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The effects of local perturbations on conservative particle systems

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Contents

Abstract

In this thesis, research was done in the area of interacting particle systems. Especially, the symmetric exclusion process with local perturbations was investigated. These perturbations, were in the form of sinks and sources, which add or take away particles at certain rates. Moreover, simulations were done for the asymmetric exclusion process. This process took place on a ring, with the addition of a source. For the symmetric exclusion process with sinks and sources, certain expressions were proven for the expected occupancy of a site. For the simulations, the main goal was to find out how the jumping rates and starting density, influenced the time to get to the fully occupied state, at different source rates. The first proof was for a source at an arbitrary site. From this expression, one could see that if the rates were recurrent, the system converged to the fully occupied state. If, however, the rates were transient, the system had a limiting density. Thereafter, it was shown that if a sink and source are placed in the same arbitrary site, the system always converged to a density, which under the Bernoulli measure, was not equal to the fully occupied state. The fact that the sink and source were in the same site, was an indispensable condition. Subsequently, the case of countably many sources was investigated. For which it was also shown that recurrent rates always yield a fully occupied state, as time tends to infinity. Whereas transient rates, once again, caused a limiting density. Moreover, the special case of a simple random walk in three dimensions or higher was investigated. If, for distances far away from the origin, the source rates could be bounded above by a certain function, then the system would not converge to the fully occupied state. Also, another proof showed that a recurrent set of source sites would always let the process converge to a fully occupied state. Lastly, similar conditions for one time dependent source at the origin were proven. Namely, it was shown for recurrent rates, that if the source dies out quick enough, as t tends to infinity, the system did not converge to the fully occupied state. For the simulation, we saw that $\langle T_f \rangle$ had the identity $b + \frac{k}{\lambda}$. The initial density had a linear effect on both the parameters b and k. The least squares error fit was applied to the found data with function $a, b = a\rho_0 + c$. This yielded the following values: $a = -417.8 \pm 85.6$ and $c = 911.2 \pm 54.94$ for b. And, we found that $a = -214.6 \pm 187.4$ and $c = 413.6 \pm 121.7$ for k. The same approach was used for finding the influence of $p - q$ on $\langle T_f \rangle$. Except the results were not that conclusive here. The same fit method with function $k, b = (p - q)a + c$, yielded the values: $a = -35.01 \pm 94.56$ and $c = 529.59 \pm 59.68$ for k. And, in this case: $a = 10.73 \pm 102.3$ and $c = 332.9 \pm 64.3$ for b. Because of the large errors and small values found, it was concluded that the relation was not linear. Finally, some concluding remarks were made on both aspects of the thesis, along with some recommendations for future projects related to this topic.

1. Introduction

In 1970 Frank Spitzer introduced the world to the phenomena of interacting Markov processes [10]. As it turns out, he laid the foundation for something nowadays known as interacting particle systems. Especially Thomas M. Ligget has contributed a lot to this field. Including, but not limited to, the description of certain models and their properties. Such as the voter model, the contact process and the exclusion process, to which he devoted an entire book[7]. These models are great examples of interacting particle systems. The voter model, for example, describes how voters, denoted by a site on a connected graph, influence each other. On the other hand: the contact process is a model that can describe the spread or dying out of an infection. In this report, the exclusion process plays a dominating role. In this case we look at particles performing a random walk on a graph where only one particle per site is allowed. In his first book, Ligget covered pretty much everything there was to know about this topic in 1985. Since then, the field has blown up to an extent where it would be impossible to cover it all in one book, which makes it an important area in probability theory.

Interacting particle systems can be used to model a lot of phenomena which have a probabilistic nature. For example, one could model traffic flow using the asymmetric simple exclusion process [3]. The subject has close links with the field of partial differential equations. These can be related via so-called hydrodynamic limits. Concretely, this means that the tools provided by interacting particle systems, is a great way to relate 'micro' to 'macro'[9], which can be used to obtain conservation laws (amongst other things). For example, it can be shown that the empirical density field of the symmetric exclusion process converges to the heat equation in the hydrodynamic limit [12]. These hydrodynamic limits are not really treated in this project, but it was mentioned to show the reader that these interacting particle systems can be extended to more than just a mathematical entity, which makes the number of applications unbounded.

As mentioned before, this project focuses on the symmetric exclusion process. We can define this process via unique Markov generators and semigroups. For this process, the conserved property is the number of particles. Subsequently, local perturbations in the form of sources and sinks will be added to these systems, which will cause the process to lose its conserved property. As it turns out, these local perturbations can have global effects on the system. Besides this, we consider the asymmetric simple exclusion process on a ring. If a source is added to this system, then it will always converge to the fully occupied state. Simulations will show how this time, to get to this fully occupied state, relates to the several parameters of the system. To get a good grasp of the concepts touched upon in this project, a preliminary knowledge of probability theory like [4] is assumed.

In chapter 2 we will treat the preliminary theory needed to work with these interacting particle systems. After the necessary preliminary theory, we will treat the processes used in Chapter 3. For these processes, we will prove the identities of the generators which describe them. We will lastly define what it means for a system to have sinks and sources in terms of the generator. In chapter 4 we will try to find the expected occupancy for a site in different scenarios. Besides that, we will also treat the results of simulations. Finally, in Chapter 5 a conclusion is given along with some remarks. This thesis was written as part of the double bachelor's degree in

Applied Mathematics and Applied Physics at the Delft University of Technology.

2. Preliminary theory

To completely understand the interacting particle systems used in this report, we start off with some preliminary theory.

2.1. Markov Processes

A stochastic process is Markov if the distribution of the stochastic process, only depends on the current state. In other words, if, conditional on its present and past states, its future value is independent of its past states. Let $\mathbb{X} = \{X_t, t \geq 0\}$ be the process to take values in a state space Ω. In this report, we will always work with processes that have right continuous with left limits trajectories (Càdlàg). Then a formal definition, equivalent to the one given in [9], for the Markov property is the following.

Definition 2.1 (The Markov property)**.** *Let Ω denote a state space which is a measurable space $(\Omega, \mathcal{A})$. Moreover, let $\{X_t, t \geq 0\}$ denote the stochastic process which takes values in Ω. Denote $\mathcal{F}_t = \sigma(\{X_r, 0 \leq r \leq t\})$, i.e. the σ-algebra generated by the random variable X_r. Now if for all $f : \Omega \to \mathbb{R}$ bounded and measurable and $0 \leq s \leq t$ we have that.*

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s] \tag{2.1}$$

Then the process is Markov.

In practice, the definition above states that the expectation, given some points in the past, only the last one is retained. In this report, we will deal with continuous time Markov jump processes. This means a random process in a certain state has a probabilistic waiting time before it jumps to the next state. The reader could wonder how these jumps are distributed. If the process continuously 'forgets' its past, this brings to mind the 'memory less' property of the exponential distribution [4]. This intuition turns out to be right, as the waiting times before jumping to a new site $x \in \Omega$ are exponentially distributed.

Proposition 2.2. *In order to ensure the Markov property of a process, the waiting times T_x for a site $x \in \Omega$ have to be exponentially distributed.*

Proof. Let the process be as described in definition 2.1, starting from $X_0 = x$. Then call T_x the time spent at x before jumping. Now, let $t \geq 0, s \geq 0$ be arbitrary such that $0 \leq s \leq t$, then we have the following.

$$\begin{aligned} \mathbb{P}(T_x > t | T_x > s) &= \mathbb{P}(T_x > t | X_s = x) \\ &= \mathbb{P}(T_x > t - s | X_0 = x) \end{aligned} \tag{2.2}$$

Where in the first equality we used the Markov property. Thus, we end up with the equality $\mathbb{P}(T_x > t | T_x > s) = \mathbb{P}_x(T_x > s)\mathbb{P}_x(T_x > t - s)$, where the subscript the denotes that both processes are starting from $X_0 = x$. Then, using the definition of conditional probability, one immediately gets the following.

$$\mathbb{P}_x(T_x > t) = \mathbb{P}_x(T_x > t)\mathbb{P}_x(T_x > t - s) \tag{2.3}$$

This is only possible if $T_x \sim exp(\lambda)$ for some $\lambda > 0$ ☐

So the previous proposition shows that starting from a site $x \in \Omega$, its waiting times are $exp(\lambda_x)$ distributed. After this, a new site $y \in \Omega$ must be chosen to jump to with probability $p(x, y)$. This process of waiting and jumping can be concealed in the parameter called rates, which are denoted by $c(x, y)$. Intuitively, this means the 'probability per unit of time'. Now for every $x \in \Omega$, the process waits exponentially with parameter $\lambda_x = \sum_y c(x, y) > 0$, [1] then it jumps to y with a probability $p(x, y) = \frac{c(x,y)}{\lambda_x}$.

2.2. Markov semigroups and generators on a finite state space Ω

For a Markov process $\{X_t, t \geq 0\}$ on a state space Ω, consider the following operator, called the semigroup.

$$S_t f(x) = \mathbb{E}[f(X_0)|X_0 = x] = \mathbb{E}_x[f(X_t)] \tag{2.4}$$

To intuitively make the concepts of semigroups clear, it makes sense to start off with a state space Ω which is finite. Then later in the more abstract theory, it will be extended to a compact metric space. For now, the finite (or countable) state space Ω is a good start. Semigroups can be seen as an operator which 'push' forward a function from x, a time step $t > 0$ further to X_t. This concept can be compared to the flow of a differential equation. But now, since the process is not deterministic, we still have to take the expectation to retrieve a value. Let $C(\Omega)$ denote the space of continuous functions $f : \Omega \to \mathbb{R}$. Note that for now, every f is continuous on Ω, since this state space is finite [1]. For the semigroup working on $f \in C(\Omega)$, one has the following properties.

Proposition 2.3 (Semigroup properties on a finite state space)**.** *Let $\{X_t, t \geq 0\}$ be a Markov process taking place on a finite state space Ω. For all $f : \Omega \to \mathbb{R}$, the family of operators $S_t : C(\Omega) \to C(\Omega)$ has the following properties.*

1. *$S_0 = I$ (Where I is the identity)*
2. *Right-continuity: the map $t \to S_t f$ is right continuous, $\forall t \geq 0$*
3. *Semigroup property: $S_{t+s} = S_t S_t$, $\forall t, s \geq 0$*
4. *Positivity: If $f \geq 0$ then $S_t f \geq 0$, $\forall t \geq 0$*
5. *Normalization: $S_t 1 = 1$, $\forall t \geq 0$*
6. *Contraction: $\sup_{x\in\Omega} |S_t f(x)| \leq \sup_{x\in\Omega} |f(x)|$, $\forall t \geq 0$*

Proof. Let $\{X_t, t \geq 0\}$ be a Markov process and let S_t be as in equation 2.4. Then for all $f \in C(\Omega)$ one has.

1. $S_0 f = \mathbb{E}[f(X_0)|X_0 = x] = f(x)$
2. Consider $S_t f(x) - f(x) = \mathbb{E}_x[f(X_t) - f(X_0)]$. $f(X_t) - f(X_0)$ is zero if no jump happened. Since the process takes place on the finite space it lasts a certain exponential time before the process makes a jump, and therefore $X_t \to X_0$ as $t \to 0$

[1] One should note that this is always defined for a finite state space. For the more general (compact) case, one needs extra conditions on the rates.

3. For the semigroup property consider the following

$$\begin{aligned} S_{t+s} &= \mathbb{E}_x[f(X_{t+s})] \\ &= \mathbb{E}_x[\mathbb{E}_x[f(X_{t+s})|\mathcal{F}_s]] \\ &= \mathbb{E}_x[\mathbb{E}_{X_s}(f(X_t)] \\ &= \mathbb{E}_x[S_t f(X_s)] = (S_s(S_t f(x))) \end{aligned} \tag{2.5}$$

   Where from the first to the second line the law of total expectation is used [13]. And from the second to the third line, the Markov property is used. One can therefore conclude that the Markov property is responsible for the semigroup property.

4. Positivity: Let $f \geq 0$, then, $S_t f(x) = \mathbb{E}_x[f(X_t)] \geq 0$ as the expectation of a positive value remains positive.

5. $S_t 1 = \mathbb{E}_x[1] = 1$

6. $|S_t f|(x) = |\mathbb{E}_x[f(X_t)]| \leq \mathbb{E}_x[|f(X_t)|] \leq \sup_{x\in\Omega} |f(x)|$, taking the supremum yields the desired result.

□

Note that for these properties, the state space being finite was only used for property (2). These properties summarize what a Markov semigroup is. As will be seen later, when the semigroup will be extended to the case where Ω is compact, it turns out that the converse reasoning is also true. So, consider a family of operators on $C(\Omega)$ that satisfy (1)—(6), then this is called a Markov semigroup and corresponds to a Markov process on Ω via $S_t f(x) = \mathbb{E}_x f(X_t)$. So one can go from a process to a semigroup and vice versa. These semigroups are linked to a generator. Property (3) suggests, namely, that there exists an operator A such that.

$$S_t f = e^{tA} f \tag{2.6}$$

This is defined as the generator. Since Ω is finite, A is a bounded operator and can therefore be defined by its Taylor series.

$$S_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = I + tA + \mathcal{O}(t^2) \tag{2.7}$$

From equation 2.7 it becomes clear how to find this generator.

$$Af = \lim_{t\to 0} \frac{S_t f - f}{t} \tag{2.8}$$

On the domain.

$$\mathbb{D}(A) = \left\{ A : \lim_{t\to 0} \frac{S_t f - f}{t} \text{converges} \right\} \tag{2.9}$$

This is a logical way to find A. Because if we look at equation (2.7), all the terms of t^2 and higher will vanish if t goes to zero. As mentioned before, all the above holds for a finite state space Ω. But for the general case where Ω is a compact state space, the semigroup is not equal to the Taylor series, as the generator typically becomes an unbounded operator. Therefore, we need a more formal theory on these subjects.

