Stellingen

behorende bij het proefschrift:

"Digital control of distributed parameter systems. A state space approach"

van

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1. Basically, *regular systems* (the Salamon-Weiss class) are representable by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \in \mathcal{X} \\ y(t) = Cx(t) + Du(t) \end{cases}$$

with A, B, C possibly unbounded operators and D bounded. If a certain condition, similar to $x_0 \in \mathcal{W}$ for Pritchard-Salamon systems, implying the smoothness of the output function, would be available for regular systems, along with a result on the spectral decomposition of the operator A, then most of the results presented in this thesis would lead to a beautiful generalization.

2. The impulse response of a Pritchard-Salamon system with an infinite-dimensional space of inputs is a tempered $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued distribution with support in $[0, \infty)$. Furthermore, if $\dim(\mathcal{U}_1) = \infty$, it is generally not possible to make sense of $CS(\cdot)B$ as a function. In particular, expressions as $CS(\cdot)B$ and $\int_0^t CS(t-\tau)Bu(\tau)$ cannot be interpreted in the usual sense for arbitrary $u \in L_2^{loc}(0, \infty; \mathcal{U})$. Even in the bounded case $\mathcal{W} = \mathcal{V}$, when $CS(t)B$ is well-defined as a function, it might not be strongly measurable. This fact implies that it is not locally integrable with respect to the uniform norm topology. We claim here that the $\mathcal{H}^2$-optimal control problem for infinite-dimensional systems with infinite-dimensional disturbance-input space is a very interesting and delicate problem.

3. Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard-Salamon system with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$ and assume that $x_0 \in \mathcal{W}$. Then the time-discretized Pritchard-Salamon system $\Sigma(\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \boldsymbol{\Theta})$ is well defined. Even if in general

$$B \notin \mathcal{L}(\mathcal{U}, \mathcal{W}) \cap \mathcal{L}(\mathcal{U}, \mathcal{V}),$$

its time-discretized counterpart satisfies

$$\boldsymbol{\Gamma} \in \mathcal{L}(\mathcal{U}, \mathcal{W}) \cap \mathcal{L}(\mathcal{U}, \mathcal{V}).$$

Since the operators defining $\Sigma(\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \boldsymbol{\Theta})$ satisfy $\boldsymbol{\Phi} \in \mathcal{L}(\mathcal{W})$, $\boldsymbol{\Gamma} \in \mathcal{L}(\mathcal{U}, \mathcal{W})$, $\boldsymbol{\Lambda} \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ and $\boldsymbol{\Theta} \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, it follows that the digital control of Pritchard-Salamon

systems is an easy problem, since the unboundedness has vanished away. This is only apparently. If one takes into account that

(a) digital control implies that the admissible set of controls is the class of piece-wise constant functions, which is a dense subset of $L_2(0, \infty; \mathcal{U})$,

(b) a C_0-semigroup perturbed by a digital feedback is no longer a C_0-semigroup,

(c) any digital control problem associated with the Pritchard-Salamon system has to be well-posed on both spaces $\mathcal{W}$ and $\mathcal{V}$,

(d) from the mathematical point of view, a solution to a well-posed digital control problem for Pritchard-Salamon systems has to imply stability on both spaces $\mathcal{W}$ and $\mathcal{V}$,

it follows that the digital control of Pritchard-Salamon systems is not easier than the control problem posed in continuous-time.

4. The number of elementary particles in the known universe is approximately 10^{80} and it represents the biggest and for sure the most complex system known at this moment. However, an extremely inventive species living on a small planet, lost somewhere on the Milky Way, is using *infinite dimensional systems* for a better modeling of their local environment.

5. There are people born to be researchers. Their intellectual curiosity pushes them digging deeper and deeper towards solving difficult problems. Their attitude with respect to the surrounding world resembles, somehow, the attitude of an actor on the stage who lives totally for the show. For this kind of people the show always goes on and on without stopping.

There are people who become researchers because they have been educated to do so. They might have done, probably even better, any other job demanding a certain level of intelligence and capacity. It is not surprising if one day those people would give up their research preoccupations and start something else, without even having the feeling of being loosers.

If

(a) we admire the genius of the first ones by wishing, secretly, to be part of their exclusive club,

(b) if we envy them sometimes when they make gigantic steps while reaching certain conclusions that strike everybody and which proves, finally, to be a big novelty,

(c) if we find ourselves everyday more than we would like to admit on the side of the second ones,

it means, at least, that we grew up sufficiently to understand that there must be a third way and it is worthwhile to do something about going that way.

6. When the robust control technique fails to meet the objectives of a control system designer, there are two left over possibilities. Either to do adaptive control or to give it up. Neither the first one nor the second one brings tremendous satisfaction since the behavior of adaptive control systems resembles somehow Chaikowsky's "Swans Lake" performed by elephants.

7. It is better to be rich, healthy, smart and handsome rather than being poor, sick, stupid and ugly.

8. One specific feature of the last century's capitalist society was the wild exploitation of man by man. In this century's socialist society it is the other way around.

9. Big scientists have names of theorems and lemmas. Sometimes they have names of streets and boulevards. Even in the latter case, the size of the street is not directly proportional to how big the scientist was.

10. It is amazing that in a country like Holland, geographically situated below the see level, where it rains more than 75% of the days in a year, peolple are still economizing on the tap water.

11. If we would spend each day only 5 seconds thinking profoundly about death, then we would enjoy the rest of 86395 seconds of the day much more and, probably, we would be more happy.

Digital control of distributed parameter systems: a state-space approach

PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus,
Prof. ir. K. F. Wakker,
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door
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geboren te Cluj, Roemenië
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Părinților mei

Foreword

This thesis represents the outgrowth of my research activity during the last three years as a guest of the Department of Technical Mathematics and Informatics at the Delft University of Technology, The Netherlands, on leave from the Research Institute in Informatics, Bucharest, Romania.

Many people have contributed to the completion of this dissertation. It is my pleasure to thank them all here.

Let me begin by acknowledging the constant support I have received from the Dutch Systems and Control Theory Network and its Director, Professor Jan van Schuppen, from the very early stage when I applied to it, the summer of 1991, and until today. In those three years, thanks to the people working in this network, many things were made possible. Among them, let me mention the attendance of high-level doctoral courses, my participation to the 1993 CDC conference and many other international workshops where I had the opportunity to meet people from whom I have learned a lot.

I wish to thank Professor Geert Jan Olsder for giving me the opportunity to pursue this research within his research group at Delft. His creativity and inspiring presence has been a constant source of "élan" in my research activity.

I wish to thank my thesis supervisor, Dr. Willem de Koning, for the many stimulating discussions that we had in the last three years as well as for the fact that the his office door has always been opened for me. His pragmatical attitude towards research has made me also drop an anchor in the engineering world. The art with which he knew to be both a project manager and a close friend has created the indispensable environment to any research work.

The first steps a child makes in life would be a burden without parental care. The first steps of an young researcher toward entering the scientific community would be impossible without guidance. I wish to thank here Professor Vlad Ionescu for the care he guided my first steps in research. His contribution to my development as researcher, starting in the years when I was his undergraduate student, is beyond price.

I would like to thank Professor Ruth Curtain from whom I've learned some beautiful applied functional analysis. I am extremely indebted to her for the generosity with which she spent a lot of time reading draft versions of my manuscripts on digital control of Pritchard-Salamon systems as well as the concept of this thesis. I wish to thank her for the opportunity she gave me to visit her group in Groningen as well as to attend several workshops on infinite-dimensional systems she has organized in the last three years. Her annotations on my manuscripts, now precious memories to me, were meant to help me grow up as a mathematician, to eliminate the nonrigorousity from the proofs I have constructed for the claims that I've made in my papers. No matter how successful she has been in trying to help me to become a veritable mathematician, I shall always be greatful to her.

Several people that I've met in those years have became close friends of mine. The one who has probably succeeded to come closer than others was Martin Weiss. I thank him for

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sharing with me all the frustrations and successes that I had, inherently, in my research activity as well as for being a good pal and research partner in all those years.

My work at T.U. Delft would have not been possible without the encouragement and moral support of my colleagues. I wish to thank to all the people from the 7th floor of the *Electotechniek* building for providing me a pleasant environment to work.

A "reading committee" formed by Professor Ruth Curtain, Professor Malo Hautus, Professor Hans van Duijn and Professor Vlad Ionescu has approved this thesis for defense. I thank them all for the careful reading of the manuscript.

There are two special people in my life to whom I owe my love for mathematics from the time I was a high-school boy. They are prof. Petre Dumitru and dr. Vasile Diţă. I thank them both.

I wish to thank my parents for the generous and incommensurate effort they have made to ensure that I got the best possible education. The philosophical discussions I had with my sister and my brother in law about science and life will always remain a precious memory to me.

Finally, I thank Nora for her numerous and various extra-professional contributions to this thesis. Especially, I thank her for her love, inspiration, support and the great confidence she has always had in me.

Delft, October 24, 1994.

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Part I

Framework

Chapter 1

Introduction

The computer control approach has become an every-day reality for more than a decade. With the advent and proliferation of microcomputers, the role played by digital control design techniques has become increasingly more important. In many industrial settings the designer has to control physical entities such as temperature, fluid flow, and an appropriate modeling often leeds to *distributed parameter systems*, systems which are defined on Hilbert spaces. It has become necessary, therefore, to develop extensions of many of the "classical" digital control strategies for finite-dimensional systems to an infinite-dimensional setting. A generalization to infinite-dimensional systems with bounded operators of the finite-dimensional results on the digital LQ-optimal control problem has been done more than 20 years ago by Lee, Chow and Barr [60]. But since then various finite-dimensional control problems have been addressed and solved and, as it is normal, various approaches and theories to and about those control problems have been taken and constructed. We refer here to the $\mathcal{H}^2$-optimal control problem when considered in the deterministic case, LQG-optimal control problem when considered in stochastic sense, and $\mathcal{H}^\infty$ suboptimal control problems. In this thesis we consider the aforementioned problems in the infinite-dimensional setting and our approach to control is a digital one. From the rather wide variety of control theories that have emerged up to this time, we shall focus our attention upon two of them.

(i) The first one is the Hyland-Bernstein control theory. This theory gives the set of necessary conditions for the existence of the LQG-optimal finite-dimensional fixed-order compensator for an infinite-dimensional plant, in the continuous-time case and finite-dimensional but higher-order plant in the discrete-time case. The main feature of this theory is that of expressing the set of first-order necessary conditions for the solution of the finite-dimensional fixed-order compensation problem via a system of four coupled operator/matrix equations, two modified Riccati equations and two modified Lyapunov equations, coupled by an oblique projection operator/matrix which is born naturally from the optimality constraints to the design process. The oblique coupling operator/matrix is a projector having the rank equal precisely to

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the order of the compensator. The separation principle breaks down and only in the full-order case, the oblique projection operator/matrix becomes the identity operator/matrix and the two modified Riccati and the two modified Lyapunov equations drop out and the set of four equations simplify to the standard pair of Riccati and Lyapunov equations.

(ii) The second control theory serving our main purpose announced in this introduction is the so called Popov theory. Let us begin by explaining the reader why we have made this option, why a discrete Popov theory approach to the digital control of DP systems is a valuable one. There are two main reasons.

  (a) Among different results emerging from Popov's positiveness theory [69], a theory with origins in the work of Kalman [55] and Yakubovitch [89], the one that establishes the connections between the properties of a quadratic cost functional and the existence of a stabilizing solution to a certain Riccati equation is probably most relevant. This is especially true since there are, probably, few control theory concepts that have been more extensively studied than Riccati equations. For more than three decades, their application to stability, LQ-optimal control and more recently $\mathcal{H}^2$-optimal control and $\mathcal{H}^\infty$-optimal control, has set a strong basis to the development of a huge control theory literature.

  (b) The second motivation for our option for the discrete Popov theory approach is the possibility to replace the positivity condition of the Popov function with a more general one, given by the invertibility of a certain Toeplitz-like operator. The reader is referred to the original work of Ionescu and Weiss [50] and its extensions [45]. This new direction has proved to be reach enough to incorporate game-theoretic situations (see [21]) and to permit one to write down the solution to the $\mathcal{H}^\infty$ control problem.

Classical optimal control theory of distributed parameter systems assumes boundedness of control and observation operators. However, most of the interesting infinite-dimensional control systems encountered in practice arise in a different way. The next example fully supports this statement.

Example 1.1 The temperature distribution of a unit length heated rod with *point heat control* and *point temperature measurement* is described by the following PDE model

$$\frac{\partial z(x,t)}{\partial t} = \frac{\partial^2 z(x,t)}{\partial x^2} + \delta_{x_0}(x)u(t), \quad z(x,0) = z_0(x) \tag{1.1}$$

$$\frac{\partial z(x,t)}{\partial t}\Big|_{(x=0)} = \frac{\partial z(x,t)}{\partial t}\Big|_{(x=1)} = 0 \tag{1.2}$$

$$y(t) = z(x_1,t), \tag{1.3}$$

where $z(x,t)$ is the temperature distribution, $x \in [0,1]$ the distance alongi the rod, $u(t)$ the control and $y(t)$ the observation. In this example the observation operator which is

the temperature measurement at a point x_1, formally defined by

$$Cz = z(x_1, \cdot),$$

is unbounded on the state space $\mathcal{X} = L_2[0,1]$; it is not defined on all of $L_2[0,1]$. Since we have modeled the control operator in terms of the delta distribution at x_0 it is clear that the control mapping is not a bounded one either and the control system from the Example 1.1 has unbounded control and observation operators. □

Let us now explain why the digital control of systems with unboundedness is a difficult and interesting problem. There are three common ways to design a digital control system.

(i) The first one is to do a continuous-time design followed by a digital implementation, i.e. the time discretized controller is connected via **A/D** and **D/A** devices to the original continuous-time plant. Besides the general disadvantage represented by the fact that in such a design process, the sampling frequency is not a designable parameter, in the case of infinite-dimensional systems with unboundedness there exists another particular disadvantage, which makes this method unapplicable. This statement will be made more clear in the next paragraph. Van Keulen [80] gave a complete generalization for the Pritchard-Salamon class of infinite-dimensional systems [70] of the early Doyle *et al.* results on $\mathcal{H}^\infty$-optimal control [37]. Besides the full generalization of the above mentioned results, he showed that the controller minimizing the $\mathcal{H}^\infty$-norm of the closed-loop system, regarded on its turn as a system, falls also in the Pritchard-Salamon class. Hence the digital implementation of such a controller implies the discretization of a Pritchard-Salamon system, a fact which, as shown in Barb, de Koning and Weiss [15], is not always possible unless certain restrictions on the initial state are imposed to guarantee a smooth output function.

(ii) The second way is to discretize the plant and to do a discrete-time design. This method is also not the best one could hope for; besides the obvious disadvantage of ignoring intersampling behavior, it presents the same drawback as the first mentioned method, that of having to discretize a Pritchard-Salamon system.

(iii) The third way is to do direct digital design, treating the control system as a sampled-data system. Since sampled-data systems are time-varying systems (actually continuous-time T-periodically systems), it is rather obvious that the design process will be more complicated than in the time-invariant case.

With respect to the direct digital design approach, let us notice that there are several sources of difficulties one has to overcome in the design process, such as

- The structural constraint on the controller $\mathbf{H}\Sigma_K\mathbf{S}$, where $\mathbf{H}$ and $\mathbf{S}$ are the zero-order hold and sampler, and Σ_K being the discrete-time controller.

- The fact that the controller $\mathbf{H}\Sigma_\mathbf{K}\mathbf{S}$ is not time-invariant despite the fact that $\Sigma_\mathbf{K}$ is.

- The hybrid nature of the overall control system; the state of the plant evolves in continuous-time, while the controller state in discrete-time and hence, they are not defined over the same time set.

One trend to approach the direct digital design is given by the general framework for periodic continuous-time systems as proposed by Bamieh and Pearson in [18]. The main tool used in this framework is the so-called *lifting technique.* [1] Essentially, lifting means to represent a linear time-varying (periodic) continuous-time system by a linear time-invariant discrete-time one such that the periodicity of the first one is reflected in the shift invariance of the second one. Working in such a framework as the one proposed in [18], one can reduce the digital control problem to an equivalent discrete-time control problem for its lifted counterpart, in the sense that a solution to the latter one is also a solution to the former one. The reader is referred for comprehensive treatment of this subject to Chapter 5 and to Chapter 6 of this thesis. We adopt in this thesis the above described framework. Unlike Tadmor who gives the solution to the $\mathcal{H}^\infty$ sampled-data control in continuous-time in terms of three Riccati equations (see [78]), we deal with a discrete-time control problem on abstract Hilbert spaces, and hence we shall give the solution in terms of discrete time Riccati equations. The price we pay, specific to lifting, is that we have to work with signals taking values in function spaces (if the original signal is an R-valued function, then its lifted counterpart is in $L_2(0,T;R)$, see for more details on lifting [18]). For such a discrete-time control problem, both discrete theories developed in this thesis, the discrete Hyland-Bernstein control theory and the discrete Popov theory, seem to be ideal tools which enables us to write down the solution to the $\mathcal{H}^\infty$ digital control problem and the fixed-order digital LQG control problem.

1.1   Three basic control problems

In the first part of this introductory chapter we have chosen the control theories we shall focus on in this thesis. We have mentioned the main control problems addressed in the literature which will be considered in this thesis. Let us now be more precise about this by formulating them mathematically. Suppose we have an infinite-dimensional plant Σ_G (two inputs and two outputs) as shown in Figure 1.1 and a controller Σ_K as in Figure 1.2.

We interconnect the controller and the plant

$$\begin{cases} \nu(\cdot) = y_2(\cdot) \\ \zeta(\cdot) = u_2(\cdot) \end{cases}$$

and we obtain the closed-loop structure depicted in Figure 1.3

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[1]The idea of lifting a continuous-time T-periodic system to a discrete-time one was independently developed by Francis and Tannenbaum [19] and a similar idea was announced in the "early" 1990-CDC paper of Yamamoto [92].
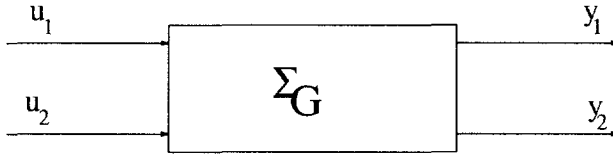
Figure 1.1: The generalized plant
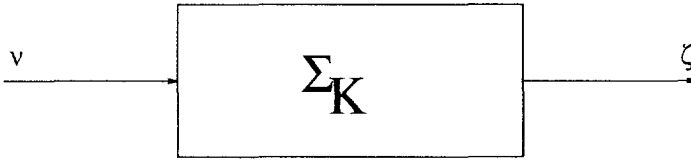
Figure 1.2: The controller

In our setup u_1 is the *disturbance input*, y_1 the *controlled output*, u_2 the *control input* and y_2 the *measured output*. The problem we want to solve is that of *finding a controller* Σ_K *which "stabilizes" the plant* Σ_G *and which makes the influence of the disturbance input on the controlled output to be minimal in some sense.* Let us be more precise about those two concepts introduced above. Let

$$(\sigma x)(t) \triangleq \begin{cases} \dot{x}(t) & , \quad t \in \mathbb{R} \\ x(t+1) & , \quad t \in \mathbb{N} \end{cases} \tag{1.4}$$

denote either the differential operator in continuous-time or the advance unit shift operator in the discrete-time case. We shall assume that we have a state space description of the
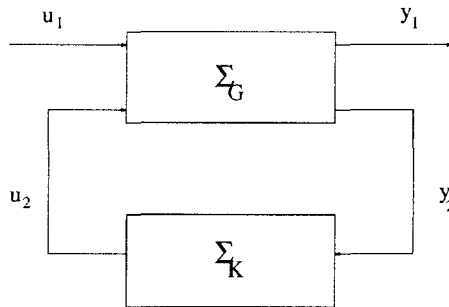
Figure 1.3: The closed loop configuration

plant

$$\Sigma_G \begin{cases} (\sigma x)(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) \\ y_1(t) = C_1 x(t) + D_{12} u_2(t) \\ y_2(t) = C_2 x(t) + D_{21} u_1(t) \end{cases} \tag{1.5}$$

and of the controller

$$\Sigma_K \begin{cases} (\sigma x_c)(t) = A_c x_c(t) + B_c y_2(t) \\ u_2(t) = C_c x_c(t) + D_c y_2(t) \end{cases} , \tag{1.6}$$

where $A \in \mathcal{L}(\mathcal{X})$, $B_1 \in \mathcal{L}(\mathcal{U}_1, \mathcal{X})$, $B_2 \in \mathcal{L}(\mathcal{U}_2, \mathcal{X})$, $C_1 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_1)$, $C_2 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_2)$, $D_{12} \in \mathcal{L}(\mathcal{U}_2, \mathcal{Y}_1)$, $D_{21} \in \mathcal{L}(\mathcal{U}_1, \mathcal{Y}_2)$, $A_c \in \mathcal{L}(\mathcal{K})$, $B_c \in \mathcal{L}(\mathcal{Y}_2, \mathcal{K})$, $C_c \in \mathcal{L}(\mathcal{K}, \mathcal{U}_2)$ and $D_c \in \mathcal{L}(\mathcal{Y}_2, \mathcal{U}_2)$ are linear time-invariant operators on the real separable Hilbert spaces $\mathcal{X}, \mathcal{U}_1, \mathcal{U}_2, \mathcal{Y}_1, \mathcal{Y}_2$ and $\mathcal{K}$, respectively.

We shall say that Σ_K stabilizes Σ_G if the linear system from u_1 to y_1 is *exponentially stable* if the continuous-time situation is considered and *power stable* in the discrete-time case. This means that the closed-loop system dynamics is governed by a resultant A-operator

$$A_R = \begin{pmatrix} A + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{pmatrix} \tag{1.7}$$

which is exponentially stable for the case $t \in \mathbb{R}$ or power stable for $t \in \mathbb{N}$.

The influence of the disturbance input on the controlled output can be estimated in several ways. Depending upon which one we choose, we deal with different control problems. Three basic control problems, matching the control set-up depicted in Figure 1.3 are introduced in the next subsections.

1.1.1   The $\mathcal{H}^\infty$ control problem

Let G and K denote the transfer functions of Σ_G and Σ_K, respectively and consider the partition on G induced by its two input/two output structure.

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \tag{1.8}$$

Let the controller be given by

$$u_2 = K y_2. \tag{1.9}$$

For such a control structure as the one depicted in Figure 1.1, the transfer function from the disturbance input to the controlled output, $G_{y_1 u_1}$, is defined as the linear fractional transformation of the plant and the controller

$$G_{y_1 u_1} \triangleq G_{11} + G_{12} K \left(I - G_{22} K \right)^{-1} G_{21}. \tag{1.10}$$

Since the controller achieves exponential stability of the closed loop system, it follows that $G_{y_1 u_1}$ belongs to the Hardy space $\mathcal{H}^\infty (\mathcal{L}(\mathcal{U}_1, \mathcal{Y}_1))$, the space of $\mathcal{L}(\mathcal{U}_1, \mathcal{Y}_1)$-valued functions which are bounded and analytic in the right half plane

$$C_+ = \{s \in \mathbb{C} | Re(s) > 0\},$$

if $t \in \mathbb{R}$ or outside the open unit disc

$$U_1(0) = \{z \in \mathbb{C} | |z| < 1\},$$

if $t \in \mathbb{N}$.

An appropriate measure of the influence of u_1 on y_1 is the $\mathcal{H}^\infty$-norm of $G_{y_1 u_1}$ defined by

$$\|G_{y_1 u_1}\|_\infty \triangleq \begin{cases} \sup_{\Re e(s) > 0} \|G_{y_1 u_1}(s)\|_{\mathcal{L}(\mathcal{U}_1, \mathcal{Y}_1)}, & t \in \mathbb{R} \\ \sup_{\|z\| > 1} \|G_{y_1 u_1}(z)\|_{\mathcal{L}(\mathcal{U}_1, \mathcal{Y}_1)}, & t \in \mathbb{N} \end{cases}. \tag{1.11}$$

Consequently, the $\mathcal{H}^\infty$-optimal control problem can be formulated as

$$\min_{\Sigma_K \text{ stabilizing}} \|G_{y_1 u_1}\|_\infty. \tag{1.12}$$

Notice that (1.12) is equivalent to the disturbance attenuation problem since the $\mathcal{H}^\infty$-norm of $G_{y_1 u_1}$ is precisely equal to the norm of the bounded linear map from $u_1(\cdot) \in L_2(0, \infty; \mathcal{U}_1)$ to $y_1(\cdot) \in L_2(0, \infty; \mathcal{Y}_1)$ defined for zero initial conditions.

When we have access to the system state, i.e. $C_2 = I$, $D_{21} = 0$ we deal with the *state-feedback $\mathcal{H}^\infty$ optimal control problem*. When the system state is not available in the controller design process we shall use the term *output-feedback $\mathcal{H}^\infty$-optimal control problem*. In both cases we shall focus on the so called *suboptimal $\mathcal{H}^\infty$ control problem*

$$\|G_{y_1 u_1}\|_\infty < \gamma, \tag{1.13}$$

where $\gamma > 0$ is a prespecified bound. Without loosing generality, we shall assume that

$$\gamma = 1.$$

This can be done by an appropriate scaling of the operators involved in the expressions of the transfer functions $G_{ij}, i, j = 1, 2$ (see [46], pp. 178). As it is well known from the literature, the solution to the $\mathcal{H}^\infty$ control problem represents the input of the γ-iterations procedure which gives, in the limiting case, the exact solution to the $\mathcal{H}^\infty$-optimal control problem. Since we shall not be concerned with the optimal case, we shall refer to the suboptimal case simply as to the $\mathcal{H}^\infty$-optimal control problem.

In this thesis we shall give a digital solution to the $\mathcal{H}^\infty$ control problem for the Pritchard-Salamon class of infinite-dimensional systems with unbounded input and output operators. The solution to the digital control problem is obtained by applying the discrete Popov theory to the equivalent discrete-time system obtained by lifting the hybrid T-periodically Pritchard-Salamon digital control system.

1.1.2   The $\mathcal{H}^2$ control problem

Let us consider the plant Σ_G given by (1.5) and controller Σ_K satisfying (1.6) in closed-loop connection as depicted in Figure 1.3. We want to determine a stabilizing controller Σ_K which minimizes the $\mathcal{H}^2$-norm of the linear system from u_1 to y_1 defined by

$$\|G_{y_1 u_1}\|_2^2 \triangleq \begin{cases} \frac{1}{2\pi} \int_0^{2\pi} \text{trace}\,\left(G_{y_1 u_1}(e^{\jmath\omega}) G_{y_1 u_1}^*(e^{\jmath\omega})\right) d\omega, & t \in \mathbb{N} \\[2mm] \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}\,\left(G_{y_1 u_1}(\jmath\omega) G_{y_1 u_1}^*(\jmath\omega)\right) d\omega, & t \in \mathbb{R} \end{cases}, \tag{1.14}$$

where $G_{y_1 u_1}(\cdot)$ is the transfer function of the closed-loop system. Hence, the $\mathcal{H}^2$ control problem can be formulated as

$$\min_{\Sigma_K \text{ stabilizing}} \|G_{y_1 u_1}\|_2^2. \tag{1.15}$$

In this thesis we shall give a solution to the discrete-time $\mathcal{H}^2$-optimal control problem in the case when the controller is assumed to be of a fixed finite order. This means that the Hilbert space $\mathcal{K}$ is now a finite-dimensional Euclidean space

$$\mathcal{K} = \mathbb{R}^n,$$

with $n \in \mathbb{N}$ a fixed integer. Such a development represents the theoretical basis for obtaining the solution to the fixed finite-order digital control problem of infinite-dimensional systems with unbounded input and bounded output operators. A Hyland-Bernstein theory based approach led us to give a set of necessary conditions for the existence of the solution. State-space formulae are given in terms of two modified Riccati equations coupled with two modified Lyapunov equations via an oblique projection operator shown to have the same rank as the dimension of the controller state space.

1.1.3   The LQG control problem

The Linear Quadratic Gaussian control problem represents the stochastic counterpart of the $\mathcal{H}^2$ control problem. Consider the plant Σ_G given by (1.5) and controller Σ_K satisfying (1.6) in closed-loop connection as depicted in Figure 1.3. Let us assume that the disturbance input u_1 is a standard Gaussian white noise (the concept of white noise will be defined in one of the next chapters in the spirit of the work of Balakrishnan [3]). We want to determine a stabilizing controller Σ_K such that the following LQG quadratic cost function is minimized over the class of stabilizing controllers

$$\mathcal{J}(\Sigma_{GK}) \triangleq \begin{cases} \lim_{t \longrightarrow \infty} \frac{1}{t} \mathbb{E}\left(\int_0^t \langle y_1(t), y_1(t) \rangle_{\mathcal{Y}_1}\right), & t \in \mathbb{R} \\[2mm] \lim_{k \longrightarrow \infty} \frac{1}{k} \mathbb{E}\left(\sum_{i=0}^{k-1} \langle y_1(i), y_1(i) \rangle_{\mathcal{Y}_1}\right), & t \in \mathbb{N} \end{cases}, \tag{1.16}$$

where $\mathbb{E}(\cdot)$ denotes the expectation operator. Thus the Linear Quadratic Gaussian control problem is stated as

$$\min_{\Sigma_K \text{ stabilizing}} \mathcal{J}(\Sigma_{GK}). \tag{1.17}$$

As in the previous subsection, the Hilbert space $\mathcal{K}$ is now an Euclidean space

$$\mathcal{K} = \mathbb{R}^n,$$

where n a fixed integer.

In this thesis we shall give a digital solution to the fixed-order LQG control problem for the Pritchard-Salamon class of infinite-dimensional systems with unbounded input and output operators. A Hyland-Bernstein approach led us to give a set of necessary conditions for the existence of the solution. State-space formulae are given in terms of two modified Riccati equations coupled with two modified Lyapunov equations via an oblique projection operator shown to have the same rank as the dimension of the controller state space. The solution to the digital control problem is given in terms of the solvability of two algebraic Riccati equations associated with the equivalent discrete-time system obtained by lifting the T-periodically Pritchard-Salamon digital control system.

1.2 The standard digital control problem

We have formulated the $\mathcal{H}^\infty$ control problem, the $\mathcal{H}^2$-optimal control problem and its stochastic counterpart, the LQG-optimal control problem, both in continuous-time as well as in discrete-time. The ultimate goal is to obtain a digital solution to the aforementioned control problems considered in continuous-time. Therefore we shall restrict ourselves to controllers of the form as depicted in Figure 1.4, where $\mathbf{A/D}$ and $\mathbf{D/A}$ represent the 'analog-to-discrete' and 'discrete-to-analog' devices, assumed to be synchronized in time and with a given sampling period $T > 0$.
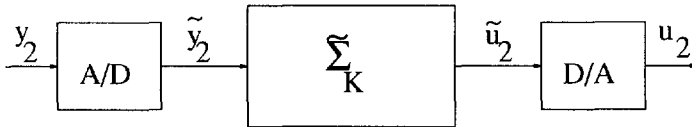
Figure 1.4: The digital controller

The designable element is now $\hat{\Sigma}_K$, the discrete-time controller and the closed-loop system is as in Figure 1.5

Here $\mathbf{S}$ and $\mathbf{H}$ denote the sampler and the zero-order hold operators, respectively, which will be rigorously introduced later on.

A few remarks should be made from the very beginning about the set-up depicted in Figure 1.5.
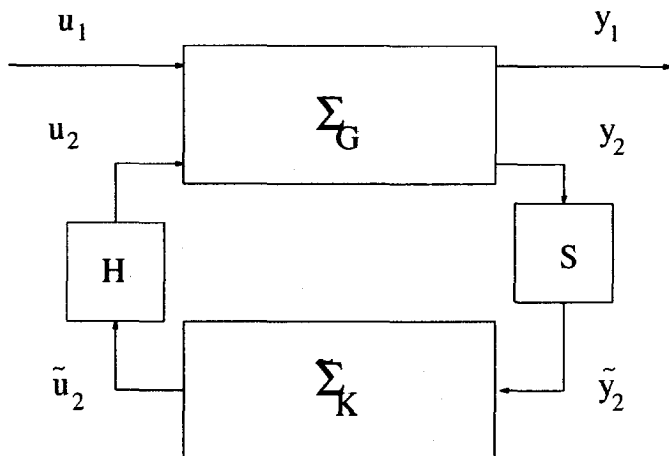
Figure 1.5: The Standard Digital Control Configuration (SDCC)

(i) We deal with a *hybrid (closed-loop) system* since the state of the plant Σ_G evolves in continuous-time and the state of the controller Σ_K in discrete-time, respectively.

(ii) The closed-loop system from u_1 to y_1 is time-invariant no longer due to the periodic characteristic of the **A/D** and **D/A** devices. Therefore, the transfer function of the linear system from u_1 to y_1.

(iii) Certain assumptions should be clearly made on the initial data. For example, we cannot allow a direct feedthrough from the disturbance input to the controlled output. This is especially true since when giving a mathematical model of the **A/D** device, we deal with the *sample operator* which is not well defined on $L_2(0, \infty; \mathcal{U}_1)$, the space of the disturbance input.

The main technique we use to overcome those difficulties is the *lifting technique*. The idea of lifting is to break up a signal defined on the real line into the sequence of signals given by the restriction of the original continuous-time one to time intervals of the form $[kT, (k + 1)T)$ where $T > 0$ is the fixed sampling step and k is in the set of integers. Such a lifting technique applied to the hybrid control system leeds to a discrete-time representation for which, the original digital control problem is converted into a discrete-time control problem. We show that a solution to the latter one is also a solution to the former one.

1.3 Structure of this thesis

The structure of the thesis is the following.

Since the infinite-dimensional systems considered in this thesis present unboundedness in control and/or observation, it is of crucial importance to begin with defining the type of unboundedness which is considered. This is done in Chapter 2 where we introduce the Pritchard-Salamon class as well as the concept of Pritchard-Salamon-Popov triples in the spirit of [70, 87]. Also in Chapter 2 we formulate in its full generality the *digital optimal control problem for Pritchard-Salamon-Popov triples*. This problem is shown to be sufficiently general to include as particular cases the digital LQ-optimal control problem for Pritchard-Salamon systems and the digital $\mathcal{H}^{\infty}$ control problem. Not willing that the unfamiliar reader remains only with an abstract picture, we end Chapter 2 with a special section where several examples of systems that fall in the Pritchard-Salamon class are discussed.

Chapter 3 is devoted to the so-called discrete Popov theory. The main result relates the existence of a stabilizing solution to the discrete-time Riccati equation on a real separable Hilbert space to the invertibility of a certain Toeplitz operator associated with the discrete Popov triple, the basic object this theory operates upon. Several applications of this discrete-time Riccati equation theory are presented, such as

 (*i*) *Discrete-time LQ optimal control,*

 (*ii*) *Discrete-time $\mathcal{H}^{\infty}$ control.*

For a more extended and complete treatment of these subjects the reader is referred to the book [46].

Chapter 4 represents one of the main theoretical contributions of this thesis. We give here the complete generalization of the so-called Hyland-Bernstein control theory for the case of infinite-dimensional discrete-time systems. The theory is built in the deterministic case, when the quadratic cost function to be minimized is given by the $\mathcal{H}^2$ norm of the transfer function associated with the linear closed-loop system $G_{y_1 u_1}$ as defined in subsection 1.1. Nevertheless, the same result can be applied in a stochastic framework with minor modification of the proofs as shown in [6, 14]. The developments from Chapter 4 represent the theoretical support for writing down the digital solution to the fixed-order compensation problem, optimal with respect to a quadratic cost function, as it was formulated in subsection 1.1.2 and subsection 1.1.3, respectively. The basic references are the papers of Barb and De Koning [6, 14, 7, 8].

Chapter 5 is entirely devoted to the *digital stability of Pritchard-Salamon systems*. The main result on exponential stability of Pritchard-Salamon systems under digital state-feedback and various consequences of this result are given in section 5.1. Since the Popov theory requires exponential stability of the system, we give a digital solution to the so-called *prestabilization problem*, both optimal with respect to a minimum-energy type cost function associated with the antistable subsystem, as well as suboptimal. Another thing

we accomplish in this chapter is the full generalization to Pritchard-Salamon systems of the finite-dimensional results on so-called *hybrid stability* from [29]. Two subsections are devoted to the *lifting technique*. The results on lifting outlined here represent the minimum requirement the reader should fulfil in order to understand in which way this digital control tool operates and how to apply such a technique to reduce the digital control problem into an equivalent discrete-time one.

In Chapter 6 we give the main digital control results of this thesis. Our attention is focused on the following control problems for systems in the Pritchard-Salamon class

(i) *Digital LQ optimal control problem,*

(ii) *Digital $\mathcal{H}^\infty$ control problem,*

(iii) *Digital fixed-order LQG-optimal compensation problem.*

We show how these digital control problems can be converted into equivalent discrete-time control problems. The main results are given by applying the discrete Popov theory and the discrete Hyland-Bernstein control theory results proved in Chapter 3 and Chapter 4.

Chapter 7 represents the application part of this thesis. We begin with a case study section where the result on the equivalent discrete-time LQ-optimal control problem of Chapter 6, namely Lemma 6.1 is applied to two examples. The first one is the general parabolic system presented in Example 2.21. The second one is a general hyperbolic system that belongs to the Pritchard-Salamon class.

Special attention is paid to *reduced-order time-discretized systems*. For the sake of simplicity most of the results are presented for finite-dimensional (high-order) systems, but we highlight the difficulties occurring while one is trying to extend those results to infinite dimensional systems. An "approximately" balanced realization of linear finite-dimensional systems is proposed here and an immediate application is to the model reduction problem for time-discretized systems. Our main references are the original papers of Barb and Weiss [4, 5, 16, 17]. In Chapter 9 we reach the conclusions of this thesis and we point out the main investigation directions and interesting open problems that arose during this study.

Some technical details and proofs and a list with symbols used throughout this thesis has been deferred to an appendix A.

Chapter 2

Motivation. Systems with unboundedness. Examples.

The aim of this chapter is to present to the reader the difficulties arising when solving the digital control problem for systems with certain unboundedness in control and observation. We consider first the digital control problem for linear time-invariant infinite-dimensional systems with bounded input operators.

Definition 2.1 *Let $\mathcal{X}$ and $\mathcal{U}$ be real separable Hilbert spaces. A triple of the form*

$$\Sigma \left(S(\cdot), B, M = \left(\begin{array}{cc} Q & L \\ L^* & R \end{array} \right) = M^* \right),$$

with $S(\cdot)$ a strongly continuous semigroup of bounded operators on $\mathcal{X}$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $Q = Q^* \in \mathcal{L}(\mathcal{X})$, $L \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ and $R = R^* \in \mathcal{L}(\mathcal{U})$ is called a Popov triple on $(\mathcal{X}, \mathcal{U})$.

We associated with the Popov triple $\Sigma(S(\cdot), B, M)$ the following objects

(i) The initial value problem

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathcal{X}, \tag{2.1}$$

where A is the infinitesimal generator of the strongly continuous semigroup of bounded operators $S(\cdot)$ on $\mathcal{X}$

(ii) The quadratic index

$$J_\Sigma \left(x_0, u(\cdot) \right) \triangleq \int_0^\infty \langle \left(\begin{array}{cc} Q & L \\ L^* & R \end{array} \right) \left(\begin{array}{c} x(t) \\ u(t) \end{array} \right), \left(\begin{array}{c} x(t) \\ u(t) \end{array} \right) \rangle_{\mathcal{X} \times \mathcal{U}} dt, \tag{2.2}$$

where $x(\cdot)$ is the mild solution to the initial value problem (2.1)

(iii) The class of admissible control functions

$$U_{\text{adm}}^{x_0} \triangleq \left\{ u(\cdot) \in L_2(0,\infty;\mathcal{U}) | x(t) = S(t)x_0 + \int_0^t S(t-\tau)Bu(\tau)d\tau \in L_2(0,\infty;\mathcal{X}) \right\},$$

(2.3)

where $\mathcal{U}$ is the space of control.

Remark 2.2 *Proposition 3.10 from [87] gives the set of necessary and sufficient conditions for which $U_{\text{adm}}^{x_0}$ is not void. To be more precise, it is shown that $u(\cdot) \in U_{\text{adm}}^{x_0}$ if and only if there exists a stabilizing feedback operator $F \in \mathcal{L}(\mathcal{X},\mathcal{U})$ and $v(\cdot) \in L_2(0,\infty;\mathcal{U})$ such that*

$$u(\cdot) = Fx(\cdot) + v(\cdot).$$

Let

$$\mathcal{S}(0,\infty;\mathcal{Z}) \triangleq \{z(k) \in \mathcal{Z}, k \geq 0\}$$

and

$$\mathcal{PC}_{\text{T}}(0,\infty;\mathcal{Z}) \triangleq \{z \in \mathcal{Z} | z(t) = z(k), kT \leq t < (k+1)T, k \geq 0\}$$

denote the spaces of $\mathcal{Z}$-valued sequences and $\mathcal{Z}$-valued piece-wise constant functions, respectively. Let

$$\mathcal{C}(0,\infty;\mathcal{Z}) \triangleq \{z \in \mathcal{Z} | z(t) \text{ is continuous w.r.t. } t\}$$

be the space of continuous $\mathcal{Z}$-valued functions. Define the sample and zero-order hold operators of period T by

$$\begin{aligned} \mathbf{S} &: \quad \mathcal{C}(0,\infty;\mathcal{Z}) \longrightarrow \mathcal{S}(0,\infty;\mathcal{Z}) \\ \mathbf{S}z &\triangleq \quad \{z(0),\cdots,z(k)=z(kT),\cdots\} \triangleq \tilde{z}, \\ \mathbf{H} &: \quad \mathcal{S}(0,\infty;\mathcal{Z}) \longrightarrow \mathcal{PC}_{\text{T}}(0,\infty;\mathcal{Z}) \\ z_{\text{step}} &\triangleq \quad \mathbf{H}\tilde{z}, \end{aligned}$$

(2.4)

(2.5)

such that

$$z_{\text{step}}(t) = z(k), \quad kT \leq t < (k+1)T, k \geq 0.$$

Remark 2.3 *The space of piece-wise constant functions*

$$\mathcal{PC}_{\text{T}}(0,\infty;\mathcal{Z}) = \mathbf{H}\mathcal{S}(0,\infty;\mathcal{Z}) \subset L_2(0,\infty;\mathcal{Z}),$$

is dense in the space of square integrable $\mathcal{Z}$-valued functions. We also have the natural relation

$$\mathcal{C}(0,\infty;\mathcal{Z}) \subset \mathcal{PC}_{\text{T}}(0,\infty;\mathcal{Z}).$$

Definition 2.4 *(digital exponential stabilizability and digital exponential stabilizing control)*

(i) *Let $\tilde{u} = u(0), u(1), \cdots, u(k), \cdots$ denote a sequence in $\mathcal{S}(0, \infty; \mathcal{U})$. A piece-wise constant control function*

$$u_{\text{step}}(t) = (\mathbf{H}\tilde{u})(t),$$

is called a digital exponential stabilizing control for the initial value problem (2.1) if, for $u(\cdot) = u_{\text{step}}(\cdot)$, the solution to (2.1) decays exponentially to zero as t tends to infinity, i.e.

$$\exists M > 1, \alpha > 0 \ s.t. \ \|x(t)\|_{\mathcal{X}} \le Me^{-\alpha t}\|x_0\|_{\mathcal{X}}. \tag{2.6}$$

(ii) *A pair $(S(\cdot), B)$ is called digital exponentially stabilizable if there exists a digital control law in feedback form, i.e. if there exists a feedback operator $F \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ such that for the control function of the form*

$$u_{\text{step}}(t) = Fx(k), \quad kT \le t < (k+1)T$$

the solution to the initial value problem decays exponentially to zero as t tends to infinity.

Then the *digital optimal control problem for infinite-dimensional systems with bounded input and bounded output operators* can be formulated as

$$\min_{u \in U_{\text{adm}}^{x_0} \cap \mathcal{PC}_T(0,\infty;\mathcal{U})} J_{\Sigma}\left(x_0, u_{\text{step}}(\cdot)\right). \tag{2.7}$$

As is well known (see Lee, Chow and Barr [60], Hager and Horowitz [43] or the recent paper of Rosen and Wang [73]), under certain circumstances the solution to this problem can be expressed in the state-feedback form, i.e. there exists a feedback gain operator $F \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ such that for all $x_0 \in \mathcal{X}$

$$u_{\text{step}}^{\text{opt}}(\cdot) = Fx(\cdot, x_0, u_{\text{step}}^{\text{opt}}(\cdot)),$$

or expressed alternatively,

$$u_{\text{step}}^{\text{opt}}(t) = (\mathbf{H}F\tilde{x}) = Fx(k), \quad kT \le t < (k+1)T, \quad k \ge 0,$$

where $u_{\text{step}}^{\text{opt}}(\cdot)$ is the input for which the minimum in (2.7) is attained and, in addition, it is an exponentially stabilizing control law.

2.1   Pritchard-Salamon systems

As it is well known, most of the interesting infinite-dimensional control systems encountered in practice do not fall in the framework of systems with bounded input and output operators. For systems formally described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{2.8}$$
$$y(t) = Cx(t) + Du(t), \tag{2.9}$$

the unboundedness of the B and the C-operators is usually considered in the sense of *admissibility*. One special class of infinite-dimensional systems with unbounded B and C-operators has recently captured the attention of specialists due to its nice structural properties especially related to the linear quadratic control problem. It is the so called Pritchard-Salamon class of systems (see Pritchard and Salamon [70]).

Definition 2.5 *(admissible input and output operators) Let $\mathcal{W}$, $\mathcal{X}$, $\mathcal{V}$, $\mathcal{U}$, $\mathcal{Y}$ be real separable Hilbert spaces and suppose that*

$$\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}.$$

Let $S^{\mathcal{V}}(\cdot)$ be C_0-semigroup with the infinitesimal generator $A^{\mathcal{V}}$ on $\mathcal{V}$ which restricts to a C_0-semigroups $S^{\mathcal{X}}(\cdot)$ and $S^{\mathcal{W}}(\cdot)$ with infinitesimal generator $A^{\mathcal{X}}$ and $A^{\mathcal{W}}$ on $\mathcal{X}$ and $\mathcal{W}$, respectively.

(i) *An operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ is called an admissible input operator for $S(\cdot)$ with respect to $(\mathcal{W}, \mathcal{V})$, if there exists some $t > 0$ and $c > 0$ such that*

$$\int_0^t S(t - \tau)Bu(\tau)d\tau \in \mathcal{W} \tag{2.10}$$

and

$$\left\| \int_0^t S(t - \tau)Bu(\tau)d\tau \right\|_{\mathcal{W}} \leq c\|u(\cdot)\|_{L_2(0,t;\mathcal{U})} \tag{2.11}$$

for all $u(\cdot) \in L_2(0, \infty\mathcal{U})$.

(ii) *An operator $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ is called an admissible output operator for $S(\cdot)$ with respect to $(\mathcal{W}, \mathcal{V})$, if there exists some $t > 0$ and $c > 0$ such that such that*

$$\|CS(\cdot)x\|_{L_2(0,t;\mathcal{Y})} \leq c\|x\|_{\mathcal{V}} \tag{2.12}$$

for all $x \in \mathcal{W}$.

Remark 2.6 *The admissibility of C implies that the linear map from $\mathcal{W}$ to $L_2(0, \infty; \mathcal{Y})$, denoted by $x \mapsto CS(\cdot)x$ has a unique bounded extension from $\mathcal{V}$ to $L_2(0, \infty; \mathcal{Y})$. We shall denote it for every $x \in \mathcal{V}$ by $x \mapsto \overline{CS}(\cdot)x$.*

Definition 2.7 *(Pritchard-Salamon systems)*
Let $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ and $C \in \mathcal{L}(\mathcal{W}, \mathcal{Y})$ be admissible input and output operators for $S^{\mathcal{V}}(\cdot)$ with respect to $(\mathcal{W}, \mathcal{V})$ and suppose that $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$. Then the linear infinite-dimensional system given by

$$x(t) = S^{\mathcal{V}}(t)x_0 + \int_0^t S^{\mathcal{V}}(t - \tau)Bu(\tau)d\tau \tag{2.13}$$

$$y(\cdot) = \overline{CS}(\cdot)x_0 + C\int_0^{\cdot} S(\cdot - \tau)Bu(\tau)d\tau + Du(\cdot), \tag{2.14}$$

where $x_0 \in \mathcal{V}$ and $u \in L_2^{\mathrm{loc}}(0, \infty; \mathcal{U})$ is called a Pritchard-Salamon system and is denoted by $\Sigma(S(\cdot), B, C, D)$. If in addition

$$D(A^{\mathcal{V}}) \hookrightarrow \mathcal{W} \tag{2.15}$$

then we call the system a smooth Pritchard-Salamon system.

Remark 2.8 *(Properties of Pritchard-Salamon systems)*
Hypotheses (2.10) and (2.11) implies that for every $x_0 \in \mathcal{W}$ and every $u(\cdot) \in L_2(0, t; \mathcal{U})$ formula (2.13) defines a continuous function $x(\cdot)$ on the interval $(0, t)$ with values in $\mathcal{W}$. The output function can be defined by

$$y(t) = CS^{\mathcal{W}}(t)x_0 + C\int_0^t S(t - \tau)Bu(\tau)d\tau \tag{2.16}$$

and it is a continuous function on the interval $(0, t)$ with values in $\mathcal{Y}$
If $x_0 \in \mathcal{V}$, then $x(\cdot)$ is only a continuous function with values in $\mathcal{V}$ and (2.16) does not make sense directly. But if hypothesis (2.12) is satisfied, then the right-hand side of (2.16) is a well-defined L_2-function with values in $\mathcal{Y}$.

Remark 2.9 *The motivation why admissible input and output operators are suitable candidates for possible unbounded control and observation operators has been given in Salamon [75, 76] and G. Weiss [82, 84, 85] in connection with the well-posedness of abstract linear control systems, abstract linear observation systems on Banach spaces and regular linear systems on Hilbert spaces, respectively. Intuitively, even when unboundedness of B is allowed as above, we would like that the solution to the state differential equation (2.8) to have sense. Due to unboundedness,*

$$\int_0^t S(t - \tau)Bu(\tau)d\tau$$

is basically a function with values in the large space $\mathcal{V}$ but if admissibility of B is assumed then it restricts its values to $\mathcal{W}$. On the other hand, the observability map $x \mapsto CS(\cdot)x$ is well defined only on the smaller space $\mathcal{W}$. Its action on the state space $\mathcal{X}$ is considered only for its continuous extension.

One can notice that the restricted controllability and extended observability mappings are acting also on $\mathcal{X}$. Their common action on $\mathcal{X}$ is referred as the overlap of control and observation. It represents a intrinsic feature of systems with unboundedness and it is a direct consequence of the admissibility of the B and C operators.

Remark 2.10 *Notice that in Definition 2.7 admissibility of B and C has been considered with respect to $(\mathcal{W}, \mathcal{V})$ and the $\mathcal{X}$-space satisfying $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$ has played no role so far. Its only purpose is to define a dual systems (see Definition A.8 from Appendix A.2). However, if B and C are admissible control and observation operators with respect to $(\mathcal{W}, \mathcal{V})$ then B and C are also admissible control and observation operators with respect to $(\mathcal{X}, \mathcal{V})$ and $(\mathcal{W}, \mathcal{X})$, respectively. There are certain applications where B and C are such that the admissibility with respect to $(\mathcal{W}, \mathcal{V})$ is not satisfied, despite the fact that admissibility condition of B and C with respect to $(\mathcal{X}, \mathcal{V})$ and $(\mathcal{W}, \mathcal{X})$, respectively, is fulfield. Figure 2.1 shows the overlap of the control and observation mappings with respect to the state space $\mathcal{X}$.*
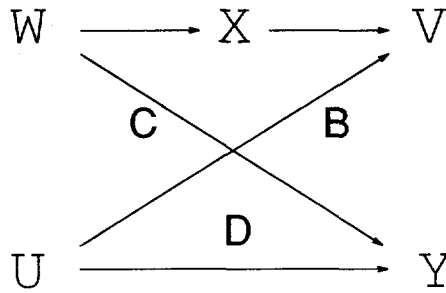
Figure 2.1: The overlap of the control and observation mappings

2.2   Pritchard-Salamon-Popov triples

We shall formulate firstly the digital control problem for Pritchard-Salamon systems in its full generality in the same spirit as we did in the bounded case at the beginning of this chapter and, after that, we shall explain why the problem is difficult and interesting and what can be done to surmount those difficulties.

The Popov triple and the associated concepts have been generalized for Pritchard-Salamon systems in M. Weiss [87]. We report here the following

Definition 2.11 *Let* $\mathcal{W}$, $\mathcal{V}$ *and* $\mathcal{U}$ *be real separable Hilbert spaces and suppose that* $\mathcal{W}$ *is contained in* $\mathcal{V}$ *with continuous dense injection* $(\mathcal{W} \hookrightarrow \mathcal{V})$. *Let* $S^{\mathcal{V}}(\cdot)$ *be a* C_0-*semigroup with the infinitesimal generator* $A^{\mathcal{V}}$ *on* $\mathcal{V}$ *which restricts to a* C_0-*semigroup* $S^{\mathcal{W}}(\cdot)$ *with infinitesimal generator* $A^{\mathcal{W}}$, *on* $\mathcal{W}$. *Let* $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ *be an admissible input operator for* $S^{\mathcal{V}}(\cdot)$ *with respect to* $(\mathcal{W}, \mathcal{V})$. *Let* $Q = Q^* \in \mathcal{L}(\mathcal{W})$ *be an admissible weighting operator for* $S^{\mathcal{V}}(\cdot)$ *with respect to* $(\mathcal{W}, \mathcal{V})$, *i.e.* Q *is an admissible output operator and there exists* $M > 0$ *and* t *such that for every* $x, y \in \mathcal{W}$

$$\int_0^t |\langle QS^{\mathcal{W}}(\tau)x, S^{\mathcal{W}}(\tau)y \rangle_{\mathcal{W}}| d\tau \leq M \|x\|_{\mathcal{V}} \|y\|_{\mathcal{V}}. \tag{2.17}$$

Then a triple of the form

$$\Sigma \left(S^{\mathcal{V}}(\cdot), B, M = \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} = M^* \right), \tag{2.18}$$

with L *bounded in such a way that* $L^* \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ *is an admissible output operator for* $S^{\mathcal{V}}(\cdot)$ *with respect to* $(\mathcal{W}, \mathcal{V})$ *and* $R = R^* \in \mathcal{L}(\mathcal{U})$ *is called a Pritchard-Salamon-Popov triple on* $(\mathcal{W} \hookrightarrow \mathcal{V}, \mathcal{U})$. *We shall call* M *defined in (2.18) the Popov index.*

In Figure 2.2 we have represented the mappings generated by a Pritchard-Salamon-Popov triple
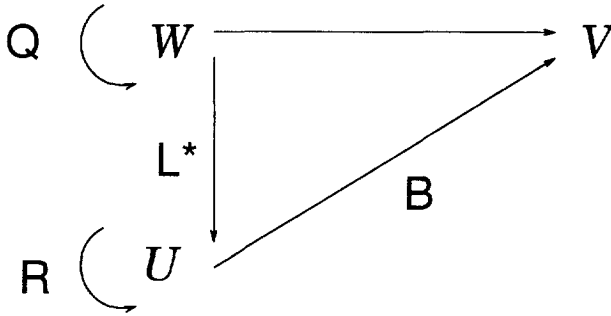
Figure 2.2: The mappings generated by a Pritchard-Salamon-Popov triple

Similar to the definition of Pritchard-Salamon systems, a Pritchard-Salamon-Popov triple on $(\mathcal{W} \hookrightarrow \mathcal{V}, \mathcal{U})$ is called *smooth* if condition (2.15) is satisfied and it shall be called *regular* if the operator R is boundedly invertible.

Definition 2.12 *Let* $\Sigma(S(\cdot), B, C, D)$ *be a Pritchard-Salamon system.*

(i) *The pair $(S(\cdot), B)$ is called admissible (boundedly) stabilizable if there exists an admissible output operator $F \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ (an operator $F \in \mathcal{L}(\mathcal{V}, \mathcal{U})$) such that the perturbed C^0-semigroup $S_F(\cdot)$, generated by $A^{\mathcal{V}} + BF$, is exponentially stable on $\mathcal{W}$ and $\mathcal{V}$.*

(ii) *The pair $(C, S(\cdot))$ is called admissible (boundedly) detectable if there exists an admissible input operator $H \in \mathcal{L}(\mathcal{Y}, \mathcal{V})$ (an input operator $H \in \mathcal{L}(\mathcal{Y}, \mathcal{W})$) such that the perturbed C^0-semigroup $S_H(\cdot)$, generated by $A^{\mathcal{V}} + HC$, is exponentially stable on $\mathcal{W}$ and $\mathcal{V}$.*

The following result [33] gives the relationship between the two concepts of stabilizability quoted above

Proposition 2.13 *The following hold*

(i) *The pair $(S(\cdot), B)$ is admissibly stabilizable if and only if is boundedly stabilizable.*

(ii) *The pair $(C, S(\cdot))$ is admissibly detectable if and only if is boundedly detectable*

Let us assume that $(S^{\mathcal{V}}(\cdot), B)$ is admissibly (boundedly) stabilizable. As in the bounded case we shall associate with the Pritchard-Salamon-Popov triple an initial value problem and a quadratic index. The first one is easy to define. It is given by

$$\dot{x}(t) = A^{\mathcal{V}} x(t) + Bu(t), \quad x(0) = x_0, \quad x_0 \in \mathcal{V}. \tag{2.19}$$

The set of admissible control functions is now

$$U_{\text{adm}}^{x_0} \triangleq \left\{ u(\cdot) \in L_2(0, \infty; \mathcal{U}) \,|\, x(t) = S^{\mathcal{V}}(t) x_0 + \int_0^t S^{\mathcal{V}}(t - \tau) Bu(\tau) d\tau \in L_2(0, \infty; \mathcal{V}) \right\} \tag{2.20}$$

Notice that if $(S^{\mathcal{V}}(\cdot), B)$ is admissibly stabilizable, then $U_{\text{adm}}^{x_0} \neq \{\emptyset\}$. Hence, for any $x_0 \in \mathcal{W}$ there exists $u(\cdot) \in U_{\text{adm}}^{x_0}$ such that the state function $x(\cdot)$ is a square integrable $\mathcal{W}$-valued function and then the (infinite-time horizon) quadratic index associated with the Pritchard-Salamon-Popov triple $\Sigma(S(\cdot), B, M)$ is well defined by the quadratic functional

$$J_\Sigma(x_0, u(\cdot)) \triangleq \int_0^\infty \langle \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}} dt, \tag{2.21}$$

where $x(\cdot)$ is the mild solution to the initial value problem (2.19). Martin Weiss showed in [87] that $J_\Sigma(x_0, u(\cdot))$ defined by (2.21) can be extended to a bounded quadratic functional

for $x_0 \in \mathcal{V}$ and $u(\cdot) \in U_{\mathrm{adm}}^{x_0}$. This was done as follows: define for a Pritchard-Salamon-Popov triple $\Sigma(S(\cdot), B, M)$ the following operators

$$\Phi^{\mathcal{Z}} \;\; : \;\; \mathcal{Z} \longrightarrow L_2(0, \infty; \mathcal{Z}), \;\; \left(\Phi^{\mathcal{Z}} x\right)(t) = S^{\mathcal{Z}}(t) x \tag{2.22}$$

$$\Psi \;\; : \;\; L_2(0, \infty; \mathcal{U}) \longrightarrow L_2(0, \infty; \mathcal{W}), \;\; (\Psi u)(t) = \int_0^t S^{\mathcal{V}}(t - \tau) B u(\tau) d\tau, \tag{2.23}$$

where $\mathcal{Z}$ is either $\mathcal{W}$ or $\mathcal{V}$. Notice that $\Phi^{\mathcal{W}} = \Phi^{\mathcal{V}}|_{\mathcal{W}}$ and the mild solution to the initial value problem (2.19) is

$$x(\cdot) = \Phi^{\mathcal{V}} x_0 + \Psi u(\cdot).$$

In this case, for any $x_0 \in \mathcal{W}$ and $u(\cdot) \in U_{\mathrm{adm}}^{x_0}$, the quadratic index (2.21) admits an equivalent operator expression given by

$$J_\Sigma\left(x_0, u(\cdot)\right) \stackrel{\triangle}{=} \langle \begin{pmatrix} \mathcal{P}_0 & \mathcal{P} \\ \mathcal{P}^* & \mathcal{R} \end{pmatrix} \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix}, \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix} \rangle_{\mathcal{W} \times L_2(0, \infty; \mathcal{U})}, \tag{2.24}$$

where

$$\begin{cases} \mathcal{P}_0 \;\; : \;\; \mathcal{W} \longrightarrow \mathcal{W} & , \;\; \mathcal{P}_0 \stackrel{\triangle}{=} \left(\Phi^{\mathcal{W}}\right)^* Q \Phi^{\mathcal{W}} = \mathcal{P}_0^* \\ \mathcal{P} \;\; : \;\; \mathcal{W} \longrightarrow L_2(0, \infty; \mathcal{U}) & , \;\; \mathcal{P} \stackrel{\triangle}{=} (\Psi^* Q + L^*) \Phi^{\mathcal{W}} \\ \mathcal{R} \;\; : \;\; L_2(0, \infty; \mathcal{U}) \longrightarrow L_2(0, \infty; \mathcal{U}) & , \;\; \mathcal{R} \stackrel{\triangle}{=} R + L^* \Psi + \Psi^* L + \Psi^* Q \Psi \end{cases} \tag{2.25}$$

Now, for any $x_0 \in \mathcal{V}$ the quadratic index is defined as

$$J_\Sigma\left(x_0, u(\cdot)\right) \stackrel{\triangle}{=} \langle \begin{pmatrix} \overline{\mathcal{P}_0} & \overline{\mathcal{P}} \\ \overline{\mathcal{P}}^* & \mathcal{R} \end{pmatrix} \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix}, \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix} \rangle_{\mathcal{V} \times L_2(0, \infty; \mathcal{U})}, \tag{2.26}$$

where

$$\begin{cases} \overline{\mathcal{P}_0} \;\; : \;\; \mathcal{V} \longrightarrow \mathcal{V} & , \;\; \overline{\mathcal{P}_0} x \stackrel{\triangle}{=} \overline{\mathcal{P}} x \\ \overline{\mathcal{P}} \;\; : \;\; \mathcal{V} \longrightarrow L_2(0, \infty; \mathcal{U}) & , \;\; \overline{\mathcal{P}} \stackrel{\triangle}{=} \overline{L^* \Phi^{\mathcal{W}}} + \Psi^* \overline{Q \Phi^{\mathcal{W}}} \end{cases} , \tag{2.27}$$

and where $\overline{P}$ is defined by the extended sesquilinear form

$$\langle \overline{P} x, y \rangle_{\mathcal{V}} \stackrel{\triangle}{=} \langle P x, y \rangle_{\mathcal{W}} = \int_0^\infty \langle Q S^{\mathcal{W}}(t) x, S^{\mathcal{W}}(t) y \rangle_{\mathcal{W}} dt, \forall x, y \in \mathcal{W}. \tag{2.28}$$

Here $\overline{L^* \Phi^{\mathcal{W}}}$ and $\overline{Q \Phi^{\mathcal{W}}}$ denote the bounded extensions to $\mathcal{V}$ of the operators $L^* \Phi^{\mathcal{W}}$ and $Q \Phi^{\mathcal{W}}$, extensions which exists due to the fact that L^* is an admissible output operator and Q is and admissible weighting operator, respectively.

2.3   Digital feedback control law

Definition 2.14 *Let $S(\cdot)$ be a C_0-semigroup on $\mathcal{W} \hookrightarrow \mathcal{V}$, $B \in \mathcal{L}(\mathcal{U},\mathcal{V})$ be an admissble input operator and consider the initial value problem on $\mathcal{W}$*

$$\dot{x}(t) = A^{\mathcal{W}} x(t) + Bu(t), \quad x_0 \in \mathcal{W}, \tag{2.29}$$

where $A^{\mathcal{W}}$ is the infinitesimal generator of $S^{\mathcal{W}}(\cdot)$. Let $F \in \mathcal{L}(\mathcal{V},\mathcal{U})$ be a feedback operator and let $\tilde{x} = \{x(0), x(1), \cdots, x(k), \cdots,\}$ be the sampled state function.

(i) A piece-wise constant control function of the form

$$u_{\text{step}}(\cdot) = (\mathbf{H}F\tilde{x})(t) = Fx(k)$$

   is called a digital feedback control law for the pair $(S(\cdot), B)$.

(ii) If the solution to the initial value problem (2.29) under the digital feedback control law decays exponentially to zero as t tends to infinity, i.e.

$$\exists M_{\mathcal{W}} > 1, \alpha_{\mathcal{W}} > 0 \ \text{s.t.} \ \|x(t)\|_{\mathcal{W}} \le M_{\mathcal{W}} e^{-\alpha_{\mathcal{W}} t} \|x_0\|_{\mathcal{W}}, \tag{2.30}$$

   then the digital feedback control law is called an exponentially stabilizing digital feedback.

We can formulate the *digital optimal control problem* for infinite-dimensional systems with unbounded input and unbounded output operators that belong to the *Pritchard-Salamon class* as follows: determine an exponentially stabilizing digital feedback $u \in U^{x_0}_{\text{adm}} \cap \mathcal{PC}_T(0,\infty;\mathcal{U})$ minimizing the quadratic index $J_\Sigma(x_0, u_{\text{step}}(\cdot))$

$$\min_{u \in U^{x_0}_{\text{adm}} \cap \mathcal{PC}_T(0,\infty;\mathcal{U})} J_\Sigma(x_0, u_{\text{step}}(\cdot)), \tag{2.31}$$

where $J_\Sigma(x_0, u_{\text{step}}(\cdot))$ is defined by

$$J_\Sigma(x_0, u(\cdot)) \overset{\triangle}{=} \left\langle \begin{pmatrix} \mathcal{P}_0 & \mathcal{P} \\ \mathcal{P}^* & \mathcal{R} \end{pmatrix} \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix}, \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix} \right\rangle_{\mathcal{W} \times L_2(0,\infty;\mathcal{U})}, \quad x_0 \in \mathcal{W}.$$

The above formulation of the digital optimal control problem for Pritchard-Salamon systems shows clearly our intention to restrict to the case

$$x_0 \in \mathcal{W} \tag{2.32}$$

and, anticipating a little bit, to be basically concerned only with stabilization on $\mathcal{W}$. Our choice is motivated by the following

Minimum energy control problem The main digital control results we shall prove in this thesis would be, consequently, meaningless if there would not exist at least one practical control problem, whith the feedback generated by sampling the output rather than the state, and which can be formulated as (2.31). Fortunately, the following minimum-energy problem which has been addressed and solved in [13] is a sufficiently strong motivation for our developments. Consider the Pritchard-Salamon system $\Sigma \left(S^{\mathcal{V}}(\cdot), B, C, 0 \right)$ described by the system of equations (2.8), (2.9) subject to the initial condition $x(0) = x_0 \in \mathcal{X}$ where $\mathcal{X}$ is either $\mathcal{W}$ or $\mathcal{V}$. Associated with $\Sigma \left(S^{\mathcal{V}}(\cdot), B, C, 0 \right)$ is the following minimum-energy cost function

$$J_\Sigma \left(x_0, u(\cdot) \right) \triangleq \int_0^\infty \left(\|y(t)\|_{\mathcal{Y}}^2 + \|u(t)\|_{\mathcal{U}}^2 \right) dt. \tag{2.33}$$

Interpreting (2.8) as the initial value problem and noticing that (2.33) is the quadratic functional (2.21) for

$$\begin{cases} Q = C^*C \\ L = 0 \\ R = I \end{cases}, \tag{2.34}$$

then the control problem (2.31) is well posed in continuous-time if certain adequate assumptions of admissible stabilizability/detectability are made. We would like to determine a piece-wise constant control function which is generated from the sampled version of the output, rather than from the discretized state function and to formulate properly the digital control problem. Recall that (see Remark 2.8) if $x_0 \in \mathcal{W}$ then $y(t)$ is continuous with respect to time and if $x_0 \in \mathcal{V}$ then the output function has an interpretation only in L_2-sense given by the following expression

$$y(\cdot) = \overline{CS^{\mathcal{W}}(\cdot)}x_0 + C \int_0^{\cdot} S^{\mathcal{V}}(\cdot - \tau)Bu(\tau)d\tau. \tag{2.35}$$

In the first case the sampled version of the output function $\tilde{y} = \mathbf{S}y$ is well defined while in the second case it is not. Hence, when $\mathcal{X} = \mathcal{W}$ the minimum-energy problem formulated above admits a digital output-measurement solution. In the second case, when $\mathcal{X} = \mathcal{V}$, a Popov theory based result might also be developed to obtain a digital exponentially stabilizing state-feedback solution, but such a construction, involving many mathematical subtleties specific when one works on $\mathcal{V}$, is beyond the main goal of this thesis. It represents a promising direction for future research.

