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Eigenvectors and eigenvalues of the effective resistance matrix of a graph

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ABSTRACT

We express the eigenvectors of the Laplacian matrix of a graph as a linear combination of the eigenvectors of the effective resistance matrix of that graph and vice versa. We find partial fraction expansions for the eigenvalues of either matrix in terms of the spectrum of the other matrix as well as the explicit characteristic polynomials and some sums of powers of eigenvalues. We also improve the bounds of the Interlacing Theorem for the eigenvalues of the Laplacian and effective resistance matrix of a same graph.

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1. Introduction

We consider an undirected, unweighted and connected graph G, whose corresponding graph-related matrices are symmetric. The graph G contains a set $\mathcal{N}$ of N nodes and a set $\mathcal{L}$ of L links. As mentioned in my book [1], I believe that, after the adjacency matrix A and the Laplacian matrix Q of a graph G, the effective resistance matrix $\mathbf{\Omega}$ is the third important matrix associated with graph G. The effective resistance matrix $\mathbf{\Omega}$ is closely related to the Laplacian matrix by

$$\mathbf{\Omega} = \zeta u^T + u\zeta^T - 2Q^\dagger \tag{1}$$

where $u = (1, 1, \ldots, 1)$ is the unity vector or all-one vector, the vector $\zeta = (Q^\dagger_{11}, Q^\dagger_{22}, \ldots, Q^\dagger_{NN})$ and $Q^\dagger$ is the pseudoinverse of the Laplacian [2], [1, Secion 4.2]. In a graph G, two different types [3] of transport are possible that lead to either "path networks", such as telecommunication and transportation networks (for e.g. IP packets, cars, trains, ships, airplanes), or 'flow networks', such as power grids and utility (water, gas, etc.) networks. In a path network, the transport of items follows a single path $\mathcal{P}_{ij}$ between a node pair (i, j), whereas in a flow network, the transport from node i to node j propagates over all possible paths from node i to node j. The effective resistance matrix $\mathbf{\Omega}$ plays a crucial role in 'flow networks', such as electrical resistor networks [1, art. 14].

The effective resistance matrix $\mathbf{\Omega}$ is a distance matrix [1, art. 8]. The spectrum of distance matrices was overviewed by Aouchiche and Hansen [4]. A fundamental relation in the

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theory of the effective resistance matrix $\mathbf{\Omega}$ is Fiedler's block matrix [5],[1, (5.17)]

$$\begin{pmatrix} 0 & u^T \\ u & \mathbf{\Omega} \end{pmatrix}^{-1} = \begin{pmatrix} -2\sigma^2 & p^T \\ p & -\frac{1}{2}Q \end{pmatrix} \quad \text{with } \mathbf{\Omega}p = 2\sigma^2 u \tag{2}$$

where the $N \times 1$ vector p, defined as

$$p = \frac{1}{2}Q\zeta + \frac{u}{N} \tag{3}$$

with norm equal to

$$p^T p = \|p\|_2^2 = \frac{1}{4}\zeta^T Q^2 \zeta + \frac{1}{N} \tag{4}$$

obeys $p^T u = 1$, and where[1]

$$\sigma^2 = \frac{R_G}{N^2} + \frac{1}{4}\zeta^T Q\zeta \tag{5}$$

and $R_G = Nu^T\zeta = \frac{1}{2}u^T\mathbf{\Omega}u = N\,\mathrm{trace}(Q^\dagger)$ is the effective graph resistance, an important graph metric [1, Section 5.2], which is also known as the Kirchhoff index. The effective resistance matrix $\mathbf{\Omega}$ possesses interesting geometric properties, recently explored by Devriendt and colleagues [6–8]. They demonstrate that the vector p in (3) has several fundamental properties and that σ^2 in (5) can be interpreted as a variance of a distribution on a graph. In particular [9], the vector p can be interpreted as a discrete scalar curvature on a graph, p is the solution of the optimization problem $\max_x x^T\mathbf{\Omega}x$ for the vector x subject to $x^T u = 1$ and that maximum equals $2\sigma^2 = p^T\mathbf{\Omega}p$, where

$$\sigma^2 = \frac{1}{4}\sum_{(i,j)\in\mathcal{L}} \left(\omega_{ik} - \omega_{jk}\right)^2$$

for any node k in the graph G and where $\omega_{ij} = \omega_{ji}$ is the effective resistance between node i and j. We also add that the vector p is the barycentric coordinate of the circumcenter of the simplex S of the graph [10] and σ is the radius of the circumcircle of the simplex S. Each component $\zeta_i = Q_{ii}^\dagger$ of the vector ζ has the property that $\frac{1}{\zeta_i}$ equals [5, Corollary 1.4.14 on p. 23] the altitude of the vertex i in the simplex S towards the face containing all other vertices[2] . Following the geometry-inspired terminology of Devriendt, the vector p is called the 'resistance curvature' and σ is the 'resistance radius' of the circumcircle of the simplex S. In comparison with the Laplacian $Q = \Delta - A$, where Δ is the diagonal matrix with element $\Delta_{ii} = d_i$ equal to the degree of node i and being a component of the degree vector $d = (d_1, d_2, \ldots, d_N)$, we emphasize that the vector ζ, which we may call the 'inverse simplex-altitude' vector, is equally important as the degree vector d, which is perhaps the most prominent characterizing vector of any graph. The inverse simplex-altitude vector ζ,

[1] Unfortunately, there is an error in [1, p. 182], where $\sigma^2 = R_G + \frac{1}{4}\zeta^T Q\zeta$ is written, whereas (5) correct is.

[2] Any undirected graph G, with N nodes and L links, possesses a simplex S in the $(N-1)$-dimensional Euclidian space, consisting of N vertices. Each vertex in the simplex S corresponds with a node in the graph G, but a link in the graph G does *not* correspond to an edge in the simplex S. To avoid confusion, we have proposed in [1, p. 2] to use nodes and links in a graph G, while vertices and edges in the simplex S of that graph G.

whose component ζ_i quantifies the spreading capacity in node i, is studied in [2], where the upper bound for the Euclidean norm $\zeta^T\zeta = \|\zeta\|_2^2$ is deduced

$$\zeta^T\zeta = \sum_{m=1}^{N} \left(Q^\dagger_{mm}\right)^2 \le \left(1 - \frac{1}{N}\right) \sum_{n=1}^{N-1} \frac{1}{\mu_n^2} \tag{6}$$

which is tight in the sense that equality is achieved if $Q^\dagger_{mm} = \frac{R_G}{N^2}$ for any node m in the graph. Such a graph, characterized by $\zeta = \frac{R_G}{N^2}u$, is called 'resistance-regular' [11].

The eigenvalue equation for the $N \times N$ symmetric Laplacian matrix Q of the graph G is

$$Qz_k = \mu_k z_k \tag{7}$$

where μ_k is the k-th eigenvalue of Q belonging to the eigenvector z_k, normalized by $z_k^T z_k = 1$. The real eigenvalues are ordered as in [1]: $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_N = 0$. Any Laplacian matrix [1, art. 102] possesses the eigenvector $z_N = \frac{u}{\sqrt{N}}$ belonging to the smallest eigenvalue $\mu_N = 0$. The eigenvalue equation [1, Section 5.5] of the $N \times N$ non-negative, symmetric, effective resistance matrix $\mathbf{\Omega}$ of the graph G is

$$\mathbf{\Omega} v_j = \rho_j v_j \tag{8}$$

where ρ_j is the j-th eigenvalue of $\mathbf{\Omega}$ belonging to the normalized eigenvector v_j, i.e. $v_j^T v_j = 1$, for $1 \le j \le N$. The real eigenvalues are ordered as $\rho_1 \ge \rho_2 \ge \cdots \ge \rho_N$. The effective resistance matrix $\mathbf{\Omega}$ in a connected graph has $N-1$ negative eigenvalues (see e.g. [1, Theorem 34], but also Theorem 1 below) and only the spectral radius $\rho_1 = -\sum_{j=2}^{N} \rho_j > 0$ is positive and $\det \mathbf{\Omega} \neq 0$.

Here, we explicitly express the eigenvectors $v_1, v_2, \ldots, v_N$ and eigenvalues $\rho_1, \rho_2, \ldots, \rho_N$ of the effective resistance matrix $\mathbf{\Omega}$ in terms of the eigenvectors $z_1, z_2, \ldots, z_N = \frac{u}{\sqrt{N}}$ and eigenvalues $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_N = 0$ of the symmetric Laplacian matrix Q of an unweighted[3] , undirected graph G, and vice versa.

The article is outlined as follows. Section 2 starts by deriving two 'generalized eigenvalue equations' that are consequences of Fiedler's fundamental block-matrix equation (2). The remainder of the article concentrates on spectral decomposition and insights deduced from those two generalized eigenvalue Equations (10) and (11). We express the eigenvectors of the effective resistance matrix $\mathbf{\Omega}$ as a linear combination of the eigenvectors of the Laplacian matrix Q and vice versa in Section 3. Section 4 deduces partial fractions of the eigenvalues of $\mathbf{\Omega}$ in terms of the spectrum (i.e. eigenvalues and eigenvectors) of the Laplacian Q and vice versa, from which the explicit scaled characteristic polynomials are derived in Section 5. The characteristic polynomial of the Laplacian Q and effective resistance matrix $\mathbf{\Omega}$ are derived in Section 6 and 7, respectively, and the coefficients of those polynomials are explicitly deduced together with some sums of powers of eigenvalues. Section 8 focuses on the computation of eigenvalues of the effective resistance matrix $\mathbf{\Omega}$, given the spectrum of the Laplacian Q, and vice versa. Section 9 concludes, while Appendix 1 proves double orthogonality of the row and column vectors of the orthogonal matrix C in (24). Appendix 2 complements the main Theorem 2 in Section 3 and tries to address – unsuccessfully though – the problem of determining the sign of the components of the resistance curvature vector p.

[3] Some results can be readily extended to weighted, undirected graphs.

2. Two generalized eigenvalue equations

We start by deducing two different almost eigenvalues Equations (10) and (11). First, we left-multiply the eigenvalue Equation (8) by Q,

$$Q\Omega v_j = \rho_j Q v_j$$

Fiedler's block matrix identity (2) leads to

$$Q\Omega = 2pu^T - 2I \tag{9}$$

Using (9)

$$\rho_j Q v_j = 2\left(u^T v_j\right) p - 2v_j$$

and the fact that each eigenvalue $\rho_j \neq 0$, for $1 \leq j \leq N$, results in, what we call, the first generalized eigenvalue equation

$$Qv_j = -\frac{2}{\rho_j} v_j + \frac{2\left(u^T v_j\right)}{\rho_j} p \tag{10}$$

Second, the eigenvalue equation $\Omega^{-1} v_j = \rho_j^{-1} v_j$ combines with $\Omega^{-1} = \frac{1}{2\sigma^2} p.p^T - \frac{1}{2}Q$, another deduction from Fiedler's block matrix identity (2) as shown in [1, (5.17)], to a slightly different generalized eigenvalue equation than (10),

$$Qv_j = -\frac{2}{\rho_j} v_j + \frac{p^T v_j}{\sigma^2} p \tag{11}$$

Equating the generalized eigenvalue Equations (10) and (11) indicates

$$p^T v_j = \frac{2\sigma^2\left(u^T v_j\right)}{\rho_j} \tag{12}$$

If $u^T v_j \neq 0$, then (12) shows that $p^T v_j \neq 0$ and the eigenvalue ρ_j of the effective resistance matrix Ω equals

$$\rho_j = 2\sigma^2 \frac{u^T v_j}{p^T v_j} \tag{13}$$

while [1, (5.27)] deduces

$$\rho_j = \frac{R_G}{N} + N\frac{\zeta^T v_j}{u^T v_j} \tag{14}$$

Clearly, if the last term in (10) or (11) vanishes, then an eigenvalue equation arises. If that last term is not too large, the generalized eigenvalue equations and the Laplacian eigenvalue equation (7) suggest that $\mu_j \approx -\frac{2}{\rho_j}$ plus some correction. At this point, we mention the beautiful Interlacing Theorem 1, proved by Fiedler [5, Corollary 6.2.9] and later by Sun *et al.* [12]:

Theorem 1: *In a connected graph, the eigenvalues $\rho_1, \rho_2, \ldots, \rho_N$ of the effective resistance matrix $\mathbf{\Omega}$ interlace with those of the Laplacian matrix as*

$$\rho_1 > 0 > -\frac{2}{\mu_1} \geq \rho_2 \geq -\frac{2}{\mu_2} \geq \cdots \geq -\frac{2}{\mu_{N-2}} \geq \rho_{N-1} \geq -\frac{2}{\mu_{N-1}} \geq \rho_N \tag{15}$$

The Interlacing Theorem 1 shows that, if $\mu_k = \mu_{k-1}$, then $\rho_k = -\frac{2}{\mu_k}$. In other words, when a graph has Laplacian eigenvalues of multiplicity larger than 1, then equality as in $\rho_k = -\frac{2}{\mu_k}$ occurs. Furthermore, if $\mu_k = -\frac{2}{\rho_k}$, then the generalized eigenvalue Equations (10) and (11) indicate that v_k is both an eigenvector of the Laplacian Q and of the effective resistance matrix $\mathbf{\Omega}$ and that $u^T v_k = p^T v_k = 0$. If the inverse simplex-altitude vector $\zeta = \frac{R_G}{N^2} u$, thus in a 'resistance-regular' graph, then [1, Theorem 35] states that $\rho_j = -\frac{2}{\mu_j}$ for all $1 < j \leq N$ and the proof indicates that the eigenvector $v_j = z_j$ for all $1 \leq j \leq N$. The definition (1) for a resistance-regular graph, with the all-one matrix $J = u.u^T$, reduces to

$$\mathbf{\Omega}_{\text{resistance-regular}} = 2\left(\frac{R_G}{N^2} J - Q^\dagger\right)$$

In the sequel, we will ignore resistance-regular graphs. Furthermore, we will confine ourselves to graphs with simple Laplacian eigenvalues, i.e. each eigenvalue μ_k is distinct with multiplicity 1. For such graphs, the Interlacing Theorem 1 then states that the eigenvalues $\{\rho_j\}_{1 \leq j \leq N}$ of $\mathbf{\Omega}$ are also simple. Those graphs are connected due to a positive algebraic connectivity $\mu_{N-1} > 0$ (else the zero eigenvalue $\mu_N = \mu_{N-1} = 0$ would have multiplicity 2) as shown in [1, Theorem 20, art. 115].

In spite of the *approximate* nature of $\mu_j \approx -\frac{2}{\rho_j}$ in Theorem 1, we show in (45) below the *exact* product

$$\prod_{m=1}^{N-1} \mu_m = -\sigma^2 N \prod_{m=1}^{N} \left(-\frac{2}{\rho_m}\right) = \frac{2\sigma^2 N}{\rho_1} \prod_{m=2}^{N} \left(-\frac{2}{\rho_m}\right)$$

We will later improve the Interlacing Theorem 1 and derive an exact governing equation in Corollary 4 for the eigenvalues $\rho_1, \rho_1, \ldots, \rho_N$ of the effective resistance matrix $\mathbf{\Omega}$.

