SYMMETRY ANALYSIS IN LINEAR COMPRESSIBLE HYDRODYNAMIC STABILITY THEORY

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Abstract We present a unifying solution framework for the linearized gas-dynamical equations for a two-dimensional (2-D) linearly-sheared unbounded homentropic compressible flow using Lie symmetry classification. The full set of symmetries that are admitted by the underlying system of equations are employed to systematically derive three distinct invariant Ansatz functions, which unify the existing ones for normal mode, Kelvin mode analysis, as well as a novel approach. The latter approach considers modes that are localized in the cross-stream and periodic in the streamwise direction and travel on parabola shaped curves at constant velocity in the cross-stream, while being accelerated constantly in streamwise direction by the underlying base flow.

INTRODUCTION

Since Reynolds' pioneering experiment in 1883 the transition from a quiescent laminar to a strongly turbulent state in non-uniform/shear flows has attracted the interest of generations of scientists. A classical approach to stability problems, aiming to analyze the onset of turbulence, consists of assuming a laminar base flow, which a small perturbation is superposed. The most famous and extensively used approach, the so-called normal mode approach, was derived by Orr [7]. Here, periodic modes are considered that travel in the streamwise direction. However, for numerous flows, especially the plane Couette and pipe flow, the results obtained by this approach leads to deceptive conclusions (see e.g. [8]).

In the 1990s the limiting nature of the classical modal mode approach was recognized and the non-normal nature of non-uniform/shear flows was revealed (see e.g. [8] and references herein) – a major breakthrough in the understanding of shear flow dynamics. In fact, the operators involved in the modal analysis of plane shear flows are non-normal, resulting in the non-orthogonality of the corresponding eigenfunctions, and hence strong interference phenomena among the eigenmodes. The so-called non-modal approach, shifting the emphasis from the asymptotic to the short-time dynamics, became a well established alternative, taking into account the shortcomings of the modal approach. In one branch of non-modal stability theory, the system's response to initial conditions is investigated, which is central to hydrodynamic stability theory. Here, the Kelvin mode approach has been extensively used since the 1990s (see e.g. [1, 3]).

RESULTS

We apply Lie point symmetry analysis, as founded by the Norwegian mathematician Sophus Lie (1842–1899) and applied successfully in the area of fluid mechanics [4, 5], to systematically derive invariant solutions in the context of linear stability of a 2-D linearly-sheared unbounded homentropic compressible flow and perform a symmetry classification of the linearized gas-dynamical equations for 2-D perturbations. The full set of generators of three-dimensional gas dynamics is given in [2]. Similar to the analysis conducted in [4] for the linearized Navier-Stokes equations, we obtain the normal mode, the Kelvin mode and a novel approach by successive symmetry reduction using the `MAPLE` symbolic platform and the symmetry package `DESOLVII` [9]. Either of the three found ansatz functions transform the initial set of partial differential equations into ordinary differential equations (ODE's), written in invariant coordinates.

The linearized equations for a base flow with a non-uniform velocity profile ($U(y) = Ay$, $A =$const.) is given by

$$\varrho_t + Ay\varrho_x + u_x + v_y = 0\,, \qquad u_t + Ayu_x + Av + c_s^2\varrho_x = 0\,, \qquad v_t + Ayv_x + c_s^2\varrho_y = 0\,, \tag{1}$$

where u and v are the perturbation velocities in streamwise (x) and cross-stream (y) direction, ϱ the density perturbations and c_s the constant speed of sound.

The most general Lie group of point transformations admitted by equations (1) is a 4-parameter group, which read

$$X_1: \;\tilde{t} = t + t_0\,, \quad X_2: \;\tilde{x} = x + x_0\,, \quad X_3: \;\tilde{u} = Cu,\; \tilde{v} = Cv,\; \tilde{\varrho} = C\varrho, \quad X_4: \;\tilde{x} = x - y_0 At,\; \tilde{y} = y - y_0 \tag{2}$$

X_1 and X_2 are time and space translation symmetries, X_3 the scaling symmetry (arbitrary constant C) and X_4 a symmetry that is restricted to the chosen base flow. The normal mode approach appears to be an invariant solution with respect to the combination of the three symmetries X_1–X_3, whereas the Kelvin mode approach to the combination of symmetries X_2–X_4, excluding time-translation. Finally, a new class of ansatz functions can be found by also taking the aforementioned neglected time-translation symmetry into account, i.e. X_1–X_4, leading to the following new ansatz functions:

$$\Psi(x, y, t) = \tilde{\Psi}(\xi)\, \mathrm{e}^{\mathrm{i}k_x\left(x - \frac{A\mu}{2}t^2\right)} \tag{3}$$

with $\Psi = [u, v, \varrho]$ and the new variable $\xi = y - \mu t$. The novel modes are periodic and accelerated with $x_{tt} = A\mu$ by the underlying base flow in x-direction, while they travel with constant speed μ in y-direction. Their paths are described by the following relation $\mathbf{x} - \mathbf{x_0} = \mu\left(At^2/2, t\right)$, with $\mathbf{x} = (x, y)$ and the initial position $\mathbf{x_0}$.

