# The Hydrodynamic Limit of The Random Waiting Time Model and The Fractional Kinetics Process

by

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# Abstract

In this thesis, the hydrodynamic limit (HDL) for two trapping models is studied, the Random Waiting Time Model (RWTM) and the fractional kinetics process (FKP), on a discrete lattice $\mathbb{Z}^d$. The RWTM is studied for dimension $d \geq 1$ and $\mathbb{E}[w_i], \infty$ where $w_i$ denotes the waiting time at position $i$. On the other hand, the FKP is studied for $d \geq 3$ and a nonexisting first finite moment. Instead, it is assumed that the waiting times $w_i$ follow a power law distribution. Two main results are presented in this thesis. Firstly, it is proven that the HDL for the RWTM converges to the solution of the heat equation. The solution is deterministic. The proof consists of showing that the expectation value of the empirical density fields converge to the aforementioned solution by using the duality property and Doob's theorem, and that the variance is finite and decays to 0. Additionally, it is proven that a rescaled random walk converges to a Brownian motion by using Lévy's characterisation of Brownian motion.

Secondly, it is proven that the HDL for the FKP converges to a random measure of the solution of the fractional heat equation, defined in the Caputo sense. Hence, the solution is random. The proof consists of using similar techniques of the proof of the RWTM and of using that the limit of a sequence of random speed measures is again random.

Before all of this, an introduction to Markov processes, their semigroups, martingales, and generators is presented. Additionally, an introduction to random walks, Brownian motion, and duality is included.

# Preface

The process of composing an undergraduate thesis is frequently underestimated and often regarded as less enjoyable within a student's academic pursuits. Long discussions with my fellow student indicate that most students tend to be more comfortable with coursework and examinations rather than undertaking the task of producing a scholarly work. However, my personal experience diverges from this prevailing perception. Upon completing this thesis, I have arrived at a distinct conclusion. Despite not having obtained groundbreaking mathematical results, the endeavour of writing this thesis has proven to be immensely fruitful in terms of deepening my understanding of the thesis topic. Additionally, and of greater significance, I have discovered a genuine affinity for writing scientific papers, an attribute that holds importance as I embark on my forthcoming master's degree. It is my sincere hope that this initial thesis will mark the beginning of a series of scientific papers to be written in the future.

In conclusion, I would like to extend my appreciation to Professor Frank Redig for his invaluable guidance and supervision throughout the past few months. It is evident that this thesis would not have come to fruition without his support.

*Floris Fokker Delft, June 2013*

# Contents

# 1
# Introduction

There is geometry in the humming of the strings. There is music in the spacing of the spheres.
Pythagoras

## 1.1. Particle Behaviour and Statistical Mechanics

Our surrounding world is composed of minuscule constituents, ranging from the bricks employed in constructing our universities to the intricate structures of molecules, atoms, and electrons. The Ancient Greek philosopher Democritus, who lived in the fifth century BCE, articulated the idea that atoms were ''indivisible,' [...] infinite in number and various and shape, and perfectly solid, with no internal gaps. They move about in an infinite void, repelling one another when they collide or combining into clusters by means of tiny hooks and barbs on their surfaces, which become entangled [6]. Since then, a multitude of philosophers and physicists have pondered the existence of fundamental building blocks, referred to as 'elementary particles', as the indivisible constituents that constitute the fundamental essence of nature.

An influential proponent of this concept was the English chemist and physicist John Dalton, who, through his research on gas behaviour and molecular interactions, postulated the existence of discrete, element-specific building blocks coined 'atoms'. Many of his contemporaries regarded his proposition as an embellishment of nature, deeming it excessively intricate and dismissing it as an act of sheer imprudence [9]. Nonetheless, subsequent scientific advancements have partially corroborated Dalton's conjecture.

Toward the end of the 19th century, the discovery of the electron by J. J. Thomson and subsequent discoveries revealed that Dalton's atoms were not the fundamental constituents of matter. Instead, atoms were found to encompass yet smaller constituents, exemplified by the electron, proton, and neutron.[1]

The domain of elementary particle physics has undergone significant transformations owing to fundamental discoveries such as Einstein's Theory of Relativity and the emergence of Quantum Mechanics. These advancements have rendered the field of (elementary) particle physics a diverse, innovative, and exhilarating realm of scientific research, enabling explorations into the intricate tapestry of the universe.

Diverse approaches exist to characterize the microscale dynamics of particles, including electrons, protons, and larger entities like molecules. While the interesting phenomena that govern the behaviour of individual particles are of great interest, there is also compelling need to investigate the collective behaviour of systems composed of these entities. Statistical Physics serves as a fundamental physical discipline to analyse macroscopic quantities such as temperature, energy, and structure based on the underlying microscopic laws that govern the behaviour of individual entities. Einstein, widely recognised by the public for his contributions to the Theory of Relativity, was awarded the Nobel Prize for his groundbreaking work on elucidating the behaviour of a system that comprise particles exhibiting *Brownian Motion*[9]. This exemplifies the interdisciplinary nature of Statistical Physics, bridging the gap between microscopic and macroscopic phenomena. Equilibrium Statistical Physics is devoted to understanding the transition from microscopic laws to macroscopic laws, known as *phase transitions*. In contrast, Non-Equilibrium Statistical Physics focuses on macroscopic transport phenomena such as particle transport and heat conduction, drawing insights from the microscopic behaviour of the individual particles. Such investigations often explore the long-term behaviour of a system or the limit of the system as the number of particles tends towards infinity.

## 1.2. Interacting Particle Systems, the Hydrodynamic Limit and Duality

One of the examples of Non-Equilibrium Statistical Physics is the study and modelling of individual particle behaviour through Interacting Particle System (IPS). While a witty student in Year 10 might correctly identify Newton's laws as the logical and straightforward approach to describing the motion of a single particle, Non-Equilibrium Statistical Physics takes a different perspective.

Instead of adhering to Newton's laws, we assume that particles follow stochastic Markovian dynamics. The particle's behaviour is akin to a random walk on a given state space. An elementary example of this model is the movement of i.i.d. particles on a one-dimensional integer line with a certain probability. A comprehensive introduction to this problem can be found in [39].

By introducing an additional 'layer' of randomness, such as assigning specific depths to traps placed at each site on the integer line, we obtain a model that encompasses two forms of randomness. This scenario is referred to as a 'Random Walk in a Random Environment'. This thesis will focus on a specific type of model that is called the Random Waiting Time Model (RWTM)(2). Instead of assuming that the particles are i.i.d., one can also assume that the particles do interact and that the number of particles per site is limited, a model referred to as the Simple (Symmetric) Exclusion Process (S(S)EP) and that was introduced by [40] and further studied by e.g. [28, 34]. Another extensively studied model is the Simple Inclusion Process (SIP), where particles attract each other and the number of particles per site is unbounded. This model was introduced by [16] and further investigated by e.g. [23, 16].

After describing the microscopic behaviour of the particles, the objective is to describe the behaviour of macroscopic quantities. For instance, consider a collection of particles that evolve in a $d$-dimensional volume $V$. If all equilibrium states of the systems are characterised by a macroscopic parameter $\rho$ (e.g. the density or temperature) within some set $P$, deviations from macroscopic equilibrium are expected to occur in small regions around each macroscopic point $x$. These deviations, known as 'local equilibrium', can be described by $\rho(x)$. For $t \geq 0$, the local equilibrium can change, leading to a local equilibrium $\rho(x,t)$. The expectation is that this parameter changes smoothly in macroscopic time and space, following a differential equation known as the hydrodynamic limit (HDL) [24]. In this thesis, making the (local) particle density $\rho(x,t)$ the considered parameter. A scaling parameter $\theta_N$ is introduced, representing the ratio between the microscopic and macroscopic length scales. The hydrodynamic limit can be viewed as a Law of large numbers for the time evolution, usually described by a partial differential equation (PDE), of empirical density fields in interacting particle systems. However, in many existing results, only a Weak Law of large numbers is established, wherein the limit is shown in probability with respect to the law of the process [3].

To derive the HDL, we employ the duality property. Specifically, for symmetric[1] versions of the aforementioned models, we translate the problem of infinitely many particles to of a single particle, simplifying the computational aspects. Duality with respect to a function first appeared in the literature in the late 1940s and early 1950s in [27]. The concept of duality finds widespread applicability across various scientific domains, encompassing diverse fields such as interacting particle systems, population genetics models, queueing theory, and stochastic partial differential equations (SPDEs). Notably, a fundamental characteristic of these processes involves the significance of temporal perspectives in both forward and backward directions. For further exploration of this topic, we refer the interested reader to [20, 30].

[1] The probabilities of jumping to all neighbouring sites is equal.

## 1.3. The Hydrodynamic Limit: Applications

This section delves into a more profound exploration of the physical applications associated with the hydrodynamic limit. Among the multitude of contexts in which the HDL finds utility, one prominent example lies in its application to the investigation of impurities. In each case, the reader is referred to the associated papers of the given examples.

One of these applications is the *dissipative linear Boltzmann equation* that describes the dynamic of a set of particles with mass $m_1$ interacting inelastically with a background gas in thermodynamical equilibrium composed of particles with mass $m \ll m_1$. For instance, the case of fine polluting impurities interacting with air or another gas. the only conserved quantity is the number of inelastic particles and as a result, a conventional hydrodynamic approach is justified. This approach results in a single equation describing the advection (the transport of a substance or quantity by bulk motion of a fluid) of inelastic particles at the velocity of the background. The reader is referred to [10].

The HDL approach is also employed in the analysis of Hamiltonian systems exhibiting a superstable pairwise potential, leading to the emergence of stochastic dynamics through the introduction of a noise term that facilitates the exchange of momenta between adjacent particles.In the scaling limit, the time conserved quantities, energy, momenta and density, satisfy the Euler equation of conservation laws[2] up to a fixed time $t$ provided that the Euler equation has a smooth solution with a given initial data up to time $t$. The strength of the noise term is chosen to be very small (but nonvanishing) so that it disappears in the scaling limit. The reader is referred to [31].

Another example is the Anderson impurity model that is a Hamiltonian that is used to describe magnetic impurities embedded in metals. The model is an IPS on $\mathbb{Z}^d$ where each site of the lattice is allowed to contain at most one particle, and particles could jump to an empty neighbouring site only under a certain constraint, conserving the total number of particles. More precisely, depending on an integer parameter $n$, every particle jumps with rate 1 to each of its neighbouring sites, provided that the particle has at least $n$ empty neighbours both before and after the jump (so for $k = 1$ we obtain the SSEP). The interested reader is referred to [38].

Prior to presenting and comprehending the principal findings, we shall look at some fundamental definitions and theorems. This preliminary discourse serves as a crucial foundation for establishing a solid conceptual framework and facilitating a thorough understanding of the subsequent results.

## 1.4. Reading Overview

The thesis is structured as follows. Section 1.5 introduces the mathematical preliminaries that serve as the foundation for this thesis. Sections 1.6, 1.7, 1.8, 1.9, and 1.10 delve into the necessary mathematical background.

Chapter 2 focuses on the Bouchaud Trap Model (2.2) and the derivation of the hydrodynamic limit (HDL) for the Random Waiting Time Model (2.3). Section 2.4 presents the derivation of the HDL for the fractional kinetics process. Lastly, chapter 3 concludes this thesis with some final remarks.

[2]The Euler equations are a set of quasilinear partial differential equations governing adiabatic and inviscid (zero-viscosity) flow. The interested reader is referred to [5]

## 1.5. Mathematical Preliminaries

In this section, we will discuss pertinent mathematical definitions and outcomes to acquaint the reader with the mathematical fundamentals of this thesis. It is important to note that this thesis falls within the realm of Markov processes and their associated findings. To adequately establish these findings, we define Markov processes as sequences of random variables that evolve on probability and measurable spaces. Hence, the mathematical framework employed in this thesis must align with the concepts found in real analysis, such as measures, measurable spaces, filtrations, and adaptations, among others. Moreover, various functions require well-defined domains and path spaces, such as Skorokhod spaces. However, for two reasons, this formalism will not be employed in this thesis. Firstly, delving into this formalism would introduce notions beyond the scope of this thesis, necessitating extensive discussions of topology and real analysis. Secondly, this rigorous mathematical formalism is not a prerequisite for deriving the key results of this thesis, as it primarily concerns Markov chains rather than their underlying definitions in real analysis or topology. Nonetheless, readers are encouraged to explore equivalent or more abstract definitions related to the definitions that will be presented.

## 1.6. Preliminaries of Probability Theory

**Definition 1.6.1** (Random Variable)**.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(E, \mathscr{E})$ a measurable space that is called the state space. Then an $(E, \mathscr{E})$-valued random variable is a measurable function $X : \Omega \to E$. $\Omega$, $\mathscr{F}$, and $P$ are called the sample space, event space, and probability function, respectively.*
*The probability that $X$ takes on a value in a measurable subset $S \subseteq E$ is given by $\mathbb{P}(X \in S) = \mathbb{P}(\{\omega \in \Omega | X(\omega) \in S\})$.*

**Remark 1.6.1.** *Throughout this thesis, we assume that $E = \mathbb{N}$ . The state space $(E, \mathscr{E})$ is known as a discrete state space. In this particular scenario, $\mathscr{E} = \mathscr{P}(\mathbb{N})$ where $\mathscr{P}$ is the power set.*

**Definition 1.6.2** (Stochastic Process)**.** *Let $X(t)$ be a random variable with index $t \in T = [0,\infty)$. Then the stochastic process is denoted by $(X(t), t \geq 0)$. In simpler terms, a stochastic process is defined as a collection of random variables that is indexed by some mathematical set. Each random variable is uniquely associated with an element within that set.*

**Definition 1.6.3** (Filtration and Adaption)**.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space and $(T, \mathscr{T})$ a time space,*

- *if $T$ is discrete ($T = \mathbb{N}$), then a non-decreasing sequence $(\mathscr{F}_n)_{n\in\mathscr{N}}$ of $\sigma$-algebras of $\mathscr{F}$,*

$$\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq ... \subseteq \mathscr{F}_n \subseteq ... \tag{1.1}$$

  *is called a discrete filtration*

- *if $T$ is continuous ($T = [0,\infty)$), then a non-decreasing sequence $(\mathscr{F}_t)_{t\in T}$ of $\sigma$-algebras of $\mathscr{F}$,*

$$\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F} \tag{1.2}$$

*Intuitively, the filtration $\mathscr{F}_t$ can be understood as the information available up to time $t$. Given a filtration $\mathscr{F}_t$, the space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in T}, P)$ is referred to as a filtered probability space. If a sequence $(X_t)_{t\in T}$ of random variables satisfies the following property: $X_t$ is $\mathscr{F}_t$-measurable for all $t \in T$. Then the sequence is said to be adapted to $(\mathscr{F}_t)_{t\in T}$.*

We now present the definition of a so-called *stopping time* that will be used in subsequent sections.

