# Effect of particle inertia and gravity on the turbulence in a suspension

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A theoretical model is presented for the effect of particle inertia and gravity on the turbulence in a homogeneous suspension. It is an extension of the one-fluid model developed by L'vov, Ooms, and Pomyalov [Phys. Rev. E **67**, 046314 (2003)], in which the effect of gravity was not considered. In the extended model the particles are assumed to settle in the fluid under the influence of gravity due to the fact that their density is larger than the fluid density. The generation of turbulence by the settling particles is described, with special attention being paid to the turbulence intensity and spectra. A comparison is made with direct numerical simulation calculations and experimental data. Also a sensitivity study is carried out to investigate at which conditions the gravity effect becomes important. © *2005 American Institute of Physics*. [DOI: 10.1063/1.2139683]

## I. INTRODUCTION

The occurrence of particle-laden turbulent flows in nature and industrial applications is abundant. Several good reviews about this topic were published during recent years; see, for instance, Hetsroni,[1] Elgobashi,[2] Crowe, Troutt, and Chung,[3] and Mashayek and Pandya.[4] It is known that when the mass loading of the particles is considerable, the two-way coupling effect of the fluid on the particles and vice versa must be taken into account. This two-way coupling effect has been studied by means of direct numerical simulations, experiments, and theoretical models. A detailed review of these studies for a homogeneous, turbulently flowing suspension is given by Poelma and Ooms.[5]

Recently L'vov, Ooms, and Pomyalov[6] developed a one-fluid theoretical model for a homogeneous, isotropic turbulent suspension, paying particular attention to the two-way coupling effect. It is based on a modified Navier-Stokes equation with a wave-number-dependent effective density of suspension and an additional damping term representing the fluid-particle friction. The statistical model is simplified by a modification of the usual closure procedure based on the Richardson-Kolmogorov picture of turbulence. A differential equation for the budget of the turbulent kinetic energy is derived. For the case of a stationary turbulent suspension L'vov *et al.* solved this equation analytically for various limiting cases and numerically for the general case. The model successfully explains observed features of numerical simulations of stationary turbulent suspensions.

In experiments the effect of gravity, due to the difference in density between the particles and the carrier fluid, is present. It causes an anisotropy in the turbulence of the suspension. This effect is not included in model of L'vov *et al.* So for a proper comparison with experiments it is necessary to extend the theoretical model by including also the gravity effect. It is the purpose of this publication to report about such an extension for a homogeneous, turbulently flowing suspension without a mean velocity gradient. Also a comparison with experimental data and a sensitivity study to investigate under which conditions the gravity effect becomes important will be given.

## II. EQUATION OF MOTION FOR ONE-FLUID MODEL WITH GRAVITY EFFECT

The following one-fluid equation of motion in wave-number space $k$ is used in this paper to describe the effect of inertia and gravity on the turbulence in a suspension:

$$\rho_{\text{eff}}(k)\left[\frac{\partial}{\partial t}+\gamma_p(k)+\gamma_0(k)\right]\mathbf{u}(t,\mathbf{k}) = -\mathbf{N}\{\mathbf{u},\mathbf{u}\}_{t,\mathbf{k}}+\mathbf{f}(t,\mathbf{k})+\frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\text{tv}}\delta_{(\mathbf{k}=\mathbf{0})}. \tag{1}$$

It is the equation as derived by L'vov, Ooms, and Pomyalov, but extended with the term $(\rho_f\phi/\tau_p)\mathbf{v}_{\text{tv}}\delta_{(\mathbf{k}=\mathbf{0})}$ to take into account the gravity effect. In the Appendix a "derivation" is given of Eq. (1). It is stressed, however, that several simplifying assumptions have to be made. Here we will explain the equation in more physical terms. $\rho_{\text{eff}}(k)$ represents the wave-number-dependent effective density of the suspension, $\gamma_p(k)$ is the viscous fluid-particle friction, $\gamma_0(k)$ is the viscous friction inside the fluid, $\mathbf{u}(t,\mathbf{k})$ is the suspension velocity, $\mathbf{N}\{\mathbf{u},\mathbf{u}\}_{t,\mathbf{k}}$ is the nonlinear term discussed in more detail in the Appendix, $\mathbf{f}(t,\mathbf{k})$ is the stirring force responsible for the maintenance of the turbulence, $\rho_f$ is the fluid density, $\phi$ is the mass fraction of the particles, $\tau_p$ is the particle response time, $\mathbf{v}_{\text{tv}}$ is the terminal velocity of a settling particle, $t$ is the time, and $\mathbf{k}$ is the wave vector.

We may interpret $\rho_{\text{eff}}(k)$ and $\gamma_p(k)$ in a simplified fashion. Denote as $f_{\text{com}}(k)$ the fraction of particles co-moving with eddies of size $k$, in the sense that their velocity is almost the same as the velocity of these eddies. These particles also participate in the motion of eddies with a smaller wave number $k' < k$, but not necessarily in the motion of eddies with $k' > k$. For small $k$ the turnover frequency $\gamma(k)$ of $k$ eddies is small in the sense that $\gamma(k)\tau_p \ll 1$. Therefore in this region of $k$ values the particle velocity is very close to that of the carrier fluid and we can describe the suspension as a single fluid with effective density $\rho_{\rm eff}$, which is very close to the density of suspension:

$$\rho_s = \rho_f(1-\psi) + c_p m_p = \rho_f(1-\psi+\phi), \tag{2}$$

in which $m_p$ is the particle mass, $c_p$ is the particle concentration, $\psi = c_p(4\pi a^3/3)$ is the particle volume fraction (with $a$ the particle diameter), and $\phi = c_p m_p/\rho_f$ is the particle mass fraction or particle mass loading parameter. However, for large $k$, when $\gamma(k)\tau_p \gg 1$, the particles cannot follow the very fast motion of the eddies and may be considered at rest. Thus, these particles do not contribute to the effective density and $\rho_{\rm eff} \to \rho_f$. In the general case $\rho_{\rm eff}(k)$ may be considered as

$$\rho_{\rm eff}(k) = \rho_f[1-\psi+\phi f_{\rm com}(k)]. \tag{3}$$

Here a statistical ensemble of all particles, partially involved in the motion of $k$ eddies, is replaced by two subensembles of "fully co-moving" particles [fraction $f_{\rm com}(k)$] (which contribute to $\rho_{\rm eff}$), and "fully at rest" particles [fraction $f_{\rm rest}(k) = 1 - f_{\rm com}(k)$] (which do not contribute to $\rho_{\rm eff}$). The particles at rest cause the fluid-particle friction. The damping frequency of a suspension $\gamma_p(k)$ may be related to the particle response time $\tau_p$ via the ratio of the total mass $M_p$ of the particles at rest and the total effective mass $M_{{\rm eff}(k)}$ of the suspension in the following way:

$$\gamma_p(k) = \frac{M_p}{\tau_p M_{\rm eff}(k)} = \frac{c_p m_p f_{\rm rest}(k)}{\tau_p \rho_{\rm eff}(k)} = \frac{\phi\rho_f f_{\rm rest}(k)}{\tau_p \rho_{\rm eff}(k)}. \tag{4}$$

In order to evaluate $\rho_{\rm eff}(k)$ and $\gamma_p(k)$ further an expression for $f_{\rm rest}(k)$ is needed. L'vov, Ooms, and Pomyalov present such an expression, partially based on a physical reasoning and partially "derived" by mathematical analysis. It has the following form:

$$f_{\rm rest}(k) = 1 - f_{\rm com}(k) = \{\tau_p\gamma(k)/[1+\tau_p\gamma(k)]\}^2. \tag{5}$$

Using Eq. (5) we can rewrite Eqs. (3) and (4) as follows:

$$\rho_{\rm eff}(k) = \rho_f\left(1 + \phi\frac{[1+2\tau_p\gamma(k)]}{[1+\tau_p\gamma(k)]^2}\right) \tag{6}$$

and

$$\gamma_p(k) = \frac{\phi\tau_p[\gamma(k)]^2}{(1+\phi)[1+2\tau_p\gamma(k)] + [\tau_p\gamma(k)]^2}. \tag{7}$$

In Eq. (6) the volume fraction $\psi$ of the particles has been omitted, because we assume the suspension to be dilute. In the Appendix it is shown that the term $\gamma_0(k)$ is equal to

$$\gamma_0(k) = \nu_{\rm eff}(k)k^2, \quad \nu_{\rm eff}(k) = \frac{\nu\rho_f}{\rho_{\rm eff}(k)}, \tag{8}$$

in which $\nu$ is the kinematic viscosity of the fluid.

As mentioned, in this section we have given a physical interpretation of the one-fluid equation of motion by considering in particular $\rho_{\rm eff}(k)$ and $\gamma_p(k)$. In the Appendix we try to provide a mathematical derivation for this equation. However, it is stressed that simplifying assumptions have to be made in order to provide such a derivation. Another type of simplification in our study is the fact that the statistics of the particles is supposed to be independent of the statistics of the turbulence. So we neglect the effect that the local concentration of particles can be significantly affected by the turbulence.