3. Eigenvectors of $\mathbf{\Omega}$ as linear combination of those of Q

3.1. Explicit expressions

Theorem 2 expresses any eigenvector of the effective resistance matrix $\mathbf{\Omega}$ in terms of the eigenvectors of the Laplacian matrix Q and vice versa:

Theorem 2: *If $\mu_k + \frac{2}{\rho_j} \neq 0$ for all $1 \leq j, k \leq N$, then the j-th eigenvector v_j of the effective resistance matrix $\mathbf{\Omega}$ is written as a linear combination of eigenvectors of the Laplacian matrix*

Q, for $1 \leq j \leq N$, *as*

$$v_j = \alpha_j \left(\frac{u}{N} + \frac{2}{\rho_j} \sum_{k=1}^{N-1} \frac{\frac{\mu_k}{2} \left(\zeta^T z_k \right)}{\frac{2}{\rho_j} + \mu_k} z_k \right) \tag{16}$$

where

$$\frac{1}{\alpha_j^2} = \frac{1}{\left(u^T v_j \right)^2} = \frac{1}{N} + \left(\frac{2}{\rho_j} \right)^2 \sum_{k=1}^{N-1} \left(\frac{\frac{\mu_k}{2} \left(\zeta^T z_k \right)}{\frac{2}{\rho_j} + \mu_k} \right)^2 \tag{17}$$

Conversely, the k-th eigenvector z_k *of the Laplacian matrix Q is written, for* $1 \leq k \leq N$, *as a linear combination of eigenvectors of the effective resistance matrix* $\boldsymbol{\Omega}$ *as*

$$z_k = \beta_k \sum_{j=1}^{N} \frac{\frac{1}{\rho_j} \left(u^T v_j \right)}{\frac{2}{\rho_j} + \mu_k} v_j \tag{18}$$

where $\beta_N = \frac{2}{\sqrt{N}}$ *and for* $1 \leq k < N$

$$\frac{1}{\beta_k^2} = \frac{1}{\left(\mu_k \zeta^T z_k \right)^2} = \sum_{j=1}^{N} \left(\frac{\frac{1}{\rho_j} \left(u^T v_j \right)}{\frac{2}{\rho_j} + \mu_k} \right)^2 \tag{19}$$

Before giving the proof, we compute the scalar product of the resistance curvature vector p and the normalized eigenvector z_k of the Laplacian Q. From the definition (3), it follows that

$$z_k^T p = \frac{1}{2} z_k^T Q \zeta + z_k^T \frac{u}{N} = \frac{\mu_k}{2} z_k^T \zeta + z_k^T \frac{u}{N}$$

Thus, if $k = N$, then $\mu_N = 0$ and $z_N = \frac{u}{\sqrt{N}}$ result in

$$z_N^T p = \frac{1}{\sqrt{N}} \tag{20}$$

else, for $1 \leq k < N$,

$$z_k^T p = \frac{\mu_k}{2} z_k^T \zeta \tag{21}$$

Proof: Left-multiplying (10) by z_k^T leads to

$$z_k^T Q v_j = -\frac{2}{\rho_j} z_k^T v_j + \frac{2\left(u^T v_j\right)}{\rho_j} z_k^T p$$

With the eigenvalue equation $Qz_j = \mu_j z_j$ in (7) of the symmetric Laplacian Q,

$$\left(\mu_k + \frac{2}{\rho_j}\right) z_k^T v_j = \frac{2\left(u^T v_j\right)}{\rho_j} z_k^T p$$

leads to the scalar product between eigenvectors of $\mathbf{\Omega}$ and Q

$$z_k^T v_j = \frac{\frac{2}{\rho_j}}{\frac{2}{\rho_j} + \mu_k} \left(u^T v_j\right) \left(p^T z_k\right)$$

because of the restriction that $\mu_k + \frac{2}{\rho_j} \neq 0$ for all $1 \leq j, k \leq N$. Invoking the scalar product $p^T z_k$ in (20) and (21) then leads to

$$z_k^T v_j = \begin{cases} \frac{\frac{2}{\rho_j}\frac{\mu_k}{2}}{\frac{2}{\rho_j}+\mu_k} \left(u^T v_j\right) \left(\zeta^T z_k\right) & \text{for } 1 \leq k < N \\ \frac{1}{\sqrt{N}} \left(u^T v_j\right) & \text{for } k = N \end{cases} \tag{22}$$

If $u^T v_j = \sum_{m=1}^N (v_j)_m = 0$ for $2 \leq j \leq N$, because the Perron-Frobenius Theorem for the non-negative matrix $\mathbf{\Omega}$ states that all components $(v_1)_m > 0$ for $1 \leq m \leq N$ in a connected graph and thus $u^T v_1 > 0$, then (22) shows that $z_k^T v_j = 0$ for all $1 \leq k \leq N$. Since a non-null vector as the eigenvector v_j cannot be orthogonal to all Laplacian eigenvectors, we must require that $\mu_k + \frac{2}{\rho_j} = 0$ for at one pair (j, k) of integers. The Interlacing Theorem 1 indicates that $\mu_k = -\frac{2}{\rho_j}$ if $j = k > 1$, which only happens if μ_k has multiplicity at least equal to 2. Hence, the restriction $\mu_k + \frac{2}{\rho_j} \neq 0$ for all $1 \leq j, k \leq N$ is sufficient to exclude Laplacian eigenvalues with multiplicity larger than 1 and to prevent that $u^T v_j = 0$ for any $1 \leq j \leq N$.

We can now compute $v_j = \sum_{k=1}^N (z_k^T v_j) z_k = \sum_{k=1}^{N-1} (z_k^T v_j) z_k + (z_N^T v_j) z_N$,

$$v_j = \left(u^T v_j\right) \left(\frac{u}{N} + \frac{2}{\rho_j} \sum_{k=1}^{N-1} \frac{\frac{\mu_k}{2}}{\frac{2}{\rho_j} + \mu_k} \left(\zeta^T z_k\right) z_k \right)$$

Normalization of the eigenvector, $v_j^T v_j = 1$, leads, with the orthogonality of eigenvectors $z_k^T z_m = \delta_{mk}$ of a symmetrix matrix [1, Section 10.3], where the Kronecker delta $\delta_{km} = 1$ if $k = m$, otherwise $\delta_{km} = 0$, to (17), from which (16) follows.

The reverse that expresses the Laplacian eigenvector $z_k = \sum_{j=1}^N (z_k^T v_j) v_j$ as a linear combination of the eigenvectors $v_1, v_2, \ldots, v_N$ of the effective resistance matrix $\mathbf{\Omega}$ follows

from (22), for $1 \leq k < N$, as

$$z_k = \left(\zeta^T z_k\right) \mu_k \sum_{j=1}^{N} \frac{\frac{1}{\rho_j}}{\frac{2}{\rho_j} + \mu_k} \left(u^T v_j\right) v_j$$

After normalization $z_k^T z_k = 1$, we obtain

$$\frac{1}{\left(\mu_k \zeta^T z_k\right)^2} = \sum_{j=1}^{N} \left(\frac{\frac{1}{\rho_j}}{\frac{2}{\rho_j} + \mu_k} \left(u^T v_j\right) \right)^2$$

If $k = N$, then $\mu_N = 0$ and $z_N = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} (u^T v_j) v_j = \frac{u}{\sqrt{N}}$, from which $\beta_N = \frac{2}{\sqrt{N}}$ in (18). Hence, (18) and (19) are demonstrated for $1 \leq k \leq N$. ■

3.2. Discussion

The idea of Theorem 2 to deduce a scalar product between eigenvectors of two different matrices does not apply to the eigenvectors of the adjacency A and Laplacian matrix $Q = \Delta - A$ of a graph G. Indeed, starting from the eigenvalue equation (7) of the Laplacian matrix Q,

$$\mu_k z_k = Q z_k = \Delta z_k - A z_k$$

left-multiplying with the m-th eigenvector x_m^T of the adjacency matrix A,

$$\mu_k x_m^T z_k = x_m^T \Delta z_k - x_m^T A z_k$$

invoking the eigenvalue equation $A x_m = \lambda_m x_m$ of the adjacency matrix leads to the scalar product

$$x_m^T z_k = \frac{x_m^T \Delta z_k}{\mu_k + \lambda_m} \tag{23}$$

In contrast to (22) for the effective resistance matrix $\mathbf{\Omega}$, the scalar product (23) is of a different and *not-separable* form, because $x_m^T \Delta z_k = \sum_{j=1}^{N} d_j (x_m)_j (z_k)_j$ is a single 'scalar product' containing x_m as well as z_k, whereas (22) is a product of scalar products, where each scalar product only involves either an eigenvector v_j of $\mathbf{\Omega}$ or an eigenvector z_k of the Laplacian Q.

3.3. An orthogonal matrix

The $N \times N$ matrix V contains in its columns the set of all eigenvector $v_1, v_2, \ldots, v_N$ of the effective resistance matrix $\mathbf{\Omega}$,

$$V = \begin{bmatrix} v_1 & v_2 & \cdots & v_N \end{bmatrix}$$

Similarly, we define the orthogonal eigenvector matrix of the Laplacian Q as

$$Z = \begin{bmatrix} z_1 & z_2 & \cdots & z_N \end{bmatrix}$$

If $\frac{2}{\rho_j} + \mu_k \neq 0$ for all $1 \leq j,k \leq N$, then the eigenvector (16) can be expressed [1, art.191] as

$$v_j = \frac{\alpha_j \frac{\mu_1}{\rho_j} \left(\zeta^T z_1\right)}{\frac{2}{\rho_j} + \mu_1} \begin{bmatrix} (z_1)_1 \\ (z_1)_2 \\ \vdots \\ (z_1)_N \end{bmatrix} + \frac{\alpha_j \frac{\mu_2}{\rho_j} \left(\zeta^T z_2\right)}{\frac{2}{\rho_j} + \mu_2} \begin{bmatrix} (z_2)_1 \\ (z_2)_2 \\ \vdots \\ (z_2)_N \end{bmatrix} + \cdots + \frac{\alpha_j}{\sqrt{N}} \begin{bmatrix} (z_N)_1 \\ (z_N)_2 \\ \vdots \\ (z_N)_N \end{bmatrix}$$

$$= Z \begin{bmatrix} \frac{\alpha_j \frac{\mu_1}{\rho_j} (\zeta^T z_1)}{\frac{2}{\rho_j} + \mu_1} \\ \frac{\alpha_j \frac{\mu_2}{\rho_j} (\zeta^T z_2)}{\frac{2}{\rho_j} + \mu_2} \\ \vdots \\ \frac{\alpha_j}{\sqrt{N}} \end{bmatrix}$$

Substitution into the matrix V yields

$$V = Z.C$$

where the $N \times N$ matrix

$$C = \begin{bmatrix} \frac{\alpha_1 \frac{\mu_1}{\rho_1} (\zeta^T z_1)}{\frac{2}{\rho_1} + \mu_1} & \frac{\alpha_2 \frac{\mu_1}{\rho_2} (\zeta^T z_1)}{\frac{2}{\rho_2} + \mu_1} & \cdots & \frac{\alpha_N \frac{\mu_1}{\rho_N} (\zeta^T z_1)}{\frac{2}{\rho_N} + \mu_1} \\ \frac{\alpha_1 \frac{\mu_2}{\rho_1} (\zeta^T z_2)}{\frac{2}{\rho_1} + \mu_2} & \frac{\alpha_2 \frac{\mu_2}{\rho_2} (\zeta^T z_2)}{\frac{2}{\rho_2} + \mu_2} & \cdots & \frac{\alpha_N \frac{\mu_2}{\rho_N} (\zeta^T z_2)}{\frac{2}{\rho_N} + \mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\alpha_1}{\sqrt{N}} & \frac{\alpha_2}{\sqrt{N}} & \cdots & \frac{\alpha_N}{\sqrt{N}} \end{bmatrix} \tag{24}$$

is also an orthogonal matrix, because both V and Z are orthogonal matrices. Hence, the orthogonal eigenvector matrix $V = ZC$ of the effective resistance matrix $\mathbf{\Omega}$ equals the product of two orthogonal matrices Z and C. Since the determinant of an orthogonal matrix is $\det C = 1$, we find that

$$1 = \frac{1}{\sqrt{N}} \prod_{j=1}^{N} \alpha_j \prod_{j=1}^{N-1} \mu_j \left(\zeta^T z_j\right) \begin{vmatrix} \frac{1}{2+\rho_1\mu_1} & \frac{1}{2+\rho_2\mu_1} & \cdots & \frac{1}{2+\rho_N\mu_1} \\ \frac{1}{2+\rho_1\mu_2} & \frac{1}{2+\rho_2\mu_2} & \cdots & \frac{1}{2+\rho_N\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{vmatrix}$$

Orthogonality of a matrix implies double orthogonality [1, art. 248] in its corresponding column and row vectors, which is proved for the orthogonal matrix C in Appendix 1.

Finally, the doubly stochastic matrix $\Xi_\Omega = V \circ V$ obeys the equation [1, art. 274],

$$\Xi_\Omega \rho = 0 \tag{25}$$

where the $N \times 1$ vector $\rho = (\rho_1, \rho_2, \ldots, \rho_N)$ contains the eigenvalues of the effective resistance matrix $\mathbf{\Omega}$. Equation (25) is an eigenvalue equation $\Xi_\Omega w_k = \xi_k w_k$, where the eigenvector ρ belongs to the eigenvalue $\xi_k = 0$, and implies that $\det \Xi_\Omega = 0$ and $\text{rank}\Xi_\Omega \leq$

$N-1$. The zero eigenvalue of Ξ_Ω has multiciplicity at least 1. The corresponding doubly stochastic matrix $\Xi_A = X \circ X$ of the adjacency matrix A and $\Xi_Q = Z \circ Z$ for the Laplacian matrix Q also possess a zero eigenvalue with multiplicity at least 1. An open problem [13, Appendix A.3] is the determination of rankΞ_A, rankΞ_Q and rankΞ_Ω of a same graph G and their meaning for the graph G.

4. Governing eigenvalue equations and quadratric forms

Theorem 2 provides us with valuable corollaries:

Corollary 3: *If $\mu_k + \frac{2}{\rho_j} \neq 0$ for all $1 \leq j,k \leq N$, then the positive eigenvalues μ of the Laplacian matrix Q satisfy the governing eigenvalue partial fractions*

$$\sigma^2 = \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}} \tag{26}$$

or

$$\frac{1}{4\sigma^2} = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{\rho_j\left(\rho_j \mu + 2\right)} \tag{27}$$

or

$$0 = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{2 + \rho_j \mu} \quad \text{for } \mu \neq 0 \tag{28}$$

Proof: After rewriting (18) with $\frac{u^T v_j}{\rho_j} = \frac{p^T v_j}{2\sigma^2}$ in (12) and left-multiplying with the transpose of the resistance curvature vector p yields

$$p^T z_k = \frac{\left(\zeta^T z_k\right)\mu_k}{2\sigma^2} \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\mu_k + \frac{2}{\rho_j}}$$

The scalar product $z_k^T p = \frac{\mu_k}{2} z_k^T \zeta$ in (21) then leads, for $1 \leq k < N$, to (26) or, again with $p^T v_j = \frac{2\sigma^2}{\rho_j} u^T v_j$ in (12), to (27). After left-multiplying (18), valid for $1 \leq k < N$, with u^T, we obtain

$$u^T z_k = \beta_k \sum_{j=1}^{N} \frac{\frac{1}{\rho_j}\left(u^T v_j\right)^2}{\frac{2}{\rho_j} + \mu_k}$$

which leads with $z_N = \frac{u}{\sqrt{N}}$, due to orthogonality of eigenvectors $z_N^T z_k = 0$ for $1 \leq k < N$, to (28). ■

From the fractional expansions (26),(27) and (28), the Interlacing Theorem 1 can be deduced. The reverse of Corollary 3 is

Corollary 4: *If $\mu_k + \frac{2}{\rho_j} \neq 0$ for all $1 \leq j,k \leq N$, then any eigenvalue ρ of the effective resistance matrix $\mathbf{\Omega}$ satisfies, with $y = \frac{2}{\rho}$, the partial fraction expansion*

$$\sigma^2 = \frac{1}{4}\sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k + y} + \frac{1}{Ny} \tag{29}$$