2.3. Semigroups and generators on a compact state space Ω

In this section, the more general case of a compact metric space Ω will be treated. The propositions and theorems will be given without proof, as they are beyond the scope of this project and do not contribute to the intuition significantly. The interested reader can find these in [6]. This section serves for the purpose of giving an intuition of how one would generate a semigroup from an unbounded operator. For this, we first have to consider the definition of a Feller process.

Definition 2.4 (Feller processes)**.** *A Markov process $\{X_t, t \geq 0\}$ on Ω is said to be a Feller process if $S_t f \in C(\Omega)$ for every $t \geq 0$ and $f \in C(\Omega)$*

So this basically means that if one has a Feller process, it still fulfils all the properties of proposition 2.3, except the right continuity now becomes an assumption. In other words, it will always be assumed that S_t maps from $C(\Omega)$ to itself. In this report, we will only consider Feller semigroups. Before finding an actual generator L, the definition of the pregenerator is given.

Definition 2.5 (Pregenerator)**.** *An operator[2] L on the domain D(L) is called a Markov pregenerator if it fulfils the following properties.*

1. *$D(L)$ is dense in $C(\Omega)$*
2. *$1 \in D(L)$, L1 =0*
3. *if $f \in D(L), \lambda \geq 0$ and $f - \lambda Lf = g$, then.*

$$\min_{\eta\in\Omega} f(\eta) \geq \min_{\eta\in\Omega} g(\eta) \tag{2.10}$$

Especially property (3) is very important to require, as the resolvent $(I - \lambda L)^{-1}$ is used to define the exponential of the operator. So it is essential to need some control over this resolvent. This is so significant because eventually one wants information on systems which may have infinite volume. At this infinite volume, the operators are in general unbounded. This means another way is needed to find the exponential of an operator. An equivalent definition on (iii) is the following, which is the maximum principle.

Proposition 2.6. *Assume L satisfies (1) and (2) from 2.5 and additionally if η_0 is such that $f(\eta_0) = \min_\eta\{f(\eta)\}$ implies that $Lf(\eta_0) \geq 0$. Then L also satisfies (3) from definition 2.5.*

From this pregenerator the closure is desired, as this is the key to finding the regular Markov generator. This is defined as follows.

Definition 2.7. *A linear operator $L : D(L) \to C(\Omega)$ is closed if its graph $G(L) = \{(f, Lf), f \in D(L)\}$ is a closed subset of $C(\Omega) \times C(\Omega)$ with measure, $||\cdot||_\infty \times ||\cdot||_\infty$*

Unbounded operators are in general not closed. Which means some kind of extension is needed to achieve this. This is exactly what being closable means.

Definition 2.8. *$L : D(L) \to C(\Omega)$ is closable if $\forall f_n \in D(L)$ such that $f_n \to 0$ and $Lf_n \to h$, then h = 0.*

[2]In this report we will always mean a linear operator

The intuition behind definition 2.8 is that in order to have a well-defined closure $\bar{L}$, we need that if some f_n converges to a limit f, then so does $\bar{L}f_n$. In other words, if one has that $f_n^{(1)} \to f$ and $f_n^{(2)} \to f$. Then if $\bar{L}f_n^{(1)} \to h$ and $\bar{L}f_n^{(2)} \to g$, we need $g = h$. And this is essentially the same as saying $\bar{L}(f_n^{(1)} - f_n^{(2)}) \to 0$. This is logical because it is desired that approaching a limit via different sequences, the operator should not yield different limits (as this would make it multivalued). For a closable operator, its closure is then defined as: $\bar{L} : D(\bar{L}) \to C(\Omega)$ where $D(\bar{L}) = \{f \in C(\Omega), \exists f_n \in D(L), f_n \to f, \exists g \in C(\Omega) : Lf_n \to g\}$. It is not always the case that an operator has a closure. Fortunately, one could prove the following proposition, which does make life a bit easier.

Proposition 2.9. *Suppose L is a Markov pregenerator. Then it is closable, and its closure $\bar{L}$ is again a Markov pregenerator.*

Moreover, it can be proved that.

Proposition 2.10. *Suppose L is a closed Markov pregenerator. Then the range of $I - \lambda L$ is a closed subset of $C(\Omega)$ for $\lambda > 0$.*

Now it is time to bring out the definition for a Markov generator.

Definition 2.11. *A Markov Generator is a closed Markov pregenerator L which satisfies.*

$$\mathcal{R}(I - \lambda L) = C(\Omega) \tag{2.11}$$

For a sufficiently small $\lambda > 0$.

So, this definition is stated in such a way that we can use the resolvent to define an operator as an exponential. Which brings us to the following proposition.

Proposition 2.12.

1. *A bounded Markov pregenerator is a Markov generator.*
2. *A Markov generator satisfies $\mathcal{R}(I - \lambda L) = C(\Omega)\ \forall \lambda > 0$.*

This result describes perfectly that in order to construct a (possibly unbounded) generator, one needs to have a bounded pregenerator. Which then finally allows us to write the one-to-one correspondence of Markov generators and semigroups on $C(\Omega)$, which is what we desired.

Theorem 2.13 (Hille-Yosida)**.** *There is a one-to-one correspondence between Markov generators and semigroups on $C(\Omega)$. This correspondence is as follows.*

1. *$D(L) = \left\{f \in C(\Omega) : \lim_{t\downarrow 0} \frac{S_t f - f}{t} \text{ exists}\right\}$ and L is then given by.*

$$Lf = \lim_{t\downarrow 0} \frac{S_t f - f}{t} \tag{2.12}$$

   Where $f \in D(L)$.

2. *$S_t f = \lim_{n\to\infty}(I - \frac{t}{n}L)^{-n} f$ for $f \in C(\Omega)$ and $t \geq 0$.*
3. *If $f \in D(L)$, then $\frac{d}{dt}S_t f = LS_t f = S_t L f$.*
4. *for $g \in C(\Omega)$ and $\lambda > 0$, the solution of $f - \lambda Lf = g$ is given by.*

$$f = \int_0^\infty e^{-t} S_{\lambda t}(g) dt \tag{2.13}$$

This gives us all the information we need to work with generators and semigroups on a compact state space Ω. Especially Hille-Yosida will (implicitly) be used quite frequently in this report.

2.4. Invariant and reversible measures

Let Ω be a compact state space and let $\mathbb{X} = \{\eta_t, t \geq 0\}$ be a Markov process on Ω with semigroup S_t. Now define the set $\mathscr{P}(\Omega) = \{\mu : \mu \text{ a probability measure on } \mathbb{X}\}$ which has a natural weak topology, defined by.

$$\mu_n \to \mu \Longleftrightarrow \int f d\mu_n \to \int f d\mu \tag{2.14}$$

For all $f \in C(\Omega)$, where $C(\Omega)$ denotes all the continuous functions $f : \Omega \to \mathbb{R}$. Then this set $\mathscr{P}(\Omega)$, equipped with this topology, forms a compact metric space. The process $\mathbb{X}$ is called invariant if a distribution μ is the same as now and a time step $t > 0$ further. Formally put in a definition, this yields.

Definition 2.14 (Invariant measures). *A probability measure $\mu \in \mathscr{P}(\Omega)$ is called invariant if.*

$$\int S_t f d\mu = \int f d\mu \tag{2.15}$$

for all $t \geq 0$, $f \in C(\Omega)$. We denote the set of all invariant measures by $\mathscr{I}$.

Remark. One could show that the set $\mathscr{I}$ is non-empty by a 'Bogolioubov-Krylov' argument. For this, we want to show that every accumulation point of the averages is an invariant measure itself. For this, consider that if.

$$\frac{1}{T_n} \int_0^{T_n} \mu S_t dt \to \mu* \tag{2.16}$$

As $T_n \uparrow \infty$, then $\mu*$ is invariant. Thus, if $T \to \infty$ along this subsequence, and this subsequence is convergent, we have that $\mu*$ is an invariant measure. Now since Ω is compact, there exists such a converging subsequence. Letting $s \geq 0$ we have that.

$$\frac{1}{T} \int_0^T \mu S_{t+s} dt = \frac{1}{T} \int_s^{T+s} \mu S_t dt = \frac{1}{T} \int_0^T \mu S_t dt - \frac{1}{T} \int_0^s \mu S_t dt + \frac{1}{T} \int_T^{T+s} \mu S_t dt \tag{2.17}$$

Now note that the last two terms converge to zero as $T \to \infty$ along this subsequence. We can then conclude that.

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \mu S_{t+s} dt = \lim_{T\to\infty} \frac{1}{T} \int_0^T \mu S_t dt \tag{2.18}$$

which is exactly what we wanted to show.

A stronger property than invariance is reversibility. This means that a distribution also remains the same under time-reversal. In other words, the process starting from η_0 has the same distribution as $\{\eta_{T-t} : 0 \leq t \leq T\}$ under the measure μ.

Definition 2.15 (reversible measures). *A measure μ is called reversible if for all f,g in $C(\Omega)$.*

$$\int (S_t f) g d\mu = \int f (S_t g) d\mu \tag{2.19}$$

It turns out that reversibility implies invariance.

Proposition 2.16. *A reversible measure is invariant.*

Proof. Let μ be reversible. This means $\forall f, g$ one has.

$$\int (S_t f) g d\mu = \int f (S_t g) d\mu \tag{2.20}$$

Letting $g = 1$ yields the desired result. □

2.5. Feynman-Kac Formula

The Feynman-Kac formula is a powerful tool that links two seemingly unrelated subjects, namely the solution of a partial differential equation to the expectation value of a random variable. In this specific case [5] it means that, even with a perturbation (which can be non-Markov) to a Markov process, the expectation can still be computed. For this, imagine the following situation: let S_t be a Markov semigroup with corresponding generator L. Then by Hille-Yosida, we have that.

$$\frac{d}{dt}S_t f(x) = LS_t f(x) \tag{2.21}$$

Now let S'_t be a non-Markov semigroup with corresponding generator $L + V$, with V being the perturbation. Due to the perturbation, this equality does no longer hold. Keeping this in mind, we would still like a solution to $\frac{d}{dt}S'_t f(x) = (L + V)S'_t f(x)$. So let $u(x, t) = (S'_t f(x))$, then the following theorem gives the solution to a partial differential equation in terms of an expectation operator.

Theorem 2.17. *Consider a bounded function $V : \mathbb{R}_+ \times \Omega \to \mathbb{R}$ and a bounded function $F_0 : \Omega \to \mathbb{R}$. Fix $T > 0$, then consider the following partial differential equation with Markov generator $L : D(L) \to C(\Omega)$.*

$$\frac{\partial}{\partial t}u(t, x) = (Lu)(t, x) + V(T - t, x)u(t, x) \tag{2.22}$$

With initial condition $u(x, 0) = F_0(x)$. Then the solution of this partial differential equation is given by.

$$u(x, t) = \mathbb{E}_x[e^{\int_0^t V(s,X_s)ds}F_0(X_t)] \tag{2.23}$$

The proof of this formula is beyond the scope of this project, but can be found in [5]. In practice, this will be used in the following way. Say there is a system of which the generator is known, like the symmetric exclusion process (as we will see in the next chapter). Now to this system a disturbance is added, say a sink/source at a certain location in $\mathbb{Z}^d$. Then, if the generator can be written as the sum of a known system and the disturbances, the Feynman-Kac formula can be used.

2.6. Duality

In the next chapter, certain interacting particle systems will be defined. It turns out that duality can be used to connect two processes via a duality function. As will be seen in chapter 3, this allows us, for example, to reduce a many-particle (possibly uncountable) problem to a single particle problem. This is due to the processes having the same generator and essentially solving the same differential equation. For duality, we use the definition as stated in [6].

Definition 2.18. *Suppose $\{\eta_t, t \geq 0\}$ and $\{X_t, t \geq 0\}$ are Markov processes with state spaces Ω_η and Ω_X respectively. And let $D(\eta, X)$ be a bounded measurable function on $\Omega_\eta \times \Omega_X$. The processes are said to be dual to one and the other with respect to D if*

$$\mathbb{E}_\eta[D(\eta_t, X)] = \mathbb{E}_x[D(\eta, X_t)] \tag{2.24}$$

for all $\eta \in \Omega_\eta$ and $X \in \Omega_X$.

Notice that the expectation operator on the left-hand side is working on the $\{\eta_t, t \geq 0\}$ process. Whereas it's working on $\{X_t, t \geq 0\}$ in the right-hand side, where X_t is a simpler process. This is exactly why duality is so powerful.

3. Processes for particles on $\mathbb{Z}^d$

3.1. Poisson process

A Poisson process is a specific Markov process used to describe the probability of a number of observations at a time t. Knowing this, yields the following distribution (as can be found, included with the proof, in [4]):

Theorem 3.1 (The Poisson distribution)**.** *for each $t > 0$, the random variable N_t has the Poisson distribution with parameter λt. that is, for $t > 0$,*

$$\mathbb{P}(N_t = k) = \frac{1}{k!}(\lambda t)^k e^{-\lambda t} \text{ for k = 0, 1, 2, 3,...} \tag{3.1}$$

3.2. Continuous time Markov chains (random walks on $\mathbb{Z}^d$)

For this report, stochastic processes are considered that take place on a compact state space Ω.

Definition 3.2 (Continuous time Markov chains)**.** *Let $\{X_t, t > 0\}$ be a Markov process on the state space $\mathbb{Z}^d$. This process waits for each site $x \in \mathbb{Z}^d$ an exponential time with parameter $\lambda_x = \sum_y c(x, y)$, and then jumps from x to y with probability $p(x, y) = \frac{c(x,y)}{\lambda_x}$. We call this process symmetric if $c(x, y) = c(y, x)$.*

Note that the definition above is equivalent to a particle performing a random walk on $\mathbb{Z}^d$, and is often how it will be called in this report. It can also be stated differently. Namely, as independent Poisson processes on the edge $\langle x, y\rangle$, with the number of jumps from x to y in a time t. In this way, we can create a path over the edges which will make the particle end up in a certain position. Since $t \in \mathbb{R}^+$, this process is also well-defined. As two Poisson clocks cannot ring simultaneously in continuous time. Intuitively, it is easy to understand that this process is Markov. Since at every $x \in \mathbb{Z}^d$, the transition probabilities only depend on the current site and not the past states. A special case of definition 3.2 is nearest neighbour walk, where $p(x, y) > 0$ if $|x - y| = 1$, and 0 otherwise. In chapter 4 this case will be considered for the asymmetric exclusion process.