## III. THE TURBULENT KINETIC ENERGY SPECTRUM FOR A SUSPENSION

### A. Derivation of equation for energy spectrum

In order to derive the equation for the turbulent energy spectrum we multiply Eq. (1) by $\mathbf{u}(t,\mathbf{k}')$ and average. This yields

$$\rho_{\rm eff}(t,k)\left(\frac{\partial \mathbf{F}(t,\mathbf{k})}{2\ \partial t} + [\gamma_0(k)+\gamma_p(k)]\mathbf{F}(t,\mathbf{k})\right) = \mathbf{J}(t,\mathbf{k}) + \mathbf{W}(t,\mathbf{k}) + \mathbf{G}(t,\mathbf{k}). \tag{9}$$

[In the case of decaying turbulence $\rho_{\rm eff}(t,k)$ is also a function of time.] The tensor $\mathbf{F}(t,\mathbf{k})$ is the second-order simultaneous velocity correlation function given by

$$(2\pi)^3\delta(\mathbf{k}+\mathbf{k}_1)F^{\alpha\beta}(t,\mathbf{k}) = \langle u^\alpha(t,\mathbf{k})u^\beta(t,\mathbf{k}_1)\rangle. \tag{10}$$

The tensor $\mathbf{J}(t,\mathbf{k})$ is related to the third-order simultaneous velocity correlation function $\mathbf{F}_3(t,\mathbf{k})$ in the following manner:

$$J^{\alpha\beta\gamma}(t,\mathbf{k}) \equiv \int \frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{(2\pi)^3}\Gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} F_3^{\alpha\beta\gamma}(t,\mathbf{k},\mathbf{k}_1,\mathbf{k}_2), \tag{11}$$

where

$$(2\pi)^3\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)F_3^{\alpha\beta\gamma}(t,\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \langle u^\alpha(t,\mathbf{k})u^\beta(t,\mathbf{k}_1)u^\gamma(t,\mathbf{k}_2)\rangle. \tag{12}$$

$\mathbf{W}(t,\mathbf{k})$ is the simultaneous $(\mathbf{u},\mathbf{f})$ cross-correlation function defined as

$$(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_1)W^{\alpha\beta}(t,\mathbf{k}) = \langle u^\alpha(t,\mathbf{k})f^\beta(t,\mathbf{k}_1)\rangle. \tag{13}$$

$\mathbf{G}(t,\mathbf{k})$ is the simultaneous $[\mathbf{u},(\rho_f\phi/\tau_p)\mathbf{v}_{\rm tv}]$ cross-correlation function defined as

$$(2\pi)^3\delta(\mathbf{k}-\mathbf{k}_1)G^{\alpha\beta}(t,\mathbf{k}) = \left\langle u^\alpha(t,\mathbf{k})\left(\frac{\rho_f\phi}{\tau_p}v^\beta_{\rm tv}(t,\mathbf{k}_1)\,\delta_{(\mathbf{k}_1=0)}\right)\right\rangle. \tag{14}$$

We apply a contraction in Eq. (9) with respect to $\alpha$ and $\beta$:

$$\rho_{\rm eff}(t,k)\left(\frac{\partial F^{\alpha\alpha}(t,\mathbf{k})}{2\ \partial t} + [\gamma_0(k)+\gamma_p(k)]F^{\alpha\alpha}(t,\mathbf{k})\right) = J^{\alpha\alpha}(t,\mathbf{k}) + W^{\alpha\alpha}(t,\mathbf{k}) + G^{\alpha\alpha}(t,\mathbf{k}). \tag{15}$$

In isotropic turbulence it is possible to express the correlation functions in terms of one scalar, namely the absolute value $k$ of the wave number. Batchelor [8] suggested doing the same in anisotropic but homogeneous turbulence by averaging the correlation functions over all directions of $k$, thus taking the mean values of the functions over spherical surfaces $k$=const. We apply the same procedure to the correlation tensors in Eq. (15), for instance,

$$F^{\alpha\alpha}(t,k) = \frac{1}{4\pi k^2}\int dA(k) F^{\alpha\alpha}(t,\mathbf{k}), \tag{16}$$

where $A(k)$ is the spherical surface in wave-number space. The same averaging procedure is applied to the other tensors. This yields

$$\rho_{\text{eff}}(t,k)\left(\frac{\partial F^{\alpha\alpha}(t,k)}{2\,\partial t} + [\gamma_0(k) + \gamma_p(k)]F^{\alpha\alpha}(t,k)\right) = J^{\alpha\alpha}(t,k) + W^{\alpha\alpha}(t,k) + G^{\alpha\alpha}(t,k). \tag{17}$$

We now introduce the spectrum $E(t,k)$ for the density of the turbulent kinetic energy of the suspension

$$E(t,k) = \frac{\rho_{\text{eff}}(t,k)}{2\pi}k^2 F^{\alpha\alpha}(t,k), \tag{18}$$

with summation convention with respect to $\alpha$. The physical meaning of $E(t,k)$ is that, when integrated over $k$, it yields the turbulence intensity averaged over all directions. In this paper we will restrict ourselves to the study of $E(t,k)$. So the investigation of the turbulence energy spectra for the three individual directions ($\alpha$=1, 2, or 3) will not be carried out here. Multiplying Eq. (17) by $k^2/2\pi$ finally gives the dynamic equation for the turbulent energy spectrum:

$$\frac{\partial E(t,k)}{2\,\partial t} + [\gamma_0(k) + \gamma_p(k)]E(t,k) = J'(t,k) + W'(t,k) + G'(t,k), \tag{19}$$

where

$$J'(t,k) = \frac{k^2}{2\pi}J(t,k), \tag{20}$$

$$W'(t,k) = \frac{k^2}{2\pi}W(t,k), \tag{21}$$

and

$$G'(t,k) = \frac{k^2}{2\pi}G(t,k). \tag{22}$$

Using the assumption that the modeled nonlinearity is conservative, the energy redistribution term $J'(t,k)$ can be written as

$$J'(t,k) = -\frac{\partial \epsilon(t,k)}{\partial k}, \tag{23}$$

in which $\epsilon(t,k)$ is the energy flux through the turbulence eddies of the suspension (see L'vov, Ooms, and Pomyalov[6]).

### B. Closure relations

In order to solve Eq. (19), closure relations are needed for $E(t,k)$ and $\gamma(t,k)$. The simple closures used in Ref. 6 will also be applied here. Dimension analysis yields the following relation for $E(t,k)$:

$$E(t,k) = C_1[\epsilon^2(t,k)\rho_{\text{eff}}(t,k)/k^5]^{1/3}. \tag{24}$$

$C_1$ is a constant of order unity. The inverse lifetime (frequency) of eddies $\gamma(t,k)$ is determined by their viscous damping and by the energy loss in the cascade process of turbulence:

$$\gamma(t,k) = \gamma_0(t,k) + \gamma_c(t,k). \tag{25}$$

The inverse lifetime due to viscous damping has already been introduced in Eq. (8). Applying dimension analysis the inverse lifetime of a $k$ eddy due to energy loss in the cascade process is given by

$$\gamma_c(t,k) = C_2[k^2\epsilon(t,k)/\rho_{\text{eff}}(t,k)]^{1/3}. \tag{26}$$

$C_2$ is again a constant. The evaluation of the energy input in the suspension via the force $W'(t,k)$ will be done in the next paragraph.

Finally we evaluate the gravity term $G'(t,k)$. As explained before, this term is due to the correlation between the velocity ($\mathbf{u}$) of the turbulent flow field and the relative (settling) velocity ($\mathbf{v}_{\text{tv}}$) of the particles with respect to the fluid. For the modeling of this term we use the study carried out by Parthasarathy and Faeth.[9] They investigated the properties of a homogeneous dilute particle-laden flow caused by (nearly) monodisperse glass particles falling in a stagnant water bath. They assume that all the work carried out by the particles on the fluid is used to generate turbulence. The rate of production of turbulence is then equal to the rate of loss of potential energy of the particles as they fall through the bath. In our nomenclature this yields the following expression for the turbulence production $\rho_f\phi v_{\text{tv}}^2/\tau_p$, so the product of the friction force $\rho_f\phi v_{\text{tv}}/\tau_p$ of the particles on the fluid (see the Appendix) and the settling velocity $v_{\text{tv}}$. We use the same assumption, therefore in our case the turbulence production is also supposed to be given by $\rho_f\phi v_{\text{tv}}^2/\tau_p$. In order to be able to solve Eq. (19) it is not only necessary to know the total turbulent energy production, but also its spectral distribution. In the paper of Parthasarathy and Faeth it is stated that the measured fluid velocity fluctuations are comparable to the Kolmogorov velocity scale. The integral length scale $\Lambda$ of the produced turbulence can then be estimated from $\Lambda = (u')^3/\epsilon_p$, in which $u'$ is the fluid velocity fluctuation of the generated turbulence and because of its comparability to the Kolmogorov velocity scale it can be estimated by $u' = (\epsilon_p\nu)^{1/4}$. $\epsilon_p$ is the dissipation of the turbulence generated by the particles which is taken equal to the turbulence production $\epsilon_p = \rho_f\phi v_{\text{tv}}^2/\tau_p$. This yields for the integral scale of the turbulence generated by the settling particles

$$\Lambda = \left(\frac{\nu^3\tau_p}{\phi v_{\text{tv}}^2}\right)^{1/4}. \tag{27}$$

The integral length scale $\Lambda$ of the turbulence generated by the settling particles is usually not equal to the integral length scale $L$ of the turbulence generated by the stirring force $f(t,k)$ or by the turbulence-generating grid (in case of decaying turbulence behind a grid). $\Lambda/L$ is, therefore, one of the parameters that determines the turbulence spectrum of the suspension. Finally we choose an analytic expression from Hinze[10] for the spectral distribution

$$G'(k,t)=\frac{\rho_f\phi v_{\mathrm{tv}}^2}{\tau_p}\left(\frac{16}{3\pi}\frac{k^4\Lambda^4}{(1+k^2\Lambda^2)^3}\right)\Lambda, \tag{28}$$

with the property that

$$\int_0^\infty\left(\frac{16}{3\pi}\frac{k^4\Lambda^4}{(1+k^2\Lambda^2)^3}\right)\Lambda dk=1. \tag{29}$$