Proof: After computing the scalar product of the eigenvector v_j in (16), where $\alpha_j = \rho_j \frac{p^T v_j}{2\sigma^2}$ by (13), with the resistance curvature vector p in (3), taking into account (20) and (21), we deduce for $\rho_j \neq 0$ that

$$1 = \frac{1}{2\sigma^2}\left(\frac{1}{2}\sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k + \frac{2}{\rho_j}} + \frac{\rho_j}{N}\right)$$

that can be rewritten as (29), by letting $y = \frac{2}{\rho}$ and excluding $\rho = 0$, because any eigenvalue $\rho_j \neq 0$ must satisfy (29). ■

The partial fractions can be rephrased as a quadratic form using $f(\mathbf{\Omega}) = \sum_{j=1}^{N} f(\rho_j) v_j v_j^T$ deduced in [1, Equation (A.88)], where $f(.)$ is a function defined for the eigenvalues. For example, the partial fraction (28)

$$0 = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{2 + \rho_j \mu} = \sum_{j=1}^{N} \frac{\left(u^T v_j\right)\left(v_j^T u\right)}{2 + \rho_j \mu} = u^T \left(\sum_{j=1}^{N} \frac{v_j v_j^T}{2 + \rho_j \mu}\right) u$$

transforms to the quadratric form, for $\mu \neq 0$,

$$0 = u^T \left(2I + \mu \mathbf{\Omega}\right)^{-1} u \tag{30}$$

Similarly, for an eigenvalue μ of the Laplacian Q, we find quadratic forms for the square σ^2 of the resistance radius from (26),

$$\sigma^2 = p^T \left(\mu I + 2\mathbf{\Omega}^{-1}\right)^{-1} p \tag{31}$$

and from (27)

$$\frac{1}{4\sigma^2} = u^T \left(\mathbf{\Omega}\left(2I + \mu \mathbf{\Omega}\right)\right)^{-1} u \tag{32}$$

For $\mu = 0$ in (31) and (32), we again find the known relations $2\sigma^2 = p^T \mathbf{\Omega} p$ and $\frac{1}{2\sigma^2} = u^T \mathbf{\Omega}^{-1} u$. For an eigenvalue ρ of the effective resistance matrix $\mathbf{\Omega}$ and $y = \frac{2}{\rho}$, we also give the quadratic form corresponding to (29)

$$\sigma^2 - \frac{1}{Ny} = \frac{1}{4}\zeta^T \left(Q^2 \left(Q + yI\right)^{-1}\right) \zeta \tag{33}$$

5. From partial fraction expansions to characteristic polynomials

The classical definition [1, art. 235] of a characteristic polynomial of an $n \times n$ matrix B with eigenvalues $\lambda_1(B), \lambda_2(B), \ldots, \lambda_n(B)$ is the determinant $c_B(\lambda) = \det(B - \lambda I)$, which can be expanded as a polynomial in λ of degree n,

$$c_B(\lambda) = \sum_{k=0}^{n} c_k \lambda^k = \prod_{k=1}^{n} (\lambda_k(B) - \lambda) \tag{34}$$

from which $c_n = (-1)^n$. Provided that $\det B \neq 0$, the transformation of the complex variable $\lambda \to \frac{a}{\lambda}$, where $a \neq 0$ is a complex constant, results in

$$\lambda^n c_B\left(\frac{a}{\lambda}\right) = \sum_{j=0}^{n} a^{n-j} c_{n-j} \lambda^j = \det B \ (-1)^n \, c_{aB^{-1}}(\lambda) \tag{35}$$

In other words, the transformation $\lambda \to \frac{a}{\lambda}$ of the complex variable λ, for any complex number $a \neq 0$, in the characteristic polynomial $c_B(\lambda)$ transforms the polynomial into a characteristic polynomial of the matrix aB^{-1} (provided $\det B \neq 0$). The transformation scales the set of coefficient $\{c_k\}_{0 \leq k \leq n}$ by a power of a resulting in the set $\{a^{n-j} c_{n-j}\}_{0 \leq j \leq n}$.

After multiplying the partial fraction expansions in Corollary 3 by $\prod_{m=1}^{N}(2 + \rho_m \mu)$ and multiplying each term in (29) with $\prod_{m=1}^{N}(y + \mu_m)$, we obtain the following *scaled* characteristic polynomials for the eigenvalues of the Laplacian matrix Q and of the effective resistance matrix $\boldsymbol{\Omega}$:

Lemma 5: *A scaled characteristic polynomial of degree N of the Laplacian eigenvalues is*

$$p_Q(\mu) = \sum_{j=1}^{N} \rho_j \left(p^T v_j\right)^2 \prod_{m=1; m \neq j}^{N} (2 + \rho_m \mu) - \prod_{m=1}^{N} (2 + \rho_m \mu) \, \sigma^2 \tag{36}$$

while a scaled characteristic polynomial of degree N−1 is

$$\widetilde{p}_Q(\mu) = \sum_{j=1}^{N} \left(u^T v_j\right)^2 \prod_{m=1; m \neq j}^{N} (2 + \rho_m \mu) \tag{37}$$

whose zeros, i.e. solutions of $p_Q(\mu) = 0$, equal the Laplacian eigenvalues $\mu_1, \mu_2, \ldots \mu_{N-1}$, $\mu_N = 0$, while the zeros of $\widetilde{p}_Q(\mu)$ are the positive Laplacian eigenvalues $\mu_1, \mu_2, \ldots \mu_{N-1} > 0$.

Another rewriting of (29) *is a scaled characteristic polynomial of degree N in $y = \frac{2}{\rho}$*

$$p_{2\Omega^{-1}}(y) = \prod_{m=1}^{N} (y + \mu_m) - \frac{1}{4\sigma^2} \sum_{k=1}^{N-1} \mu_k^2 \left(\zeta^T z_k\right)^2 \prod_{m=1; m \neq k}^{N} (y + \mu_m) - \frac{1}{N\sigma^2} \prod_{m=1}^{N-1} (y + \mu_m) \tag{38}$$

whose zeros, i.e. solutions of $p_\Omega(y) = 0$, are twice the inverse eigenvalues $\frac{2}{\rho_1}, \frac{2}{\rho_2}, \ldots, \frac{2}{\rho_N}$.

The polynomial (36) possesses both a summation representation and a product form,

$$p_Q(\mu) = \sum_{k=0}^{N} a_k(Q)\,\mu^k = a_N(Q)\prod_{m=1}^{N}(\mu-\mu_m) \tag{39}$$

and similarly for (37),

$$\widetilde{p}_Q(\mu) = \sum_{k=0}^{N-1} \widetilde{a}_k(Q)\,\mu^k = \widetilde{a}_{N-1}(Q)\prod_{m=1}^{N-1}(\mu-\mu_m) \tag{40}$$

where $a_k(Q)$ and $\widetilde{a}_k(Q)$ are the k-th coefficient of the summation form, respectively. The characteristic polynomial in (38) of degree N in the complex variable $y = \frac{2}{\rho}$ equals

$$p_{2\Omega^{-1}}(y) = \sum_{k=0}^{N} a_k\left(2\Omega^{-1}\right) y^k = a_N\left(2\Omega^{-1}\right)\prod_{m=1}^{N}\left(y-\frac{2}{\rho_m}\right) \tag{41}$$

The first term in $p_{2\Omega^{-1}}(y)$ in (38) is a polynomial of degree N in the complex variable $y = \frac{2}{\rho}$ with highest order coefficient 1, while the other terms in (38) contain polynomials of degree $N-1$ implying that $a_N(2\Omega^{-1}) = 1$.

We wrote *scaled* characteristic polynomial in Lemma 5 due to the scaling of the highest order coefficient c_n in (34). None of (39), (40) and (41) possesses the standard form (34) of a characteristic polynomial, which justifies the notation 'p' for polynomial instead of 'c' as in (34) for characteristic polynomial.

6. Polynomials of the Laplacian matrix Q

6.1. General properties of the polynomials $p_Q(\mu)$ and $\widetilde{p}_Q(\mu)$

We will now relate the coefficients $a_k(Q)$ and $\widetilde{a}_k(Q)$ of the polynomials $p_Q(\mu)$ and $\widetilde{p}_Q(\mu)$. Since $p_Q(0) = 0$, (39) indicates that

$$a_0(Q) = 0$$

while (37) and (40) show

$$\widetilde{p}_Q(0) = \widetilde{a}_0(Q) = 2^{N-1}\sum_{j=1}^{N}\left(u^T v_j\right)^2 = N2^{N-1} \tag{42}$$

The highest order coefficient of (36) is

$$a_N(Q) = -\sigma^2\prod_{m=1}^{N}\rho_m \tag{43}$$

while for (37), we find

$$\widetilde{a}_{N-1}(Q) = \sum_{j=1}^{N}\left(u^T v_j\right)^2\prod_{m=1;m\neq j}^{N}\rho_m = \prod_{m=1}^{N}\rho_m\sum_{j=1}^{N}\frac{1}{\rho_j}\left(u^T v_j\right)^2 = u^T\Omega^{-1}u\prod_{m=1}^{N}\rho_m$$

With $u^T\Omega^{-1}u = \frac{1}{2\sigma^2}$ (see [1, p. 182]) or from (32) for $\mu = 0$, it holds that

$$\widetilde{a}_{N-1}(Q) = \frac{1}{2\sigma^2}\prod_{m=1}^{N}\rho_m \tag{44}$$

Since $\widetilde{a}_0(Q) = \widetilde{a}_{N-1}(Q)\prod_{m=1}^{N-1}(-\mu_m)$, as follows from the product form in (40), we find that

$$\prod_{m=1}^{N-1}\mu_m = (-1)^{N-1}\frac{\widetilde{a}_0(Q)}{\widetilde{a}_{N-1}(Q)} = \frac{(-1)^{N-1}N2^{N-1}}{\frac{1}{2\sigma^2}\prod_{m=1}^{N}\rho_m}$$

Thus, the complexity $\xi(G) = \frac{1}{N}\prod_{m=1}^{N-1}\mu_m$ of a graph G, which is [1, Section 4.1.3] the number of all possible spanning trees in that graph G, equals

$$\xi(G) = \frac{1}{N}\prod_{m=1}^{N-1}\mu_m = \frac{2^N\sigma^2}{\prod_{m=1}^{N}|\rho_m|} \tag{45}$$

and

$$\widetilde{a}_{N-1}(Q) = (-1)^{N-1}\frac{2^{N-1}}{\xi(G)}$$

In conclusion, the definitions (39) and (40) show that both polynomials are related by

$$p_Q(\mu) = a_N(Q)\,\mu\prod_{m=1}^{N-1}(\mu-\mu_m) = \frac{a_N(Q)}{\widetilde{a}_{N-1}(Q)}\mu\widetilde{p}_Q(\mu)$$

and (43) and (44) lead to

$$p_Q(\mu) = -2\sigma^4\mu\widetilde{p}_Q(\mu) \tag{46}$$

Consequently, after introducing the series in (39) and (40) into (46) and after equating corresponding powers in μ, we find, for $1 \le k \le N$, that the coefficients of the polynomials of the polynomials $p_Q(\mu)$ and $\widetilde{p}_Q(\mu)$ satisfy

$$a_k(Q) = -2\sigma^4\widetilde{a}_{k-1}(Q) \tag{47}$$

6.2. Computation of the coefficients $\widetilde{a}_k(Q)$ of the polynomial $\widetilde{p}_Q(\mu)$

We rely on our characteristic coefficients, first defined in [14] and more elaborated in [15, Appendix A]. The Taylor series of a function $G(z)$ of a function $f(z)$, thus $G(f(z))$, can be

formally expanded around the point z_0 as

$$G(f(z)) = G(f(z_0)) + \sum_{m=1}^{\infty} \left(\sum_{k=1}^{m} \frac{1}{k!} \left. \frac{d^k G(f)}{df^k} \right|_{f=f(z_0)} s[k,m]_{f(z)}(z_0) \right) (z-z_0)^m \quad (48)$$

in always re-appearing quantities $s[k,m]_{f(z)}(z_0)$, that I have called, therefore, the characteristic coefficient of a complex function. A complex function with Taylor series $f(z) = \sum_{k=0}^{\infty} f_k z^k$ around $z_0 = 0$ possesses a combinatorial form of the characteristic coefficient

$$s_f[k,m] = \sum_{\sum_{i=1}^{k} j_i = m; j_i > 0} \prod_{i=1}^{k} f_{j_i} \quad (49)$$

as well as a derivative form

$$s[k,m]_f(z_0) = \frac{1}{m!} \left. \frac{\mathrm{d}^m}{\mathrm{d}z^m} [f(z) - f(z_0)]^k \right|_{z=z_0} \quad (50)$$

The characteristic coefficient $s_f[k,m]$ also satisfies a recursion [14] that allows us to compute all characteristic coefficients exactly. Sections 6.2 and 7.2 illustrate the power of characteristic coefficients.

Theorem 6: *The coefficients $\widetilde{a}_m(Q)$ of the polynomial form $\widetilde{p}_Q(\mu) = \sum_{m=0}^{N-1} \widetilde{a}_m(Q)\mu^m$ in* (40) *are, for $0 \le m \le N-1$,*

$$\widetilde{a}_m(Q) = 2^{N-1} \frac{(-1)^m}{2^m} \left(u^T \Omega^m u \right) + 2^{N-1} \sum_{j=1}^{m} \frac{(-1)^{m-j}}{2^{m-j}} \left(u^T \Omega^{m-j} u \right) \sum_{k=1}^{j} \frac{1}{k!} s_g[k,j] \quad (51)$$

Beside $a_0(Q) = 0$ and $a_1(Q) = -2^{N-2}(p^T \Omega^2 p)$, the coefficient $a_m(Q)$ of the polynomial $p_Q(\mu) = \sum_{m=0}^{N} a_m(Q)\mu^m$ in (39) *for $2 \le m \le N$ as*

$$a_m(Q) = 2^N \frac{(-1)^m}{2^{m+1}} \left(p^T \Omega^{m+1} p \right) + 2^N \sum_{j=0}^{m-2} \frac{(-1)^{m-j-1}}{2^{m-j}} \left(p^T \Omega^{m-j} p \right) \sum_{k=1}^{j+1} \frac{1}{k!} s_g[k, j+1] \quad (52)$$

where the function $g(\mu) = \sum_{m=1}^{N} \log(1 + \frac{\rho_m}{2}\mu)$ has Taylor coefficients $g_0 = 0$ and $g_n = \frac{(-1)^{n-1}}{n2^n} \sum_{m=1}^{N} \rho_m^n$ for $n > 0$.

Proof: We write (37) as

$$\widetilde{p}_Q(\mu) = \prod_{m=1}^{N} (2 + \rho_m \mu) \sum_{j=1}^{N} \frac{\left(u^T v_j\right)^2}{(2 + \rho_j \mu)}$$

and expand both $\prod_{m=1}^{N}(2+\rho_m\mu)$ and $\sum_{j=1}^{N} \frac{(u^T v_j)^2}{(2+\rho_j\mu)}$ in a Taylor series around $\mu = 0$. First,

$$\prod_{m=1}^{N} (2 + \rho_m \mu) = 2^N \prod_{m=1}^{N} \left(1 + \frac{\rho_m}{2}\mu\right) = 2^N e^{\sum_{m=1}^{N} \log\left(1 + \frac{\rho_m}{2}\mu\right)}$$

Taylor expansion of $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$ for $|x| < 1$ shows, for $\mu < \frac{2}{\rho_1}$, that

$$g(\mu) = \sum_{m=1}^{N} \log\left(1 + \frac{\rho_m}{2}\mu\right) = \sum_{m=1}^{N}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{\rho_m}{2}\right)^n \mu^n$$
$$= \sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{n2^n} \sum_{m=1}^{N} \rho_m^n\right)\mu^n$$

with Taylor coefficient $g_n = \frac{1}{n!}\frac{d^n g(\mu)}{d\mu^n}|_{\mu=0}$ equal to $g_0 = 0$ and, for $n > 0$, $g_n = \frac{(-1)^{n-1}}{n2^n}\sum_{m=1}^{N} \rho_m^n$. Invoking our characteristic coefficients $s[k,m]$ in [16] and the Taylor series, that follows from (48),

$$e^{g(\mu)} = 1 + \sum_{m=1}^{\infty}\left[\sum_{k=1}^{m} \frac{1}{k!} s_g[k,m]\right]\mu^m$$

we arrive at the Taylor series

$$\prod_{m=1}^{N}(2+\rho_m\mu) = 2^N\left(1 + \sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\frac{1}{k!}s_g[k,m]\right]\mu^m\right) = 2^N\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\frac{1}{k!}s_g[k,m]\right]\mu^m$$

where $s_g[k,0] = \delta_{k0} = 1_{\{k=0\}}$ and $s_g[0,m] = \delta_{0m} = 1_{\{m=0\}}$ and the indicator $1_x = 1$ if the condition x is true, else $1_x = 0$.