A random walk of a particle on $\mathbb{Z}^d$ is either transient or recurrent. It is recurrent if the particle eventually returns to its starting point with probability 1. Whereas, the walk is said to be transient if it visits its starting point with a probability smaller than one. For a simple random walk on $\mathbb{Z}^d$, we have Pólya's recurrence theorem, which states that a random walk is always recurrent in $d = 1, 2$ but transient whenever $d \geq 3$[8].

3.2.1. Generator of a symmetric random walk on $\mathbb{Z}^d$

With the use of the definition, the generator of this specific process will be calculated. As we will see later, this process has a few dual processes on $\mathbb{Z}^d$.

Lemma 3.3. *Let $\{X_t, t \geq 0\}$ be the process as described in 3.2 with rates $c(x, y)$ such that $\sup_x \sum_y c(x, y) < \infty$ and $c(x, y) = c(y, x)$. Then, for a function $f(x) : \mathbb{Z}^d \to \mathbb{R}$, the generator is given by.*

$$Lf(x) = \sum_y c(x, y)(f(y) - f(x)) \tag{3.2}$$

Proof. Let $\{X_t, t \geq 0\}$ be a continuous time Markov chain on $\mathbb{Z}^d$ with rates $c(x, y)$ starting in $X_0 = x$. We define $\lambda_x = \sum_y c(x, y)$ as the inverse waiting time and $p(x, y) = \frac{c(x,y)}{\lambda_x}$ as the transition probability from site x to y. In order to find the generator, we first have to calculate.

$$S_t f(x) - f(x) = \mathbb{E}_x[f(X_t)] - f(x) \tag{3.3}$$

In order to do this, we start by defining $N_t = \#jumps$ in a time $t > 0$ where $\mathbb{P}(N_t = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$. Since the process has to make somewhere between 0 and ∞ jumps in this time, this makes a good partition of the event space. Using the partition theorem then yields.

$$\mathbb{E}_x[f(X_t)] = \sum_{(n=0)}^{\infty} \mathbb{E}_x[f(X_t) \cap N_t = n] \tag{3.4}$$

We then realise that, since we will be dividing by t and taking t to zero, in order to calculate the generator, all the terms above t^2 will vanish. This yields.

$$\begin{aligned} S_t f(x) - f(x) =& \ \mathbb{E}_x[f(X_t) \cap N_t = 0] + \mathbb{E}_x[f(X_t) \cap N_t = 1] + \mathcal{O}(t^2) - f(x) \\ =& \ e^{-\lambda_x t} f(x) + \lambda_x t e^{-\lambda_x t} \sum_y p(x, y) f(y) + \mathcal{O}(t^2) - f(x) \\ =& \ e^{-\lambda_x t} f(x) + t e^{-\lambda_x t} \sum_y \frac{c(x, y)}{p(x, y)} p(x, y) f(y) + \mathcal{O}(t^2) - f(x) \end{aligned} \tag{3.5}$$

Where we used the fact that $\lambda_x = \sum_y c(x, y) = \frac{c(x,y)}{p(x,y)}$. Dividing by t and letting $t \to 0$, yields the following.

$$\lim_{t \to \infty} \left(\frac{S_t f - f}{t} \right) = \lim_{t \to \infty} \frac{e^{-\lambda_x t} - 1}{t} f(x) + \sum_y c(x, y) f(y) \tag{3.6}$$

Finally, using l'Hôpital on the left term of the right-hand side, yields the desired result. □

3.3. The symmetric exclusion process on $\mathbb{Z}^d$

This model is essentially a collection of random walkers $(X_t^{(1)}, X_t^{(2)}, X_t^{(3)}...)$ which are all collected in a configuration random variable $\eta_t \in \{0, 1\}^{\mathbb{Z}^d}$. Moreover, only one random walker is allowed to be at a site $x \in \mathbb{Z}^d$. This means that the particles are now interacting with each other. In a formal definition from [6], this yields.

Definition 3.4. *Let $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ be a state space and let $\eta \in \Omega$ be a configuration. Then a particle $i \in \eta$ moves on $\mathbb{Z}^d$ according to the following rules:*

(a) There is always at most one particle per site.

(b) A particle at x waits $exp(\lambda_x)$ with $\lambda_x = \sum_y c(x, y)$, and then chooses a y with probability $p(x, y) = \frac{c(x,y)}{\lambda_x}$.

(c) If $\eta(y) = 0$, then the particle moves to y. If, however $\eta(y) = 1$, then the particle will stay at the site x.

Moreover, for cylinder functions (a function depending on finitely many coordinates), we can define the generator of this process as follows[7].

$$Lf(\eta) = \sum_{x,y} c(x,y,\eta)[f(\eta^{x,y}) - f(\eta)] \tag{3.7}$$

Where $\eta^{x,y}$ denotes the particle moving from x to y. It can be seen that the rates $c(x,y,\eta)$ depend on the process as well.

3.3.1. Generator of the exclusion process

In a moment, definition 3.7 will be used to calculate the generator of the exclusion process for the specific function $\eta(x)$. It will be seen that the generator, working on this function for this process, is the same as the generator for a single particle performing a random walk. This is of course remarkable, since this gives the ability to reduce a many-particle problem, to a single particle problem.

Lemma 3.5. *Let $\{\eta_t, t \geq 0\}$ be the exclusion process on $\mathbb{Z}^d$ as described in definition 3.4 with symmetric, non-zero rates for all $\langle xy \rangle$ such that $\sup_x \sum_y c(x,y) < \infty$. Then the generator $\tilde{L}$ working on $\eta(x) : \mathbb{Z}^d \to \{0,1\}$ is given by.*

$$\tilde{L}\eta(x) = \sum_y c(x,y)[\eta(y) - \eta(x)] \tag{3.8}$$

Proof. Let $\{\eta_t, t \geq 0\}$ be the exclusion process as described in definition 3.4. Let $f(\eta) = \eta(x)$, which is a local function and thus allows us to use the generator as posed in equation 3.7. $\eta^{x,y}$ denotes switching the occupations of positions x and y.

$$\eta^{x,y}(z) = \begin{cases} \eta(y) & \text{if } z = x \\ \eta(x) & \text{if } z = y \\ \eta(z) & \text{else} \end{cases} \tag{3.9}$$

Note that with this definition for $\eta^{x,y}$, the transition rates remain irreducible, since they are still greater than zero. And in this case $\eta \to \eta^{x,y}$ with rate $c(x,y,\eta) = c(x,y)\eta(x)(1-\eta(y))$ to ensure the exclusion principle. Note that for $\eta^{x,y}(z) - \eta(z)$, we have the following.

$$\eta^{x,y}(z) - \eta(z) = \begin{cases} -1 & \text{if } z = x \\ 1 & \text{if } z = y \\ 0 & \text{else} \end{cases} \tag{3.10}$$

Plugging these identities in our expression for the generator yields.

$$\begin{aligned} \tilde{L}\eta(z) &= \sum_{x,y} c(x,y)\eta(x)(1-\eta(y))[\eta^{x,y}(z) - \eta(z)] \\ &= \sum_x c(x,z)\eta(x)(1-\eta(z)) - \sum_y c(z,y)\eta(z)(1-\eta(y)) \\ &= \sum_x c(x,z)\eta(x)(1-\eta(z)) - \sum_x c(z,x)\eta(z)(1-\eta(x)) \\ &= \sum_x c(x,y)[\eta(x) - \eta(z)] \end{aligned} \tag{3.11}$$

Where in the last step we used symmetry. This concludes the proof. □

Now we can show the duality to a single random walker.

Theorem 3.6. *Let $\{\eta_t, t \geq 0\}$ be the exclusion process on $\mathbb{Z}^d$. Moreover, let $\{X_t, t \geq\}$ denote a random walk starting from $X_0 = x$. Then if $c(x, y) = c(y, x) \forall x, y \in \mathbb{Z}^d$ with the duality function $D(\eta, x) = \eta(x)$, we have that.*

$$\mathbb{E}_\eta[D(\cdot, x)(\eta_t)] = \mathbb{E}_x D[(\eta, \cdot)(X_t)] \tag{3.12}$$

Proof. Let L be the generator from lemma 3.3 for a random walk $\{X_t, t \geq 0\}$ starting from $X_0 = x$. Let $\tilde{L}$ be the generator of the exclusion process from lemma 3.5 working on the process $\{\eta_t, t \geq 0\}$. Then taking $D(\eta, x) = \eta(x)$ one sees.

$$\tilde{L}D(\cdot, x)(\eta) = LD(\eta, \cdot)(x) \tag{3.13}$$

Then because of Hille-Yosida, the duality follows for the semigroups as well □

3.4. Addition of sinks and sources

To the process defined in the previous section, one last modification is made. Namely, the addition of sources and sinks. A source at a location x adds particles to the system with a rate λ if there is no particle at that site. A sink does exactly the opposite, it takes away a particle with a rate μ if the site is occupied. Since these sinks and sources only have a local effect on $\{0, 1\}^{\mathbb{Z}^d}$, and the fact that this is a continuous time process: the generator just consists of the sum of the exclusion process and that of sinks and sources. This yields the following definition.

Definition 3.7. *Let $\{\eta_t, t \geq 0\}$ be the exclusion process on $\mathbb{Z}^d$. Let $\lambda(x)$ and $\mu(x)$ denote functions such that $\lambda : \mathbb{Z}^d \to \mathbb{R}$ and $\mu : \mathbb{Z}^d \to \mathbb{R}$. Then the generator of this process is given by.*

$$\mathbb{L}f(\eta) = \tilde{L}f(\eta) + \sum_x \lambda(x)(1 - \eta(x))(f(\eta^x) - f(\eta)) - \sum_x \mu(x)\eta(x_j)(f(\eta^x) - f(\eta)) \tag{3.14}$$

Where η^x denotes the flipping of the occupation on a certain site and $\tilde{L}$ is the generator of the exclusion process.

This equation implies that $\mathbb{E}_\eta[f(\eta)]$ can only be found using the Feynman-Kac formula, since the generator is not Markov anymore. We are curious if the process converges to a certain steady state or that the system will 'fill up'. This will be shown in the next chapter.

4. Results

In this section, the main goal is to find an expression for $\mathbb{E}_\eta[1-\eta_t(x)]$ or $\mathbb{E}_\eta[\eta_t(x)]$ under different circumstances.

4.1. One source at an arbitrary site

Let $\{\eta_t, t \geq 0\}$ be the exclusion process taking place on $\mathbb{Z}^d$. One source at an arbitrary site $x_i \in \mathbb{Z}^d$ implies that $\lambda(x) = \lambda\delta_{x,x_i}$. Then, using definition 3.7, the following generator will be used in order to prove the theorem.

$$Lf(\eta) = \sum_{x,y} c(x,y)[f(\eta^{x,y}) - f(\eta)] - \lambda\delta_{x,x_i}(1-\eta(x))(f(\eta^x) - f(\eta)) \tag{4.1}$$

One would like to know $\mathbb{E}[\eta_t(x)]$ at a time $t > 0$, since this is the expected occupation at a location $x \in \mathbb{Z}^d$. However, plugging this function into the generator yields the following.

$$\begin{aligned} L\eta(x) &= \sum_y c(x,y)[\eta(y) - \eta(x)] + \lambda\delta_{x,y}(1-\eta(x)) \\ &= A^{RW}\eta(x) + \lambda\delta_{x,x_i} - \lambda\delta_{x,x_i}\eta(x) \end{aligned} \tag{4.2}$$

Due to the inhomogeneity $\lambda\delta_{x,x_i}$, one cannot use Feynman-Kac directly and would have to use the variation of constants method. Which is possible, but using $1-\eta(x)$ instead does simplify the task. Moreover, the start configuration η_0 will be distributed according to a Bernoulli measure ν_ρ. So, at $t = 0$, each site has a probability ρ to be occupied by a particle. This measure will prove useful to determine the expected fraction of the sites that is filled. The expectation of an exclusion process starting from η_0 distributed according to ν_ρ, can be denoted by $\mathbb{E}_{\nu_\rho}[f(\eta_t)] = \int \mathbb{E}_\eta[f(\eta_t)]\nu_\rho(d\eta)$. Where for this specific case $f(\eta) = 1-\eta(x)$. This gives enough notation for the first theorem, accompanied by its proof.