### C. Dimensionless equation for energy flux

First we define a dimensionless wave number $\kappa$ and integral-scale related parameters

$$\kappa=kL,\quad \epsilon_L=\epsilon(k=1/L),\quad \gamma_L=\gamma(k=1/L),$$
$$\rho_L=\rho_{\mathrm{eff}}(k=1/L). \tag{30}$$

Using these parameters the following dimensionless functions are defined:

$$\epsilon_\kappa=\epsilon/\epsilon_L,\quad \gamma_\kappa=\gamma/\gamma_L,\quad \rho_\kappa=\rho_{\mathrm{eff}}/\rho_L, \tag{31}$$

in which the argument $\kappa$ is written as a subscript to distinguish these functions from their corresponding dimensional functions of the dimensional argument $k$. Substituting the closure relations and the dimensionless functions in the energy equation (19) we find after lengthy but straightforward calculations the following dimensionless equation for the energy flux $\epsilon_\kappa$:

$$f(\tau,\kappa)\frac{\partial\epsilon_\kappa(\tau,\kappa)}{\partial\tau}+\frac{\partial\epsilon_\kappa(\tau,\kappa)}{\partial\kappa}+g(\tau,\kappa)$$
$$=W_\kappa(\tau,\kappa)+G_\kappa(\tau,\kappa), \tag{32}$$

where

$$f(\tau,\kappa)=\frac{1}{3}C_1\kappa^{-5/3}\rho_\kappa^{1/3}\epsilon_\kappa^{-1/3}\left(1-\frac{1}{2}\frac{\epsilon_\kappa}{\rho_\kappa}\frac{(\partial\Phi/\partial\epsilon_\kappa)}{(\partial\Phi/\partial\rho_\kappa)}\right), \tag{33}$$

with

$$\frac{\partial\Phi}{\partial\epsilon_\kappa}=-C_3\phi\left(2\delta(1+\delta\gamma_\kappa)(2+3\delta\gamma_\kappa)\frac{\partial\gamma_\kappa}{\partial\epsilon_\kappa}\right) \tag{34}$$

and

$$\frac{\partial\Phi}{\partial\rho_\kappa}=1-C_3\phi\left(2\delta(1+\delta\gamma_\kappa)(2+3\delta\gamma_\kappa)\frac{\partial\gamma_\kappa}{\partial\rho_\kappa}\right), \tag{35}$$

in which

$$\frac{\partial\gamma_\kappa}{\partial\epsilon_\kappa}=\frac{1}{3}\frac{\kappa^{2/3}}{\epsilon_\kappa^{2/3}\rho_\kappa^{1/3}} \tag{36}$$

and

$$\frac{\partial\gamma_\kappa}{\partial\rho_\kappa}=-\frac{\kappa^2}{C_4\rho_\kappa^2}-\frac{1}{3}\frac{\epsilon_\kappa^{1/3}\kappa^{2/3}}{\rho_\kappa^{4/3}}. \tag{37}$$

The functions $\rho_\kappa$ and $\gamma_\kappa$ are given by

$$\rho_\kappa=\left(1+\phi\frac{1+2\delta\gamma_\kappa}{(1+\delta\gamma_\kappa)^2}\right)\Bigg/\left(1+\phi\frac{1+2\delta}{(1+\delta)^2}\right) \tag{38}$$

and

$$\gamma_\kappa=\frac{\kappa^2}{C_2\,\mathrm{Re}_s\rho_\kappa}+\frac{\epsilon_\kappa^{1/3}\kappa^{2/3}}{\rho_\kappa^{1/3}}. \tag{39}$$

The function $g(\tau,\kappa)$ is equal to

$$g(\tau,\kappa)=C\frac{\epsilon_\kappa}{\kappa}T_\kappa+\frac{C_1}{\mathrm{Re}_s}\left(\frac{\kappa\epsilon_\kappa^2}{\rho_\kappa^2}\right)^{1/3}(1+T_\kappa), \tag{40}$$

in which

$$T_\kappa=\frac{\phi\delta\gamma_\kappa}{(1+\phi)(1+2\delta\gamma_\kappa)+(\delta\gamma_\kappa)^2}. \tag{41}$$

$C=C_1C_2$, and the constants $C_3$ and $C_4$ are equal to $C_3=[1+\phi(1+2\delta)/(1+\delta)^2]^{-1}$ and $C_4=C_2\,\mathrm{Re}_s$. $\tau$ is the dimensionless time defined as $\tau=t/\tau_c$ with $\tau_c=L^{2/3}/(\epsilon_L/\rho_f)^{1/3}$. $\delta=\tau_p\gamma_L$ is the dimensionless particle response time. The suspension Reynolds number is defined by $\mathrm{Re}_s=Lv_L/\nu_L$. $L$ is the integral length scale and $v_L$ the integral velocity scale defined by $v_L=(\epsilon_L L/\rho_L)^{1/3}$. $\nu_L$ is the effective kinematic viscosity of the suspension for $k=L^{-1}$ and is given by $\nu_L=\nu(\rho_f/\rho_L)$ with $\rho_L$ the effective density of the suspension for $k=L^{-1}$ given by $\rho_L=\rho_f[1+\phi(1+2\delta)/(1+\delta)^2]$. The fluid Reynolds number is defined by $\mathrm{Re}_f=Lv_L/\nu$ and is related to the suspension Reynolds number in the following way: $\mathrm{Re}_s=\mathrm{Re}_f\nu/\nu_L$.

The "pumping" term $W_\kappa$ is somewhat arbitrarily chosen in the following way:

$$W_\kappa=\frac{W_0}{(2\pi)^{1/2}\sigma}\exp\left(-\frac{(\kappa-1)^2}{2\sigma^2}\right), \tag{42}$$

in which $W_0$ represents the dimensionless energy input. The function $W_\kappa$ has a maximum at $\kappa=1$ (the input of energy is largest at $\kappa=1/L$), while the parameter $\sigma$ describes the characteristic width of the pumping region.

Finally the term representing the energy generation by the settling particles $G_\kappa$ is given by

$$G_\kappa=\frac{\phi\delta}{[1+\phi(1+2\delta)/(1+\delta)^2]}\frac{1}{\mathrm{Fr}^2}\left(\frac{16}{3\pi}\frac{\kappa^4(\Lambda/L)^4}{[1+\kappa^2(\Lambda/L)^2]^3}\right)$$
$$\times(\Lambda/L), \tag{43}$$

where the Froude number is defined as $\mathrm{Fr}=(\rho_p/\Delta\rho)(v_L^2/gL)$.

After $\epsilon_\kappa$ has been calculated for a certain case from Eq. (32) the energy spectrum of the suspension can be determined using the closure relation

$$E_\kappa=\epsilon_\kappa^{2/3}\rho_\kappa^{1/3}\kappa^{-5/3}. \tag{44}$$

L'vov *et al.* have shown that the energy flux of the carrier fluid can be calculated from the suspension spectrum in the following manner:

$$E_\kappa^f=E_\kappa/\rho_\kappa=\epsilon_\kappa^{2/3}\rho_\kappa^{-2/3}\kappa^{-5/3}. \tag{45}$$

($E_\kappa$ and $E_\kappa^f$ have been made dimensionless by means of their values at $\kappa=1$.) In this way it becomes possible to study the decay of the turbulent energy spectrum of the fluid as function of the relevant dimensionless groups, namely the particle mass fraction $\phi$, the dimensionless particle response time $\delta$, the fluid Reynolds number $\mathrm{Re}_f$, the Froude number Fr, and the ratio $\Lambda/L$ of the turbulence length scale generated by the settling particles and the integral length scale of the turbulence generated by the stirring force or the grid.

## IV. COMPARISON WITH DNS CALCULATIONS AND EXPERIMENTAL DATA

In the rest of this paper we will consider only stationary solutions of Eq. (32). Omitting the time-dependent term this equation becomes

$$\frac{\partial \epsilon_\kappa(\tau,\kappa)}{\partial \kappa} + g(\tau,\kappa) = W_\kappa(\tau,\kappa) + G_\kappa(\tau,\kappa). \tag{46}$$

### A. Simplification of the energy pumping term

The turbulence statistics in the energy containing range $\kappa \sim 1$ is not universal and depends on the type of energy pumping, so on the function $W_\kappa$. In order to allow a general analysis, independent of some particular type of turbulence generation, we assume that the pumping of energy takes place in a narrow shell in $k$ space. This means

$$\lim_{\sigma\to 0}\{W_\kappa\} = \delta(\kappa), \tag{47}$$

where $\delta(\kappa)$ is the Dirac $\delta$ function. In this limit and with zero boundary conditions for $\epsilon_\kappa$ and $\gamma_\kappa$ at $\kappa=0$ (and consequently $\rho_\kappa=1$ at $\kappa=0$), Eq. (46) can be solved on the interval $0 \leqslant \kappa \leqslant 1$. This gives

$$\epsilon_\kappa = 1, \quad \gamma_\kappa = 1, \quad \rho_\kappa = 1, \quad \text{at } \kappa = 1. \tag{48}$$

In the limit of Eq. (47), Eq. (46) becomes for $\kappa > 1$

$$\frac{\partial \epsilon_\kappa(\tau,\kappa)}{\partial \kappa} + g(\tau,\kappa) = G_\kappa(\tau,\kappa). \tag{49}$$

Equation (48) can be considered as the boundary conditions for Eq. (49) at $\kappa=1$.

