Reciprocation Effort Games

Polevoy, Gleb; De Weerdt, Mathijs

DOI
10.1007/978-3-319-76892-2_4

Publication date
2018

Document Version
Final published version

Published in
Artificial Intelligence

Citation (APA)
Polevoy, G., & De Weerdt, M. (2018). Reciprocation Effort Games. In B. Verheij, & M. Wiering (Eds.), *Artificial Intelligence : 29th Benelux Conference, BNAIC 2017, Revised Selected Papers* (pp. 46-60). (Communications in Computer and Information Science; Vol. 823). Springer. https://doi.org/10.1007/978-3-319-76892-2_4

Reciprocation Effort Games

Gleb Polevoy[1(✉)] and Mathijs de Weerdt[2]

[1] University of Amsterdam, Amsterdam, The Netherlands
g.polevoy@uva.nl
[2] Delft University of Technology, Delft, The Netherlands
m.m.deweerdt@tudelft.nl

Abstract. Consider people *dividing their time and effort* between friends, interest clubs, and reading seminars. These are all *reciprocal interactions,* and the reciprocal processes determine the utilities of the agents from these interactions. To advise on efficient effort division, we determine the existence and efficiency of the Nash equilibria of the game of allocating effort to such projects. When no minimum effort is required to receive reciprocation, an equilibrium always exists, and if acting is either easy to everyone, or hard to everyone, then every equilibrium is socially optimal. If a minimal effort is needed to participate, we prove that not contributing at all is an equilibrium, and for two agents, also a socially optimal equilibrium can be found. Next, we extend the model, assuming that the need to react requires more than the agents can contribute to acting, rendering the reciprocation imperfect. We prove that even then, each interaction converges and the corresponding game has an equilibrium.

1 Introduction

In many real-world situations people invest effort in several interactions, such as in discretionary daily activities [16], daily communication between school pupils, sharing files over networks, or in business cooperation. In such an interaction, people tend to reciprocate, i.e., react on the past actions of others (sometimes only if a certain minimum effort is invested) [10,12]. For example, users of various social networks (Facebook, VKontakte) repeatedly interact in those projects (networks). To recommend how to divide one's limited efforts efficiently, we aim to predict stable strategies for these settings and estimate their efficiency. We study settings with and without a threshold for minimum effort.

Dividing a budget of effort is studied in *shared effort games* [4]. In these games players contribute to various projects, and given their contributions, each project attains a value, which is subsequently divided between the contributors. In order to support decisions regarding individually and publicly good stable strategy profiles in these games, the social welfare (total utility) of strategy profiles is important, and in particular of Nash equilibria (NE). For this, the

G. Polevoy—Most of this work was done at Delft University of Technology.

B. Verheij and M. Wiering (Eds.): BNAIC 2017, CCIS 823, pp. 46–60, 2018.
https://doi.org/10.1007/978-3-319-76892-2_4

price of anarchy (PoA) [15], and stability (PoS) [1,23] are the most famous efficiency measures. The price of anarchy is the ratio of the least social welfare in an equilibrium to the optimal social welfare, and the price of stability is the ratio of the social welfare in a best NE to the optimal social welfare.

Bachrach et al. [4] bound the price of anarchy, but only when a player obtains at least a constant share of her marginal contribution to the project's value; this does not hold for a positive participation threshold. Polevoy et al. [20] have analyzed the Nash equilibria, and price of anarchy and stability also in the case with a threshold. When the threshold is equal to the highest contribution, such shared effort games are equivalent to all-pay auctions. In all-pay auctions, only one contributor benefits from the project. Its equilibria are analyzed by Baye et al. [5] and many others. Anshelevich and Hoefer [2] study graph nodes contributing to edges, which are minimum effort projects. In the literature, the utilities are based on the project values, which are directly defined by the contributions, such as in contributions to online communities, Wikipedia, political campaigns [25], and paper co-authorship [14]. Unlike the existing literature, our paper assumes contributions to the projects determine the interactions, which define utility.

We now review the reciprocation models. Existing models of reciprocation often consider why reciprocation has emerged. The following works consider the emergence of reciprocation. Axelrod [3] studies and motivates direct evolution of reciprocal behavior. Others consider a more elaborate evolution, like Bicchieri's work on norm emergence [6, Chap. 6] or [27]. Trivers [26] describes how altruism-related emotions like guilt and suspicion have evolved. There exist also other approaches to the nature of reciprocation, such as the *strong reciprocation* [11]. Works like [8,10,22] assume the reciprocal behavior and analyze the development of certain interactions, modeling them as appropriate games.

With a model inspired by works on arms races [7,28] and spouses' interaction [13], Polevoy et al. [21] formally analyze lengthy repeated reciprocation and show convergence. They define an action on an agent as a convex combination[1] between one's own last action, the considered other agent's and all the other agents' last actions. They call this the *floating* reciprocation attitude.

The main contributions of this paper comprise of the analysis of a unifying model of shared effort games with reciprocal projects and creating a basis for further analysis. We define two games: one without a threshold, and another one with a threshold. In the second game, those who are below the threshold in an interaction, are not allowed to participate in the respective interaction. We identify when Nash equilibria exist and find the prices of anarchy and stability. In addition to the main part, where the initial actions are fully reciprocated by the reciprocal agents [26], we model the situation when the budget of an agent to invest in the various projects may fall short of satisfying the requirements of every reciprocal interaction. This forces the agent to curb her investments in some interactions, complicating the process, but we prove it still converges, and therefore, generalizing the definitions to that case is well-defined. We also prove that the corresponding reciprocation effort game and its exclusive thresholded

[1] A combination is convex if it has nonnegative weights that sum up to 1.

version have an equilibrium. We consider only pure equilibria throughout the paper, even when we do not mention this explicitly. Since the strategies include all the ways to divide budget among the interactions, the set of pure strategies is already uncountably infinite.

The model of several reciprocal interactions is given in Sect. 2. Section 3 characterizes the equilibria and their efficiency in a game without a threshold. Then, we analyze the game with a threshold in Sect. 4. We prove the convergence of an interaction with insufficient budgets and the NE existence in Sect. 5. Section 6 concludes and outlines new research directions.