Theorem 4.1 (Expectation of the exclusion process with a source at an arbitrary site)**.** *Let $\{\eta_t, t \geq 0\}$ be the symmetric exclusion process taking place on $\mathbb{Z}^d$ starting from $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$, which is ν_ρ distributed. Moreover, let x_i be the site of a source with constant rate λ. Then the interacting particle system has an expectation of the following form.*

$$\mathbb{E}_\eta[1-\eta_t(x)] = \mathbb{E}_x^{RW}[e^{-\lambda\int_0^t \delta_{X_s,xi}ds}(1-\eta_0(X_t))] \tag{4.3}$$

Which under the measure ν_ρ leads to an expectation.

$$\mathbb{E}_{\nu_\rho}[1-\eta(x)] = (1-\rho)\mathbb{E}_x^{RW}[e^{-\lambda\int_0^t \delta_{X_s,x_i}ds}] \tag{4.4}$$

Proof. Let $\{\eta_t, t > 0\}$ be the exclusion process starting from η_0 which is ν_ρ distributed. Let us start by using generator 4.1 with the function $f(\eta) = 1-\eta(x)$, for the reasons mentioned before. Then, plugging this in the equation yields.

$$L[1-\eta(x)] = \sum_y c(x,y)[(1-\eta(y)) - (1-\eta(x))] - \delta_{x,x_i}\lambda(1-\eta(x)) \tag{4.5}$$

Note that the minus sign before $\delta_{x,x_i}\lambda(1-\eta(x))$ makes sense because a source for a particle, is equivalent to a sink for a vacant site. Now let $D(x,\eta) = 1-\eta(x)$, then one can write the following.

$$LD(x,\cdot)(\eta) = \sum_y c(x,y)[D(y,\cdot)(\eta) - D(x,\cdot)(\eta)] - \lambda\delta_{x,x_i}D(x,\cdot)(\eta) \tag{4.6}$$

Where the $\cdot$ denotes that the operator is explicitly working on the η-variable. Next, consider a random walker $\{X_t, t \geq 0\}$ starting from $X_0 = x$, with a source at x_i and symmetric rates $c(x,y)$. Then for a function $f : \mathbb{Z}^d \to \mathbb{R}$ one has the following generator.

$$Af(x) = \sum_y c(x,y)[f(y) - f(x)] - \delta_{x,x_i}\lambda f(x) \tag{4.7}$$

Note that this generator is the sum of a random walk plus a potential $-\delta_{x,x_i}\lambda f(x)$. Since this differential equation is homogeneous, one can use the Feynman-Kac formula, which yields the following.

$$e^{tA}f(x) = \mathbb{E}_x^{RW}[e^{-\lambda\int_0^t \delta_{X_s,x_i}ds}f(X_0)] \tag{4.8}$$

Now notice the duality $AD(\cdot,\eta)(x) = LD(x,\cdot)(\eta)$ to conclude that.

$$e^{tL}[1-\eta(x)] = \mathbb{E}_\eta[1-\eta_t(x)] = \mathbb{E}_x^{RW}[e^{-\lambda\int_0^t \delta_{X_s,x_i}ds}(1-\eta_0(X_t))] \tag{4.9}$$

Finally, we integrate over the Bernoulli measure, such that.

$$\mathbb{E}_{\nu_\rho}[1-\eta(x)] = (1-\rho)\mathbb{E}_x^{RW}[e^{-\lambda\int_0^t \delta_{X_s,x_i}ds}] \tag{4.10}$$

Which proves the result. □

Note that the term $\int_0^t \delta_{x,xi}ds$ is exactly the time spent of a random walker in this point x_i. Thus, if $t \to \infty$, the configuration will fill up if the particle spends infinite time in this point, which is the case with recurrent rates. On the other hand, it will establish a constant density if the time spent in this point is finite, which is the case when the rates are transient.

4.2. One sink and source at an arbitrary site

Calculating the expected density of the exclusion process with one sink and source at an arbitrary site, will be done similarly to the case with one sink. Except, we now have no other option than to use variation of constants. So consider the generator with functions $\lambda(x) = \lambda\delta_{x,x_i}$ and $\mu(x) = \mu\delta_{x,x_i}$.

$$\begin{aligned} Lf(\eta) = \sum_{x,y} c(x,y)[f(\eta^{x,y}) - f(\eta))] - \delta_{x,x_i}\mu\eta(x)(f(\eta^x) - f(\eta)) \\ + \delta_{x,x_i}\lambda(1-\eta(x))(f(\eta^x) - f(\eta)) \end{aligned} \tag{4.11}$$

In the following theorem, an expression for the expected density is found.

Theorem 4.2 (Expected occupancy of symmetric exclusion with a sink and source at an arbitrary site)**.** *Let $\{\eta_t, t \geq 0\}$ be the symmetric exclusion process taking place on $\mathbb{Z}^d$ starting from $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$, which is ν_ρ distributed. Moreover, let x_i be the site of a source and a sink with corresponding rates λ, μ. Then the interacting particle system has an expected density of the following form.*

$$\mathbb{E}_\eta[\eta_t(x)] = \mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}\eta_0(X_t)] + \frac{\lambda}{\lambda+\mu}(1 - \mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}]) \tag{4.12}$$

Which, under the measure ν_ρ, yields the following.

$$\mathbb{E}_{\nu_\rho}[\eta_t(x)] = \frac{\lambda}{\lambda+\mu} + (\rho - \frac{\lambda}{\lambda+\mu})\mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}]) \tag{4.13}$$

Proof. Let $\{\eta_t, t > 0\}$ be the exclusion process starting from η_0, which is ν_ρ distributed, with a sink and source at an arbitrary site x_i. Moreover, let $\psi(x,t)$ denote $\mathbb{E}_\eta[\eta_t(x)]$. By using generator 4.11 with the function $f(\eta) = \eta(x)$, we find the following equation.

$$L\eta(x) = \sum_y c(x,y)[\eta(y) - \eta(x)] - \delta_{x,x_i}\mu\eta(x) - \delta_{x,x_i}\lambda\eta(x) + \delta_{x,x_i}\lambda \tag{4.14}$$

As said before, due to the inhomogeneous nature of this equation, we cannot use Feynman-Kac directly. Instead, we have to solve using variation of constants. For this, consider a random walker $\{X_t, t \geq 0\}$, with a sink and source at the same location x_i. Then, for a generator B and a function $\phi(x,t)$, we have the following.

$$B\phi(x,t) = \frac{d}{dt}\phi(x,t) = A\phi(x,t) + \lambda\delta_{x,x_i} \tag{4.15}$$

Where $A = A^{RW} - \lambda\delta_{x,x_i} - \mu\delta_{x,x_i}$ and A^{RW} denotes the generator of a single random walker. Then the homogeneous solution is given by the Feynman-Kac formula once again.

$$\phi_h(x,t) = \mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}\eta_0(X_t)] = e^{tA}\phi(x,0) \tag{4.16}$$

The last thing we have to do is find a particular solution. For this, return to equation 4.14 and try the Ansatz $\phi(x,t) = C(t)e^{tA}$, where $C(t)$ is an unknown function depending on t. Plugging this back into the differential equation yields.

$$e^{tA}\frac{d}{dt}C(t) = \lambda\delta_{x,x_i} \longrightarrow C(t) = \lambda\int_0^t e^{-sA}\delta_{x,x_i} ds + \phi_h(x,t) \tag{4.17}$$

Where we have used the chain rule for differentiation. Then the final solution yields.

$$\phi(x,t) = \lambda\int_0^t e^{(t-s)A}\delta_{x,x_i} ds + e^{tA}\phi(x,0) \tag{4.18}$$

Now let $D(x,\eta(x)) = \eta(x)$ be a duality function. Then it can be concluded that $LD(x,\cdot)(\eta) = BD(\cdot,\eta)(x)$. Thus, letting e^{tb} work on the x variable of $\eta(x)$ yields.

$$\begin{aligned}\mathbb{E}_\eta[\eta_t(x)] &= e^{tL}D(x,\cdot)(\eta) = e^{tB}D(\cdot,\eta)(x) \\ &= e^{tA}\eta_0(x) + \lambda\int_0^t e^{(t-s)A}\delta_{x,x_i} ds \\ &= \mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}\eta_0(X_t)] + \lambda\int_0^t e^{sA}\delta_{x,x_i} ds\end{aligned} \tag{4.19}$$

Where in the last line the substitution $u = t - s$ was used. Now consider the following.

$$S_t f(x) = e^{tA} f(x) \tag{4.20}$$

Then for the derivative working on $f(x) = 1$, one has using Hille-Yosida, that.

$$\frac{d}{dt}(S_t 1) = -(\lambda + \mu)\delta_{x,x_i}(S_t 1) \tag{4.21}$$

Where we used that the $A^{RW} 1 = 0$, since it is a Markov generator. Plugging this back in 4.19, using the fundamental theorem of calculus, yields.

$$\begin{aligned} \mathbb{E}[\eta_t(x)] &= \mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}\eta_0(X_t)] - \frac{\lambda}{\lambda+\mu}\int_0^t \frac{d}{ds}(S_s 1)dt \\ &= \mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}\eta_0(X_t)] + \frac{\lambda}{\lambda+\mu}(1 - \mathbb{E}_x[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}]) \end{aligned} \tag{4.22}$$

Integrating over the measure yields.

$$\mathbb{E}_{\nu_\rho}[\eta_t(x)] = \frac{\lambda}{\lambda+\mu} + \left(\rho - \frac{\lambda}{\lambda+\mu}\right)\mathbb{E}_x^{RW}[e^{-(\lambda+\mu)\int_0^t \delta_{X_s,x_i} ds}]) \tag{4.23}$$

□

Remark. Note that if the source and sink are not in the same site, this causes trouble. Because if the source site was x_i and the sink site x_j, with $x_i \neq x_j$, the solution would not be that elegant.

$$\begin{aligned} \mathbb{E}[\eta_t(x)] &= \mathbb{E}_x^{RW}[e^{-\lambda\int_0^t \delta_{X_s,x_i} ds} e^{-\mu\int_0^t \delta_{X_s,x_j} ds}\eta_0(X_t)] \\ &- \lambda\int_0^t \mathbb{E}_x^{RW}[e^{-\lambda\int_0^s \delta_{X_r,x_i} dr - \mu\int_0^s \delta_{X_r,x_j} dr}\delta_{X_s,x_i}]ds \end{aligned} \tag{4.24}$$

Which does not have an analytical solution due to the fact that we cannot use the same 'semi-group' trick as we did in theorem 4.2 in equation 4.21.

4.3. A countable set of sources

In this section, a more general case will be considered. Namely, the case where we have a set of countable sources. So in this case, $\lambda(x)$ is just a general function that maps from $\mathbb{Z}^d$ to $[0, \infty)$. Meaning that every site has its own rate, but the true sources will have $\lambda(x) > 0$. In this section, the following generator will be used.

$$Lf(\eta) = \sum_y c(x,y)[f(\eta^{x,y}) - f(\eta)] + \sum_x \lambda(x)(1 - \eta(x))(f(\eta^x) - f(\eta)) \tag{4.25}$$

We will again be looking for an expression for the expected occupancy. This yields the following theorem.

Theorem 4.3. *Let $\{\eta_t, t \geq 0\}$ be the symmetric exclusion process on $\mathbb{Z}^d$ starting from $\eta_0 \in \{0,1\}^{\mathbb{Z}^d}$, which is ν_ρ distributed. Moreover, let $\lambda(x)$ be a function that assigns every site $x \in \mathbb{Z}^d$ a rate. Then the expected occupation is as follows.*

$$\mathbb{E}_\eta[1 - \eta_t(x)] = \mathbb{E}_x^{RW}[e^{-\int_0^t \sum_x \lambda(X_s) ds}(1 - \eta_0(X_t))] \tag{4.26}$$

Which under the measure ν_ρ yields.

$$\mathbb{E}_{\nu_\rho}[1 - \eta_t(x)] = (1 - \rho)\mathbb{E}_x^{RW}[e^{-\int_0^t \sum_x \lambda(X_s) ds}] \tag{4.27}$$

Proof. Let $\{\eta_t, t > 0\}$ be the exclusion process starting from η_0, which is ν_ρ distributed, with sources at arbitrary sites with rate $\lambda(x) > 0$. Moreover, let $\psi(x,t)$ denote $\mathbb{E}_\eta[1 - \eta_t(x)]$. By using generator 4.25 with the function $f(\eta) = 1 - \eta(x)$, we have that.

$$L[1 - \eta(x)] = \sum_y c(x,y)[(1 - \eta(y)) - (1 - \eta(x))] - \sum_x \lambda(x)(1 - \eta(x)) \tag{4.28}$$

Consider a random walker $\{X_t, t \geq 0\}$ with sources at the same sites. Then for a generator H and function $\phi(x,t)$ we have.

$$\frac{d}{dt}\phi(x,t) = H\phi(x,t) \tag{4.29}$$

Where $H = H^{RW} - \sum_x \lambda(x)(1 - \eta(x_i))$. Thus, solving with the Feynman-Kac formula.

$$\phi(x,t) = \mathbb{E}_x^{RW}[e^{\int_0^t \sum_x \lambda(X_s) ds}(1 - \phi(x,0))] \tag{4.30}$$

With the duality function $D(x,\eta) = 1 - \eta(x)$, one immediately concludes that $LD(x,\cdot)(\eta) = HD(\cdot,\eta)(x)$. Which we can exploit to find $\psi(x,t)$.

$$\begin{aligned} \mathbb{E}_\eta[1 - \eta_t(x)] &= e^{tL}D(x,\cdot)(\eta) = e^{tH}D(\cdot,\eta)(x) \\ &= \mathbb{E}_x^{RW}[e^{-\lambda\int_0^t \sum_x \delta_{X_s,x_i} ds}(1 - \eta_0(X_t))] \end{aligned} \tag{4.31}$$

Integrating over the Bernoulli measure yields the result. □

In the theorem above, it can be seen that for certain conditions, the system will 'fill up' to a configuration in which every site is occupied. This can be seen in the next corollary.

Corollary 4.4. *Let $\{\eta_t, t \geq 0\}$ be the process as described in theorem 4.3. Then depending on the rates we have the following.*

1. *If $c(x,y)$ is recurrent then $\nu_\rho(t) \to \delta_1$ as $t \to \infty$, where δ_1 denotes the fully occupied state.*

2. *If $c(x,y)$ is transient and the number of sources is finite, then the limiting density is as follows.*

$$\lim_{t\to\infty} \mathbb{E}_{\nu_\rho}[1 - \eta_t(x)] = (1 - \rho)\mathbb{E}_x^{RW}[e^{-\int_0^\infty \lambda(X_s) ds}] \tag{4.32}$$

Proof. This result follows immediately when we look at the term $\int_0^\infty \sum_x \lambda(X_s)ds$, which is an expression for the time that a random walker spends in the source sites. Thus, if the rates are recurrent, the time spent will diverge if $t \to \infty$, which results in $\mathbb{E}_{\nu_\rho}[1-\eta_t(x)]$ converging to zero. In the transient case, this integral converges, which yields $0 < \mathbb{E}_{\nu_\rho}[1-\eta_t(x)] < 1-\rho$. This concludes the proof. □

From this we can conclude that the expected occupancy converges to 1 if the random walk spends an infinite time in the source sites as t goes to infinity. The reader might wonder if in the transient case, there are certain conditions on the source function $\lambda(x)$, such that the configuration still converges to δ_1 or not. The next theorems explore lower and upper bounds for $\mathbb{E}_{\nu_\rho}[1-\eta_t(x)]$. Theorem 4.5 gives a condition for not filling up in the case where each particle performs a nearest neighbour random walk. Whereas theorem 4.7 gives the condition for convergence to δ_1 with transient rates.

