Social Power Games for Parallel Friedkin–Johnsen Models

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***Abstract*—In this article, we consider a strategic game played by a group of agents on a set of opinion dynamics models. The models are all Friedkin–Johnsen (FJ) models, which are independent of each other (we call them "parallel FJ models"). The task of an agent is to maximize her overall social power by allocating a given budget of stubbornness across the parallel FJ models. For this game, the cost function is shown to be convex in the action profile set, but discontinuous at some boundary points when for some FJ model only one agent is stubborn (i.e., assigning nonzero stubbornness in the FJ model). Despite the discontinuity, an Nash equilibrium is shown to exist, but is not necessarily unique. Some sufficient conditions that can guarantee the uniqueness are proposed, relying on the strictly monotone pseudogradient mappings associated to the game. The conditions are applied to complete graphs with rank-1 weight matrices, for which the link weights are unequal for different agents and on different FJ models. Moreover, for the complete graph case, given the actions of the other agents, the best response of each agent is analytically characterized.**

***Index Terms*—Complex networks, network systems, multi-agent systems, social dynamics.**

I. Introduction

A. Background and Related Works

CONSIDER a social network in which the agents reciprocally influence each other through their interactions on the network, and a dynamical rule (typically based on some form of averaging) is used to update the opinions of the agents. To measure the influence that each agent has in such opinion propagation process, one can for instance use different centrality measurements [48], [51], which are determined merely by the network topology. However, for a process of opinion formation, a more appropriate assessment should also consider the specific dynamical rule that is used to propagate the opinions [13], [23], [28]. For averaging-based models, a natural way to describe an agent's influence on the opinion outcome is to use the notion of social power. For the DeGroot model, for instance, the social power of an agent can be defined as the agent's weight in the weighted average that constitutes the endpoint consensus value [2], [31]. Specifically, for this case, the definition of social power coincides with that of eigenvector centrality. This definition can also be applied to other DeGroot-like models, such as the Friedkin–Johnsen (FJ) model [23], [35], in which agents are stubbornly defending their own initial opinions. For the FJ model, social power is no longer equal to eigenvector centrality, but can still be regarded as an approximation of it if multiple FJ models are concatenated in a sequence [6].

Apart from "passive" opinion propagation, a social network of interactions can be seen as the medium through which an agent can "actively" spread her influence to the other agents, and thereby try to impact the decision made collectively by the group of agents. A widespread objective is to try to steer the opinions of the agents towards that of an "influencer." For instance, a problem widely studied for opinion diffusion models is seed selection [8], [32], in which an external influencer needs to carefully choose the seed set to maximize the number of final "infected" agents. Other variants of this problem on averaging-based opinion models are also studied in literature, depending on the various possible optimization targets and actions of the influencers [25], [34], [37], [55]. The idea of opinion influence maximization can be extended to the case of two or more influencers, and a game formulation is normally used for the model description. More specifically, for some underlying opinion dynamics model, influencers compete with each other by investing their limited resource to a group of selected agents, and a zero-sum game is often constructed in this way [3], [14], [16], [26].

When it comes to modeling influence maximization, there are several limitations in the existing literature. First, to the best of the authors' knowledge, most of the papers deal with external influencers. That is, agents who want to maximize influence always keep their own opinions unchanged. However,

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it is common to see changes in people's own opinions even as they try to persuade someone else. Second, the payoff is normally computed in terms of the opinions themselves, not of more intrinsic quantities like the social power. Third, in the game-theoretical formulations, the players only deal with one playground, while competition over multiple "battlefields" exist in many real-world scenarios, such as electoral campaigns and competitive advertising [20], [40]. With the first and second perspectives in mind [5], [58] developed a strategic game for the so-called concatenated FJ model, in which the action of each agent is her stubbornness during the opinion-forming process. The idea of taking the stubbornness of an FJ model as an action is also evidenced in [1]. Unlike the aforementioned literature, the utility of each agent in [58] is her final social power, and the resulting game is called a social power game. The current paper further develops the notion of social power game, and applies it to a scenario in which multiple opinion forming processes on independent issues are considered. Interestingly, each of the opinion forming processes can be regarded as a battlefield over which the agents compete with each other.

B. Main Contributions

The setting we consider here is one of multidimensional opinions: each agent has a vector of opinions on a number of subjects that are discussed with the other agents. For each discussion, an FJ model is applied, but unlike many other works considering interdependent opinions [38], [43], [61], the topics under discussion are considered independent, and the interaction network need not be the same across the topics. Therefore, we skip the tensor product formalism adopted in [43] and treat the discussion on each topic as occurring disjointly, independently, and simultaneously. For simplicity, we refer to our current social power game as occurring on parallel FJ models. In the social power game, the action of each agent is her stubbornness over the FJ models, and the utility is a function of the social power itself (rather than of the opinions, as in the vast majority of the literature). More precisely, to accommodate the parallel nature of the FJ models, the utility is taken as the sum of the social powers accumulated in each of the parallel FJ meetings we consider. Under such a formulation, the social power game for parallel FJ models is strategically equivalent to a zero-sum game.

The main technical problem considered in this article is the existence and uniqueness of Nash equilibrium (NE) for the social power game. This is a basic problem for a strategic game, widely investigated for different games [4], [17], [42], [49]. For the social power game on parallel FJ models, it is shown that the cost function of each agent is convex in the feasible set of action profiles (Proposition 1) but discontinuous for some boundary points (see Proposition 2). The discontinuity of the cost functions, despite holding on a zero-measure set, poses an issue when proving the existence of the NE, as the standard results for continuous games cannot be applied directly. On the other hand, even for three-player zero-sum games with continuous and convex cost functions, there is no general result to guarantee the uniqueness of NE, let alone for the multiplayer games with discontinuous cost functions studied in this article. Therefore, a careful analysis tailored to the considered game is needed. Specifically, our main technical contributions are summarized as follows.

1) For a single discussion (i.e., a single FJ model), it is proved that the best strategy of each agent is to increase her stubbornness as much as possible (see Theorem 1), which means that the unique NE for the game is that all the agents are as stubborn as possible.
2) For multiple parallel discussions, we first prove the existence of NE for the social power game when its action space is constrained to be strictly contained in the interior of the positive orthant (see Proposition 3). Then, using a property of the constrained NE, namely, that it must be bounded away from the origin (see Lemma 6), the original social power game is shown to have an NE, which can be approximated by a sequence of NEs of constrained social power games (see Theorem 2). Moreover, if the underlying graphs for the parallel FJ models are the same, an NE for the game is that all agents distribute their resource of stubbornness equally over the discussions (see Proposition 4).
3) To guarantee the uniqueness of NE, we propose a variant of a widely used condition that assumes strict monotonicity of the pseudogradient mapping (see Assumption 3). With this condition, any constrained social power game is first proved to have a unique NE (see Proposition 5). Then, by using the equivalence between variational inequality (VI) problems and NE seeking problems, the uniqueness of NE for the original social power game is proved through a sensitivity analysis of the associated VIs (see Theorem 3).
4) The special case that the underlying graphs are all rank-1 complete graphs is studied to give more insights. For this case, the strong monotonicity condition holds if the edge weights of each graph are close to each other (see Lemma 10). In particular, if the graphs are homogenous and with equal edge weights, the associated social power game will have a unique NE given that each agent has enough resource of stubbornness, and the NE corresponds to each agent being equally stubborn for every FJ model (see Theorem 4). If the graphs are heterogeneous and with three agents, under some bounds of the parameters of the game, a unique NE is also proved to exist (see Theorem 5). Moreover, given the actions of the other agents, the best response of an agent is characterized in Theorem 6.