**Definition 1.6.4** (Stopping Time)**.** *Let $T : \Omega \to T$ be a random variable that is defined on a filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in T}, P)$. $T$ is called a stopping time with respect to $(\mathscr{F}_t)_{t\in T}$ if,*

$$\{T \leq t\} \in \mathscr{F} \tag{1.3}$$

**Remark 1.6.2.** *Henceforth, our focus shall be confined to the continuous adaptations of the aforementioned definitions. It is important to acknowledge that the forthcoming definitions and propositions, delineated within this chapter, possess discrete as well as continuous counterparts. Considering that the substantive content of this thesis is reliant upon the continuous formulations, our emphasis shall be placed upon those particular formulations.*

## 1.7. Markov Processes

Let us commence by introducing some nomenclature. For a countable set $E$, we call

- $\mathscr{B}(E,\cdot)$ the space of bounded and measurable real valued functions on $E$;
- $\mathscr{C}_0(E,\cdot)$ the space of continuous real valued functions on $E$ that vanish at infinity;
- $\mathscr{C}^n(E,\cdot)$ the space of continuous real valued functions on $E$ that are $n$-times differentiable;
- $\mathscr{C}_b(E,\cdot)$ the space of continuous real valued functions on $E$ that are bounded;
- $\mathscr{C}^\infty(E,\cdot)$ the space of continuous real valued functions on $E$ that are infinitely differentiable.
- $S(\mathbb{R}^n)$ the *Schwartz space* or space of rapidly decreasing functions on $\mathbb{R}^n$ that consists of all functions whose derivatives are rapidly decreasing, i.e. $S(\mathbb{R}^n) := \{f \in \mathscr{C}^\infty(\mathbb{R}^n) | \forall \alpha,\gamma \in \mathbb{N}^n, \|f\|_{\alpha,\gamma} < \infty\}$, with $\mathbb{N}^n := \underbrace{\mathbb{N}\times\cdots\times\mathbb{N}}_{n \text{ times}}$ the $n$-fold Cartesian product and

$$\|f\|_{\alpha,\gamma} := \sup_{x\in\mathbb{R}^n} \left| x^\alpha (D^\beta f)(x) \right|. \tag{1.4}$$

  where we use the notation $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ and $D^\gamma := D_1^{\gamma_1} D_2^{\gamma_2} \cdots D_n^{\gamma_n}$.

  Hence, a function $f$ in Schwartz space could be considered as a rapidly decreasing function $f(x)$ such that all its (infinitely many) derivatives exist everywhere on $\mathbb{R}^n$ and decay to zero for $x \to \pm\infty$ faster than any power of $x$.

where $'\cdot'$ denotes the codomain of the function.

Let us furthermore note that $\mathscr{C}_0(E) \subseteq \mathscr{B}(E)$. Both these spaces are equipped with the supremum norm $\|\phi\|_\infty := \sup_{x\in E}|\phi(x)|$ with $\phi \in \mathscr{B}(E)$.

**Example 1.7.1.** *The Gaussian function $g(x) := p(x)e^{-|x|^2}$, with $p(x)$ a polynomial, belongs to $S(\mathbb{R}^n)$. More generally, the space of all functions that have a compact support on a space $\mathbb{R}^n$, i.e. the functions whose closed support is a compact subset of $\mathbb{R}^n$, is contained in $S(\mathbb{R}^n)$.*

*In contrast, the function $h : \mathbb{R} \to \mathbb{R}$ given by*

$$h(x) = e^{-x^2} \sin\left(e^{x^2}\right) \tag{1.5}$$

*does not belong to $S(\mathbb{R})$ as $h'(x)$ does not decay to zero for $|x| \to \infty$.*

**Remark 1.7.1** (Skorokhod Space and Càdlàg Functions)**.** *The natural space for trajectories of $E$-valued Markov processes is referred to as the Skorokhod space. This space consists of that are càdlàg on a given domain. A function $g$ is said to be càdlàg if it is right-continuous and has a left limit.*[3] *The Skorokhod space is denoted as $\mathbb{D}(E,M)$ and consists of all càdlàg functions from $E$ to $M$. In our particular context, we define $M = \mathbb{R}{\geq}\, 0$, yielding the Skorokhod space $\mathbb{D} = \mathbb{D}(E,\mathbb{R}_{\geq 0})$.*

**Definition 1.7.1** (Markov process)**.** *Let $(\Omega, \mathscr{F}, P)$ be a probability space, $(E,\mathscr{E})$ a measurable space and $(T,\mathscr{T})$ a time space (where $\mathscr{T}$ is the $\sigma$-algebra on $T$). Let $\boldsymbol{X} = \{X_t : t \geq 0\}$ be a sequence of random variables. If,*

- *The paths $t \mapsto X_t$ are right-continuous; and*
- *The process satisfies the Markov property with respect to $\sigma(X_s, 0 \leq s \leq t)$.*

*where $\boldsymbol{\mu} = \{\mu_x : x \in E\}$ is a family of probability measures on $\mathbb{D} = \mathbb{D}(E,\mathbb{R}_{\geq 0})$. Then we call $\boldsymbol{X} = \{X_t : t \geq 0\}$ a Markov process on state space $(E,\mathscr{E})$.*

[3] Càdlàg is a French abbreviation for *continue à droite, limite à gauche*. For a comprehensive examination of càdlàg functions, we refer the reader to [37].

**Definition 1.7.2** (Markov Property)**.** *Let* $(\Omega, \mathscr{F}, P)$ *be a probability space with a continuous filtration* $(\mathscr{F}_t, t \in T)$ *and let* $(E, \mathscr{E})$ *be a measurable space. A random variable* $X : \Omega \to E$ *adapted to the filtration is said to possess the Markov property if, for every* $A \in \mathscr{E}$ *and for every* $s, t \in T$ *with* $0 \le s < t$,

$$P(X_t \in A|\mathscr{F}_s) = P(X_t \in A|X_s) \tag{1.6}$$

*Equivalently, the Markov property can be formulated as follows*

$$\mathbb{E}(f(X_t)|\mathscr{F}_s) = \mathbb{E}(f(X_t)|\sigma(X_s)) \tag{1.7}$$

*for all* $f \in \mathscr{B}(E)$. *In essence, the Markov property asserts that the probability of future states, given the present states, is solely determined by the present states themselves, rendering the history of the process leading up to the present state inconsequential and dispensable. Consequently, the knowledge of the present state provides a comprehensive description of the system's behaviour, obviating the need to consider or incorporate past states or events.*

**Remark 1.7.2.** *The time space* $(T, \mathscr{T})$ *in (1.7.1) can be either discrete* $(T = \mathbb{N})$ *or continuous* $(T = [0,\infty))$. *In the case of discrete time, the counting measure is employed, while for continuous time, the Lebesgue measure is utilized.*[4] *For further details, please refer to (1.6.2).*

**Remark 1.7.3.** *The measurable space or state space* $(E, \mathscr{E})$ *is generally assumed to be LCCB: locally compact, Hausdorff, and with a countable base. The specific details of this assumption will not be discussed within this thesis; however, interested readers are encouraged to consult [43]. Nonetheless, it should be highlighted that a state space where* $E$ *has the LCCB topology and* $\mathscr{E}$ *is the Borel* $\sigma$*-algebra, implies that, under the assumption that there is a positive measure* $\lambda$ *on the state space,* $\lambda$ *will be a Borel measure with* $\lambda(C) < \infty$ *for* $C \subseteq E$ *compact.*

**Definition 1.7.3.** *(Continuous-Time Markov Chain) Let* $(\Omega, \mathscr{F}, P)$ *be a probability space,* $(E, \mathscr{E})$ *a discrete state space and* $(T, \mathscr{T})$ *a continuous time space. Furthermore, let* $\{X_t : t \in T\}$ *be a Markov process in the sense of (1.7.1). Then* $\{X_t : t \in T\}$ *is called a continuous-time Markov chain (CTMC) and defined by*

- *a probability vector* $\mu$ *on* $E$*, that is interpreted as the initial distribution;*
- *a rate matrix* $Q$ *on* $E$*, that is, a function* $Q : E^2 \to \mathbb{R}$ *such that*
  - *for all* $i, j \in E$*, with* $i \ne j$, $Q_{i,j} \ge 0$*; and,*
  - *for all* $i \in E$, $\sum_{j\in E: j\ne i} Q_{i,j} = -Q_{i,i}$.

**Remark 1.7.4.** *There exist three distinct methods for appropriately defining* $\{X_t : t \ge 0\}$ *as a Markov chain with an initial distribution* $\lambda$ *and rate matrix* $Q$*. The first and most intuitive method, from a probabilistic standpoint, involves defining* $X_t$ *using holding times and embedded discrete-time chains. The second method employs transition probability matrices in continuous time, which, although less intuitive than the first approach, aligns closely with the treatment of many Markov processes. The third method entails defining* $X_t$ *through the use of potential matrices, which are transformations of the transition matrices. Although the least intuitive of the three approaches, it offers analytical advantages. Since the second method encompasses concepts such as semigroups and generators, which are further utilized in this thesis, it will be expounded upon in this chapter. It is worth noting that introductory books on Markov chains, such as [8], as well as the exceptional notes by [18], are recommended resources for readers seeking a comprehensive understanding of these concepts.*

### 1.7.1. Transition Probability Matrices

Unless explicitly specified, we consider $\boldsymbol{X} = \{X_t : t \in T\}$ to be a continuous-time Markov chain in accordance with the formulation provided in (1.7.3). The CTMC is defined on a given probability space $(\Omega, \mathscr{F}, P)$, with the state space $(E, \mathscr{E})$ being discrete and the time space $(T, \mathscr{T})$ being continuous.

**Definition 1.7.4** (Transition probability matrix)**.** *Let* $\boldsymbol{X} = \{X_t : t \in T\}$ *be a CTMC on* $E$*. Then the matrix* $P_t$ *of* $\boldsymbol{X}$ *corresponding to* $t$ *is called the transition probability matrix with entries given by*

$$P_t(x, y) = \mathbb{P}(X_t = y|X_0 = x), \qquad (x, y) \in E^2 \tag{1.8}$$

[4] The distinction in measures for these time spaces stems from their respective topologies: the discrete time space possesses the discrete topology, while the continuous time space adheres to the Euclidean topology.

*Particularly, $P_0 = I$ is the identity matrix on $E$. If the matrix $P_t$ does not depend on time, the Markov chain $\boldsymbol{X}$ is called time homogeneous. We can write*

$$P_t(x,y) = \mathbb{P}(X_{s+t} = y | X_s = x), \qquad (x,y) \in E^2, s \in T \tag{1.9}$$

**Theorem 1.7.1** (Markov Operator)**.** *Let $\boldsymbol{P} = \{P_t : t \in T\}$ be the family of transition matrices $P_t$ of the Markov chain $\boldsymbol{X}$. Suppose that $f : E \to \mathbb{R}$ is either non-negative or $f \in \mathscr{B}(E)$. Then for all $t \in T$*

$$P_t f(x) = \sum_{y \in E} P_t(x,y) f(y) = \mathbb{E}[f(X_t)|X_0 = x], \qquad x \in E \tag{1.10}$$

*The mapping $f \mapsto P_t f$ is a bounded, linear operator on $\mathscr{B}(E)$ and $\|P_t\| = 1$. $Pf$ defines a Markov operator $P : E \to E$.*

*Proof.* The reader is referred to [8]. □

**Remark 1.7.5.** *If the state space $(E, \mathscr{E})$ possesses a positive measure $\mu$, then the function $f : E \to [0,\infty)$ defined as $f(x) = \mu_x$ for $x \in E$ is referred to as the density function with respect to the counting measure on $(E, \mathscr{E})$. Consequently, for any subset $A \subseteq E$, we can express $\mu(A)$ as $\sum_{x \in A} f(x)$.*

**Theorem 1.7.2.** *Let $f : E \to [0,\infty)$ be the density function of a positive measure $\mu$ on the state space $(E, \mathscr{E})$. Then, $fP_t$ is the density function of the measure $\mu P_t$ given by*

$$\mu P_t(A) = \sum_{x \in A} \mu\{x\} P_t(x,A) = \sum_{x \in A} f(x) P_t(x,A), \qquad A \in E \tag{1.11}$$

*Proof.* The proof follows by conditioning on $x$. □

### 1.7.2. Semigroups and Generators

Let $\boldsymbol{P} = \{P(t) : t \geq 0\}$ be a family of Markov operators that satisfies the following conditions:

1. $P(0) = \mathrm{Id}$; and
2. $P(t+s) = P(t)P(s)$ for $s, t \geq 0$; and
3. for all $f : E \to [0,\infty)$ the function $t \mapsto P(t)f$ is continuous.

Then $\boldsymbol{P}$ is called a *Markov semigroup.*

We commence by introducing another type of semigroup

**Definition 1.7.5.** *Let $(S(t))_{t \in T}$ be a (strongly) continuous semigroup on space $E$.*[5] *The operator*

$$A\phi := \lim_{t \to 0+} \frac{S(t)\phi - \phi}{t}, \quad x \in D(A) := \left[\phi \in Y : \exists \lim_{t \to 0+} \frac{S(t)\phi - \phi}{t}\right] \tag{1.12}$$

*is called an infinitesimal generator of $(S(t))_{t \in T}$.*

**Example 1.7.2.** *A very common example of an infinitesimal generator is the case for $\phi = \mathbf{1}$ and $S(t) = e^{tA}$. We obtain $A = \lim_{t \to 0} \frac{e^{tA} - \mathbf{1}}{t}$, where $\mathbf{1}$ is the identity matrix.*

We prove the important result:

**Corollary 1.7.3.** *If $\phi \in D(A)$ then,*

$$\frac{d}{dt} S(t)\phi = S(t)A\phi = AS(t)\phi, \qquad \forall t \in T \tag{1.13}$$

*Proof.* The proof consists of showing that the left and right derivative exists and that taking the derivative of the semigroup is closed under operation. The reader is again referred to [18]. □

[5] Properly, $E$ should be a Banach space.