2 Model

This section models dividing effort between reciprocal interactions. Adopting the reciprocation model from [21] and the inspired by shared effort games models from [4,20], we define a *reciprocation effort game*. First, we define a reciprocal process and the agents' utilities in this process. Next, we define a reciprocation effort game, where agents divide their effort budgets between several such processes. We define a thresholded variation on this game, to model the minimal required investment, in Sect. 4.

We begin with the reciprocation model, based on models for arms race and arguments. Given agents $N = \{1, \ldots, n\}$, at any time $t \in T \triangleq \{0, 1, 2, \ldots\}$, every agent acts on any other agent. The action by agent $i \in N$ on another agent $j \in N$ at moment t is characterized by its weight, denoted by $\mathrm{act}_{i,j}(t) \colon T \to \mathbb{R}$. Since only the weight of an action is relevant, we usually write "action" while referring to its weight. For example, the weights of the actions of helping, nothing, or insulting are in the decreasing order.

In order to define how agents reciprocate, we need the following notation. The kindness of agent i, constant for a given reciprocal process, is denoted by $k_i \in \mathbb{R}$. Agent i's kindness models i's inherent inclination to act on any other agent: the larger the kindness, the kinder the agent acts; in particular, it determines the first action of an agent, before the others have acted. We model agent i's inclination to mimic another agent's action and the actions of all the other participants in the project by reciprocation coefficients $r_i \in [0, 1]$ and $r'_i \in [0, 1]$ respectively, both staying constant for all interactions. r_i is the fraction of $\mathrm{act}_{i,j}(t)$ that is determined by the previous action of j upon i, and r'_i is the fraction that is determined by $\frac{1}{n-1}$th of the total action on i by all the other agents at the previous time. Fractions sum up to 1, thus $r_i + r'_i \leq 1$. We denote the total received action from all the other agents at time t by $\mathrm{got}_i(t) \colon T \to \mathbb{R}$; formally, $\mathrm{got}_i(t) \triangleq \sum_{j \in N} \mathrm{act}_{j,i}(t)$.

We now define the actions. At time 0, there is nothing to react to, so the kindness determines the action: $\mathrm{act}_{i,j}(0) \triangleq k_i$.

Definition 1. *At any positive time t, agent i's action is a weighted average of her own last action (inertia), of that of the other agent j (direct reaction)* *and of the total action of all the other agents divided over all the others (social reaction):*

$$\underset{i,j}{\mathrm{act}}(t) \triangleq (1 - r_i - r'_i) \cdot \underset{i,j}{\mathrm{act}}(t-1) + r_i \cdot \underset{j,i}{\mathrm{act}}(t-1) + r'_i \cdot \frac{\mathrm{got}_i(t-1)}{n-1}.$$

We have defined how agents reciprocate. An agent's *utility* from a given reciprocation project at a given time is the action one receives minus effort to act, following [19]. This is classical (see, for example, the quasilinear preferences of auction theory [17, Chap. 9.3]). Formally, define the utility of agent i at time t, $u_{i,t} \colon \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \to \mathbb{R}$, as

$$u_{i,t}\left(\left\{\underset{i,j}{\mathrm{act}}(t)\right\}_{i,j \in N}, \left\{\underset{j,i}{\mathrm{act}}(t)\right\}_{i,j \in N}\right) \triangleq \sum_{j \in N} \underset{j,i}{\mathrm{act}}(t) - \beta_i \sum_{j \in N} \underset{i,j}{\mathrm{act}}(t)$$

where the constant $\beta_i \in \mathbb{R}$ is the importance of performing actions relatively to receiving them for i's utility. The personal price of acting is higher, equal or lower than of receiving an action if β_i is bigger, equal or smaller than 1, respectively. The minus in front of i's actions subtracts the effort of acting from one's utility (unless β_i is negative, where that is added). Since the presence of negative actions would mess up this logic (since negative actions would still take effort while increasing the above expression), we assume that actions are always non-negative, which occurs if and only if all kindness values are non-negative. We can have negative influence, but we assume having added large enough a constant to all the actions, to avoid negative actions.

Every such interaction converges, as shown in [21]. To model the utility in the long run, we define the asymptotic utility, or just the *utility*, of agent i, as the limit of her utilities as the time approaches infinity. In formulas, $u_i \colon \left(\mathbb{R}^{n-1}\right)^\infty \times \left(\mathbb{R}^{n-1}\right)^\infty \to \mathbb{R}$, as $u_i\left(\bigcup_{t'=0}^{\infty} \{\mathrm{act}_{i,j}(t), \mathrm{act}_{j,i}(t)\}\right) \triangleq \lim_{t \to \infty} u_{i,t}\left(\mathrm{act}_{i,j}(t), \mathrm{act}_{j,i}(t)\right)$. This is the utility we consider here. This definition of the utility of a process is equivalent to the discounted sum of utilities when the discounting is slow enough. The proof is omitted for the lack of space.

We now define a *reciprocation effort game*. Our agents N participate in m interactions $\Omega = \{1, 2, \ldots, m\}$. Each of the m interactions is what we have defined till now, with its own kindness values and actions. The kindness, the actions, the total received action, and the utility in a concrete interaction $\omega \in \Omega$ will be denoted, when the concrete interaction is important, by $(k_i)_\omega$ and $(\mathrm{act}_{i,j}(t))_\omega$, $(\mathrm{got}_i(t))_\omega$, and $(u_{i,t})_\omega$ or $(u_i)_\omega$, respectively. Each player's strategies are the possible contributions to the interactions at time zero (the further contributions are determined by the reciprocation and not by the player). A contribution goes to the whole interaction, not to a particular action on another agent, but it determines the kindness values of the interactions as follows.

Player i's kindness at reciprocal interaction ω is determined by her contribution to that interaction at time zero, called just "the contribution", divided by the number of other agents who participate in the interaction at ω, accounting for acting on them. This means that i's kindness at interaction j is $\frac{x_j^i(0)}{n-1}$.