Theorem 4.5 (Condition for not filling up). *Let $\{\eta_t, t \geq 0\}$ be the exclusion process on $\mathbb{Z}^d$ with $d \geq 3$. Where we assume that $c(x,y)$ is transient nearest neighbour. Let G be a countable set which contains the source sites, each site with its own rate λ_y, which is equal to zero whenever $y \notin G$. We assume that these rates are bounded above by a function $h(|y|)$ such that $0 \leq \lambda_y \leq h(|y|)$. Let r denote the distance from the origin to a point $y \in \mathbb{Z}^d$. Moreover, assume that the sources are far away and thus $r \to \infty$, then the configuration will not fill up if λ_r has the following long distance behaviour.*

$$\int_1^\infty C(d) r \lambda_r < \infty \tag{4.33}$$

Where $C(d)$ is a constant depending on the dimension in which the process takes place.

Proof. Let $\{\eta_t, t \geq 0\}$ be the exclusion process as described above. We wish to find a lower bound such that $\mathbb{E}_{\nu_\rho}[1-\eta_t(x)]$ does not converge to zero, which implies that $\nu_\rho(t)$ does not converge to δ_1. To get an expression for this lower bound, we use Jensen's inequality [4].

$$\mathbb{E}_x^{RW}[e^{-\int_0^t \lambda(X_s)ds}] \geq e^{-\int_0^t \mathbb{E}_x^{RW}[\lambda(X_s)]} \tag{4.34}$$

Then the integral can be written as.

$$\int_0^t \mathbb{E}_x^{RW}[\lambda(X_s)] = \sum_{y \in G} \lambda_y \int_0^t \mathbb{E}_x^{RW}[\delta_{X_s,y}]ds \to \sum_{y \in G} \lambda_y G(x,y) \tag{4.35}$$

As $t \to \infty$, where $G(x,y)$ is Green's function: for every random walker starting from $x \in \mathbb{Z}^d$, this denotes the expected time spent in y. According to [11], for a nearest neighbour random walk in $d \geq 3$, we have that for $|x-y| \to \infty$.

$$G(x,y) \simeq \frac{C(d)}{|x-y|^{d-2}} \tag{4.36}$$

If the sum in equation 4.35 converges, we will have a sufficient condition for the configuration not filling up. Since x is fixed, it suffices to show that the sum converges in $x = 0$.

$$\sum_{y \in G} \lambda_y G(0,y) \text{ converges } \implies \sum_{y \in G} \lambda_y G(x,y) \text{ converges} \tag{4.37}$$

Furthermore, we assumed that $0 \leq \lambda_y \leq h(|y|)$, which yields the following implication.

$$\sum_{y\in G} h(|y|)G(0,y) \text{ converges} \implies \sum_{y\in G} \lambda_y G(0,y) \text{ converges} \tag{4.38}$$

Finally, since $h(|y|)$ is a positive function, we can state the final implication.

$$\int_1^\infty \frac{C(d)h(|y|)}{|y|^{d-2}}dy \implies \sum_{y\in G} h(|y|)G(0,y) \text{ converges} \tag{4.39}$$

Then, if we switch to polar coordinates, we arrive at the following expression.

$$\int_1^\infty \frac{C(d)\lambda_r r^{d-1}}{r^{d-2}}dr = \int_1^\infty C(d)r\lambda_r dr \tag{4.40}$$

Thus, equation 4.35 will converge if λ_r behaves as follows at large r.

$$\int_1^\infty C(d)r\lambda_r < \infty \tag{4.41}$$

Which finishes the proof. □

The previous theorem is quite powerful. For example, if $\lambda_r = ke^{-cr}$ for some $k, c > 0$ when $r \to \infty$, we will have that the system does not fill up as the improper integral does not diverge in this case. More generally, every function λ_r that is smaller than $1/r^2$ will be small enough to let the configuration not converge to δ_1.
Conversely, it would be interesting to find a condition for which the configuration does converge to δ_1, even though the rates are transient. For this, we first need a definition from [11].

Definition 4.6 (A recurrent subset). *Define the hitting probability for a point $y \in \mathbb{Z}^d$ as $H(x,y) = \mathbb{P}_x(X_t = y; T < \infty)$ for $x \in \mathbb{Z}^d$. In other words, if a random walker starts from some $x \in \mathbb{Z}^d$, then $H(x,y)$ gives the probability to be at y for some $T \geq 0$. Also define $H_A(x) = \sum_{y\in A} H(x,y)$ as the entrance probability of A. We say that this set A is recurrent if $H_A(x) = 1$, $\forall x \in \mathbb{Z}^d$. The set is transient if it is not recurrent.*

Armed with this definition, we are able to formulate and prove the following theorem.

Theorem 4.7 (Condition for filling up). *Let $\{\eta_t, t \geq 0\}$ be the exclusion process on $\mathbb{Z}^d$. Moreover, let $G \subset \mathbb{Z}^d$ be a countable set which contains the source sites, each with its own rate λ_y such that $\sup_{y\in G} \lambda_y < \infty$ and $\lambda_y = 0$ whenever $y \notin G$. Moreover, assume that $\exists \delta > 0$ such that $\min_{y\in G} \lambda_y = \delta$. Then the system will fill up iff the set G is recurrent.*

Proof. Let $\{\eta_t, t \geq 0\}$ be the process as described above, with G again a countable set containing the source sites. Then the solution is of the form.

$$\mathbb{E}_\eta[1-\eta_t(x)] = \mathbb{E}_x^{RW}[e^{-\int_0^t \lambda(X_s)ds}] = \mathbb{E}_x^{RW}[e^{-\sum_{y\in G}\lambda_y l_t(x,y)}] \tag{4.42}$$

where $l_t(x,y) = \int_0^t I(X_s = y)ds$ denotes the time spent in y of a random walker which started from x. As $t \to \infty$ this converges to the following in the transient case.

$$\lim_{t\to\infty} \mathbb{E}_\eta[1-\eta_t(x)] = \mathbb{E}_x^{RW}[e^{-\sum_{y\in G} l_\infty(x,y)\lambda_y}] \tag{4.43}$$

The condition for filling up is when $\sum_{y\in G} l_\infty(x,y)\lambda_y = \infty$ almost sure. Since we assumed that $\sup_{y\in G}\lambda_y < \infty$ and the fact that $l_\infty(x,y) < \infty$ in the transient case, it must depend on the set G. Which yields the conclusion that the configuration will converge to δ_1 if and only if G is recurrent. □

This theorem also has some nice applications. If for example, the complete xy-plane is filled with sources in the three-dimensional case, which are all greater equal than $\delta > 0$. Then this is a recurrent set, which yields the conclusion that this system must fill up.

4.4. Time dependent sources

In the Feynman-Kac formula, we see that the disturbance $V(x,t)$ can also have a time dependence. So this suggests that we can add particles to the system where λ has a time dependence. So consider the exclusion process $\{\eta_t, t \geq 0\}$ with time dependent source at $x = 0$. This yields the following generator working on the function $[1-\eta(x)]$.

$$L[1-\eta(x)] = \sum_x c(x,y)[[1-\eta(0)] - [1-\eta(x)]] - \lambda(t)\delta_{x,0}(1-\eta(x)) \tag{4.44}$$

Where $\lambda(t)$ can have various identities. The Feynman-Kac formula then yields the following solution.

$$\mathbb{E}_\eta[1-\eta_t(x)] = \mathbb{E}_x^{RW}[e^{-\int_0^t \lambda_{t-s}\delta_{X_s,0}ds}(1-\eta_0(X_t))] \tag{4.45}$$

$-\int_0^t \lambda_{t-s}\delta_{X_s,0}ds$ is not an integral that has an analytical solution. That is why we will try to find a proper inequality to get a condition for converging to δ_1 or not. The next theorem gives a condition for not filling up with recurrent rates.

Theorem 4.8 (Condition for recurrent rates not filling up)**.** *Let $\{\eta_t, t \geq 0\}$ be the exclusion process on $\mathbb{Z}^d$ where $c(x,y) = c(y,x)$ is recurrent. We place a source at the origin with time dependent rate λ_t, which converges to zero as t $\to \infty$. Now let $q > \max\{2/d, 1\}$ and choose $p \in (1,\infty)$ such that $\frac{1}{q} + \frac{1}{p} = 1$. Then the configuration does not fill up if $\lambda_t \in L^p(\mathbb{R}^+)$ (i.e. $\int_0^\infty \lambda_s^p ds < \infty$).*

Proof. Let the process be as described above. The expected value of $1-\eta(x)$ is as in equation 4.45. We wish to find a lower bound of this expectation such that we can show that it does not converge to the fully occupied state. For this, consider Jensen's inequality.

$$\mathbb{E}_x^{RW}[e^{-\int_0^T \lambda_{T-s}\delta_{X_s,0}ds}] \geq e^{-\int_0^T \lambda_{T-s}\mathbb{E}_x[\delta_{X_s,0}]ds} = e^{-\int_0^T \lambda_{T-s}p_s(x,0)ds} \tag{4.46}$$

Where in the last step we used the definition of the expectation. We want the integral to be finite. We use the fact that $0 \leq p_s(x,0) \leq p_s(0,0)$ [11] to find the following inequality.

$$\int_0^T \lambda_{T-s}p_s(x,0)ds \leq \int_0^T \lambda_{T-s}p_s(0,0)ds \tag{4.47}$$

We also have the local central limit theorem, which implies that $p_s(0,0) \leq \frac{C(d)}{(s+1)^{d/2}}$[11]. If we use this, we arrive at the following inequality.

$$\int_0^T \lambda_{T-s}p_s(0,0)ds \leq \int_0^T \lambda_{T-s}\frac{C(d)}{(1+s)^{d/2}}ds \tag{4.48}$$

Now let $p, q \in (1, \infty)$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then Hölder's inequality yields the following.

$$\int_0^T \lambda_{T-s} \frac{C(d)}{(1+s)^{d/2}} ds \leq \left(\int_0^T \lambda_s^p ds \right)^{1/p} \left(\int_0^T \left(\frac{C(d)}{(1+s)^{d/2}} \right)^q ds \right)^{1/q} \tag{4.49}$$

Where for $\int_0^T \lambda_s^p ds$, we used the substitution rule. Thus, the expected occupancy has an upper bound if for $T \to \infty$ both integrals are finite. The right integral converges since $q > \max\{2/d, 1\}$. Now because $\lambda_s \in L^p(\mathbb{R}^+)$, we have that both terms converge as $T \to \infty$. This finishes the proof. □

Remark. The transient case, where $\lambda_t \to \infty$ as $t \to \infty$, does not necessarily imply that the configuration converges to δ_1. Because for this we would need to have that (looking again at the solution provided in 4.45).

$$\int_0^T \lambda_{T-s} \delta_{X_s,0} ds \to \infty \text{ a.s.} \tag{4.50}$$

To show this we could start off by taking $\epsilon > 0$ arbitrarily small, then since the rate is positive and continuous.

$$\int_0^T \lambda_{T-s} \delta_{X_s,0} ds \geq \int_{T-\epsilon}^T \lambda_{T-s} \delta_{X_s,0} ds \simeq \lambda_T \int_{T-\epsilon}^T \delta_{X_s,0} ds \tag{4.51}$$

For this to diverge, we would need to know how fast $\int_{T-\epsilon}^T \delta_{X_s,0} ds$ converges to zero as $t \to \infty$, which we do not know for the moment. Only then a statement about filling up or not can be made.

4.5. The one dimensional exclusion process with asymmetric rates

Until this point, only the symmetric exclusion process on $\mathbb{Z}^d$ was considered. But what if this is not the case? In other words: what would happen if $c(x, y) \neq c(y, x)$. Moreover, what would happen if a source is added to this asymmetric case. We will analyse these cases on a lattice, which is a ring consisting of N sites. We will assume that every particle performs a simple (but asymmetric) walk. The rules of the exclusion process are still the same as in definition 3.4. So consider a configuration on a one-dimensional lattice which has a length of N sites. If a particle moves to the right when $i = N$, it ends up at $i = 0$ again. The source will be left out for the moment. Imagine the particles moving with a rate p to the right and q to the left such that $p > q$.We denote the occupation at the site $i \in [0, N]$ by η_i. Then the generator of this process on η_i looks as follows.

$$L\eta_i = p[\eta_{i-1}(1 - \eta_i) - \eta_i(1 - \eta_{i+1})] + q[\eta_{i+1}(1 - \eta_i) - \eta_i(1 - \eta_{i-1})] \tag{4.52}$$

As can be seen immediately, duality with a single random walker does not work anymore. What does work is seeing $[\eta_{i-1}(1-\eta_i) - \eta_i(1-\eta_{i+1})]$ and $[\eta_{i+1}(1-\eta_i) - \eta_i(1-\eta_{i-1})]$as left and right discrete derivatives of $\rho(x, t)$, respectively, which is the mass density. This yields the Burgers equation[3] [2].

$$\frac{\partial}{\partial t} \rho = -p \frac{\partial \rho(1-\rho)}{\partial x} + q \frac{\partial \rho(1-\rho)}{\partial x} = (q - p) \frac{\partial \rho(1-\rho)}{\partial x} \tag{4.53}$$

[3] An equation named after the Dutch Physicist Johannes Martinus Burgers

This is a well known equation also used for gas dynamics, traffic flow problems and liquid dynamics. This equation has a property that due to the discontinuities it can create, it could create a so-called shock wave. In this context, that also intuitively makes sense: it may seem like there is a sort of 'drift' speed moving the particle density to the right. Now, adding a source term at $i = 0$ with rate λ changes the generator to.

$$L\eta_i = p[\eta_{i-1}(1-\eta_i) - \eta_i(1-\eta_{i+1})] + q[\eta_{i+1}(1-\eta_i) - \eta_i(1-\eta_{i-1})] + \lambda(1-\eta_i)\delta_{i,0} \quad (4.54)$$

Which then yields the following.

$$\frac{\partial}{\partial t}\rho = (q-p)\frac{\partial \rho(1-\rho)}{\partial x} + \lambda(1-\rho)\delta_{i,0} \quad (4.55)$$

With the model we have just defined, the Bernoulli's measure $\nu_\rho(t)$ will always converge to δ_1, since the process takes place on a closed ring. We denote this time to get to δ_1 by T_f. Since T_f itself is a random variable, we will try to approximate $\mathbb{E}[T_f]$, which depends on the initial density ρ_0, jumping rates p, q, lattice length N and source rate λ. by a sample mean $\langle T_f \rangle$, which is an average of the observations at λ, we try to approximate this expectation. Our goals this section is to see what effect the following parameters have when we plot $\langle T_f \rangle$ against λ.