This example extends generalizes the following remarks

Remark 2.15 *(i) An engineering approach to the control problem of Pritchard-Salamon systems, i.e. a digital control based on the sampled of the output function is possible only under the assumption (2.32).*

(ii) Notice that finding a solution to the digital state-feedback control problem has, in the case of systems with unboundedness, more a theoretical impact than a practical one. This is especially true since the time-discretized state function is obtained by sampling a $\mathcal{V}$-valued function, i.e. a function usually living in a Sobolev space. Thus, the time-discretized state function has to be interpreted as "data generated by a similar mechanism to practical sampling" rather than the periodical measurement of a physical entity.

Remark 2.16 *The first difficulty that arose in the digital control of Pritchard-Salamon systems due to the way the sample operator is defined, i.e. it is represented as an operator giving the pointwise value of a function. In [78] it is shown that it can be represented by a finite-rank bounded integral operator with L_2-kernel. The motivation behind such a representation relies upon the fact that the sampling process can be viewed as being done via short-time-window/wide-frequency-band filtering.*

The second difficulty that arises in the digital control of Pritchard-Salamon systems is due to the intrinsic structure of those systems. To be more clear, one feature of Pritchard-Salamon systems is that stability on $\mathcal{W}(\mathcal{V})$ does not necessarily imply stability on $\mathcal{V}(\mathcal{W})$ (see Example 2.1 from [33]). In [13, 15] the digital solution to the minimum-energy problem (2.33) was obtained in terms of the solution to the so-called equivalent discrete-time minimum-energy problem. Among different results proved therein, it is shown that the same awkward feature is transferred to the equivalent discrete-time system, i.e. the power stability of the discretized semigroup on $\mathcal{W}(\mathcal{V})$ does not necessarily imply power stability on $\mathcal{V}(\mathcal{W})$. Thus, the digital exponential stability was considered with respect to the smaller space $\mathcal{W}$. In this thesis we adopt the same line as we did in [13, 15], i.e. we assume (2.32). The difference with respect to the developments from [13, 15] is that here, we construct a Popov theory based solution to the problem. Anticipating some facts, we shall be concerned only with stabilizability under digital feedback on the smaller state space $\mathcal{W}$. This is especially true since we shall prove in one of the next chapters that an exponentially stable Pritchard-Salamon system with $x_0 \in \mathcal{W}$ has a sampled state function $\tilde{x}$ which belongs also to $\mathcal{W}$. The problem of stabilizing by digital state feedback simultaneously on $\mathcal{W}$ and $\mathcal{V}$ spaces, is left as one of the future research direction open by this thesis.

Finally, it is worth mentioning that the assumption made on $Q \in \mathcal{L}(\mathcal{W})$, that of being an admissible weighting operator, offers us the flexibility to give a digital solution to the so-called nonstandard LQ-optimal control problems [80, 87], where

$$Q = C_1^* C_1 - C_2^* C_2$$

where $C_1 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_1)$, $C_2 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_2)$ are admissible output operators ($\mathcal{Y}_1, \mathcal{Y}_2$ are arbitrary real separable Hilbert spaces).

2.4 Examples of Pritchard-Salamon systems

In the previous chapter we gave an example of a distributed parameter system with unbounded input and output operators. In this chapter we have introduced a class of infinite-dimensional systems with certain unboundedness in control and observation, the Pritchard-Salamon class. Let us treat here some examples of systems that fall in this class. We proceed by giving two examples of infinite-dimensional systems that are modeled using either *bounded* or *unbounded* input and output operators, depending on the particular control and observation that is used. The first example is a parabolic partial differential equation with either distributed or boundary control, while the second example is a delay differential equation (see [70]).

$$\Sigma_G \begin{cases} \dot{x}(t) &= Ax(t) + Bu(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) + Du(t). \end{cases} \tag{2.36}$$

Here, the state $x(\cdot)$, the control $u(\cdot)$ and the observation $y(\cdot)$ are all functions of time with values in certain Hilbert spaces. Furthermore, A is the infinitesimal generator of a C_0-semigroup, B is the (possibly unbounded) input operator, C is the (possibly unbounded) output operator and D is the feedthrough operator. The system Σ_G is usually represented by the *mild solution* to the differential equation (2.36):

$$\Sigma_G \begin{cases} x(t) &= S(t)x_0 + \int_0^t S(t-s)Bu(s)ds \\ y(t) &= Cx(t) + Du(t), \end{cases} \tag{2.37}$$

where $S(\cdot)$ is the C_0-semigroup generated by A (the reader is referred to [68] for more on the concept of mild solutions).

Example 2.17 (*A parabolic PDE. The bounded case*)
The temperature distribution in a heated rod is described by

$$\begin{cases} \dfrac{\partial z}{\partial t}(\xi, t) = \dfrac{\partial^2 z}{\partial \xi^2}(\xi, t) + b(\xi)u(t), & t > 0 \quad 0 < \xi < 1, \\ \dfrac{\partial z}{\partial \xi}(0, t) = 0, \quad \dfrac{\partial z}{\partial \xi}(1, t) = 0 & t > 0, \\ y(t) = \int_0^t c(\xi)z(\xi, t)d\xi & t > 0, \\ z(\xi, 0) = z_0(\xi) & 0 < \xi < 1 \end{cases} \quad ,$$

where $b(\cdot)$ and $c(\cdot)$ are elements of $L_2(0, 1)$. Here $z(t, \xi)$ represents the temperature profile at time t, $b(\xi)u(t)$ represents the addition of heat along the rod and $y(t)$ is some averaged measurement of the temperature. Choosing the state space $\mathcal{X} = L_2(0, 1)$ with the state function

$$x(t) = z(t, \xi), \quad t > 0 \ 0 < \xi < 1,$$

the PDE can be modeled as

$$\begin{cases} x(t) & = & S(t)x_0 + \int_0^t S(t-s)Bu(s)ds \\ y(t) & = & Cx(t). \end{cases}$$

Here $S(\cdot)$ is a C_0-semigroup on $\mathcal{X}$ with infinitesimal generator A, the input operator B and the output operator C are defined by

$$D(A) = \{x \in \mathcal{X} \mid x, \tfrac{dx}{d\xi} \text{ are absolutely continuous}, \tfrac{d^2x}{d\xi^2} \in \mathcal{X} \text{ and } \tfrac{dx}{d\xi}(0) = \tfrac{dx}{d\xi}(1) = 0\},$$
$$Ax = \frac{d^2x}{d\xi^2},$$
$$Bu = b(\cdot)u,$$
$$Cx = <c(\cdot), x(\cdot)>_{\mathcal{X}} .$$

Furthermore, the initial condition $x_0 \in \mathcal{X}$ is given by $x_0(\xi) = z(\xi, 0)$. It is easy to see that both the input and the output operator are linear and bounded: $B \in \mathcal{L}(\mathbb{R}, \mathcal{X})$ and $C \in \mathcal{L}(\mathcal{X}, \mathbb{R})$. Notice that for smooth input functions $u(\cdot)$ and for initial conditions $x_0 \in D(A)$, the state function $x(\cdot)$ is differentiable (with respect of the topology of $\mathcal{X}$).

Example 2.18 (*A delay differential equation. The bounded case*)
Consider the delay differential equation

$$\begin{cases} \dot{z}(t) & = & A_{f1}z(t) + A_{f2}z(t-2) + B_f u(t), \\ y(t) & = & C_f z(t), \end{cases} \tag{2.38}$$

where $z(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p, A_{f1}, A_{f2} \in \mathbb{R}^{n\times n}, B_f \in \mathbb{R}^{n\times m}$ and $C_f \in \mathbb{R}^{p\times n}$. This delay differential equation is a very simple example of a retarded functional differential equation and we shall show how it can be modeled as a system of the form (2.37). Define the product Hilbert space $M_2 = \mathbb{R}^n \times L_2(-2, 0 ; \mathbb{R}^n)$ (with the obvious inner product). For every initial condition $x_0 = (\eta_0, \phi_0) \in M_2$ there exists a unique solution of (2.38) satisfying

$$\lim_{t\downarrow 0} z(t) = \eta_0, \quad z(\tau) = \phi_0(\tau); \quad -2 < \tau < 0.$$

In order to give the state of the system we define the solution segment

$$z_t(\tau) = z(t+\tau) ; \quad -2 \le \tau \le 0 ;$$

(the past of $z(\cdot)$ up to 2 time units). Choosing the state space $\mathcal{X} = M_2$ with state function

$$x(t) = (z(t), z_t) \in M_2,$$

it follows that the system (2.38) can be reformulated as a system of the form (2.37), where $S(\cdot)$ is a C_0-semigroup on the Hilbert space $\mathcal{X}$, the input and output operators are given by

$$Bu = (B_f u, 0) \quad \text{for} \quad u \in \mathbb{R}^m$$
$$Cx = C_f \eta \quad \text{for} \quad x = (\eta, \phi) \in \mathcal{X}$$

and the feedthrough operator is given by $D = 0$. Hence, $B \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$ and $C \in \mathcal{L}(\mathcal{X}, \mathbb{R}^p)$: the input and output operators are bounded.

We have seen that the systems of Examples 2.17, 2.18 can be modeled as semigroup control systems of the form (2.37), with bounded input and output operators. This class of systems has nice system theoretic properties and allows for extensions of important finite-dimensional control ideas (see [35] and references therein). Furthermore, many partial differential equations can be modeled in this framework, as well as many delay equations. However, some infinite-dimensional systems are most naturally modeled using *unbounded* input and output operators. This unboundedness is usually a consequence of point or boundary control and observation. Delay equations with delays in the inputs or outputs lead to unbounded control and observation operators as well. Let us modify Examples 2.17 and 2.18 in such a way that those modeling features are captured in the new set-up. We shall consider the parabolic PDE of example 2.17 with boundary control, rather than distributed control and we show what happens to the delay equation of Example 2.17 if the output $y(\cdot)$ depends on the past $z_t(\cdot)$, rather than on $z(t)$.

Example 2.19 (*A parabolic PDE. The unbounded case*)
The temperature distribution of a heated rod with *boundary* heat control may be described by

$$\begin{cases} \dfrac{\partial z}{\partial t}(\xi, t) = \dfrac{\partial^2 z}{\partial \xi^2}(\xi, t), & t > 0 \quad 0 < \xi < 1, \\ \dfrac{\partial z}{\partial \xi}(0, t) = u(t), \quad \dfrac{\partial z}{\partial \xi}(1, t) = 0 & t > 0, \\ y(t) = \int_0^t c(\xi) z(\xi, t) d\xi & t > 0, \\ z(\xi, 0) = z_0(\xi) & 0 < \xi < 1 \end{cases}$$

where $c(\cdot) \in L_2(0, 1)$ (as before, the output $y(t)$ is some averaged measurement of the temperature). Formally, the PDE may be expressed as

$$\frac{\partial z}{\partial t}(\xi, t) = \frac{\partial^2 z}{\partial \xi^2}(\xi, t) - \delta_0 u(t),$$

where δ_0 denotes the Dirac delta impulse at $\xi = 0$ (actually, this can be done in a rigorous way, using distribution theory, see Salamon [75]). In fact, choosing the state space $\mathcal{X} = L_2(0, 1)$ with state function

$$x(t) = z(t, \xi), \quad t > 0 \ 0 < \xi < 1$$

as in Example 2.17, this equation can formally be modeled as a system of the form (2.37) with the same C_0-semigroup $S(\cdot)$, the same (bounded) output operator C and with B given by $Bu = -\delta_0 u$. The difference with Example 2.17 is that now $b(\cdot)$ is not an L_2-function, but a distribution. Hence $B \notin \mathcal{L}(\mathbb{R}, \mathcal{X})$: we say that the input operator is unbounded. In the next chapter we shall see how one can interpret now the system equations (2.37)

in a rigorous manner. The technique used is to introduce another Hilbert space $\mathcal{V}$ that contains $\mathcal{X}$ and is such that $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$. Furthermore, B and C satisfy the admissibility requirements which should be used in the interpretation of (2.37).

Example 2.20 (*A delay differential equation. The unbounded case*)
Consider the delay equation

$$
\begin{cases}
\dot{z}(t) &= A_{f1}z(t) + A_{f2}z(t-2) + B_f u(t), \\
y(t) &= C_f z(t-1),
\end{cases}
\tag{2.39}
$$

where $z(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ and the matrices A_{f1}, A_{f2}, B_f and C_f are as in Example 2.18, with a notable difference with respect to Example 2.18, that of having a delay in the output equation for $y(t)$. The state equation can still be given by

$$
x(t) = S(t)x_0 + \int_0^t S(t-s)Bu(s)ds,
$$

where $S(\cdot)$ is the same C_0-semigroup on the same state space $\mathcal{X} = M_2 = \mathbb{R}^n \times L_2(-2, 0 \,; \mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m, \mathcal{X})$ is the same (bounded) input operator and x_0 is the same initial condition as in Example 2.18. Defining $< \delta_x, \phi > = \phi(x)$ for $\phi \in C(-2, 0 \,; \mathbb{R}^n)$, we can express the delayed output equation for y as in formula

$$
y(t) = Cx(t),
$$

where C is given by

$$
Cx = C_f < \delta_{-1}, \phi > = C_f \phi(-1) \quad \text{for} \quad x = (\eta, \phi).
$$

It follows that $C \notin \mathcal{L}(\mathcal{X}, \mathbb{R}^p)$, because an element of $L_2(-2, 0 \,; \mathbb{R}^n)$ does not have any continuity properties in general: C is unbounded with respect to $\mathcal{X}$. Due to an 'admissibility' property of B, the second component of $x(t)$ is an element of $C(-2, 0 \,; \mathbb{R}^n)$ for all $t \geq 0$ if the initial condition is smooth. Furthermore, C satisfies another 'admissibility' property so that this can somehow be extended to initial conditions in $\mathcal{X} = M_2$ and formula (2.37) makes sense.

It is worthwhile, finally, to note that one of the simplest delay systems, namely the system represented by the transfer function

$$
e^{-s}C_f(sI - A_f)^{-1}B_f,
$$

where A_f, B_f and C_f are matrices of appropriate dimensions, can be described in the above formulation (take $A_{f1} = A_f$ and $A_{f2} = 0$ in formula (2.39)).

Next, we show how the systems of Examples 2.19 and 2.20 can both be formulated as Pritchard-Salamon systems. The delicate problem is to find the right choices for $\mathcal{W}$ and $\mathcal{V}$, so that the input and output operators satisfy the admissibility conditions.

Example 2.21 (*Continuation of Example* 2.19)
In order to formulate the parabolic PDE with boundary control of Example 2.19 as a
Pritchard-Salamon system, we first consider a more general type of parabolic system. Let
A be a self-adjoint operator on a separable Hilbert space $\mathcal{X}$ and suppose that it has compact
resolvent and that its spectrum consists of strictly decreasing real eigenvalues λ_n ; $n \in \mathbb{N}$
with eigenvectors $\phi_n \in \mathcal{X}$, $\|\phi_n\|_{\mathcal{X}} = 1$. In this case $\{\phi_n$; $n \in \mathbb{N}\}$ forms an orthonormal
basis of $\mathcal{X}$ so that for all $x \in \mathcal{X}$,

$$\sum_{n=0}^{\infty} < x, \phi_n >_{\mathcal{X}}^2 < \infty \quad \text{and} \quad x = \sum_{n=0}^{\infty} < x, \phi_n >_{\mathcal{X}} \phi_n.$$

A can be represented as

$$
\begin{aligned}
D(A) &= \{x \in \mathcal{X} \mid \textstyle\sum_{n=0}^{\infty} \lambda_n^2 < x, \phi_n >_{\mathcal{X}}^2 < \infty\} \\
Ax &= \textstyle\sum_{n=0}^{\infty} \lambda_n < x, \phi_n >_{\mathcal{X}} \phi_n
\end{aligned}
\tag{2.40}
$$

and the C_0-semigroup $S(\cdot)$ generated by A is given by

$$S(t)x = \sum_{n=0}^{\infty} \exp(\lambda_n t) < x, \phi_n >_{\mathcal{X}} \phi_n. \tag{2.41}$$

Now let β_n and γ_n be positive sequences satisfying $0 < \beta_n \le 1 \le \gamma_n < \infty$ and suppose
that $\mathcal{W}$ and $\mathcal{V}$ are determined by

$$\mathcal{W} = \{x \in \mathcal{X} \mid \sum_{n=0}^{\infty} \gamma_n < x, \phi_n >_{\mathcal{X}}^2 < \infty\}$$

$$\mathcal{V}^* = \{x \in \mathcal{X} \mid \sum_{n=0}^{\infty} \beta_n^{-1} < x, \phi_n >_{\mathcal{X}}^2 < \infty\},$$

with the obvious inner products. Here we assume that $\mathcal{X}$ is identified with its dual, so that
$\mathcal{V}^* \subset \mathcal{X} = \mathcal{X}^* \subset \mathcal{V}$. This means that $\mathcal{V}$ can be represented as a space of sequences

$$\mathcal{V} = \{x \in \mathbb{R}^{\mathbb{N}} \mid \sum_{n=0}^{\infty} \beta_n x_n^2 < \infty\}$$

and the injection $\mathcal{X} \subset \mathcal{V}$ is given by identifying $x \in \mathcal{X}$ with the sequence $\{< x, \phi_n >_{\mathcal{X}}$
; $n \in \mathbb{N}\}$. Finally, let $B \in \mathcal{L}(\mathbb{R}, \mathcal{V})$ and $C \in \mathcal{L}(\mathcal{W}, \mathbb{R})$ be given by

$$Bu = \{b_n u \; ; n \in \mathbb{N}\} \quad \text{and} \quad Cx = \sum_{n=0}^{\infty} c_n < x, \phi_n >_{\mathcal{X}},$$

where the sequences $\{b_n$; $n \in \mathbb{N}\}$ and $\{c_n$; $n \in \mathbb{N}\}$ are such that

$$\sum_{n=0}^{\infty} \beta_n b_n^2 < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} \gamma_n^{-1} c_n^2 < \infty. \tag{2.42}$$

It is not difficult to show that B and C are admissible with respect to $(\mathcal{W}, \mathcal{V})$ if

$$\sum_{n=n_0}^{\infty} \frac{\gamma_n b_n^2}{|\lambda_n|} < \infty \quad \text{and} \quad \sum_{n=n_0}^{\infty} \frac{c_n^2}{\beta_n |\lambda_n|} < \infty, \tag{2.43}$$

where $n_0 = 1 + \max\{n \in \mathbb{N} \mid \lambda_n \geq 0\}$. Furthermore, given sequences $b_n, c_n, \lambda_n \in \mathbb{R}$ such that λ_n is strictly decreasing and $\lambda_n \downarrow -\infty$, there exist sequences β_n, γ_n such that the inequalities (2.42)-(2.43) are satisfied if and only if

$$\sum_{n=n_0}^{\infty} \frac{b_n c_n}{|\lambda_n|^{\frac{1}{2}}} < \infty. \tag{2.44}$$

(see [70, Lemma 4.4]). This last result particularly shows that B and C cannot be 'too unbounded' with respect to $\mathcal{X}$. Furthermore, it shows that B can be 'more unbounded' as long as C is 'less unbounded' and vice versa. We mention that the condition $D(A^{\mathcal{V}}) \hookrightarrow \mathcal{W}$ is satisfied, provided that γ_n, β_n are chosen appropriately.

Finally, we can formulate the parabolic PDE with boundary control of Example 2.19 as a Pritchard-Salamon system. Choosing $\mathcal{X} = L_2(0,1)$ it follows that A is indeed self-adjoint, that it has compact resolvent and that it is of the form (2.40), where $\lambda_0 = 0$; $\phi_0 = 1$ and $\lambda_n = -n^2\pi^2$; $\phi_n(\xi) = \sqrt{2}\cos(n\pi\xi)$ for $n \geq 1$. Recall that the input and output operators were given by

$$Cx = \int_0^1 c(\xi)x(\xi)d\xi, \quad c(\cdot) \in \mathcal{X}.$$

and

$$Bu = -\delta u.$$

Hence we get $c_n = <c(\cdot), \phi_n>_\mathcal{X}$; $n \in \mathbb{N}$ and $b_0 = -1$; $b_n = -\sqrt{2}$ for $n \geq 1$ and condition (2.44) is satisfied if and only if

$$\sum_{n=1}^{\infty} \frac{|c_n|}{n} < \infty. \tag{2.45}$$

Since $c(\cdot) \in \mathcal{X}$, we have $\sum_{n=1}^{\infty} c_n^2 < \infty$ and so condition (2.45) is satisfied. In fact, we can choose the sequences γ_n and β_n by $\gamma_n = 1$ and $\beta_n = n^{-2}$, $n \geq 1$. This corresponds to the choice $\mathcal{W} = \mathcal{X}$ and $\mathcal{V}' = W^{1,2}(0,1)$ (the Sobolev space of absolute continuous functions in $L_2(0,1)$ whose derivative is in $L_2(0,1)$). Hence, we have modeled Example 2.19 as a Pritchard-Salamon system.

Finally, we note that we could allow for more unboundedness of the operator C, because c_n is only required to satisfy (2.45). However, C cannot be taken to represent a point observation, because this would correspond to $c_n = \sqrt{2}$, $n \geq 1$ and then condition (2.45) would be violated.

Example 2.22 (*Continuation of Example 2.20*)
In order to model the delay differential equation of Example 2.20 as a Pritchard-Salamon

system, we first introduce a much more general functional differential equation (cf. [70]), so that the system of Example 2.20 will be merely a special case (in fact, here we consider a system of the neutral type, rather than of the retarded type). Consider the system given by

$$
\begin{cases}
\dfrac{d}{dt}(z(t) - M z_t) &= L z_t + B_0 u(t), \\[2mm]
y(t) &= C_0 z_t,
\end{cases}
\tag{2.46}
$$

where $z(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p$ and z_t is the solution segment defined by

$$
z_t(\tau) = z(t + \tau) ; \quad -h \le \tau \le 0 ; \quad h > 0.
$$

We have $B_0 \in \mathbb{R}^{n \times m}$ and L, M, C_0 are bounded linear functionals from $C(-h, 0 \, ; \mathbb{R}^n)$ into $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively. It is easy to see that the delay differential equation of Example 2.20 can be reformulated as in (2.46), using

$$
\begin{aligned}
h &= 2 \\
M &= 0 \\
L &= A_{\text{if}1} < \delta_0, \cdot > + A_{\text{f}2} < \delta_{-2}, \cdot > \\
B_0 &= B_{\text{f}} \\
C_0 &= C_{\text{f}} < \delta_{-1}, \cdot >,
\end{aligned}
$$

where $< \delta_x, \phi > = \phi(x)$ for $\phi \in C(-1, 0 \, ; \mathbb{R}^n)$.

Under some conditions (see [70, Example 4.1]), the system given by (2.46) has a unique solution $z(t)$; $t \ge -h$, for every input $u(\cdot) \in L_2^{\text{loc}}(0, \infty; \mathbb{R}^m)$ and every initial condition satisfying

$$
\lim_{t \downarrow 0}(z(t) - M z_t) = \eta_0, \quad z(\tau) = \phi_0(\tau); \quad -h < \tau < 0,
$$

where $x_0 = (\eta_0, \phi_0) \in M_2 = \mathbb{R}^n \times L_2(-h, 0 \, ; \mathbb{R}^n)$. Moreover, the evolution of the state

$$
x(t) = (z(t) - M z_t, z_t) \in M_2
$$

of the system can be described by

$$
x(t) = S(t)x_0 + \int_0^t S(t - s)Bu(s)ds,
$$

where $B \in \mathcal{L}(\mathbb{R}^m, M_2)$ maps $u \in \mathbb{R}^m$ into the pair $Bu = (B_0 u, 0)$ and $S(t) \in \mathcal{L}(M_2)$ is the C_0-semigroup generated by the operator A given by

$$
\begin{aligned}
D(A) &= \{x = (\eta, \phi) \in M_2 \mid \phi \in W^{1,2}, \eta = \phi(0) - M\phi\}, \\[2mm]
Ax &= (L\phi, \dot{\phi}).
\end{aligned}
$$

Here $W^{1,2}$ denotes the Sobolev space $W^{1,2}(-h, 0 ; \mathbb{R}^n)$ of absolute continuous functions in $L_2(-h, 0 ; \mathbb{R}^n)$ whose derivative is in $L_2(-h, 0 ; \mathbb{R}^n)$. Now $D(A)$ can be considered as a Hilbert space by choosing the inner product

$$< (\eta, \phi), (\bar{\eta}, \bar{\phi}) >_{D(A)} = < \phi, \bar{\phi} >_{W^{1,2}}$$

and it follows that $S(\cdot)$ restricts to a C_0-semigroup on $D(A)$. The output $y(t) = C_0 z_t$ of the system may formally be described by

$$y(t) = Cx(t) = C \left(z(t) - M z_t, z_t \right),$$

where the output operator C is given by

$$C : D(A) \to \mathbb{R}^p, \quad Cx = C(\eta, \phi) = C_0 \phi.$$

We recall that by assumption C_0 is a bounded linear map from $C(-h, 0 ; \mathbb{R}^n)$ to $\mathbb{R}^p$ so that $C \in \mathcal{L}(D(A), \mathbb{R}^p)$ but in general $C \notin \mathcal{L}(M_2, \mathbb{R}^p)$. This last situation of course occurs in Example 2.20, because there $C_0 = C_f < \delta_{-1}, \cdot >$, which is not bounded on $L_2(-2, 0 ; \mathbb{R}^n)$.

Using the fact that $S(\cdot)$ restricts to a C_0-semigroup on $D(A)$, a natural choice for $\mathcal{W}$ and $\mathcal{V}$ is $\mathcal{W} = D(A)$ and $\mathcal{V} = M_2$, because then $C \in \mathcal{L}(\mathcal{W}, \mathbb{R}^p)$ and $B \in \mathcal{L}(\mathbb{R}^m, \mathcal{V})$ and we can choose $U = \mathbb{R}^m$ and $Y = \mathbb{R}^p$. In [70] it is explained that now B and C are both admissible so that the neutral functional differential equation of Example 2.20 can indeed be modeled as a Pritchard-Salamon system. Finally, we note that condition (2.15) is trivially satisfied because of $\mathcal{W} = D(A)$, and hence, we have a smooth Pritchard-Salamon system.

Remark 2.23 *We have seen two examples of infinite-dimensional systems with unbounded input and output operators which can be formulated as Pritchard-Salamon systems: a class of parabolic PDE's with Neumann boundary control in Example 2.21 and the class of neutral functional differential equations of Example 2.22. In [70] the authors show that also certain hyperbolic PDE's can be modeled as Pritchard-Salamon systems. However, these systems are in general only exponentially stabilizable if there is some internal damping. Furthermore it should be noted that Dirichlet boundary control usually leads to input operators that are 'too unbounded' for the Pritchard-Salamon framework.*

Part II

Discrete-time control theory

Chapter 3

Discrete-time Popov theory

Originally developed by Ionescu and Weiss [50] for finite-dimensional continuos and discrete-time systems, the main results on discrete Popov theory have received a complete generalization for the case of time-varying discrete-time systems on Hilbert spaces in [45, 46]. Among different results of Popov theory, the one giving the link between a quadratic cost functional and the solution to the so-called Kalman-Szego-Popov-Yakubovitch system, strongly related to Riccati equation theory, is, in our opinion, the most relevant. The central result proved in [50, 45] was an equivalent condition to the the Popov positivity condition expressed in terms of the invertibility of a certain Toeplitz-like operator. Immediate applications, probably the most important ones, were to write down the solution to the discrete-time LQ-optimal and $\mathcal{H}^\infty$ control problems. In this chapter we give the main results on discrete Popov theory from [45, 46, 50].

3.1   Discrete-time Popov triples. Basic concepts.

We shall begin this section by outlining some basic definitions that are needed for the further developments.

Definition 3.1 *(discrete semigroup)*

  (i) *A discrete semigroup on $\mathcal{X}$ is an operator valued function $A : \mathbb{N} \longrightarrow \mathcal{L}(\mathcal{X})$, which satisfies*

  (a) $A^0 = I_\mathcal{X}$,

  (b) $A^{i+j} = A^i A^j, \ \forall i, j \in \mathbb{N}.$

  (ii) *If $\rho_\mathcal{X} \triangleq \lim_{k \longrightarrow \infty} \frac{1}{k} \log \|A^k\|_\mathcal{X} < \infty$ then $\rho_\mathcal{X}$ is called the discrete growth bound of A.*

  (iii) *A discrete semigroup $A \in \mathcal{L}(\mathcal{X})$ is called power stable if there exist $M_{\rho_\mathcal{X}} > 0$ and $0 < \rho_\mathcal{X} < 1$ such that*
$$\|A^k\| < M_{\rho_\mathcal{X}} \rho_\mathcal{X}^k, \quad \forall k \geq 0.$$

(iv) *If in particular* $\|A^k\|_{\mathcal{X}} < 1, \forall k > 0$ *then the discrete semigroup is called a* contraction discrete semigroup.

Definition 3.2 *(coercive operator)*
A self-adjoint operator $T = T^* \in \mathcal{L}(\mathcal{X})$ *is called coercive and we denote this by* $T \gg 0$ *if there exists* $\delta > 0$ *such that*
$$\langle Tx, x \rangle \geq \delta \|x\|^2$$
for all $x \in \mathcal{D}(T)$.

Let $\Sigma(A, B, C, D)$ be an infinite-dimensional discrete-time system satisfying the set of equations
$$\Sigma \left\{ \begin{array}{l} (\sigma x)(k) = Ax(k) + Bu(k) \\ y(k) = Cx(k) + Du(k) \end{array} \right. , \tag{3.1}$$

with the discrete-time semigroup $A \in \mathcal{L}(\mathcal{X})$ assumed power stable, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

Definition 3.3 *(Lyapunov transformation. Lyapunov similarity.)*

(i) *A bounded operator for which the inverse exists and is also bounded is called a Lyapunov transformation.*

(ii) *Two bounded operators* A_1 *and* A_2 *are called Lyapunov similar if there exists a Lyapunov transformation* T *such that*
$$A_2 = TA_1T^{-1}.$$

(iii) *Two infinite-dimensional discrete-time systems on* $\mathcal{X}$, $\Sigma_1(A_1, B_1, C_1, D_1)$ *and* $\Sigma_2(A_2, B_2, C_2, D_2 D_1)$, *satisfying the set of equations (3.1) are called Lyapunov similar if there exists a Lyapunov transformation such that*
$$\begin{array}{rcl} A_2 & = & TA_1T^{-1}, \\ B_2 & = & TB_1, \\ C_2 & = & C_1T^{-1}. \end{array} \tag{3.2} \tag{3.3} \tag{3.4} \tag{3.5}$$

We shall denote the Lyapunov similarity of two systems by
$$\Sigma_1 \rightsquigarrow \Sigma_2.$$

Definition 3.4 *(controllable and observable system)*
Let $\Sigma(A, B, C, D)$ be an infinite-dimensional discrete-time system satisfying the set of equations (3.1) with the discrete-time semigroup $A \in \mathcal{L}(\mathcal{X})$ assumed power stable, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

   (*i*) *The pair (A, B) is called controllable if the controllability gramian $P = P^* \in \mathcal{L}(\mathcal{X})$ defined by*

$$P \overset{\triangle}{=} \sum_{i=0}^{\infty} A^i BB^*(A^*)^i \tag{3.6}$$

   is coercive.

   (*ii*) *The pair (C, A) is called observable if the observability gramian $Q = Q^* \in \mathcal{L}(\mathcal{X})$ defined by*

$$Q \overset{\triangle}{=} \sum_{i=0}^{\infty} (A^*)^i C^* C A^i \tag{3.7}$$

   is coercive.

Definition 3.5 *(power stabilizable and power detectable system)*
Let $\Sigma(A, B, C, D)$ be an infinite-dimensional discrete-time system satisfying the set of equations (3.1) with the discrete-time semigroup $A \in \mathcal{L}(\mathcal{X})$ assumed power stable, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $D \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$.

   (*i*) *The pair (A, B) is called power stabilizable on $\mathcal{X}$ if there exists $F \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ such that the discrete-time perturbed semigroup $A + BF$ is power stable on $\mathcal{X}$.*

   (*ii*) *The pair (C, A) is called power detectable $\mathcal{X}$ if there exists $H \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ such that the discrete-time perturbed semigroup $A + HC$ is power stable on $\mathcal{X}$*

Definition 3.6 *(inner system)*
Let $\Sigma(A, B, C, D)$ be an infinite-dimensional discrete-time system satisfying the set of equations (3.1) and let $T \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}), \ell_2(0, \infty; \mathcal{Y}))$ be its input/output operator. The system $\Sigma(A, B, C, D)$ is called inner if $T^*T = I$.

Definition 3.7 *(discrete-time Popov triple)*
Let $\mathcal{X}$ and $\mathcal{U}$ be real separable Hilbert spaces. A triple of the form

$$\Sigma\left(A, B, M = \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} = M^*\right),$$

with $A \in \mathcal{L}(\mathcal{X})$ a discrete-time power stable semigroup on $\mathcal{X}$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $Q = Q^* \in \mathcal{L}(\mathcal{X})$, $L \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ and $R = R^* \in \mathcal{L}(\mathcal{U})$ is called a discrete Popov triple on $(\mathcal{X}, \mathcal{U})$.

We associate to the discrete Popov triple $\Sigma(A, B, M)$ the following objects

(i) The initial value problem

$$x(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0 \in \mathcal{X}, \tag{3.8}$$

where $A \in \mathcal{L}(\mathcal{X})$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$.

(ii) The quadratic index

$$J(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \langle \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \rangle_{\mathcal{X} \times \mathcal{U}}. \tag{3.9}$$

(iii) The class of admissible control sequences

$$U_{\text{adm}}^{x_0} \triangleq \left\{ u(\cdot) \in \ell_2(0, \infty; \mathcal{U}) | x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu(i) \in \ell_2(0, \infty; \mathcal{X}) \right\}. \tag{3.10}$$

Remark 3.8 *Since A has been assumed to be power stable on $\mathcal{X}$, then*

$$U_{\text{adm}}^{x_0} = \ell_2(0, \infty; \mathcal{U})$$

for every $x_0 \in \mathcal{X}$.

Let us define the following operators

$$\Phi \quad : \quad \mathcal{X} \longrightarrow \ell_2(0, \infty; \mathcal{X}), (\Phi x)(k) = A^k x, \tag{3.11}$$

$$\Psi \quad : \quad \ell_2(0, \infty; \mathcal{U}) \longrightarrow \ell_2(0, \infty; \mathcal{X}), (\Psi u(\cdot))(k) = \sum_{i=0}^{k-1} A^{k-i-1} Bu(i). \tag{3.12}$$

Then, for every $(x_0, u(\cdot)) \in \mathcal{X} \times \ell_2(0, \infty; \mathcal{U})$, the discrete-time initial value problem (3.8) has a unique solution $x \in \ell_2(0, \infty; \mathcal{X})$ given by

$$x(x_0, u(\cdot)) = \Phi x_0 + \Psi u(\cdot). \tag{3.13}$$

Definition 3.9 *(Discrete-time Riccati Equation. Stabilizing solution. Stabilizing feedback) Let $\Sigma(A, B, M)$ be a discrete Popov triple on $(\mathcal{X}, \mathcal{U})$.*

(i) *The Riccati equation associated with $\Sigma(A, B, M)$ is defined to be the equation in the unknown $X = X^* \in \mathcal{L}(\mathcal{X})$*

$$A^*XA - X - (A^*XB + L)(R + B^*XB)^{-1}(L^* + B^*XA) + Q = 0. \qquad (3.14)$$

(ii) *An operator $X = X^* \in \mathcal{L}(\mathcal{X})$ satisfying (3.14), for which $R + B^*XB$ has a bounded inverse and for which $A + BF$ is power stable on $\mathcal{X}$ with the operator $F \in \mathcal{L}(\mathcal{X}, \mathcal{U})$, defined by*

$$F \triangleq -(R + B^*XB)^{-1}(L^* + B^*XA), \qquad (3.15)$$

is called a stabilizing solution to the Discrete-Time Riccati Equation (3.14) and the feedback operator $F \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ defined by (3.15) is called a stabilizing feedback.

Remark 3.10 *If $R + B^*XB$ has a bounded inverse, then a straightforward computation shows that the Discrete-Time Riccati Equation (3.14) is equivalent with the following so-called Kalman-Szego-Popov-Yakubovitch (KSPY) system*

$$(KSPY) \begin{cases} V^*V & = R + B^*XB \\ W^*V & = L + A^*XB \\ W^*W & = Q + A^*XA - X \end{cases}. \qquad (3.16)$$

Furthermore, the discrete-time Riccati equation (3.14) has a self-adjoint stabilizing solution $X = X^* \geq 0$ if and only if the KSPY system (3.16) has a solution (V, W, X) with V boundedly invertible, $X = X^* \geq 0$ and $A + BV^{-1}W$ power stable. We shall use the KSPY system instead of the Riccati equation when we write down the solution to the discrete-time $\mathcal{H}^\infty$ control problem since it is a more convenient manner of expressing the stabilizing feedback operator. The reader is referred for more details to Chapter 3 of [46].

The discrete quadratic functional admits an equivalent representation (see for details [45])

$$J(x_0, u) = \left\langle \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix}, \begin{pmatrix} \mathcal{P}_0 & \mathcal{P} \\ \mathcal{P}^* & \mathcal{R} \end{pmatrix} \begin{pmatrix} x_0 \\ u(\cdot) \end{pmatrix} \right\rangle_{\mathcal{X} \times \ell_2(0,\infty;\mathcal{U})}, \qquad (3.17)$$

where

$$\mathcal{P}_0 \; : \; \mathcal{X} \longrightarrow \mathcal{X}, \quad \mathcal{P}_0 = \Phi^*Q\Phi, \qquad (3.18)$$

$$\mathcal{P} \; : \; \mathcal{X} \longrightarrow \ell_2(0,\infty;\mathcal{U}), \quad \mathcal{P} = (\Psi^*Q + L)\Phi^*, \qquad (3.19)$$

$$\mathcal{R} \; : \; \ell_2(0,\infty;\mathcal{U}) \longrightarrow \ell_2(0,\infty;\mathcal{U}), \quad \mathcal{R} = R + L^*\Psi + \Psi^*L + \Psi^*Q\Psi. \qquad (3.20)$$

The main discrete Popov theory result of this chapter represents the time-invariant counterpart of Theorem 1 from [45]

Theorem 3.11 *Let $\Sigma(A, B, M)$ be a Popov triple on $(\mathcal{X}, \mathcal{U})$. Then the following assertions are equivalent*

(i) *$\mathcal{R} \gg 0$.*

(ii) *The discrete-time Riccati equation (3.14) has a stabilizing solution X such that $R + B^*XB \gg 0$ and the solution admits the following representation*

$$X = \mathcal{P}_0 - \mathcal{P}\mathcal{R}^{-1}\mathcal{P}^*. \tag{3.21}$$

(iii) *The following Kalman-Szego-Popov-Yakubovitch system*

$$(KSPY)\begin{cases} V^*V & = R + B^*XB \\ W^*V & = L + A^*XB \\ W^*W & = Q + A^*XA - X \end{cases} \tag{3.22}$$

has a stabilizing solution (X, V, W) with $X = X^*$, V boundedly invertible and for which $A - BV^{-1}W$ is power stable. Furthermore, the stabilizing feedback operator is given by $F = -V^{-1}W$.

(iv) *The quadratic index (3.9) can be expressed as*

$$J(x_0, u(\cdot)) = \|Vu + Wx\|_2^2 + \langle x_0, Xx_0 \rangle_{\mathcal{X}} \tag{3.23}$$

and it attains its minimum for the stabilizing state-feedback law

$$u = Fx = -V^{-1}W$$

and this equals $\langle x_0, Xx_0 \rangle_{\mathcal{X}}$.

For the proof of the above result the reader is referred to [46].

3.2   The discrete-time LQ-optimal control problem

An immediate consequence of Theorem 3.11 is the result on the solution to the *optimal (linear quadratic) control problem* associated with the discrete Popov triple $\Sigma(A, B, M)$

Theorem 3.12 *Let $\Sigma\left(A, B, M = \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} = M^*\right)$ with A assumed power stable on $\mathcal{X}$, $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, $Q = Q^* \in \mathcal{L}(\mathcal{X})$, $L \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ and $R = R^* \in \mathcal{L}(\mathcal{U})$ be a discrete Popov triple on $(\mathcal{X}, \mathcal{U})$ and let $U_{\mathrm{adm}}^{x_0}$ be the class of admissible control sequences defined by (3.10) and*

$$\mathcal{R} : \ell_2(0, \infty; \mathcal{U}) \longrightarrow \ell_2(0, \infty; \mathcal{U}), \quad \mathcal{R} = R + L^*\Psi + \Psi^*L + \Psi^*Q\Psi$$

is strictly positive. Then

$$\min_{u(\cdot)\in U_{\text{adm}}^{x_0}} J_\Sigma(x_0, u(\cdot)) = \min_{u(\cdot)\in U_{\text{adm}}^{x_0}} \sum_{k=0}^\infty \langle \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \rangle_{\mathcal{X}\times\mathcal{U}} \qquad (3.24)$$

exists and equals $\langle X x_0, x_0 \rangle$ where X is a stabilizing solution to the discrete-time Riccati equation (3.14). The minimum is attained for the input satisfying

$$u(\cdot) = F x(x_0, u(\cdot)),$$

with F the power stabilizing feedback operator given by (3.15).

Remark 3.13 *Formula (3.21) is another strong motivation for the Popov approach to the digital control of infinite-dimensional systems we took in this paper. It enables one to compute the stabilizing solution to the Riccati equation (3.14), and in this way, the digital exponential stabilizing feedback operator we are looking for. In the application part of this thesis , where an example of a parabolic system which belongs to the Pritchard-Salamon class of systems is considered, we show that computing the inverse of a real axis valued operator enables us to write down the solution to the associated digital optimal control problem.*

3.3 The discrete-time $\mathcal{H}^\infty$ control problem

The discrete-time $\mathcal{H}^\infty$ control problem has been addressed and solved in the finite-dimensional case by several authors (Stoorvogel [77], Iglesias [49] and Ionescu and Weiss [51]). Its extension to the general case of infinite-dimensional discrete-time systems has been recently made by Ionescu and Halanay [46] where the discrete Popov theory has proved to be an adequate tool for obtaining a state-space solution to the problem. In this section we are concerned with the discrete-time $\mathcal{H}^\infty$ control problem in its most general form. Let us make this statement more clear. The basic model considered here is the infinite-dimensional discrete-time system

$$\Sigma \left(A, \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & 0 \end{pmatrix} \right)$$

satisfying the set of equations

$$\Sigma_G \begin{cases} (\sigma x)(k) = A x(k) + B_1 u_1(k) + B_2 u_2(k) \\ y_1(k) = C_1 x(k) + D_{11} u_1(k) + D_{12} u_2(k) \\ y_2(k) = C_2 x(k) + D_{21} u_1(k) \end{cases} \qquad (3.25)$$

for $k \in \mathbb{N}$, where $A \in \mathcal{L}(\mathcal{X})$ $B_1 \in \mathcal{L}(\mathcal{U}_1, \mathcal{X})$, $B_2 \in \mathcal{L}(\mathcal{U}_2, \mathcal{X})$, $C_1 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_1)$, $D_{11} \in \mathcal{L}(\mathcal{U}_1, \mathcal{Y}_1)$, $D_{12} \in \mathcal{L}(\mathcal{U}_2, \mathcal{Y}_1)$, $D_{21} \in \mathcal{L}(\mathcal{U}_1, \mathcal{Y}_2)$. We consider $\mathcal{U}_1, \mathcal{U}_2, \mathcal{Y}_1$ and $\mathcal{Y}_2$ to be the real

separable Hilbert spaces of the *disturbance input* $(u_1(k) \in \mathcal{U}_1)$, *control input* $(u_2(k) \in \mathcal{U}_2)$, *controlled output* $(y_1(k) \in \mathcal{Y}_1)$ and *measured output* $(y_2(k) \in \mathcal{Y}_2)$, respectively. Here $\mathcal{X}$ is the real separable Hilbert state space $(x(t) \in \mathcal{X})$. Unlike in [77] or [51], no structural constraints of the form

$$D_{12}^* D_{12} \gg 0, \tag{3.26}$$

$$D_{21} D_{21}^* \gg 0 \tag{3.27}$$

are necessarily made on the plant. This general framework permits us to consider the limiting case

$$D_{21} = 0,$$

which is referred as to the *singular discrete-time $\mathcal{H}^\infty$ control problem*. The importance of having available a solution to the singular discrete-time $\mathcal{H}^\infty$ control problem becomes evident in section 6.3 where we apply the discrete-time $\mathcal{H}^\infty$ control theory results of this section in order to obtain a solution to the digital $\mathcal{H}^\infty$ control problem. The only major assumption we make on the infinite-dimensional discrete-time system is

$$A \text{ is power stable on } \mathcal{X}.$$

This is a requirement specific to the discrete-time Popov theory and details on how it can be removed are discussed later on in this thesis.

Associated with the system (3.25) is the controller defined by the following discrete-time system

$$\Sigma_{\mathrm{K}} \left\{ \begin{array}{l} (\sigma x_{\mathrm{c}})(t) = A_{\mathrm{c}} x_{\mathrm{c}}(t) + B_{\mathrm{c}} y_2(t) \\ u_2(t) = C_{\mathrm{c}} x_{\mathrm{c}}(t) + D_{\mathrm{c}} y_2(t), \quad x_{\mathrm{c}}(0) \in \mathcal{K} \end{array} \right. , \tag{3.28}$$

where $\mathcal{K}$ is the real separable Hilbert controller state space $(x_{\mathrm{c}}(t) \in \mathcal{K})$ and A_{c}, B_{c}, C_{c} and D_{c} are operators bounded on appropriate subspaces.

The resultant closed-loop system is given by

$$\Sigma_{\mathrm{GK}} \left\{ \begin{array}{l} (\sigma x_{\mathrm{R}})(t) = A_{\mathrm{R}} x_{\mathrm{R}}(t) + B_{\mathrm{R}} u_1(t) \\ y_1(t) = C_{\mathrm{R}} x_{\mathrm{R}}(t) + D_{\mathrm{R}} u_1(t) \end{array} \right. , \tag{3.29}$$

where

$$\left\{ \begin{array}{l} x_{\mathrm{R}} = \left(\begin{array}{c} x \\ x_{\mathrm{c}} \end{array} \right) \\[2mm] A_{\mathrm{R}} = \left(\begin{array}{cc} A + B_2 D_{\mathrm{c}} C_2 & B_2 C_{\mathrm{c}} \\ B_{\mathrm{c}} C_2 & A_{\mathrm{c}} \end{array} \right) \\[3mm] B_{\mathrm{R}} = \left(\begin{array}{c} B_1 + B_2 D_{\mathrm{c}} D_{21} \\ B_{\mathrm{c}} D_{21} \end{array} \right) \\[3mm] C_{\mathrm{R}} = \left(\begin{array}{c} C_1 + D_{12} D_{\mathrm{c}} C_2 \\ D_{12} C_{\mathrm{c}} \end{array} \right) \\[2mm] D_{\mathrm{R}} = D_{11} + D_{21} D_{\mathrm{c}} D_{21} \end{array} \right. .$$

The augmented state space $\mathcal{X}_R = \mathcal{X} \oplus \mathcal{K}$ is a real, separable Hilbert space under the inner product

$$\langle x_R, x_R \rangle_{\mathcal{X}_R} = \langle x, x \rangle_{\mathcal{X}} + \langle x_c, x_c \rangle_{\mathcal{K}}.$$

The system (3.25) can be written in an input-output fashion as

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $T_{ij} \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}_i), \ell_2(0, \infty; \mathcal{Y}_j))$ are the input/output operators from u_i to y_j, $i, j = 1, 2$. If T_c is the input/output operator associated to the controller then the closed-loop input/output operator is expressed as the linear fractional transformation of the system (3.25) and the controller (3.28)

$$T_{y_1 u_1} \triangleq \mathcal{F}(T, T_c) = T_{11} + T_{12} T_c (I - T_{22} T_c)^{-1} T_{21}. \tag{3.30}$$

The discrete-time $\mathcal{H}^\infty$ control problem consists in finding a controller (3.28) for the system (3.25) such that

Stability: The closed-loop system (3.29) is *internally stable* i.e. A_R is power stable on $\mathcal{X}_R$.

Attenuation: The closed-loop input/output operator is a contraction operator

$$\|T_{y_1 u_1}\| < 1.$$

Remark 3.14 *The closed-loop interconnection of a system Σ_G defined (3.25) and the controller Σ_K defined by (3.28) will be denoted in this chapter by Σ_{GK}. Its input/output operator is the linear fractional transformation of the input/output operators of the system and the controller.*

Remark 3.15 *The discrete-time $\mathcal{H}^\infty$ control problem will be referred as the state-feedback discrete-time $\mathcal{H}^\infty$ control problem when*

$$C_2 = I, \tag{3.31}$$
$$D_{21} = 0 \tag{3.32}$$

and as the singular discrete-time $\mathcal{H}^\infty$ control problem when

$$D_{21} = 0. \tag{3.33}$$

The first basic results for the development of this section are given by the following

Theorem 3.16 *Let $T_{12} \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}_1), \ell_2(0, \infty; \mathcal{Y}_2))$ and $T_{21} \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}_2), \ell_2(0, \infty; \mathcal{Y}_1))$ be the input/output operator from u_1 to y_2 and from u_2 to y_1, respectively. Assume that*

$$T_{12}^* T_{12} \gg 0, \tag{3.34}$$

$$T_{21} T_{21}^* \gg 0. \tag{3.35}$$

If Σ_K defined by (3.28) is a solution to the discrete-time $\mathcal{H}^\infty$ control problem then the following Kalman-Szego-Popov-Yakubovitch systems

$$KSPY1: \quad \begin{cases} V^*JV & = R + B^*XB \\ W^*JV & = L + A^*XB \\ W^*JW & = Q + A^*XA - X \end{cases}, \tag{3.36}$$

$$KSPY2: \quad \begin{cases} \hat{V}^*\hat{J}\hat{V} = \hat{R} + CYC^* \\ \hat{W}^*\hat{J}\hat{V} = \hat{L} + AYC^* \\ \hat{W}^*\hat{J}\hat{W} = \hat{Q} + AYA^* - Y \end{cases}, \tag{3.37}$$

associated to the following Popov triples

$$\Sigma_1 \left(A, \left(\begin{array}{cc} B_1 & B_2 \end{array} \right), M = M^* \right), \tag{3.38}$$

$$\Sigma_1 \left(A^*, \left(\begin{array}{cc} C_1^* & C_2^* \end{array} \right), \hat{M} = \hat{M}^* \right), \tag{3.39}$$

with

$$M \triangleq \left(\begin{array}{cc} Q & L \\ L^* & R \end{array} \right) = \left(\begin{array}{cc} C_1^*C_{11} & \left(\begin{array}{cc} C_1^*D_{11} & C_1^*D_{12} \end{array} \right) \\ \left(\begin{array}{c} D_{11}^*C_1 \\ D_{12}^*C_1 \end{array} \right) & \left(\begin{array}{cc} D_{11}^*D_{11} - I_{u_1} & D_{11}^*D_{12} \\ D_{12}^*D_{11} & D_{12}^*D_{12} \end{array} \right) \end{array} \right), \tag{3.40}$$

$$\hat{M} \triangleq \left(\begin{array}{cc} \hat{Q} & \hat{L} \\ \hat{L}^* & \hat{R} \end{array} \right) = \left(\begin{array}{cc} B_1B_{11}^* & \left(\begin{array}{cc} B_1D_{11}^* & B_1D_{21}^* \end{array} \right) \\ \left(\begin{array}{c} D_{11}B_1^* \\ D_{21}B_1^* \end{array} \right) & \left(\begin{array}{cc} D_{11}D_{11}^* - I_{y_1} & D_{11}D_{21}^* \\ D_{21}D_{11}^* & D_{21}D_{21}^* \end{array} \right) \end{array} \right) \tag{3.41}$$

have stabilizing solutions (X, V, W) and $(Y, \hat{V}, \hat{W})$, respectively, where $J = \left(\begin{array}{cc} -I_{u_1} & \\ & I_{u_2} \end{array} \right)$ and $\hat{J} = \left(\begin{array}{cc} -I_{y_1} & \\ & I_{y_2} \end{array} \right)$.

Furthermore, $X \geq 0$, $Y \geq 0$, V and $\hat{V}$ have bounded inverses of the form

$$V = \left(\begin{array}{cc} V_{11} & 0 \\ V_{12} & V_{22} \end{array} \right), \quad W = \left(\begin{array}{c} W_1 \\ W_2 \end{array} \right), \tag{3.42}$$

$$\hat{V} = \left(\begin{array}{cc} \hat{V}_{11} & \hat{V}_{12} \\ 0 & \hat{V}_{22} \end{array} \right), \quad \hat{W} = \left(\begin{array}{c} \hat{W}_1 \\ \hat{W}_2 \end{array} \right) \tag{3.43}$$

and $A - BV^{-1}W$ and $A - \hat{W}\hat{V}^{-1}C$ are power stable on $\mathcal{X}$, for $B = \left(\begin{array}{cc} B_1 & B_2 \end{array} \right)$ and $C = \left(\begin{array}{c} C_1 \\ C_2 \end{array} \right)$.

The proof of Theorem 3.16 is essentially based on the following

Lemma 3.17 *(Lemma 3.1, pp. 97 of [46])*
Let $\Sigma(A, B, C, D)$ be an infinite-dimensional discrete-time system satisfying the set of equations (3.1). Let $T \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}), \ell_2(0, \infty; \mathcal{Y}))$ be its input/output operator. Consider the following discrete-time Popov triple

$$\Sigma\left(A, B, M = \begin{pmatrix} C^*C & C^*D \\ DC^* & D^*D \end{pmatrix} = M^*\right). \tag{3.44}$$

Then

$$\mathcal{R} = T^*T, \tag{3.45}$$

where $\mathcal{R}$ is the operator defined by (3.20) associated with the discrete-time Popov triple (3.44).

Proof Recall that the solution to the initial value problem

$$(\sigma x)(k) = Ax(k) + Bu(k), x(0) = x_0 \in \mathcal{X}$$

is given by

$$x(k) = A^k x_0 + \sum_{i=0}^{k-1} A^{k-i-1} Bu(i) = \Phi x_0 + \Psi u,$$

where Φ and Ψ are the operators define by (3.11) and (3.12), respectively. Then the output function satisfies

$$y(k) = CA^k x_0 + C\sum_{i=0}^{k-1} A^{k-i-1} Bu(i) + Du = C\Phi x_0 + C\Psi u + Du. \tag{3.46}$$

For zero initial value, i.e. $x_0 = 0$, the input/output operator is defined by

$$T : \ell_2(0, \infty; \mathcal{U}) \longrightarrow \ell_2(0, \infty; \mathcal{Y}), \quad y = Tu.$$

A simple inspection of (3.46) shows actually that T can be expressed in the following way

$$Tu = C\Psi u + Du. \tag{3.47}$$

A straightforward computation gives

$$T^*Tu = D^*Du^* + DC^*\Psi u + \Psi^*C^*Du + \Psi^*C^*C\Psi u, \tag{3.48}$$

where

$$\Psi^* : \ell_2(0, \infty; \mathcal{X}) \longrightarrow \ell_2(0, \infty; \mathcal{U}), \quad \Psi^*x)(k) = \sum_{i=k+1}^{\infty} B^*(A^*)^{i-k-1} x(i),$$

which is precisely the operator $\mathcal{R}$ defined by (3.20) associated with the discrete-time Popov triple (3.44).

∎

Proof(of Theorem 3.16)
Apply Lemma 3.17 to $T_{12}^{*}T_{12}$ and $T_{21}T_{21}^{*}$, respectively. It follows that they are equal with the $\mathcal{R}$-operators associated with the discrete-time Popov triples (3.38) and (3.39), respectively. Apply now Theorem (3.11). It follows that the corresponding Kalman-Szego-Popov-Yakubovitch systems have stabilizing solutions such that $A - BV^{-1}W$ and $A - \hat{W}\hat{V}^{-1}C$ are power stable on $\mathcal{X}$. For the proof of the fact that $\hat{V}$ and V have upper-block and lower-block triangular forms the reader is referred for technical details to Section 5 of [46].

∎

Remark 3.18 *The result given by Theorem 3.16 represents the set of necessary conditions for the existence of the solution to the discrete-time $\mathcal{H}^{\infty}$ control problem.*

In the sequel of this section shal be concerned with the set of sufficient conditions for the existence of the solution to the discrete-time $\mathcal{H}^{\infty}$ control problem. We shall consider two modified systems that we obtain by the following change of variables

$$\tilde{u}_1 \stackrel{\triangle}{=} V_{11}u_1 + W_1 x, \tag{3.49}$$

$$\tilde{y}_1 \stackrel{\triangle}{=} V_{21}u_1 + V_{22}u_2 + W_2 x, \tag{3.50}$$

with V_{11}, W_1, V_{21}, V_{22} and W_2 as in (3.42).

Remark 3.19 *The above change of variables is a natural step and it is suggested by the form (3.23) of the quadratic cost function in Theorem 3.11.*

Substitute

$$u_1 = V_{11}^{-1}\tilde{u}_1 - V_{11}^{-1}W_1 x$$

into (3.25) and replace y_1 by $\tilde{y}_1$ (notice that V_{11} has a bounded inverse since V has a bounded inverse). We obtain the following modified system associated with (3.25)

$$\Sigma_{\mathrm{O}} \begin{cases} (\sigma x)(k) = A_{\mathrm{O}}x(k) + B_{\mathrm{O}1}\tilde{u}_1(k) + B_{\mathrm{O}2}u_2(k) \\ \tilde{y}_1(k) = C_{\mathrm{O}1}x(k) + D_{\mathrm{O}11}\tilde{u}_1(k) + D_{\mathrm{O}12}u_2(k) \\ y_2(k) = C_{\mathrm{O}2}x(k) + D_{\mathrm{O}21}\tilde{u}_1(k) \end{cases}, \tag{3.51}$$

where

$$A_{\mathrm{O}} = A - B_1 V_{11}^{-1} W_1, \tag{3.52}$$

$$B_{\mathrm{O}1} = B_1 V_{11}^{-1}, \tag{3.53}$$

$$B_{\mathrm{O}2} = B_2, \tag{3.54}$$

$$C_{\text{O1}} = W_2 - V_{21}V_{11}^{-1}W_1, \tag{3.55}$$
$$C_{\text{O2}} = C_2 - D_{21}V_{11}^{-1}W_1, \tag{3.56}$$
$$D_{\text{O11}} = V_{21}V_{11}^{-1}, \tag{3.57}$$
$$D_{\text{O12}} = V_{22}, \tag{3.58}$$
$$D_{\text{O21}} = D_{21}V_{11}^{-1}. \tag{3.59}$$

The key of the above change of variables is the following two results

Proposition 3.20 *(Proposition 5.1 pp. 161 and Proposition 6.1, pp 166 of [46])*

(*i*) *The quadratic index corresponding to the discrete-time Popov triple (3.38) associated with (3.25) can be expressed as*

$$J(x_0, u(\cdot)) = -\|u_1\|_2^2 + \|y_1\|_2^2. \tag{3.60}$$

(*ii*) *Assume that*
$$T_{12}^* T_{12} \gg 0,$$

where $T_{12} \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}_1), \ell_2(0, \infty; \mathcal{Y}_2))$ is the input/output operator from u_1 to y_2. If the change of variables (3.49) and (3.50) is performed, then

$$\|u_1\|_2^2 + \|y_1\|_2^2 = \|\tilde{u}_1\|_2^2 + \|\tilde{y}_1\|_2^2. \tag{3.61}$$

Theorem 3.21 *(Theorem 2.3 pp. 150 of [46])*
Let $T_{12} \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}_1), \ell_2(0, \infty; \mathcal{Y}_2))$ and $T_{21} \in \mathcal{L}(\ell_2(0, \infty; \mathcal{U}_2), \ell_2(0, \infty; \mathcal{Y}_1))$ be the input/output operator from u_1 to y_2 and from u_2 to y_1, respectively. Assume that

$$T_{12}^* T_{12} \gg 0, \tag{3.62}$$
$$T_{21} T_{21}^* \gg 0. \tag{3.63}$$

If Σ_K defined by (3.28) is a solution to the discrete-time $\mathcal{H}^\infty$ control problem for the system Σ_G defined by (3.25), then the Kalman-Szego-Popov-Yakubovitch system (3.37) written for the modified system Σ_O define by (3.51) has a stabilizing solution denoted $(Y_O, \hat{V}_O, \hat{W}_O)$ with $Y_O \geq 0$.

Remark 3.22 *Equality (3.61) and the results of Proposition 3.20 suggests that Σ_K defined by (3.28) represents a solution to the discrete-time $\mathcal{H}^\infty$ control problem for the system Σ_G defined by (3.25) if and only if it is a solution for the modified system Σ_O define by (3.51),*

We have seen until now that the existence of a solution to the discrete-time $\mathcal{H}^\infty$ control problem implies the existence of stabilizing solutions to two Kalman-Szego-Popov-Yakubovitch systems defined by (3.36) and (3.37), respectively. This fact implies the existence of stabilizing solutions to the first Kalman-Szego-Popov-Yakubovitch system considered above, namely the one defined by (3.36) and to another Kalman-Szego-Popov-Yakubovitch system defined by (3.37) associated with the modified counterpart of the original infinite-dimensional discrete-time system. In [46] it is shown that those conditions are also sufficient as it is stated in the following

Theorem 3.23 *(Theorem 8.1, pp. 178 of [46])*
Consider the infinite-dimensional discrete-time system

$$\Sigma\left(A,\left(\begin{array}{cc} B_1 & B_2 \end{array}\right),\left(\begin{array}{c} C_1 \\ C_2 \end{array}\right),\left(\begin{array}{cc} D_{11} & D_{12} \\ D_{21} & 0 \end{array}\right)\right)$$

and assume that the Kalman-Szego-Popov-Yakubovitch system (3.36) written for Σ_G defined by (3.25) and the Kalman-Szego-Popov-Yakubovitch system (3.37) written for the outer system Σ_O defined by (3.51) have stabilizing solutions. Then there exists a solution to the discrete-time $\mathcal{H}^\infty$ control problem.

We shall outline in the sequel of this section the main steps on which the effective construction of the solution to the discrete-time $\mathcal{H}^\infty$ control problem is based upon. Consider again the change of variables (3.49) and (3.50) but this time in connection with the infinite-dimensional system which expresses the transition

$$(u_1,\tilde{y}_1) \mapsto (y_1,\tilde{u}_1)\,. \tag{3.64}$$

A straightforward computation shows that the above mentioned system is given by

$$\Sigma_I \left\{ \begin{array}{l} (\sigma x)(k) = A_I x(k) + B_{I1}\tilde{u}_1(k) + B_{I2}\tilde{y}_1(k) \\ y_1(k) = C_{I1}x(k) + D_{I11}\tilde{u}_1(k) + D_{I12}\tilde{y}_1(k) \\ \tilde{u}_1(k) = C_{I2}x(k) + D_{I21}\tilde{u}_1(k) \end{array} \right. , \tag{3.65}$$

where

$$\begin{array}{rcll} A_I & = & A - B_2 V_{22}^{-1} W_2, & (3.66) \\ B_{I1} & = & B_1 - B_2 V_{22}^{-1} V_{21}, & (3.67) \\ B_{I2} & = & B_2 V_{22}^{-1}, & (3.68) \\ C_{I1} & = & C_1 - D_{12} V_{22}^{-1} W_2, & (3.69) \\ C_{I2} & = & W_1, & (3.70) \\ D_{I11} & = & D_{11} - D_{12} V_{22}^{-1} V_{21}, & (3.71) \\ D_{I12} & = & D_{12} V_{22}^{-1}, & (3.72) \\ D_{I21} & = & V_{11}. & (3.73) \end{array}$$

The system Σ_I defined by (3.65) is of a special type, and its structure is the cornerstone of the set of sufficient conditions for the existence of the solution to the $\mathcal{H}^\infty$ control problem.

Proposition 3.24 (*Proposition 7.1 pp. 174 of [46]*)
Let us consider Σ_I defined by (3.65) and Σ_I defined by (3.65) and consider their closed-loop connection Σ_{IO}. Then Σ_{IO} and Σ_G are Lyapunov similar

$$\Sigma_{IO} \rightsquigarrow \Sigma_G$$

modulo a power stable uncontrollable part.