Therefore, the sum of all the actions of agent i at time $t = 0$ is equal to her contributions to all the reciprocation projects, which are bounded by her budget b_i. The contribution of player $i \in N$ to interaction project $\omega \in \Omega$ at a general time $t \in T$ is defined as the sum of her actions in that interaction at that time, i.e. $x^i_\omega(t) \triangleq \sum_{j \in N \setminus \{i\}} (\mathrm{act}_{i,j}(t))_\omega$.

An agent contributes something in the beginning of a reciprocation, and from that time on the reciprocation "automatically" uncurls according to Definition 1. We assume that not only the sum of the contributions at $t = 0$, but also the sum of the contributions at any time $t > 0$ is within the acting agent's budget. Each player i has a normal budget $b_i > 0$ (or just a budget) to contribute from at $t = 0$ and an extended budget $B_i \geq b_i$ that can be used when the actions are required by the reciprocation process at $t > 0$, perhaps resulting in a higher summarized contribution than the voluntarily chosen at $t = 0$. We differentiate between these two budgets, since the need to reciprocate can urge people to act more actively [11], and we assume that B_is are high enough to allow reciprocation.

Formally, the strategy space of player i consists of her contributions (at time zero), determining her kindness values at the interactions, $\left\{x^i = (x^i_\omega)_{\omega \in \Omega} \in \mathbb{R}_+^{|\Omega|} | \sum_{\omega \in \Omega} x^i_\omega \leq b_i\right\}$. As mentioned, a "contribution" always means the contribution at $t = 0$. Since the strategy profile $x = (x^i)_{i \in N}$ determines all the interactions, the above defined utilities in a reciprocal interaction, namely $(u_{i,t})_\omega$ and $(u_i)_\omega \triangleq \lim_{t \to \infty} (u_{i,t})_\omega$, are also functions of x. The utility $u_i(x)$ of a player $i \in N$ in the game is defined to be the sum of the utilities it obtains from the various projects, $u_i(x) \triangleq \sum_{\omega \in \Omega} (u_i(x))_\omega$, completing the definition of a *reciprocation effort game*.

An agent does not have to use up all her budget, so that the inequality $\sum_{\omega \in \Omega} x^i_\omega \leq b_i$ may be strict. The strategies of all the players except i are denoted x^{-i}. We denote the vector of all the contributions by $x = (x^i_\omega)^{i \in N}_{\omega \in \Omega}$.

We now give a concrete example of the model.

Example 1. People choose between going to an interest club, meeting friends, or going to a scientific reading seminar, as illustrated in Fig. 1. A player first decides on how much she wants to invest in each interaction, determining her kindness in each one of them. Subsequently, she reciprocates. Each of these projects is an interaction; for instance, in an interest club, a positive action

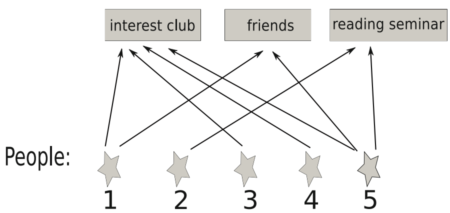

Fig. 1. People divide their own effort between interactions.

can be supporting another person, while showing contempt would be negative. Interacting, a person continues her previous course of action, represented by $(1 - r_i - r'_i) \cdot \mathrm{act}_{i,j}(t-1)$ in Definition 1, reacts on the other person's previous action, represented by $r_i \cdot \mathrm{act}_{j,i}(t-1)$, and reacts on the social climate, for which $r'_i \cdot \frac{\mathrm{got}_i(t-1)}{n-1}$ stands.

For the sake of efficiency analysis, we remind that the social welfare is defined as $\mathrm{SW} \stackrel{\Delta}{=} \sum_{i \in N} u_i(x)$, and the prices of anarchy [15] and stability [1,23] are defined as $\frac{\min\{\mathrm{SW}(x) | x \text{ is an NE}\}}{\max\{\mathrm{SW}(x) | x \text{ is a strategy}\}}$ and $\frac{\max\{\mathrm{SW}(x) | x \text{ is an NE}\}}{\max\{\mathrm{SW}(x) | x \text{ is a strategy}\}}$, respectively. We define 0/0 to be 1, because 0 from 0 means no loss occurs in the social welfare in the equilibria.

Polevoy et al. [21] prove the following theorem.

Theorem 1. *In an interaction, where for any agent i, $r'_i > 0$ and at least one agent i has $r_i + r'_i < 1$, for all pairs of agents $i \neq j$, the limit $\lim_{t \to \infty} \mathrm{act}_{i,j}(t)$ exists. The convergence is geometrically fast (exponential). All these limits are equal to each other and it is a convex combination of the kindness values, namely*

$$L = \frac{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \cdot k_i \right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \right)}. \tag{1}$$

3 Reciprocation Effort Game Without a Threshold

We first completely analyze existence of NE, and then we find all the prices of anarchy and stability. This theorem characterizes the existence of equilibria.

Theorem 2. *Assume that for any agent i, $r'_i > 0$, and in addition, either $n > 2$ or $r_1 + r'_1 + r_2 + r'_2 < 2$. The set of all the NE is exactly all the strategy profiles where every agent with $\beta_i < 1$ somehow divides all her budget among the projects $\{1, \ldots, m\}$, and every agent with $\beta_i > 1$ contributes nothing. These strategies are also dominant. In particular, there always exists an NE.*

Proof. Consider an arbitrary player l, and let her strategy (her contributions[2]) be $x^l = (x^l_1, \ldots, x^l_m)$. By Formula(s) (1), the limit of the actions at (project) interaction j is

$$\left(\frac{\left(\frac{1}{r_l + r'_l} \cdot (x^l_j) \right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \right)} + C_j \right),$$

where C_j is independent of l's strategy. This is both given and received by an agent w.r.t. the $n-1$ other agents, so we need to multiply the limit by $(n-1)(1-\beta_l)$. Summarizing, agent j's utility from this strategy is

$$(n-1)(1-\beta_l) \left(\frac{\left(\frac{1}{r_l + r'_l} \cdot (x^l_1 + \ldots + x^l_m) \right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \right)} + C \right),$$

[2] Contributions by default refer to the contributions at time zero.

for C that is independent of l's strategy. Therefore, if $\beta_l < 1$, then l's strategy is a best response to others' strategies if and only if l arbitrarily divides all her budget among the projects $\{1, \ldots, m\}$. On the other hand, if $\beta_l > 1$, then a strategy is a best response if and only if all the contributions are zero. This is true for every agent l, proving that this is an NE. Since each agent is independent of the others, these strategies are also dominant. ■

The possible variations in an NE profile are what the agents with $\beta = 1$ do. This is important for analyzing the efficiency of the NE.[3] To analyze efficiency, we define: $N^{<} \triangleq \{i \in N : \beta_i < 1\}$, $N^{\leq} \triangleq \{i \in N : \beta_i \leq 1\}$, $N^{=} \triangleq \{i \in N : \beta_i = 1\}$. We now analyze the efficiency of the most and the least efficient equilibria, comparing their social welfare to the maximum possible social welfare.