Compared with the social power game considered in [5] and [58], the setting investigated in this article is rather different, since the social power game in [5] and [58] was based on sequential ("concatenated") discussions. The differences in the model formulation imply that the technical problems investigated in this article are totally different from those of [5] and [58]. For instance, the main feature of the game in [58], i.e., early mover advantage, no longer exists for the game studied here.

C. Motivating Examples for the Model

Let us illustrate the motivation for studying the social power game for parallel FJ models through some examples.

One reason to consider parallel FJ models is that, in the real world, complex decision processes are often broken down into several smaller issues, each discussed in a separate meeting. These meetings need to maintain some independence and particularity so that the final decision can take into account different factors. A concrete example is the United Nations climate talks [6]. Every year multiple meetings are held by different committees, with each committee in charge of a specific aspect of climate change. Several of these meetings are held in parallel during a two-week window of time, usually in November and December, and contribute to the resolutions that are finalized in the so-called conference of the parties (COP) of which the committees are subsidiary bodies. By considering an FJ model for the opinion evolution in each meeting, the social power game considered in this article can capture the strategic involvement of the parties participating in the climate talks, in the sense that each party wants to exert an influence on the final resolution of the COP, which is as large as possible.

Additional motivation for why we consider parallel FJ models comes from multilayer social networks [15]. For instance, the same group of individuals might engage in both a WhatsApp group (for informal socializing) and a Teams channel (for structured work coordination). Each platform has its own interaction topology and dynamics, modeled via separate FJ models. While social power in a single network reflects sustained leadership in opinion dynamics, our social power game characterizes agents' competition for leadership across all layers simultaneously.

Another example of parallel FJ models appears in the legislative process in parliamentary democracies. Often, during its redaction, a law passes through different committees in charge of different aspects (e.g., constitutional compatibility, social impact, economic sustainability). It is common that the same members of Parliament (MPs) belong to different subcommittees (or, lumping together MPs into political parties [22], that the same parties are represented in multiple subcommittees [18]). MPs (or parties) allocate their influence across the various subcommittees in such a way to maximize their overall leadership. This phenomenon can be naturally represented within our framework of parallel FJ models applied to overlapping agent sets.

An important characteristic of the social power game is that stubbornness can be considered as the action of each agent. In the context of the UN climate negotiations, this comes from the observation that stubbornness of an agent can be linked to the number of speaking occasions in a meeting—speaking more can be considered a proxy to being more stubborn, see [5] and [6], for a quantitative analysis on real data. In this context, imposing "budget constraints," such as $K_i, i \in \mathcal{V}$, is natural, and motivated by the fact that meetings cannot go on forever and that no agent can have the monopoly of the discussion. More stubbornness intuitively leads to more social power, but the presence of constraints makes the game nontrivial.

Mathematically, the social power game is similar to the multiplayer Colonel Blotto game [7], [41], which is a resource allocation game describing the competition among a group of agents over multiple battlefields. In the social power game, each meeting or FJ model represents a battlefield, and is valued equally by all the agents. However, instead of obeying a winner-take-all rule as in standard Blotto games, the agents in each FJ model play a constant-sum game whose outcome is linked to how the opinions evolve during the dynamics. In a general sense, the social power game can be categorized as a special type of contest game [9] consisting of multiple subcontests (i.e., FJ models), with the contest success function (the probability of winning the contest) of each FJ model as the social power, which is determined by the underlying opinion formation process and highly nonlinear in the resource (i.e., stubbornness) put into that contest.

D. Organization of the Paper

The rest of this article is organized as follows. Section II gives the notations and some background material. Section III presents the model and formulates the problems of interest. The main results are included in Sections IV and V. Finally, Section VI concludes this article. All proofs are given in the Appendices.

A preliminary version of this paper appeared in ECC24 [59], and only considers the single meeting case, the constrained social power game, and the graphs with uniform weights, with all the proofs omitted. It should be noted that [59] mistakenly claims nonconvexity of the cost functions on some boundary profiles. This is not the case, instead the cost functions are convex but discontinuous on some boundary profiles.

II. Notations and Preliminaries

A. Notations

The set of real numbers and integers are denoted by $\mathbb{R}$ and $\mathbb{Z}$, respectively. All vectors are real column vectors and are denoted by bold lowercase letters, such as $\mathbf{x}$ and $\mathbf{y}$. The ith entry of a vector $\mathbf{x}$ is denoted by $[\mathbf{x}]_i$ or, if no confusion arises, x_i. The symbol $\mathrm{diag}(\mathbf{x})$ represents the square matrix with diagonal entries equal to the entries of $\mathbf{x}$ and the others equal to 0. Matrices are denoted by the normal upright letters, such as $\mathrm{A}, \mathrm{B}, \ldots,$ of entries $[\mathrm{A}]_{ij}$ or a_{ij} (here lowercase letters represent scalars). Given a vector $\mathbf{x}_k$ (or a matrix A_k) with subscript k, we use $x_{i,k}$ (or $a_{ij,k}$) to represent its ith (or ijth) entry. The identity matrix is denoted by I_n, with dimension sometimes omitted, depending on the context. The n-order vector and matrix with all entries being 0 or 1 are denoted by $\mathbf{0}_n$ or $\mathbf{1}_n$, respectively with the dimensions omitted if there is no confusion. Let $\mathbf{e}_i$ be the vector with the ith entry as 1 and all the others as 0. We use $[n]$ to represent the set $\{1, \ldots, n\}$. Given a set $\mathcal{C}$, we use $|\mathcal{C}|$ to denote its cardinality. For a finite set of vectors $\mathbf{c}_i, i \in \mathcal{V}$, use $(\mathbf{c}_i)_{i\in\mathcal{V}}$ to denote $(\mathbf{c}_1^\top, \ldots, \mathbf{c}_{|\mathcal{V}|}^\top)^\top$, and the subscript is sometimes omitted when no confusion arises, i.e., $(\mathbf{c}_i)$; for finitely many vector sets $\mathcal{C}_i, i \in \mathcal{V}$, their product is denoted as $\Pi_{i\in\mathcal{V}}\mathcal{C}_i$ or $(\mathcal{C}_i)_{i\in\mathcal{V}}$ (with the subscript sometimes omitted), i.e., $(\mathcal{C}_i) = \{(\mathbf{c}_i) : \mathbf{c}_i \in \mathcal{C}_i\}$. Given $\mathbf{x}^* \in \mathbb{R}^n$, let $\mathcal{B}_r(\mathbf{x}^*)$ be the closed ball centered in $\mathbf{x}^*$ with radius r, i.e., $\mathcal{B}_r(\mathbf{x}^*) = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{x}^*\| \le r\}$. Given two square matrices $\mathrm{A}, \mathrm{B} \in \mathbb{R}^{n\times n}$, use $\mathrm{A} \succeq \mathrm{B}$ to imply that $\mathrm{A} - \mathrm{B}$ is positive semi-definite; $\mathrm{A} \ge \mathrm{B}$ means that $\mathrm{A}_{ij} \ge \mathrm{B}_{ij}$ for all $i, j \in [n]$; A is called substochastic if $\mathrm{A} \ge \mathbf{0}$ and $\mathbf{1} \ge \mathrm{A}\mathbf{1}$, and if the equality holds, A is called stochastic, otherwise A is called strictly substochastic. Given a real number x, let $\lfloor x \rfloor$