### B. Comparison with DNS calculations

Several authors have carried out direct numerical simulation (DNS) calculations for particle-laden homogeneous, isotropic turbulent flows. For instance, Squires and Eaton[11] used DNS to study a forced (so statistically stationary) turbulent suspension. Elghobashi and Truesdell[12] examined turbulence modulation by particles in decaying turbulence. Similar DNS studies (for a stationary or decaying turbulent suspension) with more details were carried out by Boivin, Simonin, and Squires,[13] Sundaram and Collins,[14] Druzhinin,[15] and Ferrante and Elgobashi.[16] In all of these studies the influence of gravity is neglected, apart from one case in the paper by Ferrante and Elgobashi.[16] So these DNS results are not suited to test the validity of the description of the gravity effect in our model. Yet they are useful for a first check on the validity of our model for the case without gravity effect.

Some general remarks can be made about these DNS studies. The effect of turbulence generation by the particle wakes and by vortices shed by the particles was not taken into account. It would also have been difficult to include this effect of turbulence generation, as the particles were treated

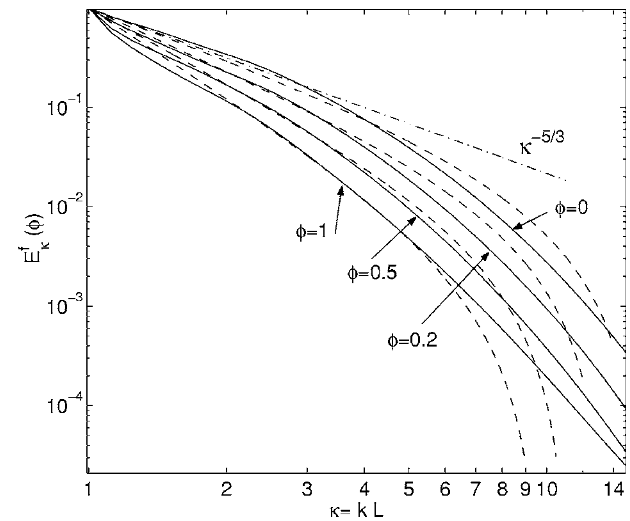


FIG. 1. Log-log plot of turbulent kinetic energy spectrum taken from Boivin, Simonin, and Squires (Ref. 13) for $\phi$=0, 0.2, 0.5, and 1 with $\delta$ =1.65 (solid lines), and numerical solution of Eq. (49) without the gravity effect ($G_\kappa$=0) for the same values of $\phi$ and $\delta$ (dashed lines).

as point particles. From the DNS calculations it can be concluded that for a suspension with particles with a response time much larger than the Kolmogorov time scale the main effect of the particles is suppression of the energy of eddies of nearly all sizes (at the same energy input into the suspension as for the particle-free case). So for such a suspension the total turbulent energy of the carrier fluid will be smaller than the total turbulent energy of the fluid for the particle-free case. However, for a suspension with particles with a response time comparable to or smaller than the Kolmogorov time, the Kolmogorov length scale will decrease and the turbulent energy of (nearly) all eddy sizes increases. In that case the total turbulent energy of the carrier fluid is larger than the total turbulent energy of the fluid for the particle-free case. For a suspension with particles with a response time in between the two limiting cases mentioned above, the energy of the larger eddies is suppressed, whereas the energy of the smaller ones is enhanced. It is important to realize that these results were found neglecting the effect of gravity and the effect of turbulence generation in the particle wakes and by the vortices shed by the particles.

Ooms and Poelma[7] have compared the theoretical predictions made by the model developed L'vov, Ooms, and Pomyalov[6] (which is the model that we have extended in this publication by including the effect of gravity) with the DNS results described above for the case of a decaying turbulent suspension and they found a reasonable agreement. Special attention was also paid by them to a physical explanation of the influence of the particles on the turbulence of the carrier fluid. For the case of a stationary turbulent suspension L'vov, Ooms, and Pomyalov[6] compare in their publication theoretical predictions made with their model for the turbulent energy spectrum of the fluid with results of DNS calculations carried out by Boivin, Simonin, and Squires.[13] The result is given in Fig. 1.

The solid line, labeled by $\phi$=0, describes the particle-free case. The dashed-dotted line gives the well-known $\kappa^{-5/3}$

law. It can be seen that only the first half of the first decade of the DNS calculations can be considered as the inertial subrange. With a chosen value for $C_1=1$, $C_2=13$ (and hence $C=C_1\,C_2=13$) the numerical solutions of Eq. (49) without gravity effect ($G_\kappa=0$) (dashed lines) approximate well all the DNS energy spectra (solid lines) $E^f_\kappa(\phi)$ for $\phi=0.2$, 0.5, and 1 in a region between $\kappa=1$ and a maximum value of $\kappa$ referred to as $\kappa_{\max}$. In this region the spectra decrease from unity (at $\kappa=1$) to some values smaller than $10^{-3}$. The value of $\kappa_{\max}$ decreases from $\kappa_{\max}=14$ for the ($\phi=0$) spectrum to $\kappa_{\max}=7$ for the ($\phi=1$) spectrum. For $\kappa>\kappa_{\max}$ the solution of Eq. (49) gives too small values for the turbulent energy. As discussed by L'vov, Ooms, and Pomyalov, this is due to the rather simple closure relations used, which is not realistic in the viscous subrange. In conclusion it can be stated that our theoretical model agrees rather well with results from DNS calculations. However, in these DNS calculations the effect of gravity is not included. Therefore, we have compared our model predictions also with experiments for which, of course, the effect of gravity is present.

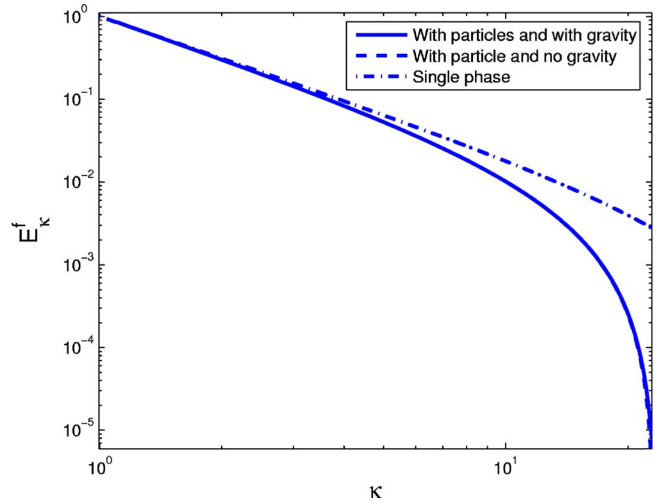


FIG. 2. Log-log plot of turbulent kinetic energy spectrum $E^f_\kappa$ of carrier fluid for the particle-free case, for the case with particles but without gravity the effect, and for the case with particles and with gravity. Simulation of the experimental results of Schreck and Gleis for plastic particles. There is no visible difference between the calculation for the case with particles and with gravity and the case with particles without gravity.

### C. Comparison with experiments

Schreck and Gleis[17] studied the two-way coupling effect in grid-generated turbulence. They used solid particles in water. There were two types of particles: glass and neutrally buoyant plastic particles. They measured, for instance, the development in turbulent kinetic energy for the particle-free case and the particle-laden case by means of the laser Doppler anemometer (LDA) technique. We compare their experimental results at a certain position in the water tunnel with predictions made with our model. Before doing that we point out that in the derivation of our model it is assumed that the particle density is significantly larger than the density of the fluid. Otherwise Eq. (A7) for the particle motion is not valid, as the Bassett history force and the virtual mass need to be accounted for if the particle density is comparable to the fluid density. Therefore, a comparison of predictions made with our model with the experimental results of Schreck and Gleis is questionable. However, the number of publications dealing with accurate experiments concerning the two-way coupling effect between particles and turbulence in a homogeneous suspension is very limited. For that reason we show the comparison with the experiments of Schreck and Gleis, although we realize that for a reliable comparison our model should have been extended first with the Bassett history force and the virtual added mass in the particle equation of motion. However, at the moment it is not clear to us how to make such an extension. In the comparison with the results of Schreck and Gleis we did not solve the time-dependent equation (32), but the stationary state equation (49) for the conditions at the chosen position in the water channel. [We plan to solve the time-dependent equation (32) in the near future to calculate the turbulence development in the channel, but that will require considerably more work.] From the publication of Schreck and Gleis we derive the following values for the relevant dimensionless groups belonging to their experiments at the chosen position in the channel: for the plastic particles $\phi=0.015$, $\delta=0.21$, $\mathrm{Re}_f=220$, $\mathrm{Fr}=0.272$, and $\Lambda/L=0.1$ and for the glass particles $\phi=0.036$, $\delta=0.38$, $\mathrm{Re}_f=220$, $\mathrm{Fr}=0.014$, and $\Lambda/L=0.1$. Schreck and Gleis measured turbulence suppression by both type of particles: for plastic particles $(u^2_{\text{particles}}/u^2_{\text{particle free}})=0.86$ and for glass particles $(u^2_{\text{particles}}/u^2_{\text{particle free}})=0.77$, in which $u^2$ represents the turbulence intensity. With the value of $C_1=1$, $C_2=13$ already chosen for the comparison with the DNS results of Boivin, Simonin, and Squires, we found the same result for the turbulence suppression. In Fig. 2 (plastic particles) and in Fig. 3 (glass particles) we show the results of our calculations for the energy spectrum $E^f_\kappa$ of the carrier fluid for the particle-free case, for the case with particles but without gravity effect, and for the case with particles and with gravity.