Proposition 1. *Under the assumptions of Theorem 2, if* $(n > \sum_{i \in N} \beta_i)$, *we have* $\text{PoA} = \frac{\sum_{i \in N^{<}} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}$, *and the* PoS *is given by the same expression, where we use* $N^{\leq}$ *instead of* $N^{<}$. *Consequently, if* $(n = \sum_{i \in N} \beta_i)$, *we have* $\text{PoA} = \text{PoS} = 1$. *If* $(n < \sum_{i \in N} \beta_i)$, *then:*

If $N^{<} \neq \emptyset$, *then we have* $\text{PoA} = \text{PoS} = -\infty$.
If $N^{<} = \emptyset$, **but** $N^{\leq} \neq \emptyset$, *then* $\text{PoA} = -\infty$, *but* $\text{PoS} = 1$.
If $N^{\leq} = \emptyset$, *then* $\text{PoA} = \text{PoS} = 1$.

The proof compares the possible social welfare in equilibria with the optimum social welfare.

Proof. The possible social welfare values that an NE can achieve are exactly

$$(n-1)(n - \sum_{i \in N} \beta_i) \frac{\sum_{i \in N^{<}} \left(\frac{1}{r_i + r'_i} \cdot b_i\right) + \sum_{i \in N^{=}} \left(\frac{1}{r_i + r'_i} \cdot x^i\right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i}\right)},$$

where $0 \leq x^i \leq b_i$. The optimum social welfare is

$$(n-1)(n - \sum_{i \in N} \beta_i) \frac{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i}\right)}$$

if $(n > \sum_{i \in N} \beta_i)$, and 0 otherwise.

Thus, if $(n > \sum_{i \in N} \beta_i)$, we have

$$\text{PoA} = \frac{\sum_{i \in N^{<}} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)} \text{ and } \text{PoS} = \frac{\sum_{i \in N^{\leq}} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}.$$

[3] $\beta_i > 1$ implies negative utilities that sometimes result in negative PoA and PoS.

If ($n = \sum_{i \in N} \beta_i$), we have PoA = PoS = 1, since the social welfare is always zero, and we define here $0/0 = 1$.

If ($n < \sum_{i \in N} \beta_i$), then we may get negative social welfare, since zero is optimal, while some NE yield a negative social welfare. Concretely, we have the following subcases:

If $N^{<} \neq \emptyset$, then we have PoA = PoS = $-\infty$, because any NE has the social welfare of at most $(n-1)(n - \sum_{i \in N} \beta_i) \frac{\sum_{i \in N^{++}} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i}\right)}$.

If $N^{<} = \emptyset$, **but** $N^{\leq} \neq \emptyset$, then PoA $= -\infty$ but PoS $= 1$. The reason is that an NE can have the social welfare from zero and down to $(n-1)(n - \sum_{i \in N} \beta_i) \frac{\sum_{i \in N^{\leq}} \left(\frac{1}{r_i + r'_i} \cdot b_i\right)}{\sum_{i \in N} \left(\frac{1}{r_i + r'_i}\right)}$.

If $N^{\leq} = \emptyset$, then PoA = PoS = 1, since an NE has the social welfare of zero.

■

In particular, we have shown that if all the agents find acting easy (i.e., all $\beta_i < 1$), or if all agents really do not like acting (i.e., all $\beta_i > 1$), then PoA = PoS = 1, so that any NE is optimum for the society. Intuitively, this is because here, all the agents have similar preferences: either everyone wants to act and receive action, or no one does. We have also shown, that if the average agent finds not contributing more important than receiving (i.e., $\sum_{i \in N} \beta_i > n$), but still $\beta_i < 1$ for some agent i, then PoA = PoS = $-\infty$, so any NE is catastrophic to the society. Intuitively, this stems from the differences in the agents' preferences. Finally, we see that if $\sum_{i \in N} \beta_i > n$, some agents have $\beta_i = 1$, but none have $\beta_1 < 1$, then PoA $= -\infty$ but PoS $= 1$, requiring regulation.

Theorem 2 implies that if all the projects have $\beta \leq 1$, then any dividing of all the budget in cooperating is always an NE. This is unintuitive, since usually, some groups are more efficient to interact with than some other groups. The reason for this is that the model assumes that all agents always interact at every project $\omega \in \{1, \ldots, m\}$, and only their kindness depends on the strategy. Basically, everyone attends all the interactions, and some people are passive.

4 Exclusive Thresholded Reciprocation Effort Game

We now define a variation on a shared effort game with reciprocation, where only the agents who contribute at least the threshold may interact. First, following [20], we define a *θ-sharing mechanism*. This models, for example, a minimum invested effort to be considered a coauthor, or a minimum effort to master a technology before working with it. Define, for every $\theta \in [0, 1]$, the players who get a share from project ω to be $N_\omega^\theta \triangleq \left\{i \in N | x_\omega^i \geq \theta \cdot \max_{j \in N} x_\omega^j\right\}$, which are those who bid at least θ fraction of the maximum contribution to ω.

We now define an *exclusive thresholded reciprocation effort game*, as a reciprocation effort game, where exclusively the agents in N_ω^θ interact. Others do not obtain utility and do not even interact. If an agent ends up participating alone at a project, he obtains zero utility from that project, since no interaction occurs. Exclusive thresholded reciprocation effort games model situations when joining an interaction requires contributing enough, like the initial effort it takes to learn the required technology to contribute to Wikipedia, the effort to become a member of a file sharing community or to start a firm.