### 1.7.3. Martingales

We now define the so-called Martingales.

**Definition 1.7.6** (Martingale). *Let $(\mathscr{F}_t)_{t\in T}$ be a filtration and $(X_t)_{t\in T}$ a sequence of integrable random variables. Then $(X_t)_{t\in T}$ is a $(\mathscr{F}_t)_{t\in T}$-martingale if*

1. *$(X_t)_{t\in T}$ is adapted to $\mathscr{F}$; and*
2. *$\mathbb{E}(X_t) < \infty$; and*
3. *$\forall t \in T, \mathbb{E}[X_t|\mathscr{F}_s] = X_s$ with $0 \leq s < t$ a.s.*

*In simpler terms, a martingale is a sequence of random variables for which, at a particular time, the conditional expectation of the next value in the sequence is equal to the present value, irrespective of the past values.*[6]

The notion of a martingale can be linked to the infinitesimal generator (1.7.5) through the application of Dynkin's Martingale. This connection can be formally established by utilizing Itô's Lemma, which can be conceptually perceived as the stochastic process equivalent of the chain rule.

In a more rigorous sense, the relationship between a martingale and the infinitesimal generator can be elucidated through the utilization of Dynkin's Martingale. By incorporating Itô's Lemma, one can formally demonstrate the correspondence between these concepts. Itô's Lemma serves as a powerful tool in analysing stochastic processes, allowing for the differentiation of stochastic functions while taking into account the associated dynamics. In essence, Itô's Lemma enables the extension of the chain rule to the realm of stochastic calculus, providing a framework for understanding and manipulating stochastic differential equations.

**Lemma 1.7.4** (Itô's Lemma). *Let $f \in \mathscr{C}^2(\mathbb{R})$ and $X_t$ a standard Brownian motion. Then, for all $t \in T$*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)ds \tag{1.14}$$

*Proof.* See [42]. □

**Proposition 1.7.5** (Dynkin's Martingale). *Let $f \in \mathscr{C}^2(\mathbb{R}^2, \mathbb{R})$ with compact support. Let the process $M : T \times \Omega \to \mathbb{R}$ be defined by*

$$M_t = f(X_t) - \int_0^t Af(X_s)ds \tag{1.15}$$

*with $A$ the infinitesimal generator of $X$. Then $M_t$ is a martingale with respect to the natural filtration* (A.1).

*Proof.* This follows from Itô's Lemma, e.g. [44]. The statement can also be proven via semigroups.

□

[6] The reader is invited to compare the definition of a martingale to [Definition 1.7.2], the Markov property.

## 1.8. Random Walks, Brownian Motion and Duality

Random walks are fundamental stochastic processes that involve the cumulative sum of random steps taken on a mathematical space. They have been extensively studied in probability theory due to their simplicity and applicability to various real-world phenomena. Let's explore some of the key aspects of random walks.

- **One-Dimensional Random Walk**: The simplest form of a random walk is a one-dimensional random walk on an integer line. It starts at position 0 and at each step, it can move either left or right with certain probabilities. The probabilities of moving left and right may or may not be equal, introducing a bias in the walk. This basic model serves as the building block for more complex random walks.

- **Random Walk on a Lattice**: Random walks can be extended to higher dimensions, such as the random walk on a lattice in d-dimensional space, denoted as $\mathbb{Z}^d$. In this case, each step of the walk corresponds to moving to one of the neighbouring lattice points. Random walks on lattices have diverse applications, including modelling diffusion processes and analysing the behaviour of particles in various physical systems.

- **Random Walks in Finance**: Random walks have been employed to model financial phenomena, such as the financial status of a gambler. In this context, the random walk represents the gambler's wealth, which can increase or decrease with each betting decision. Studying random walks in finance helps understand the behaviour of financial markets and evaluate investment strategies.

- **Brownian Motion**: Brownian motion is a specific type of random walk that has gained significant attention. It refers to the erratic movement of particles suspended in a fluid, such as the motion of pollen grains in water. Brownian motion is characterized by continuous, random fluctuations and is often modelled using a mathematical concept called the Wiener process. It has numerous applications in physics, chemistry, finance, and other fields.

Duality is another intriguing aspect related to random walks. It involves a correspondence between certain properties of a random walk and the behaviour of its inverse process. For example, the probability of a random walk reaching a specific position can be related to the probability of its inverse process starting at that position. Duality has proved to be a powerful tool in the analysis of random walks and has contributed to many interesting results.

By studying random walks and their variations, researchers have gained insights into the behaviour of stochastic processes, diffusion phenomena, financial markets, and more. These simple models continue to be an essential part of probability theory and have widespread applications in various scientific disciplines.

**Example 1.8.1.** *The Gambler's Ruin Problem is a well-studied random walk in probability theory. The set-up is as follows. Consider a gambler who starts with an initial bank account of €i. On each successive gamble, the gambler can either win €1 with probability p or lose €1 with probability q = 1 − p independent of the past. The gambler's goal is to obtain a total bank account balance of €N before being ruined, that is, before his bank account is empty. If the gambler succeeds, he is said to have won the game. It can be shown that the probability of winning the game is given by*

$$P_i = \begin{cases} \frac{1-(\frac{q}{p})^i}{1-(\frac{q}{p})^N} & \text{if} p \neq q \\ \frac{i}{N} & \text{if} p = q = \frac{1}{2} \end{cases} \tag{1.16}$$

*with i the initial bank account balance. Similarly, the probabilities that the gambler loses the game or becomes infinitely rich can be deduced. See for an elaborate discussion of random walks and the Gambler's Ruin Problem [15, 26].*

The one-dimensional random walk can also be perceived as a Markov chain with a state space given by the integer line. For some number $0 < p < 1$, the transition probabilities of moving from state $i$ to state $j$ with $|i - j| = 1$ are given by $P_{i,i+1} = p = 1 - P_{i,i-1}$.

The expectation of a random walk is an elementary result in probability theory and given by

$$\mathbb{E}_x^{\mathrm{RW}} \phi([X(t))] = \sum_y P(X(t) = y|X(0) = x)\phi(y) \tag{1.17}$$

where $\boldsymbol{X}(t)$ is a Markov process on a state space $Z$, $\phi \in S(\mathbb{R})$ a test function and $P_t$ the transition probabilities of the Markov process $\boldsymbol{X}(t)$.

Before delving into the interesting phenomena that can be observed as a result of the (scaling) limits of random walks, we commence by introducing Wiener processes. The definitions of a Wiener process and Brownian motion are equivalent, even though some physicists distinguish between the observed (Brownian motion) and theoretical (Wiener process) aspects.

**Definition 1.8.1** (Wiener Processes (Brownian Motion))**.** *Let* $\{W(t) : t \geq 0\}$ *be a stochastic process with* $W_i$ *i.i.d. random variables. If the process satisfies the following conditions:*

1. $W_0 = 0$ *a.e.; and*
2. $W$ *has independent increments, i.e. for all* $l, t, s \geq 0$*, the (future) increments* $W_{t+l} - W_t$ *are independent of the past values* $W_s$ *for* $l > t > s \geq 0$*; and*
3. $W$ *has Gaussian increments with mean* 0 *and variance* $l$*, i.e.* $W_{t+l} - W_t \sim \mathcal{N}(0, l)$*; and*
4. $W$ *has almost surely continuous paths, i.e.* $W_t$ *is almost surely continuous in* $t$*.*

*then the process is called a Wiener process.*

**Example 1.8.2.** *For* $\{W(t) : t \geq 0\}$ *a Wiener process, let* $(M_t)_{t\geq 0}$ *be a sequence of random variables on probability space* $(\Omega, (\mathscr{F}_t)_{t\geq 0}, P)$ *given by* $M(t) = e^{W_t - \frac{\lambda^2}{2}t}$*. It is easily confirmed that* $M(t)$ *is a martingale.*

**Proposition 1.8.1** (Generator of a Wiener Process)**.** *The generator* $A$ *of a Wiener process applied on a function* $f \in D(A)$ *is given by*

$$Af = \frac{1}{2}f" \tag{1.18}$$

*Proof.* A formal proof can be found in [29]. Intuitively, one applies the semigroup to the function $f$ and computes the second-order Taylor expansion. □

The investigation of the limiting behaviour of a random walk, as mentioned earlier, has garnered considerable interest among researchers. Expanding on these inquiries, Monroe D. Donsker made significant advancements in this field and presented his findings in the form of the renowned Donsker's invariance principle.

**Theorem 1.8.2** (Donsker's Invariance Principle)**.** *Let $\{X_i : i \in \mathbb{N}\}$ be a sequence of i.i.d. random variables with mean 0 and variance $\sigma^2$. Let $S_n = \sum_{i=1}^{n} X_i$,[7] then the stochastic process $S = (S_n)_{n\in\mathbb{N}}$ is known as a random walk. Let $W_t^n = \frac{S_{\lfloor nt \rfloor}}{\sqrt{n}}$ for $t \in [0,1]$ be a partial continuous sum (a rescaled random walk). Then, $(W_t^n)_{n\in\mathbb{N}}$ converges to $(W_t)_{t\in[0,1]}$ in distribution for $n \to \infty$ where $(W_t)_{t\in[0,1]}$ is a standard Brownian motion.*

*Proof.* See [35]. □

**Remark 1.8.1.** *Donsker's invariance principle is an extension of the Central Limit Theorem that states that $(W_{(1)}^n)_{n\in\mathbb{N}}$ converges to $(W_{(1)})$ in distribution for $n \to \infty$ where $(W_{(1)})$ is a standard Gaussian random variable.*

**Example 1.8.3.** *It can be checked that the random walk process $S = (S_n)_{n\in\mathbb{N}}$ with partial sums $S_n = \sum_{i=1}^{n} X_i$ is a martingale, as*

$$\mathbb{E}[S_n|\mathscr{F}_{n-1}] = \mathbb{E}\left[\sum_{i=1}^{n-1} X_i + X_n|\mathscr{F}_{n-1}\right] = \sum_{i=1}^{n-1} X_i + \mathbb{E}[X_n] = S_{n-1}$$

In order to proceed, it is essential to establish a crucial definition that serves as a prerequisite for comprehending and proving the subsequent statements.

**Definition 1.8.2** (Quadratic Variation)**.** *Let $\{X_t : t \geq 0\}$ be a stochastic process on a probability space $(\Omega, \mathscr{F}, P)$. The the quadratic variation of $X_t$ is given by*

$$\langle X \rangle_t = \lim_{\|P\| \mapsto 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})^2 \tag{1.19}$$

*where $P$ is a partition over the interval $[0, t]$ and converges in probability.*

*More generally, the covariation (or cross-variance) of two processes $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ is given by*

$$\langle X, Y \rangle_t = \lim_{\|P\| \mapsto 0} \sum_{k=1}^{n} (X_{t_k} - X_{t_{k-1}})(Y_{t_k} - Y_{t_{k-1}}) \tag{1.20}$$

**Proposition 1.8.3** (Itô Isometry)**.** *Let $\{X_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ be two stochastic processes, both adapted to the natural filtration. Let $W_t$ be a Brownian motion, then,*

$$\mathbb{E}\left[\left(\int_0^T X_t dW_t\right)\left(\int_0^T Y_t dW_t\right)\right] = \mathbb{E}\left[\int_0^T X_t Y_t dt\right] \tag{1.21}$$

*is referred to as the Itô isometry*

*Proof.* See [32]. □

[7]Again, one should actually be more precise and require that $S_n$ takes values in $\mathbb{D}([0,1],\mathbb{R})$.

It was the French mathematician Lévy who discovered the sufficient conditions for a stochastic process $X_t$ to be a Brownian motion.

**Theorem 1.8.4** (Lévy's Characterisation of Brownian Motion)**.** *Let* $\{X_t : t \geq 0\}$ *be a stochastic process defined on the probability space* $(\Omega, \mathscr{F}, P)$ *with filtration* $\mathscr{F}_t$*. If* $\{X_t : t \geq 0\}$ *satisfies the following conditions:*

1. $\mathbb{P}[X_0 = 0] = 1$*; and*
2. $X_t$ *is a continuous martingale with respect to the filtration* $\mathscr{F}_t$*; and*
3. *The quadratic variation* $< X >_t = t$ *a.s. under* $\mathbb{P}$*.*

*then* $\{X_t : t \geq 0\}$ *is a Brownian motion.*

*Proof.* The proof follows from considering the four conditions of a Brownian motion (1.8.1) separately. A detailed proof can be found in [41]. □

Lastly, we state the definition of duality. See for a more elaborate explanation (1.2) and (2.3.1).

**Definition 1.8.3** (Dual with respect to a Duality Function)**.** *Let* $Z, \tilde{Z}$ *be two state spaces of the Markov processes* $\{\eta(t) : t \geq 0\}$ *and* $\{\xi(t) : t \geq 0\}$ *evolving on them, respectively. We say that these Markov processes are dual with respect to the duality function* $D : Z \times \tilde{Z} \to \mathbb{R}$ *if* $\forall\, t \geq 0, \eta \in Z$*, and* $\xi \in \tilde{Z}$

$$\mathbb{E}_\eta[D(\xi, \eta(t))] = \mathbb{E}_\kappa[D(\xi(t), \eta)] \tag{1.22}$$

*holds. Equation (1.22) is referred to as the duality property. If* $Z = \tilde{Z}$ *we call the Markov process self-dual.*

## 1.9. General Properties

We now introduce some basic definitions that are important in the study of Markov chains and the related phenomena.