Second, the expansion, valid for $|\mu| < \frac{2}{\rho_1}$,

$$\sum_{j=1}^{N}\frac{\left(u^T v_j\right)^2}{(2+\rho_j\mu)} = \frac{1}{2}\sum_{j=1}^{N}\frac{\left(u^T v_j\right)^2}{1+\frac{\rho_j}{2}\mu} = \frac{1}{2}\sum_{j=1}^{N}\left(u^T v_j\right)^2\sum_{m=0}^{\infty}\left(-\frac{\rho_j}{2}\mu\right)^m$$

leads to the Taylor series around $\mu_0 = 0$,

$$\sum_{j=1}^{N}\frac{\left(u^T v_j\right)^2}{(2+\rho_j\mu)} = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{m+1}}\left(\sum_{j=1}^{N}\rho_j^m\left(u^T v_j\right)^2\right)\mu^m = \sum_{m=0}^{\infty}\frac{(-1)^m}{2^{m+1}}\left(u^T\Omega^m u\right)\mu^m$$

The corresponding Taylor series for the polynomial $p_Q(\mu)$ is

$$\sum_{j=1}^{N}\frac{\rho_j\left(p^T v_j\right)^2}{(2+\rho_j\mu)} - \sigma^2 = \sum_{m=1}^{\infty}\frac{(-1)^m}{2^{m+1}}\left(p^T\Omega^{m+1}p\right)\mu^m$$

The Taylor series of the polynomial $\widetilde{p}_Q(\mu)$ is found after computing the Cauchy product,

$$\widetilde{p}_Q(\mu) = 2^N\left(\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\frac{1}{k!}s_g[k,m]\right]\mu^m\right)\sum_{m=0}^{\infty}\frac{(-1)^m}{2^{m+1}}\left(u^T\Omega^m u\right)\mu^m$$
$$= 2^N\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{k=0}^{j}\frac{1}{k!}s_g[k,j]\frac{(-1)^{m-j}}{2^{m-j+1}}\left(u^T\Omega^{m-j}u\right)\right)\mu^m$$

Similarly, the Cauchy product leads to

$$p_Q(\mu) = -2^{N-2}\left(p^T\Omega^2 p\right)\mu + 2^N \sum_{m=2}^{\infty}\left(\frac{(-1)^m}{2^{m+1}}\left(p^T\Omega^{m+1}p\right) + \sum_{j=0}^{m-2}\frac{(-1)^{m-j-1}}{2^{m-j}}\left(p^T\Omega^{m-j}p\right)\sum_{k=1}^{j+1}\frac{1}{k!}s_g[k,j+1]\right)\mu^m$$

Finally, after equating corresponding powers in μ in the above and the polynomial form $\sum_{m=0}^{N-1}\widetilde{a}_m(Q)\mu^m$ in (40), we arrive, for $0 \le m \le N-1$, at

$$\widetilde{a}_m(Q) = 2^{N-1}\sum_{j=0}^{m}\frac{(-1)^{m-j}}{2^{m-j}}\left(u^T\Omega^{m-j}u\right)\sum_{k=0}^{j}\frac{1}{k!}s_g[k,j]$$

which we rewrite, by splitting-off the $j=0$ term and using $s_g[k,0]=\delta_{k0}$ and $s_g[0,m]=\delta_{0m}$, as (51). Similarly, beside $a_0(Q)=0$ and $a_1(Q)=-2^{N-2}(p^T\Omega^2 p)$, we find the coefficient $a_m(Q)$ of the polynomial $p_Q(\mu)$ in (39) for $2 \le m \le N$ as (52). ■

Invoking $a_m(Q) = -2\sigma^4\widetilde{a}_{m-1}(Q)$ in (47) yields, from (51) and (52), that

$$p^T\Omega^m p = 4\sigma^4 u^T\Omega^{m-2}u \tag{53}$$

If $m > N-1$, then $\widetilde{a}_m(Q)=0$ and the right-hand side of (51) vanishes and expresses $u^T\Omega^m u$ in terms of $u^T\Omega^j u$ with $0 \le j \le N-1$ (and similarly for $m>N$ in (52) as a consequence of the Caley-Hamilton theorem [1, art. 228]),

$$u^T\Omega^m u = \sum_{j=1}^{m}(-1)^{j-1}2^j\left(u^T\Omega^{m-j}u\right)\sum_{k=1}^{j}\frac{1}{k!}s_g[k,j] \quad \text{for } m > N-1$$

We compute the polynomial coefficient $\widetilde{a}_m(Q)$ for a few values of m. First, if $m=0$, then

$$\widetilde{a}_0(Q) = N2^{N-1}$$

If $m=1$, then (51) becomes

$$\widetilde{a}_1(Q) = 2^{N-1}\frac{-1}{2}\left(u^T\Omega u\right) + 2^{N-1}\left(u^T u\right)s_g[1,1] = -2^{N-1}R_G + 2^{N-1}N\frac{1}{2}\sum_{m=1}^{N}\rho_m$$

and, since $\sum_{m=1}^{N}\rho_m = 0$, we find

$$\widetilde{a}_1(Q) = -2^{N-1}R_G$$

We can further compute[4] the polynomial coefficient $\widetilde{a}_m(Q)$ in (51) for any desired value of the integer $m<N$ symbolically by the recursion of the characteristic coefficients. As an

[4] We add that, based on the coefficients $\widetilde{a}_m(Q)$, the algebraic connectivity μ_{N-1} can be expressed as a Lagrange series (which is explicitly listed in [1, (B.68) on p. 459] up to order 5).

example, we list

$$\widetilde{a}_2(Q) = 2^{N-3}u^T\Omega^2 u - 2^{N-4}N\sum_{m=1}^{N}\rho_m^2$$

$$\widetilde{a}_3(Q) = -2^{N-4}u^T\Omega^3 u + 2^{N-4}R_G\sum_{m=1}^{N}\rho_m^2 + \frac{2^{N-4}}{3}N\sum_{m=1}^{N}\rho_m^3$$

$$\widetilde{a}_4(Q) = -2^{N-5}u^T\Omega^4 u + 2^{N-6}u^T\Omega^2 u\sum_{m=1}^{N}\rho_m^2 + 2^{N-8}N\left(\sum_{m=1}^{N}\rho_m^2\right)^2 - \frac{2^{N-4}}{3}R_G\sum_{m=1}^{N}\rho_m^3 - 2^{N-7}N\sum_{m=1}^{N}\rho_m^4$$

7. Characteristic polynomial of the effective resistance matrix Ω

The transformation (35) indicates that $y^N c_\Omega(\frac{2}{y}) = \det\Omega\,(-1)^N c_{2\Omega^{-1}}(y)$ and the definition (34) gives $c_{2\Omega^{-1}}(y) = \prod_{k=1}^{N}(\frac{2}{\rho_m} - y) = (-1)^N p_{2\Omega^{-1}}(y)$, so that

$$c_\Omega(\rho) = \left(\frac{\rho}{2}\right)^N \det\Omega\, p_{2\Omega^{-1}}\left(\frac{2}{\rho}\right)$$

Hence, the characteristic polynomial of the effective resistance matrix Ω

$$c_\Omega(\rho) = \det(\Omega - \rho I) = \sum_{k=0}^{N} c_k(\Omega)\,\rho^n = \prod_{j=1}^{N}(\rho_j - \rho)$$

has the coefficients

$$c_k(\Omega) = c_0(\Omega)\,\frac{a_{N-k}\left(2\Omega^{-1}\right)}{2^k} \tag{54}$$

for $0 \le k \le N$ with $c_0(\Omega) = \det\Omega = \prod_{j=1}^{N}\rho_j$. Since $c_{N-1}(\Omega) = (-1)^{N-1}\sum_{j=1}^{N}\rho_j = 0$, it holds that $a_1(2\Omega^{-1}) = 0$.

Section 7.1 deduces properties of the coefficients $a_k(2\Omega^{-1})$ that follow from the definition (41) and known theory, while Section 7.2 shows that all coefficients $a_k(2\Omega^{-1})$ can be exactly computed based on properties of characteristic coefficients.

7.1. Basic properties of the polynomial $p_{2\Omega^{-1}}(y)$

The coefficient $a_{N-1}(2\Omega^{-1})$ of y^{N-1} follows from (38) in Corollary 4 as

$$a_{N-1}\left(2\Omega^{-1}\right) = \sum_{m=1}^{N}\mu_m - \frac{1}{4\sigma^2}\sum_{k=1}^{N-1}\mu_k^2\left(\zeta^T z_k\right)^2 - \frac{1}{N\sigma^2}$$

$$= \sum_{m=1}^{N}\mu_m - \frac{1}{\sigma^2}\left(\frac{1}{4}\zeta^T Q^2\zeta + \frac{1}{N}\right)$$

Relation (4) and $\sum_{m=1}^{N} \mu_m = 2L$ (see e.g. [1, art. 105]) show that

$$a_{N-1}\left(2\Omega^{-1}\right) = 2L - \frac{p^T p}{\sigma^2}$$

The Newton identities [1, art. 294] indicate that the coefficient $a_{N-1}(2\Omega^{-1}) = -2\sum_{m=1}^{N} \frac{1}{\rho_m}$ in (41). Combining both expression for $a_{N-1}(2\Omega^{-1})$ gives

$$\sum_{m=1}^{N} \frac{1}{\rho_m} = -\left(L - \frac{p^T p}{2\sigma^2}\right) \tag{55}$$

Using (A6) in Appendix 2 leads to

$$\frac{p^T p}{2\sigma^2} = \frac{1}{2}\sum_{j=1}^{N} v_j^T Q v_j + \sum_{j=1}^{N} \frac{1}{\rho_j} \tag{56}$$

Comparison with (55) indicates that the number of links in the graph G equals

$$L = \frac{1}{2}\sum_{j=1}^{N} v_j^T Q v_j \tag{57}$$

Formula (57) is an instance of the general property of the trace of an $n \times n$ matrix M and an $n \times n$ orthogonal matrix X,

$$\sum_{j=1}^{n} x_j^T M x_j = \text{trace}\left(X^T M X\right) = \text{trace}\,(M) = \sum_{i=1}^{n} m_{ii} \tag{58}$$

which follows from the cyclic permutation property of the trace-operator [1, eq, (4,15)],

$$\text{trace}\left(X^T M X\right) = \text{trace}\left(M X X^T\right) = \text{trace}\,(M)$$

because the orthogonal matrix X satisfies $XX^T = I$. Alternatively, (58) is a direct consequence of the second orthogonality relation $\sum_{j=1}^{n}(x_j)_k(x_j)_l = \delta_{kl}$ in [1, Equation (A.126)]. Since $L = \frac{1}{2}\sum_{j=1}^{N} d_j$, where d_j is the degree of node j in the graph G, it might be suggestive to propose that $d_j = v_j^T Q v_j$, which is unfortunately not true. However, if $\lceil x \rceil$ denotes the smallest integer equal to or larger than x, we found numerically for $N < 15$ that $\lceil v_1^T Q v_1 \rceil = 0$, while $\lceil v_j^T Q v_j \rceil$ for $2 \le j \le N$ is close to a degree in the graph G within an error of ± 2.

Furthermore, $p_{2\Omega^{-1}}(0) = \prod_{m=1}^{N}(-\frac{2}{\rho_m}) = -\frac{2^N}{\prod_{m=1}^{N}|\rho_m|}$. Recalling that $\mu_N = 0$, the characteristic polynomial in (38) shows that

$$p_{2\Omega^{-1}}(0) = -\frac{1}{N\sigma^2}\prod_{m=1}^{N-1} \mu_m$$

Equating both expressions for $p_{2\Omega^{-1}}(0)$ again leads to the complexity $\xi(G)$ formula in (45). The computation of the coefficient $a_k(2\Omega^{-1}) = \frac{1}{k!}\frac{d^k p_{2\Omega^{-1}}(y)}{dy^k}|_{y=0}$ for $k = 1$ results, after equating the two different ways, in the definition (5) of σ^2.

7.2. Computation of the coefficients $a_k(2\Omega^{-1})$ of the polynomial $p_{2\Omega^{-1}}(y)$

Theorem 7: *The coefficient* $a_m(2\Omega^{-1})$ *of the polynomial* $p_{2\Omega^{-1}}(y) = \sum_{k=0}^{N} a_k(2\Omega^{-1})y^k$ *in* (41) *is, for integer* $0 \leq m \leq N$,

$$a_m\left(2\Omega^{-1}\right) = N\xi\left(G\right)\left(F_m + \sum_{j=1}^{m} F_{m-j} \sum_{k=1}^{j} \frac{1}{k!}\, s_h[k,j]\right) \tag{59}$$

where $\xi(G) = \frac{1}{N}\prod_{m=1}^{N-1}\mu_m$ *is the complexity in* (45), *the Taylor coefficients are*

$$F_0 = -\tfrac{1}{N\sigma^2} \quad F_1 = \tfrac{1}{\sigma^2}\tfrac{R_G}{N^2} \quad F_2 = \tfrac{1}{4\sigma^2}\left(\zeta^T\zeta - \tfrac{R_G^2}{N^3}\right) \tag{60}$$

and for $n > 2$,

$$F_n = \frac{(-1)^n}{4\sigma^2}\zeta^T\left(Q^{\dagger}\right)^{n-2}\zeta \tag{61}$$

and where $s_h[k,j]$ *is the characteristic coefficient* (49) *of the function* $h(y) = \log\prod_{m=1}^{N-1}(1 + \frac{y}{\mu_m})$, *which has Taylor coefficients* $h_n = \frac{(-1)^{n-1}}{n}\sum_{m=1}^{N-1}\frac{1}{\mu_m^n}$ *for* $n > 0$ *around* $y_0 = 0$.