be the nearest integer that is no larger than x. For a sequence of sets $\mathcal{C}_k, k \in \mathbb{Z}_+$ and a set $\mathcal{C}$, we use $\mathcal{C}_k \uparrow \mathcal{C}$ to denote that $\mathcal{C}_k \subset \mathcal{C}_{k+1}\ \forall k \in \mathbb{Z}_+$ and $\lim_{k\to\infty} \mathcal{C}_k = \mathcal{C}$.

B. FJ Model

Consider a network with nodes (agents) indexed in $\mathcal{V} = [n]$. It is represented by a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{E}$ is a set of ordered pairs of nodes and $(i, j) \in \mathcal{E}$ represents a link from node i to node j. A (directed) *path* is a concatenation of directed links of $\mathcal{E}$. We say that node i is connected to node j if there is a directed path from i to j. The graph $\mathcal{G}$ is called *strongly connected* if any two nodes are connected to each other. Given a stochastic matrix $\mathrm{W} = [w_{ij}] \in \mathbb{R}^{n\times n}$, its associated digraph is denoted $\mathcal{G}_{\mathrm{W}} = (\mathcal{V}, \mathcal{E}_{\mathcal{V}})$, where $\mathcal{V}$ and $\mathcal{E}_{\mathcal{V}}$ are the sets of nodes and edges respectively, with $\mathcal{V} = [n]$ and $(i, j) \in \mathcal{E}_{\mathcal{V}}$ if and only if $w_{ji} > 0$.

The FJ model is a DeGroot-like model for opinion dynamics in which some agents behave stubbornly, in the sense that they defend their initial positions while discussing with the other agents [23]. If n agents participate in a discussion, the FJ model is

$$\mathbf{y}(t+1) = (\mathrm{I} - \Theta)\mathrm{W}\mathbf{y}(t) + \Theta\mathbf{y}(0), \quad t = 0, 1, \ldots \tag{1}$$

where $\mathbf{y}$ is the n-dimensional opinion vector, W is a row-stochastic matrix, and $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_n)$, with $\theta_i \in [0, 1]$ representing the stubbornness of agent i. Stubbornness here means attachment of an agent to her own opinion, represented by the initial condition $\mathbf{y}(0)$ at the beginning of the discussion ($\theta_i = 0$ means agent i is not stubborn, $\theta_i = 1$ means a totally stubborn agent). Note that stubbornness is different from the so-called self-appraisal [31] [i.e., the weights of self-loops w_{ii} in (1)], representing how much the agents support their current opinions (instead of the initial ones) in the update law.

Let $\mathcal{G}_{\mathrm{W}}$ be the graph associated to W in the FJ model.

Lemma 1 (See [44]): Assume $\theta_i \in [0, 1]$ for all $i = 1, \ldots, n$, and $\theta_i > 0$ for at least one i. If $\mathcal{G}_{\mathrm{W}}$ is strongly connected, then

a) $(\mathrm{I} - \Theta)\mathrm{W}$ is Schur stable, i.e., $\rho((\mathrm{I} - \Theta)\mathrm{W}) < 1$

b) the matrix $\mathrm{V} = (\mathrm{I} - (\mathrm{I} - \Theta)\mathrm{W})^{-1}\Theta$ is stochastic, and

c) $\mathbf{y}(\infty) = \lim_{t\to+\infty} \mathbf{y}(t) = \mathrm{V}\mathbf{y}(0)$.

Lemma 2 (See [57]): Let all the conditions of Lemma 1 hold. If $\theta_i = 0$, it holds $\mathrm{V}_{ji} = 0\ \forall j \in \mathcal{V}$. If $\theta_i > 0$, $\mathrm{V}_{ii} > 0$, and for $j \neq i$, $\mathrm{V}_{ji} > 0$ holds if and only if there exists a path $i \to j_1 \to j_2 \to \cdots \to j_H = j$ such that $\theta_{j_h} < 1\ \forall h \in [H]$.

C. Game Theory and VI

The definitions about strategic games are from [39].

Definition 1 (Strategic game): Given

1) a finite set of players $\mathcal{V}$,
2) an action set $\mathcal{A}_i \subseteq \mathbb{R}^M$ for each $i \in \mathcal{V}$, and
3) a utility function $u_i : \mathcal{A} \mapsto \mathbb{R}$ for each $i \in \mathcal{V}$, with $\mathcal{A} := \Pi_{j\in\mathcal{V}}\mathcal{A}_j \subset \mathbb{R}^{M|\mathcal{V}|}$.

The tuple $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$ is called a strategic game.

Any $\mathbf{a} = (\mathbf{a}_i)_{i\in\mathcal{V}} \in \mathcal{A}$ is called an *(action) profile*, for which each $\mathbf{a}_i$ is an *action*. Given $i \in \mathcal{V}$, we use $\mathbf{a}_{-i} = (\mathbf{a}_j)_{j\in\mathcal{V}\setminus\{i\}}$ to denote the collection of actions of all agents but i. With a slight abuse of notation, the utility function $u_i(\mathbf{a})$ is sometimes denoted $u_i(\mathbf{a}_i, \mathbf{a}_{-i})$ to emphasize its dependency on $\mathbf{a}_i$.

Let $\mathbf{u}(\mathbf{a}) = (u_i(\mathbf{a}))_{i\in\mathcal{V}} \in \mathbb{R}^{|\mathcal{V}|}$ be the vector of all utilities w.r.t. the profile $\mathbf{a}$. If the utility functions u_i are all differentiable, the pseudogradient mapping of the game $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$ is defined as $\nabla_{\mathbf{a}}\mathbf{u}(\cdot) : \mathbb{R}^{M|\mathcal{V}|} \mapsto \mathbb{R}^{M|\mathcal{V}|}$, with

$$\nabla_{\mathbf{a}}\mathbf{u}(\bar{\mathbf{a}}) = (\nabla_{\mathbf{a}_i} u_i(\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_{-i}))_{i\in\mathcal{V}} \quad \forall \bar{\mathbf{a}} \in \mathcal{A}.$$

In a strategic game, one important concept is that of NE, which is a special action profile for which no player has any incentive to change her strategy.

Definition 2 (NE): Given a strategic game $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$, a profile $\mathbf{a}^* = (\mathbf{a}_i^*)_{i\in\mathcal{V}}$ is an NE if $\mathbf{a}_i^* \in \arg\max_{\mathbf{a}_i\in\mathcal{A}_i} u_i(\mathbf{a}_i, \mathbf{a}_{-i}^*)$ holds for all $i \in \mathcal{V}$.