Proof The closed-loop connection of Σ_I and Σ_O is depicted in Figure 3.1
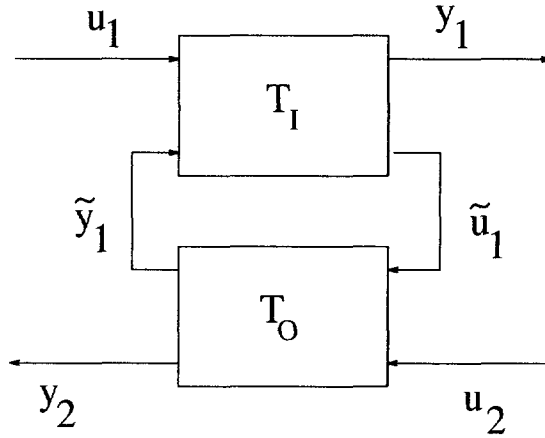
Figure 3.1: The connection of Σ_I and Σ_O

The state space realization of the resultant system from

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

is given by

$$A_{IO} = \begin{pmatrix} A + B_2 F_2 & -B_2 F_2 \\ -B_1 F_1 & A + B_1 F_2 \end{pmatrix}, \tag{3.74}$$

$$B_{IO} = \begin{pmatrix} B_1 & B_2 \\ B_1 & B_2 \end{pmatrix}, \tag{3.75}$$

$$C_{IO} = \begin{pmatrix} C_1 + D_{12} F_2 & -D_{12} F_2 \\ -D_{21} F_1 & C_2 + D_{21} F_1 \end{pmatrix}, \tag{3.76}$$

$$D_{IO} = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{3.77}$$

with $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ the power stabilizing feedback operator defined by 3.15) associated with the Kalman-Szego-Popov-Yakubovitch system defined by (3.36). Considered the following Lyapunov transformation

$$T_{\mathrm{IO}} : \mathcal{X} \oplus \mathcal{X} \longrightarrow \mathcal{X} \oplus \mathcal{X}, \quad T \begin{pmatrix} x \\ x \end{pmatrix} \triangleq \begin{pmatrix} I_{\mathcal{X}} & 0 \\ -I_{\mathcal{X}} & I_{\mathcal{X}} \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix}. \tag{3.78}$$

The state-space realization of the system Σ_{IO} transformed via (3.78) is given by

$$\tilde{A}_{\mathrm{IO}} = T A_{\mathrm{IO}} T^{-1} = \begin{pmatrix} A & -B_2 F_2 \\ 0 & A + BF \end{pmatrix}, \tag{3.79}$$

$$\tilde{B}_{\mathrm{IO}} = T B_{\mathrm{IO}} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}, \tag{3.80}$$

$$\tilde{C}_{\mathrm{IO}} = C_{\mathrm{IO}}^{-1} = \begin{pmatrix} C_1 & -D_{12} F_1 \\ C_2 & C_2 + D_{21} F_1 \end{pmatrix}, \tag{3.81}$$

$$\tilde{D}_{\mathrm{IO}} = D_{\mathrm{IO}}, \tag{3.82}$$

A simple inspection of (3.79), (3.80), (3.81) and (3.82) leads to the conclusion.

∎

A second property of the system (3.65) is given by the following

Proposition 3.25 *(Proposition 7.2 pp. 175 of [46])*
The system Σ_{I} defined by (3.65) is inner

Proof Follows by direct computation.

∎

For the inner system (3.65), Proposition 4 pp. 174 of [46] shows that the following controller

$$\tilde{\Sigma}_{\mathrm{K}} = \begin{cases} \sigma x_c = A_c x_c + B_c \tilde{u}_1 \\ \tilde{y}_1 = C_c x_c + D_c \tilde{u}_1 \end{cases} \tag{3.83}$$

achieves the attenuation property

$$T_{y_1 u_1} < 1$$

under the assumptions

Sc A_c is power stable on $\mathcal{K}$,

Ac $T_c < 1$ where T_c is the bounded input/output operator defined by (3.83).

Proposition 3.24 shows that the closed-loop connection of Σ_I defined by (3.65) and Σ_O defined by (3.51) (denoted Σ_{IO}) is Lyapunov similar to the original system Σ_G defined by (3.25) modulo a power stable uncontrollable part

$$\Sigma_{IO} \rightsquigarrow \Sigma_G.$$

Hence, if Σ_K achieves attenuation and power stability for Σ_O, the corresponding closed-loop system fulfils assumptions **Sc** and **Ac** from above. This implies furthermore that the closed-loop connection of Σ_K and Σ_O (denoted Σ_{OK}) connected in closed-loop with the inner system Σ_I (denoted Σ_{IOK}) is Lyapunov similar with the direct closed-loop connection of Σ_G and Σ_K (denoted Σ_{GK}) modulo a power stable controllable part

$$\Sigma_{IOK} \rightsquigarrow \Sigma_{GK}.$$

In Figure 3.2 we have depicted the two equivalent systems configuration in the sense given by the definition of Lyapunov similarity
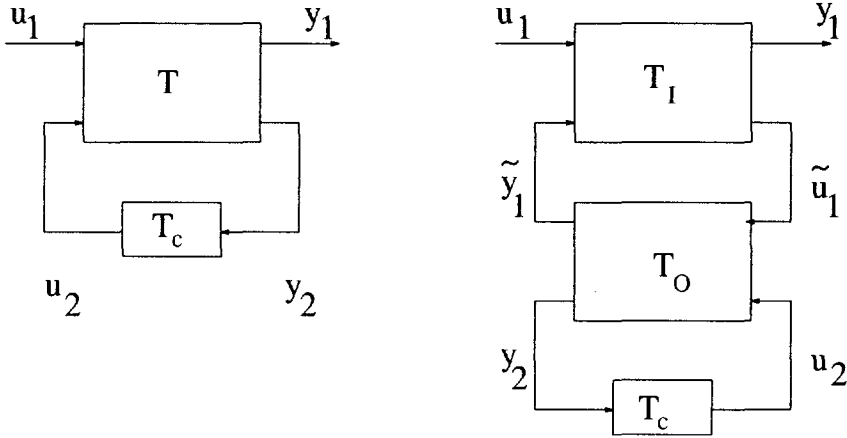
Figure 3.2: Two equivalent system configuration

At the end of this chapter a few final remarks

Remark 3.26 (*i*) *The inner systems have been considered in the $\mathcal{H}^\infty$ control theory together with the so called outer systems. Basically, a stable system is called outer is its inverse exists and is also stable. If T is the input/output operator of an arbitrary discrete-time system with power stable discrete semigroup, one can show that if*

$$T^*T \gg 0,$$

then there exists an inner system with input/output operator T_I and an outer system with input/output operator T_O such that

$$T = T_\mathrm{I} T_\mathrm{O}. \tag{3.84}$$

Relation (3.84) called the ineer-outer factorization.

(*ii*) *The system Σ_O defined by (3.51) has properties similar with an outer system in the sense that while the inner-outer factorization (3.84) involves the cascade connection of two systems, denoted by an abusive but suggestive notation,*

$$\Sigma_\mathrm{G} = \Sigma_\mathrm{I} \Sigma_\mathrm{O},$$

the "factorization"

$$\Sigma_\mathrm{G} \rightsquigarrow \Sigma_\mathrm{IO}$$

involves the closed-loop connection of the two systems. While the first one operates on the product of the input output oerators, the second one operates on their linear fractional transformation.

(*iii*) *If certain assumptions are made on the initial system Σ_G, the solution to the discrete-time $\mathcal{H}^\infty$ control problem can be effectively constructed in terms of the stabilizing solution to te KSPY system (3.37) associated with Σ_O. For details the reader is referred to Chapter 4, Section 8 of [46].*

Chapter 4

Discrete-time Hyland-Bernstein theory

4.1 Introduction

Standard engineering practice shows that reduced-order controllers are preferred over high-order ones. Various issues support this statement. It is sufficient to mention that a low-order controller would be much easier to implement and much easier to be debugged than a high-order one. If for finite-dimensional models the demand of a low-order control is actually a matter of choice, specific to the particular control problem one has to solve, in the case of infinite-dimensional models resulting from dynamics governed by partial differential equations and/or delay equations, this demand becomes a vital one. This is especially true since most of the "classical" finite-dimensional control issues have also been addressed for the infinite-dimensional case. This implies that the controller has the same dimension as the model (infinite-dimensional) and hence, will be unimplementable.

The demand of designing a finite-dimensional controller for an infinite-dimensional system can be fulfilled in several ways. One choice is to design a controller for a certain finite-dimensional approximation of the infinite-dimensional original plant. The most significant effort that has been made for obtaining finite-dimensional approximation schemes is represented by the work of Glover *et al.* [38] where the solution to the optimal Hankel-norm approximation of infinite-dimensional systems with error bounds is given. The characteristics of such an approximation method is that it guarantees optimality only in the limit, as the order of the reduced-order controller tends to infinity. As it was pointed out in Bernstein and Hyland [23] there is no reason to consider a controller for a particular approximation of the model as being optimal over the class of reduced-order controllers. Another approach is to perform a controller-order reduction. The main ideas of the "closed-loop" reduction strategies, clearly presented for the finite-dimensional case in Anderson and Liu [1] can be carried out for the infinite-dimensional case as well. The third approach is to directly design a fixed-order controller, optimal with respect to a given performance index. We refer to

the work Bernstein and Hyland [23] which is the most relevant one in the continuous-time case. The direct approach that enables the designer to avoid both model and controller reduction is to consider the controller as having a fixed, specified structure and a given fixed-order. Thus, the controller parameters are determined by solving a certain optimization problem subject to an associated performance index. In Bernstein and Hyland [23] the authors showed that the set of first-order necessary conditions for the quadratically optimal continuous-time fixed-order dynamic compensation of infinite-dimensional systems with bounded input and output operators can be transformed into a system of four coupled matrix equations, two modified Riccati equations and two modified Lyapunov equations. These equations are coupled by an idempotent operator which arises naturally from the optimality constraint to the design process. Exploiting the optimal projection they also gave a generalization of the classical LQG theory in the sense that the coupled Riccati equations preserve the form of the Riccati equations involved in the LQG synthesis, the separation principle breaks down and, only in the full-order case, the oblique projection operator becomes the identity operator and the four coupled Riccati and Lyapunov equations are "decoupled". Thus, the two modified Riccati equations reduce to the standard pair of Riccati equations and the two modified Lyapunov equations express the fact that the compensator was assumed to be minimal and strictly proper.

The developments for the discrete-time setting have been carried out in Bernstein et al. [22] where the plant was assumed to be finite-dimensional, the order of the controller strictly lower than the order of the plant. As in the continuous-time case the optimal low-order compensator state space matrices are expressed in terms of the solutions of four discrete-time coupled equations, i.e. two modified Riccati equations and two modified Lyapunov equations. A characteristic feature of the discrete-time setting is the possibility to have static feedthrough operator in both the estimator and the controller designs. The results based on the optimal projection equations for the discrete-time case were extended in Bernstein and Haddad [24] for systems with multiplicative white noise.

Starting with the work of Youla [93] the classical LQG control problem has been addressed from a frequency-domain (deterministic) point of view within the $\mathcal{H}^2$ approach. The complete solution to the continuous-time $\mathcal{H}^2$-optimal control problem was given by Doyle et al. [37]. In the discrete-time setting a representative work is that of Ionescu and Weiss [52] where the generalization to the time-varying case was derived. In this paper we are concerned with the fixed-order control of infinite-dimensional discrete-time systems. The starting point is the set of necessary conditions for optimal fixed-order dynamic compensation of infinite-dimensional continuous-time systems as developed in Bernstein and Hyland [23] and their discrete-time finite-dimensional corresponding results from Bernstein et al. [22]. One central problem we solve here is the extension of the main results from [22] in the case of discrete-time infinite-dimensional systems.

4.2   Definition and useful results

In this section we introduce the notation specific to this chapter along with basic definitions and results for use in later sections.

Definition 4.1 *Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. Then its image or range is defined as*

$$\mathrm{Im}(A) \overset{\text{notation}}{=} R(A) \overset{\triangle}{=} A\mathcal{X} = \{Ax | x \in \mathcal{X}\}.$$

If, in particular, $\mathrm{Im}(A)$ is a finite-dimensional space, then A is called finite rank operator and its rank is defined by

$$\mathrm{rank}(A) = \dim\left(\mathrm{Im}(A)\right).$$

Definition 4.2 *(projection operator)*

   (*i*) *An operator $\tau \in \mathcal{L}(\mathcal{X})$ is called a projection operator if $\tau^2 = \tau$. If in addition $\tau = \tau^*$ then it is called a orthogonal projection.*

  (*ii*) *An operator $\tau \in \mathcal{L}(\mathcal{X}, \mathbb{R}^n)$ is called an oblique projection operator if $\tau^2 = \tau$, but $\tau \neq \tau^*$.*

Definition 4.3 *(non-negative definite, positive definite and semi-simple matrices)*
Let $A \in \mathbb{R}^{n \times n}$.

   (*i*) *We shall say that A is non-negative definite if A is symmetric and $x^T A x \geq 0$, $\forall x \in \mathbb{R}^n$*

  (*ii*) *We shall say that A is positive definite if A is symmetric and $x^T A x > 0$, $\forall x \in \mathbb{R}^n$, $x \neq 0$*

 (*iii*) *We shall say that A is semisimple (see [72], pp. 13) or nondefective (see [64] pp. 375) if A has n linear independent eigenvectors, i.e. A has a diagonal Jordan form over the complex field. We shall say that A is real semisimple if A is semisimple with real eigenvalues. We shall say that A is positive semisimple if A is semisimple with positive eigenvalues. We shall say that A is negative semisimple if A is semisimple with negative eigenvalues.*

Remark 4.4 *Notice that $A \in \mathbb{R}^{n \times n}$ is real (respectively non-negative, positive) semisimple if and only if there exists invertible $S \in \mathbb{R}^{n \times n}$ such that SAS^{-1} is diagonal (respectively non-negative diagonal, positive diagonal).*

A similar definition can be given for the case of bounded operators

Definition 4.5 *(semisimple, real semisimple and non-negative semisimple operators)*
An operator $A \in \mathcal{L}(\mathcal{X})$ is called semisimple (real semisimple, non-negative semisimple) if there exists a boundedly invertible $T \in \mathcal{L}(\mathcal{X})$ such that $B \triangleq TAT^{-1}$ is normal (self-adjoint, non-negative definite) (see Definition A.3 for the definition of normal, self-adjoint and non-negative definite operators).

Definition 4.6 *(nuclear operator)*
Let $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be a compact operator and let $\{\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \geq \cdots\}$ be the at most countable sequence of eigenvalues of $(AA^*)^{\frac{1}{2}}$ with appropriate multiplicity. The operator A is called nuclear if

$$\|A\|_N \triangleq \sum_{k=0}^{\infty} \lambda_k < \infty. \tag{4.1}$$

We shall denote by

$$\mathcal{N}(\mathcal{X}, \mathcal{Y}) \triangleq \{A : \mathcal{X} \longrightarrow \mathcal{Y} \mid \|A\|_N < \infty\} \tag{4.2}$$

the set of nuclear operators. It is a Banach space equiped with the norm (4.1) (see Theorem 4.1 pp. 105 of [42]).

Proposition 4.7 *(norm properties of nuclear operators) If $A \in \mathcal{N}(\mathcal{X}, \mathcal{Y})$ and $B \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ then $BA \in \mathcal{N}(\mathcal{X}, \mathcal{Z})$ and*

$$\|BA\|_N \leq \|B\|_{\mathcal{Z}} \|A\|_N. \tag{4.3}$$

For the proof the reader is referred to the proof of Theorem 4.2 pp.106 [42].

Proposition 4.8 *(Lemma 2.6 [23])*

(*i*) *Let P and Q be two positive definite matrices. Then PQ is positive semisimple.*

(*ii*) *Let P and Q be two non-negative definite matrices. Then PQ is non-negative semisimple.*

4.3 The fixed-order ℓ^2-optimal discrete-time compensation problem

In this section we are concerned with giving the set of necessary conditions for the existence of a fixed finite-order stabilizing compensator for an infinite-dimensional discrete-time system.

4.3.1 The main control result

Let $\Sigma_G(A, B_1, B_2, C_1, C_2, D_{12}, D_{21})$ denote the linear infinite-dimensional discrete-time system

$$\Sigma_G \begin{cases} x(k+1) = Ax(k) + B_1 u_1(k) + B_2 u_2(k) \\ y_1(k) = C_1 x(k) + D_{12} u_2(k) \\ y_2(k) = C_2 x(k) + D_{21} u_1(k) \end{cases}, \tag{4.4}$$

where $A \in \mathcal{L}(\mathcal{X})$, $B_1 \in \mathcal{L}(\mathcal{U}_1, \mathcal{X})$, $B_2 \in \mathcal{L}(\mathcal{U}_2, \mathcal{X})$, $C_1 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_1)$, $C_2 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}_2)$, $D_{12} \in \mathcal{L}(\mathcal{U}_2, \mathcal{Y}_1)$ and $D_{21} \in \mathcal{L}(\mathcal{U}_1, \mathcal{Y}_2)$. Here $x(\cdot)$, $u_1(\cdot)$, $u_2(\cdot)$, $y_1(\cdot)$ and $y_2(\cdot)$ stand for the state, disturbance input, control input, controlled output and measured output. We assume that $\mathcal{X}$, $\mathcal{U}_1$, $\mathcal{U}_2$, $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are real separable Hilbert spaces.

Let $\Sigma_K^n(F, K, L)$ denote the n-dimensional controller

$$\Sigma_K^n \begin{cases} \xi(k+1) = F\xi(k) + Ky(k) \\ u(k) = L\xi(k) \end{cases}, \tag{4.5}$$

where n, the controller state-space dimension is fixed, $\xi \in \mathbb{R}^n$, and with $F \in \mathbb{R}^{n \times n}$, $K \in \mathcal{L}(\mathcal{Y}_2, \mathbb{R}^n)$ and $L \in \mathcal{L}(\mathbb{R}^n, \mathcal{U}_2)$.

The closed-loop system from u_1 to y_1, denoted Σ_{GK}^n, is described by

$$\Sigma_{GK}^n \begin{cases} x_R(k+1) = A_R x_R(k) + B_R u_1(k) \\ y_1(k) = C_R x_R(k) \end{cases}, \tag{4.6}$$

with $A_R \in \mathcal{L}(\mathcal{X}_R)$, $B_R \in \mathcal{L}(\mathcal{U}_1, \mathcal{X}_R)$ and $C_R \in \mathcal{L}(\mathcal{Y}_1, \mathcal{X}_R)$ are defined by

$$A_R \triangleq \begin{pmatrix} A & B_2 L \\ KC_2 & F \end{pmatrix}, \quad B_R \triangleq \begin{pmatrix} B_1 \\ K D_{21} \end{pmatrix}, \quad C_R \triangleq \begin{pmatrix} C_1 & D_{12} L \end{pmatrix}.$$

Definition 4.9 *The closed-loop linear infinite-dimensional discrete-time system from u_1 to y_1 is called internally stable if A_R is power-stable on $\mathcal{X}_R \triangleq \mathcal{X} \oplus \mathbb{R}^n$.*

The control problem we address is the following: find a controller $\Sigma_K^n(F, K, L)$ defined by (4.5) such that the following conditions are fulfilled

Stability The linear infinite-dimensional discrete-time system from u_1 to y_1 is internally
   stable.

Optimality The following quadratic cost function is minimized

$$\mathcal{J}(\Sigma_{\text{GK}}^n) \triangleq \begin{cases} \|G_{y_1 u_1}\|_{\mathcal{H}^2}^2 = \frac{1}{2\pi} \int_0^{2\pi} \text{trace} \left(G_{y_1 u_1}(e^{\jmath\omega})G_{y_1 u_1}^*(e^{\jmath\omega})\right) d\omega \\ \text{when the deterministic case is considered} \\ \lim_{k \longrightarrow \infty} \frac{1}{k}\mathbb{E}\sum_{i=0}^{k-1}\langle y_1(i), y_1(i)\rangle_{y_1} \\ \text{when the stochastic case is considered} \end{cases}, \qquad (4.7)$$

where $G_{y_1 u_1}(\cdot)$ is the transfer function of the closed-loop system.

We shall refer to

$$\min_{\Sigma_{\text{K}}^n \text{stabilizing}} \mathcal{J}(\Sigma_{\text{GK}}^n) \qquad (4.8)$$

with $\mathcal{J}(\Sigma_{\text{GK}}^n)$ given by (4.7) as *the finite-dimensional fixed-order $\mathcal{H}^2$-optimal control prob-
lem for the discrete-time infinite-dimensional system (4.4)* when considered in the deter-
ministic sense and to the *the finite-dimensional fixed-order LQG-optimal control problem
for the discrete-time infinite-dimensional system (4.4)* when considered in the stochastic
sense. When the stochastic or the deterministic context is irrelevant we shall refer to
it as the *finite-dimensional fixed-order ℓ^2-optimal control problem for the discrete-time
infinite-dimensional system (4.4)*.

   We shall focus on the following class of admissible controllers

$$\mathcal{A}_+ = \{(F, K, L)|A_{\text{R}} \text{ is power-stable}, (F, K, L) \text{ is minimal}\},$$

where (F, K, L) is a realization of Σ_{K}^n. This guarantees that the cost functions (4.7) is
finite, independent of the initial conditions and independent of the internal realization of
the controller. Before stating the main result of this section we do two things: firstly, we
give the structural constraints we make on the original data. Secondly, we report from
Bernstein and Hyland [23] a cornerstone result (Lemma 4.13) which enables us to prove
that the two modified Riccati equations are coupled with two modified Lyapunov equations
via an oblique projection operator having precisely the rank equal with the controller state
space dimension. Let us define the controllability and observability gramians of the closed
loop system

$$P_{\text{R}} = \sum_{k=0}^{\infty} A_{\text{R}}^k B_{\text{R}} B_{\text{R}}^* (A_{\text{R}}^*)^k, \qquad (4.9)$$

$$Q_{\text{R}} = \sum_{k=0}^{\infty} (A_{\text{R}}^*)^k C_{\text{R}}^* C_{\text{R}} A_{\text{R}}^k \qquad (4.10)$$

and partition them according to the state space decomposition $\mathcal{X}_{\text{R}} = \mathcal{X} \oplus \mathbb{R}^n$

$$P_{\text{R}} = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}, Q_{\text{R}} = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^* & Q_2 \end{pmatrix}.$$

The following proposition holds

Proposition 4.10 *Let Σ^n_{GK} defined by (4.6) be the internally stable closed-loop system obtained by interconnecting Σ_G with Σ^n_K. Assume that (F, K, L) is a minimal realization of Σ^n_K. Then $P_2 > 0$ and $Q_2 > 0$, where P_2 and Q_2 are the lower right corners of the partition of P_R and Q_R defined above.*

Proof Since Σ^n_K achieves internal stability it follows that the discrete-time semigroup $A \in \mathcal{L}(\mathcal{X}_R)$ is power stable on $\mathcal{X}_R$. It follows by Theorem 5.2 and Theorem 5.5 from Chapter 1 of [46] that P_R and Q_R are the unique positive definite solutions to the following Lyapunov equations

$$A_R P_R A_R^* - P_R + B_R B_R^* = 0, \tag{4.11}$$

$$A_R^* Q_R A_R - Q_R + C_R^* C_R = 0. \tag{4.12}$$

We shall prove only that $P_2 > 0$ since the proof of $Q_2 > 0$ is similar. The Lyapunov equation (4.11) can be written with respect to the partition of P_R as a system of four equations. The one corresponding to P_2 has the following form

$$F P_2 F^* - P_2 + K C_2 P_1 C_2^* K^* + F P_{12}^* C_2^* K^* + K C_2 P_{12} F^* + K D_{21} D_{21}^* K^* = 0. \tag{4.13}$$

If $P_2^\dagger$ denotes the Moore-Penrose or Drazin generalized inverse of P_2 a simple computation shows that (4.13) can be written in the following equivalent form

$$\left(F + K C_2 P_{12} P_2^\dagger\right) P_2 \left(F + K C_2 P_{12} P_2^\dagger\right)^* - P_2 +$$
$$+ \; K \left(C_2(P_1 - P_{12} P_2^\dagger P_{12}^*)C_2^* + D_{21} D_{21}^*\right) K^* = 0. \tag{4.14}$$

Denote by

$$W_2 \triangleq C_2(P_1 - P_{12} P_2^\dagger P_{12}^*)C_2^* + D_{21} D_{21}^*.$$

Notice that it is sufficient to prove that $W_2 \geq 0$. If this would be the case, then the minimality of the realization (F, K, L) of Σ^n_K implies the fact that (F, K) is controllable, which at its turn implies the controllability of

$$(F + K C_2 P_{12} P_2^\dagger, K W_2^{\frac{1}{2}}).$$

Applying a Lyapunov argument the conclusion $P_2 > 0$ follows immediately. In order to show that $W_2 \geq 0$, let us notice first that $P_R \geq 0$ implies $P_1 \geq 0$, $P_2 \geq 0$ and $P_{12} P_2^\dagger P_{12}^* \geq 0$. We infer that if $P_R \geq 0$ then also $P_R^\dagger \geq 0$, where $P_R^\dagger$ denoted its Drazin inverse. Exploiting the following identity written for $P_R^\dagger$

$$P_R^\dagger = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}^\dagger = \begin{pmatrix} \Delta & -\Delta P_{12} P_2^\dagger \\ -P_2^\dagger P_{12}^* \Delta & P_2^\dagger + P_2^\dagger P_{12}^* \Delta P_{12} P_2^\dagger \end{pmatrix},$$

with $\Delta \triangleq (P_{11} - P_{12} P_2^\dagger P_{12}^*)$ the conclusion $P_{11} - P_{12} P_2^\dagger P_{12}^* \geq 0$ follows. ∎

Since $P_2 > 0$ is a $n \times n$ matrix, then $P_2^\dagger = P_2^{-1}$ and the following corollary of Proposition 4.10 holds

Corollary 4.11

$$P \triangleq P_1 - P_{12}^* P_2^{-1} P_{12} \geq 0, \tag{4.15}$$

$$Q \triangleq Q_1 - Q_{12} Q_2^{-1} Q_{12}^* \geq 0. \tag{4.16}$$

The following assumptions are made on the plant

A1: $B_1 B_1^*$ is nuclear,

A2: $C_1^* C_1$ is nuclear,

A3: $D_{21} D_{21}^* + C_2 Q C_2^*$ is boundedly invertible,

A4: $D_{12}^* D_{12} + B_2^* P B_2$ is boundedly invertible.

Remark 4.12 *If we consider the finite-dimensional fixed-order ℓ_2-optimal control problem in the stochastic setting, then $B_1 B_1^*$ represents the covariance operator of the $\mathcal{U}_1$-valued random variable $u_1 \in L_2(\Omega, p, \mathcal{U}_1)$, where $(\Omega, p, \mathcal{U}_1)$ represents a complete probability space (see Chapter 5 in [34]). The covariance of u_1 defined by*

$$\mathrm{Cov}(u_1) \triangleq \mathbb{E}((u_1 - \mathbb{E}(u_1)) \circ (u_1 - \mathbb{E}(u_1))),$$

*where $x \circ y \in \mathcal{L}(\mathcal{U}_1)$ is defined for all $x, y \in \mathcal{U}_1$ by $(x \circ y)z \triangleq x\langle y, z \rangle$, $z \in \mathcal{U}_1$ is symmetric, positive definite and nuclear. It is therefore natural to assume [**A1**] and its "dual" counterpart [**A2**].*

Lemma 4.13 *Suppose $\hat{Q}, \hat{P} \in \mathcal{L}(\mathcal{X})$ have finite rank and are non-negative definite. Then $\hat{Q}\hat{P}$ is non-negative semisimple. Furthermore, if $\mathrm{rank}(\hat{Q}\hat{P}) = n$ then there exist $G, \Lambda \in \mathcal{L}(\mathcal{X}, \mathbb{R}^n)$ and a positive semi-simple $M \in \mathbb{R}^{n \times n}$ such that*

$$\hat{Q}\hat{P} = G^* M \Lambda, \tag{4.17}$$

$$\Lambda G^* = I_{n \times n}. \tag{4.18}$$

Proof Applying Theorem 2.1 pp. 240 from [41] one can conclude that there exists a finite-dimensional subspace $\mathcal{M} \subset \mathcal{X}$ such that

$$\hat{Q}\mathcal{M} \subset \mathcal{M}, \quad \hat{Q}\mathcal{M}^{\perp} = 0, \quad \hat{P}\mathcal{M} \subset \mathcal{M}, \quad \hat{P}\mathcal{M}^{\perp} = 0, \tag{4.19}$$

where $\mathcal{M}^\perp$, the orthogonal complement of $\mathcal{M}$ is closed in $\mathcal{X}$. Hence, there exists an orthonormal basis for $\mathcal{X}$ with respect to which $\hat{Q}$ and $\hat{P}$ have the infinite-matrix representation

$$\hat{Q} = \begin{pmatrix} \hat{Q}_1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{P} = \begin{pmatrix} \hat{P}_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.20}$$

where $\hat{Q}_1 \in \mathbb{R}^{r \times r}$, $\hat{P}_1 \in \mathbb{R}^{r \times r}$ with $r \overset{\triangle}{=} \dim \mathcal{M}$. By Proposition 4.8 there exist invertible $\Psi \in \mathbb{R}^{r \times r}$ such that $\tilde{\Lambda} = \Psi^{-1} \hat{Q}_1 \hat{P}_1 \Psi$ is non-negative semisimple. Then we have

$$\hat{Q}\hat{P} = \begin{pmatrix} \Psi & 0 \\ 0 & I_\infty \end{pmatrix} \begin{pmatrix} \tilde{\Lambda} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & I_\infty \end{pmatrix}, \tag{4.21}$$

which shows that $\hat{Q}\hat{P}$ is non-negative semisimple. If, furthermore, $\operatorname{rank}(\hat{Q}_1 \hat{P}_1) = n$ then it is clear that Ψ can be chosen such that $\tilde{\Lambda} = \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix}$ where $\Lambda \in \mathbb{R}^{n \times n}$ is positive diagonal. Hence, we have

$$\hat{Q}\hat{P} = \begin{pmatrix} \Psi & 0 \\ 0 & I_\infty \end{pmatrix} \left(\begin{pmatrix} I_{n \times n} \\ 0 \end{pmatrix} \right) \Lambda \left(\begin{pmatrix} I_{n \times n} & 0 \end{pmatrix} \ 0 \right) \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & I_\infty \end{pmatrix}, \tag{4.22}$$

which shows that (4.17) and (4.18) are satisfied with

$$G = \left(\begin{pmatrix} S^T & 0 \end{pmatrix} \ 0 \right) \begin{pmatrix} \Psi^T & 0 \\ 0 & I_\infty \end{pmatrix}, \tag{4.23}$$

$$M = S^{-1} \Lambda S, \tag{4.24}$$

$$\Gamma = \left(\begin{pmatrix} S^{-1} & 0 \end{pmatrix} \ 0 \right) \begin{pmatrix} \Psi^{-1} & 0 \\ 0 & I_\infty \end{pmatrix}, \tag{4.25}$$

for all invertible $S \in \mathbb{R}^{n \times n}$.

$\blacksquare$

The main result of this chapter is represented by the following

Theorem 4.14 *Let $\Sigma_K^n(F, K, L)$ be a solution to the fixed finite-order $\mathcal{H}^2$-optimal discrete-time control problem for $\Sigma_G(A, B_1, B_2, C_1, C_2, D_{12}, D_{21})$. Then there exists $Q \geq 0$, $P \geq 0$ and $\hat{Q}$ and $\hat{P}$ such that F, K and L are given by*

$$\left\{ \begin{array}{l} F = \Lambda F^\circ G^* \\ K = \Lambda K^\circ \\ L = L^\circ G^* \end{array} \right. , \quad \left\{ \begin{array}{l} L^\circ = R_{2E}^{-1} P_E \\ K^\circ = Q_E V_{2E}^{-1} \\ F^\circ = A - K^\circ C_2 - B_2 L^\circ \end{array} \right. , \quad \left\{ \begin{array}{l} Q_E = AQC_2^* \\ P_E = B_2^* P A \\ V_{2E} = C_2 Q C_2^* + D_{21} D_{21}^* \\ R_{2E} = B_2^* P B_2 + D_{12}^* D_{12} \end{array} \right. ,$$

for some (G, M, Λ)-factorization of $\hat{Q}\hat{P}$, and such that if $\tau = G^*\Lambda$, the following conditions are satisfied:

$$\text{rank}(\hat{Q}) \quad = \quad \text{rank}(\hat{P}) = \text{rank}(\hat{Q}\hat{P}) = n, \tag{4.26}$$

$$Q = AQA^* + V_1 - Q_E V_{2E}^{-1} Q_E^* + \tau_\perp \left((A - B_2 L^\circ)\hat{Q}(A - B_2 L^\circ)^* + Q_E V_{2E}^{-1} Q_E^* \right) \tau_\perp^*, \tag{4.27}$$

$$P = A^* P A + R_1 - P_E^* R_{2E}^{-1} P_E + \tau_\perp^* \left((A - K^\circ C_2)^* \hat{P}(A - K^\circ C_2) + P_E^* R_{2E}^{-1} P_E \right) \tau_\perp, \tag{4.28}$$

$$\hat{Q} = \tau \left((A - B_2 L^\circ)\hat{Q}(A - B_2 L^\circ)^* + Q_E V_{2E}^{-1} Q_E^* \right) \tau^*, \tag{4.29}$$

$$\hat{P} = \tau^* \left((A - K^\circ C_2)^* \hat{P}(A - K^\circ C_2) + P_E^* R_{2E}^{-1} P_E \right) \tau, \tag{4.30}$$

where $\tau_\perp = I_\chi - \tau$.

4.3.2   A few remarks on Theorem 4.14

Before proceeding to prove the main result, Theorem 4.14, let us discuss first a few additional features of it.

Proposition 4.15 *The following hold*

(i) *Let $\Sigma_K^n(F, K, L) \in \mathcal{A}_+$. If $S \in \mathbb{R}^{n \times n}$ is invertible, then $\Sigma_K^n(SFS^{-1}, SK, LS^{-1}) \in \mathcal{A}_+$ and*

$$\mathcal{J}(\Sigma_{GK}^n(F, K, L)) = \mathcal{J}\left(\Sigma_{GK}^n(SFS^{-1}, SK, LS^{-1})\right). \tag{4.31}$$

(ii) *If $S \in \mathbb{R}^{n \times n}$ is invertible then*

$$\bar{G} \quad \triangleq \quad S^{-T} G, \tag{4.32}$$

$$\bar{\Lambda} \quad \triangleq \quad S\Lambda, \tag{4.33}$$

$$\bar{M} \quad \triangleq \quad SMS^{-1}, \tag{4.34}$$

satisfy

$$\hat{Q}\hat{P} \quad = \quad \bar{G}^* \bar{M} \bar{\Lambda}, \tag{4.35}$$

$$\bar{\Lambda}\bar{G}^* \quad = \quad I_{n \times n}. \tag{4.36}$$

For the proof the reader is referred to [23].

Remark 4.16 *The following facts deserve to be highlighted*

(i) *In view of Proposition 4.15, one can notice that Theorem 4.14 applies also to*

$$\Sigma_K^n(SFS^{-1}, SK, LS^{-1}).$$

Indeed, this is the case since the (G, M, Λ)-factorization of $\hat{Q}\hat{P}$ is not unique. Moreover, all these factorizations are related by a nonsingular transformation in $\mathbb{R}^{n \times n}$.

(ii) *Notice that τ is invariant over the class of factorizations.*

(iii) *In general, τ is an oblique projection operator and not an orthogonal projection operator since there is no requirement that τ should be self-adjoint.*

Let $\hat{P}, \hat{Q} \in \mathcal{L}(\mathcal{X})$ have finite rank, say n. The next result shows that $\hat{Q}\hat{P}$ and τ can be simultaneously diagonalized by a nonsingular transformation

Proposition 4.17 *(Hyland and Bernstein [23])*
There exists invertible $\Phi \in \mathcal{L}(\mathcal{X})$ such that

$$\hat{Q} = \Phi^{-1} \begin{pmatrix} \Lambda_{\hat{Q}} & 0 \\ 0 & 0 \end{pmatrix} \Phi^{-*}, \hat{P} = \Phi^* \begin{pmatrix} \Lambda_{\hat{P}} & 0 \\ 0 & 0 \end{pmatrix} \Phi,$$

$$\hat{Q}\hat{P} = \Phi^{-1} \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} \Phi, \tau = \Phi^{-1} \begin{pmatrix} I_{n \times n} & 0 \\ 0 & 0 \end{pmatrix} \Phi,$$

where $\Lambda_{\hat{Q}}, \Lambda_{\hat{P}} \in \mathbb{R}^{n \times n}$ are positive diagonal and $\Lambda_{\hat{Q}} \Lambda_{\hat{P}} = \Lambda$ where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ are the eigenvalues of M.

For the proof of the above result the reader is referred to [23]. An immediate consequence is given by the following

Corollary 4.18 *The following equalities hold*

$$\hat{Q} = \tau \hat{Q}, \tag{4.37}$$
$$\hat{P} = \hat{P}\tau. \tag{4.38}$$

Remark 4.19 *The following items deserve highlighting*

(i) *To obtain a geometrical interpretation of the optimal projection equations we introduce as in [23] the so-called quasi full-order state estimator*

$$\hat{x}(\cdot) \triangleq G^* x_c(\cdot) \in \mathcal{X} \tag{4.39}$$

so that

$$\tau \hat{x}(\cdot) \;=\; \hat{x}(\cdot), \tag{4.40}$$

$$x_c(\cdot) \;=\; \Lambda \hat{x}(\cdot). \tag{4.41}$$

The state equation with the estimation based on the measured output can be written in the following form

$$x(k+1) \;=\; Ax(k) + B_1(D_{12}^* D_{12})^{-1} B_1^* P\tau \hat{x}(k) + B_2 u_2(k), \tag{4.42}$$

$$\hat{x}(k+1) \;=\; \tau(A + B_1(D_{12}^* D_{12})^{-1} B_1^* P + QC_2^*(D_{21} D_{21}^*)^{-1} C_2)\tau \hat{x}(k) + $$
$$+ \; \tau QC_2(D_{21} D_{21}^*)^{-1}(C_2 x(k) + D_{21} u_1(k)). \tag{4.43}$$

From (4.42) and (4.43) one can easily see that the geometric structure of the quasi full-order compensator (4.43) is entirely determined by the projection operator τ. Particularly, if the control input $u_2(k)$ determined by (4.43) are contained in $\mathrm{Im}(\tau)$ and sensor inputs $\tau QC_2(D_{21} D_{21}^*)^{-1} y_2(k)$ are annihilated unless they are contained in $\mathrm{Im}(\tau^*)$. Consequently,

- *$\mathrm{Im}(\tau)$ is the compensator control space,*

- *$\mathrm{Im}(\tau^*)$ is the compensator observation space.*

Since τ is not necessarily orthogonal, those spaces might be different in general.

At the end of this section we give an explicit formula for the projection operator τ in terms of $\hat{P}$ and $\hat{Q}$. Since $\hat{Q}\hat{P}$ is a finite-rank operator, its *Drazin inverse* denoted $(\hat{Q}\hat{P})^D$ exists (see Theorem 6, pp. 108 from [59]) and since

$$(\hat{Q}\hat{P})^2 = G^* M^2 \Lambda,$$

it follows that

$$\mathrm{rank}\left((\hat{Q}\hat{P})^2\right) = \mathrm{rank}(\hat{Q}\hat{P})$$

and the so-called *index* of $(\hat{Q}\hat{P})$ is equal to 1 (see [59, 28]) . It follows that the Drazin inverse is the *group inverse*, denoted by $(\hat{Q}\hat{P})^{\#}$ (see [28], pp. 124). Then the following holds

Proposition 4.20 *(Hyland and Bernstein [23])*
The optimal projection τ is given by

$$\tau = \hat{Q}\hat{P}(\hat{Q}\hat{P})^{\#}. \tag{4.44}$$

4.3.3 Proof of Theorem 4.14

In order to prove the main result we shall prove some lemmas that would enable us to calculate the Fréchet derivatives of the cost function $\mathcal{J}(\Sigma_G, \Sigma_K^n)$ with respect to the compensator parameters F, K and L. Thus, the set of first-order necessary conditions for the existence of a solution to the discrete-time compensation problem are obtained by setting to zero these derivatives. We proceed by generalizing a result regarding the $\mathcal{H}^2$-norm of finite-dimensional discrete-time systems to infinite-dimensional discrete-time systems.

Lemma 4.21 *Let $\Sigma(A, B, C)$*

$$x(k+1) \;=\; Ax(k) + Bu(k), \tag{4.45}$$
$$y(k) \;=\; Cx(k) \tag{4.46}$$

be an infinite-dimensional discrete-time system with the discrete-time semigroup $A \in \mathcal{L}(\mathcal{X})$ assumed power-stable, $B \in \mathcal{L}(R^m, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, R^p)$. Then if G denotes the transfer function of $\Sigma(A, B, C)$ the following norm relation holds

$$\|G\|_{\mathcal{H}^2}^2 = \text{trace} \, (CPC^*) = \text{trace} \, (B^*QB), \tag{4.47}$$

with P and Q the controllability and observability gramians of the system $\Sigma(A, B, C)$. Furthermore, if

$$V \;\triangleq\; BB^*, \tag{4.48}$$
$$R \;\triangleq\; C^*C, \tag{4.49}$$

then

$$\|G\|_{\mathcal{H}^2}^2 = \text{trace} \, (RP) = \text{trace} \, (QV). \tag{4.50}$$

Proof Let $\{h(k)\}_{k \geq 0} \in \mathbb{R}^{m \times p}$, $h(k) = CA^k B$ be the impulse response of the system $\Sigma(A, B, C)$. By Parseval's theorem (see Openheim [67], pp. 326) we get

$$
\begin{aligned}
\|G\|_{\mathcal{H}^2}^2 \;&=\; \frac{1}{2\pi} \int_0^{2\pi} \text{trace} \, G(e^{j\theta}) G^*(e^{j\theta}) d\theta = \sum_{k=1}^{\infty} \text{trace} \, (h(k)h(k)^*) = \\
&=\; \sum_{k=1}^{\infty} \text{trace} \, (CA^k BB^*(A^*)^k C^*) = \text{trace} \, \big(\sum_{k=1}^{\infty} CA^k BB^*(A^*)^k C^* \big) = \\
&=\; \text{trace} \, \big(C[\sum_{k=1}^{\infty} A^k BB^*(A^*)^k]C^* \big) = \text{trace} \, (CPC^*). \tag{4.51}
\end{aligned}
$$

The second part of (4.47) follows from the fact that

$$\sum_{k=1}^{\infty} \text{trace } (h(k)h(k)^*) = \sum_{k=1}^{\infty} \text{trace } (h(k)^*h(k)).$$

Since $CPC^* \in \mathbb{R}^{p \times p}$, it follows that $\text{rank}(C^*CP) = p$. It follows without difficulty that

$$\text{trace}(CPC^*) = \text{trace}(C^*CP) = \text{trace}(RP).$$

The proof is complete if we show that $\text{trace}(QV)$ is actually finite. This holds since $Q \in \mathcal{L}(\mathcal{X})$ and $V \in \mathcal{N}(\mathcal{X})$ and by Lemma 4.7

$$\|RP\|_{\mathbb{N}} \leq \|P\|_{\mathcal{L}(\mathcal{X})}\|R\|_{\mathbb{N}} < \infty.$$

■

Let us introduce the following notation

$$V_{\text{R}} \overset{\triangle}{=} B_{\text{R}}B_{\text{R}}^* = \begin{pmatrix} B_1 B_1^* & 0 \\ 0 & K D_{21} D_{21}^* K^* \end{pmatrix} = \begin{pmatrix} V_1 & 0 \\ 0 & K V_2 K^* \end{pmatrix}, \tag{4.52}$$

$$R_{\text{R}} \overset{\triangle}{=} C_{\text{R}}^* C_{\text{R}} = \begin{pmatrix} C_1^* C_1 & 0 \\ 0 & L^* D_{12}^* D_{12} L \end{pmatrix} = \begin{pmatrix} R_1 & 0 \\ 0 & L^* R_2 L \end{pmatrix}. \tag{4.53}$$

Remark 4.22 *Since $B_1 B_1^*$ and $C_1^* C_1$ have been assumed to be nuclear operators it follows that $V_1 \in \mathcal{N}(\mathcal{X})$ and $R_1 \in \mathcal{N}(\mathcal{X})$ which implies that $V_{\text{R}} \in \mathcal{N}(\mathcal{X}_{\text{R}})$ and $R_{\text{R}} \in \mathcal{N}(\mathcal{X}_{\text{R}})$.*

Remark 4.23 *In the stochastic case a similar result to the one proved in Lemma 4.47 can be also derived. Notice first that the stochastic LQG cost function can be expressed as*

$$J(\Sigma_{\text{GK}}^n) = \lim_{k \to \infty} \mathbb{E}\langle R_{\text{R}} x_{\text{R}}(k), x_{\text{R}}(k)\rangle. \tag{4.54}$$

In [6] it was proved that if

$$Q_{\text{R}}(k) \overset{\triangle}{=} \mathbb{E}\left((x_{\text{R}}(k) - \mathbb{E}x_{\text{R}}(k))(x_{\text{R}}(k) - \mathbb{E}x(k))^*\right), \tag{4.55}$$

then there exists a operator Q_{R} such that

$$\lim_{k \to \infty} Q_{\text{R}}(k) = Q_{\text{R}} \tag{4.56}$$

and furthermore,

$$J(\Sigma_{\text{GK}}^n) = \text{trace } (Q_{\text{R}} R_{\text{R}}). \tag{4.57}$$

Let us give some notation useful for computing the Fréchet derivatives of the cost function with respect to F, K and L. For any $F' \in \mathbb{R}^{n \times n}$, $K' \in \mathcal{L}(\mathbb{R}^n, \mathcal{U}_2)$ and $L' \in \mathcal{L}(\mathcal{Y}_2, \mathbb{R}^n)$ define

$$\left\{ \begin{array}{l} \delta_F = F' - F \\ \delta_K = K' - K \\ \delta_L = L' - L \end{array} \right. , \tag{4.58}$$

and $\|(\delta_F, \delta_K, \delta_L)\| = \|\delta_F\| + \|\delta_K\| + \|\delta_L\|$. Let A'_R, V'_R and R'_R denote A_R, V_R and R_R with the triple of operators (F, K, L) replaced by (F', K', L')

$$\left\{ \begin{array}{l} A'_R = \left(\begin{array}{cc} A & B_2 L' \\ K' C_2 & F' \end{array} \right) \\[2ex] V'_R = \left(\begin{array}{cc} V_1 & 0 \\ 0 & K' V_2 (K')^* \end{array} \right) \\[2ex] R'_R = \left(\begin{array}{cc} R_1 & 0 \\ 0 & L' R_2 (L')^* \end{array} \right) \end{array} \right.$$

and define

$$\left\{ \begin{array}{l} \delta_{A_R} = A'_R - A_R = \left(\begin{array}{cc} 0 & B_2 \delta_L \\ \delta_K C_2 & \delta_F \end{array} \right) \\[2ex] \delta_{V_R} = V'_R - V_R = \left(\begin{array}{cc} 0 & 0 \\ 0 & \delta_K V_2 \delta_K^* + \delta_K V_2 K^* + K V_2 \delta_K^* \end{array} \right) \\[2ex] \delta_{R_R} = R'_R - R_R = \left(\begin{array}{cc} 0 & 0 \\ 0 & \delta_L^* R_2 \delta_L + \delta_L^* R_2 L + L^* R_2 \delta_L \end{array} \right) \end{array} \right. .$$

The following Lemma represents a collection of results from [23] that enables one to compute the Fréchet derivatives of the cost function with respect to the compensator parameters.

Lemma 4.24 *Let* $P_R = \left(\begin{array}{cc} P_1 & P_{12} \\ P_{12}^* & P_2 \end{array} \right)$, $Q_R = \left(\begin{array}{cc} Q_1 & Q_{12} \\ Q_{12}^* & Q_2 \end{array} \right)$ *be the matrix decomposition of* P_R *and* Q_R *over* X_R. *Let also* Q'_R *and* P'_R *denote* Q_R *and* P_R *with the triple of operators* (F, K, L) *replaced by* (F', K', L'). *Define* $\delta_{Q_R} = Q'_R - Q_R$, $\delta_P = P'_R - P_R$ *and introduce*

$$\mathcal{B} = \{\Sigma_K(F, K, L) | A_R \text{ is power-stable }\}.$$

Then

(i) $\mathcal{B}$ *is an open set and furthermore there exists a positive constant,* γ, *and an open neighborhood of* (F, K, L), $N \subset \mathcal{B}$, *such that*

$$\|\delta_{Q_R}\| \leq \gamma \|(\delta_F, \delta_K, \delta_L)\|, \tag{4.59}$$
$$\|\delta_{P_R}\| \leq \gamma \|(\delta_F, \delta_K, \delta_L)\|. \tag{4.60}$$

for all $\Sigma_{K'}(F', K', L') \in N$.

(ii) Q_2 and P_2 are positive definite.

(iii) $\mathcal{A}_+$ is an open set.

(iv) Let

$$\delta_J(\delta_F, \delta_K, \delta_L) \triangleq \mathcal{J}(\Sigma_G, \Sigma_K^n) - \mathcal{J}(\Sigma_G, \Sigma_{K'}^n)$$

for any $\Sigma_{K'}(F', K', L') \in \mathcal{B}$. Then

$$\delta_J(\delta_F, \delta_K, \delta_L) = \mathcal{F}(\delta_J(\delta_F, \delta_K, \delta_L)) + \theta(\|(\delta_J(\delta_F, \delta_K, \delta_L)\|),$$

where

$$\mathcal{F} : \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n} \longrightarrow \mathbb{R}, \quad \mathcal{F} = \mathcal{F}(\delta_J(\delta_F, \delta_K, \delta_L)) \qquad (4.61)$$

is a bounded linear functional defined by

$$\mathcal{F}(\delta_J(\delta_F, \delta_K, \delta_L)) \;\triangleq\; 2\mathrm{trace}[P_{12}^* Q_{12} + P_2 Q_2 \delta_F + (V_2 K^* P_2 + C(Q_{12} P_2 + Q_1 P_{12}))\delta_K +$$
$$+ \; Q_2 L^* R_2 + (Q_2 P_{12}^* + Q_1 P_{12})\delta_L], \qquad (4.62)$$

such that

$$\lim_{(\delta_F, \delta_K, \delta_L) \longrightarrow 0} \|(\delta_F, \delta_K, \delta_L)\|^{-1} \theta(\|(\delta_J(\delta_F, \delta_K, \delta_L)\|) = 0. \qquad (4.63)$$

For details about the proof the reader is referred to [23]

Let $(F, K, L) \in \mathcal{A}_+$ be as it is given by the main result and consider $\delta_J(\delta_F, \delta_K, \delta_L)$ for an arbitrary $(F', K', L') \in \mathcal{A}_+$. Since $\mathcal{F}$ defined by (4.61) is a bounded linear functional and $\mathcal{A}_+$ is an open set, the limit relation (4.63) shows that $\mathcal{F}$ is precisely the Fréchet derivative of the cost function with respect to (F, K, L). Since $\mathcal{A}_+$ is open, then the optimality implies

$$\mathcal{F}(\delta_F, \delta_K, \delta_L) = 0, \quad \forall (F, K, L) \in \mathcal{A}_+. \qquad (4.64)$$

Clearly, the first-order conditions for the minimality of the cost criterion are equivalent with the following set of relations

$$\begin{aligned}
\delta_J(\delta_F) &= \delta_J(\delta_F, 0, 0) = 0, \\
\delta_J(\delta_K) &= \delta_J(0, \delta_K, 0) = 0, \\
\delta_J(\delta_L) &= \delta_J(0, 0, \delta_L) = 0.
\end{aligned}$$

A straightforward computation gives

$$\begin{aligned}
\delta_J(\delta_F) &= \mathrm{trace}\,(P_R A_R Q_R \delta_{A_R}^* + P_R \delta_{A_R} Q_R A_R^* + P_R \delta_{A_R} Q_R \delta_{A_R}^*) + \theta(\|\delta_F\|) & (4.65) \\
\delta_J(\delta_K) &= \mathrm{trace}\,(P_R A_R Q_R \delta_{A_R}^* + P_R \delta_{A_R} Q_R A_R^* + P_R \delta_{A_R} Q_R \delta_{A_R}^* + P_R \delta_{V_R}) + \theta(\|\delta_K\|) & (4.66) \\
\delta_J(\delta_L) &= \mathrm{trace}(P_R A_R Q_R \delta_{A_R}^* + P_R \delta_{A_R} Q_R A_R^* + P_R \delta_{A_R} Q_R \delta_{A_R}^* + Q_R \delta_{R_R}) + \theta(\|\delta_L\|) & (4.67)
\end{aligned}$$

Exploiting the matrix decomposition of P_R and Q_R over $\mathcal{X}_R$ we obtain

$$
\begin{aligned}
\delta_J(\delta_F) &= 2\mathrm{trace}\,[(Q_{12}^*A^*P_{12} + Q_2L^TB_2^*P_{12} + Q_{12}^*C_2^*K^TP_2 + Q_2F^TP_2)\delta_F + \\
&\quad P_2\delta_F Q\delta_F^T] + \theta(\|\delta_F\|), \tag{4.68}
\end{aligned}
$$

$$
\begin{aligned}
\delta_J(\delta_K) &= 2\mathrm{trace}\,\{[C_2(Q_1A + Q_{12}L^TB_2^*)P_{12} + C_2(Q_1C_2^*K^T + Q_{12}F^T)P_2 + \\
&\quad + V_2K^TP_2]\delta_K + [P_2C_2Q_1C_2^* + P_2V_2]\delta_K^T\delta_K\} + \theta(\|\delta_K\|), \tag{4.69}
\end{aligned}
$$

$$
\begin{aligned}
\delta_J(\delta_L) &= 2\mathrm{trace}\,\{[Q_2L^TR_2 + Q_{12}^*A^*P_1B_2 + Q_2L^TB_2^*P_1B_2 + \\
&\quad + Q_{12}^*C^*K^TP_{12}^*B_2 + Q_2F^TP_{12}^*B_2]\delta_L + R_2\delta_L\delta_L^T\} + \theta(\|\delta_L\|), \tag{4.70}
\end{aligned}
$$

and by the vanishing the expressions that stand as coefficients for $\delta_\bullet$ in (4.68), (4.69) and (4.70) we obtain

$$
0 = Q_{12}^*A^*P_{12} + Q_2L^TB_2^*P_{12} + Q_{12}^*C_2^*K^TP_2 + Q_2F^TP_2, \tag{4.71}
$$

$$
\begin{aligned}
0 &= C_2Q_1A + C_2Q_{12}L^TB_2^*P_{12} + C_2Q_1C_2^*K^TP_2 + C_2Q_{12}F^TP_2 + \\
&\quad + V_2K^TP_2, \tag{4.72}
\end{aligned}
$$

$$
\begin{aligned}
0 &= Q_2L^TR_2 + Q_{12}^*A^*P_1B_2 + Q_2L^TB_2^*P_1B_2 + \\
&\quad + Q_{12}^*C_2^*K^TP_{12}^*B_2 + Q_2F^TP_{12}^*B_2, \tag{4.73}
\end{aligned}
$$

Clearly, (4.71), (4.72) and (4.73) represent a system of three implicit necessary conditions defining the solution to the fixed finite-order $\mathcal{H}^2$-optimal discrete-time control problem. In the remaining of this section we show that they can be written in a more tractable, explicit form as presented in the main result. For this define firstly a new set of variables

$$
\hat{Q} \triangleq Q_1 - Q = Q_{12}^*Q_2^{-1}Q_{12}, \tag{4.74}
$$

$$
\hat{P} \triangleq P_1 - P = P_{12}P_2^{-1}P_{12}^*, \tag{4.75}
$$

where P and Q are defined by (4.15) and (4.16), respectively. By Proposition 4.10 $\hat{Q}$ and $\hat{P}$ are non-negative definite and furthermore have finite rank (since Q_{12} and P_{12}^* have finite rank) and Q_1, P_1, Q and P are also non-negative definite. The following results hold

Lemma 4.25 *The fixed finite-order compensator is given by*

$$
F = \Lambda F^\circ G^*, \tag{4.76}
$$

$$
K = \Lambda K^\circ, \tag{4.77}
$$

$$
L = L^\circ G^*, \tag{4.78}
$$

where

$$
L^\circ = (B_2^*PB_2 + R_2)^{-1}B_2^*PA, \tag{4.79}
$$

$$
K^\circ = AQC_2^*(C_2QC_2^* + V_2)^{-1}, \tag{4.80}
$$

$$
F^\circ = A - K^\circ C_2 - B_2L^\circ \tag{4.81}
$$

and

$$G = Q_2^{-1}Q_{12}^* \in \mathcal{L}(\mathcal{X}, \mathbb{R}^n), \tag{4.82}$$

$$M = Q_2 P_2 \in \mathbb{R}^{n \times n}, \tag{4.83}$$

$$\Lambda = P_2^{-1} P_{12}^* \in \mathcal{L}(\mathcal{X}, \mathbb{R}^n). \tag{4.84}$$

Proof From (4.71) and (4.73) we obtain

$$
\begin{aligned}
L &= \{B_2^*[P_1 - P_{12}P_2^{-1}P_{12}^*]B_2 + R_2\}^{-1}B_2^*[P_1 - P_{12}P_2^{-1}P_{12}^*]AQ_{12}Q_2^{-1} \\
&= L^\circ Q_{12}Q_2^{-1} = L^\circ G^*.
\end{aligned} \tag{4.85}
$$

From (4.71) and (4.72) we get

$$
\begin{aligned}
K &= P_2^{-1}P_{12}^*A[Q_1 - Q_{12}Q_2^{-1}Q_{12}^*]C_2^*\{C_2[Q_1 - Q_{12}Q_2^{-1}Q_{12}^*]C_2^* + V_2\}^{-1} \\
&= P_2^{-1}P_{12}^*K^\circ = \Lambda K^\circ.
\end{aligned} \tag{4.86}
$$

From (4.71) we obtain

$$
\begin{aligned}
F &= P_2^{-1}P_{12}^*\{A - A[Q_1 - Q_{12}Q_2^{-1}Q_{12}^*]C_2^*\{C_2[Q_1 - Q_{12}Q_2^{-1}Q_{12}^*]C^* + V_2\}^{-1} + \\
&\quad + B_2\{B_2^*[P_1 - P_{12}P_2^{-1}P_{12}^*]B_2 + R_2\}^{-1}B_2^*[P_1 - P_{12}P_2^{-1}P_{12}^*]A\}Q_{12}Q_2^{-1} \\
&= A - K^\circ C_2 - B_2 L^\circ = \Lambda F^\circ G^*.
\end{aligned} \tag{4.87}
$$

∎

Lemma 4.26 *The triple (G, M, Λ) defined by (4.82),(4.83) and (4.84) represents a (G, M, Λ)-factorization of $\hat{Q}\hat{P} = Q_{12}Q_2^{-1}Q_{12}^*P_{12}P_2^{-1}P_{12}^*$.*

Proof Let $\tau = G^*\Lambda \in \mathcal{L}(\mathcal{X})$ and bring the partition of P_R and Q_R as $P_R = \begin{pmatrix} P_1 & P_{12} \\ P_{12}^* & P_2 \end{pmatrix}$ and $Q_R = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^* & Q_2 \end{pmatrix}$ with $P_1, Q_1 \in \mathcal{L}(\mathcal{X})$, $Q_{12}, P_{12} \in \mathcal{L}(\mathbb{R}^n, \mathcal{X})$, $Q_2, P_2 \in \mathbb{R}^{n \times n}$, P_1, Q_1, P_2, Q_2 non-negative definite and let us write the two Lyapunov equations satisfied by P_R and Q_R with respect to its components

$$0 = Q_1 - AQ_1A^* - AQ_{12}L^TB_2^* - B_2LQ_{12}^*A^* - B_2LQ_2L^TB_2^* - V_1, \tag{4.88}$$

$$0 = Q_{12} - AQ_1C_2^*K^T - AQ_{12}F^T - B_2LQ_{12}^*C_2^*K^T - B_2LQ_2F^T, \tag{4.89}$$

$$
\begin{aligned}
0 = &\; Q_2 - KC_2Q_1C_2^*K^T - KC_2Q_{12}F^T - FQ_{12}^*C_2^*K^T - \\
&- FQ_2F^T - KV_2K^T,
\end{aligned} \tag{4.90}
$$

$$0 = P_1 - A^*P_1A - A^*P_{12}KC_2 - C_2^*K^TP_{12}^*A - C_2^*K^TP_2KC_2 - R_1, \tag{4.91}$$

$$0 = P_{12} - A^*P_1B_2L - A^*P_{12}F - C_2^*K^TP_{12}^*B_2L - C_2^*K^TP_2F, \tag{4.92}$$

$$0 = P_2 - L^TB_2^*P_1B_2L - L^TB_2^*P_{12}F - F^TP_{12}^*B_2L - F^TP_2F - L^TR_2L, \tag{4.93}$$

A straightforward computation leads to $Q_2 = \Lambda Q_{12}$, $P_2 = -GP_{12}$ which implies $P_2 Q_2 = -P_{12}^* Q_{12}$ and furthermore

$$
\begin{aligned}
\Lambda G^* &= P_2^{-1} P_{12}^* Q_{12} Q_2^{-1} = I_n, \quad &(4.94)\\
\tau^2 &= Q_{12} Q_2^{-*} P_2^{-1} P_{12}^* Q_{12} Q_2^{-*} P_2^{-1} P_{12}^* \\
&= Q_{12} Q_2^{-*} P_2^{-1} P_{12}^* \\
&= \tau, \quad &(4.95)\\
G^* M \Lambda &= Q_{12} Q_2^{-1} Q_2 P_2 P_2^{-1} P_{12}^* = \hat{Q} \hat{P}. \quad &(4.96)
\end{aligned}
$$

Notice that if $F_1 \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $F_2 \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ then

$$
\operatorname{rank} (F_1 F_2) \leq \min\{\operatorname{rank} (F_1), \operatorname{rank} (F_2)\}.
$$

Furthermore, if $\dim (\mathcal{Y}) = p < \infty$, then

$$
\operatorname{rank} (F_1) + \operatorname{rank} (F_2) - p \leq \operatorname{rank} (F_1 F_2) \leq \min\{\operatorname{rank} (F_1), \operatorname{rank} (F_2)\} \leq \operatorname{rank} (F_2).
$$

We conclude that rank $(G) = n$, rank $(\Lambda) = n$, rank $(Q_{12}) = n$, rank $(P_{12}) = n$, fact which assures rank $(\hat{Q}) = n$, rank $(\hat{P}) = n$, rank $(\hat{Q}\hat{P}) = n$ and the proof is complete. ∎

Let us introduce the following notation

$$
\begin{cases}
Q_E = AQC_2^* \\
P_E = B_2^* PA \\
V_{2E} = C_2 Q C_2^* + V_2 \\
R_{2E} = B_2^* P B_2 + R_2.
\end{cases}
$$

Notice that V_{2E} and R_{2E} are boundedly invertible (by assumptions **A3** and **A4**, respectively). Write now the components of Q_R and P_R in terms of $\hat{Q}$, $\hat{P}$, Q, P, G and Λ as follows

$$
\begin{aligned}
Q_1 &= Q + \hat{Q}, \quad Q_{12} = \hat{Q}\Lambda^*, \quad Q_2 = \Lambda\hat{Q}\Lambda^*, \quad &(4.97)\\
P_1 &= P + \hat{P}, \quad P_{12} = \hat{P}G^*, \quad P_2 = G\hat{P}G^*. \quad &(4.98)
\end{aligned}
$$

Substituting now the components into (4.88) ... (4.93) we get

$$
\begin{aligned}
0 &= AQA^* + (A - B_2 L^\circ)\hat{Q}(A - B_2 L^\circ)^* + V_1 - Q - \hat{Q}, \quad &(4.99)\\
0 &= \{Q_E V_{2E}^{-1} Q_E^* - \hat{Q} + (A - B_2 L^\circ)\hat{Q}(A - B_2 L^\circ)^*\}\Lambda^*, \quad &(4.100)\\
0 &= \Lambda\{Q_E V_{2E}^{-1} Q_E^* - \hat{Q} + (A - B_2 L^\circ)\hat{Q}(A - B_2 L^\circ)^*\}\Lambda^*, \quad &(4.101)\\
0 &= A^* PA + (A - K^\circ C_2)^*\hat{P}(A - K^\circ C_2) + R_1 - P - \hat{P}, \quad &(4.102)\\
0 &= \{P_E^* R_{2E}^{-1} P_E - \hat{P} + (A - K^\circ C_2)^*\hat{P}(A - K^\circ C_2)\}G^*, \quad &(4.103)\\
0 &= G\{P_E^* R_{2E}^{-1} P_E - \hat{P} + (A - K^\circ C_2)^*\hat{P}(A - K^\circ C_2)\}G^*. \quad &(4.104)
\end{aligned}
$$

Define the auxiliary operators

$$\bar{Q} = (A - B_2 L^\circ)\hat{Q}(A - B_2 L^\circ)^* + Q_{\mathrm{E}}V_{2\mathrm{E}}^{-1}Q_{\mathrm{E}}^* \in \mathcal{L}(\mathcal{X}). \tag{4.105}$$

$$\bar{P} = (A - K^\circ C_2)^*\hat{P}(A - K^\circ C_2) + P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} \in \mathcal{L}(\mathcal{X}). \tag{4.106}$$

$$\tau_\perp = I_{\mathcal{X}} - \tau \in \mathcal{L}(\mathcal{X}). \tag{4.107}$$

The following manipulation shows that

(*i*) From (4.99)+$G^*\Lambda$(4.100)G-(4.100)G-G^*(4.100)* = 0 we get

$$Q - AQA^* - V_1 + Q_{\mathrm{E}}V_{2\mathrm{E}}^{-1}Q_{\mathrm{E}}^* - \tau_\perp \bar{Q}\tau_\perp^* = 0. \tag{4.108}$$

(*ii*) From (4.102)+Λ^*G(4.103)Λ- (4.103)Λ - Λ^*(4.103)* = 0 we have

$$P - A^*PA - R_1 + P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} - \tau_\perp^* \bar{P}\tau_\perp = 0. \tag{4.109}$$

(*iii*) From $G^*\Lambda$(4.100)G-(4.100)G-G^*(4.100)* = 0 we obtain

$$\bar{Q} = (A - B_2 L^\circ)\tau\bar{Q}\tau^*(A - B_2 L^\circ)^* + Q_{\mathrm{E}}V_{2\mathrm{E}}^{-1}Q_{\mathrm{E}}^*. \tag{4.110}$$

(*iv*) From Λ^*G(4.103)G-(4.103)Λ - Λ^*(4.103)* = 0 we derive

$$\bar{P} = (A - K^\circ C_2)^*\tau^* \bar{P}\tau(A - K^\circ C_2) + P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}}. \tag{4.111}$$

Exploiting the fact that $\hat{Q} = \tau\bar{Q}\tau^*$ and $\hat{P} = \tau^*\bar{P}\tau$ it is a routine to show that (4.108), (4.109), (4.110) and (4.111) collapses to (4.27), (4.28), (4.29) and (4.30), respectively. Notice that (4.108), (4.109), (4.110), (4.111) represent, actually, an equivalent way of writing the two modified Riccati equations coupled with the two modified Lyapunov equations via the idempotent operator τ and the proof of Theorem 4.14 is now complete.

Remark 4.27 *The expression $D_{12}^* D_{12} + B_2^* P B_2$ plays in the modified discrete-time Riccati equation (4.28) the same role as the operator $R + B^* X B$ plays in the discrete-time classical Riccati equation (3.14). If in the latter case, the invertibility of $R + B^* X B$ can be achieved under certain assumptions (see Chapter 3), the invertibility of $D_{12}^* D_{12} + B_2^* P B_2$ had to be assumed in order to obtain the modified discrete-time Riccati equation (4.28).*

4.4   The full-order case

In this section we shall focus on the so-called *full-order case*. This case is characterized by the fact that the plant and the controller have the same order, either both are infinite-dimensional, or both are finite-dimensional. We shall derive the expression of the $\mathcal{H}^2$-optimal controller in the discrete-tieme case as a direct consequence of the main result proved in the previous subsection. We shall also discuss the continuous-time infinite-dimensional $\mathcal{H}^2$-optimal control problem under the assumption that the disturbance input and controlled output spaces are infinite-dimensional pointing out the main mathematical difficulties one has to overcome.