Proof: We mimic the method of Section 6.2 to compute only the coefficients $a_m(2\Omega^{-1}) = \frac{1}{m!}\frac{d^m p_{2\Omega^{-1}}(y)}{dy^m}|_{y=0}$ for any integer $0 \leq m \leq N$ and write (38) as

$$p_{2\Omega^{-1}}\left(y\right) = \prod_{m=1}^{N-1}\left(y + \mu_m\right)\left(y - \frac{1}{4\sigma^2}\sum_{k=1}^{N-1}\frac{y\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k + y} - \frac{1}{N\sigma^2}\right)$$

Recalling that the algebraic connectivity $\mu_{N-1} > 0$ for connected graphs, we expand the product $\prod_{m=1}^{N-1}(y + \mu_m) = \prod_{m=1}^{N-1}\mu_m\prod_{m=1}^{N-1}(1 + \frac{y}{\mu_m})$, using our characteristic coefficients with

$$h\left(y\right) = \log\prod_{m=1}^{N-1}\left(1 + \frac{y}{\mu_m}\right) = \sum_{n=1}^{\infty}\left(\frac{(-1)^{n-1}}{n}\sum_{m=1}^{N-1}\frac{1}{\mu_m^n}\right)y^n$$

into a Taylor series around $y = 0$ in terms of the complexity $\xi(G) = \frac{1}{N}\prod_{m=1}^{N-1}\mu_m$,

$$\prod_{m=1}^{N-1}\left(y + \mu_m\right) = N\xi\left(G\right)\left(1 + \sum_{m=1}^{\infty}\left[\sum_{k=1}^{m}\frac{1}{k!}\, s_h[k,m]\right]y^m\right)$$

The Taylor series of the partial fraction expansion is

$$\begin{aligned} F\left(y\right) &= y - \frac{1}{4\sigma^2}\sum_{k=1}^{N-1}\frac{y\mu_k\left(\zeta^T z_k\right)^2}{\left(1 + \frac{y}{\mu_k}\right)} - \frac{1}{N\sigma^2} \\ &= -\frac{1}{N\sigma^2} + y + \frac{1}{4\sigma^2}\sum_{n=1}^{\infty}(-1)^n y^n\sum_{k=1}^{N-1}\frac{\left(\zeta^T z_k\right)^2}{\mu_k^{n-2}} \end{aligned}$$

and

$$F(y) = -\frac{1}{N\sigma^2} + \frac{1}{\sigma^2}\frac{R_G}{N^2}y + \frac{1}{4\sigma^2}\left(\zeta^T\zeta - \left(\zeta^T z_N\right)^2\right)y^2 + \frac{1}{4\sigma^2}\sum_{n=3}^{\infty}(-1)^n y^n \zeta^T\left(\zeta^T\left(Q^\dagger\right)^{n-2}\zeta\right)\zeta$$

where (5) and the spectral decomposition [1, Equation (4.30)] of the pseudoinverse of the Laplacian $Q^\dagger = \sum_{k=1}^{N-1}\frac{1}{\mu_k}z_k z_k^T$ has been used. The Taylor coefficients of the Taylor series $F(y) = \sum_{n=0}^{\infty}F_n y^n$ are given by (60) and (61). Executing the Cauchy product results in

$$p_{2\Omega^{-1}}(y) = N\xi(G)\sum_{m=0}^{\infty}\left[\sum_{k=0}^{m}\frac{1}{k!}s_h[k,m]\right]y^m\sum_{n=0}^{\infty}F_n y^n = N\xi(G)\sum_{m=0}^{\infty}\sum_{j=0}^{m}\left[F_{m-j}\sum_{k=0}^{j}\frac{1}{k!}s_h[k,j]\right]y^m$$

Equating corresponding powers in y in the above and (41) finally yields, for $0 \le m \le N$, the general expression of the coefficients in (59). ■

Again, as an example, we list the first few coefficients in (59)

$$\sigma^2\frac{a_0\left(2\Omega^{-1}\right)}{\xi(G)} = -1$$

$$a_1\left(2\Omega^{-1}\right) = 0$$

$$\sigma^2\frac{a_2\left(2\Omega^{-1}\right)}{\xi(G)} = \frac{1}{4}\left(\frac{R_G^2}{N^2} + N\zeta^T\zeta\right) + \frac{1}{2}\sum_{m=1}^{N-1}\frac{1}{\mu_m^2}$$

$$\sigma^2\frac{a_3\left(2\Omega^{-1}\right)}{\xi(G)} = \frac{R_G^3}{12N^3} + R_G\frac{\zeta^T\zeta}{4} - N\frac{\zeta^T Q^\dagger\zeta}{4} - \frac{1}{3}\sum_{m=1}^{N-1}\frac{1}{\mu_m^3}$$

$$\sigma^2\frac{a_4\left(2\Omega^{-1}\right)}{\xi(G)} = \frac{R_G^2\zeta^T\zeta}{8N} - R_G\frac{\zeta^T Q^\dagger\zeta}{4} + N\frac{\zeta^T\left(Q^\dagger\right)^2\zeta}{4} - \frac{1}{8}\left(\frac{R_G^2}{N^2} + N\zeta^T\zeta\right)\sum_{m=1}^{N-1}\frac{1}{\mu_m^2} - \frac{1}{8}\left(\sum_{m=1}^{N-1}\frac{1}{\mu_m^2}\right)^2 - \frac{1}{4}\sum_{m=1}^{N-1}\frac{1}{\mu_m^4}$$

With $c_{N-1}(\Omega) = (-1)^{N-1}\sum_{m=1}^{N}\rho_m = 0$, the Newton identities [1, art. 294] state that $\sum_{m=1}^{N}\rho_m^2 = -2\frac{c_{N-2}(\Omega)}{c_N(\Omega)}$ and that $\sum_{m=1}^{N}\rho_m^3 = -3\frac{c_{N-3}(\Omega)}{c_N(\Omega)}$. Using (54), the above list of

coefficients $\{a_n(2\Omega^{-1})\}_{0\le n\le 4}$ leads to

$$\sum_{m=1}^{N}\rho_m^2 = 2\left(\frac{R_G^2}{N^2}+N\zeta^T\zeta\right)+4\sum_{m=1}^{N-1}\frac{1}{\mu_m^2} \tag{62}$$

and

$$\sum_{m=1}^{N}\rho_m^3 = 2\frac{R_G^3}{N^3}+6\left(R_G\zeta^T\zeta-N\zeta^TQ^{\dagger}\zeta\right)-8\sum_{m=1}^{N-1}\frac{1}{\mu_m^3} \tag{63}$$

8. Eigenvalues

Based on the explicit partial fraction expansions previously derived, Section 8.1 and Section 8.2 present new bounds for the eigenvalues of the effective resistance matrix Ω and of the Laplacian Q, respectively. Several different techniques for the zeros (i.e. eigenvalues) are applied: (i) a recursion (66) for eigenvalues, (ii) Taylor and Lagrange series in Section 8.1 and (iii) bounding in Lemma 10 and Section 8.2.

8.1. The eigenvalues of the effective resistance matrix Ω

It is instructive to rewrite eigenvalue equation (29) as the function

$$f(\rho) = 2\sigma^2-\frac{\rho}{N}-\frac{1}{2}\sum_{k=1}^{N-1}\frac{\mu_k^2\left(\zeta^Tz_k\right)^2}{\mu_k+\frac{2}{\rho}} \tag{64}$$

which equals for real ρ, with the quadratic form (33),

$$f(\rho) = 2\sigma^2-\frac{\rho}{N}-\frac{1}{2}\zeta^T\left(Q^2\left(Q+\frac{2}{\rho}I\right)^{-1}\right)\zeta$$

Since

$$\frac{df(\rho)}{d\rho} = -\frac{1}{N}-\sum_{k=1}^{N-1}\frac{\mu_k^2\left(\zeta^Tz_k\right)^2}{(\rho\mu_k+2)^2} < 0$$

the function $f(\rho)$ is monotonously decreasing for all real ρ. The function $f(\rho)$ possesses poles at $\rho=-\frac{2}{\mu_k}$ and travels from ∞ to $-\infty$ between two poles (at which $f(\rho)$ has an asymptote, i.e. vertical line), illustrating that between those poles, there is a zero of the function $f(\rho)$, in agreement with the Interlacing Theorem 1 and implying that there are $N-1$ real, negative zeros (i.e. eigenvalues of Ω) of the function $f(\rho)$.

8.1.1. Recursion for the eigenvalues ρ_j

Any eigenvalue ρ_j of the effective resistance matrix Ω is a zero of $f(\rho)$. Solving $f(\rho)=0$ for ρ in (64) or rewriting of the partial fractional expansion (29) in terms of $\rho=\frac{2}{y}$ results

in

$$\rho = \frac{2\sigma^2}{\frac{1}{2}\sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k \rho + 2} + \frac{1}{N}} \tag{65}$$

or

$$\rho = \frac{2\sigma^2}{\frac{1}{2}\zeta^T \left(Q^2 \left(\rho Q + 2I\right)^{-1}\right) \zeta + \frac{1}{N}}$$

Formula (65) can be recast into a recursion in r_m for $m > 0$, where r_0 is an initial start value and $\lim_{m\to\infty} r_m = \rho$,

$$r_m = \frac{2\sigma^2}{\frac{1}{4}\sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{1 + \frac{\mu_k}{2} r_{m-1}} + \frac{1}{N}} \tag{66}$$

which shows that if $r_0 > 0$, all subsequent iterates $r_1, r_2, \ldots$ are positive. A study on the convergence of the recursion (66) is omitted.

For $\rho \geq 0$ and for a inverse simplex-altitude vector $\zeta \neq \alpha u$ not proportional to the all-one vector u, the quadratic form $\zeta^T (Q^2(\rho Q + 2I)^{-1})\zeta = \sum_{k=1}^{N-1} \frac{\mu_k^2 (\zeta^T z_k)^2}{\mu_k \rho + 2} > 0$ in (65) is positive and decreases monotonically in ρ towards zero at $\rho \to \infty$. With (4) and $\rho \geq 0$, (65) leads to the bounds

$$\|p\|_2^2 > \frac{1}{2}\zeta^T \left(Q^2 \left(\rho Q + 2I\right)^{-1}\right) \zeta + \frac{1}{N} = \frac{2\sigma^2}{\rho} > \frac{1}{N}$$

Hence, we conclude that the largest eigenvalue satisfies the lower bound

$$\rho_1 > 2\left(\frac{\sigma}{\|p\|_2}\right)^2 \tag{67}$$

and the upper bound

$$\rho_1 < 2\sigma^2 N \tag{68}$$

The lower bound (67) also follows from the Rayleigh inequality [1, art. 251], stating that $\frac{w^T \mathbf{\Omega} w}{w^T w} \leq \rho_1$ for any vector w and that equality only is achieved if $w = v_1$, the eigenvector of $\mathbf{\Omega}$ belonging to eigenvalue ρ_1. If $w = p$ and recalling, as mentioned in the Introduction (Section 1), that $p^T \mathbf{\Omega} p = 2\sigma^2$, then the Rayleigh inequality indicates that $\rho_1 \geq \frac{2\sigma^2}{p^T p}$. Finally, Lemma 13 shows that, generally[5], $p \neq v_1$. Hence, the strict inequality in (67) is demonstrated for *non*-resistance-regular graphs, where $\zeta \neq \frac{R_G}{N^2} u$.

[5] Only in resistance-regular graphs, where $\zeta = \frac{R_G}{N^2} u$, then $p = \frac{u}{N} = \frac{v_1}{\sqrt{N}}$.

8.1.2. Smallest (in absolute value) eigenvalue ρ_2

Lemma 8: *For* $|\rho| < \frac{2}{\mu_1}$, *The Taylor series of the function* (64) *around* $\rho_0 = 0$, *convergent for* $|\rho| < \frac{2}{\mu_1}$ *is*

$$f(\rho) = 2\sigma^2 - \|p\|_2^2 \rho + \sum_{j=2}^{\infty} (-1)^j \frac{\zeta^T Q^{j+1} \zeta}{2^{j+1}} \rho^j \tag{69}$$

The Taylor series around $\rho_0 = 0$,

$$f(\rho)\left(1 + \rho\frac{\mu_1}{2}\right) = 2\sigma^2 + \rho\left(\mu_1\sigma^2 - \|p\|_2^2\right) + \frac{1}{2}\left(\frac{\zeta^T Q^3 \zeta}{4} - \mu_1 \|p\|_2^2\right)\rho^2 + \sum_{m=3}^{\infty} (-1)^m \left(\frac{\zeta^T Q^{m+1}\zeta - \mu_1 \zeta^T Q^m \zeta}{2^{m+1}}\right)\rho^m \tag{70}$$

converges for $|\rho| < \frac{2}{\mu_2}$, *provided that* $\mu_1 > \mu_2$.

Proof: We rewrite (64) as

$$f(\rho) = 2\sigma^2 - \frac{\rho}{N} - \frac{\rho}{4} \sum_{k=1}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{1 + \rho\frac{\mu_k}{2}}$$

Invoking the geometric series $\frac{1}{1+\rho\frac{\mu_k}{2}} = \sum_{j=0}^{\infty} (-1)^j (\frac{\mu_k}{2})^j \rho^j$, valid for $|\rho| < \frac{2}{\mu_k}$, results in the Taylor series of $f(\rho)$ around $\rho = 0$ and converging for $|\rho| < \frac{2}{\mu_1}$,

$$f(\rho) = 2\sigma^2 - \frac{\rho}{N} - \frac{\rho}{4} \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j} \sum_{k=1}^{N-1} \mu_k^{2+j} \left(\zeta^T z_k\right)^2 \rho^j$$

With the spectral decomposition $\sum_{k=1}^{N-1} \mu_k^{2+j} (\zeta^T z_k)^2 = \zeta^T Q^{2+j} \zeta$, the Taylor series simplifies to

$$\begin{aligned} f(\rho) &= 2\sigma^2 - \frac{\rho}{N} - \sum_{j=0}^{\infty} (-1)^j \frac{\zeta^T Q^{j+2} \zeta}{2^{j+2}} \rho^{j+1} \\ &= 2\sigma^2 - \left(\frac{1}{N} + \frac{\zeta^T Q^2 \zeta}{4}\right)\rho + \sum_{j=2}^{\infty} (-1)^j \frac{\zeta^T Q^{j+1}\zeta}{2^{j+1}} \rho^j \end{aligned}$$

Invoking $\|p\|_2^2 = \frac{1}{4}\zeta^T Q^2 \zeta + \frac{1}{N}$ in (4) leads to (69). The Taylor series (70) is similarly derived as that of (69). ■

In order to compute the zero $|\rho_2| > \frac{2}{\mu_1}$ with Lagrange series, we must enlarge the convergence radius of the Taylor series (69). The Taylor series (70) converges for $|\rho| < \frac{2}{\mu_2}$,

provided that $\mu_1 > \mu_2$, and still has the same zero ρ_2. The Lagrange series $\rho_2(K)$ (explicitly given in [14, Equation (77) in Appendix A]) with K terms converges to the zero $\rho_2 = \lim_{K\to\infty} \rho_2(K)$. For example with $K = 1$,

$$\rho_2(1) = -\frac{2\sigma^2}{\mu_1\sigma^2 - \|p\|_2^2} = -\frac{2}{\mu_1 - \frac{\|p\|_2^2}{\sigma^2}} < -\frac{2}{\mu_1}$$

reduces to the first Newton-Raphson estimate, starting at $\rho = 0$. We list a few further terms with $K = 2$ and $K = 3$,

$$\rho_2(2) = -\frac{2\sigma^2}{\mu_1\sigma^2 - \|p\|_2^2} - \frac{2\sigma^4\left(\frac{1}{4}\zeta^T Q^3\zeta - \mu_1\|p\|_2^2\right)}{\left(\mu_1\sigma^2 - \|p\|_2^2\right)^3}$$

$$\rho_2(3) = -\frac{2\sigma^2}{\mu_1\sigma^2 - \|p\|_2^2} - \frac{2\sigma^4\left(\frac{1}{4}\zeta^T Q^3\zeta - \mu_1\|p\|_2^2\right)}{\left(\mu_1\sigma^2 - \|p\|_2^2\right)^3} - \frac{\sigma^6\left(\mu_1\zeta^T Q^3\zeta - \zeta^T Q^4\zeta\right)^2}{2\left(\mu_1\sigma^2 - \|p\|_2^2\right)^4} + \frac{4\sigma^6\left(\frac{1}{4}\zeta^T Q^3\zeta - \mu_1\|p\|_2^2\right)^2}{\left(\mu_1\sigma^2 - \|p\|_2^2\right)^5}$$

Adding more terms becomes unwieldy as in most Lagrange series. The power of characteristic coefficients, that led to the Lagrange series, lies in the fact that we can compute *symbolically* any Taylor series (48) up to any desired number of terms. Numerical computations (with Mathematica) indicate that the Lagrange series converges reasonably well. For a few instances of an Erdos-Renyi graph $G_{0.7}(10)$ with $N = 10$ nodes and link density $p = 0.7$, we found[6] about 3 digits correct after $K = 2$ terms, 4 digits correct after $K = 10$ terms, 5 digits after $K = 30$ terms, 6 digits after $K = 47$ terms.