**Definition 1.9.1.** *Let $\boldsymbol{X}$ be a Markov chain. Then $\boldsymbol{X}$ is called*

- *a transient Markov chain, if $\forall x \in E\ \exists y \in E : x \to y$, but $y \nrightarrow x$;*
- *a recurrent Markov chain if it is not transient*

**Remark 1.9.1.** *It follows from Pólya's theorem that the symmetric random walk on $\mathbb{Z}^d$ is recurrent if $d = 1,2$ and transient if $d \geq 3$. The reader is referred to [17].*

**Definition 1.9.2** (Invariance Property)**.** *The semigroups $\boldsymbol{P} = \{P_t : t \geq 0\}$ are said to be invariant if $f P_t = f$ for every $t \in T$. This is equivalent to $\int P_t f \, d\mu = \int f \, d\mu$ with $\mu$ a probability measure on $\Omega$.*

**Remark 1.9.2.** *Note that if the initial distribution of $\boldsymbol{X}$ has an invariant density function ($X_0$ has an invariant density function $f$), then, by (1.7.2), $\boldsymbol{X} = \{X_t : t \geq 0\}$ has the invariant distribution $f$ as well.*

**Proposition 1.9.1.** *Let $\mu$ be a probability measure on $\Omega$ with bounded expectation. The following are equivalent:*

1. *$\int \mathscr{L} f \, d\mu = 0$ for all $f \in \mathscr{B}(\mathbb{R})$.*
2. *$\mu$ in $\mathscr{I}$, with $\mathscr{I}$ the set of invariant probability measures.*

*Proof.* The proof follows from bounding the argument of the integral and using Fubini's theorem. A detailed proof can be found in [36]. □

**Definition 1.9.3** (Ergodic Measure, I)**.** *A probability measure $\mu$ on $\Omega$ is said to be ergodic if for all $f \in \mathscr{B}(\mathbb{R})$ and $t \geq 0$, if $S_t f = f$ a.e., then $f = \int f \, d\mu$ a.e.*

**Remark 1.9.3.** *An equivalent definition of definition 1.9.3 can be stated as follows: for all $f \in \mathscr{B}(\mathbb{R})$, if $\mathscr{L} f = 0$ a.e., then $f = c$ for $c \in \mathbb{R}$ a.e.*

We now state an important theorem that will be used later in this thesis.

**Theorem 1.9.2** (Birkhoff's Ergodic Theorem, Continuous)**.** *Let $\boldsymbol{X}$ be a Markov process with invariant probability measure $\mu$ and $\mathscr{I}$ the set of invariant measures. Then, for any $f \in L^p(\Omega, \mu)$ $p \geq 1$, the following limit holds:*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(X_r) \, dr = \mathbb{E}_\mu[f(X_0)|\mathscr{I}] \qquad \mu - a.s \text{ and in } L^p(\Omega, \mu). \tag{1.23}$$

*Note that if the probability measure $\mu$ is ergodic as well (in the sense of Definition 1.9.3), we obtain*

$$\mathbb{E}_\mu[f(X_0)|\mathscr{I}] = \mathbb{E}_\mu\left[f(X_0)\right] \qquad \mu - a.s \text{ and in } L^p(\Omega, \mu). \tag{1.24}$$

*Proof.* See [19]. □

Alternatively, it can be shown that ergodic measures are the extreme points of the convex set of invariant measures of some given continuous dynamical system (i.e. a continuous Markov process). We consider the definition of ergodicity in terms of transformations.

**Definition 1.9.4** (Ergodic Measure, II)**.** *Given ($\Omega$, $\mathscr{F}$, $P$) a probability space with probability measure $\mu$. A transformation $T : \Omega \to \Omega$ is said to be ergodic if for every set $B \in \mathscr{F}$ with $T^{-1}B = B$, we either have $\mu(B) = 0$ or $\mu)B) = 1$. $\mu$ is said to be $T$-ergodic.*

**Definition 1.9.5** (Extreme Points)**.** *A point $z$ in a convex set $\mathbb{C}$ is said to be an extreme point if, given $z_1, z_2 \in \mathbb{C}$, there exists a $\lambda \in [0,1]$ such that if $z = \lambda z_1 + (1-\lambda) z_2$, then $z = z_1 = z_2$. In simpler terms, there exist no segment containing $z$ in its interior that is entirely contained in $\mathbb{C}$.*

**Theorem 1.9.3.** *Given* $(\Omega, \mathscr{F}, P)$ *a probability space with measure* $\mu$*. The following are equivalent:*

1. $\mu$ *is ergodic.*
2. $\mu$ *is an extreme point of the convex set* $\mathscr{I}$.

*Proof.* See [33]. □

**Example 1.9.1.** *Let* $\Omega_i = \{0,1\}$ *be a sample space for* $i = 1,2$ *and with probability measures given by*

$$\mu = \bigotimes \left( \frac{\alpha_i}{\alpha_i + \beta_i}, \frac{\beta_i}{\alpha_i + \beta_i} \right)$$

*where* $\alpha_i$ *and* $\beta_i$ *denote the rate to jump from state* 0 *to* 1 *and* 1 *to* 0*, respectively, on sample spaces* $\Omega_i$*. Let* $\alpha_1 = \beta_1 = 1$ *and* $\alpha_2 = 1, \beta_2 = 2$*. Hence, we obtain that*

$$\mu_1 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$$
$$\mu_2 = \frac{1}{3}\delta_0 + \frac{2}{3}\delta_1$$

*where the Kronecker delta* $\delta_i$ *denotes being in state* 0 *or* 1*.*

*The convex combination* $\mu$ *of* $\mu_1, \mu_2$ *is given by*

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2$$

*with* $\lambda \in [0,1]$*. For* $\lambda = \frac{1}{2}$*, we obtain*

$$\mu = \frac{1}{2}(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1) + \frac{1}{2}(\frac{1}{3}\delta_0 + \frac{2}{3}\delta_1) = \frac{5}{12}\delta_0 + \frac{7}{12}\delta_1$$

*which is clearly unequal to* $\mu_1, \mu_2$ *for every value of* $\delta$*. We conclude that* $\mu \neq \mu_1, \mu_2$ *and hence that* $\mu$ *is not an extreme point of the convex set* $\Omega = \Omega_1 \times \Omega_2$*. By Theorem 1.9.3, we conclude that* $\mu$ *is not ergodic.*

**Definition 1.9.6** (Reversible Markov Chain)**.** *Let be a CTMC that is irreducible and let* $\mu$ *be the probability measure of* $\boldsymbol{X} = \{X(t) : t \in T\}$ *such that* $X_0$ *has an invariant distribution* $\mu$ *. The Markov chain* $\boldsymbol{X}$ *is reversible if* $\boldsymbol{X}$ *and its time reversal* $\hat{\boldsymbol{X}}$ *have the same transition rates, that is*

$$\mu_x c_{(x,y)} = \mu_y c_{(y,x)}, \quad x, y \in E \tag{1.25}$$

*Equation (1.25) is called the detailed balance equation.*

## 1.10. Poisson Processes

In this section, we give a concise overview of Poisson (point) processes and relating phenomena. Note that all parameters of the Poisson distribution, i.e. $\lambda$, $\rho$, are bounded.

**Definition 1.10.1** (Poisson Distribution)**.** *Let* $(\Omega, \mathscr{F}, \mathbb{P})$ *be a probability space with measure* $\mu$*. The random variable* $X$ *is said to have the Poisson distribution with parameter* $\lambda \geq 0$*, i.e.* $X \sim \text{Pois}(\lambda)$*, if,*

$$\mathbb{P}(X = k) = \frac{(\lambda)^k}{k!} e^{-\lambda} \tag{1.26}$$

*with* $\mathbb{P}(X = 0) = 1$*. Furthermore, we have that* $\mathbb{E}[X] = \text{var}[X] = \lambda$

We now define the so-called *point processes.* The idea of a point process is that of a random, at most countable, collection $K$ of points in some space $E$.

Let $(E, \mathscr{E})$ be a measurable space. Let $\tilde{\mathbb{N}}_{<\infty}(E) := \tilde{\mathbb{N}}_{<\infty}$ be the space of all measures $\mu$ on $E$ such that $\mu(B) \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ for all $B \in \mathscr{E}$ and let $\tilde{\mathbb{N}}(E) := \tilde{\mathbb{N}}$ the space of all measures that can be written as a countable sum of measures from $\tilde{\mathbb{N}}_{<\infty}$. Any sequence $(x_n)_{n=1}^k$ with $k \in \tilde{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$ can be used to define a measure

$$\mu = \sum_{n=1}^{k} \delta_{x_n} \tag{1.27}$$

then $\mu \in \tilde{\mathbb{N}}$ and $\mu(B) = \sum_{n=1}^k \mathbf{1}_B(x_n)$ for $B \in \mathscr{E}$.

**Definition 1.10.2** (Point Process)**.** *Let $(\tilde{\mathbb{N}}, N_\sigma)$ be a measurable space with $N_\sigma := N_\sigma(E)$ the $\sigma$-field generated by all subsets of $\tilde{\mathbb{N}}$ of the form $\{\mu \in \tilde{\mathbb{N}} : \mu(B) = k\}$ for $B \in \mathscr{E}$. A point process on $E$ is a random variable $\eta$ on $(\tilde{\mathbb{N}}, N_\sigma)$.*

**Example 1.10.1.** *Let $X$ be a random variable in $E$. Then,*

$$\eta := \delta_X \tag{1.28}$$

*is a point process for which the measurability follows from*

$$\{\eta(B) = k\} = \begin{cases} \{X \in B\}, & \text{if} k = 1 \\ \{X \notin B\}, & \text{if} k = 0 \\ \emptyset, & \textit{otherwise} \end{cases} \tag{1.29}$$

**Definition 1.10.3.** *Let $\eta$ be a point process on $E$. The measure $\lambda$, defined by*

$$\lambda(B) := \mathbb{E}\left[\eta(B)\right], \qquad B \in \mathscr{E} \tag{1.30}$$

*is called an intensity measure.*

**Definition 1.10.4** (Poisson Point Process)**.** *Let $\lambda$ be a finite measure on $E$. A Poisson process with intensity measure $\lambda$ is a point process $\eta$ on $E$ with the following two properties:*

1. *For every $B \in \mathscr{E}$, the distribution of $\eta(B)$ is Poisson with intensity measure $\lambda(B)$.*
2. *For every $m \in \mathbb{N}$ and all pairwise disjoint sets $B_1, B_2, ..., B_m$, the random variables $\eta(B_1), \eta(B_2), ..., \eta(B_m)$ are independent.*

Hence, a Poisson point process consists of random points in $E$, such that the number of points within any measurable subset is Poisson distributed.
If a Poisson point process has a parameter of the form $\Lambda = \nu\lambda$, with $\nu$ the Lebesgue measure (it assigns length, area, or volume to sets) and $\lambda$ a constant that can be interpreted as the average number of points per some unit of length, area, volume, or time. It is also called the mean density, intensity, or rate.[8] In this instance, the Poisson point process is called a homogeneous Poisson point process. Let us define a Poisson process in terms of a Poisson point process as a *counting process* in dimension 1.

**Definition 1.10.5** (Counting Process)**.** *Let $\{\eta(t) : t \geq 0\}$ be a stochastic process on the probability space ($\Omega$, $\mathscr{F}$, $\mathbb{P}$). If*

1. *$\eta(t) \geq 0$; and,*
2. *$\eta(t) \in \mathbb{Z}$; and,*
3. *For all $s, t \geq 0$ with $s < t$, $\eta(s) \geq \eta(t)$.*

*then $\{\eta(t) : t \geq 0\}$ is called a counting process.*

**Definition 1.10.6** (Poisson Process)**.** *Let $\{\eta(t) : t \geq 0\}$ be a counting process on the probability space ($\Omega$, $\mathscr{F}$, $\mathbb{P}$) and $\lambda > 0$ the If*

1. *$\eta(0) = 0$; and,*
2. *For every $m \in \mathbb{N}$ and $t_0, t_1, ..., t_{m-1}, t_m \in T = [0, \infty)$ with $t_i < t_{i+1}$ for $i \in [0, m]$ : $(\eta(t_1) - \eta(t_0)), (\eta(t_2) - \eta(t_1)), ..., (\eta(t_m) - \eta(t_{m-1}))$ are independent, i.e. the stochastic process has independent increments; and,*
3. *The number of events (or points) in any interval of length $t \in T$ is a Poisson random variable with parameter $\lambda t$, i.e. $\eta(t) \sim$ Poisson(t).*

*then $\{\eta(t) : t \geq 0\}$ is called a Poisson process.*

[8] *Rate* is usually used when the underlying space is one-dimensional.

Hence, we conclude that a Poisson point process is a process that describes the random distribution of points in space (or time) according to a Poisson distribution. It is a multidimensional generalization of the Poisson process.
A Poisson process, on the other hand, is a specific type of Poisson point process that is defined in one dimension, typically representing time. It is a stochastic process that models the occurrence of events or arrivals over time. In a Poisson process, the events are assumed to be independent and randomly distributed in time, with a constant rate of arrival.
So, a Poisson point process is a broader concept that can be defined in multiple dimensions, while a Poisson process is a specific case of a Poisson point process, defined in one dimension (typically time) and characterized by a constant rate of event occurrences.

The last corollary that we state is an important result from the study of Markov processes and is true for the Poisson processes that we consider in this thesis. Let $\{\eta(t) : t \geq 0\}$ be independent random walkers, with $(\eta(t))_x$ the number of random walkers at time $t$ and location $x$, that starts at $t = 0$ from a configuration $\eta = \eta(0)$. If $\eta$ is distributed as a product of Poisson distributions at time $t = 0$, then $\eta(t)$ is again distributed as a product of Poisson distributions for $t > 0$, albeit with a different parameter.

**Proposition 1.10.1** (Doob's Theorem). *Let $\mu_\rho = \bigotimes_{x\in\mathbb{Z}^d} \mathrm{Pois}(\rho(x))$ on $\mathbb{N}^{\mathbb{Z}^d}$ be a product of Poisson distributions with parameter $\rho : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$. Let $\{\eta(t) : t \geq 0\}$ be independent random walkers with $(\eta(t))_x$ the number of random walkers at time t and location $x$. If $\eta = \eta(0) \sim \mu_\rho$, then $\eta = \eta(t) \sim \mu_{\rho_t}$ with*

$$\rho_t(x) = \sum_y \rho(y) P_t(y, x) \tag{1.31}$$

*where $P_t(y, x)$ is the transition probability of a single walker.*

*Proof.* The proof follows from the Random Displacement Theorem, see [25]. □

# 2

# Random Waiting Time Model and the Fractional Kinetics Process

The knowledge of which geometry aims is the knowledge of the eternal.