The following definitions come from [52].

Definition 3 (VI problem): Given a closed and convex set $\Omega \subseteq \mathbb{R}^n$ and a mapping $T : \Omega \mapsto \mathbb{R}^n$, the VI problem, denoted $\mathrm{VI}(\Omega, T)$, consists in finding a vector $\mathbf{x}^* \in \Omega$ such that

$$(\mathbf{y} - \mathbf{x}^*)^\top T(\mathbf{x}^*) \geq 0 \quad \forall \mathbf{y} \in \Omega$$

and $\mathbf{x}^*$ is called a solution of $\mathrm{VI}(\Omega, T)$. The set consisting all of solutions of $\mathrm{VI}(\Omega, T)$ is denoted $\mathcal{S}(\Omega, T)$.

Given a mapping $T : \mathbb{R}^n \to \mathbb{R}^n$, its *graph* is $\mathcal{G}(T) = \{(\mathbf{x}, T\mathbf{x}) \in \mathbb{R}^n \times \mathbb{R}^n : \mathbf{x} \in \mathcal{D}(T)\}$, where $\mathcal{D}(T)$ denotes the domain of T. The mapping T is called *monotone* if $\langle T\mathbf{x} - T\mathbf{y}, \mathbf{x} - \mathbf{y}\rangle \geq 0\ \forall \mathbf{x}, \mathbf{y} \in \Omega$, *strictly monotone* if the inequality holds strictly, and (σ-)*strongly monotone* if for some $\sigma > 0$, $\langle T\mathbf{x} - T\mathbf{y}, \mathbf{x} - \mathbf{y}\rangle \geq \sigma\|\mathbf{x} - \mathbf{y}\|^2$ holds for all $\mathbf{x}, \mathbf{y} \in \Omega$. Given a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ defined on a convex set Ω, it is called *convex* if its gradient operator ∇f is monotone on Ω, and *strictly* convex if ∇f is strictly monotone on Ω.

With these definitions, the following result is a direct application of [36, Thm. A(b)].

Lemma 3: Consider a sequence of VIs $\mathrm{VI}(\Omega_k, T_k)$, with T_k monotone and continuous, Ω_k convex and compact. Suppose there exist a convex compact set Ω and a monotone, continuous operator T such that: 1) $\Omega \subseteq \mathcal{D}(T)$; 2) $\Omega_k \uparrow \Omega \subseteq \mathbb{R}^n$; 3) $\lim_{k\to\infty} \mathcal{G}(T_k) = \mathcal{G}(T)$; 4) T_k can be continuously extended to Ω and is uniformly bounded on Ω. If for each $k > 0$, $\mathrm{VI}(\Omega_k, T_k)$ has a unique solution $\mathbf{x}_k$, and $\mathrm{VI}(\Omega, T)$ has a unique solution $\mathbf{x}^*$, then $\lim_{k\to\infty} \mathbf{x}_k = \mathbf{x}^*$.

The next lemma is an application of [11, Thm. 2.1].

Lemma 4: Let $\Omega_\psi \subset \mathbb{R}^n$ be a compact set indexed by $\psi \in \mathbb{R}_{\geq 0}$, such that $\Omega_{\psi_2} \subset \Omega_{\psi_1}$ for any $\psi_1 \leq \psi_2$. Consider the VIs $\mathrm{VI}(\Omega_\psi, T)$ for some $T : \Omega_0 \mapsto \mathbb{R}^n$. Assume that $\mathrm{VI}(\Omega_0, T)$ admits a solution $\mathbf{x}^*$. Suppose that for some $\mathcal{B}_r(\mathbf{x}^*)$ with $r > 0$, the mapping T is strongly monotone and Lipschitz continuous over $\Omega_0 \cap \mathcal{B}_r(\mathbf{x}^*)$. Then, there exists $\bar{\psi} > 0$ such that if $\psi < \bar{\psi}$, $\mathrm{VI}(\Omega_\psi, T)$ admits a unique solution $\mathbf{x}_\psi$ in $\Omega_0 \cap \mathcal{B}_r(\mathbf{x}^*)$, and $\lim_{\psi\to 0} \mathbf{x}_\psi = \mathbf{x}^*$.

The following lemma is from to (18) in [52], and gives the relation between NE and the solution of the corresponding VI.

Lemma 5: Given the game $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$, if for each $i \in \mathcal{V}$

1) the action set $\mathcal{A}_i$ is closed and convex and
2) the utility function $u_i(\mathbf{a}_i, \mathbf{a}_{-i})$ is continuously differentiable in $\mathbf{a}$ and concave in $\mathbf{a}_i$ for every $\mathbf{a}_{-i}$.

Then, a profile $\mathbf{a}^*$ is an NE of the game $\langle \mathcal{V}, \mathcal{A}, (u_i)\rangle$ if and only if $\mathbf{a}_i^* \in \mathcal{S}(\mathcal{A}_i, -\nabla_{\mathbf{a}_i} u_i(\cdot, \mathbf{a}_{-i}^*))\, \forall i \in \mathcal{V}$, or in compact form, $\mathbf{a}^* \in \mathcal{S}(\mathcal{A}, -\nabla_{\mathbf{a}} \mathbf{u}(\cdot))$.

III. Model and Problem Formulation

A. Social Power Game

Consider M independent discussions, with the same group of participants described by an agent set $\mathcal{V} = [n]$. For each discussion, say for the mth discussion, the agents interact with their neighbors over a graph $\mathcal{G}_m = (\mathcal{V}, \mathcal{E}_m, \mathrm{W}_m)$, with $\mathrm{W}_m = [w_{ij,m}]_{i,j\in\mathcal{V}} \in \mathbb{R}_{\geq 0}^{n\times n}$ a stochastic weight matrix, possibly varying with $m \in [M]$. The opinions of all agents are collected in a vector $\mathbf{x}^m(t) = (x_{i,m}(t))_{i\in\mathcal{V}}$, with t indexing the time axis in the mth discussion. Based on the interactions, each agent changes her own opinion, and the overall opinion evolution is described by a FJ model, that is

$$\mathbf{x}^m(t+1) = (\mathrm{I} - \Theta_m)\mathrm{W}_m \mathbf{x}^m(t) + \Theta_m \mathbf{x}^m(0)$$

where $\Theta_m = \mathrm{diag}(\theta_{1,m}, \theta_{2,m}, \ldots, \theta_{n,m})$ represents the stubbornness of each agent, i.e., the attachment to their initial opinions $\mathbf{x}^m(0)$, with $\theta_{i,m} \in [0,1]$ for all $i \in \mathcal{V}$ and $m \in [M]$. The following assumption is used throughout this article.

Assumption 1: For all $m \in [M]$, the graph $\mathcal{G}_m$ is strongly connected, with each node having a self loop, i.e., $\underline{w} := \min_{i\in\mathcal{V},\, m\in[M]} w_{ii,m} > 0$.