4.4.1 The discrete-time $\mathcal{H}^2$-optimal control problem

Let us consider the limiting case when the order of the compensator and the order of the plant are equal, and more precisely, they are both infinite-dimensional systems. In this limit case τ becomes the identity operator and hence, $\tau_\perp = 0$. The two modified Riccati equations (4.28) and (4.27) are decoupled

$$
\begin{align}
Q &= AQA^* + B_1B_1^* - AQC_2^*(C_2QC_2^* + D_{21}D_{21}^*)^{-1}C_2QA^*, \tag{4.112} \\
P &= A^*PA + C_1^*C_1 - A^*PB_2(B_2^*PB_2 + D_{12}^*D_{12})^{-1}B_2^*PA \tag{4.113}
\end{align}
$$

and furthermore, the compensator, $\Sigma_K^\infty(F^\infty, K^\infty, L^\infty)$, which is now infinite-dimensional is given by

$$
\begin{align}
F^\infty &= A - AQC_2^*(C_2QC_2^* + D_{21}D_{21}^*)^{-1}C_2 - \\
&\quad - B_2(B_2^*PB_2 + D_{12}^*D_{12})^{-1}B_2^*PA, \tag{4.114} \\
K^\infty &= AQC_2^*(C_2QC_2^* + D_{21}D_{21}^*)^{-1}, \tag{4.115} \\
L^\infty &= (B_2^*PB_2 + D_{12}^*D_{12})^{-1}B_2^*PA. \tag{4.116}
\end{align}
$$

Notice that $\Sigma_K^\infty(F^\infty, K^\infty, L^\infty)$ as defined by (4.114), (4.116) and (4.115) represents the infinite-dimensional generalization of the classical finite-dimensional $\mathcal{H}^2$-optimal controller. If both the plant and the controller are assumed to have the same finite order then the two modified Lyapunov equations (4.30) and (4.29) are decoupled and put into evidence the minimality of the controller.

4.4.2 The continuous-time $\mathcal{H}^2$-optimal control problem

At the end of this chapter let us consider the continuous-time counterpart of the full-order case discrete-time $\mathcal{H}^2$-optimal control problem, namely, the continuous-time $\mathcal{H}^2$-optimal control problem. We shall assume that the space of disturbance input and the space of controlled output are infinite-dimensional

$$
\begin{align}
\dim(\mathcal{U}_1) &= \infty, \\
\dim(\mathcal{Y}_1) &= \infty.
\end{align}
$$

To be more clear, the problem addressed is the following; being given a smooth Pritchard-Salamon system $\Sigma_G\left(S(\cdot), \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix}\right)$ with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$ find a continuous-time controller $\Sigma_K(K, L, M, N)$ such that the $\mathcal{H}^2$-norm of the transfer function of the linear continuous-time closed-loop system from $u_1(\cdot)$ to $y_1(\cdot)$ is minimized over the class of stabilizing controllers, i.e.

$$
\min_{\Sigma_K \text{ stabilizing}} \|G_{y_1 u_1}\|_{\mathcal{H}^2}.
$$

In the concluding chapter of [80], van Keulen suggests that the solution to the $\mathcal{H}^2$-optimal continuous-time control problem for Pritchard-Salamon systems can be obtained using the same technique that was used for the $\mathcal{H}^\infty$-control problem. Technically speaking, by letting γ to tend to infinity, one obtains without doubts the generalization for Pritchard-Salamon systems of the expressions of the filtering and control Riccati equations associated with the continuous-time $\mathcal{H}$-optimal control problem in finite dimensions as well as the corresponding $\mathcal{H}^2$-optimal controller. It is obvious that if $\dim(\mathcal{U}_1) < \infty$ such a manipulation is done correctly and we recover the solution to the $\mathcal{H}^2$-optimal control problem from the solution to the $\mathcal{H}^\infty$ control problem. Let us see if this can be done when $\dim(\mathcal{U}_1) = \infty$. In Section 2 of [33] it is shown that the impulse response of a Pritchard-Salamon system with infinite-dimensional space of inputs is a tempered $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued distribution with support in $[0, \infty)$. Furthermore, if $\dim(\mathcal{U}_1) = \infty$ it is generally not possible to make sens of $CS(\cdot)B$ as a function. In particular, expressions as $CS(\cdot)B$ and $\int_0^t CS(t-\tau)Bu(\tau)$ cannot be interpreted in the usual sense for arbitrary $u \in L_2^{loc}(0, \infty; \mathcal{U})$. Even in the bounded case $\mathcal{W} = \mathcal{V}$, when $CS(t)B$ is well-defined as a function, it might not be strongly measurable fact which implies it is not locally integrable with respect to the uniform norm topology. It follows that the $\mathcal{H}^2$-optimal control problem for infinite-dimensional systems with infinite-dimensional disturbance-input space is a very delicate problem which is open for future research. This represents the main motivation of our option for its stochastic counterpart when we consider in Section 6.5 the fixed-order digital control of Pritchard-Salamon systems, optimally with a LQG quadratic cost function rather than a $\mathcal{H}^2$ quadratic cost function.

Part III

Digital control theory

Chapter 5

Digital stability

In the previous chapter we stated some results (e.g. Theorem 3.11 and Theorem 3.12) under the assumption that the A-operator of the discrete Popov triple $\Sigma(A, B, M)$ is power stable on $\mathcal{X}$. However, for most of the examples encountered in practice the contrary holds. It is therefore necessary to cope with the situation when the discrete-time semigroup A is not power stable on $\mathcal{X}$. This can be done within the Popov theory framework on the basis of what we shall call here *feedback invariance*. Let us introduce and explain this concept. Consider first a discrete-time Popov triple $\Sigma(A, B, M)$ where the discrete-time semigroup A is no longer assumed to be power-stable on $\mathcal{X}$. It follows that the class of admissible control sequences associated with $\Sigma(A, B, M)$, $U^{x_0}_{\text{adm}}$, is no longer the whole $\ell_2(0, \infty; \mathcal{U})$. It could be empty for some pairs $(x_0, u(\cdot))$, but if the pair (A, B) is assumed to be power stabilizable, then $U^{x_0}_{\text{adm}} \neq \emptyset$. This fact is assured by Proposition 1.1, pp. 72 [46].

Definition 5.1 *(equivalent Popov triples, Definition 1.7, pp. 75 [46])*
Let $\Sigma_1(A_1, B_1, M_1)$ and $\Sigma_2(A_2, B_2, M_2)$ be two discrete-time Popov triples on $(\mathcal{X}, \mathcal{U})$. We shall say that they are equivalent if there exists $F \in \mathcal{L}(\mathcal{X}, \mathcal{U})$ and $X = X^* \in \mathcal{L}(\mathcal{X})$ such that

$$
\begin{align}
A_2 &= A_1 + B_1 F, \tag{5.1}\\
B_2 &= B_1, \tag{5.2}\\
Q_2 &= Q_1 + L_1 F + F^* L_1^* + F^* R_1 F + A_2^* X A - X, \tag{5.3}\\
L_2 &= L_1 + F^* R_1 + A_2 X B_1, \tag{5.4}\\
R_2 &= R_1 + B_1^* X B_1. \tag{5.5}
\end{align}
$$

If $X = 0$ they are called F-equivalent.

Related to Definition 5.1 we have the following result

Proposition 5.2 *(Proposition 1.8 pp. 75, [46]) Let $\Sigma_1(A_1, B_1, M_1)$ and $\Sigma_2(A_2, B_2, M_2)$ be two equivalent Popov triples on $(\mathcal{X}, \mathcal{U})$. Then if for each $x_0 \in \mathcal{X}$ the class of admissible*

control sequences is such that $U_{\text{adm}}^{x_0} \neq \emptyset$, then the following holds

$$J_{\Sigma_1}(x_0, u) = J_{\Sigma_2}(x_0, \tilde{u}) + \langle x_0, X x_0 \rangle_{\mathcal{X}}, \tag{5.6}$$

with

$$\tilde{u} \stackrel{\triangle}{=} u - Fx. \tag{5.7}$$

Remark 5.3 *Proposition 5.2 hides a very important property of F-equivalent discrete-time Popov triples, namely that the solutions to the discrete-time Ricati equations (when they exists) associated with two F-equivalent discrete-time Popov triples coincide. Furthermore, Definition 5.1 gives the transformation formulae to obtain a stable F-equivalent of an arbitrary discrete-time Popov triple.*

The situation looks very similar in the continuous-time case. In [87], Martin Weiss gave a Popov function based solution to the (continuous) LQ control problem associated with a Pritchard-Salamon-Popov triple with C_0-semigroup assumed to be exponentially stable on $\mathcal{W}$ and $\mathcal{V}$. To generalize his result, this assumption was relaxed to admissible (bounded) stabilizability of the pair $(S^{\mathcal{V}}(\cdot), B)$. In the new framework he proved that under an admissible (bounded) stabilizing feedback F, the Pritchard-Salamon-Popov triple becomes a new Pritchard-Salamon-Popov triple with respect to $(\mathcal{W} \hookrightarrow \mathcal{V}, \mathcal{U})$ defined by

$$\Sigma_F \left(S_{\text{BF}}^{\mathcal{V}}(\cdot), B, M_F = \begin{pmatrix} Q + N^*F + F^*N + F^*RF & N^* + F^*R \\ N + RF & R \end{pmatrix} = M_F^* \right), \tag{5.8}$$

with $S_{\text{BF}}^{\mathcal{V}}(\cdot)$ the perturbed C_0-semigroup generated by $A^{\mathcal{V}} + BF$. Furthermore, it has the remarkable property that the solutions to the Riccati equations associated with $\Sigma(S^{\mathcal{V}}(\cdot), B, M)$ and $\Sigma_F(S_{\text{BF}}^{\mathcal{V}}(\cdot), B, M_F)$, respectively, coincide and the new stabilizing optimal feedback is obtained from the old one by subtracting F. Such a feedback transformation permits, hence, to extend the results to Pritchard-Salamon-Popov triples with admissible (bounded) stabilizable pair $(S^{\mathcal{V}}(\cdot), B)$.

Remark 5.4 *A feedback transformation of the form (5.7) achieves only power stability of the discrete-time semigroup A, exponential stability of the strongly continuous semigroup $S(\cdot)$, respectively, when the continuous-time case is considered. Therefore we shall call it discrete-time prestabilization and continuous-time prestabilization, respectively.*

In this chapter we shall be concerned with the digital counterpart result of the above prestabilization feedback transformations. We show that a certain digital control law, generated by an adequately chosen feedback operator, achieves digital exponential stability

for the Pritchard-Salamon-Popov triple. We shall call this *digital prestabilization*. As we have announced in an earlier section of this thesis, we shall restrict to the case

$$x_0 \in \mathcal{W}.$$

This is a sufficient condition for having a continuous output function of the Pritchard-Salamon system $\Sigma(S(\cdot), B, C, D)$. In this case, the sampled version of the output function is well defined and hence, the digital control problem associated with the standard digital control configuration depicted in Figure 1.5 can be properly formulated. As we have seen earlier in this thesis, the control function considered for the digital control problem is of the form

$$u(\cdot) \in \mathcal{PC}_\mathrm{T}(0, \infty; \mathcal{U}) \cap U_\mathrm{adm}^{x_0}. \tag{5.9}$$

Since we have imposed that the initial value belongs to the smaller space $\mathcal{W}$, we would like to characterize the solution to the initial value problem

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \in \mathcal{W} \tag{5.10}$$

firstly when the control is an arbitrary L_2-function and then when it satisfies the restriction (5.9). For the first case the following result was proved in [80].

Proposition 5.5 *Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard-Salamon system and suppose that $S(\cdot)$ is exponentially stable on $\mathcal{W}$ and $\mathcal{V}$. Consider*

$$x(t) = S(t)x_0 + \int_0^t S(t - \tau)Bu(\tau)d\tau$$

and suppose that $u(\cdot) \in L_2(0, \infty; \mathcal{U})$. Then for $x_0 \in \mathcal{V}$ we have $x(\cdot) \in L_2(0, \infty; \mathcal{V})$ and for $x_0 \in \mathcal{W}$ we have $x(\cdot) \in L_2(0, \infty; \mathcal{W})$.

Since $\mathcal{PC}_\mathrm{T}(0, \infty; \mathcal{U}) \subset L_2(0, \infty; \mathcal{U})$, one can conclude immediately that if $x_0 \in \mathcal{W}$ and $S(\cdot)$ is exponentially stable on $\mathcal{W}$, then the state function satisfies

$$x(\cdot) \in L_2(0, \infty; \mathcal{W}) \cap \mathcal{C}(0, \infty; \mathcal{W}).$$

An immediate consequence of the above result is to show that the sampled of the state function is a $\mathcal{W}$-valued square summable sequence.

Corollary 5.6 *Let $\Sigma(S(\cdot), B, C, D)$ be a Pritchard-Salamon system and suppose that $S(\cdot)$ is exponentially stable on $\mathcal{W}$. Assume that the control function satisfies (5.9) and let $\tilde{x} = \{x(0), x(1), \cdots, x(k), \cdots, \}$ be the sampled state function. If $x_0 \in \mathcal{W}$ then $\tilde{x} \in \ell_2(0, \infty; \mathcal{W})$*

Proof Since
$$\mathcal{PC}_T(0, \infty; \mathcal{U}) \subset L_2(0, \infty; \mathcal{U}),$$
then the state function satisfying

$$
\begin{aligned}
x(t) &= S^{\mathcal{W}}(t)x_0 + \int_0^t S^{\mathcal{V}}(t - \tau)Bu(\tau)d\tau = \\
&= S^{\mathcal{W}}(t - kT)x(k) + \int_{kT}^t S^{\mathcal{V}}(t - \tau)Bu(k)d\tau = \\
&= S^{\mathcal{W}}(t - kT)x(k) + \int_0^{t-kT} S^{\mathcal{V}}(t - kT - \tau)Bu(k)d\tau \qquad (5.11)
\end{aligned}
$$

belongs also to $L_2(0, \infty; \mathcal{W}) \cap \mathcal{C}(0, \infty; \mathcal{W})$. But

$$\mathbf{S}\left(L_2(0, \infty; \mathcal{W}) \cap \mathcal{C}(0, \infty; \mathcal{W})\right) = \ell_2(0, \infty; \mathcal{W}).$$

This fact implies the conclusion.

■

Notice that (5.11) can be written at the sampling instants in the following way

$$x(k + 1) = S^{\mathcal{W}}(T)x_0 + \int_0^T S^{\mathcal{V}}(T - \tau)Bud\tau. \qquad (5.12)$$

Similarly, the output function at the sampling instants satisfies

$$y(k) = \Lambda x(k) + \Theta u(k), \qquad (5.13)$$

where the operators in (5.12) and (5.13) are defined by

$$\Phi : \mathcal{W} \longrightarrow \mathcal{W}, \quad \Phi x \triangleq S^{\mathcal{W}}(T)x, \quad \forall x \in \mathcal{W}, \qquad (5.14)$$

$$\Gamma : \mathcal{U} \longrightarrow \mathcal{W}, \quad \Gamma u \triangleq \int_0^T S^{\mathcal{V}}(\tau)Bud\tau, \quad \forall u \in \mathcal{U}, \qquad (5.15)$$

$$\Lambda : \mathcal{W} \longrightarrow \mathcal{Y}, \quad \Lambda x \triangleq Cx, \qquad (5.16)$$

$$\Theta : \mathcal{U} \longrightarrow \mathcal{Y}, \quad \Theta u \triangleq Du, \qquad (5.17)$$

where T is the sampling period.

Definition 5.7 *We shall call* $\Sigma(\Phi, \Gamma, \Lambda, \Theta)$ *defined by (5.14), (5.15), (5.16) and (5.17) the time-discretized Pritchard-Salomon system*

Remark 5.8 *If we replace* $\mathcal{W}$ *with* $\mathcal{V}$ *in Corollary 5.6 then the results still hold with minor modifications of the proof and, furthermore, it is possible to write down a discretized state equation of the form (5.12) with the* Φ *and* Γ *operators defined this time by*

$$\Phi : \mathcal{V} \longrightarrow \mathcal{V}, \quad \Phi x \triangleq S^{\mathcal{V}}(T)x, \quad \forall x \in \mathcal{V}, \qquad (5.18)$$

$$\Gamma : \mathcal{U} \longrightarrow \mathcal{V}, \quad \Gamma u \triangleq \int_0^T S^{\mathcal{V}}(\tau)Bud\tau, \quad \forall u \in \mathcal{U}. \qquad (5.19)$$

However, since for $x_0 \in \mathcal{V}$ the output function does not make sense pointwise, it is impossible to write down the difference equation corresponding to (5.13) and hence, the time-discretized Pritchard-Salamon system would not be well defined.

5.1 Digital exponential stability

In this section we give a characterization of the exponential stabilizability of Pritchard-Salamon systems under the assumption that the feedback control law is a digital one.

We have introduced the Pritchard-Salamon class of systems and for systems belonging to this class we have introduced the definition of exponential stability along with the definition of admissble (bounded) stabilizability. We have seen that if $x_0 \in \mathcal{W}$, then the time-discretized Pritchard-Salamon system is well defined. We would like to characterize its power stability/power stabilizability properties in terms of the exponential stability/admissble (bounded) stabilizability of the original Pritchard-Salamon system.

Remark 5.9 *Since our approach to the optimal control problem is a digital one we shall assume that the space of inputs and of outputs are finite-dimensional, i.e. we have finitely many sensors and actuators*

$$\mathcal{U} = \mathbb{R}^m, \tag{5.20}$$
$$\mathcal{Y} = \mathbb{R}^p. \tag{5.21}$$

The basic result on digital exponential stability is proved in the following

Theorem 5.10 *Let $S(\cdot)$ be a C_0-semigroup on $\mathcal{W} \hookrightarrow \mathcal{V}$ and let $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ be an admissible input operator. Then the following statements hold*

(i) *If the continuous-time C_0-semigroup $S^{\mathcal{W}}(\cdot)$ is exponentially stable on $\mathcal{W}$ then $\mathbf{\Phi}$ defined by (5.14) is a power stable discrete-time semigroup on $\mathcal{W}$.*

(ii) *The operator $\mathbf{\Gamma}$ defined by (5.15) satisfies*

$$\mathbf{\Gamma} \in \mathcal{L}(\mathcal{U}, \mathcal{W}) \cap \mathcal{L}(\mathcal{U}, \mathcal{V}). \tag{5.22}$$

(iii) *Assume that $(S(\cdot), B)$ is exponentially stabilizable on $\mathcal{W}$ and assume also that $\mathcal{U} = \mathbb{R}^m$, $m \geq 1$. Then there exists a sufficiently small sampling period T_0 such that for every $T < T_0$ the pair $(\mathbf{\Phi}, \mathbf{\Gamma})$ defined by (5.14) and (5.15) is power stabilizable on $\mathcal{W}$.*

(*iv*) *If a certain discrete-time feedback control law*

$$u(k) = Fx(k), \quad F \in \mathcal{L}(\mathcal{W}, \mathcal{U}) \tag{5.23}$$

makes the discrete-time semigroup $(\Phi + \Gamma F)$ *to be power stable on* $\mathcal{W}$, *then*

$$u_{\text{step}}(t) = Fx(k), kT \le t < (k+1)T \tag{5.24}$$

is a digital exponentially stabiling feedback control law.

Proof

(*i*) Since $S^{\mathcal{W}}$ is exponentially stable on $\mathcal{W}$, then there exist $M_{\mathcal{W}} \ge 1$ and $\omega_{\mathcal{W}} < 0$ such that

$$\|S^{\mathcal{W}}(t)\| \le M_{\mathcal{W}} e^{\omega_{\mathcal{W}} t}, t \ge 0.$$

Write now the exponentially stability condition for $t = kT$. It follows that

$$\|S^{\mathcal{W}}(T)^k\| = \|S^{\mathcal{W}}(kT)\| \le M_{\mathcal{W}} e^{\omega_{\mathcal{W}} kT} = M_{\mathcal{W}} \rho_{\mathcal{W}}(T)^k, k \ge 0,$$

where $\rho_{\mathcal{W}}(T) = e^{\omega_{\mathcal{W}} T}$. Since $\omega_{\mathcal{W}} < 0$ then $\rho_{\mathcal{W}}(T) \in (0, 1)$ for $\forall T > 0$ and hence, $S^{\mathcal{W}}(T)$ is power stable on $\mathcal{W}$ and the conclusion follows since by definition $\Phi = S^{\mathcal{W}}(T)$.

(*ii*) Let us write the condition for the input operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ to be admissible with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$ on the time-interval $[0, T), k > 0$. This means that the following controllability map $\mathcal{B}^T : L_2(0, T; \mathcal{U}) \longrightarrow \mathcal{V}$, defined by

$$\mathcal{B}^T u = \int_0^T S^{\mathcal{V}}(T - \tau) Bu(\tau) d\tau, \tag{5.25}$$

is bounded, $\mathcal{B}^T \in \mathcal{L}(L_2(0, T; \mathcal{U}), \mathcal{W})$, i.e. there exists $\alpha > 0$ such that

$$\|\mathcal{B}^T u\|_{\mathcal{W}} \le \alpha \|u(\cdot)\|_{L_2(0, T; \mathcal{U})}. \tag{5.26}$$

Let $u \in L_2(0, T; \mathcal{U})$ be a constant function on $[0, T]$ denoted by v. Since (5.26) holds also for such a control function it follows that

$$\begin{aligned}
\|\mathcal{B}^T v\|_{\mathcal{W}} &= \|\int_0^T S^{\mathcal{V}}(T - \tau) Bv d\tau\|_{\mathcal{W}} = (\text{ by definition of } \Gamma) \\
&= \|\Gamma v\|_{\mathcal{W}} \le \alpha T \|v\|_{\mathcal{U}} \tag{5.27}
\end{aligned}$$

fact which implies that $\Gamma \in \mathcal{L}(\mathcal{U}, \mathcal{W})$. Exploiting now the fact that $\mathcal{W} \hookrightarrow \mathcal{V}$ we get that $\Gamma \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ which yields the result.

(*iii*) By Lemma 3.5 from [30], if B has finite rank then $(S^{\mathcal{W}}(\cdot), B)$ is exponentially stabilizable on $\mathcal{W}$ by a feedback operator $F \in \mathcal{L}(\mathcal{V}, \mathcal{U})$ if and only if $A^{\mathcal{W}}$ has finitely many eigenvalues of finite multiplicity in

$$\mathbf{C}_+ \stackrel{\triangle}{=} \{s \in \mathbb{C} | \Re e(s) \geq 0\}$$

and with respect to the spectral decomposition

$$\left(A^{\mathcal{W}}, B\right) = \left(\left(\begin{array}{cc} A_+^{\mathcal{W}} & 0 \\ 0 & A_-^{\mathcal{W}} \end{array}\right), \left(\begin{array}{c} B_+ \\ B_- \end{array}\right)\right), \tag{5.28}$$

where $A_+^{\mathcal{W}}$ contains all the unstable eigenvalues of $A^{\mathcal{W}}$, the following holds

(*a*) $\left(A_+^{\mathcal{W}}, B_+\right)$ is controllable.

(*b*) $A_-^{\mathcal{W}}$ generates an exponentially stable C^0-semigroup $S_-^{\mathcal{W}}(\cdot)$ on $\mathcal{W}$.

The pair $(\mathbf{\Phi}, \mathbf{\Gamma})$ defined by (5.14) and (5.15) with the discretization performed on the decomposition (5.28) satisfies

$$(\mathbf{\Phi}, \mathbf{\Gamma}) = \left(\left(\begin{array}{cc} S_+^{\mathcal{W}}(T) & 0 \\ 0 & S_-^{\mathcal{W}}(T) \end{array}\right), \left(\begin{array}{c} \int_0^T S_+^{\mathcal{W}}(\tau)d\tau B_+ \\ \int_0^T S_-^{\mathcal{W}}(\tau)d\tau B_- \end{array}\right)\right) = \left(\left(\begin{array}{cc} \mathbf{\Phi}^+ & 0 \\ 0 & \mathbf{\Phi}^- \end{array}\right), \left(\begin{array}{c} \mathbf{\Gamma}^+ \\ \mathbf{\Gamma}^- \end{array}\right)\right).$$

Since $S_-^{\mathcal{W}}(t)$ is an exponentially stable C_0-semigroup on $\mathcal{W}$, then $\mathbf{\Phi}^-$ is power-stable on $\mathcal{W}$. This is obtained by applying the result proved at the first item of this lemma. We claim here that if $(\mathbf{\Phi}^+, \mathbf{\Gamma}^+)$ is power-stabilizable then $(\mathbf{\Phi}, \mathbf{\Gamma})$ is also power-stabilizable on $\mathcal{W}$. Let us proove this assertion. Notice first that since the pair $\left(A_+^{\mathcal{V}}, B_+\right)$ is assumed to be controllable then there exists a sampling step T_0 such that for every $T < T_0$, the pair $(\mathbf{\Phi}^+, \mathbf{\Gamma}^+)$, the time-discretized of $\left(A_+^{\mathcal{W}}, B_+\right)$, is also controllable. This is a classical finite-dimensional result and for its proof the reader is referred to [48, 20]. But a controllable finite-dimensional system is also power stabilizable (see [69]). Hence, if n is the dimension of the antistable subspace in the decomposition (5.28), then there exists $F^+ \in \mathcal{L}(\mathbb{R}^n, \mathcal{U})$ such that $\mathbf{\Phi}^+ + \mathbf{\Gamma}^+ F^+$ is power stable on $\mathbb{R}^n$. Let now $F \in \mathcal{L}(\mathcal{V}, \mathcal{U})$ be given by

$$\bar{F} \stackrel{\triangle}{=} \left(\begin{array}{cc} F^+ & 0 \end{array}\right).$$

The perturbed semigroup decomposed with respect to the partition (5.28) is given by

$$\bar{\mathbf{\Phi}} \stackrel{\triangle}{=} \mathbf{\Phi} + \mathbf{\Gamma}\bar{F} = \left(\begin{array}{cc} \mathbf{\Phi}^+ + \mathbf{\Gamma}^+ F^+ & 0 \\ \mathbf{\Gamma}^- F^+ & \mathbf{\Phi}^- \end{array}\right). \tag{5.29}$$

We claim that the operator defined by (5.29) is a discrete-time semigroup on $\mathcal{W}$. Furthermore, since $\mathbf{\Phi}^+ + \mathbf{\Gamma}^+ F^+$ and $\mathbf{\Phi}^-$ are power stable on $\mathbb{R}^n$ and $\mathcal{W}^-$, where $\mathcal{W}^-$

represents the stable subspace of $\mathcal{W}$, then $\bar{\Phi}$ is power stable on $\mathcal{W}$. By mathematical induction one can easy show that

$$\bar{\Phi}^i = \begin{pmatrix} (\Phi^+ + \Gamma^+ F^+)^i & 0 \\ \Gamma^- F^+ (\Phi^+ + \Gamma^+ F^+)^{i-1} + \Phi^- (\Gamma^- F^+)^{i-1} & (\Phi^-)^i \end{pmatrix}, \quad \forall i \in \mathbb{N}. \quad (5.30)$$

A simple inspection of (5.30) shows that $\bar{\Phi}^0 = I_{\mathcal{W}}$ and a straightforward computation shows that

$$\bar{\Phi}^{i+j} = \bar{\Phi}^i \bar{\Phi}^j, \quad \forall i, j \in \mathbb{N}.$$

Hence, $\bar{\Phi}$ verifies the axiomes of a discrete-time semigroup as given in Definition 3.1. It is also power stable on $\mathcal{W}$ since

$$\|\bar{\Phi}^k\| \leq \sigma_{\max}(\Phi^+ + \Gamma^+ F^+) + \|\Phi^-\| + \|\Gamma^- F^+\| \leq M\rho^k, \quad (5.31)$$

for some appropriately chosen $M > 0$ and $0 < \rho < 1$.

(iv) Let $F \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ such that $\Phi + \Gamma F$ is a power stable on $\mathcal{W}$ and let $u(t) = Fx(k), kT \leq t < (k+1)T$. Then the solution to the initial value problem (5.10) becomes for arbitrary $x_0 \in \mathcal{W}$

$$\begin{aligned} x(t) &= S^{\mathcal{W}}(t)x_0 + \int_0^t S^{\mathcal{V}}(t-\tau)BFx(k)d\tau = \\ &= S^{\mathcal{W}}(t-kT)x(k) + \int_{kT}^t S^{\mathcal{V}}(t-\tau)BFx(k)d\tau \\ &= S^{\mathcal{W}}(t-kT)x(k) + \int_0^{t-kT} S^{\mathcal{V}}(t-kT-\tau)BFx(k)d\tau \\ &= (S^{\mathcal{W}}(t-kT) + \int_0^{t-kT} S^{\mathcal{V}}(t-kT-\tau)BF)x(k)d\tau. \quad (5.32) \end{aligned}$$

It follows that

$$\begin{aligned} \|x(t)\|_{\mathcal{W}} &= \left\| S^{\mathcal{W}}(t-kT)x(k) + \int_0^{t-kT} S^{\mathcal{V}}(t-kT-\tau)BFx(k)d\tau \right\|_{\mathcal{W}} \leq \\ &\leq \left\| S^{\mathcal{W}}(t-kT)x(k) \right\|_{\mathcal{W}} + \left\| \int_0^{t-kT} S^{\mathcal{V}}(t-kT-\tau)BFx(k)d\tau \right\|_{\mathcal{W}} \leq \\ &\leq \sup_{kT \leq t < (k+1)T} \left\| S^{\mathcal{W}}(t-kT) \right\|_{\mathcal{L}(\mathcal{W})} \|x(k)\|_{\mathcal{W}} + \\ &+ \sup_{kT \leq t < (k+1)T} \left\| \int_0^{t-kT} S^{\mathcal{V}}(t-kT-\tau)BFx(k)d\tau \right\|_{\mathcal{W}}. \quad (5.33) \end{aligned}$$

Let us notice that the first term in the right-hand side of (5.33) satisfies

$$\begin{aligned} \sup_{kT \leq t < (k+1)T} \left\| S^{\mathcal{W}}(t-kT) \right\|_{\mathcal{L}(\mathcal{W})} \|x(k)\|_{\mathcal{W}} &\leq \sup_{kT \leq t < (k+1)T} M_{\mathcal{W}} e^{\omega_{\mathcal{W}}(t-kT)} \|x(k)\|_{\mathcal{W}} \leq \\ &\leq M_{\mathcal{W}} \|x(k)\|_{\mathcal{W}} \leq M_{\mathcal{W}} \rho_{\mathcal{W}}^k \|x_0\|_{\mathcal{W}}, \quad (5.34) \end{aligned}$$

where $\rho_{\mathcal{W}}$ is the spectral radius of $\boldsymbol{\Phi} + \boldsymbol{\Gamma} F$ on $\mathcal{W}$ (see [32] Lemma 2.1 (b)).

Let us consider the second term from the right-hand side of (5.33). First of all let us notice that since $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ is an admissible input operator

$$\int_0^{t-kT} \left\| S^{\mathcal{V}}(t - kT - \tau) BFx(k) \right\|_{\mathcal{W}} d\tau \leq$$

$$\int_0^{t-kT} \left\| S^{\mathcal{V}}(t - kT - \tau) \right\|_{\mathcal{L}(\mathcal{W})} \| BFx(k) \|_{\mathcal{V}} d\tau \leq$$

$$\leq \| BFx(k) \|_{\mathcal{V}} \int_0^{t-kT} M_{\mathcal{V}} e^{\omega_{\mathcal{V}}(t-kT-\tau)} d\tau \leq$$

$$\leq \frac{M_{\mathcal{V}}}{\omega_{\mathcal{V}}} \| BFx(k) \|_{\mathcal{V}} \left(1 - e^{\omega_{\mathcal{V}}(t-kT)} \right) \leq \infty. \tag{5.35}$$

It follows that

$$\left\| \int_0^{t-kT} S^{\mathcal{V}}(t - kT - \tau) BFx(k) \right\|_{\mathcal{W}} \leq \int_0^{t-kT} \left\| \int_0^{t-kT} S^{\mathcal{V}}(t - kT - \tau) BFx(k) d\tau \right\|_{\mathcal{W}} \leq$$

$$\frac{M_{\mathcal{V}}}{\omega_{\mathcal{V}}} \| BFx(k) \|_{\mathcal{V}} \left(1 - e^{\omega_{\mathcal{V}}(t-kT)} \right) \leq \frac{M_{\mathcal{V}}}{\omega_{\mathcal{V}}} \| B \|_{\mathcal{L}(\mathcal{U},\mathcal{V})} \| Fx(k) \|_{\mathcal{U}} \left(1 - e^{\omega_{\mathcal{V}}(t-kT)} \right) \leq$$

$$\frac{M_{\mathcal{V}}}{\omega_{\mathcal{V}}} \| B \|_{\mathcal{L}(\mathcal{U},\mathcal{V})} \| F \|_{\mathcal{L}(\mathcal{W},\mathcal{U})} \left(1 - e^{\omega_{\mathcal{V}}(t-kT)} \right) \| x(k) \|_{\mathcal{W}}. \tag{5.36}$$

Hence, the supremum over $t \in [kT, (k+1)T)$ of the second term from the right-hand side of (5.33) satisfies

$$\sup_{kT \leq t < (k+1)T} \left\| \int_0^{t-kT} S^{\mathcal{V}}(t - kT - \tau) BFx(k) d\tau \right\|_{\mathcal{W}} \leq N_{\mathcal{W}} \rho_{\mathcal{W}}^k \| x_0 \|_{\mathcal{W}}, \tag{5.37}$$

for

$$N_{\mathcal{W}} \triangleq \frac{M_{\mathcal{V}}}{\omega_{\mathcal{V}}} \| B \|_{\mathcal{L}(\mathcal{U},\mathcal{V})} \| F \|_{\mathcal{L}(\mathcal{W},\mathcal{U})} \left(1 - e^{\omega_{\mathcal{V}}(T)} \right).$$

From (5.34) and (5.37) we get that

$$\| x(t) \|_{\mathcal{W}} \leq (M_{\mathcal{W}} + N_{\mathcal{W}}) \rho_{\mathcal{W}}^k \| x_0 \|_{\mathcal{W}}, \tag{5.38}$$

which shows that $\| x(t) \|_{\mathcal{W}}$ is bounded by a sequence converging to zero, fact which yields the assertion.

∎

Remark 5.11 (*i*) *The claims we made in the items (i) and (iii) of Theorem 5.10 can be extended in the sense that it is possible to prove power stability and power stabilizability on the both spaces $\mathcal{W}$ and $\mathcal{V}$ with minor modifications of the proof and assuming that $(S(\cdot), B)$ is admissibly stabilizable.*

(*ii*) *The result of item (ii) holds also for infinite-dimensional input space* $\mathcal{U}$.

(*iii*) *The situation is very much different in what concernes the last item of Theorem 5.10.*
        If for the case $x(\cdot) \in \mathcal{W}$ *considered in Theorem 5.10 we allow that* F *is bounded from*
        $\mathcal{W}$ *to* $\mathcal{U}$, *then when considering the case* $x(\cdot) \in \mathcal{V}$, *the operator* F *has compulsory to be*
        bounded from $\mathcal{V}$ *to* $\mathcal{U}$. *This is especially true since* $F \in \mathcal{L}(\mathcal{V},\mathcal{U})$ *implies* $F \in \mathcal{L}(\mathcal{W},\mathcal{U})$
        (the reverse does not hold) and for feedback operators bounded from $\mathcal{W}$ *to* $\mathcal{U}$, *but are*
        unbounded from $\mathcal{V}$ *to* $\mathcal{U}$, *we cannot obtain a bound for* $\|x(t)\|_\mathcal{V}$ *in terms of* $\|x(k)\|_\mathcal{W}$.

Remark 5.12 *In Figure 2.1 we have represented the overlap of the control and observation*
mappings with respect to the $\mathcal{W}$-*space for a Pritchard-Salamon system* $\Sigma(S^\mathcal{V}(\cdot), B, C, D)$.
The mapping representation associated with the time-discretized Pritchard-Salamon system
is given in Figure 5.1. It is easy to see that sampling has a beneficial smoothing effect on
the original system, all operators are now bounded with respect to the smaller state space
$\mathcal{W}$. *If for original Pritchard-Salamon system, due to the unboundedness of* B, *in general*

$$B \notin \mathcal{L}(\mathcal{U},\mathcal{W}) \cap \mathcal{L}(\mathcal{U},\mathcal{V}),$$

notice that for the time-discretized Pritchard-Salamon system $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, \mathbf{\Theta})$ *the relation*
corresponding to the above one is assured by the second item proved in Theorem 5.10, i.e.

$$\mathbf{\Gamma} \in \mathcal{L}(\mathcal{U},\mathcal{W}) \cap \mathcal{L}(\mathcal{U},\mathcal{V}).$$
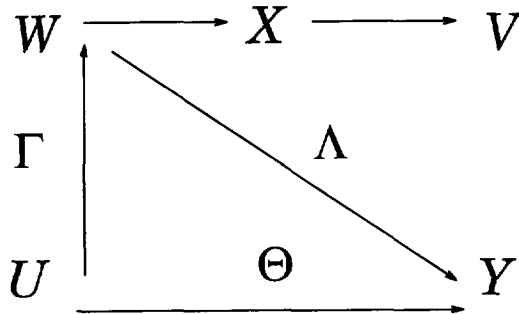
Figure 5.1: The mapping representation of the time-discretized Pritchard-Salamon system

5.2 Digital optimal prestabilization

We are able now to focus on the main goal of this chapter - removing the stability assumption on the dynamics operator required by the Popov theory results proved in chapter 3. This will be done by *digital prestabilization*. The idea is the following: assume that we have a pair $(S^{\mathcal{W}}(\cdot), B)$ which is exponentially stabilizable on $\mathcal{W}$ and B is a finite rank operator. Then the spectral decomposition (5.28) holds. We want to find a digital feedback operator such that "*the unstable poles are removed with minimum effort*". To be more precise, we want to determine a piece-wise constant function

$$u_{\text{step}}(t) = F^+ x(k), \quad kT \le t < (k+1)T \tag{5.39}$$

which provides digital exponential stability for the Pritchard-Salamon-Popov triple $\Sigma(S^{\mathcal{V}}(\cdot), B, M)$ and such that the following quadratic index associated with the antistable part of $(S^{\mathcal{V}}(\cdot), B)$ is minimized

$$J_{\Sigma}\left(x_0^+, u(\cdot)\right) \triangleq \int_0^{\infty} \langle \begin{pmatrix} Q^+ & L^+ \\ (L^+)^* & R^+ \end{pmatrix} \begin{pmatrix} x^+(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} x^+(t) \\ u(t) \end{pmatrix} \rangle_{\mathbb{R}^n \times \mathbb{R}^m} dt. \tag{5.40}$$

Here $x^+(\cdot) \in \mathbb{R}^n$ represents the state vector components corresponding to the antistable part assumed to be n-dimensional. We also assume that the following finite-dimensional Popov index

$$M^+ = \begin{pmatrix} Q^+ & L^+ \\ (L^+)^* & R^+ \end{pmatrix} = (M^+)^* \in \mathcal{L}(\mathbb{R}^n \oplus \mathbb{R}^m)$$

is positive semidefinite and that $(A_{\mathcal{W}}^+, B^+)$ is exponentially stabilizable and $(\sqrt{Q^+}, A_{\mathcal{W}}^+)$ is exponentially detectable, where

$$\dot{x}^+(t) = A_{\mathcal{W}}^+ x^+(t) + B^+ u(t), \quad x^+(0) = x_0^+ \in \mathbb{R}^n$$

is the initial value problem associated with the antistable part of the original Pritchard-Salamon-Popov triple.

We shall call this the *digital optimal prestabilization problem associated with the Pritchard-Salamon-Popov triple* $\Sigma(S^{\mathcal{V}}(\cdot), B, M)$.

The solution to this problem is obtained as follows

1. Assume that the digital prestabilization feedback has the following structure with respect to the spectral decomposition (5.28)

$$F^+ = \begin{pmatrix} F_n^+ & 0 \end{pmatrix}, \tag{5.41}$$

where F_n^+ is such that $\mathbf{\Phi}^+ + \mathbf{\Gamma}^+ F_n^+$ has all eigenvalues in the open unit disk. Such a feedback provides power stability of the perturbed discrete-time semigroup $\mathbf{\Phi} + \mathbf{\Gamma} F^+$, where $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ are defined by (5.14) and (5.15), respectively. This statement is motivated by the assertion *(iii)* of Theorem 5.10. Then a digital control las defined by (5.39) is a digital exponentially stabilizing one. This statement is supported by the fact proved in assertion *(iii)* of Theorem 5.10.

2. Assume now that F_n^+ is the power stabilizing feedback defined by the solution to the following discrete-time control problem

$$x^+(k+1) = \Phi^+ x^+(k) + \Gamma^+ u(k), \quad x^+(0) = x_0^+, \tag{5.42}$$

$$\min_{u(k)\in U_{adm}^{x_0^+}} \sum_{k=0}^{\infty} \langle \begin{pmatrix} x^+(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} Q_T^+ & L_T^+ \\ (L_T^+)^* & R_T^+ \end{pmatrix} \begin{pmatrix} x^+(k) \\ u(k) \end{pmatrix} \rangle_{\mathbb{R}^n \times \mathbb{R}^m}, \tag{5.43}$$

where the class of admissible control sequences is define by

$$U_{adm}^{x_0^+} \triangleq \left\{ u(\cdot) \in \ell_2(0, \infty; \mathbb{R}^m) | x^+(\cdot) \in \ell_2(0, \infty; \mathbb{R}^n) \right\}, \tag{5.44}$$

with $x^+(\cdot)$ the solution to the initial value problem (5.42) and where

$$Q_T^+ = \int_0^T \left(S_W^+(\tau) \right)^* Q^+ S_W^+(\tau) d\tau, \tag{5.45}$$

$$L_T^+ = \int_0^T \left(S_W^+(\tau) \right)^* [L^+ + Q^+ \Gamma^+(\tau)] d\tau. \tag{5.46}$$

$$R_T^+ = \int_0^T \left(R^+ + \left(2(L^+)^* + \left(\Gamma^+(\tau) \right)^* Q^+ \right) \Gamma^+(\tau) \right) d\tau \tag{5.47}$$

and $\Gamma^+(t) \triangleq \int_0^t S_W^+(\tau) B^+ d\tau$. Interpreting such a stabilization strategy in terms of the minimal energy problem, the stabilization is performed with the less effort (energy of the control signal) possible.

3. The latter problem (5.42)-(5.43) is a classical finite-dimensional digital control problem (see [48]). Since (A_W^+, B) and $(\sqrt{Q^+}, A_W^+)$ have been assumed exponentially stabilizable and exponentially detectable, then there exists a sufficiently small sampling step T_0 such that for all $T < T_0$ the pairs (Φ^+, Γ^+) and $\left(\sqrt{Q_T^+ - L_T^+ \left(R_T^+ \right)^{-1} L_T^{+*}}, \Phi^+ \right)$ are power stabilizable and power detectable, respectively. The optimal stabilizing feedback matrix is given by

$$F_n^+ \triangleq -(R_T^+ + \left(\Gamma^+ \right)^* X^+ \Gamma^+)^{-1} ((L_T^+)^* + \left(\Gamma^+ \right)^* X^+ \Phi^+), \tag{5.48}$$

where $X^+ = (X^+)^* \in \mathbb{R}^{n \times n}$ is the unique positive definite stabilizing solution to the following algebraic Riccati equation

$$\Phi^{+*} X^+ \Phi^+ - X^+ - (\Phi^{+*} X^+ \Gamma^+ + L_T^+)(R_T^+ + \Gamma^{+*} X^+ \Gamma^+)^{-1}(L_T^{+*} + \Gamma^{+*} X^+ \Phi^+) + Q_T^+ = 0. \tag{5.49}$$

For a digital prestabilizing feedback of the form (5.41) with F_n^+ given by (5.48), the quadratic cost function (5.43) is minimized, its minimal value is $\langle x_0^+, X^+ x_0^+ \rangle$ and this is, actually, the best achievable performance for the digital control problem associated with the antistable part of the Pritchard-Salamon-Popov triple $\Sigma(S^{\mathcal{V}}(\cdot), B, M)$. For the proof of the results stated above the reader is referred to [48].

5.2.1   Digital sub-optimal prestabilization

A sub-optimal alternative solution to the digital optimal prestabilization problem is obtained by applying the early results of Halanay and Rasvan [48]. The idea is that the optimal stabilizing solution to the discrete-time Riccati equation (5.49) enjoys some nice asymptotic properties, i.e. if regarded as a function of the sampling step T, it can be expressed as

$$X^+(T) = \hat{X}_c^+ + \hat{Y}T^2 + \theta(T^2),\tag{5.50}$$

where $\theta(T)$ is the symbol of Landau having the property that

$$\lim_{T \longrightarrow 0} \frac{\theta T}{T} = 0$$

and where $\hat{X}_c^+$ is the unique stabilizing solution to the continuous-time Riccati equation (under corresponding assumptions of stabilizability of $(A_{\mathcal{W}}^+, B^+)$ and detectability of $(\sqrt{Q^+}, A_{\mathcal{W}}^+)$)

$$A^{+*}\hat{X}^+ + \hat{X}^+A^+ + (L^+ + B^+\hat{X}^+)\left(R^+\right)^{-1}(L^{+*} + B^{+*}\hat{X}^+) + Q^+ = 0\tag{5.51}$$

and $\hat{Y}$ is some appropriate chosen positive definite matrix (see [48] for details). The degree of suboptimality of the digital prestabilizing feedback $\hat{F}^+ = \left(\begin{array}{cc} \hat{F}_n^+ & 0 \end{array}\right)$ with $\hat{F}^+$ satisfying (3.15) is of the order of T^2. Notice that the following operator

$$\tilde{F}^+ \triangleq \left(\begin{array}{cc} \tilde{F}_n^+ & 0 \end{array}\right),\tag{5.52}$$

with $\tilde{F}_n^+ \triangleq -(R^+)^{-1}(L^{+*} + B^{+*}\hat{X})$ represents a suitable candidate to continuous-time prestabilization of the Pritchard-Salamon-Popov triple since it satisfies the boundedness condition $F^+ \in \mathcal{L}(\mathcal{V}, \mathbb{R}^p)$.

The following remark proves its importance in the case when the C_0-semigroup $S(\cdot)$ is diagonal

Remark 5.13 *By Theorem 5.10 a digital state-feedback of the form*

$$u_{\text{step}}(t) = \left\{ \begin{array}{ll} \hat{F}^+x(k), & kT \le t < (k+1)T, \quad k \ge 0 \text{ (optimality)} , \\ \tilde{F}^+x(k), & kT \le t < (k+1)T, \quad k \ge 0 \text{ (suboptimality)} , \end{array}\right.\tag{5.53}$$

provides digital exponential stability of the Pritchard-Salamon-Popov triple $\Sigma(S(\cdot), B, M)$. This fact implies that the dynamics operator of the time-discretized initial value problem (2.19) has the following form

$$\Phi = \left\{ \begin{array}{l} \left(\begin{array}{cc} \Phi^+ + \Gamma^+\hat{F}^+ & 0 \\ \Gamma^-\hat{F}^+ & \Phi^- \end{array}\right) \text{ (optimality)} , \\ \left(\begin{array}{cc} \Phi^+ + \Gamma^+\tilde{F}^+ & 0 \\ \Gamma^-\tilde{F}^+ & \Phi^- \end{array}\right) \text{ (suboptimality)} , \end{array}\right.\tag{5.54}$$

Let now $\mathcal{J} = \begin{pmatrix} J & 0 \\ 0 & I_- \end{pmatrix}$ *such that* $J \in \mathbb{R}^{n \times n}$ *is invertible. Then the operator* $\mathcal{J}$ *is boundedly invertible and* $\mathcal{J}^{-1} = \begin{pmatrix} J^{-1} & 0 \\ 0 & I_- \end{pmatrix}$. *Consequently,*

$$
\mathcal{J}\Phi\mathcal{J}^{-1} = \begin{cases} \begin{pmatrix} J(\Phi^+ + \Gamma^+\hat{F}^+)^{-1} & 0 \\ \Gamma^-\hat{F}^+ & \Phi^- \end{pmatrix} & \textit{(optimality)}, \\[3mm] \begin{pmatrix} J(\Phi^+ + \Gamma^+\tilde{F}^+)J^{-1} & 0 \\ \Gamma^-\tilde{F}^+ & \Phi^- \end{pmatrix} & \textit{(suboptimality)}. \end{cases} \tag{5.55}
$$

If J is the nonsingular matrix which brings $\Phi^+ + \Gamma^+\hat{F}^+$ to Jordan canonical form, we conclude the following: if $S^{\mathcal{V}}(\cdot)$ is a diagonal semigroup, by digital prestabilization it becomes quasi-diagonal, i.e. with an infinite-dimensional diagonal part and a Jordan canonical finite-dimensional one. However, since the eigenvalues of $J(\Phi^+ + \Gamma^+\hat{F}^+)J^{-1}$ depend analytic on the matrix coefficients, we infer that there exists a perturbation of the stabilizing solution $X^+(\epsilon)$ to the discrete-time Riccati equation (5.49) such that under the feedback $F^+(\epsilon)$ the dynamics operator of the time-discretized initial value problem (2.19) remains power stable and it can be diagonalized without incurring to much loss of performance, i.e. for small $\epsilon > 0$ we have

$$
J_\Sigma(x_0^+, u(\cdot)) \approx J_\Sigma(x_0^+, u(\cdot), \epsilon).
$$

A similar fact can be obtained by performing a continuous-time prestabilization via $\tilde{F}^+(\epsilon)$ generated by $\hat{X}^+(\epsilon)$, an arbitrary small perturbation of the solution to the continuous time Riccati equation (5.51) followed by discretization. Hence, in the example treated in chapter 7, section 7.1.1, we shall assume that the C_0-semigroup is diagonal and exponentially stable on $\mathcal{W}$.

Let us consider now the cost function (5.40) for a particular choice of the penalty matrices as they appear in the minimum energy problem, i.e.

$$
Q^+ = C^+(C^+)^*, \quad L^+ = 0 \text{ and } R^+ = I_{m \times m}.
$$

Since discretization preserves the values of the C and D matrices it is tempting to perform a digital prestabilization optimally with respect to the cost function associated with the minimum energy problem for the time-discretized system. Let us see what happens in this case. The following proposition holds [10]

Proposition 5.14 *Let $(A^+, B^+, C^+) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ and let $(\Phi^+, \Gamma^+, \Lambda^+) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ be its time-discretized counterpart. If (A^+, B^+, C^+) is stabilizable and detectable then there exists a sampling step T_0 such that for every $T \le T_0$ the discrete-time Riccati equation written for $(\Phi^+, \Gamma^+, \Lambda^+)$*

$$
\Phi^{+*}X^+\Phi^+ - X^+ - \Phi^{+*}X^+\Gamma^+(I_{m\times m}^+ + \Gamma^{+*}X^+\Gamma^+)^{-1}\Gamma^{+*}X^+\Phi^+ + \Lambda^{+*}\Lambda = 0 \tag{5.56}
$$

has a stabilizing solution X_d^+ which satisfies

$$\lim_{T \to 0} T X_d^+ = X_c^+, \tag{5.57}$$

with X_c^+ the unique positive definite solution to the continuous-time Riccati equation

$$A^{+*}X^+ + X^+A^+ + X^+B^+B^{+*}X^+ + C^{+*}C^+ = 0. \tag{5.58}$$

Proof Let X_c^+ be the unique positive definite solution to the continuous-time Riccati equation (5.58) and let F_c^+ be the optimal feedback. The existence of X_c^+ is assured by the stabilizability and detectability of (A^+, B^+, C^+). Using a Taylor series expansion one can easily prove the existence of a sampling step T_0 such that for every $T \le T_0$ the triple $(\mathbf{\Phi}^+, \mathbf{\Gamma}^+, \mathbf{\Lambda}^+)$ is power stabilizable and power detectable. This fact implies the existence of a stabilizing solution X_d^+. Let F_d^+ be the optimal feedback. Notice that the discrete-time Riccati equation (5.56) admits an equivalent representation in the form of the following system

$$\begin{aligned}
0 &= \mathbf{\Phi}^{+*}X^+\mathbf{\Phi}^+ - X^+ - F_d^{+*}(I_{m \times m} + \mathbf{\Gamma}^{+*}X^+\mathbf{\Gamma}^+)^{-1}F_d^+ + \mathbf{\Lambda}^{+*}\mathbf{\Lambda}, \\
0 &= \mathbf{\Gamma}^{+*}X^+\mathbf{\Phi}^+ + (I_{m \times m} + \mathbf{\Gamma}^{+*}X^+\mathbf{\Gamma}^+)F_d^+.
\end{aligned}$$

Consider the following functions

$$\begin{aligned}
\mathcal{F}_1 &: \quad (-\infty, 0) \cup (0, \infty) \times \mathbb{R}_S^{n \times n} \times \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}_S^{n \times n}, \\
\mathcal{F}_2 &: \quad (-\infty, 0) \cup (0, \infty) \times \mathbb{R}_S^{n \times n} \times \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}^{m \times n}, \\
\mathcal{F}_3 &: \quad R \times \mathbb{R}_S^{n \times n} \times \mathbb{R}^{m \times n} \longrightarrow \mathbb{R}_S^{n \times n} \times \mathbb{R}^{m \times n},
\end{aligned}$$

defined by

$$\begin{aligned}
\mathcal{F}_1(T, X^+, F^+) &= \frac{(\mathbf{\Phi}^{+*} - I_{n \times n})X^+(\mathbf{\Phi}^+ - I_{n \times n})}{T} + \frac{(\mathbf{\Phi}^{+*} - I_{n \times n})X^+}{T} + \frac{X^+(\mathbf{\Phi}^+ - I_{n \times n})}{T} - \\
&\quad - F^{+*}\left(I + \frac{I_{m \times m} + \mathbf{\Gamma}^{+*}X^+\mathbf{\Gamma}^+}{T}\right)F^+ + C^{+*}C^+, \tag{5.59}
\end{aligned}$$

$$\mathcal{F}_2(T, X^+, F^+) = \frac{\mathbf{\Gamma}^{+*}X^+\mathbf{\Phi}^+ + \left(I + \frac{I_{m \times m} + \mathbf{\Gamma}^{+*}X^+\mathbf{\Gamma}^+}{T}\right)F^+}{T}, \tag{5.60}$$

$$\mathcal{F}_3(T, X^+, F^+) = \begin{cases} \begin{pmatrix} \mathcal{F}_1(T, X^+, F^+) \\ \mathcal{F}_2(T, X^+, F^+) \end{pmatrix} & , T \ne 0 \\ \begin{pmatrix} A^{+*}X^+ + X^+A^+ + X^+B^+B^{+*}X^+ + C^{+*}C^+ \\ B^{+*}X^+ + F^+ \end{pmatrix} & , T = 0 \end{cases}, \tag{5.61}$$

where $\mathbb{R}_S^{n\times n}$ is the space of symmetric $n\times n$ matrices. Notice that if $\tilde{X}_d^+ \stackrel{\triangle}{=} TX_d^+$ then

$$\mathcal{F}_1(T,\tilde{X}_d^+,F_d^+) = 0,$$
$$\mathcal{F}_2(T,\tilde{X}_d^+,F_d^+) = 0,$$
$$\mathcal{F}_3(T,\tilde{X}_d^+,F_d^+) = 0.$$

Notice also that $\mathcal{F}_3(T,X^+,F^+)$ is continuous with respect to T in the origin and, further-more, it is analytical as well. Clearly, it is differentiable with respect to X^+ and F^+. Notice that $\mathcal{F}_3(0,X_c^+,F_c^+)=0$ by definition of $\mathcal{F}_3$. Let us show that the differential of $\mathcal{F}_3$ with respect to X^+ and F^+ is nonsingular in $(0,X_c^+,F_c^+)$. For this consider arbitrary matrices $M\in R_S^{n\times n}$ and $N\in\mathbb{R}^{m\times n}$ and an arbitrary $\epsilon>0$. Then

$$\mathcal{F}_3(0,X_c^+ + \epsilon M + F_c^+ + \epsilon N) = \mathcal{F}_3(0,X_c^+,F_c^+) + \epsilon\begin{pmatrix} A^{+*}M + MA^* - N^*F_c^+ - F_c^{+*}N \\ B^{+*}M + N \end{pmatrix} +$$

$$+\ \epsilon^2\begin{pmatrix} -NN^* \\ 0 \end{pmatrix}. \tag{5.62}$$

Hence,

$$\frac{\partial\mathcal{F}_3(0,X_c^+,F_c^+)}{\partial(X^+,F^+)}(M,N) = \lim_{\epsilon\longrightarrow 0}\frac{(\mathcal{F}_3(0,X_c^+ + \epsilon M + F_c^+ + \epsilon N) - \mathcal{F}_3(0,X_c^+,F_c^+))}{\epsilon} =$$

$$= \begin{pmatrix} A^{+*}M + MA^* - N^*F_c^+ - F_c^{+*}N \\ B^{+*}M + N \end{pmatrix}. \tag{5.63}$$

In order to show that the linear map defined by (5.63) is nonsingular, it is sufficient to show that it is injective. Let us assume that $(M=M^*,N)$ is in its kernel. Then

$$A^{+*}M + MA^* - N^*F_c^+ - F_c^{+*}N = 0, \tag{5.64}$$
$$B^{+*}M + N = 0 \tag{5.65}$$

and it follows that

$$(A^+ + B^+F_c^+)^*M + M(A^+ + B^+F_c^+) = 0. \tag{5.66}$$

As $(A^+ + B^+F_c^+)$ is stable, by a Lyapunov function argument we conclude that the unique solution to (5.66) is $M=0$. This fact implies $N=0$ which means that the kernel is null and hence we have the injectivity of the mapping (5.63). Applying now the Implicit Function Theorem we get that there exist analytic functions

$$\tilde{X}_d^+ \ :\ \mathcal{I}\longrightarrow R_S^{n\times n}, \tag{5.67}$$
$$F_d^+ \ :\ \mathcal{I}\longrightarrow \mathbb{R}^{m\times n}, \tag{5.68}$$

where $\mathcal{I}$ is a neighbourhood of the origin, and such that

$$\tilde{X}_d^+(0) = X_c^+, \tag{5.69}$$
$$F_d^+(0) = F_c^+, \tag{5.70}$$
$$\mathcal{F}_3(T,\tilde{X}_d^+(T),F_d^+(T)) = 0, \tag{5.71}$$

for every $T \in \mathcal{I}$. The latter relation implies that $T X_d^+(T)$ is a solution to the discrete-time Riccati equation (5.56) and F_d^+ approaches F_c^+ as T approaches 0. It follows that

$$\lim_{T \to 0} \frac{\Phi^+ + \Gamma^+ F_d^+ - I_{n \times n}}{T} = A^+ + B^+ F_c^+.$$

We conclude that for a sufficiently small sampling step, say T_0, the eigenvalues of $\Phi^+ + \Gamma^+ F_d^+$ are in the open unit disc and $X_d^+(T)$ is nothing else than the stabilizing solution to (5.56) for which

$$\lim_{T \to 0} (T X_d^+(T)) = \check{X}_d^+(0) = X_c^+$$

obviously holds and the proof is complete.

■

An immediate consequence of Proposition 5.14 is represented by the following fact: let

$$J_\Sigma^c(x_0^+, u(\cdot)) = \langle X_c^+ x_0^+, x_0^+ \rangle, \tag{5.72}$$

$$J_\Sigma^{d1}(x_0^+, u(\cdot)) = \langle X_{c1}^+ x_0^+, x_0^+ \rangle, \tag{5.73}$$

$$J_\Sigma^{d2}(x_0^+, u(\cdot)) = \langle X_{d2}^+ x_0^+, x_0^+ \rangle, \tag{5.74}$$

denote the the optimal values of the cost functions associated with the continuous time LQ-optimal control problem, the digital LQ-optimal control problem and the digital LQ-suboptimal control problem from Proposition 5.14, all considered for the antistable part (A^+, B^+, C^+). On the basis of the asymptotic properties of the solutions to the discrete-time Riccati equations (5.49) and (5.56) we can represent graphically the dependence on the sampling step of (5.73) and (5.74). This is depicted in Figure 5.2
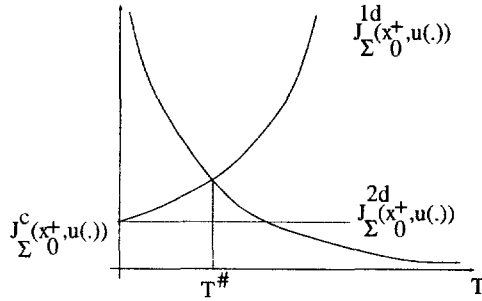
Figure 5.2: The sampling step dependence of the suboptimal values of the cost functions

The two graphs $(T, J_\Sigma^{d1}(x_0^+, u(\cdot)) = \langle X_{d1}^+ x_0^+, x_0^+ \rangle)$ and $(T, J_\Sigma^{d2}(x_0^+, u(\cdot)) = \langle X_{d1}^+ x_0^+, x_0^+ \rangle)$ have a cross point $T^{\#}$ which is computable in terms of the optimal value of the continuous time cost function by

$$J_\Sigma^c(x_0^+, u(\cdot)) + T^{\#2} J_\Sigma^{\check{Y},c}(x_0^+, u(\cdot)) \approx \frac{J_\Sigma^c(x_0^+, u(\cdot))}{T^{\#}}, \quad J_\Sigma^{\check{Y},c}(x_0^+, u(\cdot)) = \langle \check{Y} x_0^+, x_0^+ \rangle, \tag{5.75}$$

with $\hat{Y}$ defined via (5.50). If the sampling step is of the value of $T^\#$ then the loss of optimality of digital prestabilization based on the solution to the Riccati equation (5.56) is very small. This approach presents the advantage of a simpler form of the discrete-time Riccati equation (5.56). If the sampling step does not belong to a sufficient small neighbourhood of $T^\#$ then the loss of optimality becomes significant.

5.2.2   Infinite-dimensional considerations on asymptotic properties to digital Riccati equations

Let us end this section by analyzing the infinite-dimensional counterpart of the results on asymptotic properties with respect to sampling of the solutions to digital Riccati equations as presented in the previous section. Assume that one wants to write a similar expansion to (5.50) for the solution to the discrete-time Riccati equation written for the infinite dimensional discrete-time system $(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, \mathbf{\Theta})$. In [73], Rosen and Wang have proved for the class of distributed parameter systems with bounded time-varying operators that the solutions to the discrete-time Riccati equation associated with the digital LQ-optimal control problem converge strongly to the solutions of the continuous time LQ-optimal control problem. Even if an extension of those results to systems with unboundedness would be neither trivial nor uninteresting, we shall not be concerned with developing such an extension in this thesis. Instead of that let us focus our attention on the expansion (5.50) for the solution to infinite-dimensional case. In the finite-dimensional case, the central role in proving asymptotic properties of those solutions was played by the analyticity of those solutions with respect to the sampling step in a certain neighbourhood of the origin. This fact obviously do not hold in general for infinite dimensional systems since the strongly continuous semigroup $S^\nu(\cdot)$ is unbounded for negative values of the time. It remains an open question under which minimal assumption made on the system those results can be extended in infinite dimensions. We conjecture that they hold for analytic semigroups, but such a result is not of practical importance since most of the infinite-dimensional systems does not occur in this way. Another tricky aspect is that, technically speaking, for infinite dimensional systems with group instead of semigroup, we are able to write down a construction and, eventually, to use a Implicit Function Theorem argument for extending the results, but, unfortunately, for this class of systems, stabilizability is available only in the strong sense, and, hence, there are no Riccati equations involved in this theory.

5.3   Hybrid stability for Pritchard-Salamon systems

The main goal of this section is to introduce the reader the concept of hybrid stability. Several properties of hybrid stable Pritchard-Salamon systems will be explored. Since the hybrid stability is basically a stability of input/output type, we shall proceed the developments of this section by a subsection in which we collect several results on input/output

stability.

5.3.1 Input/output stability

Let $\Sigma_G(S(\cdot), B, C, D)$ be a Pritchard-Salamon system as introduces in Definition 2.7 and let $\Sigma_G(A, B, C, D)$ denote infinite-dimensional discrete-time system defined by (3.1). Let G denote the input/output operator

$$G : L_2^{loc}(0, \infty; \mathcal{U}) \longrightarrow L_2^{loc}(0, \infty; \mathcal{Y}), \quad (Gu)(\cdot) = C \int_0^{\cdot} S(\cdot - \tau) Bu(\tau) d\tau + Du(\cdot) \quad (5.76)$$

if the Pritchard-Salamon system is considered and, alternatively,

$$G : \ell_2(0, \infty; \mathcal{U}) \longrightarrow \ell_2(0, \infty; \mathcal{Y}), \quad (Gu)(\cdot) = \sum_{i=0}^{\cdot-1} C A^{\cdot-i-1} Bu(i) + Du(\cdot) \quad (5.77)$$

when $\Sigma_G(A, B, C, D)$ is considered. We shall call $\Sigma_G(S(\cdot), B, C, D)$ and $\Sigma_G(A, B, C, D)$ realizations of (5.76) and (5.77), respectively.

We consider the input/output maps as unbounded maps defined by

$$G : \mathcal{D}(G) \subset L_2(0, \infty; \mathcal{U}) \longrightarrow L_2(0, \infty; \mathcal{Y}) \quad (5.78)$$

and

$$G : \mathcal{D}(G) \subset \ell_2(0, \infty; \mathcal{U}) \longrightarrow \ell_2(0, \infty; \mathcal{Y}), \quad (5.79)$$

where

$$\mathcal{D}(G) \triangleq \{u(\cdot) \in L_2(0, \infty; \mathcal{U}) | (Gu)(\cdot) \in L_2(0, \infty; \mathcal{Y})\} \quad (5.80)$$

and

$$\mathcal{D}(G) \triangleq \{u(\cdot) \in \ell_2(0, \infty; \mathcal{U}) | (Gu)(\cdot) \in \ell_2(0, \infty; \mathcal{Y})\}, \quad (5.81)$$

respectively.

Definition 5.15 *(input/output stability)*
The Pritchard-Salamon system $\Sigma_G(S(\cdot), B, C, D)$ and the infinite-dimensional discrete-time system $\Sigma_G(A, B, C, D)$ are called input/output stable if

$$\mathcal{D}(G) = L_2(0, \infty; \mathcal{U})$$

for $\Sigma_G(S(\cdot), B, C, D)$ and

$$\mathcal{D}(G) = \ell_2(0, \infty; \mathcal{U})$$

for $\Sigma_G(A, B, C, D)$.

The following results reported from [80] and [62] give the equivalence between input/output stability and exponential (power) stability for $\Sigma_G(S(\cdot), B, C, D)$ ($\Sigma_G(A, B, C, D)$)

Lemma 5.16 *(Lemma 2.23 [80])*
Let $\Sigma_G(S(\cdot), B, C, D)$ be a Pritchard-Salamon system such that $(S(\cdot), B)$ and $(C, S(\cdot))$ are admissibly stabilizable and admissibly detectable. Then the system is input/output stable if and only if $S(\cdot)$ is exponentially stable on $\mathcal{W}$ and $\mathcal{V}$.