1. **The initial density** ρ_0**:** In both the symmetric and asymmetric case, this parameter allows for $\langle T_f \rangle$ to become smaller at every λ. However, does this effect decrease at larger λ.

2. **The jumping rates** p, q**:** in the simulation, we will always have that $p + q = 1$ for convenience. This should alter $\langle T_f \rangle$ compared to the symmetric case. Because one would expect, that in the symmetric case, the particles 'pile up' around i=0. Whereas for the asymmetric case, the difference between p, q allows the particles to 'escape' from the source site. This should allow new particles to join the ring more often.

4.5.1. The influence of ρ_0 on $\langle T_f \rangle$

For this, we take the source rates λ to be ranging from 0.1 to $5(p+q)$ with increments of 0.1. Per value of λ, the simulation is repeated ten times. The average $\langle T_f \rangle$ over these experiments is taken as we try to approximate $\mathbb{E}[T_f]$. We repeat this same process for several ρ_0. This yields the plots in figure 1.

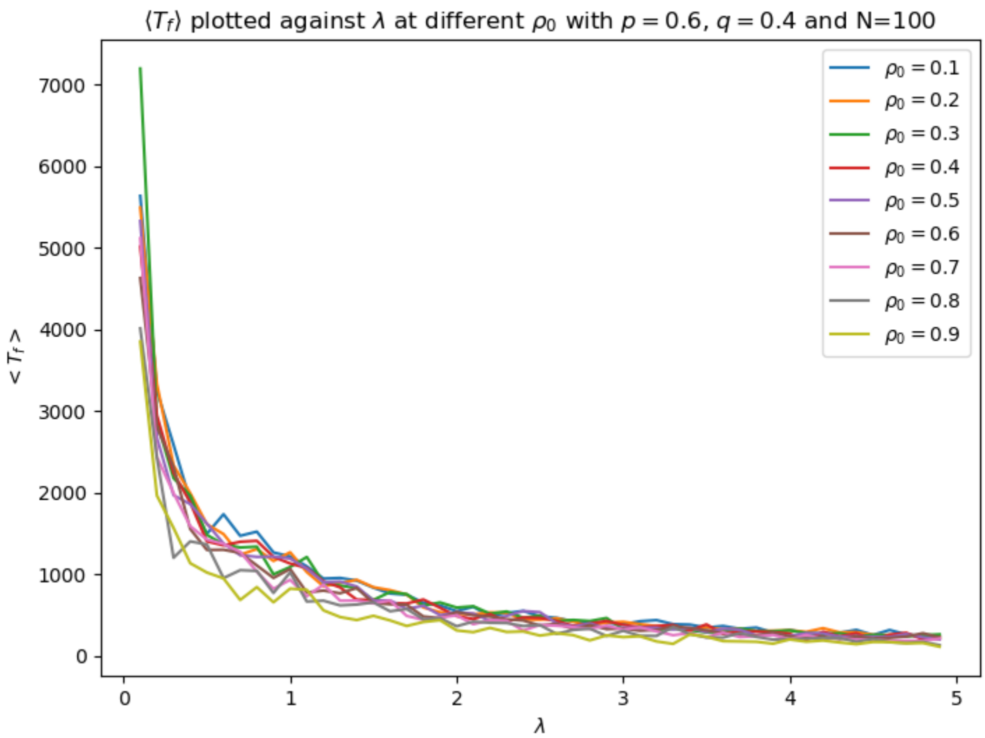

Figure 1: $\langle T_f \rangle$ plotted against λ for different ρ_0

Looking at this figure, one immediately concludes that for every ρ_0, the function $\langle T_F \rangle(\lambda)$ has the identity $b + \frac{k}{\lambda}$. It seems that in this identity, k has quite some influence from ρ_0, as it seems to get larger as ρ_0 gets smaller. Another observation is that b seems to be less influenced by ρ_0. Since at large λ, the plots seem to be very close. A last preliminary observation is that between $\lambda = 0.5$ and $\lambda = 1.0$, the plots have a lot of noise, which could imply phase-transitions. To verify these observations, a least squares curve fit is applied to every ρ_0, in which an uncertainty $\sigma = \frac{\langle T_f \rangle}{\sqrt{10}}$ is applied, since we repeated the simulation ten times. The results of these curve fits can be seen in table 1.

ρ_0	**b[-]**	**u(b)**	**k[-]**	**u(k)**
0.1	117.7	32.24	826.1	82.68
0.2	111.9	31.23	796.2	79.93
0.3	102.2	30.80	801.1	79.68
0.4	92.68	28.52	758.1	74.39
0.5	98.78	28.37	739.0	73.46
0.6	76.35	27.02	674.4	68.78
0.7	68.90	24.96	679.0	66.08
0.8	77.55	21.38	560.9	55.37
0.9	49.47	18.77	505.9	49.61

Table 1: The different coefficients relating to the function $\langle T_F \rangle(\lambda) = b + \frac{k}{\lambda}$, accompanied by their fitting errors for different ρ_0.

One can see that the fitting errors for the b coefficient are quite large relative to the actual values for b. For k this same error is smaller, but still around ten percent of the actual value. These values b and k are now plotted against ρ_0 to see the influence of this parameter on these coefficients. The influence of ρ_0 against the coefficient k can be seen in figure 2.

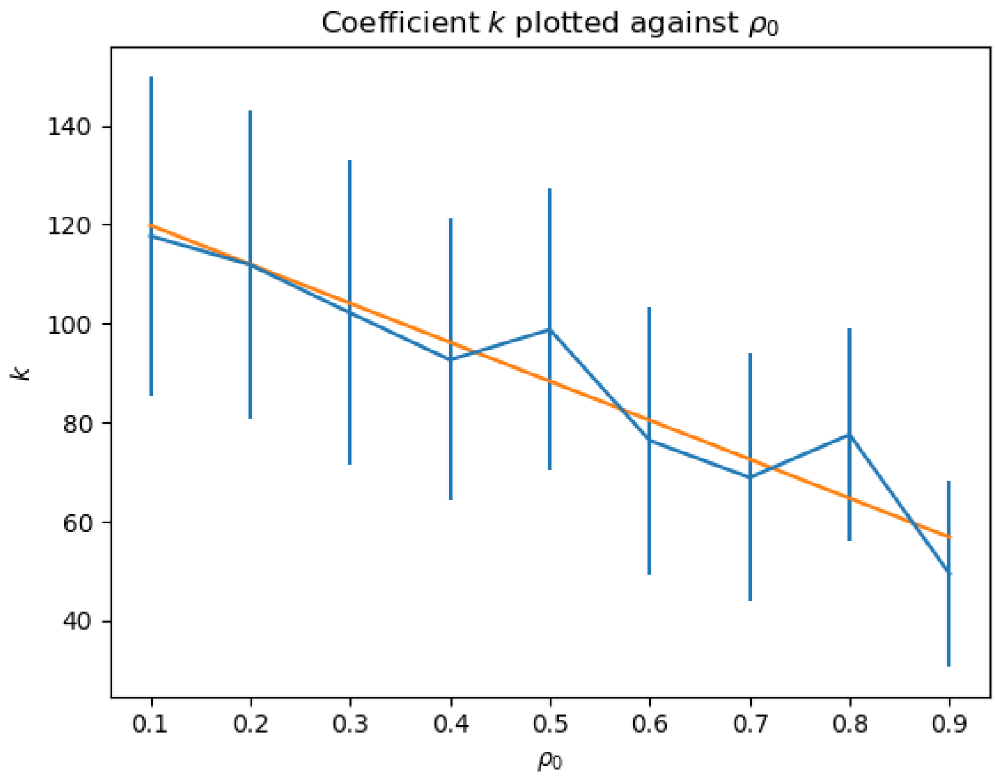

Figure 2: The values of k plotted against ρ_0 accompanied by its curve fit $a\rho_0 + c$

It is clear that the relation between k and ρ_0 is a linear one: $a\rho_0 + c$. The fit for the data goes through all the points of the error bar, which implies a proper fit. The values found for a and c are: $a = -214.6 \pm 187.4$ and $c = 413.6 \pm 121.7$. The uncertainty is most likely this large because of the uncertainty for the values of k. The most important takeaway is that the influence from ρ_0 on k is linear. This also gives an idea of how ρ_0 influences T_f.

The same analysis will now be done for the coefficient b. The results are shown in figure 3.

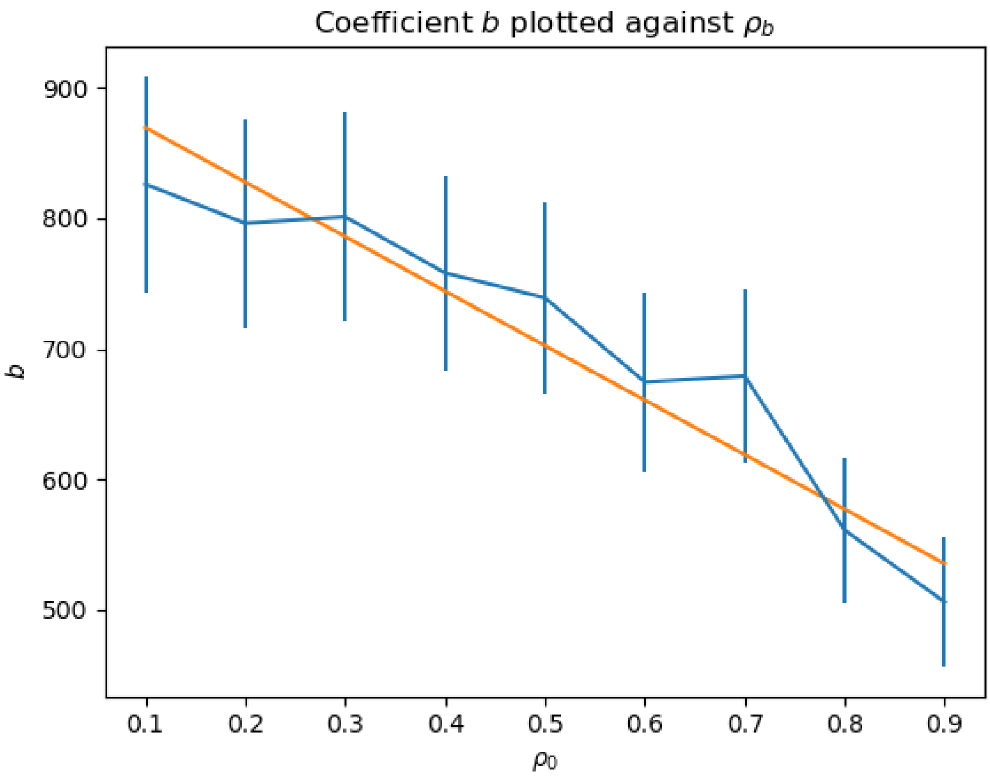

Figure 3: The values of b plotted against ρ_0 accompanied by its curve fit $a\rho_0 + c$

The first remark that can be made is that one of the initial observations, which were made based on figure 1, is contradicted in the figure above. Namely, in figure 1 it seemed that all the plots had the same horizontal asymptote. This would have to yield a constant value b. But instead a clear linear relation is found. The following values were found from the least squares fit: $a = -417.8 \pm 85.6$ and $b = 911.2 \pm 54.94$. So the biggest takeaway is that, even at large source rates, $\langle T_f \rangle$ is still influenced a great deal by ρ_0.

4.5.2. The influence of p, q on $\langle T_f \rangle$

The setup of retrieving simulation data is essentially the same as done with ρ_0. So, λ is taken ranging from 0.1 to $2(p+q)$ with increments of 0.1. This is range is smaller than before. But one can also see from figure 1 that the plot for $\lambda > 2(p+q)$ is not that interesting anymore. Repeating the simulation ten times per λ yields the plots given in figure 4.