A motivation to compute ρ_2 lies in the condition number $\kappa(\mathbf{\Omega}) = \|\mathbf{\Omega}^{-1}\|\|\mathbf{\Omega}\| \geq \|\mathbf{\Omega}^{-1}\mathbf{\Omega}\| = 1$. If we choose the Euclidean or $\|.\|_2$ norm, then $\kappa(\mathbf{\Omega}) = \frac{\rho_1}{|\rho_2|}$. The condition number $\kappa(\mathbf{\Omega})$ measures the sensitivity of $\mathbf{\Omega}^{-1}$ in numerical computations. A high condition number $\kappa(\mathbf{\Omega})$ points to numerical instability, which plays a crucial role in the inverse shortest path problem [3, Appendix] if $\mu_1 = O(N)$ and, hence, $\rho_2 = O(\frac{1}{N})$ for a large size N of the graph.

8.1.3. Largest eigenvalue ρ_1

Lemma 9 is the vehicle to deduce a set of increasingly accurate upper and lower bounds of the largest eigenvalue ρ_1:

Lemma 9: *For* $|\frac{2}{\mu_{N-1}}| < |\rho|$, *the function* (64) *can be expanded as*

$$f(\rho) = -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} + \sum_{m=2}^{\infty}\frac{(-1)^{m-1}\,2^{m-1}\zeta^T\left(Q^\dagger\right)^{m-1}\zeta}{\rho^m} \quad (71)$$

[6] We relied on numerical computations of all involved quantities μ_1, ζ, p and $\zeta^T Q^m \zeta$ and ignored the effect of their accuracy on the convergence of the Lagrange series. Most likely with exact values, the convergence may be faster.

Proof: For large $|\rho|$, the Taylor expansion of the function $f(\rho)$ in (64) is, using $\frac{1}{\mu_k+\frac{2}{\rho}} = \frac{1}{\mu_k}\sum_{m=0}^{\infty}(-\frac{2}{\mu_k})^m\frac{1}{\rho^m}$ valid for $|\frac{2}{\mu_k\rho}| < 1$ for any $1 \le k \le N-1$,

$$\begin{aligned} f(\rho) &= 2\sigma^2 - \frac{\rho}{N} - \frac{1}{2}\sum_{k=1}^{N-1}\mu_k\left(\zeta^T z_k\right)^2\sum_{m=0}^{\infty}\left(-\frac{2}{\mu_k}\right)^m\frac{1}{\rho^m} \\ &= 2\sigma^2 - \frac{\rho}{N} - \frac{1}{2}\sum_{k=1}^{N-1}\mu_k\left(\zeta^T z_k\right)^2 + \sum_{k=1}^{N-1}\left(\zeta^T z_k\right)^2\frac{1}{\rho} \\ &\quad + \sum_{m=2}^{\infty}(-1)^{m-1}2^{m-1}\sum_{k=1}^{N-1}\frac{\left(\zeta^T z_k\right)^2}{\mu_k^{m-1}}\frac{1}{\rho^m} \end{aligned}$$

Using the spectral decompositions,

$$\begin{cases} \sum_{k=1}^{N-1}\mu_k\left(\zeta^T z_k\right)^2 = \zeta^T Q\zeta \\ \sum_{k=1}^{N-1}\left(\zeta^T z_k\right)^2 = \zeta^T\zeta - \frac{\left(\zeta^T u\right)^2}{N} = \zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2 \\ \sum_{k=1}^{N-1}\frac{\left(\zeta^T z_k\right)^2}{\mu_k^{m-1}} = \zeta^T\left(Q^{\dagger}\right)^{m-1}\zeta > 0 \text{ for } m > 1 \end{cases}$$

we find, for large $\rho > \frac{2}{\mu_{N-1}}$, that

$$\begin{aligned} f(\rho) &= -\frac{\rho}{N} + 2\left(\sigma^2 - \frac{1}{4}\zeta^T Q\zeta\right) + \left(\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} \\ &\quad + \sum_{m=2}^{\infty}\frac{(-1)^{m-1}2^{m-1}\zeta^T\left(Q^{\dagger}\right)^{m-1}\zeta}{\rho^m} \end{aligned}$$

Invoking the definition (5) of $\sigma^2 = \frac{R_G}{N^2} + \frac{1}{4}\zeta^T Q\zeta$ leads to the Taylor series of $f(\rho)$, valid for $|\rho| > \frac{2}{\mu_{N-1}}$, to (71). ■

Since $f(\rho_1) = 0$ and $\rho_1 > \frac{2}{\mu_{N-1}}$, the first order in (71), ignoring the $O(\frac{1}{\rho})$ contribution, is $f(\rho) \approx \frac{1}{N}(-\rho + \frac{2R_G}{N})$ and leads to the estimate $\rho_1 \approx \frac{2R_G}{N}$, which is rather weak, because the bound $\rho_1 \ge \frac{2R_G}{N}$ holds [1, (5.33)].

After ignoring the $O(\frac{1}{\rho^2})$ terms in (71), which are negative for sufficiently large ρ, the expansion (71) becomes

$$f(\rho) \ge -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho}$$

Finding the zero, i.e. $f(\rho_1) = 0$, translates to a quadratic equation in ρ,

$$\rho^2 - \frac{2R_G}{N}\rho - \left(N\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right) = 0$$

with roots $\rho_\pm = \frac{R_G}{N} \pm \sqrt{N\zeta^T\zeta}$. The smaller root $\rho_- = \frac{R_G}{N} - \sqrt{N\zeta^T\zeta}$ violates the condition $\rho > \frac{2}{\mu_{N-1}}$, but the largest root

$$\rho_1 \le \frac{R_G}{N} + \sqrt{N\zeta^T\zeta} \tag{72}$$

seems an accurate upper bound, sharper than (68).

We can continue in sharpening the estimate for the largest eigenvalue ρ_1. After ignoring the positive $O(\frac{1}{\rho^3})$ terms (for sufficiently large ρ), we have a lower bound

$$f(\rho) < -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} - \frac{2\zeta^T Q^\dagger \zeta}{\rho^2}$$

whose right-hand side translates to a cubic polynomial in ρ,

$$\rho^3 - \frac{2R_G}{N}\rho^2 - \left(N\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\rho + 2N\zeta^T Q^\dagger \zeta = 0$$

Cardano's solution, that is involved and here omitted, can be symbolically computed. Increasing one order and ignoring $O(\frac{1}{\rho^4})$-terms,

$$f(\rho) > -\frac{\rho}{N} + \frac{2R_G}{N^2} + \left(\zeta^T\zeta - \frac{1}{N}\left(\frac{R_G}{N}\right)^2\right)\frac{1}{\rho} - \frac{2\zeta^T Q^\dagger \zeta}{\rho^2} + \frac{4\zeta^T\left(Q^\dagger\right)^2\zeta}{\rho^3}$$

is again an upper bound, that results in a quartic polynomial in ρ,

$$\rho^4 - \frac{2R_G}{N}\rho^3 - \left(N\zeta^T\zeta - \left(\frac{R_G}{N}\right)^2\right)\rho^2 + \left(2N\zeta^T Q^\dagger\zeta\right)\rho - 4N\zeta^T\left(Q^\dagger\right)^2\zeta = 0$$

which is the last polynomial whose exact zeros are still generally possible to express in closed form (e.g. by Mathematica). In summary, it is possible to find a tighter upper bound than (72), although its closed form is unattractively involved.

8.1.4. Smallest eigenvalue ρ_N

We aim to bound the smallest eigenvalue $\rho_N < 0$, which is second largest, i.e. $|\rho_N| < \rho_1$, in absolute value eigenvalue of $\mathbf{\Omega}$, because $\sum_{j=2}^N |\rho_j| = \rho_1$ and $|\rho_j| > 0$ for each $1 \le j \le N$.

Lemma 10: *The smallest eigenvalue ρ_N of the effective resistance matrix $\mathbf{\Omega}$ is upper bounded by*

$$\rho_N < -\left(\frac{1}{\mu_{N-1}} - \frac{R_G}{N}\right) - \sqrt{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2 + N\left(\zeta^T z_{N-1}\right)^2} \tag{73}$$

Proof: The definition (64) of the function $f(\rho)$ shows that

$$f(-|\rho|) = 2\sigma^2 + \frac{|\rho|}{N} - \frac{1}{2}\sum_{k=1}^{N-2}\frac{\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k - \frac{2}{|\rho|}} - \frac{1}{2}\frac{\mu_{N-1}^2\left(\zeta^T z_{N-1}\right)^2}{\mu_{N-1} - \frac{2}{|\rho|}} \tag{74}$$

and $f(-|\rho|)$ is strictly increasing in $|\rho|$. If $|\rho| > \frac{2}{\mu_{N-1}}$, which is equivalent to $\mu_k > \mu_{N-1} > \frac{2}{|\rho|}$ for $1 \le k < N-1$, then the sum in (74) is bounded by

$$\begin{aligned}
\sum_{k=1}^{N-2}\frac{\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k - \frac{2}{|\rho|}} &> \sum_{k=1}^{N-2}\frac{\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k} \\
&= \sum_{k=1}^{N-2}\mu_k\left(\zeta^T z_k\right)^2 = \sum_{k=1}^{N-1}\mu_k\left(\zeta^T z_k\right)^2 - \mu_{N-1}\left(\zeta^T z_{N-1}\right)^2 \\
&= \zeta^T Q\zeta - \mu_{N-1}\left(\zeta^T z_{N-1}\right)^2
\end{aligned}$$

Hence,

$$-\sum_{k=1}^{N-2}\frac{\mu_k^2\left(\zeta^T z_k\right)^2}{\mu_k - \frac{2}{|\rho|}} < -\zeta^T Q\zeta + \mu_{N-1}\left(\zeta^T z_{N-1}\right)^2$$

Introducing this inequality into (74) gives

$$f(-|\rho|) < 2\sigma^2 + \frac{|\rho|}{N} - \frac{1}{2}\zeta^T Q\zeta + \frac{1}{2}\mu_{N-1}\left(\zeta^T z_{N-1}\right)^2 - \frac{1}{2}\frac{\mu_{N-1}^2\left(\zeta^T z_{N-1}\right)^2}{\mu_{N-1} - \frac{2}{|\rho|}}$$

After simplification as in Lemma 9, we find, for $|\rho| > \frac{2}{\mu_{N-1}}$, the bound

$$f(-|\rho|) < \frac{|\rho|}{N} + \frac{2R_G}{N^2} - \frac{\mu_{N-1}\left(\zeta^T z_{N-1}\right)^2}{|\rho|\,\mu_{N-1} - 2} \tag{75}$$

The Interlacing Theorem 1 states that $-\rho_N \ge \frac{2}{\mu_{N-1}}$ indicating that inequality (75) is valid for $|\rho_N|$. Moreover, ρ_N satisfies $f(-|\rho_N|) = 0$ and the inequality (75) becomes, with $x = |\rho_N|$,

$$0 < \frac{x}{N} + \frac{2R_G}{N^2} - \frac{\mu_{N-1}\left(\zeta^T z_{N-1}\right)^2}{x\mu_{N-1} - 2}$$

After multiplying both sides with $x\mu_{N-1} - 2 > 0$, we find the quadratic inequality

$$0 < x^2 + 2\left(\frac{R_G}{N} - \frac{1}{\mu_{N-1}}\right)x - \left(\frac{4R_G}{N\mu_{N-1}} + N\left(\zeta^T z_{N-1}\right)^2\right)$$

which we factor as $0 < (x - x_-)(x - x_+)$ with x_- and x_+ are the negative and positive root, respectively. The positive root x_+ lower bounds $|\rho_N| > x_+$, which is equivalent to the upper bound (73). ■

We rewrite the upper bound (73) as

$$\rho_N < -\left(\frac{1}{\mu_{N-1}} - \frac{R_G}{N}\right) - \left(\frac{1}{\mu_{N-1}} + \frac{R_G}{N}\right)\sqrt{1 + \frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2}}$$

Due to orthogonality $z_{N-1}^T z_N = 0$, the eigenvector z_{N-1} must contain at least one negative component and $(\zeta^T z_{N-1})^2 < (\zeta^T z_N)^2 = \frac{1}{N}(\zeta^T u)^2 = \frac{1}{N}(\frac{R_G}{N})^2$. Hence, $\frac{N(\zeta^T z_{N-1})^2}{(\frac{R_G}{N}+\frac{1}{\mu_{N-1}})^2} < 1$, and the Taylor expansion $(1+z)^\alpha = \sum_{k=0}^{\infty}\binom{\alpha}{k} z^k$, valid for any complex α and $|z| < 1$, indicates that[7]

$$S = \sqrt{1 + \frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2}} = \sum_{k=0}^{\infty} \frac{(2k)!\,(-1)^{k+1}}{2^{2k}\,(2k-1)\,(k!)^2}\left(\frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2}\right)^k$$

$$= 1 + \frac{1}{2}\frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2} - \frac{1}{8}\left(\frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2}\right)^2 + \cdots$$

Moreover, the Taylor series S is alternating from $k > 0$ on with decreasing terms, for which it holds

$$1 + \frac{1}{2}\frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2} - \frac{1}{8}\left(\frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2}\right)^2 < S < 1 + \frac{1}{2}\frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^2}$$

Thus, we find that

$$\rho_N < -\frac{2}{\mu_{N-1}} - \frac{1}{2}\frac{N\left(\zeta^T z_{N-1}\right)^2}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)} + \frac{1}{8}\frac{N\left(\zeta^T z_{N-1}\right)^4}{\left(\frac{R_G}{N} + \frac{1}{\mu_{N-1}}\right)^3}$$

which improves the Interlacing Theorem 1. Numerical evaluations shows that the upper bound (73) is sharp.

Since $\rho_1 = -\sum_{j=2}^{N}\rho_j$, the upper bound (68) implies that all eigenvalues of $\mathbf{\Omega}$ lie in the interval $[-2\sigma^2 N, 2\sigma^2 N]$. A more precise determination of the interval $[\rho_N, \rho_1]$ is obtained

[7] Indeed,

$$\binom{\frac{1}{2}}{k} = \frac{\Gamma(\frac{1}{2}+1)}{k!\,\Gamma(\frac{1}{2}+1-k)} = (-1)^{k+1}\,\frac{\frac{1}{2}\Gamma(\frac{1}{2})\Gamma\left(k-\frac{1}{2}\right)}{\pi\,k!}$$

$$= (-1)^{k+1}\,\frac{\frac{1}{2}\Gamma(\frac{1}{2})}{\pi\,k!}\,\frac{4\sqrt{\pi}\,\Gamma(2k-1)}{2^{2k}\,(k-1)!} = (-1)^{k+1}\,\frac{(2k-2)!}{2^{2k-1}k!\,(k-1)!}$$

$$= (-1)^{k+1}\,\frac{(2k-2)!}{2^{2k-1}k!\,(k-1)!} = (-1)^{k+1}\,\frac{(2k)!}{2^{2k}\,(2k-1)\,(k!)^2}$$

where in the first line the reflection formula $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin\pi z}$ for the Gamma function and in the second the duplication formula $\Gamma(2z) = \frac{1}{2\sqrt{\pi}}2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})$ of the Gamma function (see e.g. [15]) have been used.

with the upper bound (72) for ρ_1 and the accurate (68), although (68) is an upper bound, while a lower bound would have been 'safer'.