Lemma 5.17 *(Theorem 2 [62])*
Let $\Sigma_G(A, B, C, D)$ be an infinite-dimensional discrete-time system such that (A, B) and (C, A) are power stabilizable and power detectable. Then the system is input/output stable if and only if A is power stable.

Let us consider now the following Pritchard-Salamon system

$$\Sigma\left(S(\cdot), \left(\begin{array}{cc} B_1 & B_2 \end{array} \right), \left(\begin{array}{c} C_1 \\ C_2 \end{array} \right), \left(\begin{array}{cc} D_{11} & D_{12} \\ D_{21} & 0 \end{array} \right)\right) \tag{5.82}$$

with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$ and the following infinite-dimensional discrete-time system

$$\Sigma\left(A, \left(\begin{array}{cc} B_1 & B_2 \end{array} \right), \left(\begin{array}{c} C_1 \\ C_2 \end{array} \right), \left(\begin{array}{cc} D_{11} & D_{12} \\ D_{21} & 0 \end{array} \right)\right) \tag{5.83}$$

on $\mathcal{X}$ for which the following state-space description is defined

$$\Sigma_G \begin{cases} (\sigma x)(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t) \\ y_1(t) = C_1 x(t) + D_{12} u_2(t) \\ y_2(t) = C_2 x(t) + D_{21} u_1(t) \end{cases}, \tag{5.84}$$

where $(\sigma x)(t) \stackrel{\triangle}{=} \begin{cases} \dot{x}(t) & , \quad t \in \mathbb{R} \\ x(t+1) & , \quad t \in \mathbb{N} \end{cases}$ denote either the differential operator in continuous-time or the advance unit shift operator in the discrete-time case.

If $x_0 = 0$ then either (5.82) or (5.83) can be expressed in input/output fashion as

$$\left(\begin{array}{c} y_1(\cdot) \\ y_2(\cdot) \end{array} \right) = \left(\begin{array}{cc} G_{11} & G_{12} \\ G_{21} & G_{22} \end{array} \right) \left(\begin{array}{c} u_1(\cdot) \\ u_2(\cdot) \end{array} \right), \tag{5.85}$$

where

$$\begin{aligned} G_{11} &: \quad L_2^{loc}(0, \infty; \mathcal{U}_1) \mapsto L_2^{loc}(0, \infty; \mathcal{Y}_1), \\ (G_{11} u_1)(\cdot) &= C_1 \int_0^{\cdot} S^{\mathcal{V}}(\cdot - \tau) B_1 u_1(\tau) d\tau + D_{11} u_1(\cdot), \\ G_{12} &: \quad L_2^{loc}(0, \infty; \mathcal{U}_2) \mapsto L_2^{loc}(0, \infty; \mathcal{Y}_1), \end{aligned} \tag{5.86}$$

$$(G_{12}u_2)(\cdot) \;=\; C_1 \int_0^{\cdot} S^{\mathcal{V}}(\cdot - \tau) B_2 u_2(\tau) d\tau + D_{12} u_2(\cdot), \tag{5.87}$$

$$G_{21} \;\;:\;\; L_2^{loc}(0,\infty;\mathcal{U}_1) \mapsto L_2^{loc}(0,\infty;\mathcal{Y}_2),$$

$$(G_{21}u_1)(\cdot) \;=\; C_2 \int_0^{\cdot} S^{\mathcal{V}}(\cdot - \tau) B_1 u_1(\tau) d\tau + D_{21} u_1(\cdot), \tag{5.88}$$

$$G_{22} \;\;:\;\; L_2^{loc}(0,\infty;\mathcal{U}_2) \mapsto L_2^{loc}(0,\infty;\mathcal{Y}_2),$$

$$(G_{22}u_2)(\cdot) \;=\; C_2 \int_0^{\cdot} S^{\mathcal{V}}(\cdot - \tau) B_2 u_2(\tau) d\tau, \tag{5.89}$$

if the Pritchard-Salamon system (5.82) is considered, or

$$G_{11} \;\;:\;\; \ell_2(0,\infty;\mathcal{U}_1) \mapsto \ell_2(0,\infty;\mathcal{Y}_1),$$

$$(G_{11}u_1)(\cdot) \;=\; \sum_{i=0}^{\cdot-1} C_1 A^{\cdot-i-1} B_1 u_1(i) + D_{11} u_1(\cdot), \tag{5.90}$$

$$G_{12} \;\;:\;\; \ell_2(0,\infty;\mathcal{U}_2) \mapsto \ell_2(0,\infty;\mathcal{Y}_1),$$

$$(G_{12}u_2)(\cdot) \;=\; \sum_{i=0}^{\cdot-1} C_1 A^{\cdot-i-1} B_2 u_2(i) + D_{12} u_2(\cdot), \tag{5.91}$$

$$G_{21} \;\;:\;\; \ell_2(0,\infty;\mathcal{U}_1) \mapsto \ell_2(0,\infty;\mathcal{Y}_2),$$

$$(G_{21}u_1)(\cdot) \;=\; \sum_{i=0}^{\cdot-1} C_2 A^{\cdot-i-1} B_1 u_1(i) + D_{21} u_1(\cdot), \tag{5.92}$$

$$G_{22} \;\;:\;\; \ell_2(0,\infty;\mathcal{U}_2) \mapsto \ell_2(0,\infty;\mathcal{Y}_2),$$

$$(G_{22}u_2)(\cdot) \;=\; \sum_{i=0}^{\cdot-1} C_2 A^{\cdot-i-1} B_2 u_2(i), \tag{5.93}$$

if the infinite-dimensional discrete-time system (5.83) is considered.

Consider now the controllers, $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ which is a Pritchard-Salamon system with respect to $\mathcal{W}_c \hookrightarrow \mathcal{V}_c$ and the discrete-time infinite-dimensional system $\Sigma_K(A_c, B_c, C_c, D_c)$ on $\mathcal{X}_c$ satisfying the set of equations

$$\Sigma_K \begin{cases} (\sigma x_c)(t) = A_c x_c(t) + B_c y_2(t) \\ u_2(t) = C_c x_c(t) + D_c y_2(t) \end{cases} \tag{5.94}$$

and assume that they are connected in closed loop with (5.82) and (5.83), respectively.

For $x(0) = 0$, $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ and $\Sigma_K(A_c, B_c, C_c, D_c)$ are represented in an input output fashion as

$$G_c : L_2^{loc}(0,\infty;\mathcal{Y}_2) \longrightarrow L_2^{loc}(0,\infty;\mathcal{U}_2), (G_c y_2)(\cdot) = C_c \int_0^{\cdot} S_c(\cdot - \tau) B_c y_2(\tau) + D_c y_2(\cdot) \tag{5.95}$$

for the Pritchard-Salamon system $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ and

$$G_c : \ell_2(0,\infty;\mathcal{Y}_2) \longrightarrow \ell_2(0,\infty;\mathcal{U}_2), (G_c y_2)(\cdot) = \sum_{i=0}^{\cdot-1} C_c A_c^{\cdot-i-1} B_c y_2(i) + D_c y_2(\cdot) \tag{5.96}$$

for the discrete-time system $\Sigma_K(A_c, B_c, C_c, D_c)$.

Associated with the control set-up depicted in Figure 1.3 is the following configuration, we shall call the *extended control loop configuration*
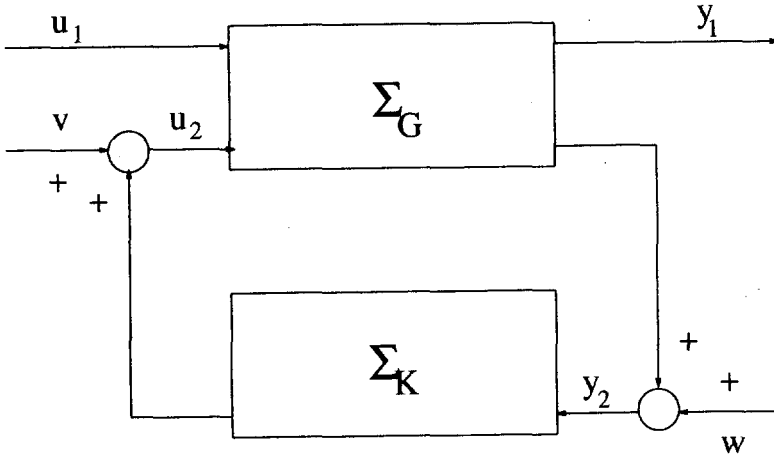

Figure 5.3: The extended closed loop configuration

We consider two additional signals

$$v \in L_2(0, \infty; \mathcal{U}_2), \quad w \in L_2(0, \infty; \mathcal{Y}_2),$$

for the extended closed loop configuration associated with the Pritchard-Salamon system (5.82), and

$$v \in \ell_2(0, \infty; \mathcal{U}_2), \quad w \in \ell_2(0, \infty; \mathcal{Y}_2),$$

when the it is associated with (5.83). Consider the mapping

$$\begin{pmatrix} u_1 \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ u_2 \\ y_2 \end{pmatrix} \tag{5.97}$$

defined from

$$L_2(0, \infty; \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2)$$

to

$$L_2(0, \infty; \mathcal{Y}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2)$$

in the Pritchard-Salamon system case and its discrete-time counterpart defined from

$$\ell_2(0, \infty; \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2)$$

to

$$\ell_2\left(0, \infty; \mathcal{Y}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2\right),$$

when the system (5.83) is considered.

From Figure 5.3 we derive the following set of equations

$$
\begin{align}
y_1 &= G_{11}u_1 + G_{12}u_2, \tag{5.98}\\
y_2 &= G_{21}u_1 + G_{22}u_2 + w, \tag{5.99}\\
u_2 &= G_c y_2 + v. \tag{5.100}
\end{align}
$$

Then the equation relating $\begin{pmatrix} u_1 \\ v \\ w \end{pmatrix}$ and $\begin{pmatrix} y_1 \\ u_2 \\ y_2 \end{pmatrix}$ is

$$
\begin{pmatrix} I & -G_{12} & 0 \\ 0 & I & -G_c \\ 0 & -G_{22} & I \end{pmatrix} \begin{pmatrix} y_1 \\ u_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ G_{21} & 0 & I \end{pmatrix} \begin{pmatrix} u_1 \\ v \\ w \end{pmatrix}. \tag{5.101}
$$

We introduce here the following

Definition 5.18 *(closed loop stability)*

(i) *We say that* $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ *is a stabilizing controller for (5.82) if*

$$
\begin{pmatrix} I & -G_{12} & 0 \\ 0 & I & -G_c \\ 0 & -G_{22} & I \end{pmatrix} : (L_2(0, \infty; \mathcal{Y}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow (L_2(0, \infty; \mathcal{Y}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2) \tag{5.102}
$$

is boundedly invertible.

(ii) *We say that* $\Sigma_K(A_c, B_c, C_c, D_c)$ *is a stabilizing controller for (5.83) if*

$$
\begin{pmatrix} I & -G_{12} & 0 \\ 0 & I & -G_c \\ 0 & -G_{22} & I \end{pmatrix} : (\ell_2(0, \infty; \mathcal{Y}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow (\ell_2(0, \infty; \mathcal{Y}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2) \tag{5.103}
$$

is boundedly invertible.

Remark 5.19 *The following hold*

(i) *Notice that a sufficient condition for (5.102) and (5.103) to be boundedly invertible is that* G_{22} *is boundedly invertible.*

(*ii*) *The motivation why Definition 5.18 is appropriate in the framework of input/output*
stability given in this subsection is the following: assume that (5.102) and (5.103)
are boundedly invertible and assume that $I - G_{22}G_c$ *is boundedly invertible. Define*

$$F_c \triangleq (I - G_{22}G_c)^{-1}.$$

Then the input/output operator associated with the extended closed loop configuration
is defined by

$$\begin{pmatrix} y_1 \\ u_2 \\ y_2 \end{pmatrix} \triangleq G^e \begin{pmatrix} u_1 \\ v \\ w \end{pmatrix}, \tag{5.104}$$

where

$$G^e \triangleq \begin{pmatrix} G_{11} + G_{12}G_cF_cG_{21} & G_{12}G_cF_cG_{22} + I & G_{12}G_cF_c \\ G_cF_cG_{21} & I + G_cF_cG_{22} & G_cF_c \\ F_cG_{21} & F_cG_{22} & F_c \end{pmatrix}.$$

One can notice immediately that the input output operator from $u_1(\cdot)$ *to* $y_1(\cdot)$ *is*
defined by

$$G_R \triangleq G_{11} + G_{12}G_c(I - G_{22}G_c)^{-1}G_{21}. \tag{5.105}$$

Since the input/output operator G^e *is well defined and bounded, it follows that* G_R
is also well defined and bounded from $L_2(0,\infty;\mathcal{U})$ *to* $L_2(0,\infty;\mathcal{Y})$, *if the Pritchard-*
Salamon system is considered and from $\ell_2(0,\infty;\mathcal{U})$ *to* $\ell_2(0,\infty;\mathcal{Y})$, *if the discrete-time*
system is considered. Notice hence, that the closed loop stability definition (5.18) is
general enough to imply the boundedness of G_R *which represents the stability require-*
ment of the $\mathcal{H}^\infty$ *control problem.*

(*iii*) *Let us notice finally that the time domain properties of input/output operators have*
their frequency domain counterparts, since by Paley-Wiener theorem L_2*-spaces and*
$\mathcal{H}^2$*-spaces are isomorphic under the Laplace transform (see Appendix A for the def-*
inition of the Hardy space $\mathcal{H}^2(\mathcal{Z})$*). Hence, if the transfer function* G *is well de-*
fined as the Laplace transform of G, *then a system is called input/output stable if*
$G \in \mathcal{H}^\infty(\mathcal{L}(\mathcal{U},\mathcal{Y}))$ *and furthermore*

$$\|G\|_{\mathcal{L}(L_2(0,\infty;\mathcal{U}),L_2(0,\infty;\mathcal{Y}))} = \|G(\cdot)\|_{\mathcal{H}^\infty(\mathcal{L}(\mathcal{U},\mathcal{Y}))}. \tag{5.106}$$

A particular set-up that we are finally considering in this section is the one depicted in
Figure 5.4
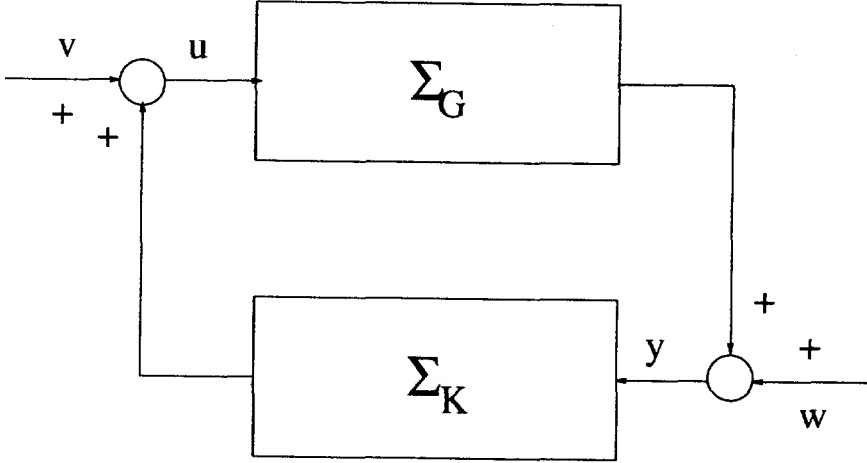
Figure 5.4: A particular a particular version of the extended closed loop configuration

Then the equation relating $\begin{pmatrix} v \\ w \end{pmatrix}$ and $\begin{pmatrix} u_2 \\ y_2 \end{pmatrix}$ is

$$\begin{pmatrix} I & -G_c \\ -G_{22} & I \end{pmatrix} \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix}. \tag{5.107}$$

Such a structure, occurring while setting $u_1 = 0$ in the extended closed loop control configuration in Figure 5.3 is called input/output stable if the operator

$$\begin{pmatrix} I & -G_c \\ -G_{22} & I \end{pmatrix} : L_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow L_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2),$$

for Pritchard-Salamon systems and

$$\begin{pmatrix} I & -G_c \\ -G_{22} & I \end{pmatrix} : \ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow \ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2),$$

for infinite-dimensional discrete-time systems are boundedly invertible. The following results holds

Lemma 5.20 $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ *is a stabilizing controller for the Pritchard-Salamon system (5.82) and* $\Sigma_K(A_c, B_c, C_c, D_c)$ *is a stabilizing controller for the infinite-dimensional discrete-time system (5.83) if* $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ *and* $\Sigma_K(A_c, B_c, C_c, D_c)$ *are stabilizing controllers for* $\Sigma_G(S(\cdot), B_2, C_2, 0)$ *and* $\Sigma_G(A, B_2, C_2, 0)$, *respectively.*

Proof Let us consider for the simplicity of the proof only the case of Pritchard-Salamon system. Since $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ stabilizes $\Sigma_G(S(\cdot), B_2, C_2, 0)$ it follows that

$$\begin{pmatrix} I & -G_c \\ -G_{22} & I \end{pmatrix} : L_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow L_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \tag{5.108}$$

is boundedly invertible. Then let us define the following partition on (5.102)

$$\begin{pmatrix} I & -G_{12} & 0 \\ 0 & I & -G_c \\ 0 & -G_{22} & I \end{pmatrix} = \begin{pmatrix} I \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} -G_{12} & 0 \\ I & -G_c \\ -G_{22} & I \end{pmatrix} \end{pmatrix}. \tag{5.109}$$

Since (5.108) is boundedly invertible it follows by direct computation and by exploiting the special upper-block triangular structure of (5.109) that

$$\begin{pmatrix} I & -G_{12} & 0 \\ 0 & I & -G_c \\ 0 & -G_{22} & I \end{pmatrix}^{-1} = \begin{pmatrix} I & \begin{pmatrix} -G_{12} & 0 \end{pmatrix}\begin{pmatrix} I & -G_c \\ -G_{22} & I \end{pmatrix}^{-1} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} I & -G_c \\ -G_{22} & I \end{pmatrix}^{-1} \end{pmatrix}. \tag{5.110}$$

Notice that it is sufficient to show that $\mathcal{D}(G_{12}) = L_2(0, \infty; \mathcal{U}_2)$ since this implies immediatedly that the operator

$$\begin{pmatrix} -G_{12} & 0 \end{pmatrix}\begin{pmatrix} I & -G_c \\ -G_{22} & I \end{pmatrix}^{-1} : L_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow L_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2$$

is bounded. The fact that $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ stabilizes $\Sigma_G(S(\cdot), B_2, C_2, 0)$ implies that the state function of $\Sigma_G(S(\cdot), B_2, C_2, 0)$ decays exponentially to zero as t tends to infinite. But the state function of $\Sigma_G(S(\cdot), B_2, C_1, D_{12})$ is the same as the one of $\Sigma_G(S(\cdot), B_2, C_2, 0)$ and hence, under controls from $L_2(0, \infty; \mathcal{U}_2)$ the output $y_1(\cdot)$ is element if $L_2(0, \infty; \mathcal{Y}_1)$ fact which shows that $\mathcal{D}(G_{12}) = L_2(0, \infty; \mathcal{U}_2)$ which completes the proof.

∎

5.3.2 Hybrid stability

The concept of hybrid stability was originally introduced for finite-dimensional systems in [29]. In this section we shall extend it to Pritchard-Salamon systems. Let us consider the smooth Pritchard-Salamon system

$$\Sigma\left(S(\cdot), \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix}\right) \tag{5.111}$$

with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$ and the discrete-time controller $\Sigma_K(A_c, B_c, C_c, D_c)$ interconnected in closed loop via a sampler and a zero-order holder as depicted in Figure 1.5.

Remark 5.21 *Notice that it is compulsory to have* $D_{21} = 0$. *Then, for* $x_0 \in \mathcal{W}$ *the solution to the measured output equation is given by*

$$y_2(t) \overset{\Delta}{=} C_2 S^{\mathcal{W}}(t)x_0 + \int_0^t C_2 S^{\mathcal{W}}(t - \tau)(B_1 u_1(\tau) + B_2 u_2(\tau))\,d\tau.$$

Assuming that $D_{21} \neq 0$, *it follows that the above holds only in* L_2-*sense*

$$y_2(\cdot) \overset{\Delta}{=} C_2 S^{\mathcal{W}}(\cdot)x_0 + \int_0^{\cdot} C_2 S^{\mathcal{W}}(\cdot - \tau)(B_1 u_1(\tau) + B_2 u_2(\tau))\,d\tau + D_{21} u_1(\cdot)$$

and hence, it cannot be sampled since the sampler is not well defined on L_2 *spaces. Standard engineering practice shows that if a low-pass filter is used prior to the sampler, then the modified sampler operator - original sampler + filter is well defined on* L_2-*spaces. However, since we do not want to modify the definition of the sampler, we shall assume for well posedness that* $D_{21} = 0$.

Since for a chosen sampling period T, the operators **S** and **H** are fixed, we shall move them around the loop and we include them into the plant obtaining the closed loop configuration depicted in Figure 5.5. The system $\Sigma_{G_{sd}}$ is the so called sampled-data system and $\tilde{y}_2 = \mathbf{S}y_2$ is the discretized measured output. By an abuse of notation we shall denote the control sequence representing the output of the controller by $\tilde{u}_2$.
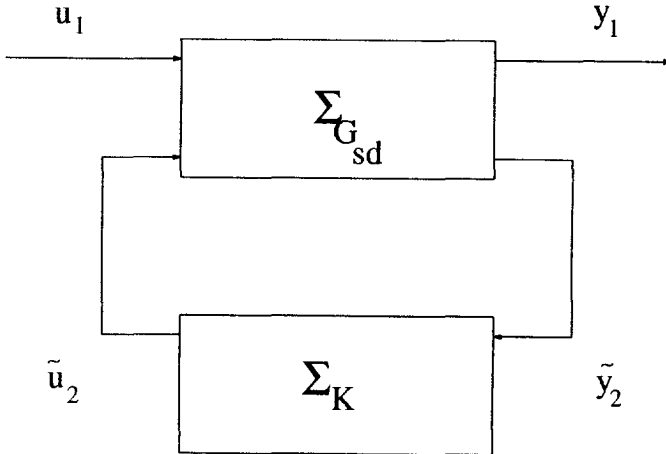
Figure 5.5: The modified digital control configuration

Let us consider

$$G \overset{\Delta}{=} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} : \mathcal{D}(G) \subset L_2(0, \infty; \mathcal{U}_1 \oplus \mathcal{U}_2) \longrightarrow L_2(0, \infty; \mathcal{Y}_1 \oplus \mathcal{Y}_2)$$

as an unbounded map with

$$\mathcal{D}(G) \triangleq \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in L_2(0,\infty;\mathcal{U}_1 \oplus \mathcal{U}_2) \mid \left(\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right)(\cdot) \in L_2(0,\infty;\mathcal{Y}_1 \oplus \mathcal{Y}_2) \right\}.$$

It represents the input/output operator associated with the Pritchard-Salamon system (5.111) and it maps

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We consider the input/output operator associated with $\Sigma_{G_{sd}}$

$$G_{sd} \triangleq \begin{pmatrix} G_{11} & G_{12}\mathbf{H} \\ \mathbf{S}G_{21} & \mathbf{S}G_{22}\mathbf{H} \end{pmatrix},$$

defined by

$$G_{sd} : \mathcal{D}(G_{sd}) \subset L_2(0,\infty;\mathcal{U}_1) \oplus \ell_2(0,\infty;\mathcal{U}_2) \longrightarrow L_2(0,\infty;\mathcal{Y}_1) \oplus \ell_2(0,\infty;\mathcal{Y}_2),$$

as an unbounded map with the domain given by

$$\mathcal{D}(G_{sd}) \triangleq \left\{ \begin{pmatrix} u_1 \\ \tilde{u}_2 \end{pmatrix} \in L_2(0,\infty;\mathcal{U}_1) \oplus \ell_2(0,\infty\mathcal{U}_2) \mid \right.$$
$$\left. \left(\begin{pmatrix} G_{11} & G_{12}\mathbf{H} \\ \mathbf{S}G_{21} & \mathbf{S}G_{22}\mathbf{H} \end{pmatrix} \begin{pmatrix} u_1 \\ \tilde{u}_2 \end{pmatrix} \right)(\cdot) \in L_2(0,\infty;\mathcal{Y}_1) \oplus \ell_2(0,\infty;\mathcal{Y}_2) \right\} \quad (5.112)$$

It maps

$$\begin{pmatrix} u_1 \\ \tilde{u}_2 \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \tilde{y}_2 \end{pmatrix}.$$

Similarly to the way we defined the extended closed loop configuration in the previous section, we shall two compatible exogenous signals $v \in \ell_2(0,\infty;\mathcal{U}_2)$ and $w \in \ell_2(0,\infty;\mathcal{Y}_2)$ and add them to the closed-loop system. We obtain the extended digital control configuration depicted in Figure 5.6.

Let us consider the mapping

$$\begin{pmatrix} u_1 \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \tilde{u}_2 \\ \tilde{y}_2 \end{pmatrix}$$

from $L_2(0,\infty;\mathcal{U}_1) \oplus \ell_2(0,\infty;\mathcal{Y}_2 \oplus \mathcal{U}_2))$ to $L_2(0,\infty;\mathcal{Y}_1) \oplus \ell_2(0,\infty;\mathcal{U}_2 \oplus \mathcal{Y}_2)$ described by the following equation

$$\begin{pmatrix} I & -G_{12}\mathbf{H} & 0 \\ 0 & I & -G_c \\ 0 & -\mathbf{S}G_{22}\mathbf{H} & I \end{pmatrix} \begin{pmatrix} y_1 \\ \tilde{u}_2 \\ \tilde{y}_2 \end{pmatrix} = \begin{pmatrix} G_{11} & 0 & 0 \\ 0 & I & 0 \\ \mathbf{S}G_{21} & 0 & I \end{pmatrix} \begin{pmatrix} u_1 \\ v \\ w \end{pmatrix}. \quad (5.113)$$
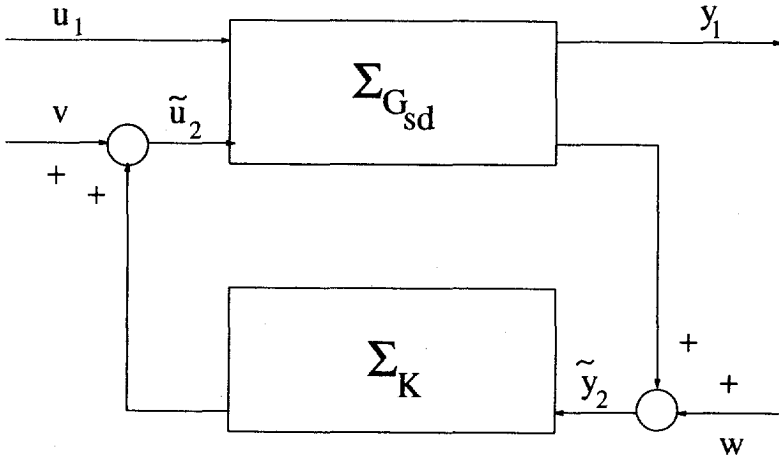
Figure 5.6: The extended digital control configuration

Definition 5.22 *(hybrid stability)*
The digital control closed loop system $\Sigma_{G_{sd}K}$ *is called hybrid stable the following operator*

$$\begin{pmatrix} I & -G_{12}\mathbf{H} & 0 \\ 0 & I & -G_c \\ 0 & -\mathbf{S}G_{22}\mathbf{H} & I \end{pmatrix} \ : \ (L_2(0,\infty;\mathcal{U}_1) \oplus \ell_2(0,\infty;\mathcal{Y}_2 \oplus \mathcal{U}_2)) \longrightarrow$$

$$(L_2(0,\infty;\mathcal{U}_1) \oplus \ell_2(0,\infty;\mathcal{Y}_2 \oplus \mathcal{U}_2)) \tag{5.114}$$

is boundedly invertible for a given sampling step $T > 0$.

Definition 5.23 *Let Σ_G defined by (5.111) be a smooth Pritchard-Salamon system with respect to $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$. We shall say that it is hybrid stabilizable if there exists a controller Σ_K such that the digital control system $\Sigma_{G_{sd}K}$ is hybrid stable.*

Remark 5.24 *The following hold*

(i) Let $T > 0$ be such that (5.114) holds. Then a sufficient condition for the boundedly invertibility of (5.114) is that $\mathbf{S}G_{22}\mathbf{H}$ is boundedly invertible.

(ii) Assume that $I - \mathbf{S}G_{22}\mathbf{H}G_c$ is boundedly invertible and let

$$F_c \triangleq (I - \mathbf{S}G_{22}\mathbf{H}G_c)^{-1}.$$

Then the input/output operator associated with the extended digital closed loop configuration is defined by

$$\begin{pmatrix} y_1 \\ \tilde{u}_2 \\ \tilde{y}_2 \end{pmatrix} \triangleq G^e_{sd} \begin{pmatrix} u_1 \\ v \\ w \end{pmatrix}, \tag{5.115}$$

where

$$G_{sd}^e \triangleq \begin{pmatrix} G_{11} + G_{12}HG_cF_cSG_{21} & G_{12}HG_cF_cSG_{22}H + I & G_{12}HG_cF_c \\ G_cF_cSG_{21} & I + G_cF_cSG_{22}H & G_cF_c \\ F_cSG_{21} & F_cSG_{22}H & F_c \end{pmatrix}.$$

Elementary computation shows that for such a $T > 0$ chosen as above, then the digital closed loop system input/output operator from $u_1(\cdot)$ to $y_1(\cdot)$ defined by

$$G_{R_{sd}} \triangleq G_{11} + G_{12}HG_c(I - SG_{22}HG_c)^{-1}SG_{21} \tag{5.116}$$

is bounded from $L_2(0, \infty; \mathcal{U}_1)$ to $L_2(0, \infty; \mathcal{Y}_1)$. The boundedness of $G_{R_{sd}}$ represents the stability condition for the digital $\mathcal{H}^\infty$ control problem. Notice hence, that the definition of hybrid stability (5.22) is enough general to cover the stability requirement from the digital $\mathcal{H}^\infty$ control problem that we shall address and solve in the next chapter.

(iii) Notice finally that the time domain properties of input/output operator $G_{R_{sd}}$ have no frequency domain counterparts, since the digital control system $\Sigma_{G_{sd}K}$ is no longer time invariant due to the periodic characteristic of the sampler.

5.3.3   Lifting the continuous-time system $\Sigma(S(\cdot), B, C, D)$

In this section we give the main results on lifting a continuous-time T-periodical system to a discrete-time time-invariant one. The idea is to rearrange the original periodic system such a way that its periodicity is reflected by shift invariance in the new set-up. Let us begin with defining lifting for signals. The idea is to represent a continuous time signal by a sequence with values in a function space. In Figure 5.7 we have represented schematically the original continuous time signal and its lifed counterpart

Let us define first the lifting operator. Let $\mathcal{Z}$ be a Banach space and let $L_2(0, \infty; \mathcal{Z})$ be the space of square integrable $\mathcal{Z}$-valued functions and $\ell_2(0, \infty; \mathcal{Z})$ the space of square summable $\mathcal{Z}$-valued sequences. Notice that $L_2(0, \infty; \mathcal{Z})$ and $\ell_2(0, \infty; \mathcal{Z})$ are Hilbert spaces with respect to the norms induced by the inner products. Let $T > 0$ be a fixed positive constant. Let

$$\Omega : L_2^{loc}(0, \infty; \mathcal{Z}) \longrightarrow \mathcal{S}(0, \infty; L_2(0, T; \mathcal{Z})), \quad \Omega\zeta = \hat{\zeta}, \tag{5.117}$$

where $\mathcal{S}(0, \infty; \mathcal{Z})$ is the space of $\mathcal{Z}$-valued sequences, $\hat{\zeta} = \{\hat{\zeta}_0, \hat{\zeta}_1, \cdots, \hat{\zeta}(k), \cdots\} = \{\hat{\zeta}(k)\}_{k \geq 0}$ and $\hat{\zeta}(k) \stackrel{\text{notation}}{=} \hat{\zeta}k(\cdot) = \zeta(kT + \cdot) \in L_2(0, T; \mathcal{X})$. It is easy to show (see [18, 92] for technical details of the proof) that Ω is a linear bijective isometry from $L_2^{loc}(0, \infty; \mathcal{Z})$ to $\mathcal{S}(0, \infty; L_2(0, T; \mathcal{Z}))$. Notice also that if $\zeta(\cdot)$ is a piece-wise constant function, $\zeta(t) =$
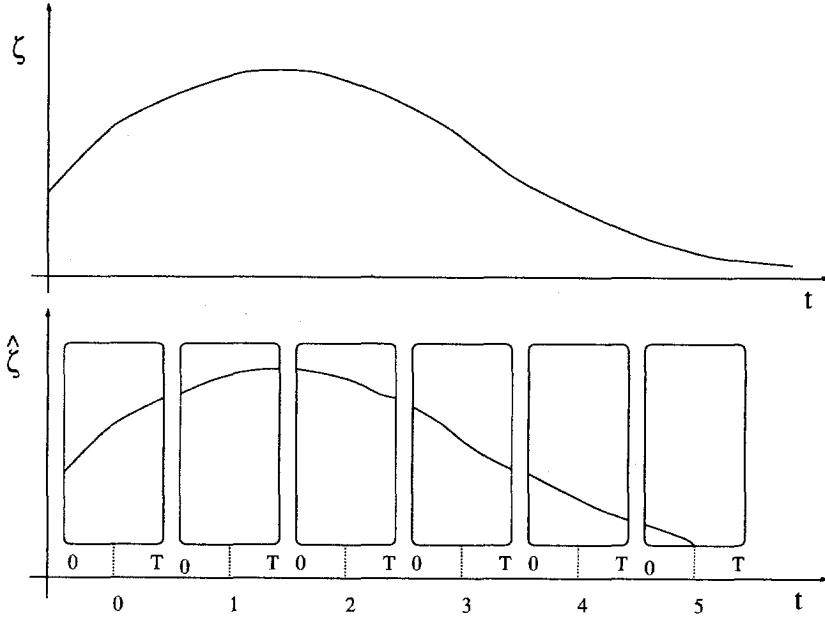
Figure 5.7: Lifting a $L_2(0, \infty; \mathcal{Z})$ signal to a $\ell_2(0, \infty; L_2(0, T, \mathcal{Z}))$ sequence

$\zeta(k)$, $t \in [kT, (k+1)T)$, then $\tilde{\zeta} = \{\zeta_0, \zeta_1, \cdots, \zeta(k) \cdots\}$ stands for the sampled of $\zeta(\cdot)$ and furthermore $\hat{\zeta} = \tilde{\zeta}$.

Let us focus on state space formulae for lifted systems. Consider the continuous-time system $\Sigma(S(\cdot), B, C, D)$ on a real separable Hilbert space $\mathcal{X}$ given by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.118}$$
$$y(t) = Cx(t) + Du(t), \tag{5.119}$$

where $S(\cdot)$ is a strongly continuous semigroup generated by A. Let $\mathcal{U}$ and $\mathcal{Y}$ be the real separable Hilbert spaces of the inputs and outputs respectively such that $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ and $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, respectively. The state equation can be reformulated as

$$x(kT + \tau) = S(\tau)x(kT) + \int_{kT}^{kT+\tau} S(kT + \tau - \eta)Bu(\eta)d\eta$$

for $\leq \tau < T$. For $\eta = kT + \theta$ equation (5.118) becomes

$$x(kT + \tau) = S(\tau)x(kT) + \int_0^\tau S(\tau - \theta)Bu(kT + \theta)d\theta.$$

Let $u(kT + \theta) \triangleq u_k(\theta)$. Then, at the sampling instants, the state equation (5.118) becomes

$$x(k + 1) = S(t)x(k) + \int_0^T S(\tau - \theta)Bu_k(\theta)d\theta. \tag{5.120}$$

Let then

$$\boldsymbol{\Phi} \; : \; \mathcal{X} \longrightarrow \mathcal{X}, \quad \boldsymbol{\Phi} = S(T), \tag{5.121}$$

$$\hat{\boldsymbol{\Gamma}} \; : \; L_2(0, T; \mathcal{U}) \longrightarrow \mathcal{X}, \quad \hat{\boldsymbol{\Gamma}}\hat{u}(k) = \int_0^T S(T - \tau)Bu_k(\tau)d\tau. \tag{5.122}$$

From (5.119) we finally obtain

$$\begin{aligned}
\hat{y}(k) &= CS(\cdot)x(k) + C\int_0^{\cdot} S(\cdot - \tau)Bu_k(\tau)d\tau + D\hat{u}(k) = \\
&= CS(\cdot)x(k) + C\int_0^{\cdot} S(\cdot - \tau)Bu_k(\tau)d\tau + D\hat{u}(k) = \\
&= \hat{\boldsymbol{\Lambda}}x(k) + \hat{\boldsymbol{\Theta}}\hat{u}(k),
\end{aligned} \tag{5.123}$$

where

$$\hat{\boldsymbol{\Lambda}} \; : \; \mathcal{X} \longrightarrow L_2(0, T, \mathcal{Y}), \quad \hat{\boldsymbol{\Lambda}}x(k) = CS(\cdot)x(k), \tag{5.124}$$

$$\hat{\boldsymbol{\Theta}} \; : \; L_2(0, T; \mathcal{U}) \longrightarrow L_2(0, T, \mathcal{Y}), \quad \hat{\boldsymbol{\Theta}}\hat{u}(k) = C\int_0^{\cdot} S(\cdot - \tau)Bu(k)(\tau)d\tau + D\hat{u}(k) \tag{5.125}$$

Then the original system admits at the sampling instants the following representation

$$x(k + 1) = \boldsymbol{\Phi}x(k) + \hat{\boldsymbol{\Gamma}}\hat{u}(k), \tag{5.126}$$

$$\hat{y}(k) = \hat{\boldsymbol{\Lambda}}x(k) + \hat{\boldsymbol{\Theta}}\hat{u}(k), \tag{5.127}$$

with $\boldsymbol{\Phi}$, $\hat{\boldsymbol{\Gamma}}$, $\hat{\boldsymbol{\Lambda}}$, $\hat{\boldsymbol{\Theta}}$ defined via (5.121), (5.122), (5.124), (5.125) and where $x \in \mathcal{S}(0, \infty; \mathcal{X})$, $\hat{u} \in \ell_2(0, \infty; L_2(0, T; \mathcal{U}))$ and $\hat{y} \in \mathcal{S}(0, \infty; L_2(0, T; \mathcal{Y}))$, respectively.

Remark 5.25 *The following issues deserve highlighting:*

(*i*) *Notice that the lifting operator (5.117) is defined on locally square integrable $\mathcal{Z}$-valued functions and its values are sequences of square integrable $\mathcal{Z}$-valued functions on $[0, T]$. However, if we restrict to square integrable $\mathcal{Z}$-valued functions then the lifted signal $\hat{\zeta}$ is a square summable sequence of square integrable $\mathcal{Z}$-valued functions on $[0, T]$.*

(*ii*) *Assume that in (5.118) the A-operator is the infinitesimal generator of a stable C_0-semigroup on $\mathcal{X}$. Then for any $u \in L_2(0, \infty; \mathcal{U})$ the state is also square integrable, i.e. $x \in L_2(0, \infty; \mathcal{X})$ fact which implies $y \in L_2(0, \infty; \mathcal{Y})$. Let $\hat{u}$ and $\hat{y}$ be the lifted of u and y. It is a routine to show that $\hat{u} \in \ell_2(0, \infty; L_2(0, T; \mathcal{U}))$ and $\hat{y} \in \ell_2(0, \infty; L_2(0, T; \mathcal{Y}))$*

(*iii*) *If we assume stability then the input output operator associated with the linear system (5.118) and (5.119) is a well defined and bounded map from $L_2(0, \infty; \mathcal{U})$ to $L_2(0, \infty; \mathcal{Y})$. It is well known (see [18]) that the lifted G is then*

$$\hat{G} = \Omega G \Omega^{-1} \in \mathcal{L}(\ell_2(0, \infty; L_2(0, T; \mathcal{U})), \ell_2(0, \infty; L_2(0, T; \mathcal{Y}))) \tag{5.128}$$

and furthermore $\|\hat{G}\| = \|G\|$.

5.3.4   Lifting the $\Sigma \left(S(\cdot), (B_1 B_2), (C_1, C_2)^*, (D_{11} D_{12}) \right)$ Pritchard- Salamon system

The theory developed in subsection 5.3.3 is now applied to lifting the following smooth Pritchard-Salamon system

$$\Sigma \left(S(\cdot), \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix} \right)$$

with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$. Assume that $S(\cdot)$ is exponential stable on $\mathcal{W}$ and consider

$$x_0 \in \mathcal{W}.$$

This fact implies that the measured output function is continuous with respect to the time t. We assume that u_2 is a constant function on $[kT(k+1)T)$, $\forall k \in \mathbb{N}$. Then the lifted $u_2(\cdot)$ is the same as its sampled $\tilde{u}_2$ and hence $\hat{u}_2 = \tilde{u}_2 \in L_2(0, \infty; \mathcal{U}_2)$.

Let us apply the lifting technique developed in section 5.3.3. The state equation written at the sampling instants becomes

$$x(k+1) = \Phi x(k) + \hat{\Gamma}_1 \hat{u}_1(k) + \Gamma_2 u_2(k), \tag{5.129}$$

with

$$\Phi \; : \; \mathcal{W} \longrightarrow \mathcal{W}, \quad \Phi \overset{\triangle}{=} S^{\mathcal{W}}(T), \tag{5.130}$$

$$\hat{\Gamma}_1 \; : \; L_2(0, T; \mathcal{U}_1) \longrightarrow \mathcal{W}, \quad \hat{\Gamma}_1 \hat{u}_1(k) \overset{\triangle}{=} \int_0^T S^{\mathcal{V}}(T - \tau) B_1 u_{1,k}(\tau) d\tau, \tag{5.131}$$

$$\Gamma_2 \; : \; \mathcal{U}_2 \longrightarrow \mathcal{W}, \quad \Gamma_2 u_2 \overset{\triangle}{=} \int_0^T S^{\mathcal{V}}(T - \tau) B_2(\tau) d\tau u_2. \tag{5.132}$$

The equation for the controlled output is now

$$\hat{y}_1(k) = \hat{\Lambda}_1 x(k) + \hat{\Theta}_{11} \hat{u}_1(k) + \hat{\Theta}_{12} u_2(k), \tag{5.133}$$

with

$$\hat{\Lambda}_1 \ : \ \mathcal{W} \longrightarrow L_2(0,T;\mathcal{Y}_1), \tag{5.134}$$

$$\hat{\Theta}_{11} \ : \ L_2(0,T;\mathcal{U}_1) \longrightarrow L_2(0,T;\mathcal{Y}_1), \tag{5.135}$$

$$\hat{\Theta}_{12} \ : \ \mathcal{U}_2 \longrightarrow L_2(0,T;\mathcal{Y}_1), \tag{5.136}$$

defined by

$$\hat{\Lambda}_1 \ \triangleq \ C_1 S^{\mathcal{W}}(\cdot), \tag{5.137}$$

$$\hat{\Theta}_{11}\hat{u}_1(k) \ \triangleq \ C_1 \int_0^\cdot S^{\mathcal{V}}(\cdot - \tau)B_1\hat{u}_{1,k}(\tau)d\tau, \tag{5.138}$$

$$\hat{\Theta}_{12}u_2(k) \ \triangleq \ C_1 \int_0^\cdot S^{\mathcal{V}}(\cdot - \tau)B_2 u_2(k)d\tau + D_{12}\hat{u}_2(k). \tag{5.139}$$

With this we have obtained the following discrete-time representation for the original system

$$x(k+1) \ = \ \Phi x(k) + \hat{\Gamma}_1\hat{u}_1(k) + \Gamma_2 u_2(k), \tag{5.140}$$

$$\hat{y}_1(k) \ = \ \hat{\Lambda}_1 x(k) + \hat{\Theta}_{11}\hat{u}_1(k) + \hat{\Theta}_{12}u_2(k), \tag{5.141}$$

$$y_2(k) \ = \ \Lambda_2 x(k), \tag{5.142}$$

where

$$\begin{cases} \hat{u}_1 \ \in \ \ell_2(0,\infty; L_2(0,T;\mathcal{U}_1)), \\ u_2 \ \in \ \ell_2(0,\infty;\mathcal{U}_2), \\ x \ \in \ \ell_2(0,\infty\mathcal{W}), \\ \hat{y}_1 \ \in \ \ell_2(0,\infty; L_2(0,\infty;\mathcal{Y}_1)), \\ y_2 \ \in \ \ell_2(0,\infty;\mathcal{Y}_2). \end{cases}$$

Corollary 5.26 *The operators in (5.140) and (5.141) satisfy the following boundedness conditions*

$$\begin{cases} \Phi \ \in \ \mathcal{L}(\mathcal{W}), \\ \hat{\Gamma}_1 \ \in \ \mathcal{L}(L_2(0,T;\mathcal{U}_1),\mathcal{W}), \\ \Gamma_2 \ \in \ \mathcal{L}(\mathcal{U}_2,\mathcal{W}), \\ \hat{\Lambda}_1 \ \in \ \mathcal{L}(\mathcal{W},L_2(0,T;\mathcal{Y}_1)), \\ \Lambda_2 \ \in \ \mathcal{L}(\mathcal{W},\mathcal{Y}_2), \\ \hat{\Theta}_{11} \ \in \ \mathcal{L}(L_2(0,T;\mathcal{U}_1),L_2(0,T;\mathcal{Y}_1)), \\ \hat{\Theta}_{12} \ \in \ \mathcal{L}(\mathcal{U}_2,L_2(0,T;\mathcal{Y}_1)). \end{cases}$$

Proof Notice first that the proof of the second item of Proposition 5.28 can be with minor modifications adjusted such that we have $\hat{\Gamma}_1 \in \mathcal{L}(L_2(0,T;\mathcal{U}_1),\mathcal{W})$. Applying the result proved in the above mentioned item to the pair $(S(\cdot),B_2)$ we also have the boundedness $\Gamma_2 \in \mathcal{L}(\mathcal{U}_2,\mathcal{W})$ which yields the result immediately by the definitions of $\hat{\Lambda}_1$, $\hat{\Theta}_{11}$, $\hat{\Theta}_{12}$. ∎

Remark 5.27 *Notice that after lifting the compensator input and output spaces remain unaltered, i.e. $\mathcal{Y}_2$ and $\mathcal{U}_2$. The lifting operator has modified only the space to which the new disturbance input, $\hat{u}_1$, and the new controlled output, $\hat{y}_1$ belong. They are now $L_2(0, T; \mathcal{U}_1)$ and $L_2(0, T; \mathcal{Y}_1)$, respectively. Notice also that only if finite-dimensionality of $\mathcal{U}_1$ and/or $\mathcal{Y}_1$ was originally assumed then, for the new problem, this fact does no longer hold, the new spaces becoming infinite-dimensional.*

5.3.5 The relationship with power stability

Let us give first the main result on hybrid stabilizability of Pritchard-Salamon systems by generalizing Theorem 1 from [29] for this class of systems.

Proposition 5.28 *Consider the smooth Pritchard-Salamon system (5.111) and assume that the following hold*

(i) *$(S(\cdot), B_2)$ is admissibly stabilizable,*

(ii) *$(C_2, S(\cdot))$ is admissibly detectable,*

(iii) *$\mathcal{U}_2 = \mathbb{R}^{m_2}$,*

(iv) *$\mathcal{Y}_2 = \mathbb{R}^{p_2}$.*

Then a digital controller $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ of sampling period T provides hybrid stability for (5.111) if it provides power stability for the discrete-time system from $\tilde{u}_2 \mapsto \tilde{y}_2$.

Proof Since $(S(\cdot), B_2)$ is admissibly stabilizable and $(C_2, S(\cdot))$ is admissibly detectable it follows that there exists a sufficiently small sampling period T_0 such that for any $T < T_0$, the time-discretized version of $\Sigma_G(S(\cdot), B_2, C_2, 0)$, denoted $\Sigma_G(\Phi, \Gamma_2, \Lambda_2, 0)$ is power stabilizable and power detectable on $\mathcal{W}$. This is a consequence of the third item of Theorem 5.10. Let $T > 0$ be such that the aforementioned implication holds and let $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ be a stabilizing compensator for $\Sigma_G(\Phi, \Gamma_2, \Lambda_2, 0)$. Let G_c denote the input/output operators associated with $\Sigma_K(K, L, M, N)$ and let $\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ be the input/output operator of (5.111). It follows that

$$\begin{pmatrix} I & -G_c \\ -SG_{22}\mathbf{H} & I \end{pmatrix} : \ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow \ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \qquad (5.143)$$

is boundedly invertible. Then we define the following partition

$$\begin{pmatrix} I & -G_{12}\mathbf{H} & 0 \\ 0 & I & -G_c \\ 0 & -SG_{22}\mathbf{H} & I \end{pmatrix} = \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} I & \begin{pmatrix} -G_{12}\mathbf{H} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} I & -G_c \\ -SG_{22}\mathbf{H} & I \end{pmatrix} \end{pmatrix}. \qquad (5.144)$$

Since (5.143) is boundedly invertible for the chosen $T > 0$, it follows by direct computation and by exploiting the special upper-block triangular structure of (5.144) that

$$
\begin{pmatrix} I & -G_{12}\mathbf{H} & 0 \\ 0 & I & -G_c \\ 0 & -SG_{22}\mathbf{H} & I \end{pmatrix}^{-1} = \begin{pmatrix} I & \left(-G_{12}\mathbf{H} \ \ 0 \right) \begin{pmatrix} I & -G_c \\ -SG_{22}\mathbf{H} & I \end{pmatrix}^{-1} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} I & -G_c \\ -SG_{22}\mathbf{H} & I \end{pmatrix}^{-1} \end{pmatrix} \tag{5.145}
$$

Notice that it is sufficient to show that

$$
\mathcal{D}(G_{12}) \supseteq \mathcal{PC}_T(0, \infty; \mathcal{U}_2)
$$

since this implies immediatedly that the operator

$$
\left(-G_{12}\mathbf{H} \ \ 0 \right) \begin{pmatrix} I & -G_c \\ -SG_{22}\mathbf{H} & I \end{pmatrix}^{-1} : \ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2) \longrightarrow \ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2
$$

is bounded. The fact that $\Sigma_K(S_c(\cdot), B_c, C_c, D_c)$ stabilizes $\Sigma_G(\mathbf{\Phi}, \mathbf{\Gamma}_2, \mathbf{\Lambda}_2, 0)$ implies that the state function of $\Sigma_G(\mathbf{\Phi}, \mathbf{\Gamma}_2, \mathbf{\Lambda}_2, 0)$, denoted $\tilde{x}$, which is the sampled version of the state function $x(\cdot)$ associated with the Pritchard-Salamon system $\Sigma_G(S(\cdot), B_2, C_2, 0)$, satisfies

$$
\lim_{k \longrightarrow \infty} \|x(k)\|_W \leq \rho_W^{k-1} \|x_0\|_W.
$$

This fact implies that

$$
\lim_{t \longrightarrow \infty} \|x(t)\|_W \leq M_W e^{\omega_W t} \|x_0\|_W.
$$

Since $\Sigma_G(S(\cdot), B_2, C_2, 0)$ and $\Sigma_G(S(\cdot), B_2, C_1, D_{12})$ have common state function it follows that for any entries from $L_2(0, \infty; \mathcal{U}_2)$ the controlled output $y_1(\cdot)$ is element of $L_2(0, \infty; \mathcal{Y}_1)$. Since $\mathcal{PC}_T(0, \infty; \mathcal{U}_2)$ is a dense subset of $L_2(0, \infty; \mathcal{U}_2)$ it follows that $y_1(\cdot)$ is element of $L_2(0, \infty; \mathcal{Y}_1)$ also for entries from $\mathcal{PC}_T(0, \infty; \mathcal{U}_2)$. But

$$
\mathcal{PC}_T(0, \infty; \mathcal{U}_2) = \mathbf{H}\ell_2(0, \infty; \mathcal{U}_2).
$$

We conclude then that $\mathcal{D}(G_{12}) \supseteq \mathcal{PC}_T(0, \infty; \mathcal{U}_2)$ and the proof is complete.

∎

   The digital control system, hybrid in nature, becomes after lifting a discrete-time one. The resulting closed loop system is then

$$
\begin{aligned}
x_R(k+1) &= \mathbf{\Phi}_R x_R(k) + \mathbf{\Gamma}_R u_R(k), & (5.146) \\
y_R(k) &= \mathbf{\Lambda}_R x_R(k) + \mathbf{\Theta}_R u_R(k), & (5.147)
\end{aligned}
$$

where

$$
\left\{
\begin{aligned}
x_R &= \begin{pmatrix} x \\ \xi \end{pmatrix}, \\
\Phi_R &= \begin{pmatrix} \Phi + \Gamma_2 D_c C_2 & \Gamma_2 C_c \\ B_c \Lambda_2 & A_c \end{pmatrix}, \\
\Gamma_R &= \begin{pmatrix} \hat{\Gamma}_1 \\ 0 \end{pmatrix}, \\
\Lambda_R &= \begin{pmatrix} \hat{\Lambda}_1 + \hat{\Theta}_{12} D_c \Lambda_2 & \hat{\Theta}_{12} C_c \end{pmatrix}, \\
\Theta_R &= \hat{\Theta}_{11}
\end{aligned}
\right.
\tag{5.148}
$$

It is rather straightforward that if $\mathcal{X}_R = \mathcal{W} \oplus \mathcal{K}$ stands for the augmented state space for the resultant closed loop system then the following boundedness conditions are satisfied

$$
\left\{
\begin{aligned}
\Phi_R &\in \mathcal{L}(\mathcal{X}_R), \\
\Gamma_R &\in \mathcal{L}(L_2(0,T;\mathcal{U}_1), \mathcal{X}_R), \\
\Lambda_R &\in \mathcal{L}(\mathcal{X}_R, L_2(0,T;\mathcal{Y}_1)), \\
\Theta_R &\in \mathcal{L}(L_2(0,T;\mathcal{U}_1), L_2(0,T;\mathcal{Y}_1)).
\end{aligned}
\right.
\tag{5.149}
$$

The following proposition establishes the equivalence between the hybrid stability of Pritchard-Salamon systems and the power stability of its lifted counterpart.

Proposition 5.29 *Let us consider smooth Pritchard-Salamon system (5.111) such that the hypothesis of Proposition 5.28 hold. Then the system (5.111) it is hybrid stabilizable if and only if the lifted closed-loop system dynamics operator*

$$
\Phi_R = \begin{pmatrix} \Phi + \Gamma_2 N \Lambda_2 & \Gamma_2 M \\ L \Lambda_2 & K \end{pmatrix}
$$

is power stable on $\mathcal{W}_R = \mathcal{W} \oplus \mathcal{K}$.

Proof
   "if":
   Suppose that Φ_R is power stable on $\mathcal{W}_R$. Then Σ_K is a stabilizing compensator for the lifted system

$$
\Sigma\left(\Phi, \begin{pmatrix} \hat{\Gamma}_1 & \Gamma_2 \end{pmatrix}, \begin{pmatrix} \hat{\Lambda}_1 \\ \Lambda_2 \end{pmatrix}, \begin{pmatrix} \hat{\Theta}_{11} & \hat{\Theta}_{12} \\ 0 & 0 \end{pmatrix}\right),
\tag{5.150}
$$

which has the input output operator defined by

$$
\hat{G}_{sd} \triangleq \begin{pmatrix} \Omega & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} G_{11} & G_{12}\mathbf{H} \\ \mathbf{S}G_{21} & \mathbf{S}G_{22}\mathbf{H} \end{pmatrix} \begin{pmatrix} \Omega^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} \Omega G_{11}\Omega^{-1} & \Omega G_{12}\mathbf{H} \\ \mathbf{S}G_{21}\Omega^{-1} & \mathbf{S}G_{22}\mathbf{H} \end{pmatrix}
$$

(see [18] for technical details about lifting $\mathbf{S}G$, $G\mathbf{H}$ and $\mathbf{S}G\mathbf{H}$ operators). It follows that Σ_K stabilizes the discrete-time system $\Sigma_G(\Phi, \Gamma_2, \Lambda_2, 0)$. This fat implies by Proposition 5.28 that the Pritchard-Salamon system (5.111) is hybrid stable.

"only if":

Let the Pritchard-Salamon system (5.111) be hybrid stable. It follows that the following mapping

$$\begin{pmatrix} u_1 \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ \tilde{u}_2 \\ \tilde{y}_2 \end{pmatrix}$$

is well defined and bounded from

$$(L_2(0, \infty; \mathcal{U}_1) \oplus (\ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2))$$

to

$$(L_2(0, \infty; \mathcal{Y}_1) \oplus (\ell_2(0, \infty; \mathcal{U}_2 \oplus \mathcal{Y}_2)).$$

This fact implies that the mapping

$$\begin{pmatrix} \hat{u}_1 \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \hat{y}_1 \\ \tilde{u}_2 \\ \tilde{y}_2 \end{pmatrix}$$

is well defined and bounded from

$$\ell_2\left(0, \infty; L_2(0, T; \mathcal{U}_1 \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2)\right)$$

to

$$\ell_2\left(0, \infty; L_2(0, T; \mathcal{Y}_1) \oplus \mathcal{U}_2 \oplus \mathcal{Y}_2\right)$$

This follows from the fact that the lifting operator is an isometry of spaces. We conclude that the discrete-time system (5.140) closed under output feedback via the controller (6.31) is input/output stable. But Lemma 5.17 shows that a power stabilizable and power detectable system the input/output stability and power-stability are equivalent. We conclude that the closed loop system is internally stable, i.e. Φ_R is power stable on $\mathcal{W}_R$.
∎

Having available results on lifting a Pritchard-Salamon system, we conclude this section by proving a result which shows that the lifting is endeed an appropriate tool for solving the digital $\mathcal{H}^\infty$ control problem. Clearly, the following holds

Proposition 5.30 *Let*

$$T_{y_1 u_1} : L_2^{\text{loc}}(0, \infty; \mathcal{U}_1) \longrightarrow L_2^{\text{loc}}(0, \infty; \mathcal{Y}_1)$$

be the input-output operator from u_1 to y_1 and let

$$T_{\hat{y}_1 \hat{u}_1} : \ell_2(0, \infty; L_2(0, T; \mathcal{U}_1)) \longrightarrow \ell_2(0, \infty; L_2(0, T; \mathcal{Y}_1))$$

be the input-output operator from $\hat{u}_1$ to $\hat{y}_1$. Then

$$\|T_{y_1 u_1}\| = \|T_{\hat{y}_1 \hat{u}_1}\|. \tag{5.151}$$

Proof Notice that the input-output operator from u_1 to y_1 can be expressed as a linear fractional transformation of the hybrid generalized plant and the controller

$$T_{y_1 u_1} = \mathcal{F}\left(\begin{pmatrix} G_{11} & G_{12}\mathbf{H} \\ \mathbf{S}G_{21} & \mathbf{S}G_{22}\mathbf{H} \end{pmatrix}, G_c\right) \tag{5.152}$$

and the input-output operator from $\hat{u}_1$ to $\hat{y}_1$ can be expressed as a linear fractional transformation of the lifted hybrid generalized plant and the controller

$$T_{\hat{y}_1 \hat{u}_1} = \mathcal{F}\left(\begin{pmatrix} \Omega G_{11}\Omega^{-1} & \Omega G_{12}\mathbf{H} \\ \mathbf{S}G_{21}\Omega^{-1} & \mathbf{S}G_{22}\mathbf{H} \end{pmatrix}, G_C\right). \tag{5.153}$$

Exploiting the fact that the lifting operator preserves the norms we get

$$\begin{aligned} \|T_{\hat{y}_1 \hat{u}_1}\| &= \left\|\Omega\left(G_{11} + G_{12}\mathbf{H}G_c(I - \mathbf{S}G_{22}\mathbf{H}G_c)^{-1}\mathbf{S}G_{21}\right)\Omega^{-1}\right\| \\ &= \left\|G_{11} + G_{12}\mathbf{H}G_c(I - \mathbf{S}G_{22}\mathbf{H}G_c)^{-1}\mathbf{S}G_{21}\right\| = \|T_{y_1 u_1}\|. \end{aligned} \tag{5.154}$$

∎

Chapter 6

The equivalent discrete-time control problem

Taking into account the prestabilization results proved in the previous chapter we can assume, without loss of generality that the Pritchard-Salamon-Popov triple $\Sigma(S^{\mathcal{V}}(\cdot), B, M)$ has the C_0-semigroup exponentially stable on $\mathcal{W}$. In this chapter we show how the digital control problem can be converted into an equivalent discrete-time control problem.

6.1 The equivalent discrete-time LQ-control problem

The following results holds

Lemma 6.1 *Let $\Sigma(S^{\mathcal{V}}(\cdot), B, M)$ be a smooth Pritchard-Salamon-Popov triple with respect to $(\mathcal{W} \hookrightarrow \mathcal{V}, \mathcal{U})$. Assume that $\mathcal{U} = \mathbb{R}^m$ and let $x_0 \in \mathcal{W}$. Let $\tilde{x} = \{x_0, x_1, \cdots, x_k = x(kT), \cdots\}$ be the sampled state function and let $u_{\text{step}} \in L_2(0, \infty; \mathcal{U})$ be a piece-wise constant control function $u_{\text{step}}(t) = \mathbf{H}u(k)$, $kT \le t < (k+1)T$. Then the smooth Pritchard-Salamon-Popov triple $\Sigma\left(S^{\mathcal{V}}(\cdot), B, M = \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} = M^*\right)$ defined by (2.18) and its associated initial value problem*

$$\dot{x}(t) = A^{\mathcal{W}}x(t) + Bu(t), \quad x(0) = x_0 \in \mathcal{W}$$

admits, at sampling instants $\{0, \cdots, kT, \cdots\}$, a discrete-time representation via the equivalent discrete-time Popov triple $\Sigma\left(\mathbf{\Phi}, \mathbf{\Gamma}, M_{\mathrm{T}} = \begin{pmatrix} Q_{\mathrm{T}} & L_{\mathrm{T}} \\ L_{\mathrm{T}}^* & R_{\mathrm{T}} \end{pmatrix} = M_{\mathrm{T}}^*\right)$ and the difference equation

$$x(k+1) = \mathbf{\Phi}x(k) + \mathbf{\Gamma}u(k), \quad x(0) = x_0 \in \mathcal{W},$$

where $(\mathbf{\Phi}, \mathbf{\Gamma})$ *are define by (5.14) and (5.15) and*

$$M_T = \begin{pmatrix} Q_T & L_T \\ L_T^* & R_T \end{pmatrix} = M_T^* \in \mathcal{L}(\mathcal{W} \oplus \mathcal{U}),$$

with $Q_T \in \mathcal{L}(\mathcal{W})$, $L_T^* \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ *and* $R_T \in \mathcal{L}(\mathcal{U})$ *defined by*

$$\langle Q_T x, x \rangle_\mathcal{W} \triangleq \int_0^T \langle Q S^\mathcal{W}(t)x, S^\mathcal{W}(t)x \rangle_\mathcal{W} dt, \tag{6.1}$$

$$\langle L_T^* x, u \rangle_\mathcal{W} \triangleq \int_0^T \langle L^* S^\mathcal{W}(t)x, u \rangle_\mathcal{W} dt +$$

$$+ \int_0^T \langle Q S^\mathcal{W}(t)x, \int_0^t S^\mathcal{V}(t-\tau) B u d\tau \rangle_\mathcal{W} dt, \tag{6.2}$$

$$\langle R_T u, u \rangle_\mathcal{U} \triangleq T \langle R u, u \rangle_\mathcal{U} + 2 \int_0^T \langle L^* \int_0^t S^\mathcal{V}(t-\tau) B u d\tau, u \rangle_\mathcal{U} dt +$$

$$+ \int_0^T \langle Q \int_0^t S^\mathcal{V}(t-\tau) B u d\tau, \int_0^t S^\mathcal{V}(t-\tau) B u d\tau \rangle_\mathcal{U} dt. \tag{6.3}$$

Proof Notice that for any $t \in [kT, (k+1)T)$ the state function can be written as

$$x(t) = S^\mathcal{W}(t-kT)x(k) + \int_{kT}^t S^\mathcal{V}(t-\tau) B d\tau u(k). \tag{6.4}$$

From (6.4) and the expression of the quadratic index (2.2) we obtain

$$J_\Sigma(x_0, u(\cdot)) = \int_0^\infty \langle \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}} dt =$$

$$= \underbrace{\int_0^\infty \langle Q x(t), x(t) \rangle_\mathcal{W} dt}_{(A1)} + 2 \underbrace{\int_0^\infty \langle L^* x(t), u(t) \rangle_\mathcal{W} dt}_{(A2)} + \underbrace{\int_0^\infty \langle R u(t), u(t) \rangle_\mathcal{U} dt}_{(A3)}. \tag{6.5}$$

Let us evaluate the expressions (A1), (A2) and (A3) separately. Using (6.4) we obtain

$$(A1): \quad \int_0^\infty \langle Q x(t), x(t) \rangle_\mathcal{W} dt =$$

$$= \sum_{k=0}^\infty \int_0^T \langle Q S^\mathcal{W}(t)x(k), S^\mathcal{W}(t)x(k) \rangle_\mathcal{W} dt + \tag{6.6}$$

$$+ 2 \sum_{k=0}^\infty \int_0^T \langle Q S^\mathcal{W}(t)x(k), \int_0^t S^\mathcal{V}(t-\tau) B u(k) d\tau \rangle_\mathcal{W} dt + \tag{6.7}$$

$$+ \sum_{k=0}^\infty \int_0^T \langle Q \int_0^t S^\mathcal{V}(t-\tau) B u(k) d\tau, \int_0^t S^\mathcal{V}(t-\tau) B u(k) d\tau \rangle_\mathcal{W} dt. \tag{6.8}$$

In a similar way we obtain

$$(A2): \qquad \int_0^\infty \langle L^* x(t), u(t) \rangle_{\mathcal{W}} =$$

$$= \sum_{k=0}^\infty \int_0^T \langle L^* S^{\mathcal{W}}(t) x(k), u(k) \rangle_{\mathcal{W}} dt + \tag{6.9}$$

$$+ \sum_{k=0}^\infty \int_0^T \langle L^* \int_0^t S^{\mathcal{V}}(t-\tau) B u(k) d\tau \rangle_{\mathcal{W}} dt \tag{6.10}$$

and

$$(A3): \qquad \int_0^\infty \langle R u(t), u(t) \rangle_{\mathcal{W}} dt = \sum_{k=0}^\infty \int_0^T \langle R u(k), u(k) \rangle_{\mathcal{W}} dt. \tag{6.11}$$

From (6.6) we get

$$\langle Q_T x(k), x(k) \rangle_{\mathcal{W}} \;\; \triangleq \;\; \int_0^T \langle Q S^{\mathcal{W}}(t) x(k), S^{\mathcal{W}}(t) x(k) \rangle_{\mathcal{W}} dt. \tag{6.12}$$

From (6.7) and (6.9) we get

$$\langle L_T^* x(k), u(k) \rangle \;\; \triangleq \;\; \int_0^T \langle L^* S^{\mathcal{W}}(t) x(k), u(k) \rangle_{\mathcal{W}} dt +$$

$$+ \int_0^T \langle Q S^{\mathcal{W}}(t) x(k), \int_0^\eta S^{\mathcal{V}}(t-\tau) B u(k) d\tau \rangle_{\mathcal{W}} dt. \tag{6.13}$$

From (6.8), (6.10) and (6.11) we get

$$\langle R_T u(k), u(k) \rangle_{\mathcal{U}} \;\; \triangleq \;\; T \langle R u(k), u(k) \rangle_{\mathcal{U}} + 2 \int_0^T \langle L^* \int_0^t S^{\mathcal{V}}(t-\tau) B u(k) d\tau, u(k) \rangle_{\mathcal{U}} dt +$$

$$+ \int_0^T \langle Q \int_0^t S^{\mathcal{V}}(t-\tau) B u(k) d\tau, \int_0^t S^{\mathcal{V}}(t-\tau) B u(k) d\tau \rangle_{\mathcal{U}} dt. \tag{6.14}$$

Combine now (6.12), (6.13) and (6.14) to obtain

$$J_\Sigma(x_0, u(\cdot)) = \sum_{k=0}^\infty \langle \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} Q_T & L_T \\ L_T^* & R_T \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}} \tag{6.15}$$

and the proof is complete.