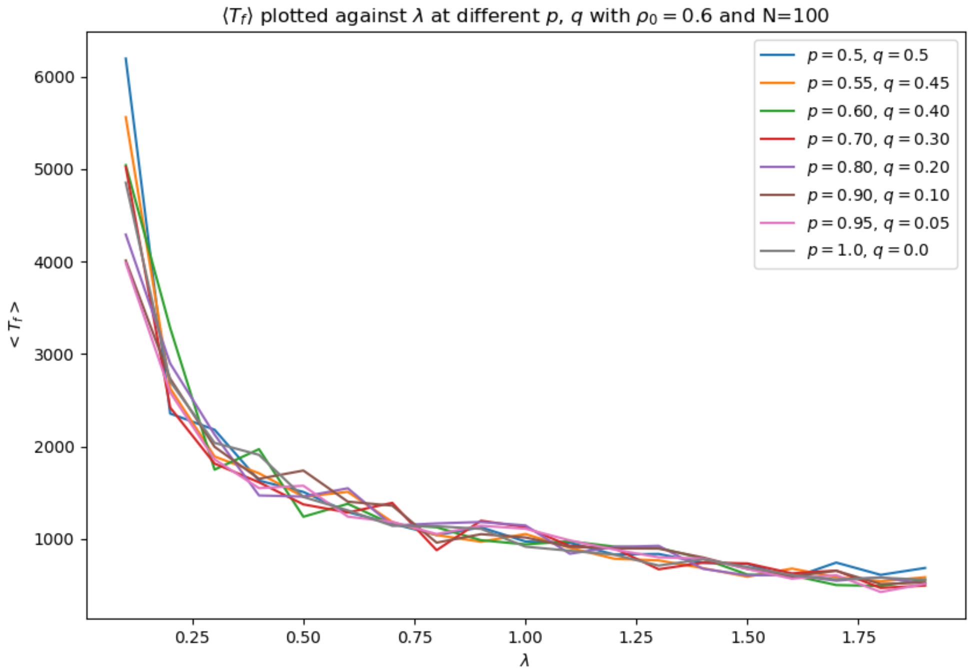

Figure 4: $\langle T_f \rangle$ plotted against λ for different p, q

Some observations can be made when looking at the plots. The trend seems to be that as the rates become more asymmetric, $\langle T_f \rangle$ becomes smaller at small λ. One outlier is the case where $p = 1.0$ and $q = 0$. Or in other the words: the case where the rates are totally asymmetric. As the plot for $< T_f >$ seems to be just below the plot for the $p = 0.55, q = 0.45$ case at these low rates. Maybe the physics of the system changes that drastically when the option to move to the left, is taken away. At larger λ, the plots again seem to be converging to the same horizontal asymptote. However, as seen before with the initial observations for different ρ_0, we cannot state this yet until the coefficients for the functions are calculated. For this, a curve fit of the function $b + \frac{k}{\lambda}$ is applied to the data. The results can be found in table 2.

$p - q$	**b[-]**	**u(b)**	**k[-]**	**u(k)**
0	392.3	112,0	510.0	100.8
0.1	315.2	104.5	539.4	98.53
0.2	289.6	101.7	550.5	97.53
0.4	330.7	99.12	500.2	91.52
0.6	340.8	99.65	510.2	92.44
0.8	375.5	99.32	484.8	89.86
0.9	338.7	93.91	332.9	86.04
1	331.3	103.5	529.7	96.78

Table 2: The different coefficients relating to the function $\langle T_F \rangle(\lambda) = b + \frac{k}{\lambda}$, accompanied by their fitting errors for different $p - q$.

Again, the errors resulting from the least squares fit are relatively larger for b. For the found values of k, these fitting errors lie around 20 percent. We again apply a least squares error fit to the coefficients found with the fit function $b = a(p - q) + c$. This yields the result in figure 5.

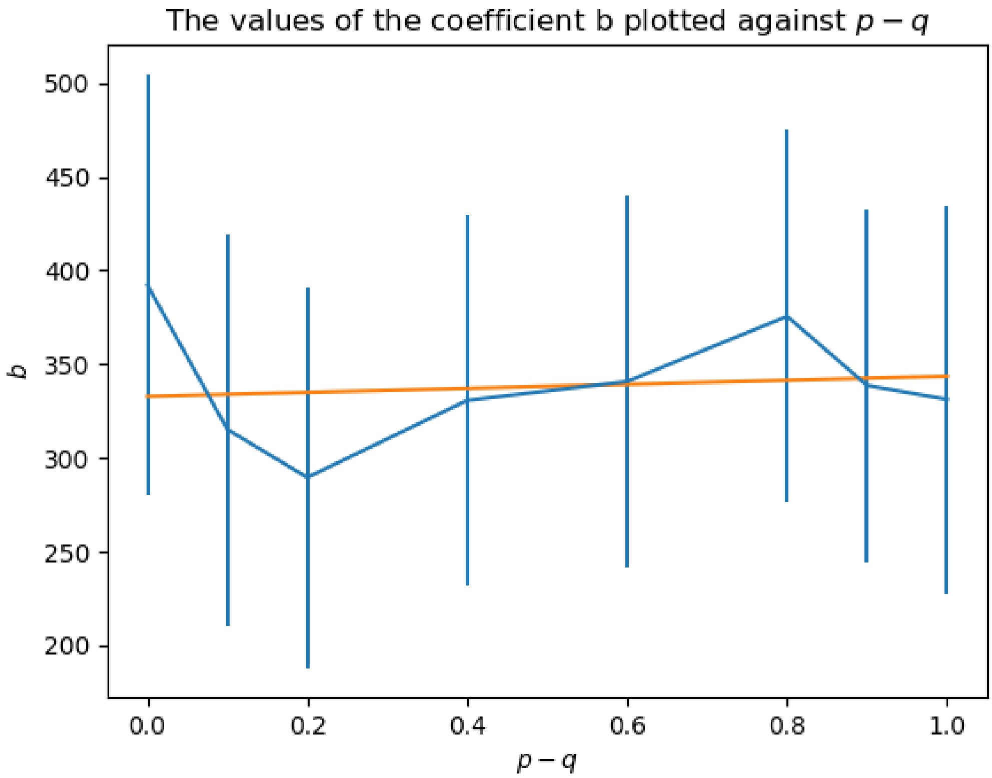

Figure 5: The values of b plotted against $p - q$ accompanied by its curve fit $a(p - q) + b$

This plot shows what we were expecting: not that much influence of $p - q$ for high source rates λ. This results in all $p - q$ having the same asymptotic behaviour. From the least squares curve fit, we retrieved the following values: $a = 10.73 \pm 102.3$ and $c = 332.9 \pm 64.3$. We now use the same method to find the relation with respect to k. The result can be seen in figure 6.

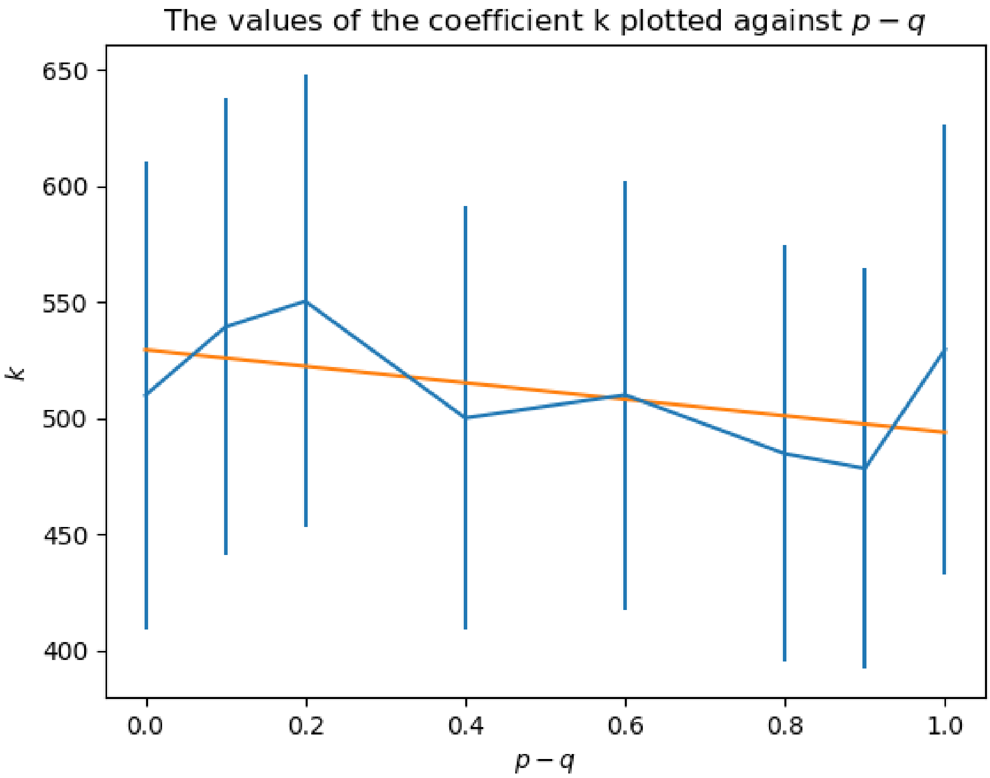

Figure 6: The values of k plotted against $p - q$ accompanied by its curve fit $a(p - q) + b$

From this we can see a slight downward trend, but it is not that convincing as figure 4 implied. The values retrieved from the least squares curve fit are as follows: $a = -35.01 \pm 94.56$ and $b = 529.59 \pm 59.68$.

5. Concluding remarks and acknowledgements

5.1. Concluding remarks with respect to the proofs

In this report, we proved several properties of the symmetric exclusion process with sinks and sources. We were interested in the solution of $\mathbb{E}_\eta[\eta_t(x)]$ or $\mathbb{E}_\eta[1-\eta_t(x)]$. We will briefly touch upon the most important results of the proofs and give some recommendations for further research. For all the proofs, the existence of a dual process and the Feynman-Kac formula were essential.

Firstly, we proved that one source at an arbitrary site, leads to a fully occupied state as $t \to \infty$, if the rates were recurrent. Whereas for the transient case, the expected occupancy was determined by the expected time, that a random walker spent in the source site. Moreover, the initial Bernoulli measure of the system would influence the value of $\mathbb{E}_\eta[1-\eta_t(x)]$ in this case. Thus, for the transient case the system would not 'fill up' as it were.

After this, we looked at the case of one source and sink, which were at the same arbitrary site. The proof was essentially the same as for one source. Except we had to use variation of parameters and a neat semigroup trick. The neat semigroup trick did not work, if the site of the sink and source were not the same. Further research could focus on the latter, because this might be interesting. Intuitively, one might think that you get some sort of 'flow' from the source to the sink site. But then again: this is just guessing for the moment.

Moreover, we looked at the more general case of countable sources. This proof did not differ much from the one source case. After this, we proceeded to prove conditions on whether the configuration would converge to δ_1 . As it turns out, for the nearest neighbour random walk in three dimensions or higher, we could find a condition for not filling up. We could show that, if the source had a certain long-distance behaviour, $\mathbb{E}_\eta[1-\eta_t(x)]$ would converge to a value greater than zero. We did this by using Jensen's inequality as a lower bound and exploiting the long distance behaviour of Green's function. Furthermore, it was shown that even with transient rates, the system would fill up as long as the set of source sites were recurrent.

Finally, we returned to the case of one source, placed at the origin. Except this time, the source had a time dependence. Here we proved that even if the rates were recurrent, the system would not converge to the fully occupied state as long as the source went to zero quick enough over time. There also was an attempt to show that with transient rates and a source rate that would 'blow up', the system would converge to δ_1. But for this, we would have to know how fast the term $\int_{T-\epsilon}^{T} \delta_{X_s,0} ds$ goes to zero. This is also something that could be investigated in the future.

5.2. Conclusion with respect to the simulations

The behaviour of $\langle T_f \rangle$ was investigated with respect to ρ_0 and $p-q$, in an attempt to estimate $\mathbb{E}[T_f]$. $\langle T_f \rangle$ had a relation equal to: $b + \frac{k}{\lambda}$. The parameter ρ_0 had a linear effect on both b and k. Thus $b, k = a\rho_0 + c$. The least squares error curve fit returned the values: $a = -417.8 \pm 85.6$ and $c = 911.2 \pm 54.94$ for b. Whereas $a = -214.6 \pm 187.4$ and $c = 413.6 \pm 121.7$ for k. We also did the same analysis for $p - q$. The curve fit showed a weak linear relation with respect to k. The least squares error fit yielded the values: $a = -35.01 \pm 94.56$ and $c = 529.59 \pm 59.68$. The same method showed that $p - q$ had an even less convincing linear effect on b. In this case: $a = 10.73 \pm 102.3$ and $c = 332.9 \pm 64.3$. Due to the large error, we cannot conclude with full certainty that $p - q$ has a linear effect on these coefficients. More about this in the next.

5.3. Remarks with respect to the simulations

The simulations clearly gave a relation between $\langle T_f \rangle$ and the parameter ρ_0. The exact values of a, c from the curve fit are not that important. The main point is that, for all the values within the range of the errors, it showed a clear linear relation. And this goes for the influence of ρ_0 on both b, k. However, the different plots from figure 1 show a lot of noise in between $\lambda = 0.5$ and $\lambda = 1$. Of course, due to the probabilistic nature of the process, this could be just that: noise. However, it could be that these discontinuities for the plots of different ρ_0, are caused by phase transitions. To confirm these suspicions, one could start by repeating this experiment, but then with smaller increments of λ and more simulations per rate. If discontinuities are now observed more clearly, one could formulate a hypothesis of a connection between ρ_0, λ and p, q with respect to, these phase transitions. The code in the appendix should then be altered, such that one could more clearly observe the behaviour of the occupancy around the source site during a simulation. Around these phase transitions, drastic change of this behaviour should be observed. This could be the topic of further research.

The simulations did not give a clear influence of $p - q$ on $\langle T_f \rangle$. This did not coincide with what we expected, before the experiment. The relation found was not that convincing, to conclude a linear relation with the coefficients b, k. However, figure 4 raised the suggestion, that at small λ, there was a distinction between symmetric and asymmetric. We might have obtained these conflicting results because λ was not taken small enough. So in future projects, one could repeat the simulations, but make the smallest value of λ even closer to zero. Then, using again smaller increments with respect to λ, this could show this distinction between symmetric and asymmetric, more clearly. Moreover: maybe the effect of $p - q$ was not linear to begin with. Different functions for fitting to the data can be considered in the future. Also, the value for $\langle T_f \rangle$ was a bit different from expected when the process was totally asymmetric. Maybe this also is some sort of phase transition, when the system converges from ASEP to TASEP. Lastly, the same arguments as the last paragraph can be made with respect to phase transitions in general.

A more general recommendation is the addition of a sink to the code provided in the appendix. This is interesting because this could lead to finding an invariant measure. One could then investigate the behaviour of this invariant measure with respect to the distance between the sink and source. With such an invariant measure, one could calculate a correlation function to investigate relations between sites.

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Appendix

Inequalities used

In the proofs, we used the following inequalities without stating them completely. The theorems and definitions are from [1] and [4].