Define $\bar{w} := \max_{i,j\in\mathcal{V},\, m\in[M]} w_{ij,m}$ as the maximum link weight over all the graphs. Under Assumption 1, it must be $\bar{w} < 1$. Note that the assumed strong connectivity and existence of self-loops are commonly used in literature [13], [44], [45].

Under Assumption 1, the opinion vector for each FJ model converges [23], [44]. Moreover, if there exists some $\theta_{i,m} > 0$, from Lemma 1, the solution of the FJ model will be

$$\mathbf{x}^m(\infty) := \lim_{t\to\infty} \mathbf{x}^m(t) = \mathrm{P}_m \mathbf{x}^m(0)$$

with

$$\mathrm{P}_m = \mathrm{Q}_m^{-1}\Theta_m, \quad \mathrm{Q}_m = \mathrm{I} - (\mathrm{I} - \Theta_m)\mathrm{W}_m. \tag{2}$$

According to Lemma 1, P_m is a stochastic matrix. Let

$$\mathrm{P}_m = [\mathbf{p}_{1,m}, \ldots, \mathbf{p}_{n,m}], \quad \mathrm{Q}_m^{-1} = [\mathbf{q}_{1,m}, \ldots, \mathbf{q}_{n,m}] \tag{3}$$

with $\mathbf{p}_{i,m}, \mathbf{q}_{i,m}$ being the ith columns of P_m and Q_m^{-1}, respectively. Each entry $p_{ji,m}$ of $\mathbf{p}_{i,m}$ represents the influence of agent i's initial opinion on the final opinion of agent j. In this sense, the matrix P_m encodes the social power of each agent [31], [56]. In this article, we follow the definition of social power from [56]. Given an agent $i \in \mathcal{V}$, her social power in the mth discussion is

$$\mathrm{sp}_{i,m} := \frac{1}{n}\mathbf{1}^\top \mathbf{p}_{i,m}. \tag{4}$$

Remark 1: In the mathematical sociology literature, the term social power can be given different meanings from the one of this article [21]. The definition (4) provides a metric strictly tailored to the FJ model and available in closed form once $\mathbf{p}_m$ is given. Intuitively, the social power defined in (4) represents an agent's average influence on the opinion outcome of the group, including the impact on her own opinion.

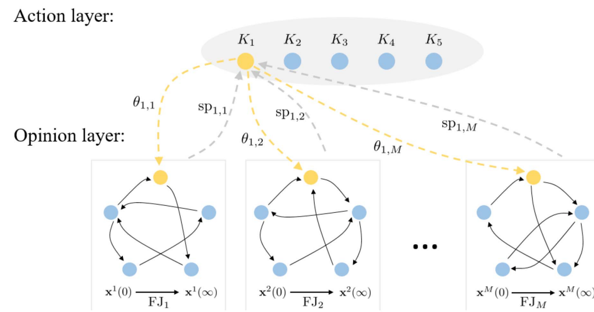

Fig. 1. Hierarchical representation of a five-player social power game.

In this article, it is considered that the agents can decide how much stubbornness will be devoted to each discussion, in order to obtain the highest total social power over all the M discussions. To avoid trivial solutions, it is assumed that the total amount of stubbornness of each agent i is bounded by a constant $K_i \in (0, M)$, i.e., $\sum_{m=1}^M \theta_{i,m} \leq K_i$. In more detail, we investigate a strategic game defined as follows.

1) *Players*: The agents are involved in all the M discussions, i.e., $\mathcal{V} = [n]$.
2) *Actions*: For each agent $i \in \mathcal{V}$, the feasible set of her actions is

$$\mathcal{A}_i = \{\boldsymbol{\theta}_i = (\theta_{i,1}, \theta_{i,2}, \ldots, \theta_{i,M})^\top \text{ s.t.} \quad \sum_{m=1}^{M} \theta_{i,m} \leq K_i \text{ and } \theta_{i,m} \in [0,1] \quad \forall m \in [M]\}.$$

3) *Utility functions*: The utility of each agent $i \in \mathcal{V}$ is the corresponding total social power, i.e.,

$$u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}) = \sum_{m=1}^{M} \mathrm{sp}_{i,m}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}).$$

The strategic game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ belongs to a class of games which we call *social power games*, as they have the social power as utility. Note that each graph $\mathcal{G}_m$ is fixed.

Remark 2: The social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ adopts a two-layer hierarchical structure, with an action layer determining the allocation of stubbornness, and an opinion layer reflecting the opinion evolution described by the FJ models. This structure is shown in Fig. 1

for a five-player special case. In fact, an FJ model can be regarded as the best-response dynamics of a quadratic game [24]. The formulation of social power game thus resembles that of hierarchical games [10], [27], with each FJ model in the opinion layer corresponding to a quadratic subgame, affected by the actions of all agents (i.e., $\{\boldsymbol{\theta}_i\}$) in the action layer. Different from classic hierarchical games, such as leader–follower games [30], the players in both layers of the social power game are the same, and the cost functions in the action layer are not affected by the equilibrium states of the opinion layer directly, but determined by the social power that is independent of specific opinion values.

Remark 3: From the definition of social power in (4), it is easy to see that $\sum_{i\in\mathcal{V}} u_i(\boldsymbol{\theta}) = M$ holds for any $\boldsymbol{\theta} \in (\mathcal{A}_i)$. This means that the social power game is a constant-sum game, and hence, it is strategically equivalent to a zero-sum game.

B. Problem of Interest

For the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$, a first problem of interest is the existence and uniqueness of the NE. As will be shown in the following, the cost functions of the game are discontinuous on some boundary points of the feasible set of action profiles. Therefore, standard analysis for convex games cannot be applied directly to our social power game. For this reason, we propose below some conditions that guarantee the existence and uniqueness of the NE. Another related problem is to understand the best response of each agent i, that is, given the actions of the other agents, what action should agent i take in order to maximize her total social power? Here, action means allocation of her budget of stubbornness across the M discussions. Written in a mathematical form, the following optimization problem needs to be solved

$$\begin{aligned} \underset{\boldsymbol{\theta}_i}{\text{Minimize}} \quad & -\sum_{m=1}^{M} \text{sp}_{i,m}(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}) \\ \text{s.t.} \quad & \sum_{m=1}^{M} \theta_{i,m} \le K_i; \quad 0 \le \theta_{i,m} \le 1 \quad \forall m \in [M]. \end{aligned} \tag{5}$$

Here, $\boldsymbol{\theta}_{-i}$ is fixed, and $\text{sp}_{i,m}$ is a function of $\boldsymbol{\theta}_i$, or more specifically, $\theta_{i,m}$.

IV. Existence and Uniqueness of the NE

In this section, first we deal with the social power optimization problem (5), then the results are applied to investigate the properties of NE for the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$.

A. Single Meeting

Before going to more general cases, consider a single FJ model as the simplest case, i.e., $M = 1$. A straightforward intuition is that more stubbornness will result in a higher social power. The intuition is proved rigorously in this section.

For the simplicity of notation, we omit m in the subscripts, and use W, Θ, sp_i to represent the weight matrix, stubbornness matrix and social power of agent i, respectively. Moreover, we use θ_i instead of $\boldsymbol{\theta}_i$ to indicate that the action of agent i is a scalar.