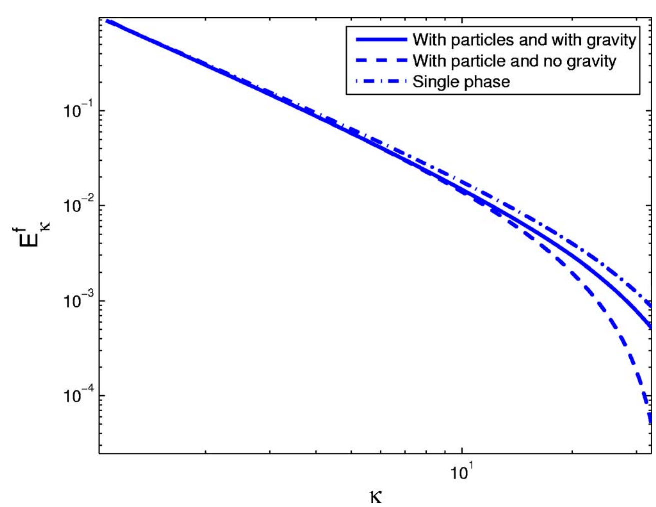


FIG. 3. Log-log plot of turbulent kinetic energy spectrum $E^f_\kappa$ of carrier fluid for the particle-free case, for the case with particles but without the gravity effect, and for the case with particles and with gravity. Simulation of the experimental results of Schreck and Gleis for glass particles.

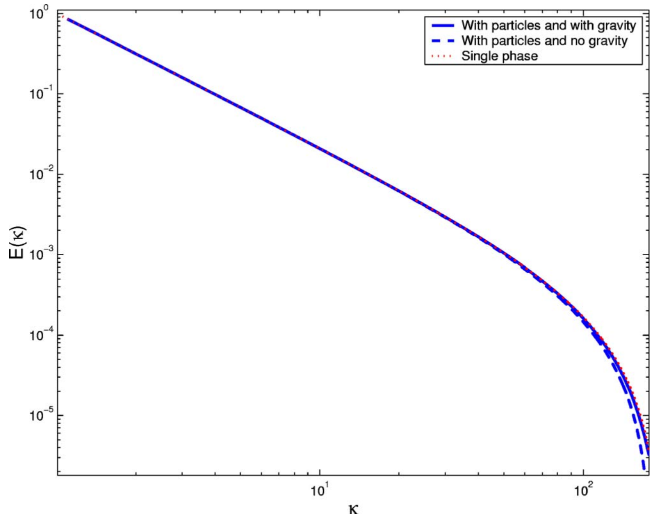


FIG. 4. Log-log plot of turbulent kinetic energy spectrum $E_\kappa^f$ of carrier fluid for the particle-free case, for the case with particles but without the gravity effect, and for the case with particles and with gravity. Simulation of the experimental results of Hussainov *et al.* There is no visible difference between the single-phase calculation and the calculation for the case with particles and with gravity. The calculation for the case with particles and without gravity is the lowest in the figure.

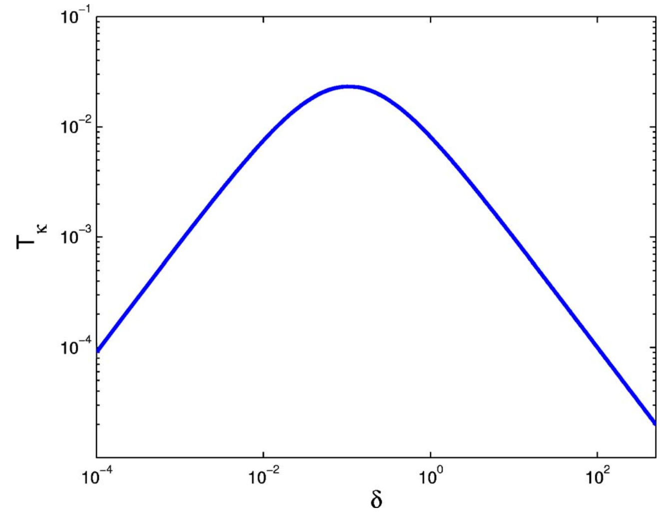


FIG. 5. Influence of particle response time on turbulence; $T_\kappa$ as a function of $\delta$.

There is almost no distinction between the case with particles without gravity effect and the case with particles with gravity effect. (Although for the glass particles there is some effect for the smallest eddies.) So it is clear that according to our model calculations the turbulence generation due to the settling of the particles is negligible (also for the glass particles). It is also found that at all wave numbers there is a suppression of turbulence.

Hussainov *et al.*[18] used particles similar to those used by Schreck and Gleis, yet instead of a water channel they used a wind tunnel. This led to larger values of the mass fraction and of the particle response time. Measurements were again carried out using the LDA technique. The effect of the particles on the turbulence intensity was found to be negligible: $(u^2_{\text{particles}}/u^2_{\text{particle free}}) \sim 1$. From their publication we derived again the values of the relevant parameters: $\phi=0.1$, $\delta=250$, $\mathrm{Re}_f=1060$, $\mathrm{Fr}=5.20$, and $\Lambda/L=0.04$. Using these values we confirmed that indeed, also according to our model, the effect of the particles on the turbulence is negligible.

In Fig. 4 we show the results of our calculations. The two-way coupling effect is negligible. The influence of gravity is only visible at high wave numbers, but the effect is very small. It seems rather strange that particles with a very large value of the dimensionless response time $\delta$ have no influence on the turbulence. So we have studied this case in some more detail. The influence on the turbulence can, for instance, be studied from the function $T_\kappa$ in Eq. (41). It can be concluded that for very small values of $\delta$ the function $T_\kappa$ becomes very small; and this holds also for very large values of $\delta$. In Fig. 5 $T_\kappa$ is shown as function of $\delta$ for $\phi=0.1$ and $\gamma_\kappa=10$. Indeed for very small and very large values of $\delta$ the influence of turbulence (via $T_\kappa$) becomes negligibly small. Moreover, from Eq. (38) it can be seen that for very large values of $\delta$ the effective density $\rho_\kappa$ approximates the density of the fluid $\rho_f$. From Eq. (43) it can be concluded that the relevant dimensionless group for the importance of the gravity effect is $\phi\delta(\Lambda/L)/\mathrm{Fr}^2$. For the experiments of Hussainov *et al.* this parameter has a value of about $\phi\delta(\Lambda/L)/\mathrm{Fr}^2 \sim 0.04$. So it is also negligibly small.

We have to conclude that the experiments by Schreck and Gleis and by Hussainov *et al.* are not suited to test the validity of the description of the gravity effect in our theoretical model, as this effect is of negligible importance in their experiments.

## D. Sensitivity study

In order to find out at which values of the parameters the gravity effect becomes important we carried out calculations for the following values of the parameters: $\phi=1$, $\delta=10^{-2}$, $\mathrm{Re}_f=10^3$, and $\Lambda/L=0.05$ with decreasing values of the Froude number $\mathrm{Fr}=1$, $10^{-1}$, $10^{-2}$, and $10^{-3}$ (and hence increasing importance of the gravity effect). We found that for $\mathrm{Fr}=1$ and $10^{-1}$ there is a slight suppression of turbulence: $(u^2_{\text{particles}}/u^2_{\text{particle free}})=0.937$ and $(u^2_{\text{particles}}/u^2_{\text{particle free}})=0.943$. However, with decreasing value of the Froude number the turbulence generation by settling particles grows in importance and the turbulence in the carrier fluid is enhanced. We calculated for $\mathrm{Fr}=10^2$ and $10^{-3}$: $(u^2_{\text{particles}}/u^2_{\text{particle free}})=1.059$ and $(u^2_{\text{particles}}/u^2_{\text{particle free}})=3.029$.

In Fig. 6 we show the two-way coupling effect on the turbulence spectrum of the carrier fluid for the four values of the Froude number. As can be seen the turbulence generation effect becomes first noticeable at large wave numbers. With increasing gravity effect the influence on the spectrum grows and becomes significant also at smaller wave numbers. As mentioned earlier, from Eq. (43) it can be concluded that the relevant dimensionless group for the importance of the gravity effect is $\phi\delta(\Lambda/L)/\mathrm{Fr}^2$. When $\phi\delta(\Lambda/l)/\mathrm{Fr}^2<1$ the gravity effect is still negligible; only for values $\phi\delta(\Lambda/L)/\mathrm{Fr}^2>1$ does this effect grow quickly in importance.

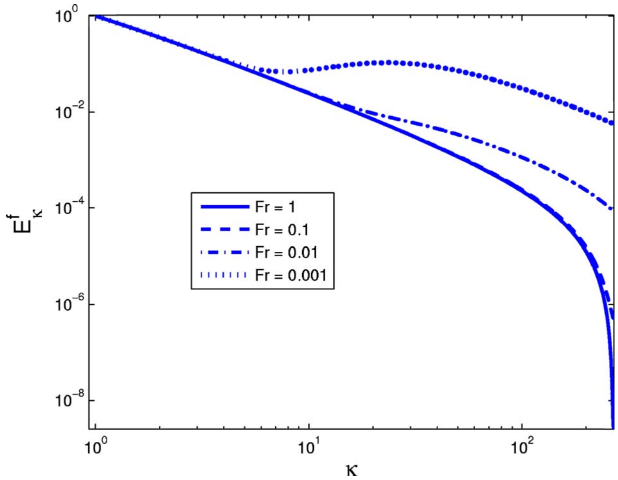


FIG. 6. Log-log plot of turbulent kinetic energy spectrum $E_\kappa^f$ of carrier fluid for the case with particles and with gravity and for four values of the Froude number. To Fr=1, Fr=0.1, Fr=0.01, Fr=0.001 correspond respectively to the following values for the relevant parameter $\phi\delta(\Lambda/L)/\mathrm{Fr}^2=5\times10^{-4}$, $\phi\delta(\Lambda/L)/\mathrm{Fr}^2=5\times10^{-2}$, $\phi\delta(\Lambda/L)/\mathrm{Fr}^2=5$, and $\phi\delta(\Lambda/L)/\mathrm{Fr}^2=5\times10^{2}$.