Plato, *The Republic*

## 2.1. Introduction

The Bouchaud trap model (BTM) is a mathematical framework that combines elements of random walks and random media. In this model, the medium is depicted as a landscape comprising traps, where each trap possesses a distinct depth. The dynamics of the model involve particles undergoing random walks within this random medium.

The key characteristic of the BTM is that when a particle encounters a trap, it becomes trapped for a certain duration of time. The length of this entrapment period is exponentially distributed, with the mean duration proportional to the depth of the specific trap at that particular site within the landscape.

In this thesis, we consider all particles to be independent and identically distributed (i.i.d.). This assumption implies that each particle's behaviour and interaction with the random medium are statistically indistinguishable from one another. This simplifying assumption allows for a more tractable analysis and exploration of the properties and phenomena arising in the BTM.

By studying the BTM, researchers aim to gain insights into the behaviour of particles navigating through complex and disordered environments. The interplay between random walks and the random medium in this model provides a fertile ground for investigating diverse phenomena, such as diffusion, trapping, and anomalous transport.

The examination of the BTM within the context of this thesis holds the promise of unveiling intriguing aspects of particle dynamics and transport processes in the presence of disorder. Through rigorous analysis and mathematical modelling, this research aims to deepen our understanding of complex systems and contribute to the broader field of statistical physics.

## 2.2. Definition of the Bouchaud Trap Model

Consider a continuous-time Markov chain (CTMC) denoted by $\boldsymbol{X} = \{X(t) : t \in T\}$, which operates on a probability space $(\Omega, \mathscr{F}, P)$. The CTMC has a discrete state space $(E, \mathscr{E})$ and a continuous time space $(T, \mathscr{T})$. In this context, we define the state space $E$ as the $d$-dimensional lattice on which $x, y$ are neighbours, i.e. $E = \tilde{\mathbb{Z}}^d := \{(x, y) \in \mathbb{Z}^d : |x - y| = 1\}$, where $d$ denotes the dimensionality of the lattice.

The system under consideration consists of a set of $W$ particles, initially located at positions $x_1, x_2, \ldots, x_W$ at time $t = 0$. To describe the behaviour of the system, we examine the translation probability $P_t(x, y)$, which represents the likelihood that a particle starting from site $x$ will transition to a target site $y$ at time $t$.

For $x, y \in \mathbb{Z}^d$ We assign to every unordered pair of sites $(x, y)$ the *weights* or *jump rates* $c_{(x,y)}$ of $X_t$ by

$$c_{(x,y)} = \begin{cases} \nu\tau_x^{-(1-a)}\tau_y^a, & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} = \begin{cases} \nu(\frac{1}{w_x})^{-(1-a)}(\frac{1}{w_y})^a, & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} \tag{2.1}$$

where $\nu$ is a constant in units of time, which we set to 1. We define $\tau_x : \mathbb{Z}^d \to \mathbb{R}$ as positive i.i.d. random variables that represent the depths of a trap located at position $x$ in the lattice $\mathbb{Z}^d$. The collection of trap depths is denoted as the "trapping landscape" $\tau = \sum_{x \in \mathbb{Z}^d} \tau_x \delta_x$, where $\delta_x$ represents the Kronecker delta. The trapping landscape characterizes the distribution of individual trap depths. In the context of the rates given by equation (2.1), we have $\tau_i = \frac{1}{w_i}$ for $i = x, y$. Here, $w_i$ can be interpreted as the inverse depth of the trap located at position $i$. When the CTMC $X(t)$ visits a site $x$, it remains there for an exponentially-distributed waiting time with mean $\tau_x$ before transitioning to either the trap at $x-1$ or $x+1$. In this chapter, we require the first moments of $w_i$ to be finite, i.e. $\mathbb{E}(w_i) < \infty$ and $w_i$ to be bounded from below. The parameter $a$ takes values in the range $[0,1]$ and describes the symmetry of the model, which will be further examined in subsequent sections of this chapter.

We can prove the reversibility of $\tau_x$ by using Definition 1.9.6. Note that the reversibility of $\tau_x$ is equivalent to the reversibility of $w_x$.

**Corollary 2.2.1.** *The trapping depths $\tau_x$ are reversible for the Markov chain $\boldsymbol{X} = \{X(t) : t \in T\}$ on $\mathbb{Z}^d$.*

*Proof.* We write down both the right-hand side and the left-hand side of (1.25) and compare the results. Filling in the expression for $c_{(x,y)}$ and $c_{(y,x)}$ (2.1) in the right-hand side and left-hand side, respectively, we obtain for the right-hand side

$$\tau_x c_{(x,y)} = \tau_x \tau_x^{-(1-a)} \tau_y^a = (\tau_x \tau_y)^a$$

Similarly, we obtain for the left-hand side

$$\tau_y c_{(y,x)} = \tau_y \tau_y^{-(1-a)} \tau_x^a = (\tau_y \tau_x)^a$$

from which it follows that both sides of (1.25) are equal. Hence, we conclude that $\tau_x$ is a reversible measure for the Markov chain $\boldsymbol{X} = \{X(t) : t \in T\}$ on $\mathbb{Z}^d$. □

**Remark 2.2.1.** *The proof of Corollary 2.2.1 demonstrates that the reversibility of $\tau_x$ is unaffected by the choice of $a \in [0,1]$. This observation significantly simplifies various computations in this chapter.*

**Remark 2.2.2.** *By utilizing the following identities: $w_x = \frac{c_x}{c_0}$, where $c_0 = \sum_x c_x$ represents a normalization factor, and $P_t(x,y) = \frac{c_{(x,y)}}{c_x}$ (and vice versa for $y$), the detailed balance equation (1.25) can be expressed equivalently as follows:*

$$w_x P_t(x,y) = w_y P_t(y,x) \tag{2.2}$$

*for* $(x,y) \in \tilde{\mathbb{Z}}^d$.

The infinitesimal generator [Definition 1.7.5] of $\boldsymbol{X} = \{X(t) : t \in T\}$ of the continuous-time Markov chain $\boldsymbol{X} = \{X(t) : t \in T\}$ can be defined as follows:

$$\mathscr{L} f(x) = \sum_{y:(x,y) \in \tilde{\mathbb{Z}}^d} c_{(x,y)}(f(y) - f(x)) \tag{2.3}$$

for $f : \mathbb{Z}^d \to \mathbb{R}$ local functions. Equation (2.3) corresponds to the generator of an independent random walk (IRW) on $\mathbb{Z}^d$ without reservoirs. The specific form of the generator determines the nature of the underlying process. It is worth noting that there are other processes, such as the SIP or the SSEP, which are discussed in greater detail in works like [11].

The BTM can now be formally defined based on the previously established concepts and results.

**Definition 2.2.1.** *Let $\tau$ and $a \in [0,1]$ be defined as above. The standard continuous-time Markov chain $\boldsymbol{X} = \{X(t) : t \in T\}$ on the given spaces*[1] *is a Bouchaud Trap Model BTM($\mathbb{Z}^d$, $w_x$, $a$) if its dynamics are given by (2.1) and (2.3).*

It is now convenient to examine the different values of $a$ in the BTM and explore the implications for the dynamics of the model.

- **$a = 0$ (symmetric case I)**
  If we set $a = 0$, equation (2.1) reduces to,

$$c_{(x,y)} = \begin{cases} \nu\tau_x^{-1}, & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} = \begin{cases} \nu w_x, & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} = c_x \tag{2.4}$$

  In this case the rate at which $X(t)$ hops from site $x$ to site $y$ is independent of the depth of the trap at $y$.

  **Remark 2.2.3.** *It was Bouchaud himself that considered this symmetric case in his original paper [7]. By defining $\tau_x = \exp(-\beta E_x)$ (the so-called Gibbs measure), we obtain rates $c_{(x,y)}$ given by $(x,y) \in \tilde{\mathbb{Z}}^d$*

$$c_{(x,y)} = \nu e^{E_x(1-a)-aE_y} \overset{\nu=1,a=0}{=} e^{E_x} \tag{2.5}$$

  *with $c_{(x,y)} = 0$ for $(x,y) \notin \tilde{\mathbb{Z}}^d$. From equation (2.5) it follows that $X(t) \sim \exp(\frac{1}{d_x w_x})$, where $d_x$ is the degree of $x$ in $\mathbb{Z}^d$. Thus, after waiting an exponentially-distributed time, $X(t)$ randomly jumps to one of its neighbours with rates given by (2.5).*

- **$a \in (0,1)$ (asymmetric case)**
  When considering the Bouchaud Trap Model with $a \in (0,1)$, the rates $c_{(x,y)}$ in equation (2.1) still depend on the depths of the traps $x$ and $y$. Let us assume, without loss of generality, that $\tau_x \gg \tau_y$. This implies that trap $x$ is significantly deeper than trap $y$, resulting in a much longer mean waiting time for $X(t)$ at trap $x$ compared to trap $y$.

  We can discuss two opposing scenarios. Firstly, if $X(t)$ is currently at trap $x$, increasing the value of $a$ leads to a decrease in the mean waiting time at trap $x$, while the waiting time at trap $y$ increases. In other words, as $a$ increases, the rates $c_{(x,y)}$ at which $X(t)$ randomly jumps from $x$ to $y$ also increase.

  Secondly, considering the case where $X(t)$ is located at trap $y$, increasing $a$ leads to higher rates $c_{(y,x)}$ at which $X(t)$ randomly jumps from $y$ to $x$. Consequently, $X(t)$ becomes more attracted to trap $x$.

  Taking both cases into account, it can be concluded that, assuming $\tau_x \gg \tau_y$, as $a$ increases, the mean waiting time at trap $x$ decreases. However, after leaving trap $x$ and arriving at target site $y$, the probability of $X(t)$ returning to trap $x$ increases.

- **$a = 1$ (symmetric case II)**
  If we set $a = 1$, equation (2.1) reduces to

$$c_{(x,y)} = \begin{cases} \nu\tau_y, & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} = \begin{cases} \nu(\frac{1}{w_y}), & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} = c_y \tag{2.6}$$

  In this scenario, the rate at which $X(t)$ transitions from a trap at $x$ to a trap at $y$ is solely determined by the depth of the trap at the target site $y$.

For the remainder of this chapter, we will focus on the case where $a = 0$, which corresponds to the symmetric instance of the Bouchaud Trap Model. In the literature, this particular case is referred to as the Random Waiting Time Model (RWTM). The motivation for considering the symmetric case is that the subsequent computations are less computationally intensive and do not require extensive additional background information beyond what has been introduced in the thesis. The asymmetric case, with $a \neq 0$, falls under the category of a Random Conductance Model and has been further explored in works such as [22].

[1] These are the probability space, state space and time space.

## 2.3. Derivation of the Hydrodynamic Limit for the Random Waiting Time Model

Having defined the Random Waiting Time Model (RWTM), we can now proceed to describe the individual dynamics of particles on the lattice $\mathbb{Z}^d$. Due to the independence of each particle, it is more insightful to study the collective behaviour of the total number of particles on each side of the lattice. This leads us to investigate the hydrodynamic limit (HDL), which characterizes the macroscopic behaviour of the system. This section is organized as follows. Firstly, we present the main result of this section and outline the two conditions that must be satisfied to prove the result (2.3.1). Secondly, we provide a definition of the HDL for the RWTM (2.3.1). Lastly, we prove the HDL for the RWTM (2.3.2). Throughout this section, we assume $d \geq 1$, $\mathbb{E}(w_i) < \infty$ and $w_i$ bounded from below.

In this section, we will obtain the following result:

**Theorem 2.3.1** (Hydrodynamic Limit for the RWTM)**.** *Let $\{X(t) : t \geq 0\}$, $\{Y(t) : t \geq 0\}$, $\{\eta(t) : t \geq 0\}$ be continuous-time Markov chains. Suppose that $\mu_\rho^{(N)} = \bigotimes_{x \in \mathbb{Z}^d} \mathrm{Pois}(\rho(\frac{x}{N}))$ on $\mathbb{N}^{\mathbb{Z}^d}$ with parameter $\rho : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ smooth and bounded, such that $\int d\mu_\rho^{(N)}(\eta)\eta_y = \rho(\frac{y}{N})$ and $\phi \in S(\mathbb{R}^d)$ a test function in Schwartz space. Let the empirical density field (2.12) be given by*

$$X_t^N(\phi) = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \phi\left(\frac{x}{N}\right) \eta_{tN^2}(x), \quad t \geq 0 \tag{2.7}$$

*If the limit of the double variance over $\mu_N$ and $\eta$ of the empirical density distribution goes to 0, and provided that the rescaled Markov chain $\{Y(t) : t \geq 0\}$ converges to a Brownian motion, i.e.*

$$\frac{Y(tN^2)}{N} \xrightarrow[D]{} B(\mathscr{D}t) \tag{2.8}$$

*then,*

$$\mathbb{E}_{\mu_\rho^{(N)}} \mathbb{E}_\eta \left[ \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta_{tN^2}(x) \phi\left(\frac{x}{N}\right) \right] \xrightarrow[D]{} \int dy \mathbb{E}_y \left[ \phi(B(\mathscr{D}t)) \right] \rho_t(y) \tag{2.9}$$

*where $\rho_t = \rho(x,t) = \mathbb{E}_x^{\mathrm{HE}}[X(t), 0]$, i.e. the expectation value of a random walk $X(t)$ starting from position $x$, is the solution to the following Cauchy problem:*

$$\begin{cases} \partial_t \rho = \frac{\mathscr{D}}{2} \partial_y^2 \rho \\ \rho_0 = \tilde{\rho} \end{cases} \tag{2.10}$$

*with diffusion constant $\mathscr{D} = \frac{1}{M}$ and $\mathbb{E}[w_0] = M$ the mean waiting time. The upper equation in (2.10) is called the heat equation.*

### 2.3.1. Definition of the Hydrodynamic Limit for the RWTM

Let us recall the system setup described in the initial section of this chapter, which involves a Markov chain $\boldsymbol{X} = \{X(t) : t \geq 0\}$ on a $\mathrm{BTM}(\mathbb{Z}^d, \tau, a)$. In this context, we define $\eta : \mathbb{Z}^d \to \{0, 1, 2, ..., W\}$ as the total number of particles at site $x$, denoted by $\eta(x) = \sum_{i=1}^W \mathbf{1}_{\{x_i = x\}}$. Consequently, $\eta(x)$ represents the number of particles at site $x$ within a given configuration $\eta$. The system's evolution is described by the collection $\{\eta(x) : x \in \mathbb{Z}^d\}$. It is worth noting that we can express the Markov chain, as introduced in the initial section, as $\{X(t) : t \geq 0\} = \{X_t^i : t \geq 0, i \in 1, 2, ..., W\}$. Hence, the time-dependent configuration of this system can be written as $\eta_t(x) = \sum_{i=1}^W \mathbf{1}_{\{X_t^i = x\}}$. In simple terms, this configuration denotes the number of particles at site $x$ and time $t$. Given that $X_t^i$ adheres to the Markov property, the configuration $\eta_t(x)$ also abides by this property.