Theorem 1: Consider the social power optimization problem (5) with $M = 1$. Let Assumption 1 hold. If there exists $j \ne i$ such that $\theta_j > 0$, the solution to the problem (5) is $\theta_i = K_i$; otherwise any $\theta_i \in (0, K_i]$ is the solution of (5).

The proof of Theorem 1 is given in Appendix A.

Remark 4: Theorem 1 implies that $\theta_i = K_i$ is the only dominant strategy of agent i. Therefore, the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ has a unique NE, i.e., $\boldsymbol{\theta}^* = (K_i)_{i\in\mathcal{V}}$, regardless of the underlying network topology.

B. Multiple Meetings: Existence of the NE

Assume $M > 1$. It is hard to give an analytical solution to the problem (5). However, as shown in the following, the problem (5) is a convex optimization problem.

Proposition 1: Consider the social power optimization problem (5) with $M > 1$. Let Assumption 1 hold. The cost function $-u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})$ is convex in the argument $\boldsymbol{\theta}_i \in [0,1]^M$. Moreover, if for all $m \in [M]$, there exists $j \ne i$ such that $\theta_{j,m} > 0$, then the cost function is strictly convex.

The proof of Proposition 1 is given in Appendix B.

According to the well-known result in [49], if a strategic game has continuous and convex cost functions, then an NE must exist. However, for the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$, the cost function $-u_i(\cdot, \boldsymbol{\theta}_{-i})$ is discontinuous on the boundary profiles corresponding to all the other agents taking zero stubbornness in some FJ model.

Proposition 2: Consider a profile $\boldsymbol{\theta}^0 \in (\mathcal{A}_i)$ such that $\theta^0_{i,m_0} = 0 \ \forall i \in \mathcal{V}$ for a given $m_0 \in [M]$. For any $i \in \mathcal{V}$, the cost function $-u_i(\cdot, \boldsymbol{\theta}^0_{-i})$ is discontinuous at $\boldsymbol{\theta}^0_i$.

The proof of Proposition 2 is given in Appendix B.

Remark 5: In general, for a strategic game, discontinuity of the cost functions on even one boundary point could lead to inexistence of (pure) NE, see Sion's example in [54]. Toward this problem, various existence conditions have been proposed [12], [46], [47]. However, none of them can be applied directly to the game in this article. For instance, the condition in [12] requires the lower semi-continuity of the so-called "value function," which is hard to verify for the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$. The condition in [46] instead requires that for all $\boldsymbol{\theta}_i \in \mathcal{A}_i$, $u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i})$ is lower semi-continuous in the argument $\boldsymbol{\theta}_{-i}$, which does not hold when we take $\boldsymbol{\theta} = \mathbf{0}$.

To prove the existence of the NE in this article, we first consider a constrained game $\langle \mathcal{V}, (\mathcal{A}^\delta_i), (u_i)\rangle$, and then use it to approximate the original social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$. Specifically, $\langle \mathcal{V}, (\mathcal{A}^\delta_i), (u_i)\rangle$ is a social power game with the same utility (or cost) functions as those in $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$, but the action profile $\mathcal{A}^\delta$ is a subset of $\mathcal{A}$, defined as

$$\mathcal{A}^\delta := (\mathcal{A}^\delta_i), \quad \mathcal{A}^\delta_i := [\delta, 1]^M \cap \mathcal{A}_i \quad \delta > 0.$$

This means that each agent should have some stubbornness, which is different from $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$.

From [49, Thm. 1], the following proposition holds.

Proposition 3: Consider the constrained social power game $\langle \mathcal{V}, (\mathcal{A}^\delta_i), (u_i)\rangle$. Under Assumption 1, an NE exists.

Furthermore, any NE of the constrained social power game is bounded away from $\mathbf{0}$ even for an arbitrarily small δ.

Lemma 6: Consider the constrained social power game $\langle \mathcal{V}, (\mathcal{A}^\delta_i), (u_i)\rangle$. Let Assumption 1 hold. For any of its NEs, denoted as $\boldsymbol{\theta}^{\delta*} = (\boldsymbol{\theta}^{\delta*}_i)$, there exists an $\bar{\epsilon} > 0$ such that for all $\delta \in (0,1)$ and $m \in [M]$, it holds $\max_{i\in\mathcal{V}} \theta^{\delta*}_{i,m} \ge \bar{\epsilon} > 0$.

The proof of Lemma 6 is given in Appendix B.

Remark 6: The idea of the proof is to show that if all agents take small stubbornness at some meeting, there must be an agent who can increase her overall social power by being more stubborn at the meeting. From the proof, $\bar{\epsilon}$ can take any value such that $\frac{1-\bar{w}}{n}\mathbf{1}^\top \mathbf{q}_{i,m} > \frac{M}{\min_{j\in\mathcal{V}} K_j}$ $\forall m \in [M]$, which does not depend on δ. Moreover, the proof still holds for $\delta = 0$, that is, any NE $\boldsymbol{\theta}^*$ of the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ satisfies $\max_{i\in\mathcal{V}} \theta^*_{i,m} \geq \bar{\epsilon} > 0$ $\forall m \in [M]$.

With Lemma 6, the following theorem can be obtained.

Theorem 2 (Existence of NE): Let $M > 1$. Under Assumption 1, the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ admits an NE.

The proof of Theorem 2 is given in Appendix C.

From the proof of Theorem 2, it can be seen that as $\delta \to 0$, any accumulation point of the NEs of the constrained social power games $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$ becomes an NE of the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$.

If the underlying graph is fixed, i.e., $\mathrm{W}_1 = \cdots = \mathrm{W}_M$, an NE for the game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ can be that each agent allocates her stubbornness equally over all the meetings, as in the following Proposition 4.

Proposition 4: Consider the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ with a fixed graph $\mathcal{G}$ for all $m \in [M]$. Let Assumption 1 hold. The game has the NE $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_i^*)$, with $\boldsymbol{\theta}_i^* = \frac{K_i}{M}\mathbf{1}$ for all $i \in \mathcal{V}$.

The proof of Proposition 4 is given in Appendix C.

In Proposition 4, the equilibrium strategy of each agent is to allocate her budget of stubbornness evenly over all the FJ meetings. From the proof, the NE is generated by the convexity of the cost functions. This can also be understood intuitively: given that the network structure is issue-invariant and all the other agents take the equilibrium strategies, there is no particular benefit to be gained from assigning a greater degree of stubbornness to any particular issue.

C. Multiple Meetings: Uniqueness of the NE

Given a strategic game, we are also interested in whether the NE is unique or not, if it exists. For the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ with general and heterogeneous underlying topologies, the answer should be no, as indicated by the following example.