Finally, by Taylor expansion of (29) in $w = y + \mu_j$ around $w = 0$, a tedious computation (neglecting[8] $O(w)$ terms) shows that

$$\rho_j < -\frac{2}{\mu_j}\left(1 + \frac{\mu_j \left(\zeta^T z_j\right)^2}{4\left(\sigma^2 + \frac{1}{N\mu_j}\right) - \sum_{k=1;k\neq j}^{N-1} \frac{\mu_k^2 \left(\zeta^T z_k\right)^2}{\mu_k - \mu_j}}\right)^{-1}$$

which improves upon the estimate of Interlacing Theorem 1.

8.2. The eigenvalues of the Laplacian matrix Q

Similarly as the function $f(\rho)$ in (64) in Section 8.1, we define from $\sigma^2 = \sum_{j=1}^{N} \frac{(p^T v_j)^2}{\mu + \frac{2}{\rho_j}}$ in (26) the monotonously increasing function in μ,

$$h(\mu) = \sigma^2 - \sum_{j=1}^{N} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}} \tag{76}$$

that satisfies $h'(\mu) = \sum_{j=1}^{N} \frac{(p^T v_j)^2}{(\mu + \frac{2}{\rho_j})^2} > 0$. Similarly as the transformation of the partial fraction (28) into the quadratic form (30), we find

$$h(\mu) = \sigma^2 - p^T \left(\mu I + 2\Omega^{-1}\right)^{-1} p \tag{77}$$

which illustrates that the rational function $h(\mu)$ has N poles at the eigenvalues of $-2\Omega^{-1}$, while the N zeros of $h(\mu)$ are the Laplacian eigenvalues. For example, if $\mu = 0$, then (77) simplifies to

$$h(0) = \sigma^2 - \frac{1}{2} p^T \Omega p$$

Since $2\sigma^2 = p^T \Omega p$, we find $h(0) = 0$ and, indeed, $\mu = 0$ is an eigenvalue of the Laplacian Q. Thus, we can reform (77) as

$$h(\mu) = \frac{\mu}{4} p^T \Omega^2 \left(\frac{\mu}{2}\Omega + I\right)^{-1} p$$

Without poles on the positive real axis, the function $h(\mu)$ increases continuously from 0 to σ^2. However, at $\mu = -\frac{2}{\rho_k}$ for an integer k satisfying $2 \leq k \leq N$, the function $h(\mu)$ has a simple[9] pole at which $h(\mu)$ switches from $+\infty$ for $\mu = -\frac{2}{\rho_k} - \varepsilon$ to $-\infty$ for $\mu = -\frac{2}{\rho_k} + \varepsilon$ for arbitrary small $\varepsilon > 0$. The increasing nature of $h(\mu)$ indicates that $h(\mu)$ can only have one zero in the interval $(-\frac{2}{\rho_k}, -\frac{2}{\rho_{k+1}})$. For example, the function $h(\mu)$ increases

[8] If we neglect $O(w^2)$, then also an analytic result as roots of a quadratic polynomial can be found, which is, however, unwieldy and omitted.

[9] Recall that we assume simple eigenvalues.

from $h(\mu)=0$ at $\mu=0$ until $h(\mu)>0$ for $\mu<-\frac{2}{\rho_2}$. At $\mu=-\frac{2}{\rho_2}$, the function $h(\mu)$ switches from $+\infty$ for $\mu=-\frac{2}{\rho_2}-\varepsilon$ to $-\infty$ for $\mu=-\frac{2}{\rho_2}+\varepsilon$ and $\varepsilon>0$ is arbitrary small. Hence, $h(\mu)$ can only have a zero for $\mu>-\frac{2}{\rho_2}$, implying that the algebraic connectivity $\mu_{N-1}>-\frac{2}{\rho_2}$, which is in agreement with the Interlacing Theorem 1. Although there is a pole at $\mu=-\frac{2}{\rho_1}<0$, the function $h(\mu)$ has only non-negative zeros (in agreement to the Laplacian eigenvalues): $h(\mu)$ increases from $\mu=-\infty$ at σ^2 towards $+\infty$ at the pole $\mu=-\frac{2}{\rho_1}-\varepsilon$, switches to $-\infty$ at $\mu=-\frac{2}{\rho_1}+\varepsilon$, from which $h(\mu)$ increases towards its first zero at $h(0)=0$.

8.2.1. Estimate of a Laplacian eigenvalue

Theorem 11 sharpens the Interlacing Theorem 1 if all eigenvalues are distinct, i.e. simple.

Theorem 11: *A simple Laplacian eigenvalue μ_k, but not the largest μ_1 nor the smallest $\mu_N=0$, is upper bounded by*

$$\mu_k<-\frac{2}{\rho_k}+\frac{\left(\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(-\frac{2}{\rho_k}\right)}\right)-\sqrt{\left(\delta_k+\frac{\left(p^Tv_{k+1}\right)^2-\left(p^Tv_k\right)^2}{r_k\left(-\frac{2}{\rho_k}\right)}\right)^2+\frac{4\left(p^Tv_k\right)^2\left(p^Tv_{k+1}\right)^2}{r_k^2\left(-\frac{2}{\rho_k}\right)}}}{2} \tag{78}$$

and lower bounded by

$$\mu_k>-\frac{2}{\rho_k}+\frac{\left(\delta_k+\frac{\left(p^Tv_k\right)^2+\left(p^Tv_{k+1}\right)^2}{r_k\left(-\frac{2}{\rho_{k+1}}\right)}\right)-\sqrt{\left(\delta_k+\frac{\left(p^Tv_{k+1}\right)^2-\left(p^Tv_k\right)^2}{r_k\left(-\frac{2}{\rho_{k+1}}\right)}\right)^2+\frac{4\left(p^Tv_k\right)^2\left(p^Tv_{k+1}\right)^2}{r_k^2\left(-\frac{2}{\rho_{k+1}}\right)}}}{2} \tag{79}$$

where $\delta_k=-\frac{2}{\rho_{k+1}}+\frac{2}{\rho_k}>0$ *and* $r_k(\mu)=h(\mu)+\frac{\left(p^Tv_k\right)^2}{\mu+\frac{2}{\rho_k}}+\frac{\left(p^Tv_{k+1}\right)^2}{\mu+\frac{2}{\rho_{k+1}}}$ *with the function* $h(\mu)$ *defined in* (76).

Proof: We write

$$h(\mu)=r_k(\mu)-\frac{\left(p^Tv_k\right)^2}{\mu+\frac{2}{\rho_k}}-\frac{\left(p^Tv_{k+1}\right)^2}{\mu+\frac{2}{\rho_{k+1}}}$$

where

$$r_k(\mu) = \sigma^2 - \sum_{j=1}^{k-1} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}} - \sum_{j=k+2}^{N} \frac{\left(p^T v_j\right)^2}{\mu + \frac{2}{\rho_j}} = \frac{\mu}{4} p^T \Omega^2 \left(\frac{\mu}{2}\Omega + I\right)^{-1} p + \frac{\left(p^T v_k\right)^2}{\mu + \frac{2}{\rho_k}} + \frac{\left(p^T v_{k+1}\right)^2}{\mu + \frac{2}{\rho_{k+1}}}$$

is regular at any point μ in the interval $[-\frac{2}{\rho_k}, -\frac{2}{\rho_{k+1}}]$, if we exclude μ_1 because $\rho_1 > 0$. Since each term $\frac{-(p^T v_j)^2}{\mu + \frac{2}{\rho_j}}$ in the defining sum (76) of $h(\mu)$ is strictly increasing, also $r_k(\mu)$ is a strictly increasing function in μ, so that $0 < r_k(-\frac{2}{\rho_k}) < r_k(\mu) < r_k(-\frac{2}{\rho_{k+1}})$. The zero $\mu^* \in (-\frac{2}{\rho_k}, -\frac{2}{\rho_{k+1}})$ satisfies

$$r_k\left(\mu^*\right) - \frac{\left(p^T v_k\right)^2}{\mu^* + \frac{2}{\rho_k}} - \frac{\left(p^T v_{k+1}\right)^2}{\mu^* + \frac{2}{\rho_{k+1}}} = 0$$

where $-\frac{(p^T v_k)^2}{\mu^* + \frac{2}{\rho_k}} < 0$ and $r_k(\mu^*) - \frac{(p^T v_{k+1})^2}{\mu^* + \frac{2}{\rho_{k+1}}} > 0$ and which is equivalent to

$$r_k\left(\mu^*\right)\left(\mu^* + \frac{2}{\rho_k}\right)\left(\mu^* + \frac{2}{\rho_{k+1}}\right) - \left(p^T v_k\right)^2\left(\mu^* + \frac{2}{\rho_{k+1}}\right) - \left(\mu^* + \frac{2}{\rho_k}\right)\left(p^T v_{k+1}\right)^2 = 0 \tag{80}$$

Let $y = \mu^* + \frac{2}{\rho_k}$, then

$$r_k\left(y - \frac{2}{\rho_k}\right) y \left(y + \frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) - \left(p^T v_k\right)^2 \left(y + \frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) - y\left(p^T v_{k+1}\right)^2 = 0$$

After simplification, we find the quadratic equation in y,

$$r_k\left(y - \frac{2}{\rho_k}\right) y^2 + \left\{ r_k\left(y - \frac{2}{\rho_k}\right)\left(\frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) - \left(p^T v_k\right)^2 - \left(p^T v_{k+1}\right)^2 \right\} y - \left(p^T v_k\right)^2 \left(\frac{2}{\rho_{k+1}} - \frac{2}{\rho_k}\right) = 0$$

Since $r_k(y - \frac{2}{\rho_k})$ does not contain a zero for $y \in [0, -\frac{2}{\rho_{k+1}} + \frac{2}{\rho_k}]$, we arrive, with $\delta_k = -\frac{2}{\rho_{k+1}} + \frac{2}{\rho_k} > 0$ at the quadratic equation in y

$$y^2 - \left\{ \delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)} \right\} y + \left(p^T v_k\right)^2 \frac{\delta_k}{r_k\left(y - \frac{2}{\rho_k}\right)} = 0 \tag{81}$$

with roots

$$y_\pm = \frac{\left(\delta_k + \frac{(p^T v_k)^2+(p^T v_{k+1})^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right) \pm \sqrt{\left(\delta_k + \frac{(p^T v_k)^2+(p^T v_{k+1})^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)^2 - \frac{4(p^T v_k)^2\delta_k}{r_k\left(y-\frac{2}{\rho_k}\right)}}}{2}$$

Since $r_k(y-\frac{2}{\rho_k}) > 0$ in $[0, -\frac{2}{\rho_{k+1}} + \frac{2}{\rho_k}]$, the discriminant

$$\begin{aligned}\Lambda &= \left(\delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)^2 - \frac{4\left(p^T v_k\right)^2\delta_k}{r_k\left(y-\frac{2}{\rho_k}\right)} \\ &= \left(\delta_k + \frac{\left(p^T v_{k+1}\right)^2 - \left(p^T v_k\right)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)^2 + \frac{4\left(p^T v_k\right)^2\left(p^T v_{k+1}\right)^2}{r_k^2\left(y-\frac{2}{\rho_k}\right)} > 0\end{aligned}$$

is positive and the quadratic equation has two positive zeros,

$$y_\pm = \frac{\left(\delta_k + \frac{(p^T v_k)^2+(p^T v_{k+1})^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right) \pm \sqrt{\left(\delta_k + \frac{(p^T v_{k+1})^2-(p^T v_k)^2}{r_k\left(y-\frac{2}{\rho_k}\right)}\right)^2 + \frac{4(p^T v_k)^2(p^T v_{k+1})^2}{r_k^2\left(y-\frac{2}{\rho_k}\right)}}}{2}$$

It remains to determine the zero lying in the interval $[0, \delta_k = -\frac{2}{\rho_{k+1}} + \frac{2}{\rho_k}]$. It holds that $\delta_k + \frac{(p^T v_k)^2+(p^T v_{k+1})^2}{r_k(y-\frac{2}{\rho_k})} > \delta_k$, but $\delta_k + \frac{(p^T v_{k+1})^2-(p^T v_k)^2}{r_k(y-\frac{2}{\rho_k})}$ can be smaller than δ_k, in which case there are two zeros in the interval, which is not possible. We are certain that the minus sign always lies in $[0, \delta_k]$. Hence, we translate back with $y = \mu^* + \frac{2}{\rho_k}$ to

$$\mu^* = -\frac{2}{\rho_k} + \frac{\left(\delta_k + \frac{(p^T v_k)^2+(p^T v_{k+1})^2}{r_k(\mu^*)}\right) - \sqrt{\left(\delta_k + \frac{(p^T v_{k+1})^2-(p^T v_k)^2}{r_k(\mu^*)}\right)^2 + \frac{4(p^T v_k)^2(p^T v_{k+1})^2}{r_k^2(\mu^*)}}}{2}$$

which is a kind of self-consistent equation in μ^*. We now invoke the bounds $r(-\frac{2}{\rho_k}) < r(\mu) < r(-\frac{2}{\rho_{k+1}})$ and arrive at (78) and (79). ■

Numerical results indicate that the upper (78) and lower bound (79) are quite tight and improve the bounds of the Interlacing Theorem 1.

Example: Figure 1 exemplifies the function $h(x)$ in (76), where the zeros of $h(x) = 0$ correspond to eigenvalues of the Laplacian matrix, which are

$$\mu = \{9.60807, 8.72603, 8.3247, 7.53112, 6.87167, 5.83041, 5.1015, 4.34396, 3.66254, 0\}$$

The vector $-2/\rho$, without $-\frac{2}{\rho_1} = -0.664381$ and $-\frac{2}{\rho_2} = 9.38741$,

$$-2/\rho = \{8.70826, 8.12108, 7.50029, 6.70122, 5.7162, 5.09825, 4.33186, 3.6209\}$$

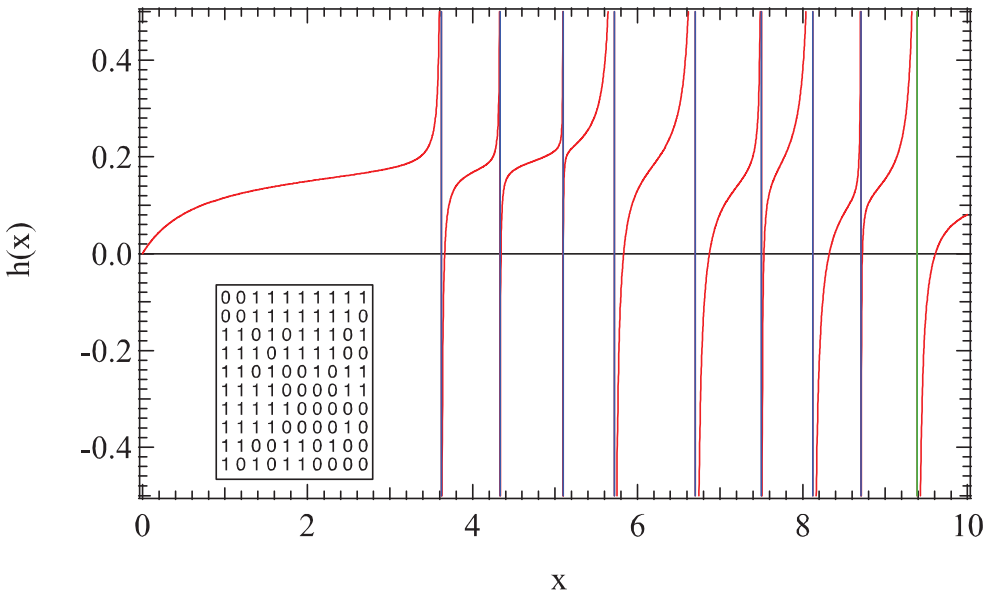


Figure 1. The function $h(x)$ in (76) versus x for an instance of an Erdos-Renyi random graph $G_p(N)$ with link density p on $N = 10$ nodes, specified by the shown adjacency matrix in the rectangle with zero-one entries. The vertical asymptotes are shown in blue; only the largest vertical asymptote is drawn in green.

corresponds to the vertical asymptotes in blue in Figure 1. The lower bounds, computed by (79)

$$\{8.72203, 8.3001, 7.52689, 6.84159, 5.82106, 5.10121, 4.34182, 3.65939\}$$

and the upper bounds (78),

$$\{8.72631, 8.33856, 7.53147, 6.87952, 5.83147, 5.1015, 4.34399, 3.66272\}$$

are closer to the eigenvalues μ than the vertical asymptotes at $-2/\rho$.