### E. Discussion

The interaction between particles and carrier fluid causes many complex phenomena in a homogeneous, turbulent suspension. Without the gravity effect (due to the difference in density between particles and fluid) the particles will dampen the fluid turbulence when their response time is larger (but not very much larger) than the Kolmogorov time scale. For particles with a response time comparable to or smaller than the Kolmogorov time scale, the turbulence will be enhanced and the Kolmogorov length scale will decrease. For a suspension with particles with a response time in between the two limiting cases mentioned above, the energy of the large eddies is suppressed, whereas the energy of the smaller ones is enhanced. For very large values of the response time the effect of particles on the turbulence disappears. These phenomena are also found by our theoretical model. When the gravity effect is taken into account turbulence will be generated by the settling particles. The overall behavior of the turbulent suspension depends on the relative importance of all these two-way coupling effects. With our theoretical model it is possible to calculate the significance of the different effects as a function of some dimensionless groups, and also an explanation in physical terms is given. Of course, many simplifications had to be made in the derivation of the model and many improvements are possible. We think, however, that the model supports the understanding of turbulent suspensions and the design of practical applications. So far we have neglected the influence of a mean velocity gradient on the turbulence of a suspension. We have only considered the "direct" effect of the particles on the turbulence. When a mean shear in the flow field is present, also the "indirect" influence of the particles on the turbulence generation mechanism via the velocity gradient have to be included. It is our intention to extend our model in this direction.


## ACKNOWLEDGMENT

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## APPENDIX: "DERIVATION" OF THE EQUATION OF MOTION FOR A ONE-FLUID MODEL WITH THE GRAVITY EFFECT

We start from the Navier-Stokes equation for the fluid in the suspension

$$\rho_f\left[\frac{\partial}{\partial t}+(\mathbf{u}\cdot\boldsymbol{\nabla}-\nu\boldsymbol{\nabla}^2)\right]\mathbf{u}+\boldsymbol{\nabla}p=\mathbf{f}_p+\mathbf{f}, \tag{A1}$$

in which $\rho_f$ is the fluid density, $\mathbf{u}$ is the fluid velocity, $\nu$ is the fluid viscosity, $p$ is the pressure, $\mathbf{f}_p$ is the force exerted by the particles on the fluid, and $f$ represents the stirring force responsible for the maintenance of the turbulent flow. For $\mathbf{f}_p$ the following expression is used:

$$f_p(t,\mathbf{r})=\sum_j \mathbf{F}_p(t,\mathbf{r}_j)\,\delta(\mathbf{r}-\mathbf{r}_j), \tag{A2}$$

where $\mathbf{F}_p(t,\mathbf{r}_j)$ is the force between the fluid and the $j$ particle positioned at $\mathbf{r}=\mathbf{r}_j$. We assume that the statistics of the particles is independent of the statistics of the turbulence and, moreover, that their distribution in space is homogeneous. In that case we can replace the sum over the positions of particles by a space integration

$$\sum_j \rightarrow \frac{1}{l^3}\int d\mathbf{r}_j, \tag{A3}$$

where $l^3$ is the average (fluid) volume per particle. In this approximation

$$\mathbf{f}_p(t,\mathbf{r})=\mathbf{F}_p(t,\mathbf{r})/l^3. \tag{A4}$$

We compute $\mathbf{F}_p(t,\mathbf{r})$ for particles that are small enough that Stokes' law may be used:

$$\mathbf{F}_p(t,\mathbf{r})=\zeta[\mathbf{v}_p(t)-\mathbf{u}(t,\mathbf{r})], \tag{A5}$$

in which $\mathbf{v}_p$ is the particle velocity. $\zeta$ the particle friction coefficient given by

$$\zeta=6\pi\rho_f\nu a, \tag{A6}$$

in which $\nu$ is the kinematic viscosity of the fluid and $a$ is the particle radius.

The equation of motion for a particle reads

$$\begin{aligned} m_p\frac{d\mathbf{v}_p(t)}{dt}&=-\mathbf{F}_p(t,\mathbf{r})+\frac{\Delta\rho}{\rho_p}m_p\mathbf{g}\\ &=-\zeta[\mathbf{v}_p(t)-\mathbf{u}(t,\mathbf{r})]+\frac{\Delta\rho}{\rho_p}m_p\mathbf{g}, \end{aligned} \tag{A7}$$

where $\Delta\rho=(\rho_p-\rho_f)$ is the difference between the density of the particles and the fluid density, $m_p$ is the particle mass, and $\mathbf{g}$ is the acceleration due to gravity. A formal solution of this equation is

$$\mathbf{v}_p(t) = \left(\tau_p \frac{d}{dt} + 1\right)^{-1} \left(\mathbf{u}(t,\mathbf{r}) + \frac{\Delta\rho}{\rho_p}\tau_p \mathbf{g}\right), \tag{A8}$$

with $\tau_p = m_p/\zeta$ the particle response time. We define

$$\mathbf{A} = \left(1 + \tau_p \frac{d}{dt}\right)^{-1} \tag{A9}$$

and

$$\mathbf{v}_{tv} = \frac{\Delta\rho}{\rho_p}\tau_p \mathbf{g}. \tag{A10}$$

Substituting Eq. (A8) with Eqs. (A9) and (A10) in Eq. (A7) gives the following expression:

$$\mathbf{F}_p = -m_p \frac{d}{dt}[\mathbf{A}(\mathbf{u} + \mathbf{v}_{\mathrm{tv}})] + \frac{\Delta\rho}{\rho_p} m_p \mathbf{g}. \tag{A11}$$

Because of Galilean invariance we can use the following relation:

$$\frac{d}{dt}[\mathbf{A}(\mathbf{u} + \mathbf{v}_{\mathrm{tv}})] = \frac{d}{dt}[\mathbf{A}(\mathbf{u})]. \tag{A12}$$

This yields the following expression for the force $F_p(t,\mathbf{r})$:

$$F_p(t,\mathbf{r}) = -m_p \frac{d}{dt}\left(\tau_p \frac{d}{dt} + 1\right)^{-1} \mathbf{u}(t,\mathbf{r}) + \frac{\Delta\rho}{\rho_p} m_p \mathbf{g} \tag{A13}$$

or

$$f_p(t,\mathbf{r}) = -\rho_f \phi \frac{d}{dt}\left(\tau_p \frac{d}{dt} + 1\right)^{-1} \mathbf{u}(t,\mathbf{r}) + \frac{\Delta\rho}{\rho_p} \rho_f \phi \mathbf{g}, \tag{A14}$$

where $\phi = m_p/\rho_f l^3$ is the particle mass loading parameter. The total time derivative $(d/dt)$ takes into account the time dependence of the coordinate $\mathbf{r}$ of the particles

$$\frac{d}{dt} = \left(\frac{\partial}{\partial t} + \mathbf{v}_p(t) \cdot \boldsymbol{\nabla}\right). \tag{A15}$$

Due to their inertia the particles do not follow the Lagrangian trajectories of fluid particles. Therefore, $d/dt$ does not coincide with the Lagrangian time derivative of the fluid $D/Dt$, which is given by

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \mathbf{u}(t,\mathbf{r}) \cdot \boldsymbol{\nabla}\right). \tag{A16}$$

Because of Eqs. (A15) and (A16) the following relation holds between $D/Dt$ and $d/dt$:

$$\frac{D\mathbf{u}(t,\mathbf{r})}{Dt} = \frac{d\mathbf{u}(t,\mathbf{r})}{dt} - [\mathbf{v}_{\mathrm{p}} - \mathbf{u}(t,\mathbf{r})] \cdot \nabla \mathbf{u}(t,\mathbf{r}). \tag{A17}$$

In order to derive an expression in Eq. (A17) for $[\mathbf{v}_{\mathrm{p}} - \mathbf{u}(t,\mathbf{r})]$ we use Eq. (A7), which yields

$$[\mathbf{v}_p - \mathbf{u}(t,\mathbf{r})] = -\tau_p \frac{d\mathbf{v}_p(t)}{dt} + \frac{\Delta\rho}{\rho_p}\tau_p \mathbf{g}, \tag{A18}$$

and after substitution of Eqs. (A8)–(A10)

$$[\mathbf{v}_p - \mathbf{u}(t,\mathbf{r})] = -\tau_p \frac{d}{dt}\{\mathbf{A}[\mathbf{u}(t,\mathbf{r}) + \mathbf{v}_{\mathrm{tv}}]\} + \frac{\Delta\rho}{\rho_p}\tau_p \mathbf{g}. \tag{A19}$$

Applying Eq. (A12) gives

$$[\mathbf{v}_p - \mathbf{u}(t,\mathbf{r})] = -\tau_p \frac{d}{dt}\{\mathbf{A}[\mathbf{u}(t,\mathbf{r})]\} + \frac{\Delta\rho}{\rho_p}\tau_p \mathbf{g}. \tag{A20}$$