Example 1: Consider a two-player game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ with $\mathcal{V} = \{1, 2\}$. Let $K_1 = 3.4$, $K_2 = 0.45$, and $M = 5$. The weight matrices are $\mathrm{W}_1 = \mathrm{W}_2 = \mathrm{W}_3 = 0.5\mathbf{1}\mathbf{1}^\top$, and $\mathrm{W}_4 = \mathrm{W}_5 = \begin{pmatrix} 0.95 & 0.05 \\ 0.95 & 0.05 \end{pmatrix}$. It can be verified that the two profiles $\boldsymbol{\theta}^*$ and $\bar{\boldsymbol{\theta}}^*$ are both NEs, where

$$\boldsymbol{\theta}_1^* = (1, 1, 1, 0.2, 0.2)^\top, \quad \boldsymbol{\theta}_2^* = (0.15, 0.15, 0.15, 0, 0)^\top;$$
$$\bar{\boldsymbol{\theta}}_1^* = (1, 1, 1, 0.19, 0.21)^\top, \quad \bar{\boldsymbol{\theta}}_2^* = (0.15, 0.15, 0.15, 0, 0)^\top.$$

In fact, all convex combinations of $\boldsymbol{\theta}^*$ and $\bar{\boldsymbol{\theta}}^*$ are NEs.

Remark 7: For the NEs shown in Example 1, agent 2 always chooses to be nonstubborn for the last two meetings. Intuitively, this is because agent 1 takes a significant advantage (in the sense of gaining more social power) from the overall stubbornness budget and the edge weights for the meetings 4 and 5, which forces the agent 2 to put all her resource into the meetings for which it is easier for her to get more social power (i.e., the meetings 1–3). This advantage of agent 1 also gives her more "freedom" to act in the meetings 4 and 5, and results in multiple NEs. □

Example 1 indicates that for the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ to admit a unique NE, one needs to exclude the case that for some meetings only one agent is stubborn, while all the others stay silent (i.e., non-stubborn). Technically, the assumption is stated as follows, and will be discussed in more detail in the following Subsection IV-D.

Assumption 2: Let $\boldsymbol{\theta}^*$ be any of the NEs of the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$. For any $m \in [M]$, there exist at least two agents i_1 and i_2 such that $\theta^*_{i_1,m}, \theta^*_{i_2,m} > 0$.

Assumption 2 is not enough to guarantee the uniqueness of NE. We need to consider the pseudo-gradient of the cost functions in (5), i.e., $-\nabla_{\boldsymbol{\theta}}\mathbf{u}(\boldsymbol{\theta}) = -(\nabla_{\boldsymbol{\theta}_i} u_i(\boldsymbol{\theta}))_{i\in\mathcal{V}}$, where $\nabla_{\boldsymbol{\theta}_i} u_i(\boldsymbol{\theta}) = (\frac{\partial \mathrm{sp}_{i,1}(\boldsymbol{\theta})}{\partial \theta_{i,1}}, \ldots, \frac{\partial \mathrm{sp}_{i,M}(\boldsymbol{\theta})}{\partial \theta_{i,M}})^\top$ $\forall i \in \mathcal{V}$. Note that $-\nabla_{\boldsymbol{\theta}} u(\boldsymbol{\theta})$ is well-defined for all $\boldsymbol{\theta} \in \mathcal{A}^o$, where $\mathcal{A}^o$ is the set of action profiles with at least two agents being stubborn in each meeting, i.e,

$$\mathcal{A}^o := \{\boldsymbol{\theta} \in \mathcal{A} : 0 < \max_{i\in\mathcal{V}} \theta_{i,m} < \sum_{i\in\mathcal{V}} \theta_{i,m} \quad \forall m \in [M]\}.$$

Denote $\mathrm{J}(\boldsymbol{\theta}) \in \mathbb{R}^{nM\times nM}$ the Jacobian of the pseudogradient $-\nabla_{\boldsymbol{\theta}}\mathbf{u}$ at $\boldsymbol{\theta} \in \mathcal{A}^o$. The following assumption is made in this section.

Assumption 3: For all $\boldsymbol{\theta} \in \mathcal{A}^o$, there exists $\sigma(\boldsymbol{\theta}) > 0$ such that $\frac{\mathrm{J}(\boldsymbol{\theta})+\mathrm{J}(\boldsymbol{\theta})^\top}{2} \succeq \sigma(\boldsymbol{\theta})\mathrm{I}$.

Note that Assumption 3 is a technical condition that is widely used in uniqueness analysis of NE, see [19] and [52] for instance. Under Assumption 3, the pseudogradient $-\nabla_{\boldsymbol{\theta}} u(\boldsymbol{\theta})$ on any constrained action set $(\mathcal{A}_i^\delta)$ becomes strongly monotone, as shown in the following lemma.

Lemma 7: Consider the constrained social power game $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$. Let Assumptions 1 and 3 hold. The pseudo-gradient mapping $-\nabla_{\boldsymbol{\theta}}\mathbf{u}$ is σ_δ-strongly monotone, with $\sigma_\delta = \min_{\boldsymbol{\theta}\in(\mathcal{A}_i^\delta)} \sigma(\boldsymbol{\theta}) > 0$.

The proof of Lemma 7 is given in Appendix D.

Given that $-\nabla_{\boldsymbol{\theta}}\mathbf{u}$ is strongly monotone, the VI problem $\mathrm{VI}(\mathcal{A}^\delta, -\nabla_{\boldsymbol{\theta}} u)$ has a unique solution, which, combined with Lemma 5, directly gives the following proposition.

Proposition 5: Let Assumptions 1 and 3 hold. For any $\delta \in (0, 1)$, the constrained social power game $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$ admits a unique NE.

With Proposition 5, the uniqueness of NE for the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ can be obtained.

Theorem 3 (Uniqueness of NE): Let Assumptions 1–3 hold. The social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ has a unique NE.

The proof of Theorem 3 is given in Appendix D.

The proof of Theorem 3 relies heavily on Assumptions 2 and 3, which are however challenging to be verified for general network topologies. One might ask in what sense Assumptions 2 and 3 are valid. For this question, we will first propose some sufficient conditions for achieving Assumption 2 in the next subsection, and then study some special networks as applications of Assumptions 2 and 3 in Section V.

D. Sufficient Conditions for Assumption 2

Assumption 2 is a requirement on the properties of NEs, which is an indirect assumption, in the sense that it cannot be verified directly from the parameters of the game. Nevertheless, Assumption 2 will be satisfied if the following assumption is added to the game.

Assumption 4: For the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$, at least one of the following conditions holds:

1) there are two agents $i_1, i_2 \in \mathcal{V}$ with $K_{i_1}, K_{i_2} > M-1$;
2) there are two agents $i_1, i_2 \in \mathcal{V}$ with $K_{i_1}, K_{i_2} > \max_{i\in\mathcal{V}} K_i - \bar{\epsilon}$, where $\bar{\epsilon}$ is the constant defined in Lemma 6.

Lemma 8: For the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$, if Assumptions 1 and 4 hold, Assumption 2 must hold.

The proof of Lemma 8 is given in Appendix E.

Assumption 4 only makes constraints on the stubbornness budgets $\{K_i : i \in \mathcal{V}\}$. As a special case, if all the agents have a same stubbornness budget, i.e, $K_i = K_j$ for all $i, j \in \mathcal{V}$, Assumption 4 will be satisfied.