Let $N \in \mathbb{N}$ be a scaling parameter that will be taken to infinity. Additionally, let us choose a sequence of discrete spaces $\mathbb{V}_N$ such that for every $x_N \in \mathbb{V}_N$, we have $\frac{x_N}{N} \to x \in \mathbb{R}$ for $N \to \infty$. In other words, we divide the points of the discrete space $\mathbb{V}_N$ by the scaling parameter $N$ and observe the resulting non-integer points, which we refer to as *macroscopic points* $(x)$, in contrast to the original *microscopic points* $(x_N)$. Moreover, we define a sequence of *empirical density fields* $\{X_t^N : N \in \mathbb{N}\}$ associated with the relevant IPS linked to the right discrete space as

$$X_t^N := \frac{1}{|\mathbb{V}^d|} \sum_{x \in \mathbb{Z}^d} \delta\left(\frac{x}{N}\right) \eta_{t\theta_N}(x), \quad t \geq 0 \tag{2.11}$$

with $\delta(x)$ the Kronecker delta on $\mathbb{R}$ and $\theta_N$ a time-scaling function that depends on the system and dimension that is considered. In our case $\theta_N = N^2$. An intuitive substantiation can be found in (A.2.1).
Now, in our case, for all $N \in \mathbb{N}$ and $\phi \in S(\mathbb{R}^d)$, the empirical density field $\{X_t^N : t \in T\}$ is a process on $\mathbb{Z}^d$ and is given by

$$X_t^N(\phi) = \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \phi\left(\frac{x}{N}\right) \eta_{tN^2}(x), \quad t \geq 0 \tag{2.12}$$

comparable to (2.12). Let us introduce a measurable function $\rho : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ which is referred to in the literature as the *macroscopic density* or *density profile*. We assign to this density a family of probability measures $\boldsymbol{\mu} = \{\mu^{(N)} : N \in \mathbb{N}\}$ on $\mathbb{Z}^d$. $\boldsymbol{\mu}$ is the family of probability measures of the Markov chain $\boldsymbol{X}$. If the initial distribution of a Markov chain $\mu$ is invariant, then all the distributions for later times $t > 0$ are distributed according to the same initial distribution. This follows from (1.9.2).

**Definition 2.3.1.** *A sequence of configurations* $(\eta^{(N)})_{N \in \mathbb{N}}$ *is said to be compatible with the density profile* $\rho$ *if for all functions* $\phi \in S(\mathbb{R}^d)$ *and* $\forall \delta > 0$,

$$\mu_\rho^{(N)} \left[ \left| \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta^{(N)}(x) \phi\left(\frac{x}{N}\right) - \int_{\mathbb{R}^d} \phi(x)\rho(x)dx \right| > \delta \right] = 0 \tag{2.13}$$

*Equivalently, a sequence of probability distributions* $\{\mu^{(N)} : N \in \mathbb{N}\}$ *is compatible with the density profile* $\rho : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$, *if,*

$$\lim_{N \to \infty} \mathbb{E}_{\mu_\rho^{(N)}} \left[ \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta^{(N)}(x) \phi\left(\frac{x}{N}\right) - \int_{\mathbb{R}^d} \phi(x)\rho(x)dx \right]^2 = 0 \tag{2.14}$$

*and*

$$\mathrm{var}_{\mu_\rho^{(N)}} \left[ \frac{1}{N^d} \sum_{x \in \mathbb{Z}^d} \eta^{(N)}(x) \phi\left(\frac{x}{N}\right) \right] \to 0 \tag{2.15}$$

*Note that this is convergence in* $P$*-law.*

It follows from Doob's Theorem [Theorem 1.10.1] that when the sequence of configurations $\eta^{(N)}$ satisfies compatibility with the density profile $\rho$, then at later macroscopic times $tN^2$, the sequence of configurations $\eta_{tN^2}^{(N)}$, satisfies compatibility with the density profile $\rho_t$, where $\rho_t$ satisfies a PDE with initial condition $\tilde{\rho}$, which is called the hydrodynamic limit. In this section, we consider the heat equation. In section 2.4, we will consider the fractional heat equation. We need to prove both conditions (2.14, 2.15). This will be the subject of the next paragraph.

### 2.3.2. Proof of the Hydrodynamic Limit for the RWTM

Before delving into the proof of the aforementioned conditions, it is essential to examine the specific duality property of the Random Waiting Time Model. Let us revisit the definition of the RWTM, where the rates $c_{(x,y)}$ are defined as follows:

$$c_{(x,y)} = \begin{cases} \frac{1}{\tau_x}, & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} = \begin{cases} w_x, & (x,y) \in \tilde{\mathbb{Z}}^d \\ 0, & (x,y) \notin \tilde{\mathbb{Z}}^d \end{cases} \tag{2.16}$$

Note that we use $\eta_x(t) = \eta_t(x)$ in the proof below.

**Proposition 2.3.2** (Duality Property for the RWTM)**.** *Let $\{X(t) : t \geq 0\}, \{\eta(t) : t \geq 0\}$ be continuous-time Markov chains. Let $w_x$ be the (inverse) depths of the RWTM. Then for all $x \in \mathbb{Z}^d$, $t \in T$,*

$$\mathbb{E}_\eta \left[ \frac{\eta_x(t)}{w_x} \right] = \mathbb{E}_x^{RW} \left[ \frac{\eta_{X(t)}}{w_{X(t)}} \right] \tag{2.17}$$

*in the sense of Definition 1.8.3 with $D(X, \eta(x)) = \frac{\eta_x(t)}{w_x}$ .*

*Proof.* We start by taking the expectation over $\eta$[2] of the process $\eta_x(t)$

$$\begin{aligned} \mathbb{E}_\eta[\eta_x(t)] &= \sum_y P_t(y,x)\eta_y \\ &\overset{(2.2.2)}{=} \sum_y P_t(x,y)\frac{w_x}{w_y}\eta_y \\ &= w_x \sum_y P_t(x,y)\frac{\eta_y}{w_y} \end{aligned}$$

where we used the detailed balance equation in the second step (2.2.2).

It follows that

$$\mathbb{E}_\eta \left[ \frac{\eta_x(t)}{w_x} \right] = \sum_y P_t(x,y)\frac{\eta_y}{w_y} \tag{2.18}$$

If we define $D(X, \eta(t)) = \frac{\eta_x(t)}{w_x}$ and combine (2.18) and (1.17), we obtain

$$\begin{aligned} \mathbb{E}_\eta[D(x, \eta(t))] &= \sum_y P_t(x,y) D(y, \eta(0)) \\ &= \mathbb{E}_x^{\mathrm{RW}}[D(X(t), \eta)] \end{aligned}$$

where $D(X,\eta)$ is the duality function as in (1.22) and the where the expectation is equal to the expectation of a random walk (1.17) starting from site $x$.

Plugging in the definition of $D(X,\eta)$, we obtain

$$\mathbb{E}_\eta \left[ \frac{\eta_x(t)}{w_x} \right] = \mathbb{E}_x^{\mathrm{RW}} \left[ \frac{\eta_{X(t)}}{w_{X(t)}} \right] \tag{2.19}$$

□

**Remark 2.3.1.** *This property provides a significant simplification to our problem by introducing the concept of duality. With the duality property, we gain the ability to determine the total number of particles at a specific site $x$ and time $t \geq 0$ using only the initial configuration $\eta$ and a single random walk $X(t)$. This implies that despite the potentially infinite number of particles in the initial configuration $\eta$, we can utilise the duality property to extract information about the number of particles at a particular site and time through the relationship between the initial configuration and the random walk.*

[2]Where we use the convention $\eta_y = \eta_y(t=0)$.

**Proof of the Hydrodynamic Limit for the RWTM**

Having established the duality property for the Random Waiting Time Model, we can now proceed to prove the convergence of both the expectation and variance. In this paragraph, we will prove the HDL for the RWTM. The proof consists of three parts: convergence of the expectation, convergence of the variance, and convergence to Brownian motion.

*Proof of Theorem 2.3.1.* Our first step is to prove that, under the assumption that the rescaled Markov chain $\{Y(t) : t \geq 0\}$ converges to a Brownian motion, the double expectation over $\mu_\rho^{(N)}$ and $\eta$ of the empirical density field converges to an integral with integrand $\rho_t$ that is the solution to the heat equation. We proceed to prove that the variance is finite and that its limit approaches zero as $N$ tends to infinity.

**Convergence of the expectation of the RWTM**

Consider the double expectation over $\mu_\rho^{(N)}$ and $\eta$ of the empirical density distribution

$$\mathbb{E}_{\mu_\rho^{(N)}}\mathbb{E}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_{tN^2}(x)\phi\left(\frac{x}{N}\right)\right] \tag{2.20}$$

Evaluating this expression gives

$$\begin{aligned}
\mathbb{E}_{\mu_\rho^{(N)}}\mathbb{E}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_{tN^2}(x)\phi\left(\frac{x}{N}\right)\right] &= \int d\mu_\rho^{(N)}\mathbb{E}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_{tN^2}(x)\phi\left(\frac{x}{N}\right)\right]\\
&= \int d\mu_\rho^{(N)}\mathbb{E}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\frac{\eta_{tN^2}(x)}{w_x}\phi\left(\frac{x}{N}\right)w_x\right]\\
&\overset{(2.2.2)}{=} \int d\mu_\rho^{(N)}\frac{1}{N^d}\sum_{x,y\in\mathbb{Z}^d}P_{tN^2}(x,y)\frac{\eta_y}{w_y}\phi\left(\frac{x}{N}\right)w_x\\
&= \int d\mu_\rho^{(N)}\frac{1}{N^d}\sum_{x,y\in\mathbb{Z}^d}P_{tN^2}(y,x)\eta_y\phi\left(\frac{x}{N}\right)
\end{aligned}$$

where we used in the first step $\mathbb{E}_{\mu_\rho^{(N)}}[f] = \int_{\mu_\rho^{(N)}} f = \int d\mu_\rho^{(N)} f$ for $f \in \mathscr{B}(\mathbb{R})$ and the detailed balance equation in the third step.

$$\begin{aligned}
\int d\mu_\rho^{(N)}\frac{1}{N^d}\sum_{x,y\in\mathbb{Z}^d}P_{tN^2}(y,x)\eta_y\phi\left(\frac{x}{N}\right) &= \frac{1}{N^d}\sum_{x,y\in\mathbb{Z}^d}P_{tN^2}(y,x)\phi\left(\frac{x}{N}\right)\rho(\frac{y}{N})\\
&= \frac{1}{N^d}\sum_{y\in\mathbb{Z}^d}\rho(\frac{y}{N})\sum_{x\in\mathbb{Z}^d}P_{tN^2}(y,x)\phi\left(\frac{x}{N}\right)\\
&\overset{(2.19)}{=} \frac{1}{N^d}\sum_{y\in\mathbb{Z}^d}\rho(\frac{y}{N})\mathbb{E}_y^{\mathrm{RW}}\left[\frac{Y(tN^2)}{N}\right]
\end{aligned} \tag{2.21}$$

We now use the assumption that the rescaled Markov chain converges to a Brownian motion,

$$\frac{Y(tN^2)}{N} \underset{D}{\longrightarrow} B(\mathscr{D}t) \tag{2.22}$$

where $B(\mathscr{D}t)$ is a standard Brownian motion in the sense of Definition 1.8.1. Note that this is convergence in distribution.

It then follows that

$$\frac{1}{N^d}\sum_{y\in\mathbb{Z}^d}\rho(\frac{y}{N})\mathbb{E}_y^{\mathrm{RW}}\left[\frac{Y(tN^2)}{N}\right] \underset{N\to\infty}{\longrightarrow} \frac{1}{N^d}\sum_{y\in\mathbb{Z}^d}\rho_t(\frac{y}{N})\mathbb{E}_{\frac{Y}{N}}^{\mathrm{BM}}[\phi(B(\mathscr{D}t))] \tag{2.23}$$

Observe that (2.23) is just a Riemann sum that converges

$$\frac{1}{N^d}\sum_{y\in\mathbb{Z}^d}\rho_t(\frac{y}{N})\mathbb{E}^{\mathrm{BM}}_{\frac{y}{N}}[\phi(B(\mathscr{D}t))] \underset{N\to\infty}{\longrightarrow} \int dy\,\mathbb{E}_y[\phi(B(\mathscr{D}t))]\rho_t(y) \tag{2.24}$$

Hence, we conclude that

$$\mathbb{E}_{\mu_\rho^{(N)}}\mathbb{E}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_{tN^2}(x)\phi\left(\frac{x}{N}\right)\right] \underset{N\to\infty}{\longrightarrow} \int dy\,\mathbb{E}_y[\phi(B(\mathscr{D}t))]\rho_t(y) \tag{2.25}$$

where $\rho_t = \rho(x,t) = \mathbb{E}^{\mathrm{HE}}_x\left[\rho(X(t),0)\right]$ is a solution to (2.26) [Theorem 1.10.1], with $X(t)$ the diffusion process.

$$\begin{cases}\partial_t\rho = \frac{\mathscr{D}}{2}\partial_y^2\rho\\ \rho_0 = \tilde{\rho}\end{cases} \tag{2.26}$$

that is called the heat equation. □

**Finite variance of the RWTM**

The double variance over $\mu_\rho^{(N)}$ and $\eta$ of the empirical density field is given by

$$\mathrm{var}_{\mu_\rho^{(N)}}\left[\mathrm{var}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_{tN^2}(x)\phi(\frac{x}{N})\right]\right] \tag{2.27}$$

From Doob's Theorem [Theorem 1.10.1], it follows that $\eta_{tN^2}$ is an independent Poisson processes with parameter

$$\rho_t(x) = \sum_{y\in\mathbb{Z}^d}\rho(\frac{y}{N})P_{tN^2}(y,x) \tag{2.28}$$

using detailed balance (2.2.2), we obtain

$$\rho_t(x) = \sum_{y\in\mathbb{Z}^d}\rho(\frac{y}{N})P_{tN^2}(x,y)\frac{w_x}{w_y} \tag{2.29}$$

Now, as $0 < A_1 < w_x, 0 < A_2 < w_y$, i.e. they are bounded from below, and $\rho(y) < A_3$ for $A_1, A_2, A_3 \in \mathbb{R}\setminus\{0\}$, it follows that

$$\begin{aligned}\rho_t(x) = \sum_{y\in\mathbb{Z}^d}\rho\left(\frac{y}{N}\right)P_{tN^2}(x,y)\frac{w_x}{w_y} &\le \frac{1}{A_2}\sum_{y\in\mathbb{Z}^d}\rho\left(\frac{y}{N}\right)P_{tN^2}(x,y)w_x\\ &\le \frac{A_3}{A_2}\frac{1}{w_x}\end{aligned}$$

where we use that the particles are i.i.d. and that the expectation value and variance of a Poisson distribution are equal [Definition 1.10.1].