Theorem 5.1 (Jensen's inequality)**.** *Let X be a random variable taking values in the (possibly infinite) interval (a, b) such that $\mathbb{E}(x)$ exists, and let $g : (a, b) \to \mathbb{R}$ be a convex function such that $\mathbb{E}[g(X)] < \infty$. Then*

$$\mathbb{E}[g(X)] \geq g[\mathbb{E}[X] \tag{5.1}$$

For the next inequality, first consider the following definition.

Definition 5.2. *For $p \in [1, \infty)$ let $(S, \mathcal{A}, \mu)$ be a measure space. Let*

$$L^p(S) = \left\{ f : S \to \mathbb{R} : f \textit{ is measurable and } \int_S |f|^p d\mu < \infty \right\} \tag{5.2}$$

For $f \in L^P(S)$ let

$$||f||_p = \left(\int_S |f|^p d\mu \right)^{\frac{1}{p}} \tag{5.3}$$

Then we are able to give Hölder's inequality.

Theorem 5.3 (Hölder's inequality)**.** *Let $p, q \in (1, \infty)$ satisfy $\frac{1}{q} + \frac{1}{p} = 1$. If $f \in L^p(S)$ and $g \in L^q(S)$, then $fg \in L^1(S)$ and*

$$||fg|| \leq ||f||_p ||g||_q \tag{5.4}$$

Code of the simulation

On the next page, the code is provided that was used to run the simulations. It also explains how we limit the error due to discretion of time.

ASEP1D_new (2)

February 19, 2023

```
[ ]: import numpy as np
     import matplotlib.pyplot as plt
     import math
     import pandas as pd
     from scipy.optimize import curve_fit
```

As we have seen the number of jumps N in a time dt is poisson distributed.

$$P(N_{dt} = n) = \frac{(\lambda_x dt)^n}{n!} e^{-\lambda_x dt} \tag{1}$$

Per time step dt we however assume that a particle only makes one jump.

$$P(i+1|i) = P(N_{dt} \geq 1) * p_{i+1} = (1 - P(N_{dt} = 0)) * p_{i+1} \tag{2}$$

$$P(i-1|i) = P(N_{dt} \geq 1) * p_{i-1} = (1 - P(N_{dt} = 0)) * p_{i_1} \tag{3}$$

$$P(i|i) = P(N_{dt} = 0) \tag{4}$$

Which means the error comes from the fact that in reality one would want to make more then one jump. From this we retrieve the rest term (or error term).

$$R(dt) = P(N_{dt} = 2) = \frac{(\lambda_x dt)^2}{2} = 0.01 \tag{5}$$

We have the same reasoning for the source rate λ_r.

```
[ ]: #eta: an array of 0,1 which represents the occupancy of each site of the lattice
     #pos: an array which represents the locations of the particles
     #N: The length of the lattice (which is a circle)
     #cr, cl: rate to the right and left respectively
     #rso: the rate of the source

     def ASEP_1D(eta, pos, N, cr, cl, rso, xso, dt):
         lambda_x = cr + cl #waiting time per site
         pr = cr/lambda_x
         pl = cl/lambda_x
         p = [(1-np.exp(-lambda_x*dt))*pl, (1-np.exp(-lambda_x*dt))*pr, np.
      ↪exp(-lambda_x*dt)] #assumption is max 1 jump per dt
         dpos = np.random.choice([-1,1,0],len(pos), p)
         pos_new =(pos + dpos)%((len(pos))*[N])
```

```
    eta_new = np.zeros(len(eta))
    rows = np.arange(0, smallestrows(len(pos)))
    jump = len(rows)
    for j in rows:
        pos_new[j::jump] = np.where(np.isin(pos_new[j::jump],␣
↪list(set(pos)-set(pos[j::jump]))), pos[j::jump], pos_new[j::jump])
        pos[j::jump] = pos_new[j::jump]
    eta_new[pos] = 1
    if not(np.isin(xso,pos)) and np.random.exponential(1/rso)<dt:
        eta_new[xso] = 1
        pos.append(xso)
    pos.sort()
    return eta_new, pos

def smallestrows(n): #find minimal number of rows with minimal of 3
    if n<4:
        return 3
    elif n%3==0 or n%3==2:
        return 3
    else:
        i = 4
        res = n%i
        while not(res == 0 or res > 1):
            i+=1
            res = n%i
        return i

def execute2(N, cr, cl, rso, xso, dt, rho):
    eta = np.random.choice([0, 1], N, p=[1-rho, rho])#fill up according to␣
↪bernoulli measure
    eta_lst = [eta]
    pos = [j for j, e in enumerate(eta) if e == 1]
    i=0
    while len(pos) < N:
        eta, pos = ASEP_1D(eta, pos, N, cr, cl, rso, xso, dt)
        #print(list(map(int, eta)))
        eta_lst.append(eta)
        i+=1
    endtime = dt*i
    now = time.time()
    return endtime
```

The function in the following cell is used to determine $\langle T_f \rangle$ for different λ. This allows us to find the relation between $\langle T_f \rangle$ and λ for different $p - q$ and ρ_0

[ ]:
```
def phase_trans_rso(N, cr, cl, xso, rho):
    tries=100
```

2

```
    dt= np.sqrt(0.005)/(cr+cl)
    rso = np.arange(0.1, 2*(cr+cl),0.01)
    res = np.zeros((len(rso),tries))

    for i in range(len(rso)):
        for j in range(tries):
            dt= min(np.sqrt(0.005)/(cr+cl), np.sqrt(0.005)/(rso[i]))
            fullt = execute2(N,cr,cl,rso[i],xso,dt,rho)
            res[i][j] = fullt
    print(res)
    res_avg=np.mean(res, axis=1)

    plt.figure()
    plt.xlim(0,rso[-1])
    plt.ylim(0,max(res_avg))
    plt.plot(rso,res_avg)
    plt.show()
    return res_avg

#result = phase_trans(10000, 1, 0.5, 0, 0.6)
```

The following program gives an idea of how the coefficients k and b were obtained. The case below is for obtaining the coefficients at different ρ_0. When the coefficients were found for ρ_0 ranging from 0.1 to 0.9, another curvefit is applied. This time a linear one to find out how ρ_0 influences these coefficients. The case for $p-q$ is equivalent to the one below in terms of code. Filenames like ptr100_06_04_0_01.csv are local files saved from the simulations.

[ ]:
```
p01 = (pd.read_csv("ptr100_06_04_0_01.csv", usecols = [1]).values).tolist()
p02 = pd.read_csv('ptr100_06_04_0_02.csv', usecols = [1]).values
p03 = pd.read_csv('ptr100_06_04_0_03.csv', usecols = [1]).values
p04 = pd.read_csv('ptr100_06_04_0_04.csv', usecols = [1]).values
p05 = pd.read_csv('ptr100_06_04_0_05.csv', usecols = [1]).values
p06 = pd.read_csv('ptr100_06_04_0_06.csv', usecols = [1]).values
p07 = pd.read_csv('ptr100_06_04_0_07.csv', usecols = [1]).values
p08 = pd.read_csv('ptr100_06_04_0_08.csv', usecols = [1]).values
p09 = pd.read_csv('ptr100_06_04_0_09.csv', usecols = [1]).values
p01_new = []
p02_new = []
p03_new = []
p04_new = []
p05_new = []
p06_new = []
p07_new = []
p08_new = []
p09_new = []
```

```
for i in range(len(p01)):
    p01_new.append(p01[i][0])
for i in range(len(p02)):
    p02_new.append(p02[i][0])
for i in range(len(p03)):
    p03_new.append(p03[i][0])
for i in range(len(p04)):
    p04_new.append(p04[i][0])
for i in range(len(p05)):
    p05_new.append(p05[i][0])
for i in range(len(p05)):
    p06_new.append(p07[i][0])
for i in range(len(p05)):
    p07_new.append(p07[i][0])
for i in range(len(p08)):
    p08_new.append(p08[i][0])
for i in range(len(p09)):
    p09_new.append(p09[i][0])
rso= (np.arange(0.1,5,0.1)).tolist()
plt.figure()
plt.title(r'$T_f$ plotted against $\lambda$ at different $\rho_0$ with $p = 0.6$, $q=0.4$ and N=100')
plt.xlabel(r'$\lambda$')
plt.ylabel(r'$<T_f>$')
plt.plot(rso,p01, label=r'$\rho_0  = 0.1$')
plt.plot(rso,p02, label=r'$\rho_0  = 0.2$')
plt.plot(rso,p03, label=r'$\rho_0  = 0.3$')
plt.plot(rso,p04, label=r'$\rho_0  = 0.4$')
plt.plot(rso,p05, label=r'$\rho_0  = 0.5$')
plt.plot(rso,p06, label=r'$\rho_0  = 0.6$')
plt.plot(rso,p07, label=r'$\rho_0  = 0.7$')
plt.plot(rso,p08, label=r'$\rho_0  = 0.8$')
plt.plot(rso,p09, label=r'$\rho_0  = 0.9$')
def func(x,b,k):
    return b+ k/x
sig1=np.zeros(len(p01))
for i in range(len(p01)):
    sig1[i]=(p01[i]/(np.sqrt(10)))
popt01, pcov01 = curve_fit(func,list(rso), np.array(p01_new), sigma = sig1, absolute_sigma=True, maxfev=10000)
perr1 = np.sqrt(np.diag(pcov01))

sig2=np.zeros(len(p02))
for i in range(len(p02)):
    sig2[i]=(p02[i]/(np.sqrt(10)))
```

4

```
popt02, pcov02 = curve_fit(func,list(rso), np.array(p02_new), sigma = sig2,␣
↪absolute_sigma=True, maxfev=10000)
perr2 = np.sqrt(np.diag(pcov02))

sig3=np.zeros(len(p03))
for i in range(len(p03)):
    sig3[i]=(p03[i]/(np.sqrt(10)))
popt03, pcov03 = curve_fit(func,list(rso), np.array(p03_new), sigma = sig3,␣
↪absolute_sigma=True, maxfev=10000)
perr3 = np.sqrt(np.diag(pcov03))

sig4=np.zeros(len(p04))
for i in range(len(p04)):
    sig4[i]=(p04[i]/(np.sqrt(10)))
popt04, pcov04 = curve_fit(func,list(rso), np.array(p04_new), sigma = sig4,␣
↪absolute_sigma=True,maxfev=10000)
perr4 = np.sqrt(np.diag(pcov04))

sig5=np.zeros(len(p05))
for i in range(len(p05)):
    sig5[i]=(p05[i]/(np.sqrt(10)))
popt05, pcov05 = curve_fit(func,list(rso), np.array(p05_new), sigma = sig5,␣
↪absolute_sigma=True, maxfev=10000)
perr5 = np.sqrt(np.diag(pcov05))

sig6=np.zeros(len(p06))
for i in range(len(p06)):
    sig6[i]=(p06[i]/(np.sqrt(10)))
popt06, pcov06 = curve_fit(func,list(rso), np.array(p06_new), sigma = sig6,␣
↪absolute_sigma=True, maxfev=10000)
perr6 = np.sqrt(np.diag(pcov06))

sig7=np.zeros(len(p07))
for i in range(len(p07)):
    sig7[i]=(p07[i]/(np.sqrt(10)))
popt07, pcov07 = curve_fit(func,list(rso), np.array(p07_new), sigma = sig7,␣
↪absolute_sigma=True, maxfev=10000)
perr7 = np.sqrt(np.diag(pcov07))

sig8=np.zeros(len(p08))
for i in range(len(p08)):
    sig8[i]=(p08[i]/(np.sqrt(10)))
popt08, pcov08 = curve_fit(func,list(rso), np.array(p08_new), sigma = sig8,␣
↪absolute_sigma=True, maxfev=10000)
perr8 = np.sqrt(np.diag(pcov08))
```

```
sig9=np.zeros(len(p09))
for i in range(len(p09)):
    sig9[i]=(p09[i]/(np.sqrt(10)))
popt09, pcov09 = curve_fit(func,list(rso), np.array(p09_new), sigma = sig9,␣
 ↪absolute_sigma=True, maxfev=10000)
perr9 = np.sqrt(np.diag(pcov09))

def func2(x,a,b):
    return a*x+b
plt.legend()
plt.show()

plt.figure()
plt.xlabel(r'$\rho_0$')
plt.ylabel(r'$k$')
plt.title(r'Coefficient $k$ plotted against $\rho_0$')
k = [popt01[0], popt02[0], popt03[0], popt04[0], popt05[0], popt06[0],␣
 ↪popt07[0], popt08[0], popt09[0]]
uk = [perr1[0], perr2[0], perr3[0], perr4[0], perr5[0], perr6[0], perr7[0],␣
 ↪perr8[0], perr9[0]]

rho = np.arange(0.1,1,0.1)
plt.errorbar(rho,k,yerr=uk, label='errorbar fits')

poptk, pcovk = curve_fit(func2,rho, k, sigma = uk, absolute_sigma=True)
perrk = np.sqrt(np.diag(pcovk))
f = func2(rho,poptk[0],poptk[1])
plt.plot(rho,f, label='fit: ax+c')

plt.figure()
plt.xlabel(r'$\rho_0$')
plt.ylabel(r'$b$')
plt.title(r'Coefficient $b$ plotted against $\rho_b$')
b = [popt01[1], popt02[1], popt03[1], popt04[1], popt05[1], popt06[1],␣
 ↪popt07[1], popt08[1], popt09[1]]
ub = [perr1[1], perr2[1], perr3[1], perr4[1], perr5[1], perr6[1], perr7[1],␣
 ↪perr8[1], perr9[1]]

rho = np.arange(0.1,1,0.1)
plt.errorbar(rho,b,yerr=ub, label='errorbar fits')

poptb, pcovb = curve_fit(func2,rho, b, sigma = ub, absolute_sigma=True)
perrb = np.sqrt(np.diag(pcovb))
f = func2(rho,poptb[0],poptb[1])
plt.plot(rho,f, label='fit: ax+c')

plt.show()
```