problem which is of primary interest in the study of forced oscillations of harbors is that of three-dimensional motions. In fact the motivation for this two-dimensional study was to develop a method which could be applied to the more realistic problem of determining the response characteristics of an arbitrary shaped harbor connected to the open-sea. An approach to this problem is presented in this section; however, no solutions to specific cases have been attempted as yet.

With reference to Fig. 14 for three-dimensional long period fluid motions, the continuity equation is:

$$\frac{\partial}{\partial x}(u\,h\,\delta y)\,\delta x + \frac{\partial}{\partial y}(v\,h\,\delta x)\,\delta y = -\frac{\partial}{\partial t}\left[(\eta + h)\,\delta x\,\delta y\right] \qquad (34)$$

where u is the water particle velocity in the x-direction, v is the water particle velocity in the y direction, h is the depth which is a function of x and y, and δx and δy are the dimensions of an element in the horizontal plane. Applying the small amplitude, shallow water wave assumptions, the equations of motion in the x-direction and the y-direction are:

$$\frac{\partial u}{\partial t} = -g\frac{\partial \eta}{\partial x} \qquad (35a)$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \qquad (35b)$$

Assuming that η is periodic with time and of the form:

$$\eta = \eta(x, y)\cos\sigma t \qquad (36)$$

Then Eqs. 34, 35, and 36 can be combined to give the following expression for the variation of the water surface amplitude η in the x and y directions which is a maximum with respect to time:

$$h\left[\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right] + \frac{\partial h}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial h}{\partial y}\frac{\partial \eta}{\partial y} + \lambda\eta = 0 \qquad (37)$$

where λ is a particular eigenvalue and is equal to $\frac{\sigma^2}{g}$.

Using difference quotients similar to Eqs. 12 and the notation of Fig. 15 where $\Delta_x = \Delta_y = \Delta$, Eq. 37 reduces to:

$$-\left[2\,h_{m,n} + h_{m-1,n} + h_{m,n-1}\right]\eta_{m,n} + h_{m,n}\eta_{m+1,n} + h_{m-1,n}\eta_{m-1,n} + h_{m,n}\eta_{m,n+1} + h_{m,n-1}\eta_{m,n-1} + \lambda\Delta^2\eta_{m,n} = 0 \qquad (38)$$

wherein h is the depth at the indicated mesh point.

Instead of the square mesh a rectangular mesh can be used ($\Delta_x \neq \Delta_y$) so that a better definition of the amplitude can be realized in either the x or y-direction. If this is desired the x-coordinate or the y-coordinate can be transformed in such a way that the mesh is square once again and the final solution can then be transformed back to the actual coordinate system. However, for the sake of this discussion Eq. 38 is sufficient.

The same boundary conditions can be employed here for the closed basin as were used for the two-dimensional oscillations. Eq. 38 then becomes an m times n set of equations, where the product of m and n

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