∎

Since $S^{\mathcal{V}}(\cdot)$ was assumed to be exponentially stable on $\mathcal{W}$ it follows by Theorem 5.10 that Φ is power-stable on $\mathcal{W}$. Hence, we can state now the main digital control result of this section

Theorem 6.2 *Let*

$$\Sigma \left(S^\nu(\cdot), B, M = \begin{pmatrix} Q & L \\ L^* & R \end{pmatrix} = M^* \right)$$

be a Pritchard-Salamon-Popov triple on $(\mathcal{W} \hookrightarrow \mathcal{V}, \mathcal{U})$. *Assume that* $\mathcal{U} = \mathbb{R}^m$ *and let* $x_0 \in \mathcal{W}$. *Consider*

$$\mathcal{R}_T \ : \ \ell_2(0, \infty; \mathcal{U}) \longrightarrow \ell_2(0, \infty; \mathcal{U}), \ \ \mathcal{R}_T \stackrel{\triangle}{=} R_T + L_T^* \Psi + \Psi^* L_T + \Psi^* Q_T \Psi, \quad (6.16)$$

$$\Psi \ : \ \ell_2(0, \infty; \mathcal{U}) \longrightarrow \mathcal{W}, \ \ (\Psi u)(k) \stackrel{\triangle}{=} \sum_{i=0}^{k-1} \Phi^{k-i-1} \Gamma u(i), \quad (6.17)$$

with Φ, Γ *and* $M_T = \begin{pmatrix} Q_T & L_T \\ L_T^* & R_T \end{pmatrix}$ *defined by (5.14), (5.15) and (6.1), (6.2) and (6.3), respectively. If* $\mathcal{R}_T^{-1}$ *is well defined and bounded then the optimal digital control problem admits a solution in the sense that*

$$\min_{u_{\text{step}}(\cdot) \in U_{\text{adm}}^{x_0}} J_\Sigma(x_0, u(\cdot)) = \min_{u_{\text{step}}(\cdot) \in U_{\text{adm}}^{x_0}} \sum_{k=0}^{\infty} \langle \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} Q_T & L_T \\ L_T^* & R_T \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}} =$$

$$= \langle X x_0, x_0 \rangle_{\mathcal{W}}$$

where $X \in \mathcal{L}(\mathcal{W})$ *is a stabilizing solution to the following discrete-time Riccati equation*

$$\Phi^* X \Phi - X - (\Phi^* X \Gamma + L_T)(R_T + \Gamma^* X \Gamma)^{-1}(L_T^* + \Gamma^* X \Phi) + Q_T = 0. \quad (6.18)$$

The minimum is attained for inputs of the form

$$u_{\text{step}}(t) = (R_T + \Gamma^* X \Gamma)^{-1} (L_T^* + \Gamma^* X \Phi) x(k), \ \ kT \le t < (k+1)T. \quad (6.19)$$

Remark 6.3 *Notice that discretization has a "smoothing" effect on the discrete Popov triple* $\Sigma(\Phi, \Gamma, M_T)$ *as shown by the mapping representation in Figure 6.1*

6.2   Comments on the case $x_0 \in \mathcal{V}$

From the very beginning of this thesis we have considered that the initial state x_0 belongs to the smaller space $\mathcal{W}$ and we believe that we have succeeded to offer the reader a reasonable motivation for such an option. From the mathematical point of view, a counterpart of Lemma 6.1, derived if $x_0 \in \mathcal{V}$ is assumed, would represent the most beautiful and interesting part. This is especially true since working on the bigger space $\mathcal{V}$ one has to overcome certain difficulties and to make use of specific techniques. For a comprehensive treatment of the LQ-optimal control problem in the case $x_0 \in \mathcal{V}$ the reader is referred to [33, 80, 87]. In this
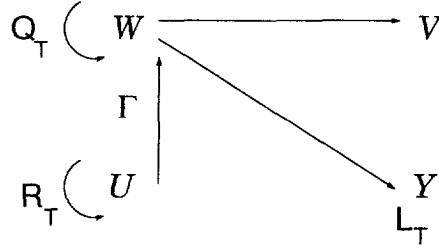
Figure 6.1: The mappings generated by the discrete Popov triple $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, M_\mathrm{T})$

section we use the technique developed in [33, 87] to extend the result proved in Lemma 6.1 to the case when the initial state belongs to the bigger state-space $\mathcal{V}$.

Let us consider the Pritchard-Salamon-Popov triple (2.18). Associated with it are the initial value problem (2.19), the class of admissible control function (2.20) and the quadratic qost function (2.26). Recall that if $x_0 \in \mathcal{V}$, then the mild solution to (2.19) is a $\mathcal{V}$-valued function which is continuous with respect to the time t. Furthermore, as we have noticed earlier in Remark 5.8, the time-discretized state function is well defined as a $\mathcal{V}$-valued sequence and, at the sampling instants $\{0, T, \cdots, kT \cdots\}$ the initial value problem (2.19) admits a discrete-time representation of the form

$$x(k+1) = \mathbf{\Phi}x(k) + \mathbf{\Gamma}u(k), \quad x(0) = x_0 \in \mathcal{V},$$

but this time $\mathbf{\Phi}$ and $\mathbf{\Gamma}$ are defined by (5.18) and (5.19), respectively. Our goal is to derive an equivalent discrete-time Pritchard-Salamon-Popov triple when $x_0 \in \mathcal{V}$. That means we want to construct an operator

$$\overline{M_\mathrm{T}} = \left(\begin{array}{cc} \overline{Q_\mathrm{T}} & \left(\overline{L_\mathrm{T}^*}\right)^* \\ L_\mathrm{T}^* & R_\mathrm{T} \end{array} \right) = M_\mathrm{T}^* \in \mathcal{L}(\mathcal{V} \oplus \mathcal{U}),$$

with $\overline{Q_\mathrm{T}} \in \mathcal{L}(\mathcal{V})$, $\overline{L_\mathrm{T}^*} \in \mathcal{L}(\mathcal{V}, \mathcal{U})$ and $R_\mathrm{T} \in \mathcal{L}(\mathcal{U})$ such that

$$J_\Sigma(x_0, u(\cdot)) = \sum_{k=0}^{\infty} \langle \left(\begin{array}{c} x(k) \\ u(k) \end{array} \right), \left(\begin{array}{cc} \overline{Q_\mathrm{T}} & \left(\overline{L_\mathrm{T}^*}\right)^* \\ L_\mathrm{T}^* & R_\mathrm{T} \end{array} \right) \left(\begin{array}{c} x(k) \\ u(k) \end{array} \right) \rangle_{\mathcal{V} \times \mathcal{U}}, \tag{6.20}$$

where $J_\Sigma(x_0, u(\cdot))$ is the quadratic index defined by (2.26).

Since $Q \in \mathcal{L}(\mathcal{W})$ is an admissible weighting operator, it follows that for any $x, y \in \mathcal{W}$,

$$|\langle Q_\mathrm{T}x, y\rangle_\mathcal{W}| = \left| \int_0^T \langle QS^\mathcal{W}(t)x, S^\mathcal{W}(t)y\rangle_\mathcal{W} \right| \leq \|x\|_\mathcal{V}\|y\|_\mathcal{V}. \tag{6.21}$$

Since $\mathcal{W} \hookrightarrow \mathcal{V}$, it follows by applying Theorem 1 from pp. 12 in [2] that the sesquilinear form (6.21) can be extended to a bounded sesquilinear form on $\mathcal{V}$. Let $\overline{Q_\mathrm{T}} = \left(\overline{Q_\mathrm{T}}\right)^* \in \mathcal{L}(\mathcal{V})$

denote the self-adjoint operator associated with the extended sesquilinear form, i.e.

$$\langle Q_T x, y\rangle_{\mathcal{W}} = \langle \overline{Q_T} x, y\rangle_{\mathcal{V}}. \tag{6.22}$$

In a similar way we construct the extension of $L_T^* \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ to $\mathcal{V}$, denoted $\overline{L_T^*} \in \mathcal{L}(\mathcal{V}, \mathcal{U})$. Notice first that the following hold

(i) Since $L_T^* \in \mathcal{L}(\mathcal{W}, \mathcal{U})$ is an admissible output operator for $S(\cdot)$ w.r.t. $\mathcal{W} \hookrightarrow \mathcal{V}$, then there exists $\alpha > 0$ such that

$$\left| \int_0^T \langle L^* S^{\mathcal{W}}(t)x, u\rangle_{\mathcal{W}} dt \right| \le \|x\|_{\mathcal{V}} \|u\|_{\mathcal{U}}. \tag{6.23}$$

This is true since the left-hand side of (6.23) defines a continuous bilinear mapping from $\mathcal{W} \oplus \mathcal{U}$ to $\mathbb{R}$.

(ii) Since $B \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ is an admissible input operator for $S(\cdot)$ w.r.t. $\mathcal{W} \hookrightarrow \mathcal{V}$, then there exists $\beta > 0$ such that

$$\left\| \int_0^t S^{\mathcal{V}}(t - \tau) B u d\tau \right\|_{\mathcal{W}} \le \beta(t) \|u\|_{\mathcal{U}}, \tag{6.24}$$

for $t \in [0, T)$. It follows that

$$\left| \int_0^T \langle Q S^{\mathcal{W}}(t)x, \int_0^t S^{\mathcal{V}}(t - \tau) B u d\tau \rangle_{\mathcal{W}} \right| \le \beta(t) \|x\|_{\mathcal{V}} \|u\|_{\mathcal{U}}. \tag{6.25}$$

From (6.23) and (6.25) it follows that

$$|\langle L_T^* x, u\rangle_{\mathcal{W}}| = \left| \int_0^T \langle L^* S^{\mathcal{W}}(t)x, u\rangle_{\mathcal{W}} dt + \langle Q S^{\mathcal{W}}(t)x, \int_0^t S^{\mathcal{V}}(t - \tau) B u d\tau \rangle_{\mathcal{W}} \right| \le$$

$$\left| \int_0^T \langle L^* S^{\mathcal{W}}(t)x, u\rangle_{\mathcal{W}} dt \right| + \left| \int_0^T \langle Q S^{\mathcal{W}}(t)x, \int_0^t S^{\mathcal{V}}(t - \tau) B u d\tau \rangle_{\mathcal{W}} \right| \le$$

$$\le \gamma \|x\|_{\mathcal{V}} \|u\|_{\mathcal{U}}, \tag{6.26}$$

for some appropriately chosen $\gamma > 0$. The last inequality implies that the bilinear form $\langle L_T^* x, u\rangle_{\mathcal{W}}$ has a continuous extension to a bilinear form on $\mathcal{V}$. Let $\overline{L_T^*} \in \mathcal{L}(\mathcal{V}, \mathcal{U})$ denote the operator associated with this extended bilinear form, i.e.

$$\langle L_T^* x, u\rangle_{\mathcal{W}} = \langle \overline{L_T^*} x, u\rangle_{\mathcal{V}}.$$

It remains to prove that R_T is well defined and bounded from $\mathcal{U}$ to $\mathcal{U}$. Exploiting the fact that L^* is an admissible output operator and B is an admissible input operator, it follows

that

$$\begin{aligned}
|\langle R_T u, u\rangle_\mathcal{U}| &\leq |T\langle Ru, u\rangle_\mathcal{U}| + 2\left|\int_0^T \langle L^* \int_0^t S^\mathcal{V}(t-\tau)Bud\tau, u\rangle_\mathcal{U} dt\right| + \\
&+ \left|\int_0^T \langle Q \int_0^t S^\mathcal{V}(t-\tau)Bud\tau, \int_0^t S^\mathcal{V}(t-\tau)Bud\tau\rangle_\mathcal{U} dt\right| \leq \\
&\leq \|M\|\|u\|_\mathcal{U}^2,
\end{aligned} \tag{6.27}$$

for some appropriately chosen $M > 0$.

An immediate fact which is implied by the construction of $\overline{Q_T} = \left(\overline{Q_T}\right)^* \in \mathcal{L}(\mathcal{V})$ and $\overline{L_T^*} \in \mathcal{L}(\mathcal{V}, \mathcal{U})$ is the following

Proposition 6.4 *The quadratic functional $J_\Sigma(x_0, u(\cdot))$ associated with the equivalent discrete-time Pritchard-Salamon-Popov triple $\Sigma\left(\Phi, \Gamma, M_T = \begin{pmatrix} Q_T & L_T \\ L_T^* & R_T \end{pmatrix} = M_T^*\right)$, with the Popov index M_T defined by (6.1), (6.2) and (6.3), has a unique bounded extension to a quadratic functional for $x_0 \in \mathcal{V}$ which can be effectively constructed. Furthermore,*

$$\begin{aligned}
J_\Sigma(x_0, u(\cdot)) &= \sum_{k=0}^\infty \langle \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} Q_T & L_T \\ L_T^* & R_T \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}\rangle_{\mathcal{W}\times\mathcal{U}} = \\
&= \sum_{k=0}^\infty \langle \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}, \begin{pmatrix} \overline{Q_T} & \left(\overline{L_T^*}\right)^* \\ \overline{L_T^*} & R_T \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}\rangle_{\mathcal{V}\times\mathcal{U}}.
\end{aligned} \tag{6.28}$$

Let us also notice that

$$\Sigma\left(\Phi, \Gamma, \overline{M_T} = \begin{pmatrix} \overline{Q_T} & \left(\overline{L_T^*}\right)^* \\ \overline{L_T^*} & R_T \end{pmatrix} = M_T^* \in \mathcal{L}(\mathcal{V} \oplus \mathcal{U})\right),$$

with Φ and Γ defined by (5.18) and (5.19) and $\overline{M_T}$ constructed as before is a well defined discrete-time Popov triple on $(\mathcal{V}, \mathcal{U})$. Let us consider then the digital control problem on $\mathcal{V}$. By modifying appropriately the definition of the stabilizing digital control law, i.e. Definition 2.14, by considering as set of admissible control functions the one defined by (2.20) and finally by considering the quadratic cost function defined by the extended quadratic functional (2.26), then the digital control problem on $\mathcal{V}$ is well-posed. This is especially true since assuming stability of $S(\cdot)$ also on $\mathcal{V}$, we can write down without anu difficulty the counterpart of the digital control result from Theorem 6.2 for the case $x(\cdot) \in \mathcal{V}$. Notice that this time the solution $\overline{X} \in \mathcal{L}(\mathcal{V})$ to the following digital Riccati equation is involved in the expression of the digital stabilizing control law

$$\Phi^* X \Phi - X - \left(\Phi^* X \Gamma + \left(\overline{L_T^*}\right)^*\right)(R_T + \Gamma^* X \Gamma)^{-1}(\overline{L_T^*} + \Gamma^* X \Phi) + \overline{Q_T} = 0, \tag{6.29}$$

with Φ and Γ defined by (5.18) and (5.19). Furthermore, the minimum of the quadratic cost functional is attained and its value is

$$\min_{u(\cdot)\in U^{x_0}_{adm}\cap\mathcal{PC}_T(0,\infty;\mathcal{U})} J_\Sigma(x_0, u(\cdot)) = \langle \overline{X}x_0, x_0\rangle_\mathcal{V}.$$

If we regard $x(\cdot) \in \mathcal{W}$ as element of $\mathcal{W}$, then the above minimum of the quadratic functional is attained at $\langle Xx_0, x_0\rangle_\mathcal{W}$. If we regard $x(\cdot) \in \mathcal{W}$ as element of $\mathcal{V}$, then it is attained attained at $\langle \overline{X}x_0, x_0\rangle_\mathcal{V}$. Hence, by exploiting the result of Proposition 6.4 we conclude that

$$\langle Xx_0, x_0\rangle_\mathcal{W} = \langle \overline{X}x_0, x_0\rangle_\mathcal{V}.$$

This fact implies that $\overline{X} = \overline{X}^* \in \mathcal{L}(\mathcal{V})$ is the unique bounded extension for $x_0 \in \mathcal{V}$ of $X = X^* \in \mathcal{L}(\mathcal{W})$.

6.3   The digital $\mathcal{H}^\infty$ control problem

Consider the continuous dense injection of spaces $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$ and assume that there exist two other real separable Hilbert spaces $\mathcal{U}_1$ and $\mathcal{Y}_1$ and let $B_1 \in \mathcal{L}(\mathcal{U}_1, \mathcal{V})$, $B_2 \in \mathcal{L}(\mathcal{U}_2, \mathcal{V})$, $C_1 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_1)$, $C_2 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_2)$, $D_{12} \in \mathcal{L}(\mathcal{U}_2, \mathcal{Y}_1)$. We consider the smooth Pritchard-Salomon system

$$\Sigma\left(S(\cdot), (B_1, B_2), \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix}\right)$$

satisfying

$$\begin{cases} x(t) = S^\mathcal{W}(t)x_0 + \int_0^t S^\mathcal{V}(t-\tau)(B_1u_1(\tau) + B_2u_2(\tau))d\tau \\ y_1(t) = C_1x(t) + D_{11}u_1(t) + D_{12}u_2(t) \\ y_2(t) = C_2x(t) \end{cases} \tag{6.30}$$

In our set-up $x(\cdot) \in \mathcal{W}$ is the state function, $u_1(\cdot) \in \mathcal{U}_1$ is the disturbance input $u_2(\cdot) \in \mathcal{U}_2$ is the control input, $y_1(\cdot) \in \mathcal{Y}_1$ the controlled output and $y_2(\cdot) \in \mathcal{Y}_2$ is the measured output. The following assumptions are made on the Pritchard-Salomon system

A1 $\mathcal{U}_2 = \mathbb{R}^m$ (we have m-actuators),

A2 $\mathcal{Y}_2 = \mathbb{R}^p$ (we have p-sensors),

A3 $x_0 \in \mathcal{W}$ (the output measurement is a continuous $\mathcal{Y}$-valued function),

A4 $(S(\cdot), B_2)$ is admissibly stabilizable,

A5 $(C_2, S(\cdot))$ is admissibly detectable ,

A6 $D_{11} = 0$,

A7 $D_{12}^* D_{12}$ is coercive.

Remark 6.5 *Let us explain a little bit why the above mentioned assumptions have been made*

- *It is rather obvious that when we control a system digitally, then it is compulsory to have finite many sensors and actuators. This represents the motivation why we have made assumptions* **A1, A2**.

- *Assumption* **A3** *guarantees that the output measurement function is continous in time and hence it can be sampled.*

- *Assumptions* **A4** *and* **A5** *are necessary for giving a sufficient condition for the hybrid stability of the digital control system*

- *Assumption* **A6** *is made for simplicity. For details how this assumption can be removed the reader is referred to section 5.4 of [80].*

- *The last assumption guarantees the well-posedness of the Pritchard-Salamon system.*

If $x_0 = 0$ then we can express the Pritchard-Salamon system in an input/output fashion as defined by (5.85), (5.86), (5.87), (5.88) and (5.89), respectively. Let $\mathcal{K}$ be another real separable Hilbert space and consider a controller $\Sigma_c\,(A_c, B_c, C_c, D_c)$ of the form

$$\begin{cases} (\sigma\xi)(k) = A_c\xi(k) + B_c\nu(k) \\ \zeta(k) = C_c\xi(k) + D_c\nu(k) \end{cases}, \tag{6.31}$$

where $A_c \in \mathcal{L}(\mathcal{K})$, $B_c \in \mathcal{L}(\mathbb{R}^p, \mathcal{K})$, $C_c \in \mathcal{L}(\mathcal{K}, \mathbb{R}^m)$ and $D_c \in \mathbb{R}^{p \times m}$ and the initial state of the controller, ξ_0, is given. If $\xi_0 = 0$ then (6.31) can be expressed in an input-output fashion as

$$\zeta(k) = G_c\eta(k) \tag{6.32}$$

where

$$G_c : \ell_2(0, \infty; \mathbb{R}^p) \longrightarrow \ell_2(0, \infty; \mathbb{R}^m), \quad G_c\eta(k) \triangleq \sum_{i=0}^{\infty} C_c A_c^i B_c \eta(k - i - 1) + D_c\eta(k). \tag{6.33}$$

Since the output measurement is a $\mathbb{R}^p$-valued continuous function we can define its time-discretized version as

$$\tilde{y}_2 = \mathbf{S}y_2. \tag{6.34}$$

We want to make sense of the following feedback connection

$$\begin{cases} \nu = \mathbf{S}y_2, & u_2(\cdot) = \mathbf{H}\tilde{\zeta} \end{cases}. \tag{6.35}$$

Our goal is to design a controller (6.31) such that

(*i*) **Attenuation**: The input/output operator from u_1 to y_1,

$$T_{y_1 u_1} : L_2^{\text{loc}}(0, \infty; \mathcal{U}_1) \mapsto L_2^{\text{loc}}(0, \infty; \mathcal{Y}_1), \tag{6.36}$$

expressed as a linear fractional transformation of the hybrid generalized plant $\Sigma_{\mathbf{G}_{sd}}$ with input/output operator $G_{sd} = \begin{pmatrix} G_{11} & G_{12}\mathbf{H} \\ SG_{21} & SG_{22}\mathbf{H} \end{pmatrix}$ and the controller Σ_K, with input/output operator G_c

$$T_{y_1 u_1} = \mathcal{F}\left(\begin{pmatrix} G_{11} & G_{12}\mathbf{H} \\ SG_{21} & SG_{22}\mathbf{H} \end{pmatrix}, G_c\right) \triangleq G_{11} + G_{12}\mathbf{H}G_c(I - SG_{22}\mathbf{H}G_c)^{-1}SG_{21} \tag{6.37}$$

is well defined and is a contraction

$$\|T_{y_1 u_1} u_1(\cdot)\| < 1. \tag{6.38}$$

(*ii*) **Stability**: The closed-loop system $\Sigma_{\mathbf{G}_{sd}\mathbf{G}_c}$, obtained by the feedback interconnection of the hybrid generalized plant G_{sd} and the controller G_c is *hybrid stable*.

The following main conversion results hold (see [12, 9, 1, 2])

Theorem 6.6 *Let* $\Sigma\left(S(\cdot), \begin{pmatrix} B_1 & B_2 \end{pmatrix}, \begin{pmatrix} C_1 \\ C \end{pmatrix}, \begin{pmatrix} 0 & D_{12} \\ 0 & 0 \end{pmatrix}\right)$ *be a smooth Pritchard-Salamon system with respect to* $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$ *satisfying*

$$\Sigma_{\mathbf{G}} \begin{cases} x(t) = S(t)x_0 + \int_0^t S(t - \tau)(B_1 u_1(\tau) + B_2 u_2(\tau))d\tau \\ y_1(t) = C_1 x(t) + D_{12} u_2(t) \\ y_2(t) = C_2 x(t), \quad x_0 \in \mathcal{W} \end{cases}, \tag{6.39}$$

Then a discrete-time controller $\Sigma(K, L, M, N)$ *satisfying*

$$\Sigma_K \begin{cases} (\sigma\xi)(k) = A_c\xi(k) + B_c\eta(k), \\ \zeta(k) = C_c\xi(k) + D_c\eta(k), \end{cases} \tag{6.40}$$

solves the digital $\mathcal{H}^\infty$ *control problem if it is a solution to the discrete-time* $\mathcal{H}^\infty$ *control problem for the lifted discrete-time system (5.150) satisfying*

$$\Sigma_{\hat{\mathbf{G}}} \begin{cases} x(k + 1) = \mathbf{\Phi}x(k) + \hat{\mathbf{\Gamma}}_1\hat{u}_1(k) + \mathbf{\Gamma}_2 u_2(k) \\ \hat{y}_1(k) = \hat{\mathbf{\Lambda}}_1 x(k) + \hat{\mathbf{\Theta}}_{11}\hat{u}_1(k) + \hat{\mathbf{\Theta}}_{12} u_2(k) \\ y_{2,k} = \mathbf{\Lambda}_2 x(k), \quad x_0 \in \mathcal{W} \end{cases}, \tag{6.41}$$

where

$$\begin{cases} \mathbf{\Phi} & \triangleq S^{\mathcal{V}}(T)|_{\mathcal{W}}, \\ \hat{\mathbf{\Gamma}}_1\hat{u}_1(k) & \triangleq \int_0^T S^{\mathcal{V}}(T - \tau)B_1 u_1(kT + \tau)d\tau, \\ \mathbf{\Gamma}_2 & \triangleq \int_0^T S^{\mathcal{V}}(T - \tau)B_2 d\tau, \\ \hat{\mathbf{\Lambda}}_1 & \triangleq C_1 S^{\mathcal{V}}(\cdot), \\ \mathbf{\Lambda}_2 & \triangleq C_2, \\ \hat{\mathbf{\Theta}}_{11}\hat{u}_1(k) & \triangleq C_1 \int_0^{\cdot} S^{\mathcal{V}}(\cdot - \tau)B_1 u_1(kT + \tau)d\tau, \\ \hat{\mathbf{\Theta}}_{12} & \triangleq C_1 \int_0^{\cdot} S^{\mathcal{V}}(\cdot - \tau)B_2 d\tau + D_{12}. \end{cases} \tag{6.42}$$

Proof Since the controller (6.31) is a solution to the discrete-time $\mathcal{H}^\infty$ control problem written for the lifted system (5.150), it provides $\Sigma(K, L, M, N)$ provides power stabilizability and power detectability for $\left(\hat{\Phi}, \Gamma_2, \Lambda_2\right)$ with on $\mathcal{W}$. By Proposition 5.28 then (6.31) provides hybrid stability for the original Pritchard-Salamon system. The input/output operator from $\hat{u}_1$ to $\hat{y}_1$ is a contraction operator from $\ell_2(0, \infty; L_2(0, T; \mathcal{U}_1))$ to $\ell_2(0, \infty; L_2(0, T; \mathcal{Y}_1))$. It follows by Proposition 5.30 that the input output operator from u_1 to y_1 is also a contraction operator from $L_2(0, \infty; \mathcal{U}_1)$ to $L_2(0, \infty; \mathcal{Y}_1)$. We conclude that (6.31) is then also a solution to the digital $\mathcal{H}^\infty$ control problem.

∎

Exploiting now the results on necessary and sufficient conditions for the existence of the solution to the discrete-time $\mathcal{H}^\infty$ control problem, stated and proved in Chapter 3, we can formulate the Popov theory based solution to the digital $\mathcal{H}^\infty$ control problem

Theorem 6.7 *The following statements hold*

(i) *If the digital $\mathcal{H}^\infty$ control problem admits a solution then the Kalman-Szego-Popov-Yakubovitch systems (3.36) and (3.37) associated with the following discrete-time Popov triples*

$$\Sigma\left(\Phi, \left(\begin{array}{cc}\hat{\Gamma}_1 & \Gamma_2\end{array}\right), \left(\begin{array}{cc}\hat{\Lambda}_1^*\hat{\Lambda}_1 & \left(\begin{array}{cc}\hat{\Lambda}_1^*\hat{\Theta}_{11} & \hat{\Lambda}_1^*\hat{\Theta}_{12}\end{array}\right) \\ \left(\begin{array}{c}\hat{\Theta}_{11}^*\hat{\Lambda}_1 \\ \hat{\Theta}_{12}^*\hat{\Lambda}_1\end{array}\right) & \left(\begin{array}{cc}\hat{\Theta}_{11}^*\hat{\Theta}_{11} - I & \hat{\Theta}_{11}^*\hat{\Theta}_{12} \\ \hat{\Theta}_{12}^*\hat{\Theta}_{11} & \hat{\Theta}_{12}^*\hat{\Theta}_{12}\end{array}\right)\end{array}\right)\right), \quad (6.43)$$

$$\Sigma\left(\Phi, \left(\begin{array}{cc}\hat{\Lambda}_1^* & \Lambda_2^*\end{array}\right), \left(\begin{array}{cc}\hat{\Gamma}_1\hat{\Gamma}_1^* & \left(\begin{array}{cc}\hat{\Gamma}_1\hat{\Theta}_{11}^* & 0\end{array}\right) \\ \left(\begin{array}{c}\hat{\Theta}_{11}\hat{\Gamma}_1^* \\ 0\end{array}\right) & \left(\begin{array}{cc}\hat{\Theta}_{11}\hat{\Theta}_{11}^* - I & 0 \\ 0 & 0\end{array}\right)\end{array}\right)\right), \quad (6.44)$$

have stabilizing solutions.

(ii) *Consider the following discrete-time Popov triple*

$$\Sigma_0\left(\Phi_0, \left(\begin{array}{cc}\hat{\Lambda}_{10}^* & \Lambda_{20}^*\end{array}\right), \left(\begin{array}{cc}\hat{\Gamma}_{10}\hat{\Gamma}_{10}^* & \left(\begin{array}{cc}\hat{\Gamma}_{10}\hat{\Theta}_{110}^* & 0\end{array}\right) \\ \left(\begin{array}{c}\hat{\Theta}_{110}\hat{\Gamma}_{10}^* \\ 0\end{array}\right) & \left(\begin{array}{cc}\hat{\Theta}_{110}\hat{\Theta}_{110}^* - I & 0 \\ 0 & 0\end{array}\right)\end{array}\right)\right) \quad (6.45)$$

corresponding to the modified system defined by (3.51) associated with the discrete-time system (5.150). If the Kalman-Szego-Popov-Yakubovitch systems (3.36) and (3.37) written for the discrete-time Popov triples (6.43) and (6.45) have stabilizing solutions, then there exists a controller which is a solution to the digital $\mathcal{H}^\infty$ control problem.

6.4   Shortcomings of the method

In this section we focus on the shortcomings of the Popov theory based solution to the $\mathcal{H}^\infty$-control problem as developed in the previous section. Recall that in Section 3.3 the minimum set of assumptions made on the discrete-time system was

S A is power stable.

CO The infinite dimensional discrete-time system (3.25) satisfies

$$T_{12}^* T_{12} \gg 0, \tag{6.46}$$
$$T_{21} T_{21}^* \gg 0. \tag{6.47}$$

We call the proerties (6.46) and (6.47) as the 12-coercivity and the 21-coercivity of the system (3.25). The above set of assumptions represented a guarantee that the discrete filter and control $\mathcal{H}^\infty$ Riccati equations, the Kalman-Szego-Popov-Yakubovitch systems, respectively, admit stabilizing solutions. Notice that this case includes also the so-called singular case when the R-operator is null. This is extremely important since the digital $\mathcal{H}^\infty$ control problem is a singular control problem ($D_{21} = 0$) and hence, the equivalent discrete-time $\mathcal{H}^\infty$ control problem associated with the lifted system would be a singular discrete-time $\mathcal{H}^\infty$ control problem. Clearly, we would like to give a set of assumptions that, when made on the original data, the smooth Pritchard-Salamon system

$$\Sigma \left(S(\cdot), (B_1, B_2), \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix} \right)$$

would guarantee that **S**, **D** and **CO** hold for the lifted system

$$\Sigma \left(\Phi, \begin{pmatrix} \hat{\Gamma}_1 & \Gamma_2 \end{pmatrix}, \begin{pmatrix} \hat{\Lambda}_1 \\ \Lambda_2 \end{pmatrix}, \begin{pmatrix} \hat{\Theta}_{11} & \hat{\Theta}_{12} \\ 0 & 0 \end{pmatrix} \right)$$

and hence, Theorem 6.7 is applicable. Let us notice that admissible or boundedly stabilizability and admissible or bounded detectability of $(S(\cdot), B_2, C_2)$ and a sufficiently small chosen sampling step, $T > 0$, assures that $(\Phi, \Gamma_2, \Lambda_2)$ is power stabilizable and power detectable with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$. This is a natural constraint which is always made when doing digital control. Let us focus now on the third item regarding 12-coercivity and 21-coercivity of the discrete-time system $\Sigma \left(A, (B_1, B_2), \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix} \right)$. The following statement is true

Proposition 6.8 *Assume that*

(i) $D_{12}^* D_{12}$ *is coercive.*

(ii) $(M_{12}C_1, A - B_2 D_{12}^\dagger C_1)$ *is power detectable, where* $D_{12}^\dagger = (D_{12}^* D_{12})^{-1} D_{12}^*$ *and* $M_{12} = I - D_{12}D_{12}^\dagger$.

Then $\Sigma \left(A, (B_1, B_2), \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix} \right)$ *is 12-coercive.*

Proof The assumptions made in Proposition 6.8 are sufficient conditions for the existence of the positive semidefinite stabilizing solution to the discrete time Riccati equation with

$$Q = C_1^* C_1, \quad L = C_1^* D_{12}, \quad R = D_{12}^* D_{12}.$$

Let $F_2 = -(R + B_2^* X B_2)^{-1}(B_2^* X A + L^*)$ be the corresponding stabilizing feedback. Then the discrete-time Riccati equation can be re-written as

$$\tilde{A}^* X \tilde{A} - X + \tilde{Q} = 0, \tag{6.48}$$

where $\tilde{A} = A + B_2 F_2$ and $\tilde{Q} = Q + F_2 L + L^* F_2^* + F_2^* R F_2$. Simple computation performed on (6.48) implies

$$R + B_* X B = \tilde{T}_{12}^* \tilde{T}_{12}, \tag{6.49}$$

where $\tilde{T}_{12} = \tilde{C}_1(\sigma I - \tilde{A})^{-1} B_2 + D_{12}$, $\tilde{C}_1 = C_1 + D_{12}F_2$. As $R = D_{12}^* D_{12} \gg 0$ and $X \geq 0$ it follows from (6.49) that $\tilde{T}_{12}^* \tilde{T}_{12} \gg 0$ which completes the proof. ∎

Let us see now what assumptions shoud one make on the original Pritchard-Salamon system such that 1. and 2. from Proposition 6.8 hold for the lifted system (5.150) From the expression of $\hat{\Theta}_{12}$ one can easy notice that if $D_{12}^* D_{12} \gg 0$ for the Pritchard-Salamon system then this property also hods for its lifted counterpart, i.e. $\hat{\Theta}_{12}^* \hat{\Theta}_{12} \gg 0$. Some straightforward computation shows that if $(C_1, S(\cdot))$ is assumed admissible or boundedly detectable then $(\mathbf{M}_{12}\hat{\Lambda}_1, \Phi - \Gamma_2 \hat{\Theta}_{12}^\dagger \hat{\Lambda}_1)$ is power detectable, where $\hat{\Theta}_{12}^\dagger = (\hat{\Theta}_{12}^* \hat{\Theta}_{12})^{-1} \hat{\Theta}_{12}^*$ and $\hat{M}_{12} = I - \hat{\Theta}_{12}\hat{\Theta}_{12}^\dagger$ is power detectable. Unfortunately Proposition 6.8 does not apply for the dual problem since $D_{21} = 0$ and hence, $D_{21}D_{21}^*$ is no longer coercive. The best we can prove is the following

Proposition 6.9 *Let* $\Sigma \left(S(\cdot), (B_1, B_2), \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \begin{pmatrix} D_{11} & D_{12} \\ 0 & 0 \end{pmatrix} \right)$ *such that* $(S(\cdot), B_1)$ *is exactly controllable on* $[0, T]$ *and* $C_2^* C_2 \gg 0$. *Then the lifted system (5.150) is 21-coercive.*

Proof Since $(S(\cdot), B_1)$ is exactly controllable on $[0, T]$ it follows that the controllability gramian

$$P_T \triangleq \int_0^T S(\tau) B_1 B_1^* S^*(\tau) d\tau \tag{6.50}$$

is coercive (see Theorem 4.1.7 from Curtain and Zwart [35]). But $P_T = \hat{\Gamma}_1 \hat{\Gamma}_1^*$ and hence we have that $\hat{\Gamma}_1 \hat{\Gamma}_1^* \gg 0$. Let us prove that this implies 21-coercivity of the lifted system.

Indeed, let K_2 be a stabilizing injection. Then the updated system is now power stable and a realization for $T_{21}^* = \mathbf{\Gamma}_1(I - \sigma\mathbf{\Phi}^*)^{-1}\sigma\mathbf{\Lambda}_2^*$ is

$$x = \mathbf{\Phi}^*\sigma x + \mathbf{\Lambda}_2^*\mu, \tag{6.51}$$

$$\eta = \mathbf{\Gamma}_1^*\sigma x, \tag{6.52}$$

with $\mu \in \ell_2(0,\infty;\mathbb{R}^p)$, $x \in \ell_2(0,\infty;\mathcal{V})$ and $\eta \in \ell_2(0,\infty;L_2(0,T;\mathcal{U}_1))$. Exploiting the coercivity of the controllability gramian (6.50) as well as the fact that $C_2^*C_2 \gg 0$ we get

$$\|\mathbf{\Gamma}_1^*\sigma x\|_2 \geq \delta_1\|\sigma x\|_2, \tag{6.53}$$

$$\|\mathbf{\Lambda}_2^*\mu\|_2 \geq \delta_2\|\mu\|_2, \tag{6.54}$$

for some $\delta_1 > 0$ and $\delta_2 > 0$. Hence from (6.51) we obtain by exploiting (6.53) and (6.54)

$$\begin{aligned}\delta_2^2\|\mu\|_2^2 &\leq \|\mathbf{\Lambda}_2^*\mu\|_2^2 = \|x - \mathbf{\Phi}^*\sigma x\|_2^2 \leq 2\|x\|_2^2 + 2\|\mathbf{\Phi}^*\sigma x\|_2^2 \leq \\ &\leq 2(1 + \|\mathbf{\Phi}\|^2)\|x\|_2^2 \leq \frac{2(1 + \|\mathbf{\Phi}\|^2)}{\delta_1^2}\|\mathbf{\Gamma}_1^*\sigma x\|_2^2,\end{aligned} \tag{6.55}$$

where from we obtain $\delta_3\|\mu\|_2^2 \leq \|\eta\|_2^2$ for $\delta_3 = \frac{\delta_1\delta_2}{\sqrt{2}(1+\|\Phi\|^2)^{\frac{1}{2}}}$. Thus $\|T_{21}^*\mu\|_2 \geq \delta_3\|\mu\|_2$ fact which shows that $T_{21}^*T_{21} \gg 0$.

∎

Let us give some comments on Proposition 6.9.

Remark 6.10   (*i*) *In general it is hard to prove that a system is exactly controllable and large classes of partial differential and delay systems are not exactly controllable, but only approximately controllable. In particular, when $\mathcal{U}_1$ is finite-dimensional, then for any $T > 0$, the pair $(S(\cdot), B)$ is not exactly controllable. Thus, checking the exact controllability of $(S(\cdot), B_1)$ is generally not a sensible way of checking whether $T_{y_2u_1}^*T_{y_2u_1}$ is coercive. However, it is well known [31, 38] that for certain classes of linear, continuous-time, infinite-dimensional systems, an input normal realization of the Hankel operator is exactly controllable and approximately observable, i.e. if P and Q are the controllability and observability gramians associated with the linear systems, then*

$$P = I > 0, \text{ (implies exact controllability)}, \tag{6.56}$$

$$Q = \Sigma^2 \geq 0, \text{ (implies only approximately observability)}. \tag{6.57}$$

This idea induced the following (possible) sufficient condition for the 21-coercivity of the lifted system associated with the original Pritchard-Salamon system: in [65] it is shown that transfer functions $G(s) \in \mathcal{L}(C,\mathcal{L}(\mathcal{U},\mathcal{Y}))$, for which analytic functions $F(s) \in \mathcal{L}(C,\mathcal{L}(\mathcal{U},\mathcal{Y}))$ exist such that $(G+F)(s)$ is essentially bounded (i.e. extends to a norm continuous function on the imaginary axis), admit parabalanced realizations, i.e. realization for which

$$P = Q.$$

The result regarding this fact is proved for infinite-horizon and hence, it holds for finite horizon as well. The idea of the proof is based on applying a bilinear transformation, which preserves the controllability and observability gramians, to a parabalanced realization of a transfer function of an infinite-dimensional discrete-time system for which exists an analytic function such that the sum is essentially bounded on the unit circle. The reader is referred for more details to Ober and Montgomery-Smith [65]. We believe that these result can be extended for input normal realizations of discrete-time and continuous-time systems. Such an extension is far away of being trivial. This subject is left open for a future investigation. If this is the case, we conjecture that an input normal realization of $(\mathbf{\Phi}, \hat{\mathbf{\Gamma}}_1, \hat{\mathbf{\Lambda}}_1)$ on $[0, T]$ (if it exists) implies 21-coercivity of the lifted Pritchard-Salamon system.

(ii) *The coercivity of $C_2^* C_2$ is, actually, not the worst test possible for the coercivity of $T_{21}^* T_{21}$. By an adequate choice of the measurement sensors this can be achieved for large classes of systems.*

(iii) *If we restrict to the finite-dimensional case then the exact controllability of $(S(\cdot), B_1)$ is nothing else than the trivial coercivity of $\hat{\mathbf{\Gamma}}_1 \hat{\mathbf{\Gamma}}_1^*$ which, by a suitable choice of the design configuration, can also be achieved.*

6.5 The fixed-order digital LQG control problem

Consider now three Hilbert spaces with continuous dense injection $\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$ and assume that there exist other real separable Hilbert spaces $\mathcal{U}_1$, and $\mathcal{Y}_1$ and let $B_1 \in \mathcal{L}(\mathcal{U}_1, \mathcal{V})$, $B_2 \in \mathcal{L}(\mathcal{U}_2, \mathcal{V})$, $C_1 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_1)$, $C_2 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_2)$, $D_{11} \in \mathcal{L}(\mathcal{U}_1, \mathcal{Y}_1)$, $D_{12} \in \mathcal{L}(\mathcal{U}_2, \mathcal{Y}_1)$. Assume also that

(i) $\mathcal{U}_1 = \mathbb{R}^{m_1}$, $\mathcal{U}_2 = \mathbb{R}^{m_2}$,

(ii) $\mathcal{Y}_1 = \mathbb{R}^{p_1}$, $\mathcal{Y}_2 = \mathbb{R}^{p_2}$,

(iii) $D_{12}^* D_{12}$ is coercive,

(iv) $B_1 B_1^* \in \mathcal{N}(\mathcal{W})$,

and consider the smooth Pritchard-Salamon system $\Sigma \left(S^{\mathcal{V}}(\cdot), \left(\begin{array}{cc} B_1 & B_2 \end{array} \right), C_1, D_{12} \right)$ satisfying

$$\begin{aligned} x(t) &= Ax(t) + B_1 u_1(t) + B_2 u_2(t)), \quad x_0 \in \mathcal{W} \\ y_1(t) &= C_1 x(t) + D_{12} u_2(t) \end{aligned}$$

$$\begin{aligned} &(6.58) \\ &(6.59) \end{aligned}$$

and the discrete-time output measurement equation

$$y_2(k) = C_2 x(k) + D_{21} u_3(k) \tag{6.60}$$

We shall assume that the C_0-semigroup $S^{\mathcal{W}}(\cdot)$ is exponentially stable on $\mathcal{W}$.

In our set-up $x(t)$ is the state, $u_1(t) \in \mathcal{U}_1$ is the disturbance input assumed to be *standard Gaussian white noise of unit intensity*, $u_2(t) \in \mathcal{U}_2$ is the control input, $y_1(t) \in \mathcal{Y}_1$ the regulated output and $y_2(k) \in \mathcal{Y}_2$ the sampled measured output corrupted by a discrete-time Gaussian white noise of unit intensity, $u_3(k) \in \mathcal{U}_3$. We assume that the initial state $x_0 \in \mathcal{W}$ is a random variable independent of the noise $u_1(\cdot)$ and independent of the noise $u_3(\cdot)$.

The state equation in (6.58) is defined by the stochastic differential equation

$$dx(t) = Ax(t) + d\beta(t) + B_2 u_2(t)dt, \quad x(0) = x_0 \in \mathcal{W}, \tag{6.61}$$

where $\{\beta(t); t \geq 0\}$ is a Wiener process on $\mathcal{W}$ with variance operator $B_1 B_1^*$, i.e.

$$\mathbb{E}(\beta(t)) = 0 \tag{6.62}$$

$$\mathrm{cov}\,(d\beta, d\beta) = B_1 B_1^* dt \tag{6.63}$$

Notice that $B_2 \in \mathcal{L}(\mathcal{U}_2, \mathcal{W})$ was assumed an admissible input operator for $S(\cdot)$. This fact implies that a solution to (6.61) of the form

$$x(t) = S^{\mathcal{W}}(t)x_0 + \int_0^t S^{\mathcal{V}}(t-\tau)d\beta(\tau) + \int_0^t S^{\mathcal{V}}(t-\tau)B_2 u_2(\tau)d\tau, \tag{6.64}$$

assumed to exist, is well-posed on $\mathcal{W}$. For necessary and sufficient conditions on the existence of the solution (6.64) on $\mathcal{W}$, the reader is referred to Section 5.3 from [34].

Let now $\mathbb{R}^{n \times n}$ be the controller state space and suppose that we have a controller $\Sigma(K, L, M)$ of the form

$$\Sigma_{\mathrm{K}}^n : \begin{cases} \zeta(k+1) = A_c\zeta(k) + B_c\nu(k), \\ \xi(k) = C_c\zeta(k), \end{cases} \tag{6.65}$$

where $\zeta \in \mathbb{R}^{n \times n}$ is the controller state, $\eta \in \mathbb{R}^p$ is what the controller input and $\xi \in \mathbb{R}^m$ is the controller output. We want to interconnect the system and the discrete-time controller via **A/D** and **D/A** devices and to perform a digital control.

Clearly, we want to make sense of the following feedback interconnection:

$$\nu = \mathbf{S}y_2, \quad u_2 = \mathbf{H}\tilde{\xi}. \tag{6.66}$$

Our goal is to design a controller $\Sigma_{\mathrm{K}}(A_c, B_c, C_c, 0)$ meeting the following specifications

(*i*) **Optimality:** The influence of the disturbance input u_1 on the regulated output y_1 is minimized in the sense that the following LQG quadratic cost function is minimized

$$J(K, L, M) = \min_{\Sigma_{\mathrm{K}}\,\mathrm{stabilizing}} \lim_{t \to \infty} \frac{1}{t}\mathbb{E}\left(\int_0^t \langle y_1(\tau), y_1(\tau)\rangle d\tau\right). \tag{6.67}$$

(ii) **Stability**: The linear system from u_1 to y_1 is hybrid stable in the sense given in Definition 5.22.

In order to guarantee that (6.67) is independent on the internal realization of the controller we shall focus on the following class of controllers

$$\mathcal{A} = \{\Sigma_K \text{ stabilizing } | (A_c, B_c, C_c, 0) \text{ is minimal } \}. \tag{6.68}$$

6.5.1 The equivalent discrete-time fixed-order LQG control problem. Case I: $x_0 \in \mathcal{W}$

As in the case of the digital LQ-optimal control problem, we shall construct an equivalent discrete-time LQG control problem. Let us notice first that

$$
\begin{aligned}
\langle y_1(t), y_1(t) \rangle_{\mathcal{Y}_1} &= \langle C_1 x(t) + D_{12} u_2(t), C_1 x(t) + D_{12} u_2(t) \rangle_{\mathcal{Y}_1} = \\
&= \langle \begin{pmatrix} Q & L^* \\ L & R \end{pmatrix} \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2},
\end{aligned}
\tag{6.69}
$$

where $Q \in \mathcal{L}(\mathcal{W})$, $L^* \in \mathcal{L}(\mathcal{U}, \mathcal{W})$ and $R \in \mathcal{L}(\mathcal{U})$

$$Q \triangleq C_1^* C_1, \tag{6.70}$$

$$L^* \triangleq C_1^* D_{12}, \tag{6.71}$$

$$R \triangleq D_{12}^* D_{12}. \tag{6.72}$$

Notice that Q is an admissible weighting operator since $C_1 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_1)$ is admissible output operator for $S(\cdot)$, and from the same reason L^* is also an admissible output operator for $S(\cdot)$. Then

$$
\lim_{t \longrightarrow \infty} \frac{1}{t} \mathbb{E} \left(\int_0^t \langle y_1(\tau), y_1(\tau) \rangle d\tau \right) = \lim_{t \longrightarrow \infty} \frac{1}{t} \mathbb{E} \left(\int_0^t \langle \begin{pmatrix} Q & L^* \\ L & R \end{pmatrix} \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2} \right).
$$

Consider the Pritchard-Salamon-Popov triple

$$
\Sigma \left(S(\cdot), \begin{pmatrix} B_1 & B_2 \end{pmatrix}, M = \begin{pmatrix} Q & L^* \\ L & R \end{pmatrix} = M^* \right) \tag{6.73}
$$

on $(\mathcal{W} \hookrightarrow \mathcal{V}, \mathcal{U}_1 \oplus \mathcal{U}_2)$ with the associated initial value problem (6.58) and the associated quadratic cost function

$$
J(x_0, u(\cdot)) = \lim_{t \longrightarrow \infty} \frac{1}{t} \mathbb{E} \left(\int_0^t \langle \begin{pmatrix} Q & L^* \\ L & R \end{pmatrix} \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2} \right). \tag{6.74}
$$

Here $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ has to be in the following class of admissible controls

$$U_{\text{adm}}^{x_0} \triangleq \left\{ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} (\cdot) \in \mathcal{PC}_T(0, \infty; \mathcal{U}_2) \oplus L_2(\Omega, \mathrm{p}; \mathcal{U}_1) \mid x(t) = S^W(t)x_0 + \int_0^t S^V(t - \tau)d\beta(\tau) + \right.$$

$$\left. + \int_0^t S^V(t - \tau)B_2u_2(\tau)d\tau \in L_2(\Omega, \mathrm{p}; \mathcal{W}) \right\}, \tag{6.75}$$

where $(\Omega, \mathcal{P}, \mathrm{p})$ is a complete probability space. We want to give a discrete-time representation of the Pritchard-Salamon-Popov triple (6.73) at the sampling instants $\{0, T, \cdots, kT, \cdots\}$ and to determine an equivalent expression for the quadratic cost function (6.74).

Notice firstly that for any $t \in [kT, (k+1)T)$ the state function (6.64) can be written as

$$\begin{aligned} x(t) &= S^W(t)x_0 + \int_0^t S^V(t - \tau)d\beta(\tau) + \int_0^t S^V(t - \tau)B_2u_2(\tau)d\tau = \\ &= S^W(t - kT)x(k) + \int_{kT}^t S^V(t - \tau)d\beta(\tau) + \int_{kT}^t S^V(t - \tau)B_2u_2(\tau)d\tau. \end{aligned}$$

It follows that

$$x(k+1) = S^W(T)x(k) + \int_{kT}^{(k+1)T} S^V((k+1)T - \tau)d\beta(\tau) + \int_0^T S^V(T - \tau)B_2u_2(\tau)d\tau \tag{6.76}$$

Let $v(\cdot) \in \mathcal{W}$ denote the second term in (6.76). Using Lemma 5.28 from [34] one can show immediately that it is a discrete-time Wiener process with self-adjoint non-negative definite covariace operator $V_T \in \mathcal{L}(\mathcal{V})$ defined by

$$V_T \triangleq \int_0^T S^V(\tau)B_1B_1^* \left(S^V(\tau)\right)^* d\tau. \tag{6.77}$$

Consider its restriction to $\mathcal{W}$ and by an abuse of notation denote it with the same symbol. Since V_T is self-adjoint and non-negative definite and since $\mathcal{U}_1$ and $\mathcal{W}$ are isomorphical spaces, it follows that there exists $\Gamma_1^T \in \mathcal{L}(\mathcal{U}_1, \mathcal{W})$ such that $V_T = \Gamma_1^T(\Gamma_1^T)^*$. Let $u_1^d(\cdot)$ be a discrete-time white noise of unit intensity. Then the following relation

$$v(k) = \Gamma_1^T u_1^d(k) \tag{6.78}$$

makes sense and we conclude that the state equation (6.58) defined by the stochastic differential equation (6.61) admits at the sampling instant am equivalent representation given by the following difference stochastic equation

$$x(k+1) = \mathbf{\Phi}x(k) + \mathbf{\Gamma}_1^T u_1^d(k) + \mathbf{\Gamma}_2 u_2(k), \tag{6.79}$$

where

$$\mathbf{\Gamma}_2 u_2 \triangleq \int_0^T S^V(T - \tau)B_2u d\tau.$$

Following the procedure developed in the case of finite-dimensional systems in [57], we shall express the cost function

$$J(x_0, u(\cdot)) = \lim_{t \to \infty} \frac{1}{t} \mathbb{E} \left(\int_0^t \langle \begin{pmatrix} Q & L^* \\ L & R \end{pmatrix} \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix}, \begin{pmatrix} x(t) \\ u_2(t) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2} \right) \qquad (6.80)$$

in the form

$$J(x_0, u(\cdot)) = \lim_{k \to \infty} \frac{1}{k} \mathbb{E} \left(\sum_{i=0}^{k-1} \langle \begin{pmatrix} Q_\mathrm{T} & L_\mathrm{T}^* \\ L_\mathrm{T} & R_\mathrm{T} \end{pmatrix} \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix}, \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2} \right) + \gamma, (6.81)$$

where Q_T, L_T^* and R_T are defined by (6.1), (6.2) and (6.3), respectively and where γ_i is defined by

$$\gamma = \mathbb{E} \int_0^T \text{trace} \left(\int_0^t S^\mathcal{V}(\tau) B_1 B_1^* \left(S^\mathcal{V}(\tau) \right)^* d\tau Q \right) dt. \qquad (6.82)$$

This is done as follows. Notice that $x(i)$ depends only on the increments $d\beta(\tau)$ for $s \in [0, iT]$, while $\int_{iT}^{iT+t} S^\mathcal{V}(iT + t - \tau) d\beta(\tau)$ only depends on the increments $d\beta(\tau)$ with $\tau \in [iT, iT + t]$. Hence, $x(i)$ and $\int_{iT}^{iT+t} S^\mathcal{V}(iT + t - \tau) d\beta(\tau)$ are independent. Since $u_2(i)$ depends only on the sequence $\{y_2(0), \cdots, y_2(i-1)\}$ and $v(i)$ and $u_3(k)$ are independent, it follows that $u_2(i)$ and $\int_{iT}^{iT+t} S^\mathcal{V}(iT + t - \tau) d\beta(\tau)$ are also independent. This fact implies that if we use the expression of the state function (6.76) in the cost function $J(x_0, u(\cdot))$ defined above, we obtain null cross terms from the aforementioned terms. Basically, there are only the terms from the expression of the equivalent discrete-time Popov triple for the digital LQ-optimal control problem plus an extra term due to the expectation of

$$\langle Q \int_{kT}^t S^\mathcal{V}(t - \tau) d\beta(\tau), \int_{kT}^t S^\mathcal{V}(t - \tau) d\beta(\tau) \rangle_\mathcal{W},$$

which, without difficulty, can be shown to be equal to (6.82).

Define now the modified cost function

$$J^\mathrm{m}(x_0, u(\cdot)) \triangleq J(x_0, u(\cdot)) - \gamma. \qquad (6.83)$$

Let us write down similar factorizations to the one defined for V_T, this time for the operators involved in the expression of the discrete Popov index determined above. Since $D_{12}^* D_{12}$ was assumed coercive, it follows that R_T is boundedly invertible and hence, by performing the change of variables

$$u_2^\mathrm{m}(k) = F_\mathrm{T} x(k) = R_\mathrm{T}^{-1} L_\mathrm{T}^* x(k)$$

we get

$$\begin{aligned} J^\mathrm{m}(x_0, u(\cdot)) &= \lim_{k \to \infty} \frac{1}{k} \mathbb{E} \left(\sum_{i=0}^{k-1} \langle \begin{pmatrix} Q_\mathrm{T} & L_\mathrm{T}^* \\ L_\mathrm{T} & R_\mathrm{T} \end{pmatrix} \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix}, \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2} \right) = \\ &= \lim_{k \to \infty} \frac{1}{k} \mathbb{E} \left(\sum_{i=0}^{k-1} \langle \begin{pmatrix} Q_\mathrm{T}^\mathrm{m} & 0 \\ 0 & R_\mathrm{T} \end{pmatrix} \begin{pmatrix} x(i) \\ u_2^\mathrm{m}(i) \end{pmatrix}, \begin{pmatrix} x(i) \\ u_2^\mathrm{m}(i) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2} \right), \quad (6.84) \end{aligned}$$

where

$$Q_{\mathrm{T}}^{\mathrm{m}} \triangleq Q_{\mathrm{T}} - L_{\mathrm{T}} R_{\mathrm{T}}^{-1} L_{\mathrm{T}}^{*}. \qquad (6.85)$$

Since $Q_{\mathrm{T}}^{\mathrm{m}}$ and $R_{\mathrm{T}}^{\mathrm{m}}$ are self-adjoint and non-negative definite operators and since $\mathcal{U}_1$, $\mathcal{Y}_1$ and $\mathcal{W}$ are isomrphical spaces, it follows that there exists $\Lambda_1 \in \mathcal{L}(\mathcal{W}, \mathcal{Y}_1)$ and $\Theta_{12} \in \mathcal{L}(\mathcal{U}_2, \mathcal{Y}_1)$ such that

$$
\begin{aligned}
Q_{\mathrm{T}}^{\mathrm{m}} &= (\Lambda_1)^* \Lambda_1, & (6.86)\\
R_{\mathrm{T}} &= (\Theta_{12})^* \Theta_{12}. & (6.87)
\end{aligned}
$$

We define the following infinite-dimensional discrete-time system

$$
\begin{aligned}
x(k+1) &= \Phi x(k) + \Gamma_1^{\mathrm{T}} u_1^{\mathrm{d}}(k) + \Gamma_2 u_2(k), & (6.88)\\
y_1(k) &= \Lambda_1 x(k) + \Theta_{12} u_2(k), & (6.89)\\
y_2(k) &= C_2 x(k) + D_{21} u_3(k). & (6.90)
\end{aligned}
$$

If, by an abuse of notation

$$u_1 \triangleq \begin{pmatrix} u_1^{\mathrm{d}} \\ u_3 \end{pmatrix} \in \mathcal{U}_1 \oplus \mathcal{U}_3$$

stands for the augmented disturbance sequence, then the system defined by (6.88), (6.89) and (6.90) can be re-written as

$$\Sigma_{\mathrm{sd}} : \left\{ \begin{array}{l} x(k+1) = \Phi x(k) + \Gamma_1 u_1(k) + \Gamma_2 u_2(k) \\ y_1(k) = \Lambda_1 x(k) + \Theta_{12} u_2(k) \\ y_2(k) = \Lambda_2 x(k) + \Theta_{21} u_1 \end{array} \right. , \qquad (6.91)$$

with

$$
\begin{aligned}
\Gamma_1 &\triangleq \begin{pmatrix} \Gamma_1^{\mathrm{T}} & 0 \end{pmatrix}, & (6.92)\\
\Lambda_2 &\triangleq C_2, & (6.93)\\
\Theta_{21} &\triangleq \begin{pmatrix} 0 & D_{21} \end{pmatrix}, & (6.94)
\end{aligned}
$$

We can formulate now the main control result of this section

Theorem 6.11 *The fixed-order digital LQG control problem for Pritchard-Salamon systems has a solution if and only if the finite-dimensional fixed-order ℓ^2-optimal control problem for the discrete-time infinite-dimensional system (6.91) has a solution. Furthermore, the compensator can be effectively constructed, i.e. there exists $X, Y \in \mathcal{L}(\mathcal{W})$, $X \geq 0$, $Y \geq 0$, and finite rank operators $\hat{X}, \hat{Y} \in \mathcal{W}$ such that A_c, B_c and C_c are defined by*

$$
\left\{ \begin{array}{l} A_{\mathrm{c}} = \Lambda A_{\mathrm{c}}^{\mathrm{o}} G^* \\ B_{\mathrm{c}} = \Lambda B_{\mathrm{c}}^{\mathrm{o}} \\ C_{\mathrm{c}} = C_{\mathrm{c}}^{\mathrm{o}} G^* \end{array} \right. ,
\left\{ \begin{array}{l} C_{\mathrm{c}}^{\mathrm{o}} = R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} \\ B_{\mathrm{c}}^{\mathrm{o}} = Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} \\ A_{\mathrm{c}}^{\mathrm{o}} = \Phi - B_{\mathrm{c}}^{\mathrm{o}} \Lambda_2 - \Gamma_2 C_{\mathrm{c}}^{\mathrm{o}} \end{array} \right. ,
\left\{ \begin{array}{l} Q_{\mathrm{E}} = \Phi Y \Lambda_2^* \\ P_{\mathrm{E}} = \Gamma_2^* X \Phi \\ V_{2\mathrm{E}} = \Lambda_2 Y \Lambda_2^* + \Theta_{21} \Theta_{21}^* \\ R_{2\mathrm{E}} = \Gamma_2^* X \Gamma_2 + \Theta_{12}^* \Theta_{12} \end{array} \right. ,
$$

for some (G, M, Λ)-factorization of $\hat{Y}\hat{X}$, and such that if $\tau = G^*\Lambda$, the following conditions are satisfied:

$$\text{rank}(\hat{X}) = \text{rank}(\hat{Y}) = \text{rank}(\hat{X}\hat{Y}) = n,$$

$$
\begin{aligned}
Y &= \Phi Y \Phi^* + V_{\mathrm{T}} - Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} Q_{\mathrm{E}}^* + \tau_\perp \left((\Phi - \Gamma_2 C_{\mathrm{c}}^{\mathrm{o}}) \hat{Y} (\Phi - \Gamma_2 C_{\mathrm{c}}^{\mathrm{o}})^* + Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} Q_{\mathrm{E}}^* \right) \tau_\perp^*, \\
X &= \Phi^* X \Phi + Q_{\mathrm{T}} - P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} + \tau_\perp^* \left((\Phi - B_{\mathrm{c}}^{\mathrm{o}} \Lambda_2)^* \hat{X} (\Phi - B_{\mathrm{c}}^{\mathrm{o}} \Lambda_2) + P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} \right) \tau_\perp, \\
\hat{Y} &= \tau \left((\Phi - \Gamma_2 C_{\mathrm{c}}^{\mathrm{o}}) \hat{Y} (\Phi - \Gamma_2 C_{\mathrm{c}}^{\mathrm{o}})^* + Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} Q_{\mathrm{E}}^* \right) \tau^*, \\
\hat{X} &= \tau^* \left((\Phi - B_{\mathrm{c}}^{\mathrm{o}} \Lambda_2)^* \hat{X} (\Phi - B_{\mathrm{c}}^{\mathrm{o}} \Lambda_2) + P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} \right) \tau,
\end{aligned}
$$

where $\tau_\perp = I_{\mathcal{W}} - \tau$.

Remark 6.12 *Notice that since $\{d\beta(t); t \geq 0\}$ was assumed to be a Wiener process on $\mathcal{W}$, then we had to assume compulsory that $B_1 B_1^*$ is a nuclear operator (see Definition 5.2 from [34]). It follows that V_{T} defined by (6.77) is also nuclear. Hence $\Gamma_1 \Gamma_1^*$ is nuclear and the first two assumptions from Theorem 4.14 are fulfilled. In order to apply it suuccessfully to the digital control problem, one has only to check for the bounded invertibility of $V_{2\mathrm{E}} = \Lambda_2 Y \Lambda_2^* + \Theta_{21} \Theta_{21}^*$ and $R_{2\mathrm{E}} = \Gamma_2^* X \Gamma_2 + \Theta_{12}^* \Theta_{12}$.*

6.5.2 The equivalent discrete-time fixed-order LQG control problem. Case II: $x_0 \in \mathcal{V}$

In the previous subsection the fixed-order digital LQG control problem for Pritchard-Salamon systems was addressed and solved for smooth Pritchard-Salamon systems with initial state in the smaller space $x_0 \in \mathcal{W}$. As in the case of digital LQ-optimal control problem, we can also consider the fixed-order digital LQG control problem on $\mathcal{V}$. We proceed as in section 6.2. Consider the modified quadratic cost functional (6.83). By Proposition 6.4 has a unique bounded extension to a quadratic functional for $x(\cdot) \in \mathcal{V}$ such that

$$
\begin{aligned}
J^{\mathrm{m}}(x_0, u(\cdot)) &= \lim_{k \to \infty} \frac{1}{k} \mathbb{E} \left(\sum_{i=0}^{k-1} \langle \begin{pmatrix} Q_{\mathrm{T}} & L_{\mathrm{T}}^* \\ L_{\mathrm{T}} & R_{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix}, \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix} \rangle_{\mathcal{W} \times \mathcal{U}_2} \right) = \\
&= \lim_{k \to \infty} \frac{1}{k} \mathbb{E} \left(\sum_{i=0}^{k-1} \langle \begin{pmatrix} \overline{Q_{\mathrm{T}}} & \overline{L_{\mathrm{T}}^*} \\ \overline{(L_{\mathrm{T}}^*)}^* & R_{\mathrm{T}} \end{pmatrix} \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix}, \begin{pmatrix} x(i) \\ u_2(i) \end{pmatrix} \rangle_{\mathcal{V} \times \mathcal{U}_2} \right), (6.95)
\end{aligned}
$$

with the Popov index

$$\overline{M_{\mathrm{T}}} \triangleq \begin{pmatrix} \overline{Q_{\mathrm{T}}} & \overline{L_{\mathrm{T}}^*} \\ \overline{(L_{\mathrm{T}}^*)}^* & R_{\mathrm{T}} \end{pmatrix} = \left(\overline{M_{\mathrm{T}}} \right)^* \in \mathcal{L}(\mathcal{V} \oplus \mathcal{U}_2). \tag{6.96}$$

As in the previous section, we can write down factorizations for

$$\overline{Q_{\mathrm{T}}}^{\mathrm{m}} \triangleq \overline{Q_{\mathrm{T}}} - \left(\overline{L_{\mathrm{T}}^{*}}\right)^{*} R_{\mathrm{T}} \overline{L_{\mathrm{T}}^{*}} \tag{6.97}$$

as

$$\overline{Q_{\mathrm{T}}}^{\mathrm{m}} = \left(\overline{\Lambda_1}\right)^{*} \overline{\Lambda_1} \tag{6.98}$$

and for R_{T} as defined by (6.87). It follows that the following infinite-dimensional system on $\mathcal{V}$ is well posed

$$\Sigma_{\mathrm{sd}} : \begin{cases} x(k+1) = \Phi x(k) + \Gamma_1 u_1(k) + \Gamma_2 u_2(k) \\ y_1(k) = \overline{\Lambda_1} x(k) + \Theta_{12}^{\mathrm{T}} u_2(k) \\ y_2(k) = \Lambda_2 x(k) + \Theta_{21} u_1 \end{cases} \tag{6.99}$$

We can formulate the main Hyland-Bernstein control result of his section as

Theorem 6.13 *The fixed-order digital LQG control problem for Pritchard-Salamon systems on $\mathcal{V}$ has a solution if and only if the finite-dimensional fixed-order ℓ^2-optimal control problem for the discrete-time infinite-dimensional system (6.91) has a solution on $\mathcal{V}$. Furthermore, the compensator can be effectively constructed, i.e. there exists $X, Y \in \mathcal{L}(\mathcal{V})$, $X \geq 0$, $Y \geq 0$, and finite rank operators $\hat{X}, \hat{Y} \in \mathcal{V}$ such that A_c, B_c and C_c are defined by*

$$\begin{cases} A_c = \Lambda A_c^{\circ} G^* \\ B_c = \Lambda B_c^{\circ} \\ C_c = C_c^{\circ} G^* \end{cases} , \begin{cases} C_c^{\circ} = R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} \\ B_c^{\circ} = Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} \\ A_c^{\circ} = \Phi - B_c^{\circ} \Lambda_2 - \Gamma_2 C_c^{\circ} \end{cases} , \begin{cases} Q_{\mathrm{E}} = \Phi Y \Lambda_2^* \\ P_{\mathrm{E}} = \Gamma_2^* X \Phi \\ V_{2\mathrm{E}} = \Lambda_2 Y \Lambda_2^* + \Theta_{21} \Theta_{21}^* \\ R_{2\mathrm{E}} = \Gamma_2^* X \Gamma_2 + \Theta_{12}^* \Theta_{12} \end{cases} ,$$

for some (G, M, Λ)-factorization of $\hat{X}\hat{Y}$, and such that if $\tau = G^*\Lambda$, the following conditions are satisfied:

$$\mathrm{rank}(\hat{X}) = \mathrm{rank}(\hat{Y}) = \mathrm{rank}(\hat{X}\hat{Y}) = n,$$

$$\begin{aligned} Y &= \Phi Y \Phi^* + V_{\mathrm{T}} - Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} Q_{\mathrm{E}}^* + \tau_{\perp} \left((\Phi - \Gamma_2 C_c^{\circ}) \hat{Y} (\Phi - \Gamma_2 C_c^{\circ})^* + Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} Q_{\mathrm{E}}^* \right) \tau_{\perp}^*, \\ X &= \Phi^* X \Phi + \overline{Q_{\mathrm{T}}} - P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} + \tau_{\perp}^* \left((\Phi - B_c^{\circ} \Lambda_2)^* \hat{X} (\Phi - B_c^{\circ} \Lambda_2) + P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} \right) \tau_{\perp}, \\ \hat{Y} &= \tau \left((\Phi - \Gamma_2 C_c^{\circ}) \hat{Y} (\Phi - \Gamma_2 C_c^{\circ})^* + Q_{\mathrm{E}} V_{2\mathrm{E}}^{-1} Q_{\mathrm{E}}^* \right) \tau^*, \\ \hat{X} &= \tau^* \left((\Phi - B_c^{\circ} \Lambda_2)^* \hat{X} (\Phi - B_c^{\circ} \Lambda_2) + P_{\mathrm{E}}^* R_{2\mathrm{E}}^{-1} P_{\mathrm{E}} \right) \tau, \end{aligned}$$

where $\tau_{\perp} = I_{\mathcal{V}} - \tau$.