A slightly easier, but less accurate estimate of a Laplacian eigenvalue μ^* than Theorem 11 is obtained for small y, by ignoring y^2 in the quadratic equation (81), leading to the inequality,

$$-\left\{\delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)}\right\} y + \left(p^T v_k\right)^2 \frac{\delta_k}{r_k\left(y - \frac{2}{\rho_k}\right)} = -y^2 < 0$$

from which

$$y < \delta_k \frac{\frac{\left(p^T v_k\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)}}{\delta_k + \frac{\left(p^T v_k\right)^2 + \left(p^T v_{k+1}\right)^2}{r_k\left(y - \frac{2}{\rho_k}\right)}} < \delta_k$$

9. Conclusion

Given the eigenvalues and eigenvectors of either the effective resistance matrix $\mathbf{\Omega}$ or the Laplacian matrix Q of a graph G allows the computation of the eigenvectors of the other matrix. Hence, there is a nice one-to-one relation, which is, unfortunately, not existent for the adjacency matrix and the Laplacian matrix. Several new formulae, e.g. in Theorem 2, Corollary 4 and 3 and several bounds on eigenvalues in Section 8, are derived from the one-to-one correspondence between $\mathbf{\Omega}$ and Q. Theorems 6 and 7 provide the coefficients $a_k(Q)$ and $a_m(\mathbf{\Omega})$ of the characteristic polynomials $p_Q(\mu)$ and $p_\Omega(\rho)$. Finally, we improve the Interlacing Theorem 1 by bounds in Theorem 11.

We have confined the analysis to connected graphs that obey the restriction $\mu_k + \frac{2}{\rho_j} \neq 0$ for all $1 \leq j, k \leq N$. An extension of the Theorems to graphs that violate that restriction stands on the agenda for future work. The functions $f(\rho)$ in (64) and $h(\mu)$ in (76), that encode the entire set of eigenvalues, enable that still sharper bounds than those presented here can be found.

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Author contributions

CRediT: **Piet Van Mieghem:** Conceptualization, Formal analysis, Methodology, Project administration, Validation, Visualization, Writing – original draft, Writing – review & editing

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Appendices

Appendix 1. double orthogonality of the row and column vectors of the orthogonal matrix *C* in (eqn24)

Orthogonality of a matrix implies double orthogonality [1, art. 248] in its corresponding column vectors, which applied to the matrix C in (24) implies that

$$v_k^T v_m = \alpha_k \alpha_m \left(\sum_{j=1}^{N-1} \frac{\left(\mu_j \left(\zeta^T z_j\right)\right)^2}{\left(2+\rho_k \mu_j\right)\left(2+\rho_m \mu_j\right)} + \frac{1}{N} \right) = \delta_{km}$$

and for the row vectors

$$\mu_k \left(\zeta^T z_k\right) \mu_m \left(\zeta^T z_m\right) \sum_{j=1}^{N} \frac{\alpha_j^2}{\left(2+\rho_j \mu_k\right)\left(2+\rho_j \mu_m\right)} = \delta_{km} \text{for } m \neq N$$

$$\frac{\mu_k \left(\zeta^T z_k\right)}{\sqrt{N}} \sum_{j=1}^{N} \frac{\alpha_j^2}{2+\rho_j \mu_k} = 0 \text{for } k \neq N$$

In the sequel, we prove both orthogonality relations. First, for the columns of the orthogonal matrix C, partial fraction expansion yields

$$\frac{1}{(2+\rho_k x)(2+\rho_m x)} = \frac{\frac{\rho_k}{2}}{(\rho_k - \rho_m)(2+\rho_k x)} + \frac{\frac{\rho_m}{2}}{(\rho_m - \rho_k)(2+\rho_m x)} \tag{A1}$$

so, for $k \neq m$,

$$v_k^T v_m = \alpha_k \alpha_m \left(\frac{\frac{\rho_k}{2}}{(\rho_k - \rho_m)} \sum_{j=1}^{N-1} \frac{\left(\mu_j \left(\zeta^T z_j\right)\right)^2}{\left(2+\rho_k \mu_j\right)} - \frac{\frac{\rho_m}{2}}{(\rho_k - \rho_m)} \sum_{j=1}^{N-1} \frac{\left(\mu_j \left(\zeta^T z_j\right)\right)^2}{\left(2+\rho_m \mu_j\right)} + \frac{1}{N} \right)$$

Each eigenvalue ρ_m obeys the eigenvalue partial fraction (29)

$$\frac{\rho_m}{2} \sum_{j=1}^{N-1} \frac{\mu_j^2 \left(\zeta^T z_j\right)^2}{\rho_m \mu_j + 2} = 2\sigma^2 - \frac{\rho_m}{N}$$

showing that $v_k^T v_m = 0$.

Similarly, for the rows of the orthogonal matrix C, partial fraction expansion

$$\frac{1}{(2+x\mu_k)(2+x\mu_m)} = \frac{\frac{\mu_k}{2}}{(\mu_k-\mu_m)(2+\mu_k x)} + \frac{\frac{\mu_m}{2}}{(\mu_m-\mu_k)(2+\mu_m x)}$$

yields

$$\sum_{j=1}^{N} \frac{\alpha_j^2}{(2+\rho_j\mu_k)(2+\rho_j\mu_m)} = \frac{\frac{\mu_k}{2}}{(\mu_k-\mu_m)} \sum_{j=1}^{N} \frac{\alpha_j^2}{(2+\mu_k\rho_j)} - \frac{\frac{\mu_m}{2}}{(\mu_k-\mu_m)} \sum_{j=1}^{N} \frac{\alpha_j^2}{(2+\mu_m\rho_j)}$$

and the eigenvalue partial fraction (28) shows that the right-hand side equals zero.

Appendix 2. the resistance curvature vector p

It seems interesting[10] to determine the sign of the component p_i in the resistance curvature vector $p = (p_1, p_2, \ldots, p_N)$. In the following subsections, we derive several decompositions of the vector p, apart from

$$p = \frac{1}{2} \sum_{k=1}^{N-1} \left(\mu_k \zeta^T z_k\right) z_k + \frac{u}{N}$$

which directly follows from the definition $p = \frac{1}{2}Q\zeta + \frac{u}{N}$ in (3) and spectral decomposition $Q = \sum_{k=1}^{N-1} \mu_k z_k z_k^T$.

We start with two lemma's.

Lemma 12: *The all-one or unity vector u is not an eigenvector of the effective resistance matrix Ω of a non-resistance-regular graph.*

Proof: It follows (see also [1, Equation (5.30), p. 186]) from the definition (1) and $\zeta^T u = \frac{R_G}{N}$ that

$$\Omega u = N\zeta + \frac{R_G}{N} u$$

Only if the graph is resistance-regular, then the vector $\zeta = \frac{R_G}{N^2} u$ and then the vector u is an eigenvector of Ω. ■

Lemma 13: *The resistance curvature vector p is not an eigenvector of the effective resistance matrix Ω of a non-resistance-regular graph.*

Proof: Suppose that the resistance curvature vector p is an eigenvector of effective resistance matrix Ω, belonging to eigenvalue ρ_k, then p satisfies the eigenvalue equation $\Omega p = \rho_k p$. The orthogonality of the eigenvectors $v_k^T v_j = \delta_{kj}$ indicates that the vector p obeys $v_j^T p = 0$ for all $1 \le j \le N$, except for $j = k$. It suffices to show that $v_j^T p = 0$ for $1 \le j \le N$ is not possible in a non-resistance-regular graph. We give four different arguments. (i) for a non-resistance-regular graph, the condition that $v_j^T p = 0$ in (A2) or (A6) would imply that $v_j^T Q v_j = -\frac{2}{\rho_j} = v_j^T \Omega v_j$, which is only possible in 'resistance-regular graphs'. (ii) The case $v_k^T p = 0$ in (A3) is only possible if $v_k^T Q v_j = 0$, which means that either $v_k = \frac{u}{\sqrt{N}}$ or $v_j = \frac{u}{\sqrt{N}}$. The latter is, again, only possible in 'resistance-regular graphs'. (iii) Since the resistance curvature vector p is the eigenvector of the matrix $Q\Omega$ belonging to the zero eigenvalue [1, p. 181-182], it holds that $Q\Omega p = 0$. The fact that $\Omega p = \rho_k p$ would be equivalent to $Qp = 0$, which implies that $p = \alpha z_N = \frac{\alpha u}{\sqrt{N}}$ for some non-zero constant α. (iv) By the definition (2), it holds that $\Omega p = 2\sigma^2 u$. If p were an eigenvector satisfying $\Omega p = \rho_k p$ for some eigenvalue ρ_k

[10] Private communication with Karel Devriendt.

of Ω, then $p = \frac{2\sigma^2}{\rho_k} u$. Lemma 12 shows, for a non-resistance-regular graph, that p cannot be an eigenvector of Ω. ■

A.1 Signs of three scalar products

(a) We use $p^T v_j = 2\sigma^2 \frac{u^T v_j}{\rho_j}$ in (12) into $p = \sum_{j=1}^{N} (p^T v_j) v_j$, then

$$p = 2\sigma^2 \sum_{j=1}^{N} \frac{u^T v_j}{\rho_j} v_j = 2\sigma^2 \frac{u^T v_1}{\rho_1} v_1 - 2\sigma^2 \sum_{j=2}^{N} \frac{u^T v_j}{|\rho_j|} v_j$$

The Perron-Frobenius Theorem [1, art. 269] of non-negative matrices indicates that the components of the principal eigenvector v_1 are all positive in a connected graph G. Thus, only if $j = 1$, the $\text{sign}(p^T v_1) = \text{sign}(u^T v_1) > 0$, else $\text{sign}(p^T v_j) = -\text{sign}(u^T v_j)$.

(b) Introducing the definition $p = \frac{1}{2} Q\zeta + \frac{u}{N}$ in (3) shows that

$$p^T v_j = \frac{1}{2} \zeta^T Q v_j + \frac{u^T v_j}{N}$$

Relation $p^T v_j = \frac{2\sigma^2}{\rho_j} u^T v_j$ in (13) then indicates that

$$\zeta^T Q v_j = 2 \left(\frac{2\sigma^2}{\rho_j} - \frac{1}{N} \right) u^T v_j$$

The upper bound $\frac{2\sigma^2}{\rho_1} > \frac{1}{N}$ in (68) illustrates that $\text{sign}(\zeta^T Q v_1) > 0$, while $\text{sign}(\zeta^T Q v_j) = -\text{sign}(u^T v_j)$ for $2 \le j \le N$.

(c) Combining $\frac{\rho_j}{2\sigma^2} p^T v_j = u^T v_j$ in (13) and $(\rho_j - \frac{R_G}{N}) u^T v_j = N \zeta^T v_j$ in (14) shows, for $u^T v_j \neq 0$, that

$$\left(\rho_j - \frac{R_G}{N} \right) \rho_j = 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j}$$

We deduce for any $1 \le j \le N$ that $\text{sign}(\zeta^T v_j) = \text{sign}(p^T v_j)$ and $\frac{\zeta^T v_j}{p^T v_j} > 0$.

Moreover, the quadratic equation

$$\rho_j^2 - \frac{R_G}{N} \rho_j - 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j} = 0$$

has two zeros

$$\rho_j = \frac{R_G}{2N} \pm \sqrt{\left(\frac{R_G}{2N} \right)^2 + 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j}}$$

Since $\frac{\zeta^T v_j}{p^T v_j} > 0$, only for $j = 1$ we must choose the positive sign, while for $j > 2$, the negative sign. Hence,

$$\rho_1 = \frac{R_G}{2N} + \sqrt{\left(\frac{R_G}{2N} \right)^2 + 2\sigma^2 N \frac{\zeta^T v_1}{p^T v_1}}$$

and, for $2 \le j \le N$,

$$\rho_j = -\sqrt{\left(\frac{R_G}{2N} \right)^2 + 2\sigma^2 N \frac{\zeta^T v_j}{p^T v_j}} + \frac{R_G}{2N}$$

A.2 Decomposition of p in the basis of eigenvectors of Ω

We further derive expressions for the scalar product $v_k^T p$. Left-multiplying the first generalized eigenvalue Equation (10) by v_k^T

$$v_k^T Q v_j = -\frac{2}{\rho_j} v_k^T v_j + \frac{2\left(u^T v_j\right)}{\rho_j} v_k^T p$$

leads, after invoking orthogonality of eigenvectors $v_k^T v_j = \delta_{kj} = 1_{\{k=j\}}$, to

$$v_k^T p = \frac{\rho_j v_k^T Q v_j}{2\left(u^T v_j\right)} + \frac{1_{\{k=j\}}}{u^T v_j}$$

Thus, if $k = j$, then the scalar product is

$$v_j^T p = \frac{\frac{1}{2}\rho_j v_j^T Q v_j + 1}{u^T v_j} \tag{A2}$$

else, for $k \neq j$,

$$v_k^T p = \frac{\rho_j v_k^T Q v_j}{2\left(u^T v_j\right)} \tag{A3}$$

The decomposition $p = \sum_{j=1}^{N} (p^T v_j) v_j$ of the resistance curvature vector p in the basis of eigenvectors $v_1, v_2, \ldots, v_N$ follows from (A2) and (A3) as

$$p = \left(\frac{\frac{1}{2}\rho_j v_j^T Q v_j + 1}{u^T v_j} \right) v_j + \frac{\rho_j}{2\left(u^T v_j\right)} \sum_{k=1;k\neq j}^{N} \left(v_k^T Q v_j \right) v_k \tag{A4}$$

Since $\rho_1 > 0 > \rho_2$, it is convenient to choose $j = 1$ and (A4) becomes

$$p = \left(\frac{\frac{1}{2}\rho_1 v_1^T Q v_1 + 1}{u^T v_1} \right) v_1 + \frac{\rho_1}{2\left(u^T v_1\right)} \sum_{k=2}^{N} \left(v_k^T Q v_1 \right) v_k \tag{A5}$$

where the prefactors $\frac{\frac{1}{2}\rho_1 v_1^T Q v_1 + 1}{u^T v_1}$ and $\frac{\rho_1}{2(u^T v_1)}$ are positive, by the Perron-Frobenius Theorem and the positive semi-definiteness [1, art. 102], i.e. $x^T Q x \geq 0$ for any vector x, of the Laplacian Q.

Either after using (13) in (A2) and (A3) or after left-multiplying the second generalized eigenvalue equation (11) by v_k^T leads to

$$v_k^T Q v_j = -\frac{2}{\rho_j} v_k^T v_j + \frac{1}{\sigma^2} \left(p^T v_j \right) \left(v_k^T p \right)$$

we obtain, for $k = j$,

$$\left(v_j^T p \right)^2 = \sigma^2 \left(v_j^T Q v_j + \frac{2}{\rho_j} \right) \tag{A6}$$

else

$$v_k^T Q v_j = \frac{1}{\sigma^2} \left(p^T v_j \right) \left(v_k^T p \right) \tag{A7}$$

Equation (A7) determines the sign of $p^T v_j$ and $p^T v_k$ in relation to that of $v_k^T Q v_j$.