In this section, we assume w.l.o.g. that $b_n \geq \ldots \geq b_1$. The existence of an equilibrium is easy, since no-one contributing constitutes an NE. Then, we show that also less trivial equilibria exist. Finally, the harder question of equilibrium efficiency is answered for two agents. We first notice a trivial equilibrium.

Observation 1. *The profile where all agents contribute nothing is an NE.*

Proof. In this profile, any agent who deviates by contributing a positive amount to a project will be the only one to interact there, so her utility will still be zero. ∎

We call an NE where at any project, at most one agent interacts (reaches the threshold) and positively contributes there, a Zero NE. There may be multiple Zero NE. We have just shown that a Zero NE always exists. A natural question is whether there exist non-Zero NE as well. They do.

Theorem 3. *Assume that all agents have $\beta_i \leq 1$. Assume that for any agent i, $r'_i > 0$ and in addition, for any pair of agents i, j we have $r_i + r'_i + r_j + r'_j < 2$. There exists a non-Zero NE.*

Proof. Consider the profile where all agents $1, \ldots, n-1$ contribute their whole respective budgets to project 1, and agent n contributes $\min\left\{b_n, \frac{b_{n-1}}{\theta}\right\}$ to project 1, and nothing to other projects.

This is an NE, for the following reasons. Any agent would be alone at any project other than 1 if it contributed to such a project, and therefore, it will not contribute there. At project 1, the only agent who perhaps can increase her contribution is n, but she will stay alone, if she does, so no deviation is profitable. ∎

The next question is the efficiency of the equilibria. Since we always have the Zero NE, and by contributing to the same project the same positive amounts we achieve a positive social welfare, we always have PoA $= 0$. Regarding the price of stability, we immediately know that it is positive, since there always exists a non-Zero NE. We now show that the price of stability for two agents is 1, meaning that there exists a socially optimal NE.

Proposition 2. *For $n = 2$ and under the assumptions of Theorem 3,* PoS $= 1$.

Proof. When we have only two players, we can assume w.l.o.g. that in a profile with maximum social welfare, a project that receives a positive contribution, receives it from both agents. Therefore, social welfare is maximized by maximizing the total contribution to the projects where interaction occurs.

Then, the following profile maximizes the social welfare. Agent 1 spreads her budget equally between all the projects. If $b_1 \geq \theta b_2$, then agent 2 divides her budget equally between all the projects, and otherwise, she contributes $\frac{1}{\theta}\frac{b_1}{m}$ to every project. Since this profile constitutes an NE, we conclude that PoS $= 1$. ∎

5 Insufficient Budgets

Till now, we have been assuming that there is enough extended budget to allow the agents make the contributions required by the sum of the reciprocal actions at any time. In this section, we consider dividing effort between reciprocal projects, where the extended budgets B_i may not suffice to reciprocate at some positive time t, and therefore the actions have to be curbed, such that the total action at any time is bounded by the B_i. In Example 1, this can happen if people are unable to keep up with the others because their free time is strictly limited. In order to justify studying the asymptotic behavior here, we prove that for any curbing, the actions in all interactions converge, as time approaches infinity. Then, we study the equilibria of the corresponding game.

The convergence of normal reciprocation is proven in [21], and we now prove the convergence of curbed reciprocation. Consider the undirected interaction graph $G = (N, E)$ of an interaction project, such that agent i can act on j and vice versa if and only if $(i, j) \in E$. Our model assumes that this graph is a clique, meaning that everyone interacts, but this is not necessary for the following theorem. At a given time, let the reciprocation from Definition 1 require actions denoted by the column vector $\boldsymbol{q} \in \mathbb{R}_+^{|E|}$, in the sense that its (i, j)th coordinate contains $\mathrm{act}_{i,j}$ (for $(i, j) \in E$). Then, the curbing is denoted by $D_{\boldsymbol{q}} \cdot \boldsymbol{q}$, where $D_{\boldsymbol{q}}$ is the diagonal curbing matrix. We omit the subscript $\boldsymbol{q}$ when the vector on which we act is clear. We denote the curbing matrix at time t by $D(t)$.

Theorem 4. *Consider dividing effort between reciprocal interactions, where every interaction has some connected interaction graph, and for all agents i, $r'_i > 0$. At every interaction, if there exists a cycle of an odd length in the interaction graph, or at least one agent i has $r_i + r'_i < 1$, then, for all pairs of agents $i \neq j$, the limit $L_{i,j} \stackrel{\Delta}{=} \lim_{t\to\infty} \mathrm{act}_{i,j}(t)$ exists.*

In our model, we assume a completely connected graph, so if at least 3 agents interact, we have an odd cycle, namely a triangle. Therefore, then we only need to assume that for all agents i, $r'_i > 0$.

The proof expresses reciprocation as matrix multiplication. Without curbing, the convergence is proven using the Perron-Frobenius theorem. Keeping convergence when curbing can occur uses the following definition and lemma.

Definition 2. *We remind that a square non-negative matrix A is called* primitive*, if there exists a positive l, such that $A^l > 0$ (see [24, Definition 1.1]).*

The following lemma, used to prove the theorem, has a value of its own as well. Given a convergent sequence of primitive matrices, the lemma shows that arbitrarily squeezing the matrices keeps the convergence.