In the case of digital LQ-optimal control problem we have shown that the solutions to the digital Riccati equation on $\mathcal{W}$ has a continuous extension to the solution of the digital Riccati equation on $\mathcal{V}$. Let us see what happens to the solutions to the modified Riccati and modified Lyapunov equations on $\mathcal{W}$ and $\mathcal{V}$. Apply Lemma 4.21 to the closed loop system from u_1 to y_1 first on $\mathcal{W}$ and then on $\mathcal{V}$. It follows that

$$
\begin{aligned}
J^m(x_0, u(\cdot)) &= \text{trace } (Q_{\text{T}} X_1 + C_c^* \Theta_{12}^* \Theta_{12} C_c X_{22}) = \\
&= \text{trace } \left(\overline{Q_{\text{T}} X_1} + C_c^* \Theta_{12}^* \Theta_{12} C_c X_{22} \right).
\end{aligned}
\tag{6.100}
$$

It follows that the functional

$$
\text{trace } (Q_{\text{T}} X_1) : \mathcal{W} \longrightarrow \mathbb{R}
$$

is finite and has a unique bounded extension to the functional on $\mathcal{V}$

$$
\text{trace } \left(\overline{Q_{\text{T}} X_1} \right) : \mathcal{V} \longrightarrow \mathbb{R}.
$$

We conclude that $\overline{X}_1 \in \mathcal{L}(\mathcal{V})$ represents the unique bounded extension to $\mathcal{V}$ of $X_1 \in \mathcal{L}(\mathcal{W})$, where X_1 and $\overline{X}_1$ represent the 11-block in the decomposition of the closed loop controllability gramians on $\mathcal{W}_{\text{R}} \overset{w}{\oplus} \mathbb{R}^n$ and $\mathcal{V}_{\text{R}} \overset{v}{\oplus} \mathbb{R}^n$, respectively. Let $\overline{X} \overset{\triangle}{=} \overline{X_1 - X_{12}^* X_2 X_{12}} \in \mathcal{L}(\mathcal{V})$ denote the unique bounded extension to $\mathcal{V}$ of $X \overset{\triangle}{=} X_1 - X_{12}^* X_2 X_{12} \in \mathcal{L}(\mathcal{W})$. Then the following holds

Proposition 6.14 $\overline{X} \in \mathcal{L}(\mathcal{V})$ *is the unique bounded extension to* $\mathcal{V}$ *of* $X \in \mathcal{L}(\mathcal{W})$ *if and only* $\overline{X_{12}^* X_2 X_{12}} \in \mathcal{L}(\mathcal{V})$ *is the unique bounded extension to* $\mathcal{V}$ *of* $X_{12}^* X_2 X_{12} \in \mathcal{L}(\mathcal{W})$.

It remains an open question whether in general $\overline{X} \in \mathcal{L}(\mathcal{V})$ is indeed the unique bounded extension to $\mathcal{V}$ of $X \in \mathcal{L}(\mathcal{W})$. This would imply that the condition in the above proposition is generic for Pritchard-Salamon systems.

Part IV

Applications

Chapter 7

Implementation issues

As was mentioned in the introduction chapter, most of the interesting applications occur with unboundedness of input and /or output operators. In this case study we shall focus our attention on a class of Pritchard-Salamon systems of parabolic and hyperbolic type.

7.1 A case study

In this section we shall be concerned with two examples of infinite-dimensional Pritchard-Salamon systems with a special property, namely they have a pure discrete spectrum. Those systems are commonly referred to systems with diagonal semigroup.

7.1.1 Parabolic systems

Consider the general type of parabolic system as presented in Example 2.21. The time-discretized Pritchard-Salamon system is then given by (Φ, Γ, Λ) with

$$
\Phi x \;=\; \sum_{n=0}^{\infty} e^{\lambda_n T} \langle x, \phi_n \rangle_W \phi_n =
\begin{pmatrix}
e^{\lambda_0 T} & & & \\
& \ddots & & \\
& & e^{\lambda_n T} & \\
& & & \ddots
\end{pmatrix} x,
\tag{7.1}
$$

$$
\Gamma u \;=\; \int_0^T S^V(\eta) B d\eta = \int_0^T \sum_{n=0}^{\infty} e^{\lambda_n \eta} \langle \{b_n u, n \geq 0\}, \phi_n \rangle_V \phi_n =
$$

$$
=\; \sum_{n=0}^{\infty} \frac{e^{\lambda_n T} - 1}{\lambda_n} \langle \{b_n u, n \geq 0\}, \phi_n \rangle_V \phi_n = \left\{ \frac{e^{\lambda_n T} - 1}{\lambda_n} b_n u \right\}_{n \geq 1} =
$$

$$
=\;
\begin{pmatrix}
\vdots \\
\frac{e^{\lambda_n T} - 1}{\lambda_n} b_n \\
\vdots
\end{pmatrix},
\tag{7.2}
$$

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$$\mathbf{\Lambda} x \;=\; Cx = \sum_{n=0}^{\infty} c_n \langle x, \phi_n \rangle = \Big(\; \cdots \quad c_n \quad \cdots \; \Big), \tag{7.3}$$

for all $x \in \mathcal{W}$ and $u \in \mathbb{R}$.

The impulse response of the time-discretized Pritchard-Salamon system $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, 0)$ is characterized by the following

Proposition 7.1 *Assume that (2.44) holds. Then the (causal) impulse response associated with $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda})$ belongs to $\ell_1(0, \infty; \mathbb{R})$.*

Proof Let

$$h(N) \stackrel{\triangle}{=} \mathbf{\Lambda}\mathbf{\Phi}^N\mathbf{\Gamma} = \sum_{k=0}^{\infty} \frac{c_k b_k e^{\lambda_k NT} \left(e^{\lambda_k T - 1} \right)}{\lambda_k} \tag{7.4}$$

define the impulse response associated with time-discretized Pritchard-Salamon system $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda})$. Then its ℓ_1-norm is given by

$$
\begin{aligned}
\|h\|_{\ell_1(0,\infty;\mathbb{R})} &\stackrel{\triangle}{=} \sum_{N=1}^{\infty} \sum_{k=0}^{\infty} \frac{c_k b_k e^{\lambda_k NT} \left(e^{\lambda_k T - 1} \right)}{\lambda_k} = \\
&= \sum_{k=0}^{\infty} \frac{c_k b_k \left(e^{\lambda_k T - 1} \right)}{\lambda_k} \sum_{N=0}^{\infty} e^{\lambda_k NT} = \\
&= -\sum_{k=0}^{\infty} \frac{c_k b_k}{\lambda_k}.
\end{aligned}
\tag{7.5}
$$

But we have assumed that (2.44) holds, i.e

$$\sum_{k=0}^{\infty} \frac{c_k b_k}{\sqrt{|\lambda_k|}} < \infty$$

and the sequence of real neagtive eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$$

is strictly decreasing. It follows that the right-hand side of (7.5) is finite and hence, $\|h\|_{\ell_1(0,\infty;\mathbb{R})} < \infty$.

∎

The parabolic system considered in this example presents the advantage of being characterized by a diagonal C_0-semigroup. Taking into account the discussion made on prestabilization, we shall assume it exponentially stable on $\mathcal{W}$. Since it is diagonal, it admits the

following matrix representation

$$Ax = \begin{pmatrix} \lambda_0 & & & & \\ & \ddots & & & \\ & & \lambda_n & & \\ & & & \ddots & \end{pmatrix} x, x \in \mathcal{D}\left(A^{\chi}\right),$$

$$S^{W}(t)x = \begin{pmatrix} e^{\lambda_0 t} & & & & \\ & \ddots & & & \\ & & e^{\lambda_n t} & & \\ & & & \ddots & \end{pmatrix} x, x \in \mathcal{X}.$$

Notice that C^*C has the following double infinite-dimensional matrix representation

$$C^*Cx = \begin{pmatrix} & \vdots & \\ \cdots & c_i c_j & \cdots \\ & \vdots & \end{pmatrix}_{i,j \geq 1} x, \quad x \in \mathcal{W}. \tag{7.6}$$

Then the matrix representations for $Q_T = Q_T^* \in \mathcal{L}(\mathcal{W})$, $L_T \in \mathcal{L}(\mathcal{W},\mathcal{U})$ are given by

$$Q_T = \begin{pmatrix} & \vdots & \\ \cdots & \frac{c_i c_j \left(e^{(\lambda_i + \lambda_j)T} - 1\right)}{\lambda_i + \lambda_j} & \cdots \\ & \vdots & \end{pmatrix}_{i,j \geq 1}, \tag{7.7}$$

$$L_T = \begin{pmatrix} & \vdots & \\ \sum_{i=0}^{\infty} \frac{\lambda_i \left(e^{(\lambda_i+\lambda_j)T}-1\right)-(\lambda_i+\lambda_j)e^{\lambda_i T}-\lambda_j}{\lambda_i \lambda_j (\lambda_i+\lambda_j)} c_i c_j b_j & \\ & \vdots & \end{pmatrix}_{j \geq 1}. \tag{7.8}$$

Let us calculate the expression of $R_T \in \mathcal{L}(\mathcal{U})$.

$$
\begin{aligned}
R_T &= T + \int_0^T \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \frac{\left(e^{(\lambda_i+\lambda_j)\tau} - e^{\lambda_i \tau} - e^{\lambda_j \tau} + 1\right)}{\lambda_i \lambda_j} b_i b_j c_i c_j d\tau = \text{ by theorem of Fubini} \\
&= T + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^T \frac{\left(e^{(\lambda_i+\lambda_j)\tau} - e^{\lambda_i \tau} - e^{\lambda_j \tau} + 1\right)}{\lambda_i \lambda_j} b_i b_j c_i c_j d\tau = \\
&= T + \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left(\frac{e^{(\lambda_i+\lambda_j)T} - 1}{\lambda_i \lambda_j (\lambda_i + \lambda_j)} - \frac{e^{\lambda_i T} - 1}{\lambda_i^2 \lambda_j} - \frac{e^{\lambda_j T} - 1}{\lambda_i \lambda_j^2} + \frac{T}{\lambda_i \lambda_j} \right) b_i b_j c_i c_j. \tag{7.9}
\end{aligned}
$$

Having available the operators involved in the expression of Popov index of the equivalent discrete-time Popov triple, we proceed to express the solution $X \in \mathcal{W}$ to the digital

Riccati equation (6.18). We make here use of the operatorial expression of the solution to a discrete-time Riccati equation

$$X = \mathcal{P}_0 - \mathcal{P}\mathcal{R}^{-1}\mathcal{P}^*,$$

with $\mathcal{P}_0$ $\mathcal{P}$ and $\mathcal{R}$ defined by (3.18). We begin by calculating $\mathcal{P}_0$. The following holds

Lemma 7.2 *The operator $\mathcal{P}_0 : \mathcal{W} \longrightarrow \mathcal{W}$ satisfies the following Lyapunov equation*

$$\mathbf{\Phi}^*\mathcal{P}_0\mathbf{\Phi} - \mathcal{P}_0 + Q_{\mathrm{T}} = 0. \tag{7.10}$$

Proof By direct computation.

■

Exploiting the diagonal form of the semigroup and using the expression of Q_{T} it is a routine to show that (see also [4, 16, 5])

$$\mathcal{P}_0 = \begin{pmatrix} & \vdots & \\ \cdots & \frac{c_i c_j \left(e^{(\lambda_i+\lambda_j)T}-1 \right)}{(\lambda_i+\lambda_j)(1-e^{\lambda_i+\lambda_j}T)} & \cdots \\ & \vdots & \end{pmatrix}_{i,j\geq 1}. \tag{7.11}$$

We calculate now the $\mathcal{P}$-operator. We have

$$\mathcal{P} = \sum_{i=0}^{k-1} \mathbf{\Phi}^{k-i-1}\Gamma Q_{\mathrm{T}} + L_{\mathrm{T}}\mathbf{\Phi}^i, \tag{7.12}$$

or equivalently, exploiting the diagonal form of the semigroup and the expressions of Q_{T} and L_{T}, we get

$$\mathcal{P} = \sum_{i=0}^{k-1} \begin{pmatrix} & \vdots & \\ \cdots & e^{(k-i-1)\lambda_i T} & \cdots \\ & \vdots & \end{pmatrix} \begin{pmatrix} \vdots \\ \frac{e^{\lambda_i T}-1}{\lambda_i} \\ \vdots \end{pmatrix} \begin{pmatrix} & \vdots & \\ \cdots & \frac{c_i c_j \left(e^{(\lambda_i+\lambda_j)T}-1 \right)}{\lambda_i+\lambda_j} & \cdots \\ & \vdots & \end{pmatrix} +$$

$$+ \begin{pmatrix} \vdots \\ \sum_{i=0}^{\infty} \frac{\lambda_i \left(e^{(\lambda_i+\lambda_j)T}-1 \right)-(\lambda_i+\lambda_j)e^{\lambda_i T}-\lambda_j}{\lambda_i\lambda_j(\lambda_i+\lambda_j)}c_i c_j b_j \\ \vdots \end{pmatrix} \begin{pmatrix} & \vdots & \\ \cdots & e^{\lambda_i T} & \cdots \\ & \vdots & \end{pmatrix}. \tag{7.13}$$

Let us calculate now the $\mathcal{R}$-operator. Simple manipulations show that

$$\mathcal{R} = R_{\mathrm{T}} + \sum_{i=0}^{k-1} L_{\mathrm{T}}^*\mathbf{\Phi}^{k-i-1}\Gamma + \sum_{i=k+1}^{\infty} \Gamma^*(\mathbf{\Phi}^*)^{i-k-1} L_{\mathrm{T}} + \sum_{i=k+1}^{\infty} \Gamma^*(\mathbf{\Phi}^*)^{i-k-1} Q\mathbf{\Phi}^k. \tag{7.14}$$

If $\mathcal{P}_0$ was, losely speaking, easy to calculate, in the case of $\mathcal{P}$ and $\mathcal{R}$ one has to handle expression of higher complexity. It is, therefore, of paramount importance to be able to approximate the solution in the form $X = \mathcal{P}_0 - \mathcal{P}\mathcal{R}^{-1}\mathcal{P}^*$ rather then trying to compute it analytically, even in such a simple case, as the one of diagonal semigroup is. One possible way to approximate X is to determine a certain approximation of $\mathcal{R}^{-1}$. Since $\mathcal{R}$ has a special structure, it is a discrete Toeplitz operator, we believe that a very promising direction of research would be defined by: *application of invertibility theory for Toeplitz operators to approximate stabilizing solutions of Riccati equations.*

7.1.2 Hyperbolic systems

Consider the infinite-dimensional continuous-time, time invariant system

$$\ddot{x}(t) = Ax(t) + Bu(t), \tag{7.15}$$
$$y(t) = Cx(t), \tag{7.16}$$

where A is a self-adjoint operator on a Hilbert space $\mathbf{X}$. Assume that A has a compact resolvent operator and it has a set of simple, negative eigenvalues $\lambda_n = -\omega_n^2$ satisfying

$$\omega_1 \geq \delta, \quad \omega_{n+1} - \omega_n \geq \delta, \quad n > 0,$$

for some positive δ. Denote by $\{\phi_n\}_{n>0}$ the corresponding orthonormal set of eigenvectors. Let us assume that we have two spaces

$$\mathbf{W}_0 = \{x \in \mathcal{X} | \sum_{n=1}^{\infty} \gamma_n |\lambda_n| \langle x, \phi_n \rangle^2 < \infty\}, \tag{7.17}$$

$$\mathbf{V}_1^* = \{x \in \mathcal{X} | \sum_{n=1}^{\infty} \frac{1}{\beta_n} \langle x, \phi_n \rangle^2 < \infty\} \tag{7.18}$$

Let $\{b_n\}_{n>0} \in \mathbb{R}^m$ and $\{c_n\}_{n>0} \in \mathbb{R}^p$ are such that

$$\sum_{n=1}^{\infty} \beta_n \|b_n\|^2 < \infty, \quad \sum_{n=1}^{\infty} \frac{1}{\gamma_n |\lambda_n|} \|c_n\|^2 < \infty$$

and the control and observation operators $B \in \mathcal{L}(\mathbb{R}^m, \mathbf{V}_1)$ and $C \in \mathcal{L}(\mathbf{W}_0, \mathbb{R}^p)$ are given by

$$B^* u = \sum_{n=1}^{\infty} \langle b_n, u \rangle \phi_n, \tag{7.19}$$

$$Cx = \sum_{n=1}^{\infty} c_n \langle x, \phi_n \rangle. \tag{7.20}$$

Define $\mathbf{V} \subset \mathbf{X}$ by $\mathbf{V} = D((-A)^{\frac{1}{2}}) = \{x \in \mathbf{X} | \sum_{n=1}^{\infty} |\lambda_n| \langle x, \phi_n \rangle^2 < \infty\}$ and identify $\mathbf{X} = \mathbf{X}^*$. This fact implies $\mathbf{V} \subset \mathbf{X} \subset \mathbf{V}^*$ and A extends to $A \in \mathcal{L}(\mathbf{V}, \mathbf{V}^*)$. We want to transform

(7.15) and (7.16) into a first order system. For this define $\mathcal{X} = \mathbf{V} \times \mathbf{X}$ and provide the inner product

$$\langle x, y \rangle = \sum_{n=1}^{\infty} [|\lambda_n| \langle x_0, \phi_n \rangle \langle y_0, \phi_n \rangle + \langle x_1, \phi_n \rangle \langle y_1, \phi_n \rangle].$$

Then consider the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathcal{X} \mapsto \mathcal{X}$

$$\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}, \quad D(\mathcal{A}) = D(A) \times \mathbf{V}.$$

It is straightforward to prove that $\mathcal{A}$ is the generator of the C_0-semigroup $\{S(t)\}_{t \geq 0} \in \mathcal{L}(\mathcal{X})$ which is defined by

$$S(t)x = \begin{bmatrix} \sum_{n=1}^{\infty} [\cos \omega_n t \langle x_0, \phi_n \rangle + \frac{1}{\omega_n} \sin \omega_n t \langle x_1, \phi_n \rangle] \phi_n \\ \sum_{n=1}^{\infty} [\omega_n \sin \omega_n t \langle x_0, \phi_n \rangle + \frac{1}{\omega_n} \cos \omega_n t \langle x_1, \phi_n \rangle] \phi_n \end{bmatrix}. \tag{7.21}$$

Introduce the spaces

$$\mathcal{W} = \{x \in \mathcal{X} | \sum_{n=1}^{\infty} \gamma_n [|\lambda_n| \langle x_0, \phi_n \rangle^2 + \langle x_1, \phi_n \rangle^2] < \infty\}, \tag{7.22}$$

$$\mathcal{V}^* = \{x \in \mathcal{X} | \sum_{n=1}^{\infty} \frac{1}{\beta_n} [|\lambda_n| \langle x_0, \phi_n \rangle^2 + \langle x_1, \phi_n \rangle^2] < \infty\}, \tag{7.23}$$

and the operators $\mathcal{B} \in \mathcal{L}(\mathbb{R}^m, \mathcal{V})$ and $\mathcal{C} \in \mathcal{L}(\mathcal{W}, \mathbb{R}^p)$ defined by

$$\mathcal{B}^* = [0 \ B^*], \quad \mathcal{C} = [C \ 0].$$

In this way we have re-written (7.15), (7.16) into an equivalent Cauchy problem

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{B}u(t), \tag{7.24}$$

$$y(t) = \mathcal{C}x(t). \tag{7.25}$$

Identify $\mathcal{X} = \mathbf{V} \times \mathbf{X} = \mathcal{X}^*$. Lemma 4.8 from [70] gives the sufficient conditions on $\gamma_n, \beta_n, \lambda_n, b_n$ and c_n such that $\mathcal{B}$ and $\mathcal{C}$ are admissible input and admissible output operators. Lemma 4.9 from [70] shows under which conditions $\mathcal{W}$ and $\mathcal{V}$ can be chosen in such a way that $\Sigma(\Phi(\cdot), \mathcal{B}, \mathcal{C})$ is a Pritchard-Salamon system with respect to $\mathcal{W} \hookrightarrow \mathcal{V}$. We shall assume those conditions fulfilled and we proceed to calculate the time-discretized Pritchard-Salamon system.

(i) Computation of $\Phi = \Phi^{\mathcal{W}}(T)$

Let $\xi \in \mathcal{W}$, $\xi = (\xi_0, \xi_1)$. Then

$$\Phi\xi = \begin{bmatrix} \sum_{n=1}^{\infty} [\cos \omega_n T \langle \xi_0, \phi_n \rangle + \frac{1}{\omega_n} \sin \omega_n T \langle \xi_1, \phi_n \rangle] \phi_n \\ \sum_{n=1}^{\infty} [\omega_n \sin \omega_n T \langle \xi_0, \phi_n \rangle + \frac{1}{\omega_n} \cos \omega_n T \langle \xi_1, \phi_n \rangle] \phi_n \end{bmatrix}.$$

$$\tag{7.26}$$

(ii) Computation of Γ

Let $u(\cdot) \in L_2(0, T; \mathbb{R}^m)$. Then

$$
\begin{aligned}
\Gamma u &= \begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^T \sin \omega_n t \langle b_n, u(\cdot) \rangle d\tau \phi_n \\ \sum_{n=1}^{\infty} \frac{1}{\omega_n} \int_0^T \sin \omega_n t \langle b_n, u(\cdot) \rangle d\tau \phi_n \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2} \cos \omega_n t |_{t=0}^{t=T} \langle b_n, u(\cdot) \rangle d\tau \phi_n \\ -\sum_{n=1}^{\infty} \frac{1}{\omega_n^2} \sin \omega_n t |_{t=0}^{t=T} \langle b_n, u(\cdot) \rangle d\tau \phi_n \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=1}^{\infty} \frac{1}{\omega_n^2} (\cos \omega_n T - 1) \langle b_n, u(\cdot) \rangle d\tau \phi_n \\ -\sum_{n=1}^{\infty} \frac{1}{\omega_n^2} \sin \omega_n T \langle b_n, u(\cdot) \rangle d\tau \phi_n \end{bmatrix}.
\end{aligned}
$$

(iii) Computation of Λ

$$
\Lambda \xi = \mathcal{C}\xi = [\sum_{n=1}^{\infty} c_n \langle x, \phi_n \rangle \quad 0], \quad \xi = [x \quad 0]. \tag{7.27}
$$

If in the case of the parabolic system considered in the previous section, we have taken advantage of the diagonality of the strongly continuous semigroup $S^{\mathcal{V}}(\cdot)$, and we have performed in a easy way the computation of the equivalent discrete-time Popov triple, in the case of the hyperbolic system considered in this section such a computation is considerable heavier to pe done due to the expressions of the Φ, Γ and Λ operators.

Chapter 8

Model reduction of distributed parameter systems

8.1 Approaches to finite-order control of DPSs

There are three basic ways to design a finite-order controller for a infinite-dimensional linear system. In Figure 8.1 we have depicted the principles of such a design process.
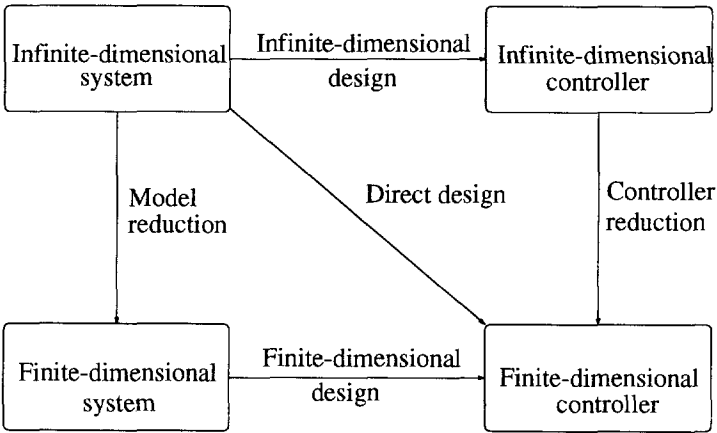
Figure 8.1: Basic principle of finite-order control design of DPS's

The finite-order control design methods for infinite-dimensional systems are broadly divided into two classes

(i) direct methods

(ii) indirect methods

If we constrain the control design process to be a digital one, then the diagram depicted in Figure 8.1 becomes
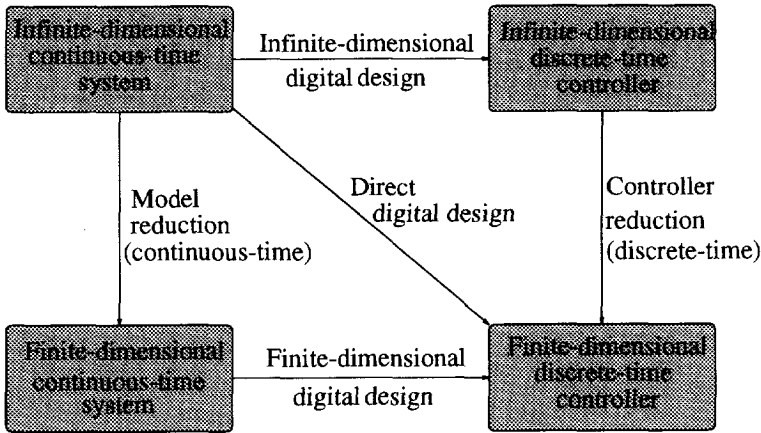
Figure 8.2: Basic principle of finite-order digital control design of DPS's

There are a few important ideas emerging from Figure 8.2.

(i) If indirect digital design is performed, then it is compulsory to have adequate tools and strategies developed for digital design, both in finite-dimensional and infinite-dimensional settings. If in the finite-dimensional setting there are many contributions to set-up strategies of digital control, in infinite-dimensional setting the literature on digital control has not been so extensively developed. The Popov theory based approach we took in this thesis offered us the right framework for writing down the digital solutions to various modern control problems, such as LQ-optimal control problem, LQG control problem, $\mathcal{H}^2$-optimal control problem and $\mathcal{H}^\infty$-(sub)optimal control problem.

(ii) If indirect digital design is performed, then besides the fact that adequate tools for digital design are needed, there is also a need to have available model reduction and controller reduction techniques. As suggested by the diagram we need to know how to obtain

- finite-order continuous-time models from infinite-dimensional continuous-time ones,

- finite-order discrete-time controllers from infinite-dimensional discrete-time ones.

Beside the above mentioned two items, there is another one, not of a less importance

- analysis of time discretized infinite-dimensional systems.

This statement is based upon the fact that these systems arise naturally in the digital design process. Recall that

(i) The solution to the digital LQ-optimal control problem for Pritchard-Salamon-Popov triples was based on the solution to the discrete-time LQ-optimal control problem written for the equivalent discrete-time Popov triple. In the case when we solve the so-called minimum energy control problem then the time-discretized Pritchard-Salamon system is the natural object we operate with.

(ii) The concept of time-discretized Pritchard-Salamon system plays a foundamental role in the hybrid stability theory built in Section 5.3.

A delicate problem one has to cope with when designing a digital controller for a infinite-dimensional continuous-time system is the implementation problem. One alternative to the design process is given by the robust stability synthesis. The idea is to design a digital stabilizing controller for a finite-dimensional approximation of the continuous-time infinite-dimensional plant which stabilizes a whole family of linear systems which contains the original infinite-dimensional model, rather than only the approximation. In order to do this one needs to obtain a discrete-time, finite-dimensional approximation of a continuos-time infinite-dimensional system. There are two basic ways to obtain such a model as it is shown in Figure 8.3
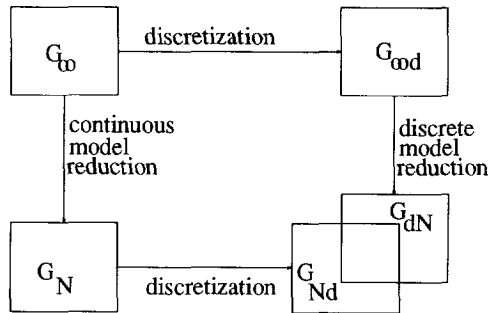
Figure 8.3: Two ways to obtain a discrete-time approximation

We have denoted here

- G_∞ the transfer function of the original continuous-time, infinite-dimensional system,

- $G_{\infty d}$ the transfer function of the time-discretized infinite-dimensional system,

- G_N the N-dimensional approximation of G_∞,

- G_{Nd} the time-discretized of the N-dimensional approximation of G_∞,

- G_{dN} the N-dimensional approximation of $G_{\infty d}$.

Apparently, the two operations

(i) model reduction,

(ii) discretization

do not intertwine. If this is the case, then discrete controllers designed for G_{dN} and G_{Nd} might be also very different and, which is the worst, might behave differently in close-loop connection with the original plant. We are going to study the problem "how far are G_{dN} and G_{Nd} one from each other" in Section 8.2. For the sake of simplicity of proofs, we shall assume that G_∞ is, actually, a high-order, but finite-dimensional system and we show that there exist realizations of G_{dN} and G_{Nd} coming close one to each other. This is done by introducing the concept of "approximately balancing" and by proving certain convergence results. The extension of those results to infinite-dimensional systems is not trivial at all. At this stage we are able to perform such an extension only for systems with diagonal semigroup. This is what we do in section 8.3.2 where we also point out the difficulties that occur when one tries to extend those results to more general classes of infinite-dimensional systems.

8.2 Reduced-order finite-dimensional discretized systems

A legitimate complaint directed toward dynamic systems modeling literature is whether a high-order model can be replaced by a low-order model, without incurring too much error. A wide variety of approaches to the model reduction problem have been proposed over the years. Despite the serious efforts that have been made, the status of the problem has changed only after the theories of balanced realizations and optimal Hankel-norm approximations have been developed [40].

Beginning with the work of Moore [63], the balanced realization have been successfully used for developing model reduction techniques. There has been a great deal of interest in the last decade to devise computational algorithms for obtaining balanced realizations of both continuous-time and discrete-time systems. After the classical algorithm of Laub [58], a major improvement was brought by an algorithm proposed by Hammarling [44] who obtains the controllability and observability gramians directly Cholesky factorized. Efficient implementations of this method are available for both continuous-time and discrete-time systems [81].

The difficulty to compute the gramians of sampled-data models for small sampling periods arises from the ill-conditioning of the corresponding discrete-time Lyapunov equations. Under a certain limit for the sampling step, the numerically results are corrupted by errors. Our paper aims to circumvent this drawback by proposing an algorithm which

provides an "approximately" balanced realization of the time-discretized system, obtained directly from the balanced realization of its continuous-time counterpart. We show that this realization comes "close" to the exactly balanced realization when the sampling period decreases to zero. As the "approximately" balancing technique is more accurate than the true balancing method for "very small" sampling steps (i.e. much smaller than the systems's time constants), we believe that the method proposed in this paper, besides its theoretical meaning, has certain importance even as a computational procedure for balancing time-discretized systems.

8.2.1 Mathematical background

We shall consider linear time invariant finite-dimensional systems $\Sigma_G(A, B, C, D)$ over the Euclidean space

$$\mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{p \times n} \times \mathbb{C}^{p \times m}$$

and we shall assume that

(i) (A, B) is controllable,

(ii) (C, A) is observable.

Let

$$S(t) \triangleq e^{At}$$

be the exponentially (power) stable continuous-time (discrete-time) semigroup generated by A on the Euclidean space $\mathbb{R}^n$ and let

$$
\begin{aligned}
(\sigma x)(t) &= Ax(t) + Bu(t),\ x(0) = x_0, \\
y(t) &= Cx(t) + Du(t),
\end{aligned}
$$

be the state space representation of $\Sigma_G(A, B, C, D)$, where $(\sigma x)(\cdot)$ defined by (1.4) denotes either the differential operator if $t \in \mathbb{R}$ or the advance unit shift operator if $t \in \mathbb{N}$. We interpret $x \in \mathbb{C}^n$ as being the state vector, $u \in \mathbb{C}^m$ the control and $y \in \mathbb{C}^p$ the measured output. A minimal stable realization of $\Sigma_G(A, B, C, D)$ will be denoted by (A_b, B_b, C_b, D_b).

The controllability and observability gramians defined by

$$P = \int_0^\infty e^{At} BB^* e^{A^*t} dt,\ t \in \mathbb{R}, \tag{8.1}$$

$$P = \sum_{i=0}^\infty A^i BB^* (A^*)^i,\ t \in \mathbb{N}, \tag{8.2}$$

$$Q = \int_0^\infty e^{A^*t} C^* C e^{At} dt,\ t \in \mathbb{R}, \tag{8.3}$$

$$Q = \sum_{i=0}^\infty (A^*)^i C^* C A^i,\ t \in \mathbb{N}, \tag{8.4}$$

are the unique hermitian solutions of the following Lyapunov equations

$$AP + PA^* + BB^* = 0, \quad t \in \mathbb{R}, \tag{8.5}$$
$$APA^* - P + BB^* = 0, \quad t \in \mathbb{N}, \tag{8.6}$$
$$A^*Q + QA + C^*C = 0, \quad t \in \mathbb{R}, \tag{8.7}$$
$$A^*QA - Q + C^*C = 0, \quad t \in \mathbb{N}, \tag{8.8}$$

where $\mathbb{N}$ and $\mathbb{R}$ stands for the set of positive integer numbers, real numbers, respectively.

Let $\ell_2(0, \infty; \mathbb{C}^n)$ and $L_2(0, \infty; \mathbb{C}^n)$ denote the space of sequences (functions) over the complex Euclidean space $\mathbb{C}^n$ which are square summable (integrable) over the set of positive integer (real) numbers. Let

$$\Gamma \;:\; \left\{ \begin{array}{c} L_2(0, \infty; \mathbb{C}^m) \\ \ell_2(0, \infty; \mathbb{C}^m) \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} L_2(0, \infty; \mathbb{C}^m) \\ \ell_2(0, \infty; \mathbb{C}^m) \end{array} \right\} \text{ if } \left\{ \begin{array}{c} t \in \mathbb{R} \\ t \in \mathbb{N} \end{array} \right\}, \tag{8.9}$$

$$(\Gamma u)(t) \;\triangleq\; \left\{ \begin{array}{l} \int_0^\infty CS(t + \tau)Bu(\tau)d\tau, t \in \mathbb{R} \\[2mm] \sum_{i=0}^\infty CA^{i+t}B(t), t \in \mathbb{N} \end{array} \right. \tag{8.10}$$

be the Hankel operators in continuous-time and discrete-time, respectively. It is well known [40] that

$$\sigma(PQ) = \sigma(\Gamma\Gamma^*), \tag{8.11}$$

where $\sigma(A)$ denotes the spectrum of A. Recall that being given a stable system $\Sigma_G(A, B, C, D)$ there exists a minimal stable realization (A_b, B_b, C_b, D_b) (called balanced) such that

$$P = Q = \Sigma = \text{diag}(\sigma_1 \geq \dots \geq \sigma_n), \tag{8.12}$$

where $(\sigma_i)_{1 \leq i \leq n}$ are the Hankel singular values of the system.

A balanced realization can be obtained as following: Let P have a Cholesky factorization $P = S^*S$. Then $SQS^* > 0$ can be diagonalized as $0 < SQS^* = V\Sigma^2 V^*$ with $VV^* = V^*V = I$ and a balancing transformation is

$$T = (S^*)^{-1}V\sqrt{\Sigma}. \tag{8.13}$$

Assume that there exists $1 < p < n$ such that $\sigma_p > \sigma_{p+1}$ and let us consider the following partition induced on the state space matrices

$$\left(\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right), \quad \left(\begin{array}{c} B_1 \\ B_2 \end{array} \right), \quad \left(\begin{array}{cc} C_1 & C_2 \end{array} \right), \quad \left(\begin{array}{cc} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{array} \right),$$

where

$$\Sigma_1 = \text{diag}(\sigma_1 \geq \dots \geq \sigma_p)$$

and
$$\Sigma_2 = \mathrm{diag}(\sigma_{p+1} \geq ... \geq \sigma_n).$$
Then $[A_{11}, B_1, C_1, D]$ and $[A_{22}, B_2, C_2, D]$ are stable systems [6] in balanced form and the first one represents the reduced order model obtained by truncating the balanced realization of the original system. State-space matrices of the singular perturbation approximation of balanced realizations of $[A, B, C, D]$ are given by [61]

$$\bar{A} = A_{11} + A_{12}(\alpha I_{p \times p} - A_{22})^{-1} A_{21}, \tag{8.14}$$
$$\bar{B} = B_1 + A_{12}(\alpha I_{p \times p} - A_{22})^{-1} B_2, \tag{8.15}$$
$$\bar{C} = C_{11} + C_2(\alpha I_{p \times p} - A_{22})^{-1} A_{21}, \tag{8.16}$$
$$\bar{D} = D + C_2(\alpha I_{p \times p} - A_{22})^{-1} B_2, \tag{8.17}$$

where
$$\alpha = \begin{cases} 1 & \text{if } t \in \mathbb{N} \\ 0 & \text{if } t \in \mathbb{R} \end{cases}.$$

If $G_p(s)$ is the transfer function of either (A_{11}, B_1, C_1, D) or $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ and $G(s) = D + C(sI - A)^{-1}B$ then the following error bound result is reported from [40]

$$\|G - G_p\|_\infty \leq 2\mathrm{trace}(\Sigma_2). \tag{8.18}$$

8.2.2 An "approximately" balanced realization of time-discretized systems.

As in infinite-dimensions, we introduce the class of time-discretized finite-dimensional systems

Definition 8.1 *Let $\Sigma_G(A, B, C, D)$ be a finite-dimensional continuous-time system. Then we shall call $\Sigma_G(\Phi, \Gamma, \Lambda, \Theta)$ its time-discretized counterpart with sampling step $T > 0$ if it is defined over the euclidean spaces*

$$\mathbb{C}^{n \times n} \times \mathbb{C}^{n \times m} \times \mathbb{C}^{p \times n} \times \mathbb{C}^{p \times m}$$

and it satisfies

$$x(k+1) = \Phi x(k) + \Gamma u(k), \tag{8.19}$$
$$y(k) = \Lambda x(k) + \Theta u(k), \tag{8.20}$$

for

$$\Phi \triangleq S(T), \tag{8.21}$$
$$\Gamma \triangleq \int_0^T S(\tau)B d\tau, \tag{8.22}$$
$$\Lambda \triangleq C, \tag{8.23}$$
$$\Theta \triangleq D. \tag{8.24}$$

Remark 8.2 *Notice that the Lyapunov equation (8.6) written for $A = \Phi$ and $B = \Gamma$ becomes progressively ill-conditioned as the sampling step approaches zero. The sampled controllability gramian P_d regarded as a real function*

$$P_d : \mathbb{R}_+ \longrightarrow \mathbb{R}^{n\times n}, \quad P_d \;=\; \sum_{i=0}^{\infty} \Phi^i \Gamma \Gamma^*(\Phi^*)^i, \tag{8.25}$$

becomes numerically singular and so will its Cholesky factor S_d. Then, under a certain limit of the sampling step $S_d Q_d S_d^*$ cannot be anymore diagonalized as

$$0 < S_d Q_d S_d^* = V_d \Sigma_d^2 V_d^*,$$

with $V_d V_d^* = V_d^* V_d = I$ and a balancing transformation for the time-discretized system

$$T_d = (S_d^*)^{-1} V_d \sqrt{\Sigma_d}$$

cannot be computed since S_d becomes numerically uninvertible.

The next result shows the way the sampled controllability and observability gramians behave for small steps.

Lemma 8.3 *Let P, Q, P_d and Q_d be the controllability and observability gramians of the continuous and time-discretized systems. Then the following relations are true*

$$\frac{P_d}{T} \;=\; P + \theta(T), \tag{8.26}$$

$$T Q_d \;=\; Q + \frac{T}{2} C^* C + \theta(T). \tag{8.27}$$

Proof Denote by

$$\mathcal{A}_d = \frac{\Phi - I}{T}$$

and notice that

$$\lim_{T \to 0} \mathcal{A}_d = A.$$

Let us consider the following function $\mathcal{F} : \mathbb{R} \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ given by,

$$\mathcal{F}(T, X) = \begin{cases} T \mathcal{A}_d^* X \mathcal{A}_d + \mathcal{A}_d^* X + X \mathcal{A}_d + \Lambda^* \Lambda, & \text{if } T \neq 0, \\ A^* X + X A + C^* C & \text{if } T = 0. \end{cases}$$

Notice that $(0, Q)$ is a solution to $\mathcal{F}(T, X) = 0$ and

$$(\frac{\partial \mathcal{F}}{\partial X})_{(0,Q)} X = A^* X + X A \tag{8.28}$$

is an invertible operator with respect to X. Applying the Implicit Function Theorem it follows that there exists a solution $X(T)$ which is analytic in a certain neighbourhood of the origin and $X(0) = Q$. Consequently, $X(T)$ has an asymptotic expansion

$$X(T) = Q + T\frac{dX(T)}{dT} + \theta(T). \tag{8.29}$$

Applying the rule of differentiating implicit defined functions

$$\frac{\partial \mathcal{F}}{\partial T} + \frac{dX}{dT}\frac{\partial \mathcal{F}}{\partial X} = 0 \tag{8.30}$$

and since $\frac{\partial F}{\partial X}$ is invertible with respect to X, we obtain after a straightforward calculation that

$$\frac{dX(T)}{dT} = \frac{C^* C}{2}. \tag{8.31}$$

It follows that

$$X(T) = Q + T\frac{C^* C}{2} + \theta(T). \tag{8.32}$$

Taking advantage that, in fact, $\mathcal{F}(T, TQ_d) = 0$ is the discrete-time Lyapunov equation (8.8) written for $A = \Phi$ and $C = \Lambda$, and of the unicity of solution $X(T)$ in the appropriate chosen neighbourhood of the origin, it follows that $X(T) = TQ_d$ and relation (8.27) is proved.

In order to prove that (8.26) also holds, denote first by $\mathcal{B}_d = \frac{\Gamma}{T}$ and let us then consider the following function $\mathcal{G} : \mathbb{R} \times \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{n \times n}$ given by

$$\mathcal{G}(T, X) = \begin{cases} T^2 \mathcal{A}_d X \mathcal{A}_d^* + \mathcal{A}_d X + X \mathcal{A}_d^* + \mathcal{B}_d \mathcal{B}_d^*, & \text{if } T \neq 0, \\ AX + XA^* + BB^* & \text{if } T = 0. \end{cases}$$

In a very similar way we proved (8.27), we proceed by exploiting the fact that $(0, P)$ is a solution to $\mathcal{G}(T, X) = 0$ and

$$\left(\frac{\partial \mathcal{G}}{\partial X}\right)|_{(0,P)} X = AX + XA^*, \tag{8.33}$$

which is invertible with respect to X. Applying the Implicit Function Theorem in a similar way as we did before, it follows that a solution $X(T)$ which is analytic in a certain

neighbourhood of the origin and $X(0) = P$. Consequently, $X(T)$ has an asymptotic expansion

$$X(T) = P + T \frac{dX(T)}{dT} + \theta(T). \tag{8.34}$$

By some simple computation we show that $\frac{dX(T)}{dT} \big|_{T=0} = 0$ and then $X(T) = P + \theta(T)$. Since $\mathcal{G}(T, \frac{P_d}{T}) = 0$ is the discrete-time Lyapunov equation written for $A = \Phi$ and $B = \Gamma$, and taking advantage of the unicity of the solution $X(T)$ in the appropriate choosen neighbourhood of the origin, it follows that $X(T) = \frac{P_d}{T}$, and relation (8.26) is also proved.

∎

The following immediate consequence of the Lemma 8.3 is

Corollary 8.4 *The sampled Hankel singular values recover their continuous-time counterparts when the sampling step approaches zero*

$$\lim_{T \to 0} \Sigma_d = \Sigma. \tag{8.35}$$

Proof From (8.26) and (8.27) we derive that

$$P_d Q_d = PQ + \frac{T}{2} PC^* C + \theta(T), \tag{8.36}$$

which implies that

$$\lim_{T \to 0} \sigma(P_d Q_d) = \sigma(PQ), \tag{8.37}$$

the latter relation being equivalent with (8.35).

∎

From Lemma 8.3 we conclude that, for a sufficiently small sampling step, the sampled controllability and observability gramians can be approximated as $P_d \approx TP$ and $Q_d \approx \frac{Q}{T}$. In this way, instead of balancing P_d and Q_d, we shall balance their approximations. Thus, the "approximately" balancing approach consists in finding a similarity transformation that would make TP and $\frac{Q}{T}$ be equal with the same diagonal matrix. The following result holds

Lemma 8.5 *Let S be a balancing transformation for the continuous-time system $\Sigma_G(A, B, C, D)$. Then the following similarity transformation*

$$\hat{S}_d = \frac{S}{\sqrt{T}} \tag{8.38}$$

makes TP and $\frac{Q}{T}$ be equal with the diagonal matrix of the Hankel singular values of the continuous system.

Proof Notice that if S is a balancing transformation for the continuous time system then $\hat{S}_d = \frac{S}{\sqrt{T}}$ is a similarity transformation. From (8.38) it follows that

$$\hat{S}_d T P \hat{S}_d^* = SPS^* = \Sigma \tag{8.39}$$

and

$$\hat{S}_d^{-*} \frac{Q}{T} \hat{S}_d^{-1} = S^{-*} Q S^{-1} = \Sigma. \tag{8.40}$$

$\blacksquare$

Lemma 8.5 suggests the way we construct the "approximately" balancing algorithm:

"Approximately" balancing

Step 1: Perform a continuous balancing

$$(A, B, C, D) \longrightarrow (A_b, B_b, C_b, D_b).$$

Step 2: Perform a continuous-to-discrete transformation

$$(A_b, B_b, C_b, D_b) \longrightarrow (A_{bd}, B_{bd}, C_{bd}, D_{bd}).$$

Step 3: Update the state space matrices as

$$B_{bd} \longleftarrow \frac{B_{bd}}{\sqrt{T}},$$

$$C_{bd} \longleftarrow C_{bd}\sqrt{T}.$$

Indeed, if $(A_{bd}, B_{bd}, C_{bd}, D_{bd})$ is a minimal stable "approximately" balanced realization of the time-discretized system, then a straightforward calculation shows that its state space matrices are given by

$$A_{bd} = S\boldsymbol{\Phi}S^{-1}, \tag{8.41}$$

$$B_{bd} = \frac{1}{\sqrt{T}}S\boldsymbol{\Gamma}, \tag{8.42}$$

$$C_{bd} = \boldsymbol{\Lambda}S^{-1}\sqrt{T}, \tag{8.43}$$

$$D_{bd} = \boldsymbol{\Theta}. \tag{8.44}$$

Clearly, the above algorithm does not provide the balanced realization of the time-discretized system. Nevertheless, the following result shows that, when the sampling period decreases to zero $(A_{bd}, B_{bd}, C_{bd}, D_{bd})$ is a good approximation of the true balanced realization.

Theorem 8.6 *Let* $(A_{bd}, B_{bd}, C_{bd}, D_{bd})$ *be a stable minimal "approximately" balanced realization of the time-discretized system* $\Sigma_G(\Phi, \Gamma, \Lambda, \Theta)$. *If* $(A_{db}, B_{db}, C_{db}, D_{db})$ *is a minimal balanced realization of it then*

$$\lim_{T \to 0} A_{bd} - A_{db} = 0_{n \times n}, \tag{8.45}$$

$$\lim_{T \to 0} B_{bd} - B_{db} = 0_{n \times m}, \tag{8.46}$$

$$\lim_{T \to 0} C_{bd} - C_{db} = 0_{p \times n}, \tag{8.47}$$

$$\lim_{T \to 0} D_{bd} - D_{db} = 0_{p \times m}, \tag{8.48}$$

where $0_{i \times j}$ *denotes the i by j null matrix.*

For proving Theorem 8.6 we need

Lemma 8.7 *Let* S_d *and* S *be the balancing transformations of the time-discretized and continuous-time systems. Then*

$$\lim_{T \to 0} S_d - \frac{S}{\sqrt{T}} = 0_{n \times n}. \tag{8.49}$$

Proof Let S_d and S be the Cholesky factors of the sampled and continuous controllability gramians P_d and P. Since (8.26) holds it follows by some simple computation that

$$\frac{S_d}{\sqrt{T}} = S + \theta(\sqrt{T}). \tag{8.50}$$

As S_d and S are invertible it is a routine to show that

$$\lim_{T \to 0} (S_d^*)^{-1} - \frac{S^{-1}}{\sqrt{T}} = 0_{n \times n}. \tag{8.51}$$

Exploiting the sampled Hankel singular values recovery property we obtain

$$\lim_{T \to 0} \sqrt{\Sigma_d} V_d - \sqrt{\Sigma} V = 0_{n \times n}, \tag{8.52}$$

as V_d and V are both from the set of orthonormal mappings given by the Singular Value Decomposition Theorem and the proof is complete.

∎

Proof(of Theorem 8.6) It is a direct consequence of the above Lemma since the state space matrices of the true balanced realization of the time-discretized system are satisfying

$$A_{db} = S_d \Phi S_d^{-1}, \tag{8.53}$$

$$B_{db} = S_d \Gamma, \tag{8.54}$$

$$C_{db} = \Lambda S_d^{-1}, \tag{8.55}$$

$$D_{db} = \Theta. \tag{8.56}$$

Substract (8.41), (8.42), (8.43) and (8.44) from (8.53), (8.54), (8.55) and (8.56) we obtain (8.45), (8.46) and (8.47) and the theorem is proved.

∎

8.2.3 An "approximately" balancing based approach to the model reduction problem for time-discretized systems

For continuous-time systems, Moore's classical truncation of balanced realizations consisting in the deletion of the most uncontrollable and unobservable part of the system (usually called the "weak" subsystem), was shown to be always convenient since the reduced order model obtained this way is internally stable, it is in balanced form and, furthermore, it retains the Hankel singular values of the most controllable and observable part of the system (usually called the "strong" subsystem).

This simple model reduction technique, usually known as the "weak" subsystem elimination, cannot be extended to the discrete-time systems case due to the fact that truncation of balanced realizations of a discrete-time model is neither in balanced form nor preserves the Hankel singular values of the "strong" subsystem.

In order to circumvent this disadvantage, a singular perturbational approximation of balanced realizations was developed (see [61]). The main advantage over the classical truncation of balanced realizations, besides the better approximation property at low frequencies and the exact preservation of the DC gain, is that the reduced-order model is internally balanced in both continuous-time and discrete-time cases.

Further we shall apply the singular perturbation model reduction scheme to the state space matrices of the true balanced realization of the time-discretized system as well as to its "approximately" balanced realization's state space matrices. Let $(A_{bdk}, B_{bdk}, C_{bdk}, D_{bdk})$ be the realization of the k-order reduced-order model obtained as the singular perturbation approximation of the "approximately" balanced realization and let $(A_{dbk}, B_{dbk}, C_{dbk}, D_{dbk})$ be the corresponding k-ordered singular perturbation approximation of the true balanced realization of time-discretized system. Then the following limit relations hold

Theorem 8.8

$$\lim_{T \longrightarrow 0} A_{bdk} - A_{dbk} = 0_{k \times k}, \tag{8.57}$$

$$\lim_{T \longrightarrow 0} B_{bdk} - B_{dbk} = 0_{k \times m}, \tag{8.58}$$

$$\lim_{T \longrightarrow 0} C_{bdk} - C_{dbk} = 0_{p \times k}, \tag{8.59}$$

$$\lim_{T \longrightarrow 0} D_{bdk} - D_{dbk} = 0_{p \times m}, \tag{8.60}$$

$$\tag{8.61}$$

where $0_{i \times j}$ denotes the i by j null matrix.

The proof is an immediate consequence of Theorem 8.6.

Remark 8.9 *The type of convergence that Theorem 8.8 shows to hold for the two reduced-order models is a strong one in the sense that in the common $\mathcal{H}_2$-norm, $\mathcal{H}_\infty$-norm or Hankel-norm, the error system defined by*

$$G_e(z) = G_{\text{bdk}}(z) - G_{\text{dbk}}(z)$$

is satisfying

$$\lim_{T \longrightarrow 0} \|G_e\|_{2,\infty,\text{H}} = 0.$$

It is well known [40, 39] that for the linear time invariant continuous-time systems an error bound result as given by relation (8.18) holds. The next Lemma represent the discrete-time version of Glover's error bound result.

Lemma 8.10 *Let $F(z)$ and $F_k(z)$ be the transfer functions of a discrete-time finite-dimensional systems and of its subsystem obtained by singular perturbation approximation. Then the following error bound result holds*

$$\|F(z) - F_k(z)\|_\infty < 2 \sum_{i=k+1}^{n} \sigma_i. \tag{8.62}$$

Proof We shall consider the linear time invariant continuous-time system obtained from the linear time invariant discrete-time system via the following bilinear mapping

$$z \mapsto \frac{1+s}{1-s}. \tag{8.63}$$

Let us notice that if A is stable in discrete-time (i.e. having all its eigenvalues in the open unit disk), then A_c is also stable in continuous-time and its transfer function is

$$G(s) = D_c + C_c(sI - A_c)^{-1}B_c, \tag{8.64}$$

where from [40] we have

$$A_c = (I + A)^{-1}(A - I), \tag{8.65}$$
$$B_c = \sqrt{2}(I + A)^{-1}B, \tag{8.66}$$
$$C_c = \sqrt{2}C(I + A)^{-1}, \tag{8.67}$$
$$D_c = D - C(I + A)^{-1}B. \tag{8.68}$$

An important property of the continuous-time equivalent system defined above is that the controllability and observability gramians as well as the Hankel singular values of the two systems will be the same (see [39, 40]). Furthermore we have

$$F(e^{j\theta}) = G(j\tan\frac{\theta}{2}). \tag{8.69}$$

Hence, if $F_k(z) = D_k + C_k(zI - A_k)^{-1}B_k$ is a k-order approximation of $F(z)$ then

$$G_k(s) = F_k\left(\frac{1+s}{1-s}\right) \tag{8.70}$$

represents a k-order approximation of the continuous-time equivalent system which will satisfy

$$\sigma_i(F(z) - F_k(z)) = \sigma_i(G(s) - G_k(s)) \tag{8.71}$$

$$F(e^{j\theta}) - F_k(e^{j\theta}) = G(j\tan\frac{\theta}{2}) - G_k(j\tan\frac{\theta}{2}). \tag{8.72}$$

Hence, a solution to the discrete-time problem generates a solution to the equivalent continuous-time problem in the sense that for both systems, the discrete-time one and its continuous-time equivalent, the Hankel singular values of the error will be the same as will the frequency responses. Notice also that the continuous-time equivalent system obtained by the bilinear mapping $z \rightarrow (1+s)/(1-s)$ is in balanced form iff the original discrete-time system was also in balanced form (see [40]). Applying now the L^∞-error bound result as given by relation (8.18) we obtain that

$$\|F(z) - F_k(z)\|_\infty = \|G(s) - G_k(s)\|_\infty < 2\sum_{i=k+1}^{n}\sigma_i. \tag{8.73}$$

∎

The following direct consequence of the above result is given by

Corollary 8.11 *Let* $(A_{bdk}, B_{bdk}, C_{bdk}, D_{bdk})$ *be the realization of the k-order reduced-order model obtained as the singular perturbational approximation of the "approximately" balanced realization and let* $(A_{dbk}, B_{dbk}, C_{dbk}, D_{dbk})$ *be the corresponding k-ordered singular perturbational approximation of the true balanced realization of the time-discretized system. Let* $G_{bdk}(z), G_{dbk}(z)$ *and* $G_d(z)$ *be their corresponding transfer functions. Then the following error bound result holds*

$$\lim_{T\to 0}\|G_d(z) - G_{bdk}(z)\|_\infty \le 2\sum_{i=k+1}^{n}\sigma_i, \tag{8.74}$$

$$\lim_{T\to 0}\|G_d(z) - G_{dbk}(z)\|_\infty \le 2\sum_{i=k+1}^{n}\sigma_i. \tag{8.75}$$

At the end of this section a few remarks should be made on the "approximately" balancing and its induced model reduction scheme presented above.

Remark 8.12 *It is well known that (see [74]) before proceeding to design a controller meeting certain robust performance specifications, a balancing step is recommended in advance the design process. This fact is motivated by the improvement brought by balancing to some of the system's properties, i.e. minimizing the condition number with respect to pointwise state control and zero input state observation (see [63]), etc.*

In sampled-data control, when "on-line" computations of state space matrices are required, the "approximately" balancing algorithm has an obvious advantage over the true balancing one, especially when the sampling step is changed by the performance specification restrictions. The discrete Lyapunov equations for time-discretized systems are replaced with the corresponding pair of continuous equations which are solved only one time, at the very first step of the design process. The sampled gramians which we want to balance are replaced by their approximations. Thus, each time the sampling step changes, instead of solving the discrete Lyapunov equation we update the continuous-time controllability and observability gramians using

$$P_{\mathrm{d}} \approx TP, \tag{8.76}$$

$$Q_{\mathrm{d}} \approx \frac{Q}{T}. \tag{8.77}$$

Remark 8.13 *The lower bounds for which the "approximately" balancing algorithm still works is influenced only by the way the continuous-to-discrete routine is implemented. Avoiding to solve the discrete Lyapunov equations for the time-discretized system in the case when the sampling step is much smaller than the minimum of the absolute values of the system's poles, we decrease the lower bound of the sampling step for which the true balancing algorithm fails.*

8.2.4   Example

Let us consider the following continuous time system having the state space matrices

$$A = \begin{pmatrix} -1.0000 & -0.5000 & -0.3333 & -0.2500 \\ -0.5000 & -0.3333 & -0.2500 & -0.2000 \\ -0.3333 & -0.2500 & -0.2000 & -0.1667 \\ -0.2500 & -0.2000 & -0.1667 & -0.1429 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix},$$

where the eigenvalues of the system are

$$\sigma(A) = (-1.5002, \ -0.1691, \ -0.0067, \ -0.0001).$$

A balanced realization state space matrices are

$$A_b = \begin{pmatrix} -0.0001 & -0.0004 & 0.0006 & -0.0005 \\ 0.0004 & -0.0047 & 0.0168 & -0.0124 \\ 0.0006 & -0.0168 & -0.1325 & 0.2306 \\ 0.0005 & -0.0124 & -0.2306 & -1.5389 \end{pmatrix},$$

$$B_b = \begin{pmatrix} 0.1010 \\ -0.2731 \\ -0.4431 \\ -0.3632 \end{pmatrix},$$

$$C_b = \begin{pmatrix} 0.0010 & 0.2731 & 0.4431 & -0.3632 \end{pmatrix}.$$

The controllability and observability gramians are equal to the diagonal matrix of the Hankel singular values

$$\Sigma = (77.2412, \ 7.9392, \ 0.7408, \ 0.0429).$$

The most suitable candidate for a good approximation is the singular perturbational approximation having the McMillan degree $k = 2$

Further, we will focus on comparing the "approximately" balanced algorithm with the true balancing one for the case when the sampling step is very small comparing to the system's modes. Thus, for $h = 1.e^{-13}$, the state space matrices of the "approximately" balanced and correct balanced realizations are

$$A_{bd} = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 1.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 1.0000 \end{pmatrix},$$

$$B_{bd} = 1.e^{-6} \begin{pmatrix} -0.0319 \\ -0.0864 \\ -0.1401 \\ -0.1149 \end{pmatrix},$$

$$C_{bd} = 1.e^{-6} \begin{pmatrix} 0.0319 & -0.0864 & 0.1401 & -0.1149 \end{pmatrix},$$

and

$$A_{db} = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 1.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 1.0000 \end{pmatrix},$$

$$B_{db} = 1.e^{-6} \begin{pmatrix} -0.0639 \\ -0.0090 \\ -0.1304 \\ 0.1141 \end{pmatrix},$$

$$C_{db} = 1.e^{-6} \begin{pmatrix} -0.0639 & 0.0090 & 0.1304 & 0.1141 \end{pmatrix}.$$

The "distance" between the two realizations, the balanced one and the "approximately" balanced is given by

$$A_{bd} - A_{db} = 1.e^{-12} \begin{pmatrix} 0 & 0.0012 & 0.0004 & 0.0014 \\ -0.0015 & 0.0003 & 0.0345 & 0.0010 \\ -0.0001 & -0.0359 & -0.0004 & 0.4628 \\ -0.0008 & 0.0025 & -0.4613 & 0.0004 \end{pmatrix},$$

$$B_{bd} - B_{db} = 1.e^{-6} \begin{pmatrix} -0.0201 \\ -0.5362 \\ -0.0021 \\ -0.7262 \end{pmatrix},$$

$$C_{bd} - C_{db} = 1.e^{-6} \begin{pmatrix} 0.0201 & -0.5362 & 0.0021 & -0.7262 \end{pmatrix}.$$

If $h < 1.e_{-13}$ the first algorithm fails due to the impossibility of computing the solutions of the sampled Lyapunov equations. For such a small sampling step, the solutions are losing their positivity. The "approximately" still can be computed even for smaller steps. The smallest step for which the algorithm fails is influenced only by the precision of the machine implementation of the continue-to-discrete procedure. An example of how far can we extend the lower limit of the sampling step for which the "approximately" balancing algorithm still works, let us consider the case when $h = 1.e^{-300}$. Then the state space matrices of the "approximately" balanced realization of the time-discretized system considered are

$$A_{bd} = \begin{pmatrix} 1.0000 & 0.0000 & 0.0000 & 0.0000 \\ -0.0000 & 1.0000 & 0.0000 & -0.0000 \\ -0.0000 & 0.0000 & 1.0000 & 0.0000 \\ -0.0000 & 0.0000 & -0.0000 & 1.0000 \end{pmatrix},$$

$$B_{bd} = 1.e^{-150} \begin{pmatrix} -0.1010 \\ -0.2731 \\ -0.4431 \\ -0.3632 \end{pmatrix},$$

$$C_{bd} = 1.e^{-150} \left(\begin{array}{cccc} 0.1010 & -0.2731 & 0.4431 & -0.3632 \end{array} \right).$$

Let $(A_{bd2}, B_{bd2}, C_{bd2}, D_{bd2})$ be the 2×2 singular perturbational approximation of the "approximately" balanced realization of $[A_d, B_d, C_d, D_d]$ with the state space matrices

$$A_{bd2} = \left(\begin{array}{cc} 1.0000 & 0.0000 \\ -0.0000 & 1.0000 \end{array} \right),$$

$$B_{bd2} = 1.e^{-6} \left(\begin{array}{c} -0.1029 \\ 0.0314 \end{array} \right),$$

$$C_{bd2} = 1.e^{-6} \left(\begin{array}{cc} -0.1029 & -0.0314 \end{array} \right),$$

$$D_{bd2} = -1.3896,$$

and let $(A_{db2}, B_{db2}, C_{db2}, D_{db2})$ be the 2×2 singular perturbational approximation of the true balanced realization of $\Sigma_G(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, \mathbf{\Theta})$ with the state space matrices

$$A_{db2} = \left(\begin{array}{cc} 1.0000 & 0.0000 \\ -0.0000 & 1.0000 \end{array} \right),$$

$$B_{db2} = 1.e^{-7} \left(\begin{array}{c} -0.9288 \\ -0.0437 \end{array} \right),$$

$$C_{db2} = 1.e^{-7} \left(\begin{array}{cc} -0.9019 & -0.2190 \end{array} \right),$$

$$D_{db2} = -1.2291.$$

The "error" between the above approximations of the time-discretized system is given by the following measure

$$A_{bd2} - A_{db2} = 1.e^{-14} \left(\begin{array}{cc} -0.0111 & -0.1216 \\ -0.0368 & -0.0777 \end{array} \right),$$

$$B_{bd2} - B_{db2} = 1.e^{-7} \left(\begin{array}{c} 0.1006 \\ -0.3574 \end{array} \right),$$

$$C_{bd2} - C_{db2} = 1.e^{-7} \left(\begin{array}{cc} 0.1276 & 0.0947 \end{array} \right),$$

$$D_{bd2} - D_{bd2} = 0.1604.$$

8.3   Balanced realizations of infinite-dimensional systems

In this section we extend the results from the previous section to a class of infinite dimensional systems with diagonal semigroup. For the sake of simplicity we shall consider only the SISO case, i.e. $m = 1$ and $p = 1$.