Hence, the variance over the empirical density field (2.27) is bounded from above by

$$\frac{A_3}{A_2}\frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}w_x\phi^2\left(\frac{x}{N}\right) \tag{2.30}$$

By the Law of large numbers,

$$\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}w_x\phi^2\left(\frac{x}{N}\right)\longrightarrow M\int\phi^2(x)dx \tag{2.31}$$

for $M = \mathbb{E}[w_0]$. We obtain that

$$\frac{A_3}{A_2}\frac{1}{N^{2d}}\sum_{x\in\mathbb{Z}^d}w_x\phi^2\left(\frac{x}{N}\right)\longrightarrow 0 \tag{2.32}$$

Hence, we conclude that

$$\text{var}_{\mu_\rho^{(N)}}\left[\text{var}_\eta\left[\frac{1}{N^d}\sum_{x\in\mathbb{Z}^d}\eta_{tN^2}(x)\phi(\frac{x}{N})\right]\right]\longrightarrow 0 \tag{2.33}$$

As a consequence, the sequence of configurations $\eta^N$, distributed as $\mu_\rho^{(N)}$ satisfies compatibility with the density profile $\rho$ in the sense of Definition 2.3.1.

**Convergence to Brownian motion**

In the preceding theorem [Theorem 2.3.1] we assumed that the rescaled Markov chain converges to Brownian motion, i.e.

$$\frac{Y(tN^2)}{N}\underset{D}{\longrightarrow}B(\mathscr{D}t) \tag{2.34}$$

The conventional method to establish this convergence involves demonstrating the convergence of the respective infinitesimal generators. By establishing this convergence, we can infer that the underlying processes also converge. However, a more comprehensive discussion and reasoning for why this approach is not applicable to equation 2.34 can be found in (A.2.2).

Instead, we will adopt an alternative approach by employing martingales, as discussed in [21]. To facilitate this approach, we will utilize Lévy's characterisation [Theorem 1.8.4]. The first condition is evident, as $\frac{1}{N}X_{tN^2}$ is a continuous-time Markov chain that conforms to standard properties. The second condition is straightforward to establish by employing the definition of a martingale and utilizing Dynkin's martingale (1.15).

However, the third condition necessitates further steps and involves more complex mathematical reasoning. We must demonstrate that $\langle X,Y\rangle_t\longrightarrow ct$ with $c\in\mathbb{R}$. Let us first introduce the process $w_t$ on the state space $[0,\infty)^{\mathbb{Z}^d}$ that is called the environment process and is given by $w_t(x)=w(X_t+x)$. The *environment process* equivalent to the process of waiting times as seen from the position of the walker. Before proving the convergence of the quadratic variation, we express the convergence expression in a slightly different form.

$$\langle\frac{1}{N}X_{tN^2},\frac{1}{N}X_{tN^2}\rangle_t\longrightarrow ct$$
$$\frac{1}{N^2}\int_0^{tN^2}\frac{1}{w(X_r)}dr\longrightarrow ct$$
$$\frac{1}{tN^2}\int_0^{tN^2}\frac{1}{w(X_r)}dr\longrightarrow c$$

By introducing a function $f(w_t)=\frac{1}{w_t(x)}$ and writing $w_t(0)=w(X_t)=w_t$, we obtain

$$\frac{1}{T}\int_0^T f(w_r)dr\longrightarrow c \tag{2.35}$$

for $T=tN^2$, the macroscopic time (see 2.3.1). This convergence is obtained by Birkhoff's Ergodic Theorem [Theorem 1.9.2]

Let $(\Omega,\mathscr{F},P)$ be a probability space with measure $\mu$.

**Proposition 2.3.3.** *The measure $\nu$ is time-invariant for all $t\in T$.*

*Proof.* By Proposition 1.9.1, it follows that that $\int\mathscr{L}f\,d\nu=0$ for all $f\in\mathscr{B}(\mathbb{R})$. The generator of the process (as seen from the position of the walker) is equal to

$$\mathscr{L}f(w)=\sum_{e:|e|=1}\frac{1}{w(0)}\big(f(\tau_e w)-f(w)\big) \tag{2.36}$$

Indeed, jumps of the walker correspond to shifts of the environment.

We now prove that $d\nu:=\frac{w(0)}{M}d\mu$ is an invariant measure for this process with $d\mu$ the joint distribution of the waiting times, i.e. the joint law of $\{w(x):x\in\mathbb{Z}^d\}$. Recall that $M=\frac{1}{\mathscr{D}}$, the inverse of the diffusion constant. By using the assumption $\int\mathscr{L}f\,d\nu=0$ for all $f\in\mathscr{B}(\mathbb{R})$, we obtain

$$\int \mathscr{L} f d\nu = \int \sum_{e:|e|=1} \frac{1}{w(0)} \big(f(\tau_e w) - f(w)\big) \frac{w(0)}{M} d\mu$$
$$= \frac{1}{M} \sum_{e:|e|=1} \int \big(f(\tau_e w) - f(w)\big) d\mu = 0 \tag{2.37}$$

where we used that $\mu$ is i.i.d. and thus invariant under shifts.

It now follows that

$$\int f(\tau_e w) d\mu = \int f(w) d\mu \tag{2.38}$$

we conclude that $\nu \in \mathscr{I}$. □

**Proposition 2.3.4.** *The time-invariant measure $\nu$ is ergodic for all $t \in T$.*

*Proof.* We use Proposition 1.9.3. Assume that $f$ is $\nu$-integrable and time-invariant, then by Proposition 1.9.1, it follows that $\mathscr{L} f = 0$ a.e. for all $f \in \mathscr{B}(\mathbb{R})$. Clearly $\int f \mathscr{L} f d\nu = 0$ a.e. as well. By Itô symmetry [Proposition 1.21], it follows that

$$0 = \int f \mathscr{L} f d\nu = \frac{1}{2} \frac{1}{M} \int \sum_{e:|e|=1} \big(f(\tau_e w) - f(w)\big)^2 d\mu \tag{2.39}$$

We obtain

$$f(\tau_e, w) = f(w) \qquad \text{a.s. for all } e \in \mathbb{Z}^d \text{ with } |e| = 1 \tag{2.40}$$

As a consequence by ergodicity of $\mu$ under spatial shifts (remember $\mu$ is i.i.d.), we conclude that $f$ is $\mu$-a.s. constant, and hence also $\nu$-a.s. constant. We obtain that $\nu$ is indeed invariant and ergodic. □

From Birkhoff's Ergodic Theorem, it follows that

$$\frac{1}{T} \int_0^T \frac{1}{w(X_r)} dr = \frac{1}{T} \int_0^T f(w_r) dr \longrightarrow \int f(w) d\nu(w)$$

where $f(w) = 1/(w(0))$, so we find

$$\int f(w) d\nu(w) = \int \frac{1}{w(0)} \left(\frac{w(0)}{M}\right) d\mu(w) = \frac{1}{M}$$

By Lévy's characterisation, we have proven that

$$\frac{Y(tN^2)}{N} \underset{D}{\longrightarrow} B(\mathscr{D} t) \tag{2.41}$$

## 2.4. Derivation of the Hydrodynamic Limit for the Fractional Kinetics Process

In the previous chapter, we focused on the scenario where $d \geq 1$ and $\mathbb{E}(w_i) < \infty$. We successfully established the convergence of the empirical density field to a Brownian motion, thereby verifying that the hydrodynamic limit follows the heat equation. In this chapter, we will explore a slightly different case. Specifically, we will assume that $d \geq 3$ and that the expectations are not finite but follow a power law distribution. The structure of this chapter is as follows. Firstly, the fractional kinetics equation will be introduced [2.1] together with the main result in this section [Theorem 2.4.3]. Secondly, the general setup will be stated [2.4.1]. Thirdly, the proof of the main theorem will be given [2.4.3].

Let $\rho : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ be the macroscopic density and $\mu_\rho^{(N)} = \bigotimes_{x\in\mathbb{Z}^d} \mathrm{Pois}(\rho\left(\frac{x}{N}\right) w_x)$ be the probability measure that initialises the process $\eta_t$ at $t = 0$. Then, under $\mathbb{P}_\eta$, the empirical density field $X_t^N$ converges to $W$, where

1. $W = W_\beta := \sum_i v_i \delta_{x_i}$ is called the *speed measure* for $(v_i, x_i)_i$ the support of the Poisson point process on $(0,\infty) \times \mathbb{R}^d$ with intensity measure $\beta v^{-(1-\beta)} dv$. Note that $W$ is random;

2. The (deterministic) macroscopic density $\rho_t$ solves the following Cauchy problem:

$$\begin{cases} \partial_t^\beta \rho = \mathscr{D}\partial_y^2 \rho \\ \rho_0 = \tilde{\rho} \end{cases} \tag{2.42}$$

with $\rho_t = \rho(x,t) = \mathbb{E}_x^{\text{FKP}}\left[\rho(X(t),0)\right]$. This system describes the so-called *fractional kinetics process.* The fractional equation (upper equation in (2.42)) is known as the fractional kinetics equation (FKE) meant in the Caputo sense, that is,

$$\frac{\partial^\beta}{\partial t^\beta} f(t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{1}{(t-s)^\beta} f'(s)\, ds, \qquad t \geq 0, f \in \mathscr{C}^1(\mathbb{R}) \tag{2.43}$$

where $\Gamma$ is the Gamma function and $\beta \in (0,1]$ the sub-difussive parameter. Note that for $\beta = 1$, equation (2.43) reduces to a normal first order derivative. It should also be noted that the FKP is a memory-preserving process, that is, the process depends on its past behaviour due to the integral from 0 to $t$. The reader is referred to [14, 34] for more details.

In this section, we will obtain the following result:

**Theorem 2.4.1** (Hydrodynamic Limit for the FKP). *For $d \geq 3$ and $\beta \in (0,1]$, let $\{X(t) : t \geq 0\}$ and $\{\eta(t) : t \geq 0\}$ be continuous-time Markov chains. For all $N \in \mathbb{N}$, let $\mu_\rho^{(N)} = \bigotimes_{x\in\mathbb{Z}^d} \mathrm{Pois}(\rho(\frac{x}{N}) w_x)$ be the initial distribution of the particle system with $\rho : \mathbb{R}^d \to \mathrm{R}_{\geq 0}$ and $\phi \in S(\mathbb{R}^d)$ a test function. Let the empirical density field (2.48) be given by*

$$X_t^N(\phi) = \frac{1}{N^{\frac{d}{\beta}}} \sum_{x\in\mathbb{Z}^d} \phi\left(\frac{x}{N}\right) \eta_{t\theta_N}(x), \quad t \geq 0 \tag{2.44}$$

*if the limit of the double variance over $\mu_N$ and $\eta$ of the empirical density function goes to 0, and provided that Theorem 2.4.2 and Theorem 2.4.3 hold, then,*

$$X_t^N(\phi) \underset{N\to\infty}{\longrightarrow} W(\phi_t \phi) \tag{2.45}$$

*where $\rho_t$ is the solution to (2.42).*

### 2.4.1. Definition of the Hydrodynamic Limit for the FKP

Recall the setup of our system that was given in the previous subsection of this chapter, which involves a Markov chain $\boldsymbol{X} = \{X(t) : t \geq 0\}$ on a BTM($\mathbb{Z}^d, \tau, a$). In this context, we denote by $\eta_x : \mathbb{Z}^d \to \{0, 1, 2, ..., W\}$ the total number of particles at site $x$ given by $\eta_x = \sum_{i=1}^{W} \mathbf{1}_{\{x_i = x\}}$ that describes the evolution of the system. In this sense, this setup is equivalent to the setup of the RWTM. However, in this scenario, we assume that the waiting times $w_i$ have a non-existent first moment, resulting in the scenario that no particles will move (their waiting times are unbounded). Therefore, the integral over $\eta$ does not exist. Therefore, we introduce the function $D_t^N(\frac{x}{N}) = \frac{\eta_{t\theta_N}(x)}{w_x}$ that homogenizes convergence to $\rho_t$, while the empirical density $(X_t^N w_x)$ does not converge, due to the unboundedness of $w_x$[12]. As we divide by $w_x$, the system becomes convergent again.

Let $\beta \in (0, 1]$ be the parameter that describes the diffusivity of the system. Let $\mathbb{P}$ be a given product measure that satisfies

$$\mathbb{P}(w_x > n) = n^{-\beta}(1 + o(1)), \qquad n \to \infty \tag{2.46}$$

for $n \in \mathbb{N}$ We further assume that $d \geq 3$. Note that equation (2.46) is equivalent to the fact that $\mathbb{E}(w_i) < \infty$ does not hold.