Lemma 2. *Given a vector $\boldsymbol{p}(0) \in \mathbb{R}^d$, a primitive matrix $A \in \mathbb{R}^{d^2}$, such that $\lim_{t\to\infty} A^t$ exists, and a sequence of diagonal matrices $\{D(t)\}_{t=0}^{\infty}$, $D(t) = \mathrm{diag}(\lambda_1(t), \ldots, \lambda_d(t))$, where each $\lambda_i(t) \in (0, 1]$, define the sequence $\{\boldsymbol{p}(t)\}_{t=0}^{\infty}$ by $\boldsymbol{p}(t) \stackrel{\Delta}{=} D(t)AD(t-1)A \ldots D(1)A\boldsymbol{p}(0)$. Then, $\lim_{t\to\infty} \boldsymbol{p}(t)$ exists.*

Proof. Assume to the contrary, that $\{\boldsymbol{p}(t)\}$ diverges. Define the sequence $\{\boldsymbol{p}'(t)\}_{t=0}^{\infty}$ by $\boldsymbol{p}'(t) \triangleq A^t \boldsymbol{p}(0)$. Since $\{\boldsymbol{p}(t)\}$ diverges and $\{\boldsymbol{p}'(t)\}$ converges, they differ at some point, intuitively speaking. We now formalize this argument. Since $\{\boldsymbol{p}(t)\}$ diverges and the space is complete, it is not a Cauchy sequence, and so there exists a positive ϵ, such that for each $N > 0$ there exist $n, m > N$, such that $|\boldsymbol{p}(n) - \boldsymbol{p}(m)| > \epsilon$ ($||$ is the Euclidean norm). Since $\{\boldsymbol{p}'(t)\}$ converges, it is a Cauchy sequence, so there exists $N > 0$, such that for all $n, m > N$ we have $|\boldsymbol{p}'(n) - \boldsymbol{p}'(m)| < \epsilon/2$. If $|\boldsymbol{p}(n) - \boldsymbol{p}(m)| > \epsilon$ and $|\boldsymbol{p}'(n) - \boldsymbol{p}'(m)| < \epsilon/2$, we cannot both have $|\boldsymbol{p}(n) - \boldsymbol{p}'(n)| < \epsilon/4$ and $|\boldsymbol{p}(m) - \boldsymbol{p}'(m)| < \epsilon/4$. Therefore, for some integer l, $|\boldsymbol{p}(l) - \boldsymbol{p}'(l)| > \delta$, for some $\delta > 0$, depending solely on ϵ. Since the product defining $\boldsymbol{p}(l)$ is like that of $\boldsymbol{p}'(l)$, but with more $D(t)$ matrices, and $D(t) = \text{diag}(\lambda_1(t), \ldots, \lambda_d(t))$, where each $\lambda_i(t) \in (0, 1]$, we have $\mathbf{0} \le \boldsymbol{p}(l) \le \boldsymbol{p}'(l)$. Remembering this, and that matrix A is primitive, thereby propagating a change of an entry to every entry, we can choose l such that every coordinate of $\boldsymbol{p}(l)$ will be at most α fraction of the corresponding coordinate of $\boldsymbol{p}'(l)$, for some $\alpha < 1$. The α can be made to depend solely on ϵ, because of the boundedness of all the relevant vectors. So, we have $\boldsymbol{p}(l) \le \alpha A^l \boldsymbol{p}(0)$.

By reiterating the same argument with $\boldsymbol{p}_1'(t) \triangleq A^t \boldsymbol{p}(l)$ and $\boldsymbol{p}_1(t) \triangleq \boldsymbol{p}(t + l)$, we find $l_1 > 0$, such that $\boldsymbol{p}_1(l_1) \le \alpha A^{l_1} \boldsymbol{p}(l)$. Thus, $\boldsymbol{p}(l_1 + l) = \boldsymbol{p}_1(l_1) \le \alpha A^{l_1} \boldsymbol{p}(l) \le \alpha A^{l_1} \alpha A^l \boldsymbol{p}(0) = \alpha^2 A^{l_1 + l} \boldsymbol{p}(0)$.

Continuing in this manner, and using the boundedness of the converging $\{A^t \boldsymbol{p}(0)\}$, we prove that $\{p(t)\}$ converges to zero. A contradiction. [4] ■

We can now prove Theorem 4.

Proof. We extend the proof of the Theorem 1 from [21], which proves the convergence without the curbing. We show that the curbing still keeps the convergence. We recapitulate the used properties from there, to stay self-contained.

We express the dynamics of interaction in a matrix, and prove the theorem by applying the Perron–Frobenius theorem [24, Theorems 1.1 and 1.2], using the above lemma to handle the curbing of actions. Denoting the neighbors of i as $\text{N}(i)$, we define the dynamics matrix $A \in \mathbb{R}_+^{|E| \times |E|}$ as

$$A((i,j),(k,l)) \triangleq \begin{cases} (1 - r_i - r_i') & k = i, l = j; \\ r_i + r_i' \frac{1}{|\text{N}(i)|} & k = j, l = i; \\ r_i' \frac{1}{|\text{N}(i)|} & k \ne j, l = i; \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Assume that for each time $t \in T$, the column vector $\boldsymbol{p}(t) \in \mathbb{R}_+^{|E|}$ describes the actions at time t. Then, $\boldsymbol{p}(t+1) = D(t) A \boldsymbol{p}(t)$, where $D(t)$ is the diagonal matrix, describing the curbing. We call $\boldsymbol{p}(t)$ an action vector. Initially, $\boldsymbol{p}(0)_{(i,j)} = k_i$.

We will use the Perron-Frobenius theorem for primitive matrices. We now prepare to use it, and first we show that A is primitive. In the proof of Theorem 1,

[4] The actual limit does not have to be zero; zero is just the result from the contradictory assumption.

it is shown that A is irreducible and aperiodic, and therefore primitive by [24, Theorem 1.4]. Since the sum of every row is 1, the spectral radius is 1.

According to the Perron-Frobenius theorem for primitive matrices [24, Theorem 1.1], the absolute values of all the eigenvalues except one eigenvalue of 1 are strictly less than 1. The eigenvalue 1 has unique right and left eigenvectors, up to a constant factor. Both these eigenvectors are strictly positive. Therefore, [24, Theorem 1.2] implies that $\lim_{t\to\infty} A^t = \mathbf{1}\boldsymbol{v}'$, where $\boldsymbol{v}'$ is the left eigenvector of the value 1, normalized such that $\boldsymbol{v}'\mathbf{1} = 1$.

Now, Lemma 2 implies that $L_{i,j}$ exists. ■

We have proven the reciprocation effort game where curbing can occur is well defined, because all the reciprocation processes converge. We now prove the existence of equilibria in such a game. In the exclusive thresholded model, Observation 1 holds in the curbed case as well, so contributing nothing is an NE. From now on, assume that no threshold exists. Since the curbing renders finding a formula for the actions in the limit unlikely, we take an abstract approach.