Therefore Eq. (A17) can be written as

$$\frac{D\mathbf{u}(t,\mathbf{r})}{Dt} = \frac{d\mathbf{u}(t,\mathbf{r})}{dt} + \frac{d}{dt}\frac{\tau_p}{1+\tau_p \frac{d}{dt}}\mathbf{u}(t,\mathbf{r}) \cdot \boldsymbol{\nabla}\mathbf{u} - \frac{\Delta\rho}{\rho_p}\tau_p \mathbf{g} \cdot \boldsymbol{\nabla}\mathbf{u}(t,\mathbf{r}). \tag{A21}$$

Equation (A21) can also be formulated in the following way:

$$\frac{D\mathbf{u}(t,\mathbf{r})}{Dt} = \frac{d}{dt}\frac{1}{1+\tau_p \frac{d}{dt}}\left(1 + \tau_p \frac{\partial}{\partial t} + L\right)\mathbf{u}(t,\mathbf{r}) - \frac{\Delta\rho}{\rho_p}\tau_p \mathbf{g} \cdot \boldsymbol{\nabla}\mathbf{u}(t,\mathbf{r}), \tag{A22}$$

in which the operator $L$ is defined by

$$L\mathbf{u} \equiv \tau_p[(\mathbf{v}_p \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}]. \tag{A23}$$

For the derivation of Eq. (A22) from Eq. (A21), we take the first two terms of Eq. (A21) together in the following way:

$$\frac{d}{dt}\left(\mathbf{u} + \frac{\tau_p}{1+\tau_p \frac{d}{dt}}\mathbf{u} \cdot \nabla \mathbf{u}\right). \tag{A24}$$

This can be rewritten as

$$\frac{d}{dt}\left(\mathbf{u} + \frac{\tau_p}{1+\tau_p \frac{d}{dt}}\mathbf{u} \cdot \nabla \mathbf{u}\right) = \frac{d}{dt}\left\{\frac{1}{1+\tau_p \frac{d}{dt}}\left[\left(1 + \tau_p \frac{d}{dt}\right)\mathbf{u} + \tau_p \mathbf{u} \cdot \nabla \mathbf{u}\right]\right\}, \tag{A25}$$

and using Eqs. (A15) and (A23) we get directly the first term on the right-hand side of Eq. (A22):

$$\frac{d}{dt}\left[\frac{1}{1+\tau_p \frac{d}{dt}}\left(\mathbf{u} + \tau_p \frac{\partial}{\partial t}\mathbf{u} + \tau_p \mathbf{v}_p \cdot \nabla \mathbf{u} + \tau_p \mathbf{u} \cdot \nabla \mathbf{u}\right)\right]. \tag{A26}$$

Equation (A22) yields the following relation:

$$\frac{d}{dt}\left(\tau_p \frac{d}{dt} + 1\right)^{-1} \mathbf{u}(t,\mathbf{r}) = \left(\frac{D}{Dt}\frac{1}{1+\tau_p \frac{\partial}{\partial t} + L}\right)\mathbf{u}(t,\mathbf{r}) + \left(\frac{\Delta\rho/\rho_p \tau_p \mathbf{g} \cdot \boldsymbol{\nabla}}{1+\tau_p \frac{\partial}{\partial t} + L}\right)\mathbf{u}(t,\mathbf{r}). \tag{A27}$$

Substitution in Eq. (A14) gives the following expression for the force $f_p$:

$$f_p(t,\mathbf{r}) = -\rho_f\phi\left(\frac{D}{Dt}\frac{1}{1+\tau_p\frac{\partial}{\partial t}+L}\right)\mathbf{u}(t,\mathbf{r}) - \left(\frac{\rho_f\phi\Delta\rho/\rho_p\tau_p\mathbf{g}\cdot\boldsymbol{\nabla}}{1+\tau_p\frac{\partial}{\partial t}+L}\right)\mathbf{u}(t,\mathbf{r}) + \frac{\Delta\rho}{\rho_p}\rho_f\phi\mathbf{g}. \quad (A28)$$

For particles with a small response time, Ferry and Balachandar[19] show that the particle velocity depends only on the local fluid quantities (the velocity and its spatial and temporal derivatives). They derive an expansion of the particle velocity in terms of the particle response time that generalizes those of previous researchers. Neglecting gravity and for large values of the ratio of the particle density and the fluid density and for small values of the particle response time our equation for the force $\mathbf{F}_p(=l^3\mathbf{f}_p)$ on a particle gives the same equation for the particle velocity as derived in Ref. 19.

Substitution of Eq. (A28) in the Navier-Stokes equation (A1) yields

$$\rho_f\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\left(1+\frac{\phi}{1+\tau_p\frac{\partial}{\partial t}+L}\right)\mathbf{u}+\boldsymbol{\nabla}p = \rho_f\nu\nabla^2\mathbf{u}+\mathbf{f}-\left(\frac{\rho_f\phi\Delta\rho/\rho_p\tau_p\mathbf{g}\cdot\boldsymbol{\nabla}}{1+\tau_p\frac{\partial}{\partial t}+L}\right)\mathbf{u}(t,\mathbf{r})+\frac{\Delta\rho}{\rho_p}\rho_f\phi\mathbf{g}. \quad (A29)$$

The inverse operator in Eq. (A29) may be understood as a Taylor expansion with respect to the nonlinearity $\mathbf{u}\cdot\nabla$:

$$\frac{1}{1+\tau_p\frac{\partial}{\partial t}+L} = \frac{1}{1+\tau_p\frac{\partial}{\partial t}} - \frac{L}{\left(1+\tau_p\frac{\partial}{\partial t}\right)^2} + \cdots. \quad (A30)$$

This expansion produces nonlinear terms of higher order in $(\mathbf{u}\cdot\nabla)$ in Eq. (A29). Now we assume (without proof) that the term $L/(1+\tau_p\partial/\partial t)$ is much smaller than unity and neglect the higher-order terms. The following approximate reasoning can be given for this simplification. The higher-order terms are not important for large eddies with $\tau_p\gamma(k)\ll 1$ for which the operator $[1+\phi/(1+\tau_p\partial/\partial t+L)]$ is close to the factor $1+\phi$. [$\gamma(k)$ represents the frequency of eddies of wave number $k$.] In the opposite case, for small eddies with $\tau_p\gamma(k)\gg 1$, the operator is close to unity. Both limiting cases one easily gets from the first term in the Taylor expansion of Eq. (A30) in which there is no contribution from $L$. This means that only for intermediate scales with $\tau_p\gamma(k)\sim 1$ this operator $L$ may be quantitatively important. For a qualitative description of the transient process between these two regimes we propose to take into account only the first term of the expansion of Eq. (A30). A more detailed study of this simplification is necessary. With this approximation Eq. (A29) becomes

$$\rho_f\left(1+\frac{\phi}{1+\tau_p\frac{\partial}{\partial t}}\right)\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}+\boldsymbol{\nabla}p = \rho_f\nu\nabla^2\mathbf{u}+\mathbf{f}-\left(\frac{\rho_f\phi\Delta\rho/\rho_p\tau_p\mathbf{g}\cdot\boldsymbol{\nabla}}{1+\tau_p\frac{\partial}{\partial t}}\right)\mathbf{u}(t,\mathbf{r})+\frac{\Delta\rho}{\rho_p}\rho_f\phi\mathbf{g}. \quad (A31)$$

In the derivation of Eq. (A31) from Eq. (A29) we assume that the operators $[1+\phi/(1+\tau_p\partial/\partial t)]$ and $(\partial/\partial t+\mathbf{u}\cdot\boldsymbol{\nabla})$ commute. This assumption needs further study. Introducing the terminal velocity of the particle

$$\mathbf{v}_{\mathrm{tv}} = \frac{\Delta\rho}{\rho_p}\tau_p\mathbf{g}, \quad (A32)$$

Eq. (A31) becomes

$$\rho_f\left(1+\frac{\phi}{1+\tau_p\frac{\partial}{\partial t}}\right)\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}+\boldsymbol{\nabla}p = \rho_f\nu\nabla^2\mathbf{u}+\mathbf{f}-\left(\frac{\rho_f\phi\mathbf{v}_{\mathrm{tv}}\cdot\boldsymbol{\nabla}}{1+\tau_p\frac{\partial}{\partial t}}\right)\mathbf{u}(t,\mathbf{r})+\frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\mathrm{tv}}, \quad (A33)$$

which can also be written as

$$\rho_f\left(1+\frac{\phi}{1+\tau_p\frac{\partial}{\partial t}}\right)\left(\frac{\partial}{\partial t}+(\mathbf{u}+\mathbf{v}_{\mathrm{tv}})\cdot\boldsymbol{\nabla}\right)(\mathbf{u}+\mathbf{v}_{\mathrm{tv}})+\boldsymbol{\nabla}p = \rho_f\nu\nabla^2(\mathbf{u}+\mathbf{v}_{\mathrm{tv}})+\mathbf{f}+\frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\mathrm{tv}}. \quad (A34)$$

As the equation of motion is invariant under a Galilean transformation (Frisch[20]), we may choose as a new reference system a coordinate system that is moving with velocity $\mathbf{v}_{\mathrm{tv}}$ with respect to the original one. With respect to this new reference system the equation of motion then becomes

$$\rho_f\left(1+\frac{\phi}{1+\tau_p\frac{\partial}{\partial t}}\right)\left(\frac{\partial}{\partial t}+\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u}+\boldsymbol{\nabla}p = \rho_f\nu\nabla^2\mathbf{u}+\mathbf{f}+\frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\mathrm{tv}}. \quad (A35)$$