The elegant results on balanced realizations obtained for finite-dimensional systems aroused interest in the problem of the extension of the concept of balanced realization to infinite-dimensional systems. In [38] it is shown that for *nuclear systems*, i.e systems with finite-dimensional inputs and outputs whose impulse response satisfies

$$h(t) \stackrel{\triangle}{=} CS(t)B \in L_1 \cap L_2(0, \infty; C^{p \times m}) \tag{8.78}$$

and induces the nuclear Hankel operator, the balanced realization always exists and their truncations converge to the original system in various topologies. Furthermore, explicit L_∞ bounds on the transfer functions errors, L_1 and L_2 bounds on the impulse response errors and Hilbert-Schmidt and nuclear bounds on the Hankel operator errors can be obtained. If in finite-dimensional case the transition between continuous-time and discrete-time cases is straightforward, the situation in infinite-dimensional case is different. For infinite-dimensional discrete-time systems with the impulse response satisfying

$$h(k) \stackrel{\triangle}{=} CA^k B \in \ell_1, \tag{8.79}$$

the balanced realization can be explicitly be given in terms of the singular values and Schmidt pairs of the hankel operator. If the Hankel operator is Hilbert-Schmidt, truncations of balanced realization generate a sequence of finite-dimensional impulse responses which converge in ℓ_2 sense to the original one and the transfer functions converge pointwise. If an extra nuclearity assumption is made on the Hankel operator then the transfer functions converge in the L_∞ norm as well.

Let us consider first the continuos-time case. Our object is represented by linear infinite-dimensional systems defined by the following input-output map

$$y(t) = \int_0^\infty h(t - s)u(s)ds, \tag{8.80}$$

where the outputs and the inputs are square integrable, i.e. $y \in L_2(0, \infty; C^p)$ and $u \in L_2(0, \infty; C^m)$, and the impulse response satisfies (8.78). Corresponding to (8.78) is the Hankel operator

$$\Gamma : L_2(0, \infty; C^m) \longrightarrow L_2(0, \infty; C^p), \quad (\Gamma u)(t) \stackrel{\triangle}{=} \int_0^\infty h(t + s)u(s)ds. \tag{8.81}$$

We notice that the condition (8.78) implies that Γ defined by (8.81) is compact and if we consider

$$\Gamma : C_1(0, \infty; C^m) \longrightarrow C_1(0, \infty; C^p), \quad (\Gamma u)(t) \stackrel{\triangle}{=} \int_0^\infty h(t + s)u(s)ds, \tag{8.82}$$

then Γ defined by (8.82) satisfies the same compactness property, where C_1 is the space of absolute continuos functions with derivatives existing in L_1 sense. In [38] is proved that $\Gamma^*\Gamma$ is compact and positive on $L_2(0,\infty; C^m)$ and so it has countably many positive eigenvalues

$$\sigma_1^2 \geq \cdots \geq \sigma_n^2 \geq \cdots \geq 0, \tag{8.83}$$

which are the singular values of Γ. If v_i and w_i, $i \geq 1$ are the corresponding normaliezed eigenvectors, the (v_i, w_i), $i \geq 1$ are called Schmidt pairs of Γ

$$\Gamma v_i = \sigma_i w_i, \tag{8.84}$$
$$\Gamma^* w_i = \sigma_i v_i, \tag{8.85}$$

An important property of the Schmidt pairs is given by

$$w_1 \in L_1(0,\infty; C^p) \cap L_2(0,\infty; C^p) \cap C_1(0,\infty; C^p), \tag{8.86}$$
$$v_1 \in L_1(0,\infty; C^m) \cap L_2(0,\infty; C^m) \cap C_1(0,\infty; C^m). \tag{8.87}$$

The following result on balanced realizations of continuous-time infinite-dimensional systems is reported from [38]

Theorem 8.14 *Consider the infinite-dimensional system with the input/output map defined by (8.80) and let $h(t) = CS(t)B$ be its impulse response satisfying (8.78). Assume that the associated Hankel operator is nuclear.*

(i) The following realization of $h(\cdot)$ is well defined

$$A_{ij} = \begin{pmatrix} \ddots & & \\ & \sqrt{\frac{\sigma_i}{\sigma_j}}\langle \dot{v}_i, v_j \rangle & \\ & & \ddots \end{pmatrix}_{i,j\geq 1}, \tag{8.88}$$

$$B_i^* = \begin{pmatrix} \cdots & \sqrt{\sigma_i}w_i(0) & \cdots & \cdots \end{pmatrix}, \tag{8.89}$$

$$C_i = \begin{pmatrix} \cdots & \sqrt{\sigma_i}v_i(0) & \cdots & \cdots \end{pmatrix}, \tag{8.90}$$

Furthermore, it is a balanced realization, i.e. if P and Q are the controllability and observability gramians defined by

$$P \triangleq \int_0^\infty S(t)BB^*S^*(t)dt, \tag{8.91}$$

$$Q \triangleq \int_0^\infty S^*(t)C^*CS(t)dt, \tag{8.92}$$

then

$$P = Q = \Sigma \triangleq \text{diag}\{\sigma_1, \cdots, \sigma_n, \cdots\}.$$

(ii) *The A-operator defined by (8.88) is the infinitesimal generator of a strongly continuos*
    semigroup of operators $S(\cdot)$ satisfying

$$S(t)_{ij} = \sqrt{\frac{\sigma_i}{\sigma_j}} \int_0^\infty w_i^*(t+s)w_i(s)ds, \tag{8.93}$$

which is a ℓ_2-contraction

$$\|S(t)x\|_2^2 \le \|x\|_2^2. \tag{8.94}$$

(iii) *The realization (A, B, C) defined by (8.88), (8.89) and (8.90) is approximately con-*
    trollable and approximately observable on ℓ_2.

Let us consider now the discrete-time case. Our object is represented by linear infinite-dimensional systems defined by the following input-output map

$$y(k) = \sum_{i=0}^{k-1} h(i)u(k-i-1), \tag{8.95}$$

where the outputs and the inputs are square summable, i.e. $y \in \ell_2(0, \infty; C^p)$ and $u \in \ell_2(0, \infty; C^m)$, and the impulse respone satisfies (8.79). Corresponding to (8.79) is the discrete-time Hankel operator

$$\Gamma : \ell_2(0, \infty; C^m) \longrightarrow \ell_2(0, \infty; C^p), \quad (\Gamma u)(k) \overset{\Delta}{=} \sum_{i=0}^\infty h(i+k)u(i). \tag{8.96}$$

We notice that the condition (8.79) implies that Γ defined by (8.96) is compact. In [31] is proved that $\Gamma^*\Gamma$ is compact and positive on $\ell_2(0, \infty; C^m)$ and so it has countably many positive eigenvalues

$$\sigma_1^2 \ge \cdots \ge \sigma_n^2 \ge \cdots \ge 0, \tag{8.97}$$

which are the singular values of Γ. If v_i and w_i, $i \ge 1$ are the corresponding normaliezed eigenvectors, the (v_i, w_i), $i \ge 1$ are called Schmidt pairs of Γ

$$\Gamma v_i = \sigma_i w_i, \tag{8.98}$$
$$\Gamma^* w_i = \sigma_i v_i. \tag{8.99}$$

The following result on balanced realizations of discrete-time infinite-dimensional systems is reported from [31, 26]

Theorem 8.15 *Consider the infinite-dimensional system with the input/output map defined by (8.95) and let $h(k) = CA^k B$ be its impulse response satisfying (8.79). Then the following realization of $h(\cdot)$ is well defined*

$$A_{ij} = \begin{pmatrix} \ddots & & \\ & \sqrt{\frac{\sigma_i}{\sigma_j}}\langle v(i+1), v(j)\rangle & \\ & & \ddots \end{pmatrix}_{i,j\geq 1}, \qquad (8.100)$$

$$B_i^* = \begin{pmatrix} \cdots & \sqrt{\sigma_i}w_i(0) & \cdots & \cdots \end{pmatrix}, \qquad (8.101)$$

$$C_i = \begin{pmatrix} \cdots & \sqrt{\sigma_i}v_i(0) & \cdots & \cdots \end{pmatrix}. \qquad (8.102)$$

Furthermore, it is a balanced realization, i.e. if P and Q are the controllability and observability gramians defined by

$$P \triangleq \sum_{k=0}^{\infty} A^k B B^* A^{*k}, \qquad (8.103)$$

$$Q \triangleq \sum_{k=0}^{\infty} A^{*k} C^* C A^k, \qquad (8.104)$$

then

$$P = Q = \Sigma \triangleq \text{diag}\{\sigma_1, \cdots, \sigma_n, \cdots\}$$

and A defined by (8.100) is a contraction on $\ell_2(0, \infty; \mathcal{X})$.

8.3.1 Truncations of balanced realization

The realizations (8.88), (8.89), (8.90) and (8.100), (8.101), (8.102) suggest the way one should approximate the infinite-dimensional system, both in continuous-time as well as in discrete-time, by its truncations, that is, by a system

$$\Sigma_n(A_n, B_n, C_n)$$

such that

$$A_{ij}^n = A_{ij}, \quad i,j = 1, \cdots, n, \qquad (8.105)$$

$$B_{ij}^n = B_i, \quad i = 1, \cdots, n, \qquad (8.106)$$

$$C_j^n = C_j, \quad j = 1, \cdots, n. \qquad (8.107)$$

Such a n-dimensional truncation has an impulse response defined by

$$h^n(t) \triangleq \begin{cases} C^n e^{A^n t} B^n & , \quad t \in \mathbb{R} \\ C^n (A^n)^t B^n & , \quad t \in \mathbb{N} \end{cases} \qquad (8.108)$$

and a corresponding associated Hankel operator, say Γ^n. Let $G(\cdot)$ and $G^n(\cdot)$ denote the transfer functions of the original infinite-dimensional system and of its n-dimensional approximation. Then the following result holds

Theorem 8.16

$$\lim_{n \to \infty} \|G - G^n\|_\infty = 0, \tag{8.109}$$

$$\lim_{n \to \infty} \|\Gamma - \Gamma^n\|_N = 0, \tag{8.110}$$

$$\lim_{n \to \infty} \|h - h^n\|_1 = 0, \tag{8.111}$$

where

$$\| \cdot \|_1 \triangleq \left\{ \begin{array}{ll} \| \cdot \|_{\ell_1(0,\infty;\mathbb{R}^{p \times m})} & , \quad t \in \mathbb{N} \\ \| \cdot \|_{L_1(0,\infty;\mathbb{R}^{p \times m})} & , \quad t \in \mathbb{R} \end{array} \right. .$$

8.3.2   Asymptotic properties of $P_{\mathrm d}$ and $Q_{\mathrm d}$ for infinite-dimensional systems

As in the case of digital Riccati equations for infinite-dimensional systems, the asymptotic properties of the sampled controllability and observability gramians can be extended from finite to infinite-dimensional case only under certain assumptions made on the original system. If one would like to prove a similar result to Lemma 8.3 in infinite dimensions, then the same drawback, the fact that $P_{\mathrm d}$ and $Q_{\mathrm d}$ are generally not analytic functions in a neighbourhood of the origin, would represent a insurmountable obstacle in applying a Implicit Function Theorem argument. However, for certain class of infinite-dimensional systems which, in our opinion, are important in practice, Lemma 8.3 can be readily extended. Let us consider the parabolic system from subsection 7.1.1 and assume for simplicity that the control and observation operators are bounded with respect to the state space $\mathcal{X}$. Since the semigroup is diagonal and B and C operators are ℓ_2 sequences, it is a routine to show that $P = P^* \in \mathcal{L}(\mathcal{X})$ and $Q = Q^* \in \mathcal{L}(\mathcal{X})$ have the following expression

$$P x = \left(-\frac{\sum_{k=1}^\infty b_{ik} b_{kj}}{\lambda_i + \lambda_j} \right)_{i,j=1,\cdots,n,\cdots} x, \quad \forall x \in \mathcal{X}, \tag{8.112}$$

$$Q x = \left(-\frac{\sum_{k=1}^\infty c_{ki} c_{kj}}{\lambda_i + \lambda_j} \right)_{i,j=1,\cdots,n,\cdots} x, \quad \forall x \in \mathcal{X}. \tag{8.113}$$

Exploiting the state space formulæ for the time-discretized counterpart we get after elementary manipulation the expressions for $P_{\mathrm d} = P_{\mathrm d}^* \in \mathcal{L}(\mathcal{X})$ and $Q_{\mathrm d} = Q_{\mathrm d}^* \in \mathcal{L}(\mathcal{X})$ as

$$P_{\mathrm d} x = \left(\frac{\sum_{k=1}^\infty b_{ik} b_{kj}}{\frac{\lambda_i \lambda_j (1 - e^{(\lambda_i + \lambda_j)T})}{(1 - e^{\lambda_i T})(1 - e^{\lambda_j T})}} \right)_{i,j=1,\cdots,n,\cdots} x, \quad \forall x \in \mathcal{X}, \tag{8.114}$$

$$Q_{\mathrm d} x = \left(\frac{\sum_{k=1}^\infty c_{ki} c_{kj}}{1 - e^{(\lambda_i + \lambda_j)T}} \right)_{i,j=1,\cdots,n,\cdots} x, \quad \forall x \in \mathcal{X}. \tag{8.115}$$

Let us define the families of functions

$$\mathcal{D}_{i,j}^1 : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathcal{D}_{i,j}^1(T) = \frac{(1 - e^{\lambda_i T})(1 - e^{\lambda_j T})}{\lambda_i \lambda_j (1 - e^{(\lambda_i + \lambda_j) T})}, \quad i, j = 1, \cdots, n, \cdots, \quad (8.116)$$

$$\mathcal{D}_{i,j}^2 : \mathbb{R} \longrightarrow \mathbb{R}, \quad \mathcal{D}_{i,j}^2(T) = \frac{1}{1 - e^{(\lambda_i + \lambda_j) T}}, \quad i, j = 1, \cdots, n, \cdots. \quad (8.117)$$

Clearly $\mathcal{D}_{i,j}^1$ and $\mathcal{D}_{i,j}^2$ are analytic since they are obtained by basic mathematical operations applied to the exponential function. Exploiting the Taylor series expansion of the exponential we immediately derive the following element-by-element expansion

$$\frac{P_{\mathrm{d}}^{i,j}}{T} = P^{i,j} + \theta(T), \quad (8.118)$$

$$T Q_{\mathrm{d}}^{i,j} = Q^{i,j} + \frac{T}{2} c_i c_j + \theta(T). \quad (8.119)$$

Let now $x = \{x_1, \cdots, x_i, \cdots, x_j, \cdots\}$ be an arbitrary element of the state space $\mathcal{X}$.

Lemma 8.17 *The following hold*

$$\frac{1}{T} \langle P_{\mathrm{d}} x, y \rangle_{\mathcal{X}} = \langle P x, y \rangle_{\mathcal{X}} + o(T), \quad \forall x, y \in \mathcal{X}, \quad (8.120)$$

$$T \langle Q_{\mathrm{d}} x, y \rangle_{\mathcal{X}} = \langle Q x, y \rangle_{\mathcal{X}} + \frac{T}{2} \langle C x, C y \rangle_{\mathcal{X}} + o(T), \quad \forall x, y \in \mathcal{X}. \quad (8.121)$$

The *input normal realization* and the *output normal realization* have always been considered together with the concept of *balanced realization* in the model-reduction literature. The first one is characterized by

$$P = I, \quad (8.122)$$

$$Q = \Sigma^2, \quad (8.123)$$

while the second one is characterized by

$$P = \Sigma^2, \quad (8.124)$$

$$Q = I. \quad (8.125)$$

State-space formulae of input-normal, output-normal and balanced realizations are given in [38]. In this subsection we shall extend the concept of *approximately balanced realization* introduced in [4, 5] to infinite-dimensional systems. We also introduce the concepts of *approximately input-normal realization* and *approximately output-normal realization* for infinite-dimensional time-discretized systems. We give here the following

Definition 8.18 *Let* $\Sigma(S^{\mathcal{V}}(\cdot), B, C, \star)$ *be an infinite-dimensional Pritchard-Salamon system with respect to* $\mathcal{W} \hookrightarrow \mathcal{V}$ *with impulse response* $h(\cdot) = \star + CS(\cdot)B \in L_1 \cap L_2(0, \infty; \mathbb{C}^{m \times p})$ *and let* $\Sigma(\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \star)$ *its time-discretized counterpart.*

(*i*) *A realizations of* $\Sigma(\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \star)$ *is called approximately balanced if*

$$P = T\Sigma, \tag{8.126}$$

$$Q = \frac{\Sigma}{T}. \tag{8.127}$$

(*ii*) *A realizations of* $\Sigma(\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \star)$ *is called approximately input-normal if*

$$P = TI, \tag{8.128}$$

$$Q = \frac{\Sigma^2}{T}. \tag{8.129}$$

(*iii*) *A realizations of* $\Sigma(\boldsymbol{\Phi}, \boldsymbol{\Gamma}, \boldsymbol{\Lambda}, \star)$ *is called approximately output-normal if*

$$P = T\Sigma^2, \tag{8.130}$$

$$Q = \frac{I}{T}. \tag{8.131}$$

We claim here that the approximately balanced realization of infinite-dimensional time discretized systems is obtainable via the same algorithm as in the finite-dimensional case (see for details [4, 5]).

<div align="center">ALGORITHM</div>

Step 1: Write down a balanced realization $(S_{\mathrm{b}}(\cdot), B_{\mathrm{b}}, C_{\mathrm{b}}, \star)$ of the impulse response

$$h(\cdot) = \star + CS(\cdot)B \in L_1 \cap L_2(0, \infty; \mathbb{C}^{m \times p})$$

as

$$S_{\mathrm{b}}^{i,j}(t) = \sqrt{\frac{\sigma_i}{\sigma_j}} \int_0^\infty v_i^*(t + \tau) v_j(\tau) d\tau, \tag{8.132}$$

$$A_{\mathrm{b}}^{i,j} = \sqrt{\frac{\sigma_i}{\sigma_j}} \int_0^\infty \dot{v}_i^*(t + \tau) v_j(\tau) d\tau, \tag{8.133}$$

$$B_{\mathrm{b}}^i = \sqrt{\sigma_i} v_i(0), \tag{8.134}$$

$$C_{\mathrm{b}}^i = \sqrt{\sigma_i} w_i(0), \tag{8.135}$$

for all $i, j = 1, 2, \cdots,$.

Step 2: The time-discretized counterpart of $(S_b(\cdot), B_b, C_b)$, denoted by $(\Phi_b, \Gamma_b, \Lambda_b, \star)$ is given by

$$\Phi_b^{i,j} = S_b^{i,j}(T) = \sqrt{\frac{\sigma_i}{\sigma_j}} \int_0^\infty v_i^*(T + \tau)v_j(\tau)d\tau, \tag{8.136}$$

$$\Gamma_b^i = \int_0^T S_b(t)B_b dt =$$

$$= \sqrt{\sigma_i} \sum_{j=0}^\infty \int_0^T \int_0^\infty v_i^*(t + \tau)v_j(\tau)d\tau dt, \tag{8.137}$$

$$\Lambda_b^i = C_b^i = \sqrt{\sigma_i}w_i(0). \tag{8.138}$$

Step 3: Update the Γ_b and Λ_b as

$$\Gamma_b \longleftarrow \frac{\Gamma_b}{\sqrt{T}}, \tag{8.139}$$

$$\Lambda_b \longleftarrow \sqrt{T}\Lambda_b. \tag{8.140}$$

Furthermore, the following proposition holds

Proposition 8.19 *Let* $\Sigma(S^V(\cdot), B, C, \star)$ *be an infinite-dimensional continuous-time system on* $\mathcal{X}$ *with impulse response* $h(\cdot) = \star + CS(\cdot)B \in L_1 \cap L_2(0, \infty; \mathbb{C}^{m \times p})$ *and let* $\Sigma(\Phi, \Gamma, \Lambda, \star)$ *its time-discretized counterpart. Then the following realizations of* $\Sigma(\Phi, \Gamma, \Lambda, \star)$

(i) $(A_{AB}, B_{AB}, C_{AB}, \star)$ *defined by*

$$A_{AB}^{i,j} = \sqrt{\frac{\sigma_i}{\sigma_j}} \int_0^\infty v_i^*(T + \tau)v_j(\tau)d\tau, \tag{8.141}$$

$$B_{AB}^i = \sqrt{\frac{\sigma_i}{T}} \sum_{j=0}^\infty \int_0^T \int_0^\infty v_i^*(t + \tau)v_j(\tau)d\tau dt, \tag{8.142}$$

$$C_{AB}^i = \sqrt{T\sigma_i}w_i(0), \tag{8.143}$$

(ii) $(A_{IN}, B_{IN}, C_{IN}, \star)$ *defined by*

$$A_{IN}^{i,j} = \sqrt{\frac{\sigma_i}{\sigma_j}} \int_0^\infty v_i^*(T + \tau)v_j(\tau)d\tau, \tag{8.144}$$

$$B_{IN}^i = \sqrt{\frac{1}{T}} \sum_{j=0}^\infty \int_0^T \int_0^\infty v_i^*(t + \tau)v_j(\tau)d\tau dt, \tag{8.145}$$

$$C_{IN}^i = \sqrt{T\sigma_i}w_i(0), \tag{8.146}$$

(*iii*) $(A_{ON}, B_{ON}, C_{ON}, \star)$ *defined by*

$$A_{ON}^{i,j} = \sqrt{\frac{\sigma_i}{\sigma_j}} \int_0^\infty v_i^*(T+\tau)v_j(\tau)d\tau, \tag{8.147}$$

$$B_{ON}^i = \frac{\sigma_i}{\sqrt{T}} \sum_{j=0}^\infty \int_0^T \int_0^\infty v_i^*(t+\tau)v_j(\tau)d\tau dt, \tag{8.148}$$

$$C_{ON}^i = \sqrt{T}w_i(0), \tag{8.149}$$

are approximately balanced, approximately input-normal and approximately output-normal realizations of $\Sigma(\Phi, \Gamma, \Lambda, \star)$.

Proof It is sufficient to prove that the controllability and observability gramians associated with (A_{AB}, B_{AB}, C_{AB}) are equal with $T\Sigma$ and $\frac{\Sigma}{T}$, respectively. Let us prove the property for the observability gramian, since the proof for the controllability gramian goes in the same way. Notice that

$$C_{AB}C_{AB}^* = \left(\cdots, \sqrt{T\sigma_i}w_i(0), \cdots \right) \left(\begin{array}{c} \vdots \\ \sqrt{T\sigma_i}w_i^*(0) \\ \vdots \end{array} \right). \tag{8.150}$$

Exploiting the orthonormality of the Schimdt vectors, one can easily see that the above double infinite-dimensional matrix is diagonal. It follows that the extradiagonal elements of

$$A_{AB}^* X A_{AB} - X$$

are nill, otherwise the Lyapunov equation that gives the observability gramian associated with (A_{AB}, B_{AB}, C_{AB}) is not solvable. A straightforward computation of the diagonal elements of $A_{AB}^* X A_{AB} - X$ shows that $X = \frac{\Sigma}{T}$. Similar argument hold for the second and the third items of this proposition and the proof is complete.
∎

Notice that the expressions of the discrete semigroups A_{AB}, A_{IN} and A_{ON} coincide with Φ_b. The following lemma shows that they are contracting semigroups on $\ell_2(0, \infty; W)$.

Lemma 8.20 Φ_b *is a contraction discrete semigroup on* $\ell_2(0, \infty; \mathcal{X})$.

An immediate consequence of Lemma 8.17 is given by the following infinite-dimensional generalization of Corollary 8.4

Corollary 8.21 *Let* $\Sigma(S^\nu(\cdot), B, C, \star)$ *be a nuclear continuous-time infinite-dimensional system with finite-dimensional input and output spaces and diagonal semigroup on* ℓ_2 *and*

let x_0 be arbitrary in $\mathcal{X}$. Then the Hankel singular values of $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, \star)$, the time-discretized of $\Sigma(S^V(\cdot), B, C, \star)$ recover their continuous-time counterparts as the sampling period approaches zero

$$\lim_{T \to 0} \Sigma_{\mathrm{d}} = \Sigma. \tag{8.151}$$

At the end of this subsection let us notice that the following weaker counterpart of the result proved in Lemma 8.17 hold generally for arbitrary infinite-dimensional systems. Indeed, let us relax the diagonality assumption made on the semigroup and consider the more general case when $S^{\mathcal{X}}(\cdot)$ has not only point spectrum. Then the following holds

Proposition 8.22 *Let $\Sigma(S^{\mathcal{X}}(\cdot), B, C, \star)$ be a nuclear infinite-dimensional system with finite-dimensional input and output spaces and diagonal semigroup on ℓ_2 and let x_0 be arbitrary in $\mathcal{X}$. Assume that $\Sigma(S^{\mathcal{X}}(\cdot), B, C, \star)$ is exponentially stable on $\mathcal{X}$ and let $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, \star)$ denote the time-discretized of $\Sigma(S^{\mathcal{X}}(\cdot), B, C, \star)$. Let P, Q, P_d and Q_d denote the controllability and observability gramians associated with $\Sigma(S^{\mathcal{X}}(\cdot), B, C, \star)$ and $\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, \star)$, respectively. Assume that the following limits exists*

$$\exists \lim_{T \to 0} \langle \frac{P_{\mathrm{d}}}{T} x, y \rangle_{\mathcal{X}}, \quad \forall x, y \in \mathcal{X}, \tag{8.152}$$

$$\exists \lim_{T \to 0} \langle T Q_{\mathrm{d}} x, y \rangle_{\mathcal{X}}, \quad \forall x, y \in \mathcal{X}, \tag{8.153}$$

Then the following hold

$$\lim_{T \to 0} \frac{\langle \frac{P_{\mathrm{d}}}{T} x, y \rangle_{\mathcal{X}} - \langle Px, y \rangle_{\mathcal{X}}}{T} = 0, \quad \forall x, y \in \mathcal{X}, \tag{8.154}$$

$$\lim_{T \to 0} T \left(\langle Q_{\mathrm{d}} x, y \rangle_{\mathcal{X}} - \langle Cx, Cy \rangle_{\mathcal{X}} \right) = \langle Qx, y \rangle_{\mathcal{X}}, \quad \forall x, y \in \mathcal{X}. \tag{8.155}$$

Proof Let us define the following auxiliar operators

$$\mathcal{A}_{\mathrm{d}} x = \frac{\mathbf{\Phi} - I_{\mathcal{X}}}{T} x, \quad x \in D(A^{\mathcal{X}}), \tag{8.156}$$

$$\mathcal{B}_{\mathrm{d}} u = \frac{1}{T} \int_0^T S^V(\tau) B d\tau, \quad u \in \mathcal{U}. \tag{8.157}$$

Notice that

$$\lim_{T \to 0} \mathcal{A}_{\mathrm{d}} x = A^{\mathcal{X}} x, \quad x \in D(A^{\mathcal{X}}), \tag{8.158}$$

$$\lim_{T \to 0} \mathcal{B}_{\mathrm{d}} u = Bu, \quad u \in \mathcal{U}. \tag{8.159}$$

Consider now the following function

$$\mathcal{G} : \mathbb{R} \times \mathcal{X} \longrightarrow \mathbb{R}$$

defined by

$$\mathcal{G}(T, X) = \begin{cases} T^2 \langle X \mathcal{A}_d^* x, \mathcal{A}_d^* y \rangle_{\mathcal{X}} + \langle X x, \mathcal{A}_d^* y \rangle_{\mathcal{X}} + \langle \mathcal{A}_d^* x, X y \rangle_{\mathcal{X}} + \langle \mathcal{B}_d^* x, \mathcal{B}_d^* y \rangle_{\mathcal{X}} & , \; T \neq 0, \\ \langle X x, \left(A^{\mathcal{X}}\right)^* y \rangle_{\mathcal{X}} + \langle \left(A^{\mathcal{X}}\right)^* x, X y \rangle_{\mathcal{X}} + \langle B^* x, B^* y \rangle_{\mathcal{X}} & , \; T = 0. \end{cases} \quad (8.160)$$

Notice that $\mathcal{G}(T, X)$ is continuous in the origin with respect to T, and this is true since (8.156) and (8.157) hold. We claim that

$$\lim_{T \longrightarrow 0} \langle \frac{P_d}{T} x, y \rangle_{\mathcal{X}} = \langle P x, y \rangle_{\mathcal{X}}. \quad (8.161)$$

Indeed, suppose that (8.161) does not hold and let $\mathbf{P} \neq P$ denote the corresponding limit, i.e.

$$\lim_{T \longrightarrow 0} \langle \frac{P_d}{T} x, y \rangle_{\mathcal{X}} = \langle \mathbf{P} x, y \rangle_{\mathcal{X}}. \quad (8.162)$$

A simple manipulation shows that

$$\begin{aligned} 0 &= \lim_{T \longrightarrow 0} \mathcal{G}\left(T, \frac{P_d}{T}\right) - \mathcal{G}(0, P) = \\ &= \langle \mathbf{P} x, \left(A^{\mathcal{X}}\right)^* y \rangle_{\mathcal{X}} + \langle \left(A^{\mathcal{X}}\right)^* x, \mathbf{P} y \rangle_{\mathcal{X}} + \langle B^* x, B^* y \rangle_{\mathcal{X}} - \\ &\quad - \langle P x, \left(A^{\mathcal{X}}\right)^* y \rangle_{\mathcal{X}} + \langle \left(A^{\mathcal{X}}\right)^* x, P y \rangle_{\mathcal{X}} + \langle B^* x, B^* y \rangle_{\mathcal{X}} = \\ &= \langle (\mathbf{P} - P) x, \left(A^{\mathcal{X}}\right)^* y \rangle_{\mathcal{X}} + \langle \left(A^{\mathcal{X}}\right)^* x, (\mathbf{P} - P) y \rangle_{\mathcal{X}}. \end{aligned} \quad (8.163)$$

But the original system was assumed exponentially stable on $\mathcal{X}$. It follows by exploiting the uniqueness of the solution to the Lyapunov equation that

$$\langle (\mathbf{P} - P) x, \left(A^{\mathcal{X}}\right)^* y \rangle_{\mathcal{X}} + \langle \left(A^{\mathcal{X}}\right)^* x, (\mathbf{P} - P) y \rangle_{\mathcal{X}} = 0 \Longleftrightarrow \mathbf{P} = P. \quad (8.164)$$

In order to prove (8.154) define the function

$$\mathcal{M} : \mathbb{R} \times \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$$

defined by

$$\mathcal{M}(T, X, Y) \triangleq \begin{cases} \frac{\mathcal{G}(T,X) - \mathcal{G}(0,Y)}{T} & , \quad T \neq 0, \\ 0 & , \quad T = 0. \end{cases} \quad (8.165)$$

Exploiting the continuity of $\mathcal{G}(\cdot, \cdot)$ in the origin, one can immediately show that a similar continuity property is enjoied by $\mathcal{M}$, fact which shows that

$$\lim_{T \to 0} \mathcal{M}\left(T, \frac{P_{\mathrm{d}}}{T}, P\right) = 0, \tag{8.166}$$

which is exactly (8.154). Dual constructions and manipulations imply (8.155) and the proof is complete.

∎

Remark 8.23 *If in the case of systems with diagonal semigroup on ℓ_2 the limits (8.152) and (8.153) always exist, it remains an open question whether these limit relations hold in the general case.*

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Part V

Conclusions

Chapter 9

Conclusions

9.1 Looking back

Two modern discrete-time control theories represent the theoretical framework for this thesis where we have approached the problem of digital control of infinite-dimensional systems with possible unbounded input and/or output operators. The first one is the so-called *discrete Popov theory* giving the necessary and sufficient conditions for the existence of a stabilizing solution to the discrete-time Riccati equation in terms of the invertibility of a Toeplitz operator associated with the discrete-time system. The discrete Popov theory framework proves to be rich enough to permit one to write down the solutions to various discrete control problems, such as the LQ-optimal control problem, and the $\mathcal{H}^\infty$ control problem output measuremet feedback, respectively. The second one is the so-called *discret Hyland-Bernstein theory* giving the set of necessary conditions for the existence of the solution to the discrete-time fixed-order compensation problem, optimally with respect to a quadratic cost function.

In this thesis we have outlined the main Popov theory based results on the discrete-time $\mathcal{H}^\infty$ control problem and we have given a full extension to discrete-time infinite-systems of the classical results of discrete Hyland-Bernstein theory known in the literature. Having available the framework offered by these two discrete control theories, we have approached the problem of *digital control of linear infinite-dimensional systems with unbounded input and/or output operators*. The type of unboundedness considered here was the one commonly accepted for systems that fall in the so-called Pritchard-Salamon class. Severeal concepts, specific to digital control, such as *digital exponential stability* and *hybrid stability*, have been analysed and discussed in the framework offered by systems with unboundedness. Consequently, they have been enriched with a new meaning, specific to such a framework, and have attained a fairly high degree of generality. Depending upon the specific control problem considered in this thesis, the digital solutions were alternatively constructed on the basis of the discrete Popov theory or the discrete Hyland-Bernstein theory. They have been obtained by using a modern concept, the so-called *lifting technique*,

which enables one to obtain a discrete-time control problem, equivalent with the digital control problem in the sense that a solution to the former one is a solution to the latter one as well. A step of paramount importance in the preocess of applying the lifting technique to digital control was represented by Theorem 5.10 where, among other results, it is shown that sampling and lifting have benefical effects on the control structure in the sense that all the operators of the equivalent control structure are bounded on appropriate function spaces.

Another feature that has been captured in the picture offered by this dissertation is represented by the fact that the two discrete control theories we have applied are suitable for coping with so-called *singular control problems*. These problems occur naturally in digital control, mostly due to the definition of the sample operator which is not well defined over L_2 spaces of functions. As an immediate example of difficulty that we had to overcome, is the singularity $D_{21} = 0$ in the digital $\mathcal{H}^\infty$ control with measurement feedback. One of the most important contributions of this thesis is given by the expression of the necessary and sufficient conditions for the existence of the subpotimal controller, achieving both *hybrid stability* and disturbance attenuation, in terms of *coercivity* of input/output operators associated with the hybrid generalized control plant.

The last part of this thesis was concerned with the applications. Two special Pritchard-Salamon systems, a parabolic system and a hyperbolic one were considered. For both of them we have computed the time-discretized counterparts. For the parabolic system, exploiting its semigroup diagonality we have computed also the equivalent discrete-time Pritchard-Salamon-Popov triple. We have pointed out the difficluties arising in a possible implementation process, dificulties that are mainly due to the fact that one has to calculate the inverse of a Toeplitz operator in order to express the stabilizing solution to a Riccati equation. Since replacing a time-discretized infinite-dimensional system with a finite-dimensional approximation of it represents a legitimate complaint directed toward system modeling literature, we have devoted our efforts in the last part of the applications to the asymptotic analysis with respect to high sampling frequency of the time-discretized infinite-dimensional systems. Most of the results have originally been proved for finite-dimensional systems, but we also show here how and in which extent they can be generalized in infinite dimensions.

9.2   Looking ahead

Let us try to point out the directions for future research opened by this study.

(i) First of all notice that the Pritchard-Salamon class of systems with unboundedness is not the most general one possible. It is sufficient to mention that Dirichlet boundary control usually leads to input operators that are 'too unbounded' for the Pritchard-Salamon framework. This is obviously limiting the generality of the digital control theory built in [13, 12, 9]. In a series of papers [75, 76, 85] Salamon and Weiss intro-

duced the concepts of *abstract linear systems* and *well posedness* for linear systems and *regular systems*, respectively. Basically, regular systems are represented by

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & x(0) = x_0 \in \mathcal{X}, \\ y(t) = Cx(t) + Du(t), \end{cases}$$

with A, B, C unbounded with respect to the Hilbert state space $\mathcal{X}$, and D bounded from the space of inputs, say $\mathcal{U}$ to the space of outputs, $\mathcal{Y}$. An extension of the results of this thesis such that they become applicable to the class of *regular systems* would be, in our opinion, not a trivial one. This statement is motivated as follows

(a) In order to be sampled the output function has to enjoy certain smoothness properties with respect to time. A sufficient condition (such as $x_0 \in \mathcal{W}$ in the case of Pritchard-Salamon systems) for smooth output function is not known yet for regular systems.

(b) The main result on digital exponentially stabilizability of this paper, Theorem 5.10 was proved on the basis of the structural decomposition (5.28). Such a decomposition might hold for regular systems with finite-dimensional input and output operators as well (we dare to state it as a conjecture that this is indeed true), but for this class of systems the result is not available yet.

(c) A general Riccati theory for regular systems is not available yet either; the latest development of George Weiss [86] is about *admissible feedback operators for regular systems*. Hence, another reason why for this class of systems we cannot extend the main digital stabilizability result of this paper, Theorem 5.10.

(ii) Secondly, even if the results presented in this thesis have, in appeerence, only a theoretical importance, they have, in our opinion, nevertheless, a practical impact as well. Let us make this statement more clear. A feature of paramount importance for implementation is the fact that the controller, solving a certain optimization control problem, is also a infinite-dimensional system. Such a controller, besides the fact that presents the great disadvantage to be a system with unboundedness, when obtained via a continuous-time synthesis, can be analytically determined only for particular examples. Even in this case, the physical process, the plant, has to be controlled, and the most convenient solution nowadays seems to be the computer control one. This implies that we have to represent the unbounded operators in a finite stack computer memory, a typical discrete device. Such a representation would imply two operations made on the controller; approximation of the controller by a finite-dimensional one and discretization. If for finite-dimensional systems those two operations can intertwine, as was shown in [5], this is not necessarily applicable for infinite-dimensional systems generally. Besides the lose of optimality generated by approximating the controller, the fact that the two operation do not intertwine generates an additional uncertainty of the degree of suboptimality of the real, implementable controller. The

digital control approach we proposed in this thsis eliminates these disadvantages. The controller is born naturally from a digital control design process where the designer manipulates only bounded operators and hence, the outgrowth of such a design process is a discrete-time infinite-dimensional controller which is a system with boundedness in operators. The only leftover major difficulty is the controller approximation. With this respect, the Popov theory based approach we took in this paper leeds to approximation of a certain discrete Toeplitz operator. We believe that this is opening a new and promising research direction, *inverse Toeplitz operator theory applied to approximate stabilizing solutions to Riccati equations*. There are basically two reasons why this direction is a promising one. The first one is represented by the fact that, usually, the approximation of infinite-dimensional Riccati equations is done via ad-hoc methods (see Delfour [36] and the references therein) and hence, there is no available theory, only some receipes. The second one, eliminating the disadvantage presented by the ad-hoc nature of those methods, is represented by its intrinsic nature, that of having to approximate a Toeplitx operator, task for which a very well developed literature is already available (see [54] and the references therein).

If we have succeeded to define two major directions opened by this research, the extension of the main digital results to regular systems and the application of Toeplitz operator theory to obtainining approximations of solutions to digital Riccati equations, let us end this concluding chapter by summarizing a some of the technical difficulties encountered that represents in our opinion, another source of interesting research.

(i) In Chapter 4 a general discrete Hyland-Bernstein theory has been developed. The main result, Theorem 4.14 gives the set of necessary conditions for the existence of a fixed-order compensator for a infinite-dimensional discrete-time system, optimally with respect to a quadratic cost function. The proof is based on vanishing the Fréchet the first derivatives of the cost function with respect to the compensator parameters. The set of necessary conditions is not necessarily the set of sufficient ones. It would be extremely important to prove this fact.

(ii) In Chapter 6, section 6.3 we have proved that a sufficient condition for the existence of the singular filtering $\mathcal{H}^\infty$ Riccati equation is represented by the coercivity of the "1st-input-2nd-output operator" T_{21}. We have conjectured that a so-called *normal input* realization of the lifted counterpart of $\Sigma(S^\nu(\cdot), B_1, C_1)$ generates always a coercive T_{21} operator. Such a realization exists for infinite-dimensional systems with the impulse response in $\ell_1(0, \infty; \mathbb{R}^{p \times m})$, but it is not obvious under which assumptions it exists for infinite-dimensional discrete-time systems with arbitrary infinite-dimensional input and output spaces. For such systems, Ober and Montgomery-Smith succeeded to prove only the existence of *parabalanced realizations* (see [65] for details). The development of such a realization theory for the general case of infinite-dimensional discrete-time systems with infinite-dimensional input and output spaces, would give,

besides the insight in this part of realization theory, an adequate framework to formulate in terms of initial-data the sufficient conditions for the existence of the digital controller solving the $\mathcal{H}^\infty$ control problem.

(*iii*) In the application part of this thesis, Chapter 7, we have derived asymptotic proerties with respect to the sampling step for controllability and observability gramians of finite-dimensional time-discretized systems and we have shown in which extent they can be generalized for infinite-dimensional system case. Summarizing, the results can be readily extended for systems with diagonal semigroup, and in the general case, they are in the form of "necessary conditions", i.e. "if the limit exists", then

$$\text{"if the limit exists"} \Rightarrow \begin{cases} \lim_{T \longrightarrow 0} \left\| \frac{\frac{P_d}{T} x - P x}{T} \right\|_{\mathcal{X}} = 0, \quad \forall x \in \mathcal{X}, \\ \lim_{T \longrightarrow 0} \left\| \frac{T Q_d y - Q y}{T} - C^* C y \right\|_{\mathcal{X}} = 0, \quad \forall y \in \mathcal{X}. \end{cases}$$

Since the solution to the digital Riccati equation converges in the strong sense to its the solution to the continuous-time Riccati equation as the sampling step approaches zero (see [73]), we believe that this type of convergence holds also for the solutions to Lyapunov equations and then we have

$$\lim_{T \longrightarrow 0} \| P_d x - P x \|_{\mathcal{X}} = 0. \tag{9.1}$$

However, this does not imply generally $\lim_{T \longrightarrow 0} \left\| \frac{\frac{P_d}{T} x - P x}{T} \right\|_{\mathcal{X}} = 0$, and it is an interesting fact to be able to state something whether this implication holds or not.

(*iv*) An extension of the concept of approximately balanced realization of time-discretized Pritchard-Salamon systems would be also an interesting direction for future research.

(*v*) Finally, the continuous-time full-order/fixed-order $\mathcal{H}^2$-optimal control problem for infinite-diemensional systems seems to be an interesting problem for both cases, the bounded case as well as the unbounded case.

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Appendix A

Notations and basic definitions

A.1 Notations

$\mathbb{N}$	-	the set of positive integers
$\mathbb{C}$	-	the field of complex numbers
$\mathbb{R}$	-	the field of real numbers
$\mathbb{R}_+$	-	the set of positive real numbers
$j\mathbb{R}$	-	the imaginary axis
$\mathbb{R}^{n \times n}$	-	$n \times n$ real matrices
$\mathbb{R}_S^{n \times n}$	-	$n \times n$ real symmetric matrices
$\mathbb{R}^n$	-	n-dimensional real array
C	-	the complex plane
$\Re e(s)$	-	the real part of $s \in C$
$\Im m(s)$	-	the imaginary part of $s \in C$
C_+	-	$\{s \in C | \Re e(s) > 0\}$, the right half plane
$U_1(0)$	-	$\{z \in C | |z| < 1\}$, the open unit disc
$\mathcal{U}, \mathcal{X}, \mathcal{Y}, \mathcal{W}, \mathcal{V}, \mathcal{Z}$	-	real separable Hilbert spaces
$\mathcal{X} \oplus \mathcal{Y}$	-	the direct sum of $\mathcal{X}$ and $\mathcal{Y}$
$\| \cdot \|_{\mathcal{X}}$	-	the norm on $\mathcal{X}$
$\langle \cdot, \cdot \rangle_{\mathcal{X}}$	-	the inner product on $\mathcal{X}$
$\mathcal{L}(\mathcal{X}, \mathcal{Y})$	-	the space of linear bounded operators from $\mathcal{X}$ to $\mathcal{Y}$.
$\mathcal{L}(\mathcal{X})$	-	the space of linear bounded operators on $\mathcal{X}$
$\mathcal{N}(\mathcal{X}, \mathcal{Y})$	-	the space of linear nuclear operators from $\mathcal{X}$ to $\mathcal{Y}$.
$\mathcal{N}(\mathcal{X})$	-	the space of linear nuclear operators on $\mathcal{X}$
Σ_G	-	the infinite-dimensional linear plant with transfer function G
Σ_K	-	the infinite-dimensional linear controller with transfer function K
Σ_K^n	-	the n-dimensional linear controller with transfer function K

$\Sigma_{\mathrm{G}}(S(\cdot), B, C, D)$	-	the infinite-dimensional continuous-time plant Σ_{G} in state space form
$\Sigma_{\mathrm{G}}(A, B, C, D)$	-	the infinite-dimensional discrete-time plant Σ_{G} in state space form
$\Sigma_{\mathrm{K}}^{\infty}(A, B, C, D)$	-	the infinite-dimensional discrete-time controller Σ_{K} in state space form
$\Sigma_{\mathrm{K}}^{n}(F, K, L)$	-	the n-dimensional controller Σ_{K}^{n} in state space form
$S^{\mathcal{X}}(\cdot)$	-	C_0-semigroup on $\mathcal{X}$
$A^{\mathcal{X}}$	-	the infinitesimal generator of $S^{\mathcal{X}}(\cdot)$ on $\mathcal{X}$
$D(A^{\mathcal{X}})$	-	the domain of A
$\overline{A}$	-	the unique extension of A
A^*	-	the adjoint of A
$\omega_{\mathcal{X}}$	-	the growth bound of $S^{\mathcal{X}}(\cdot)$ on $\mathcal{X}$
$\mathbf{H}$	-	the zero-order hold operator
$\mathbf{S}$	-	the sampler operator
T	-	sampling period
$\stackrel{\triangle}{=}$	-	equal by definition
$x \in \mathcal{X}$	-	x element of $\mathcal{X}$
$(\sigma x)(t)$	-	$\begin{cases} \dot{x}(t) & , \quad t \in \mathbb{R} \\ x(t+1) & , \quad t \in \mathbb{N} \end{cases}$
$\tilde{x}$	-	the sampled $x(\cdot)$
$\hat{x}$	-	the lifted $x(\cdot)$ unless otherwise else depending on the context
$\Sigma(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{\Lambda}, \mathbf{\Theta})$	-	the time-discretized $\Sigma_{\mathrm{G}}(S(\cdot), B, C, D)$
$\mathcal{X} \subset \mathcal{Y}$	-	$\mathcal{X}$ is contained as a set in $\mathcal{Y}$
$\mathcal{X} \hookrightarrow \mathcal{Y}$	-	$\mathcal{X}$ is contained in $\mathcal{Y}$ with continuous dense injection
G	-	transfer function of Σ_{G}
K	-	transfer function of Σ_{K}
$G_{y_1 u_1}$	-	transfer function of the closed-loop system from u_1 to y_1
$T_{y_1 u_1}$	-	the I/O operator of the closed-loop system from u_1 to y_1
$\mathcal{C}(0, \infty; \mathcal{X})$	-	the space of $\mathcal{X}$-valued piece-wise constant functions
$C(0, \infty; \mathcal{X})$	-	the space of continuous $\mathcal{X}$-valued functions
$\mathcal{Z}(0, \infty; \mathcal{X})$	-	the space of $\mathcal{X}$-valued sequences with integer index
$L_2^{\mathrm{loc}}(0, \infty; \mathcal{X})$	-	the space of locally square (Bochner) integrable functions from $[0, \infty)$ to $\mathcal{X}$
$\ell_2(0, \infty; \mathcal{X})$	-	the space of square summable functions from $[0, \infty)$ to $\mathcal{X}$
$\mathcal{H}^{\infty}(\mathcal{L}(\mathcal{U}, \mathcal{Y}))$	-	the Hardy space of $\mathcal{L}(\mathcal{U}, \mathcal{Y})$-valued functions which are bounded and analytic on C_+ or $U_1(0)$
$\mathcal{H}^2(\mathcal{Z})$	-	the Hardy space of complex-valued functions $G : C \longrightarrow \mathcal{Z}$ which are holomorphic in C_+ and for which
	-	$\|G(s)\|_2^2 \stackrel{\triangle}{=} \frac{1}{\sqrt{2\pi}} \sup_{\omega > 0} \left(\int_{\infty}^{\infty} \|G(x + \jmath y)\|_{\mathcal{Z}}^2 < \infty \right)$

$\mathbb{E}$	-	the expectation
$\mathcal{J}(\Sigma_{\mathrm{G}}, \Sigma_{\mathrm{K}})$	-	the LQG cost function
u_1	-	disturbance input
y_1	-	regulated output
u_2	-	control input
y_2	-	measured output
A/D	-	analog-to-discrete device
D/A	-	discrete-to-analog device
$u_{\mathrm{step}}(\cdot)$	-	piece-wise constant control function
$\Sigma(S(\cdot), B, M = M^*)$	-	continuous-time Popov triple
$\Sigma(A, B, M = M^*)$	-	discrete-time Popov triple
$J_{\Sigma}(x_0, u(\cdot))$	-	quadratic index associated to the Popov triple
$U_{\mathrm{adm}}^{x_0}$	-	the admissible control class of a Popov triple
$\forall x \in \mathcal{X}$	-	arbitrary x in $\mathcal{X}$
$\exists$	-	there exists
$\mathcal{B}^t$	-	the controllability map
$\mathcal{C}^t$	-	the observability map
$W^{1,2}(0,1)$	-	the Sobolev space
δ_x	-	the distribution function
$\mathcal{R} \gg 0$	-	$\mathcal{R}$ is coercive
Σ_{I}	-	inner system
Σ_{O}	-	outer system
$I_{\mathcal{X}}$	-	the identity operator on $\mathcal{X}$
$I_{n \times n}$	-	the identity $n \times n$ matrix
$\beta_{\mathcal{X}}$	-	discrete growth constant
Ω	-	the lifting operator
$\mathcal{F}\left(\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, K \right)$	-	the linear fractional transformation of $\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}$ and K.
$\theta(\cdot)$	-	the symbol of Landau

A.2 Basic definitions

Definition A.1 *(C_0-semigroup)*

(i) *A strongly continuous semigroup (or a C_0-semigroup) is an operator valued function $S^{\mathcal{X}}$: $\mathbb{R}_+ \longrightarrow \mathcal{L}(\mathcal{X})$, which satisfies*

(a) $S^{\mathcal{X}}(t + \tau) = S^{\mathcal{X}}(t)S^{\mathcal{X}}(\tau),\ \forall t, \tau \geq 0$,

(b) $S^{\mathcal{X}}(0) = I_{\mathcal{X}}$,

(c) $\lim_{\substack{t \longrightarrow 0 \\ t>0}} \|S^{\mathcal{X}}(t)x - x\|_{\mathcal{X}} = 0,\ \ \forall x \in \mathcal{X}$.

(ii) *If $\omega_{\mathcal{X}} = \inf_{t>0} \frac{1}{t} \log \|S^{\mathcal{X}}(t)\|_{\mathcal{X}}$, then $\omega_{\mathcal{X}}$ is called the growth bound of $\mathcal{X}$ on $S^{\mathcal{X}}(\cdot)$.*

(iii) *The infinitesimal generator of $\mathcal{X}$ on $S^{\mathcal{X}}(\cdot)$ is an operator $A^{\mathcal{X}} : D(A^{\mathcal{X}}) \longrightarrow \mathcal{Z}$ defined by*

$$A^{\mathcal{X}} x \stackrel{\triangle}{=} \lim_{\substack{t \longrightarrow 0 \\ t>0}} \frac{1}{t}(S^{\mathcal{X}}(t) - I_{\mathcal{X}})x,$$

whenever the limit exists; the domain of $A^{\mathcal{X}}$, $D(A^{\mathcal{X}})$, being the set of elemnts in $\mathcal{X}$ for which the above limit exists.

(iv) *The C_0-semigroup $S^{\mathcal{X}}(\cdot)$ on $\mathcal{X}$ is called exponentially stable if $\exists M_{\omega^{\mathcal{X}}} > 0$ such that $\forall t \geq 0$*

$$\|S^{\mathcal{X}}(t)\|_{\mathcal{X}} \leq M_{\omega^{\mathcal{X}}} e^{\omega^{\mathcal{X}}}$$

and its growth bound is negative, $\omega_{\mathcal{X}} < 0$.

(v) *If in particular $\|S^{\mathcal{X}}(\cdot)\|_{\mathcal{X}} < 1$ then the semigroup is called a contraction C_0-semigroup.*

Definition A.2 *(separable spaces, continuous dense injection, dual spaces, reflexive spaces)*

(i) *A normed linear space $(X, \|\cdot\|_X)$ is called separable if it contains a countable dense subset. In particular a Hilbert space is called separable if it contains an orthonormal basis.*

(ii) *Let $\mathcal{X}$ be a Banach space with norm $\|\cdot\|_{\mathcal{X}}$ and let $\mathcal{W}$ be a linear subspace of $\mathcal{X}$ and assume that another norm, $\|\cdot\|_{\mathcal{W}}$, is defined on $\mathcal{W}$ and in this way $\mathcal{W}$ is itself a Banach space. Consider the linear operator $\jmath : \mathcal{W} \longrightarrow \mathcal{X},\ \jmath w = w$.*

(a) *We call this a continuous embedding if $\jmath \in \mathcal{L}(\mathcal{W}, \mathcal{X})$*

(b) *If $\mathcal{W}$ is dense in $\mathcal{X}$ with respect to the norm in $\mathcal{X}$ (i.e. its closure is equal to $\mathcal{X}$) we shall call $\jmath$ a dense injection. In the case when $\jmath$ is also a continuous application we shall use the notation $\mathcal{W} \hookrightarrow \mathcal{X}$ to denote the continuous dense injection*

(iii) *The topological dual space of $\mathcal{X}$ is the space of all bounded linear functionals on $\mathcal{X}$ with domain $\mathcal{X}$*

$$\mathcal{X}^* \stackrel{\triangle}{=} \{f : \mathcal{X} \longrightarrow \mathbb{R} | f \text{ is linear and bounded}\}.$$

(iv) *The bidual space of $\mathcal{X}$ is the space of all bounded functionals on $\mathcal{X}^*$ with domain $\mathcal{X}^*$*

$$\mathcal{X}^{**} \triangleq \{f : \mathcal{X}^* \longrightarrow \mathbb{R} | f \text{ is linear and bounded }\}$$

and the isomorphism from $\mathcal{X}$ to $\mathcal{X}^{**}$

$$x \mapsto f_x^*, \;\; f_x^*(f) \triangleq f(x), \;\; \forall f^* \in \mathcal{X}^*$$

is called the natural embedding of $\mathcal{X}$ in $\mathcal{X}^{**}$.

(v) *A space $\mathcal{X}$ is reflexive if its bidual $\mathcal{X}^{**}$ is isometrically isomorphic to $\mathcal{X}$ under the natural embedding*

Definition A.3 *(bounded, invertible, symmetric, adjoint, normal, self-adjoint, positive definite, non-negative definite, coercive, semisimple, compact and closed operators)*

(i) *An operator $A : \mathcal{X} \longrightarrow \mathcal{Y}$ is said to be bounded if*

$$\|A\|_{\mathcal{Y}} \triangleq \sup_{\|x\| \leq 1} \|Ax\| < \infty, \;\; \forall x \in \mathcal{X}.$$

(ii) *A densely defined operator A is called symmetric if $\forall x, y \in D(A)$*

$$\langle Ax, y \rangle = \langle x, Ay \rangle.$$

(iii) *An operator $A : \mathcal{X} \longrightarrow \mathcal{Y}$ is said to be invertible if there exists $A^{-1} : \mathcal{Y} \longrightarrow \mathcal{X}$ such that*

$$\begin{align} A^{-1}Ax &= x, \forall x \in \mathcal{X}, \tag{A.1} \\ AA^{-1}y &= y, \forall y \in \mathcal{Y}. \tag{A.2} \end{align}$$

Then A^{-1} is called the inverse of A. In particular, if $A : \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $A^{-1} : \mathcal{L}(\mathcal{Y}, \mathcal{X})$ then A is called boundedly invertible.

(iv) *Let $A : D(A) \subset \mathcal{X} \longrightarrow \mathcal{Y}$ be a linear operator such that the domain of A is dense in $\mathcal{X}$. Then the adjoint operator $A^* : D(A^*) \subset \mathcal{Y} \longrightarrow \mathcal{X}$ is defined by*

$$\exists y^* \in \mathcal{X}, \text{ such that } \langle Ax, y \rangle = \langle x, yA^* \rangle, \;\; \forall y \in \mathcal{Y}. \tag{A.3}$$

(v) *If $A \in \mathcal{L}(\mathcal{X})$ and $AA^* = A^*A$ then A is called normal.*

(vi) *If $A = A^*$ then A is self-adjoint. In particular all bounded symmetric operators are self-adjoint operators.*

(vii) *A self-adjoint operator for which $\langle Ax, x \rangle \geq 0, \forall x \in D(A^{\mathcal{X}})$ is called non-negative definite.*

(viii) *A self-adjoint operator for which $\langle Ax, x \rangle > 0, \forall x \in D(A^{\mathcal{X}})$ is called positive definite.*

(ix) *A self-adjoint operator for which* $\langle Ax, x \rangle \geq \epsilon \|x\|_{\mathcal{X}}^2, \forall x \in D(A^{\mathcal{X}})$ *is called coercive.*

(x) *An operator* $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *is said to be compact if A maps bounded sets of $\mathcal{X}$ onto compact sets of $\mathcal{Y}$).*

(xi) *A linear operator* $A \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *is said to be closed if its graph*

$$\mathcal{G}(A) \triangleq \{(x, Ax) | x \in D(A)\}$$

is a closed linear subspace of $\mathcal{X} \times \mathcal{Y}$. In particular every self-adjoint operator is closed since the adjoint of any operator is closed.

Definition A.4 *Let A be a closed densely defined operator on a complex normed linear space X. We say that λ is in the resolvent set of A, denoted $\rho(A)$, if the resolvent operator defined by*

$$R(\lambda, A) \triangleq (\lambda I - A)^{-1} \tag{A.4}$$

exists and is a bounded linear operator on a dense domain of X.

Let

$$\sigma(A) \triangleq \mathbb{C} \setminus \rho(A) \tag{A.5}$$

denote the spectrum *of A, while the* point spectrum *of A is defined by*

$$\sigma_p(A) \triangleq \{\lambda \in \mathbb{C} \mid (\lambda I - A)\text{is not injective }\}, \tag{A.6}$$

with $\rho(A)$ the resolvent set of A.

Definition A.5 *A point $\lambda \in \sigma_p(A)$ is called eigenvalue, and $x \neq 0$ such that $(\lambda I - A)x = 0$ an eigenvector.*

Definition A.6 *Let $\Sigma(S(\cdot), B, C, C)$ be an infinite-dimensional system and let G be its transfer function. Let $\{\sigma_1, \cdots, \sigma_n, \cdots\}$ be the Hankel singular values. Then the Hankel norm of $\Sigma(S(\cdot), B, C, C)$ is defined by*

$$\|G\|_{\mathrm{H}} = \max_{n \geq 1} \sigma_n. \tag{A.7}$$

If

$$\sum_{n=0}^{\infty} \sigma_n < \infty, \tag{A.8}$$

then the system $\Sigma(S(\cdot), B, C, C)$ is called nuclear system and (A.8) defines the nuclear norm of the system Σ

$$\|G\|_{\mathrm{N}} \triangleq \sum_{n=0}^{\infty} \sigma_n. \tag{A.9}$$

If

$$\sum_{n=0}^{\infty} \sigma_n^2 < \infty, \tag{A.10}$$

then the system $\Sigma(S(\cdot), B, C, C)$ has a Hilbert-Schmidt Hankel operator and (A.10) defines the Hankel-Frobenius norm of the system Σ

$$\|G\|_{\mathrm{HF}} \stackrel{\Delta}{=} \sum_{n=0}^{\infty} \sigma_n^2. \tag{A.11}$$

Definition A.7 *(sesquilinear form) A sesquilinear form on the complex linear space X is a complex valued function defined on $X \times X$ which is linear in the first argument and antilinear in the second argument i.e.*

$$Q(\alpha_1 x_1 + \alpha_2 x_2, y) = \alpha_1 Q(x_1, y) + \alpha_2 Q(x_2, y)$$

and

$$Q(x, \alpha_1 y_1 + \alpha_2 y_2) = \bar{\alpha}_1 Q(x, y_1) + \bar{\alpha}_2 Q(x, y_2).$$

($\bar{\alpha}$ is the complex conjugate of the complex number α)

A common example of sesquilinear form is the inner product on a complex Hilbert space.

Definition A.8 *(The dual system associated with a Pritchard-Salamon system)*
Let $\Sigma(S^\mathcal{V}(\cdot), B, C, D)$ be a Pritchard-Salamon system with respect to the continuous dense injection

$$\mathcal{W} \hookrightarrow \mathcal{X} \hookrightarrow \mathcal{V}$$

and consider the topological dual spaces of $\mathcal{W}$, $\mathcal{X}$ and $\mathcal{V}$. Identify $\mathcal{X}$, $\mathcal{U}$ and $\mathcal{Y}$ with their dual counterparts. Then we have

$$\mathcal{V}^* \hookrightarrow \mathcal{X}^* = \mathcal{X} \hookrightarrow \mathcal{W}^*$$

The dual of B, denoted B^*, is defined by $B^* \in \mathcal{L}(\mathcal{V}^*, \mathcal{U})$ such that B^* is an admissible output operator for $S^*(\cdot)$ with respect to $\mathcal{V}^* \hookrightarrow \mathcal{X}$. The dual of C, denoted C^*, is defined by $C^* \in \mathcal{L}(\mathcal{Y}, \mathcal{W}^*)$ such that C^* is an admissible output operator for for $S^*(\cdot)$ with respect to $\mathcal{X} \hookrightarrow \mathcal{W}^*$. Then the infinite-dimensional system $\Sigma(S^{\mathcal{W}^*}(\cdot), B^*, C^*, D^*)$ is called the dual system associated with a Pritchard-Salamon system. It is also a Pritchard-Salamon system with respect to

$$\mathcal{V}^* \hookrightarrow \mathcal{X}^* = \mathcal{X} \hookrightarrow \mathcal{W}^*.$$

List of Figures

.

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Samenvatting

Een belangrijke klasse van oneindig-dimensionale systemen met onbegrensde ingangs- en uitgangsoperatoren wordt door Pritchard en Salamon geïntroduceerd. Gewone voorbeelden van systemen die tot deze klasse behoren zijn gedistribueerde-parameter systemen beschreven door partiële differentiaalvergelijkingen met puntbesturing en/of randbesturing en, respectievelijk, met puntobservatie en/of randsobservatie en verdragende systemen.

Deze dissertatie behandelt de digitale-regeling van Pritchard-Salamon systemen. De resultaten en de oplossingen van de digitale-regeling problemen die hebben we in deze dissertatie beschouwen

- Het digitale LQ-optimalregelprobleem

- Het vaste-orde eindig-dimensionale dynamische LQG-optimalcompensatieprobleem

- Het $\mathcal{H}^\infty$-optimalregelprobleem

wordt via twee moderne regeltheories gekregen.

(*i*) De eerste regeltheorie is de zogenaamde Hyland-Bernstein regeltheorie. Het specifiec van de Hyland-Bernstein regeltheorie is de volgende: door deze theorie is het mogelijke de eerste-orde noodzakelijke voorwaarden voor de existentie van de vaste-orde eindig-dimensionale dynamische compensatoren voor oneindig-dimensionale systemen te geven met behulp van een systeem van twee gemodificeerde Riccati vergelijkingen en twee gemodificeerde Lyapunov vergelijkingen.

(*ii*) De tweede regeltheorie is de zogenaamde discrete Popov theorie. Door deze theorie kunnen we bewijzen dat de noodzakelijke en voldoende voorwaarden voor de existentie van de stabiliserende oplossing van de discrete Riccati vergelijkingen is de inverteerbaarheid van een bepaalde Toeplitz operator van het systeem.

Dit proefschrift bevat vijf delen. De eerste deel is het kader en bevat twee hoofdstuken waarin presenteren we de algemene problematiek van moderne regeltheories- en regelproblemen die wordt in de volgende delen geanalisered, het motivatie van deze studie en geven we de elementaiere wiskundige begrippen en resultaten over oneindig-dimensionale systemen met onbegrensde ingangs- en uitgangsoperatoren en een paar voorbelden van systemen die in de Pritchard-Salamon klasse behoren.

In de tweede deel de meest balangrijke discrete-tijd regeltheorie resultaten worden bewezen. De originele bewijs van het resultaat over $\mathcal{H}^\infty$-optimalregelprobleem met toestand- en uitgang-reactie, de extensie van de eindig-dimensionale resultaten van Hyland-Bernstein regeltheorie tot

oneindig-dimensionale discrete-tijd systemen zijn de belangrijkste teoretische bijdragen van deze dissertatie.

In de derde deel gebruiken we de discrete-tijd regeltheorie resultaten die in de tweede deel worden bewezen om te beschouwen de digitale-oplossing van de eerder opmerking regelproblemen.

De vierde deel is de toepassing deel van dit proefschrift en bevat een studievoorbeeld en eindig-dimensionale resultaten over zogenaamde "bijna gebalanseerde realisaties" van digitale regelsystemen.

Ten slotte, de vijfde deel bevat de aanbevelingen voor verder onderzoek en conclusies.

Curriculum Vitae

Florin Dan Barb werd geboren op 8 juni 1964 te Cluj-Napoca, Roemenië.

In 1982 behaalde hij het diploma "*Diplomă de bacalaureat*" aan het lyceum der wisckunde en natuurkunde "Liceul de Matematică şi Fizică No. 4", te Boekarest, Roemenië.

In 1989 was hij afgestudeerd aan de Technische Universiteit Boekarest "Institutul Politehnic Bucureşti", en behaalde hij het diploma "*Diplomă de inginer*" in automatisering en technische informatica, met specialisatie in besturingtheorie.

Sedert september 1989 was hij weerkzaam bij verschilende bedrijven, zoals het instituut voor kernenergie reactoren "Institutul de Reactori Nucleari Energetici" te Piteşti, Roemenië en het onderzoek instituut voor informatica "Institutul de Cercetari in Informatica" te Boekarest, Roemenië, bedrijven die zich toelegt op ontwikkeling, fabricage en exploiteren van kernenergie reactoren en, respectievelijk, onderzoek in toegepaste informatica. Binnen deze bedrijven bekleedt hij en positie als onderzoeker ingineur waar hij zich bezighoudt met simulatie en digitale-regeling van lineare systemen, programa ontwikkeling voor verschillende praktische applicaties in natuurkunde.

Sinds december 1991 was hij tevens part-time als assistent in dients van de Technische Universiteit Boekarest "Institutul Politehnic Bucureşti", faculteit der Automatisering en Technische Informatics, waar begon hij te werken ter verkrijging van de grad van doctor.

Sinds mei 1992 is hij full-time assistent in opleiding in dients van de Technische Universiteit Delft, faculteit der Technische Wiskunde en Informatica, vakgroep Systeemtheorie waar hij zich bezighoudt met onderzoek in digitale-regeling van oneindigdimensionale systemen. In deze periode heeft hij verschilende curssusen in regeltechniek gekregen van de Universiteit Boekarest en de Technische Universiteit Boekarest "Institutul Politehnic Bucureşti", en ook heeft hij de curssusen van de tweede-fase onderzoekopleiding systeem- en regeltheorie gekregen van het Nederlandse Netwerk Systeem- en Regeltheorie.

Gedurenede zijn promotietijd bezocht hij verschilende conferenties in binnenland en buitenland en heeft hij meer dan 20 artiekelen gepubliceerd in belangrijke internationale regeltheorie tijdschriften en proceedings van conferenties.