Theorem 5. *Consider dividing effort between reciprocal interactions, where for all agents i, $r'_i > 0$. Assume that $n \geq 3$ or at least one agent i has $r_i + r'_i < 1$. Assume that the curbing function $D\colon \mathbb{R}^n \to \mathbb{R}^n$ is a weak contraction w.r.t. norm L_∞, i.e. $||D_{\boldsymbol{x}}\boldsymbol{x} - D_{\boldsymbol{y}}\boldsymbol{y}||_\infty \leq ||\boldsymbol{x} - \boldsymbol{y}||_\infty$.*

Then, there exist small enough β_is such that an NE exists.

Proof. By Theorem 4, the reciprocation processes converge and so the game is well defined. We prove the existence using Proposition 20.3 from [18]. The strategy set of every player consists of all the possible divisions of the budgets between the projects, which is a nonempty compact convex set.

The continuity of the utility functions follows from the action limits depending continuously on the total contributions of agents to projects. To this end, we can inductively show that at ant time t, the change in the action is $||\boldsymbol{p}'(t) - \boldsymbol{p}(t)||_\infty \leq ||\boldsymbol{\Delta}x||_\infty$, where $\boldsymbol{p}'(t)$ represents the actions at time t if $\boldsymbol{p}'(0) = \boldsymbol{p}(0) + \boldsymbol{\Delta}x$. This boundedness keeps holding in the limit of time approaching infinity as well, implying continuity. For the quasi-concavity of an agent's strategy space, notice that for small enough β_i, agent i would like to increase its contribution exactly till it can increase the limit of the actions of at least one another agent. Finally, Proposition 20.3 implies the theorem. ■

We also prove that when agents react identically, the game boils down to a single reciprocal interaction.

Proposition 3. *Assume that the curbing is determined by the sum of the actions of an agent and that all the reciprocation coefficients are equal among the agents, i.e. $r_i = r_j$ and $r'_i = r'_j, \forall i \neq j$. Then, the total contribution of agent i at any time $t \in T$, i.e. $x^i(t) \stackrel{\Delta}{=} \sum_{\omega\in\Omega} x^i_\omega(t)$, and the total received action, i.e. $\sum_{\omega\in\Omega} (\text{got}_i(t))_\omega$, are fully determined by the total contributions and the total received actions at time zero of the agents (i and others), regardless how the actions were divided between the projects.*

Proof. We prove this by induction on time. At the *basis*, $t = 0$ and the statement is trivial. At the *induction step*, assume that $\sum_{\omega \in \Omega} x^i_\omega(t-1)$ and $\sum_{\omega \in \Omega} (\text{got}_i(t-1))_\omega$ are fully determined by the total contributions and the total received actions at time zero and prove this determinancy for $\sum_{\omega \in \Omega} x^i_\omega(t)$ and $\sum_{\omega \in \Omega} (\text{got}_i(t))_\omega$. Indeed, $x^i_\omega(t)$ is equal to

$$\begin{aligned}\sum_{j \neq i} \underset{i,j}{\text{act}} &= (1 - r_i - r'_i) \sum_{j \neq i} (\underset{i,j}{\text{act}}(t-1))_\omega + r_i \sum_{j \neq i} (\underset{j,i}{\text{act}}(t-1))_\omega + r'_i (\underset{i}{\text{got}}(t-1))_\omega \\ &= (1 - r_i - r'_i) x^i_\omega(t-1) + r_i (\underset{i}{\text{got}}(t-1))_\omega + r'_i (\underset{i}{\text{got}}(t-1))_\omega.\end{aligned}$$

Sum it up over all the projects to obtain

$$\sum_{\omega \in \Omega} x^i_\omega(t) = (1 - r_i - r'_i) \sum_{\omega \in \Omega} x^i_\omega(t-1) + r_i \sum_{\omega \in \Omega} (\underset{i}{\text{got}}(t-1))_\omega + r'_i \sum_{\omega \in \Omega} (\underset{i}{\text{got}}(t-1))_\omega.$$

Since everything on the right hand side is, by the induction hypothesis, determined by the total contribution and the total received action at time zero, the actions on time t before curbing are determined by them as well. Furthermore, curbing is determined by the total action of the agents, and thus, the curbed actions are also determined by the total contribution and the total received action at time zero.

Regarding the total received action, the derivation of the step is analogous, but it requires moving $1 - r_j - r'_j, r_j$ and r'_j out of the parentheses, where we use the equality of these parameters across the agents. ■

6 Conclusions and Further Research

In order to predict investing effort in several reciprocal interactions, we define a game that models dividing efforts between several reciprocal projects. We include an analysis of a model both with and without a contribution threshold.

When no contribution threshold exists, there always exists an equilibrium, and if acting is easy to everyone (for all i, $\beta_i < 1$) or hard to everyone (for all i, $\beta_i > 1$), then every NE is socially optimal. We also show that any dividing of all the budget when acting is easy to everyone is a Nash equilibrium. The result may seem surprising. Intuitively, this happens because everyone participates in each interaction, and the concrete division of the budget does not matter to the social welfare. However, life does not often provide such situations. We also characterize when both efficient and inefficient equilibria exist, calling for regulation.

If a minimum contribution is necessary to participate in interaction, we show that the situation where no-one contributes is an equilibrium. This models the case where people are very passive, and this continues since no-one can start an interaction project on his own. In addition to this trivial equilibrium, we find an equilibrium where all the agents contribute to the same project, like Facebook, instead of participating in the other social networks. This describes the case when people interact with each other on the same topic. Such a situation is clearly not the only option, since people often have many friendships [9]. For two agents, there exists an equilibrium which is socially optimal.

The choices of strategies by the agents who are indifferent can significantly influence the social welfare. For instance, this happens in the case without threshold to agents for whom acting and receiving action are equally important. Making such agents do what benefits the society can increase the social welfare.

We also model the case when the extended budgets are not big enough and curbing is required. We show that any way of dividing effort between reciprocal interactions results in converging interactions, regardless of how actions are curbed to fit the budgets. We also prove that the resulting reciprocation effort game possesses an equilibrium, with and without threshold.