The terms on the left-hand side and the first two terms on the right-hand side of Eq. (A35) are identical to the effective Navier-Stokes equation derived by L'vov, Ooms, and Pomyalov.[6] However, the last term on the right-hand side is new. It represents the force on the fluid due to the settling of the particles with respect to the fluid as the particle density is different from the fluid density. Since we are interested in incompressible flows, we can project the potential components out of the equation of motion. This is done by means of the projection operator $\mathbf{P}$, defined via its kernel $P^{\alpha\beta}(\mathbf{r})$:

$$P^{\alpha\beta}(\mathbf{r}) \equiv \int \frac{d^3k}{(2\pi)^3} P^{\alpha\beta}(\mathbf{k})\exp[-i\mathbf{k}\cdot\mathbf{r}], \tag{A36}$$

with

$$P^{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta / k^2. \tag{A37}$$

Applying $\mathbf{P}$ to Eq. (A35) we find

$$\rho_{\text{eff}}(t)\left(\frac{\partial}{\partial t} + \mathbf{P}\cdot\mathbf{u}\cdot\boldsymbol{\nabla}\right)\mathbf{u} = \rho_f \nu \nabla^2 \mathbf{u} + \mathbf{f} + \frac{\rho_f \phi}{\tau_p}\mathbf{v}_{\text{tv}}, \tag{A38}$$

where $\rho_{\text{eff}}(t)$ may be considered as the effective density of the suspension, given by

$$\rho_{\text{eff}}(t) \equiv \rho_f\left(1 + \frac{\phi}{1 + \tau_p \frac{\partial}{\partial t}}\right). \tag{A39}$$

For the coming derivation of the balance equation for the turbulent kinetic energy of the suspension it is convenient to Fourier transform Eq. (A38) in space and time. In $\mathbf{k}$, $\omega$ representation the equation becomes

$$\begin{aligned} &i\omega\tilde{\rho}_{\text{eff}}(\omega)\tilde{\mathbf{u}}(\omega,\mathbf{k}) + \tilde{\mathbf{N}}\{\mathbf{u},\mathbf{u}\}_{\omega,\mathbf{k}} \\ &\quad = -\rho_f \nu k^2 \tilde{\mathbf{u}}(\omega,\mathbf{k}) + \tilde{\mathbf{f}}(\omega,\mathbf{k}) + \frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\text{tv}}\delta_{(\omega=0,\mathbf{k}=\mathbf{0})}, \end{aligned} \tag{A40}$$

in which the quantities with a tilde are Fourier transformed quantities; for instance,

$$\tilde{\mathbf{u}}(\omega,\mathbf{k}) = \int dt d\mathbf{r}\, \mathbf{u}(t,\mathbf{r})\exp(i\omega t + i\mathbf{k}\cdot\mathbf{r}). \tag{A41}$$

$\tilde{\rho}_{\text{eff}}(\omega)$ is given by

$$\tilde{\rho}_{\text{eff}}(\omega) = \rho_f\left(1 + \frac{\phi}{1 + i\omega\tau_p}\right). \tag{A42}$$

$\tilde{\mathbf{N}}\{\mathbf{u},\mathbf{u}\}_{\omega,\mathbf{k}}$ represents the Fourier transform of the nonlinear term in Eq. (A38):

$$\tilde{\mathbf{N}}\{\mathbf{u},\mathbf{u}\}_{\omega,\mathbf{k}} \equiv [\tilde{\rho}_{\text{eff}}(\omega)\mathbf{P}\cdot(\tilde{\mathbf{u}}\cdot\boldsymbol{\nabla})\tilde{\mathbf{u}}]_{\omega,\mathbf{k}}. \tag{A43}$$

The term $\rho_f[1+\phi/(1+i\omega\tau_p)]$ is now split in its real and imaginary parts. This allows Eq. (A40) to be written as follows:

$$\begin{aligned} &i\omega\left[\tilde{\rho}'_{\text{eff}}(\omega) - i\frac{\rho_f}{\omega}\tilde{\gamma}_p(\omega)\right]\tilde{\mathbf{u}}(\omega,\mathbf{k}) + \tilde{\mathbf{N}}\{\mathbf{u},\mathbf{u}\}_{\omega,\mathbf{k}} \\ &\quad = -\rho_f \nu k^2 \tilde{\mathbf{u}}(\omega,\mathbf{k}) + \tilde{\mathbf{f}}(\omega,\mathbf{k}) + \frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\text{tv}}\delta_{(\omega=0,\mathbf{k}=\mathbf{0})}, \end{aligned} \tag{A44}$$

in which

$$\tilde{\rho}'_{\text{eff}}(\omega) = \rho_f\left(1 + \frac{\phi}{1 + (\omega\tau_p)^2}\right) \tag{A45}$$

and

$$\tilde{\gamma}_p(\omega) = \left(\frac{\phi\omega^2\tau_p}{1 + (\omega\tau_p)^2}\right). \tag{A46}$$

Equation (A44) can be considered as a one-fluid equation of motion for the suspension. It involves a frequency-dependent effective density $\tilde{\rho}'_{\text{eff}}(\omega)$ of the suspension and an additional frequency-dependent internal friction coefficient $\tilde{\gamma}_p(\omega)$ for the suspension due to the friction between the particles and the fluid. For the derivation of the balance equation for the turbulent kinetic energy of the suspension we will use standard closure procedures for the statistical description of turbulence. To that purpose we need frequency-independent coefficients in the equation of motion. The closure procedures may be applied to an equation of motion with wave-number-dependent coefficients. Therefore for the further analysis we will use the wave-number-dependent effective density $\rho_{\text{eff}}(k)$ and the wave-number-dependent friction coefficient $\gamma_p(k)$ as derived from $\tilde{\rho}'_{\text{eff}}(\omega)$ and $\tilde{\gamma}_p(\omega)$ by L'vov, Ooms, and Pomyalov.[6] (As this derivation is described in detail by L'vov *et al.*, it will not be repeated here.) We realize very well that the derivation has been carried out under the assumption of a homogeneous and isotropic turbulent suspension. Due to the effect of gravity the turbulent flow is not isotropic for the case studied in the present publication. We think, however, that for a qualitative description of the effect of particle inertia and gravity on the turbulence in a suspension the expressions derived in Ref. 6 are sufficient. It is important to realize that the occurrence of anisotropy due to the gravity effect in case of a difference in density between the particles and the fluid remains included in our model via the "gravity term" in Eq. (A44). Using the wave-number-dependent coefficients, Eq. (A44) becomes

$$\begin{aligned} &i\omega\left(\rho_{\text{eff}}(k) - i\frac{\rho_{\text{eff}}(k)}{\omega}\gamma_p(k)\right)\tilde{\mathbf{u}}(\omega,\mathbf{k}) + \tilde{\mathbf{N}}\{\mathbf{u},\mathbf{u}\}_{\omega,\mathbf{k}} \\ &\quad = -\rho_f \nu k^2 \tilde{\mathbf{u}}(\omega,\mathbf{k}) + \tilde{\mathbf{f}}(\omega,\mathbf{k}) + \frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\text{tv}}\delta_{(\omega=0,\mathbf{k}=\mathbf{0})}, \end{aligned} \tag{A47}$$

in which

$$\rho_{\text{eff}}(k) = \rho_f\left(1 + \phi\frac{[1 + 2\tau_p\gamma(k)]}{[1 + \tau_p\gamma(k)]^2}\right) \tag{A48}$$

and

$$\gamma_p(k) = \frac{\phi\tau_p[\gamma(k)]^2}{(1+\phi)[1 + 2\tau_p\gamma(k)] + [\tau_p\gamma(k)]^2}. \tag{A49}$$

As mentioned earlier, $\gamma(k)$ is the frequency of an eddy of size $1/k$. Fourier-transforming Eq. (A47) back to time we finally find the one-fluid equation of motion in the form that we need to derive the balance equation for the turbulent kinetic energy:

$$\begin{aligned} &\rho_{\text{eff}}(k)\left(\frac{\partial}{\partial t} + \gamma_p(k) + \gamma_0(k)\right)\mathbf{u}(t,\mathbf{k}) \\ &\quad = -\mathbf{N}\{\mathbf{u},\mathbf{u}\}_{t,\mathbf{k}} + \mathbf{f}(t,\mathbf{k}) + \frac{\rho_f\phi}{\tau_p}\mathbf{v}_{\text{tv}}\delta_{(\mathbf{k}=\mathbf{0})}, \end{aligned} \tag{A50}$$

where

$$\gamma_0(k) = \nu_{\text{eff}}(k)k^2, \quad \nu_{\text{eff}}(k) = \frac{\nu\rho_f}{\rho_{\text{eff}}(k)}. \tag{A51}$$

As shown in Ref. 6, the nonlinear term is given by

$$N\{\mathbf{u},\mathbf{u}\}^{\alpha}_{\mathrm{t,k}} = \int \frac{d^3\mathbf{k}_1 d^3\mathbf{k}_2}{(2\pi)^3}\Gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} u^*_{\beta}(t,\mathbf{k_1})u^*_{\gamma}(t,\mathbf{k_2}), \tag{A52}$$

in which the vertex $\Gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ is equal to

$$\Gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \rho_{\mathrm{eff}}(k)\left(\frac{2k_1k_2k_3}{k_1^2+k_2^2+k_3^2}\right)\frac{\gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}}{\rho_f}, \tag{A53}$$

where $\gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2}$ is the standard vertex of the Navier-Stokes equation

$$\gamma^{\alpha\beta\gamma}_{\mathbf{k}\mathbf{k}_1\mathbf{k}_2} = \frac{\rho_f}{2}[P^{\alpha\beta}(\mathbf{k})k^{\gamma} + P^{\alpha\gamma}(\mathbf{k})k^{\beta}]\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2). \tag{A54}$$


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