The rescaled empirical density field over $\eta_t$ is given by

$$X_t^N = \frac{1}{N^{\frac{d}{\beta}}} \sum_{x \in \mathbb{Z}^d} \delta\left(\frac{x}{N}\right) \eta_{t\theta_N}(x), \quad t \geq 0 \tag{2.47}$$

where we set $\theta_N = N^{\frac{2}{\beta}}$.

For $\phi \in S(\mathbb{R}^d)$ a test function. The empirical density field is given by

$$X_t^N(\phi) = \frac{1}{N^{\frac{d}{\beta}}} \sum_{x \in \mathbb{Z}^d} \phi\left(\frac{x}{N}\right) \eta_{t\theta_N}(x), \quad t \geq 0 \tag{2.48}$$

or,

$$X_t^N(\phi) = \frac{1}{N^{\frac{d}{\beta}}} \sum_{x \in \mathbb{Z}^d} w_x D_t^N(\frac{x}{N}) \phi\left(\frac{x}{N}\right), \quad t \geq 0 \tag{2.49}$$

for $D_t^N(\frac{x}{N}) = \frac{\eta_{t\theta_N}(x)}{w_x}$ the duality function.

### 2.4.2. Fractional Kinetics Equation

### 2.4.3. Proof of the Hydrodynamic Limit for the FKP

To prove that the empirical density field converges to a random measure, i.e. the HDL itself is random, we need to consider several preliminary results.

**Theorem 2.4.2.** *Let $X_N(t) = \frac{X(tN^{\frac{2}{\beta}})}{N}$ be a rescaled, continuous random walk. Suppose that equation (2.46) holds. Then $X_N(t)$ converges to the fractional kinetic process (2.42) in distribution.*

*Proof.* See [4]. □

**Theorem 2.4.3.** *For $\phi \in S(\mathbb{R}^d)$ a test function and $w_x$ the random waiting times corresponding to sub-difussive behaviour, i.e. $\mathbb{E}(w_i) < \infty$. Let $\beta \in (0, 1]$ be the sub-difussive parameter. Let $W_N$ be the sequence of random speed measures given by*

$$W_N = \frac{1}{N^{\frac{d}{\beta}}} \sum_{x \in \mathbb{Z}^d} w_x \phi\left(\frac{x}{N}\right) \tag{2.50}$$

*Then,*

$$W_N \xrightarrow[D]{} W(\phi) \tag{2.51}$$

*in distribution. Note that $W$ is a random measure as well.*

*Proof.* See [13]. □

**Remark 2.4.1.** *The fact that the limit $W(\phi)$ is still random is a consequence of the factor $w_x$ that is random.*

*Proof of Theorem 2.4.1.* We first prove that the expectation value over $\mu_\rho^{(N)}$ and $\eta$ converges to a random speed measure by using Theorem 2.4.2 Theorem 2.4.3. Subsequently, we proceed to prove that the double variance of the empirical density field goes to 0 for $N \to \infty$.

**Convergence of the expectation of the FKE**

Consider the double expectation over $\mu_\rho^{(N)}$ and $\eta$. As $\mu_N = \bigotimes_{x\in\mathbb{Z}^d} \text{Pois}(\rho(\frac{x}{N})w_x)$, we have $\int d\mu_\rho^{(N)} \frac{\eta(x)}{w_x} = \rho(\frac{x}{N})$ which, by Doob's Theorem [Theorem 1.10.1], implies that $\mathbb{E}_{\mu_\rho^{(N)}}\left(\frac{\eta_{t\theta_N}(x)}{w_x}\right) = S_t^N \rho(\frac{x}{N}) = \rho_t(\frac{x}{N})$, with $S_t^N$ the semigroup corresponding to the BTM($\mathbb{Z}^d, w_x, a$) with $a \in (0,1)$, for $f \in \mathscr{C}_b(\mathbb{R}^d)$, $t \geq 0$, and $x \in \mathbb{Z}^d$.[3]

Evaluating the expectation gives,

$$\begin{aligned}
\mathbb{E}_{\mu_\rho^{(N)}}\mathbb{E}_\eta\left[\frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}\phi\left(\frac{x}{N}\right)\eta_{t\theta_N}(x)\right] &= \mathbb{E}_{\mu_\rho^{(N)}}\mathbb{E}_\eta\left[\frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}w_x\phi\left(\frac{x}{N}\right)\frac{\eta_{t\theta_N}(x)}{w_x}\right] \\
&= \mathbb{E}_{\mu_\rho^{(N)}}\left[\frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}w_x\phi\left(\frac{x}{N}\right)\mathbb{E}_\eta(D(x,\eta))\right] \\
&= \mathbb{E}_{\mu_\rho^{(N)}}\left[\frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}w_x\phi\left(\frac{x}{N}\right)S_t^N D(x,\eta)\right] \\
&= \frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}w_x\phi\left(\frac{x}{N}\right)\rho_t(\frac{x}{N})
\end{aligned} \tag{2.52}$$

where $D(x,\eta) := \frac{\eta_{t\theta_N}(x)}{w_x}$ and $\mathbb{E}_\eta\left[D(x,\eta(x)\right] = S_t^N D(x,\eta)$. Note that we implicitly used Theorem 2.4.2 in the last step.

Then, by Theorem 2.4.3,

$$\tilde{W}_N = \frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}w_x\phi\left(\frac{x}{N}\right)\rho_t(\frac{x}{N}) \underset{N\to\infty}{\longrightarrow} W(\rho_t\phi) \tag{2.53}$$

where $\rho_t = \rho(x,t) = \mathbb{E}_x^{\text{FKP}}\left[\rho(X(t),0)\right]$ is a solution to (2.42), with $X(t)$ the fractional kinetics process. Equation (2.53) signifies that, even though we started with a deterministic (i.e. non-random) density $\rho_t$, the hydrodynamic limit is random due to the randomness of $w_x$.

**Finite variance of the FKE**

Consider the double variance over $\mu_\rho^{(N)}$ and $\eta$. By using that the variance of a Poisson distribution is equal to its variance and that the particles are i.i.d., we obtain,

$$\begin{aligned}
\text{var}_{\mu_\rho^{(N)}}\text{var}_\eta\left[\frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}\phi\left(\frac{x}{N}\right)\eta_{t\theta_N}(x)\right] &= \frac{1}{N^{\frac{2d}{\beta}}}\text{var}_{\mu_\rho^{(N)}}\text{var}_\eta\left[\sum_{x\in\mathbb{Z}^d}\phi\left(\frac{x}{N}\right)\eta_{t\theta_N}(x)\right] \\
&\overset{(2.52)}{=} \frac{1}{N^{\frac{2d}{\beta}}}\text{var}_{\mu_\rho^{(N)}}\text{var}_\eta\left[N^{\frac{d}{\beta}}\frac{1}{N^{\frac{d}{\beta}}}\sum_{x\in\mathbb{Z}^d}\phi\left(\frac{x}{N}\right)\eta_{t\theta_N}(x)\right] \\
&\underset{N\to\infty}{\overset{(2.53)}{\longrightarrow}} 0
\end{aligned}$$

where we the last convergence is due to the factor $\frac{1}{N^{\frac{d}{\beta}}}$ squared. □

**Remark 2.4.2.** *Theorem 2.4.1 only holds for $d \geq 3$. Nonetheless, similar results can be derived for $d = 1,2$. For $d = 1$, the limiting processes are the so-called Fontes-Isopi-Newman (FIN) diffusion processes. The interested*

[3] $S_t^N f(\frac{x}{N}) := \mathbb{E}_x^a\left[f\left(\frac{X(tN^{\frac{2}{\beta}})}{N}\right)\right]$ for $f \in \mathscr{C}_b(\mathbb{R}^d)$, $t \geq 0$, and $x \in \mathbb{Z}^d$. See [1.7.2].

*reader is referred to [2]. The case $d = 2$ is a FKP as well, but with a different scaling limit. For more details, see [12].*

It is convenient to briefly analyse the proof that precedes this paragraph. The expectation value of the empirical density field (2.52) is a modification of the random speed measure given by equation (2.50). Namely, equation 2.52 is equation 2.50 evaluated in $\phi\left(\frac{x}{N}\right)\rho_t\left(\frac{x}{N}\right)$. Hence, by Theorem 2.4.3, the empirical density field converges to a random measure evaluated in $\phi\left(\frac{x}{N}\right)\rho_t\left(\frac{x}{N}\right)$. The result that this limit is still random is due to the 'extra' randomness that was incorporated. As observed in the introduction to this subsection, the duality function that was introduced (2.49), homogenizes the convergence to $\rho_t$,i.e. as $w_x$ follows a power law, the waiting times are non-integrable. By introducing the appropriate duality function, one obtains a function that is integrable and (still) depends on $w_x$ . As a result, incorporating this duality function into the empirical density field resulted in an extra factor of $w_x$ in the density field (2.49), and consequently, the emergence of an extra factor of randomness. It was possible to apply Theorem 2.4.3 to the density field. The final result followed.

# 3

# Concluding Remarks

It is of significance to underscore certain distinctions between the two derived equations in this thesis: the HDL solutions for the RWTM and for the FKP. Firstly, it is crucial to acknowledge that despite the determinism of the density $\rho$ in both cases, the HDL solution for the RWTM is deterministic and remains constant, whereas the HDL solution for the FKP exhibits a stochastic, non-deterministic nature attributable to the presence of the additional factor $w_x$. Secondly, the past behaviour of the process, known as *memory*, does not constitute a factor in the RWTM as opposed to the FKP in which the Fractional kinetics equation is defined in terms of a fractional derivative that does take its past behaviour into account, i.e. the derivative is defined in terms of an integral up to time $t$. Conclusively, memory is also a distinguishable characteristic between these two processes.

For both cases, the specification of the initial distribution is necessary, with the assumption of a Poisson distribution as the initial condition, in order to simplify the derivations. This choice is motivated by the utilization of the convergence result for time $t \geq 0$, wherein, for a Poisson distribution, the distribution at $t = 0$ serves as the reference. For prospective research endeavours, it is worthwhile to explore alternative initializations and assess their implications.
Additionally, a more in-depth analysis of scenarios where all waiting times $w_i$ possess a finite first moment, except for one, holds promise in shedding light on the system's evolution for future times $t$. It is expected that such investigations may reveal an intermediary variant that encompasses features from both the RWTM and the FKP. While this line of inquiry may serve as a suitable topic for a master's thesis, a simulation-based exploration within an undergraduate thesis is also feasible.

Undoubtedly, the composition of this thesis has been an enriching experience. As written in the introduction, it is my sincere hope to engage in numerous research endeavours in the future.

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# A

# Appendix

## A.1. Introduction

**Definition A.1.1** (Natural Filtration)**.** *The filtration associated to the process which records its "past behaviour" at each time is called the generated filtration or natural filtration. In a sense, it is the simplest filtration possible. All the information concerning the given process is contained, and only that information. Let $I$ be a totally ordered index set and $X$ a random variable, then the natural filtration of $\mathscr{F}$ with respect to $X$ is defined to be the filtration $\mathscr{F}_\cdot = (\mathscr{F}_i^X)_{i\in I}$, with*

$$\mathscr{F}_i^X = \sigma(X_j^{-1}(A) \,|\, j \in I, j \leq i, A \in \mathscr{E}) \tag{A.1}$$

*Clearly, this definition holds for either the discrete and continuous filtrations. In words, the natural filtration is the smallest $\sigma$-algebra on $\Omega$ that contains all pre-images of $\mathscr{E}$-measurable subsets of $E$ for 'times' $j$ up to $i$.*

## A.2. Random Waiting Time Model

### A.2.1. Scaling Parameter

In (2.3) it is proven that the double expectation of the empirical density distribution converges to an integral with an integrand that contains $\rho$, the solution to the heat equation. By performing a time and space transformation, we can deduce the appropriate scaling parameter.

The heat equation is given by

$$\frac{\partial \rho}{\partial t} = c\frac{\partial^2 \rho}{\partial x^2} \tag{A.2}$$

for $c \in \mathbb{R}$. If we apply the following transformations:

$$t' = at + l \tag{A.3}$$

$$x' = bx + p \tag{A.4}$$

where $a, b \in \mathbb{R}$, $l$ a coordinate in units of time and $p$ a coordinate of units of space. We obtain the transformed heat equation

$$\frac{\partial \rho}{\partial t'} = c\frac{\partial^2 \rho}{\partial x'^2} \tag{A.5}$$

and in our original coordinates

$$\frac{1}{a}\frac{\partial \rho}{\partial t} = c\frac{1}{b^2}\frac{\partial^2 \rho}{\partial x'^2} \tag{A.6}$$

hence we conclude that $a = b^2$. In words, a transformation in space by a factor $b$ results in a transformation in time by a factor $b^2$. Thus, for the RWTM, we have a transformation in space by a factor of $N$ and, hence, a transformation in time by a factor $N^2$. We conclude $\theta_N = N^2$.

It should be noted that if $\rho$ does obey a different PDE, the scaling parameters will be different.

### A.2.2. Convergence of Generators

Consider the RWTM with $w_x = 1$ and thus $c_{(x,y)} = 1$.[1] The infinitesimal generator is then given by

$$\mathscr{L} f(x) = \sum_{y:(x,y)\in\mathbb{Z}^d} (f(y) - f(x)) \tag{A.7}$$

Then the nth infinitesimal generator is given by

$$\mathscr{L}_n f(x) = \frac{n^2}{2}(f(x+\frac{1}{n}) + f(x-\frac{1}{n}) - 2f(x)) \tag{A.8}$$

It can be shown that this generator converges to the infinitesimal generator of a Brownian motion (1.8.1). Hence, the underlying processes converge as well and we can conclude that the model converges to a Brownian motion.

In our case, $w_x \neq 1$ and thus $c_{(x,y)} \neq 1$. The infinitesimal generator is then given by

$$\mathscr{L} f(x) = \sum_{y:(x,y)\in\mathbb{Z}^d} (f(y) - f(x)) \tag{A.9}$$

Then the nth infinitesimal generator is given by

$$\mathscr{L}_n f(x) = 2\frac{n^2}{w_{nx}}(f(x+\frac{1}{n}) + f(x-\frac{1}{n}) - 2f(x)) \tag{A.10}$$

which clearly does not converge to the infinitesimal generator of a Brownian motion due to the fact that $w_{nx}$ depends on $n$.

[1] The waiting times are constant, so there is no randomness due to the waiting times.