For future research, we are curious about the efficiency of the equilibria in the game with curbing. Consecutive decisions can be modeled by the agents first contributing to the interactions and then deciding on their reciprocation parameters. Additionally, looking at interactions in large groups where not everyone can act on everyone else would be a natural generalization of our work. Another point is that we assumed that two agents who interact in multiple projects, interact in these projects independently. Modeling the dependency between these interactions is realistic. Analyzing a mixed set of projects, only some of which are interaction projects, would model reality better. Also modeling and analyzing voting to approve who else may participate in an interaction seems promising.

This work models and analyzes a ubiquitous class of interactions and lays the basis for further research, aimed to provide more advice to the agents and to the manager who wants to maximize the social welfare.

Acknowledgments. We thank Prof. Orr M. Shalit from the Technion, Israel for the useful discussions. This work has been supported by SHINE, the flagship project of DIRECT (Delft Institute for Research on ICT at the TU Delft).

References

1. Anshelevich, E., DasGupta, A., Kleinberg, J., Tardos, E., Wexler, T., Roughgarden, T.: The price of stability for network design with fair cost allocation. In: Proceedings of the 45th Annual IEEE Symposium on Foundations of Computer Science, pp. 295–304, October 2004
2. Anshelevich, E., Hoefer, M.: Contribution games in social networks. In: de Berg, M., Meyer, U. (eds.) ESA 2010. LNCS, vol. 6346, pp. 158–169. Springer, Heidelberg (2010). https://doi.org/10.1007/978-3-642-15775-2_14
3. Axelrod, R.: The Evolution of Cooperation. Basic Books, New York (1984)
4. Bachrach, Y., Syrgkanis, V., Vojnović, M.: Efficiency and the Redistribution of Welfare. Technical report, May, Microsoft Reserach (2012)
5. Baye, M.R., Kovenock, D., de Vries, C.G.: The all-pay auction with complete information. Econ. Theor. **8**(2), 291–305 (1996)
6. Bicchieri, C.: The Grammar of Society: The Nature and Dynamics of Social Norms. Cambridge University Press, Cambridge (2006)
7. Dixon, W.J.: Reciprocity in United States-Soviet relations: multiple symmetry or issue linkage? Am. J. Polit. Sci. **30**(2), 421–445 (1986)

8. Dufwenberg, M., Kirchsteiger, G.: A theory of sequential reciprocity. Games Econ. Behav. **47**(2), 268–298 (2004)
9. Ennett, S.T., Bauman, K.E.: Adolescent social networks: friendship cliques, social isolates, and drug use risk, pp. 83–92. Tanglewood Research Inc., Greensboro, North Carolina (2000)
10. Falk, A., Fischbacher, U.: A theory of reciprocity. Games Econ. Behav. **54**(2), 293–315 (2006)
11. Fehr, E., Fischbacher, U., Gächter, S.: Strong reciprocity, human cooperation, and the enforcement of social norms. Hum. Nat. **13**(1), 1–25 (2002)
12. Fehr, E., Gächter, S.: Fairness and retaliation: the economics of reciprocity. J. Econ. Perspect. **14**(3), 159–181 (2000)
13. Gottman, J., Swanson, C., Murray, J.: The mathematics of marital conflict: dynamic mathematical nonlinear modeling of newlywed marital interaction. J. Fam. Psychol. **13**, 3–19 (1999)
14. Kleinberg, J., Oren, S.: Mechanisms for (mis)allocating scientific credit. In: Proceedings of the 43rd Annual ACM Symposium on Theory of Computing, pp. 529–538. ACM, New York (2011)
15. Koutsoupias, E., Papadimitriou, C.: Worst-case equilibria. In: 16th Annual Symposium on Theoretical Aspects of Computer Science, Trier, pp. 404–413, 4–6 March 1999
16. Nie, N.H., Hillygus, D.S.: Where does internet time come from? A reconnaissance. IT Soc. **1**, 1–20 (2002)
17. Nisan, N., Roughgarden, T., Tardos, E., Vazirani, V.: Algorithmic Game Theory. Cambridge University Press, Cambridge (2007)
18. Osborne, M.J., Rubinstein, A.: A Course in Game Theory, vol. 1. MIT Press, Cambridge (1994)
19. Polevoy, G., de Weerdt, M., Jonker, C.: The game of reciprocation habits. In: Proceedings of the 2016 European Conference on Artificial Intelligence (ECAI 2016). Frontiers in Artificial Intelligence and Applications, vol. 285, pp. 417–425 (2016)
20. Polevoy, G., Trajanovski, S., de Weerdt, M.M.: Nash equilibria in shared effort games. In: Proceedings of the 2014 International Conference on Autonomous Agents and Multi-agent Systems (AAMAS 2014), pp. 861–868. International Foundation for Autonomous Agents and Multiagent Systems, Richland (2014)
21. Polevoy, G., de Weerdt, M., Jonker, C.: The convergence of reciprocation. In: Proceedings of the 2016 International Conference on Autonomous Agents and Multiagent Systems (AAMAS 2016), pp. 1431–1432. International Foundation for Autonomous Agents and Multiagent Systems, Richland (2016)
22. Rabin, M.: Incorporating fairness into game theory and economics. Am. Econ. Rev. **83**(5), 1281–1302 (1993)
23. Schulz, A.S., Moses, N.S.: On the performance of user equilibria in traffic networks. In: Proceedings of the Fourteenth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA 2003), pp. 86–87. Society for Industrial and Applied Mathematics, Philadelphia (2003)
24. Seneta, E.: Non-negative Matrices and Markov Chains. Springer Series in Statistics. Springer, New York (2006). https://doi.org/10.1007/0-387-32792-4
25. Siegel, R.: All-pay contests. Econometrica **77**(1), 71–92 (2009)
26. Trivers, R.L.: The evolution of reciprocal altruism. Q. Rev. Biol. **46**, 35–57 (1971)
27. Van Segbroeck, S., Pacheco, J.M., Lenaerts, T., Santos, F.C.: Emergence of fairness in repeated group interactions. Phys. Rev. Lett. **108**, 158104 (2012)
28. Ward, M.D.: Modeling the USA-USSR arms race. Trans. Soc. Model. Simul. Int. **43**, 196–203 (1984)