A first result thereof is the derivation a conserved quantity $\mathcal{C}$ that is admitted by the system written in the new variable $\tilde{\Psi}(\xi)$, and has been validated numerically:

$$\mathcal{C} = \left[Ak_x\xi\tilde{u}(\xi) - (\mathrm{i}A + k_x\mu)\tilde{v}(\xi) + \left(\mathrm{i}A\mu + k_x c_s^2\right)\tilde{\rho}(\xi)\right]\frac{\mu}{Ak_x}\mathrm{e}^{-\frac{\mathrm{i}}{2}\frac{Ak_x}{\mu}\xi^2} . \tag{4}$$

with the streamwise wavenumber k_x. It appears that this combination is conserved along parabola shaped curves, which are convected by the base flow in time and is connected to the linearized potential vorticity. Following the isolines of $\mathcal{C}$ in figure 1a), one can see how these are convected along the parabola-shaped trajectory.

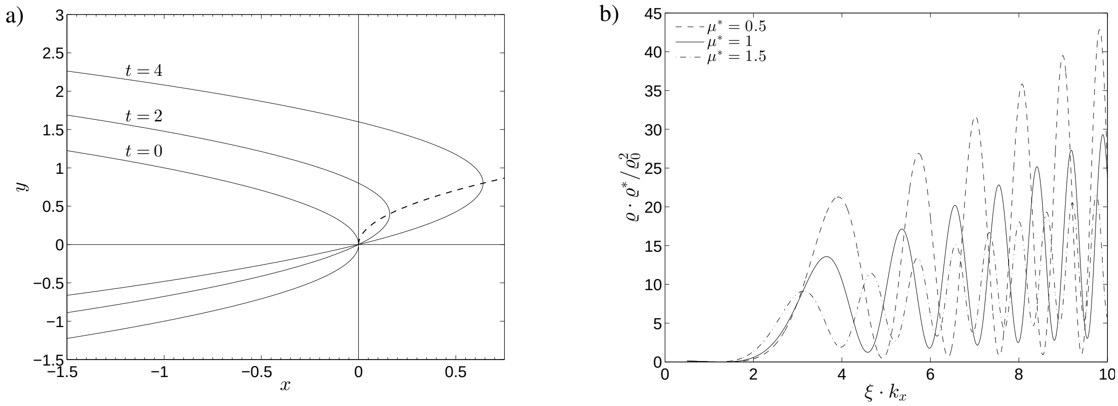

Figure 1. a) The convected conserved quantity $\mathcal{C}$ (–) along the mode path (- -) at three different times, with $\mu/c_s = 0.2$ and $A/(k_x c_s) = 0.4$ and b) the evolution of the density perturbations on the right for different ratios of $\mu^* = \mu k_x/A$.

Further, we demonstrate that the system of linearized first-order ODE's in the new similarity variable ξ can be re-written in a third-order ODE, only in any of the three primitive variables (e.g. $\tilde{u}$), whereas the remaining two (i.e. $\tilde{v}$, $\tilde{\varrho}$) can then be determined in terms of the former. Subsequently, the third-order ODE can be integrated once, giving a second-order inhomogeneous ODE with the conserved quantity $\mathcal{C}$ on the right-hand side:

$$\left[\left({c_s}^2 - \mu^2\right)\frac{\mathrm{d}^2}{\mathrm{d}\xi^2} + 2\mathrm{i}Ak_x\mu\,\xi\frac{\mathrm{d}}{\mathrm{d}\xi} + \left(k_x^2\left(A^2\xi^2 - {c_s}^2\right) + \mathrm{i}Ak_x\mu\right)\right]\tilde{\tilde{u}}(\xi) = \mathcal{C}\frac{Ak_x}{\mu}c_s^2\,\xi\,\mathrm{e}^{\frac{\mathrm{i}}{2}\frac{Ak_x}{\mu}\xi^2} , \tag{5}$$

which can be solved in terms of Kummer's functions (see e.g. [6]) for $\mu \neq 0$. We shown that $\tilde{u}$ rapidly decays, while $\tilde{v}$ grows similar to $\tilde{\varrho}$, which evolution along $\xi \cdot k_x$ is shown in Figure 1b), $(\cdot)^*$ denoting the complex conjugated, for three different parameters along the paths indicated in Figure 1a). As the mode leaves its initial location traveling with μ in y-direction, compressibility effects become more important due to the increasing velocity of the base flow, which can be seen by the onset of heavy oscillations. Hereby, the ratio of $\mu^* = \mu k_x/A$ plays a crucial role, determining the ability of the perturbation to extract compressible energy from the base flow. Especially, μ^* (solid line) separates two regimes, from which follows that subsonic modes inhere a greater potential of extraction of compressible energy, which can, in turn, lead to stronger sound emission.

Having an analytical solution of the linearized gasdynamical equations in ξ at our fingertips we believe to gain further valuable insight in the mechanisms of stability in shear flows and the propagation and conservation of perturbations.

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