When taking also the network topologies into account, we need to define $i' = \arg\max_{i\in\mathcal{V}} K_i$ (here we consider the case that only one agent has the maximum stubbornness budget) and for $i \neq i'$

$$\begin{aligned}\bar{q}_i = \max_{m\in[M]} \{\mathbf{1}^\top \mathbf{q}_{i,m} : \theta_{i,m} = \frac{K_i}{M-1},\\ \theta_{i',m} = 1 \text{ and } \theta_{j,m} = 0 \quad \forall j \neq i, i'\}\\ \underline{q}_i = \min_{m\in[M]} \{\mathbf{1}^\top \mathbf{q}_{i,m} : \theta_{i,m} = 0,\\ \theta_{i',m} = 1 \text{ and } \theta_{j,m} = 0 \quad \forall j \neq i, i'\}. \qquad (6)\end{aligned}$$

As $\theta_{i',m} = 1$, the matrix $(I - \Theta_m)W_m$ is strictly substochastic. Therefore, for each $i \neq i'$ and $m \in [M]$, $\mathbf{q}_{i,m}$ is well-defined in (3). Due to the fact that $[M]$ is a finite set, both $\bar{q}_i$ and $\underline{q}_i$ are well-defined.

Assumption 5: For the weight matrices $\{\mathrm{W}_m : m \in [M]\}$ and stubbornness budgets $\{K_i : i \in \mathcal{V}\}$, it holds $\bar{q}_i < \underline{q}_i$ for all $i \in \mathcal{V}$.

Lemma 9: For the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$, if Assumptions 1 and 5 hold, Assumption 2 must hold.

The proof of Lemma 9 is given in Appendix E.

Lemmas 8 and 9 give two concrete cases in which Assumption 2 holds. Intuitively, Assumption 4 requires that no agent has resource of stubbornness that is overwhelmingly larger than the others'. Assumption 5 is rather technical. In fact, it is easy to see that the entries of each $\mathbf{q}_{i,m}$ are monotonically decreasing with $\theta_{i,m}$ (given $\theta_{j,m}\ \forall j \neq i$). Therefore, $\bar{q}_i < \underline{q}_i$ implicitly requires that for each $m \in [M]$, $\mathbf{1}^\top \mathbf{q}_{i,m}$ should decrease sufficiently when $\theta_{i,m}$ changes from 0 to $\frac{K_i}{M-1}$.

V. Rank-1 Complete Graphs

In this section, we focus on complete graphs to give more insights of the conditions in the last section. The best-response dynamics for each agent is also investigated. The complete structure can be used to describe the interactions in many real-world meetings (see [5] and [6] for applications on the UN climate talks), in which all attendees make speeches and are heard by the others. Moreover, for the simplicity of technical analysis, we assume that the weight matrices are all of rank-1.

Specifically, let the rows of each W_m be identical, i.e., $\mathrm{W}_m = \mathbf{1c}_m^\top$ with $\mathbf{c}_m = [c_{i,m}] \in \mathbb{R}^n_{>0}$ satisfying $\mathbf{c}_m^\top \mathbf{1} = 1\ \forall m \in [M]$. According to the Sherman–Morrison formula [53], given $\boldsymbol{\theta} \in (\mathcal{A}_i)$ and $\Theta_m = \operatorname{diag}(\theta_{1,m}, \ldots, \theta_{n,m})$, the matrices P_m and Q_m can be calculated as follows:

$$\mathrm{P}_m = \Theta_m + \frac{(\mathrm{I} - \Theta_m)\mathbf{1c}_m^\top \Theta_m}{\mathbf{c}_m^\top \Theta_m \mathbf{1}},\ \mathrm{Q}_m^{-1} = \mathrm{I} + \frac{(\mathrm{I} - \Theta_m)\mathbf{1c}_m^\top}{\mathbf{c}_m^\top \Theta_m \mathbf{1}}. \qquad (7)$$

From (6), we can further obtain

$$\bar{q}_i = \max_{m\in[M]} \left\{1 + \frac{(n-2)c_{i',m} + \left(1 - \frac{K_i}{M-1}\right)c_{i,m}}{c_{i',m} + \frac{c_{i,m}K_i}{M-1}}\right\},\quad \underline{q}_i = n$$

which gives $\underline{q}_i - \bar{q}_i = \max_{m\in[M]} \left\{\frac{c_{i',m} + (\frac{nK_i}{M-1} - 1)c_{i,m}}{c_{i',m} + \frac{c_{i,m}K_i}{M-1}}\right\}$. Therefore, Assumption 5 is satisfied if

$$K_i \geq \frac{M-1}{n}, \quad i \in \mathcal{V}.$$

This will be made as an assumption throughout this section.

A. NE Properties

First, we consider the constrained social power game $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$. The following lemma can be obtained.

Lemma 10: Consider the constrained social power game $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$ with $\mathrm{W}_m = \mathbf{1c}_m^\top\ \forall m \in [M]$. If $\mathbf{c}_m \geq \frac{\bar{\alpha}}{n}\mathbf{1}$ holds for all $m \in [M]$, with $\bar{\alpha}$ satisfying

$$\bar{\alpha} > \frac{1}{1 + \frac{(n-1)(2\bar{\alpha}\delta - \bar{\alpha}^2\delta^2)}{n^3}} \qquad (8)$$

the pseudogradient mapping $-\nabla_{\boldsymbol{\theta}}\mathbf{u}$ is $\frac{\beta(\bar{\alpha})}{n}$-strongly monotone on $(\mathcal{A}_i^\delta)$, where $\beta(\cdot)$ is defined as in (22), with $\bar{b} = \bar{\alpha}\delta$.

The proof of Lemma 10 is given in Appendix F.

From the proof of Lemma 10, we know that Assumption 3 is satisfied for $\mathcal{A}^\delta$. Applying Proposition 5, the following corollary is immediately obtained.

Corollary 1: Consider the constrained social power game $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$ with $\mathrm{W}_m = \mathbf{1c}_m^\top\ \forall m \in [M]$. If $\mathbf{c}_m \geq \frac{\bar{\alpha}}{n}\mathbf{1}$ holds for all $m \in [M]$, with $\bar{\alpha}$ satisfying (8), the social power game $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$ has a unique NE.

Now we turn to the original social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$. It is easy to see that if $\bar{\alpha} = 1$, the inequality (8) is satisfied for any $\delta \in (0, 1)$. This, combined with Proposition 4 and Theorem 3, implies that if the underlying network is a fixed complete graph with uniform weights over all the agents, the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ admits a unique NE, as shown in the following theorem.

Theorem 4: Consider the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ with $K_i \geq \frac{M-1}{n}\ \ \forall i \in \mathcal{V}$ and $\mathrm{W}_m = \frac{1}{n}\mathbf{1}\mathbf{1}^\top\ \ \forall m \in [M]$. A unique NE $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_i^*)$ exists, with $\boldsymbol{\theta}_i^* = \frac{K_i}{M}\mathbf{1}\ \forall i \in \mathcal{V}$.

Uniform weights mean that the agents have the same network eigencentrality. In this sense, Theorem 4 indicates that if the agents are of the same importance in the network, the unique NE of the social power game must be that each agent assigns her resource of stubbornness equally over all the meetings.

Again from the proof of Lemma 10, it can be seen that for Assumption 3 to be satisfied, it suffices to let each $\frac{\mathrm{J}_m+\mathrm{J}_m^\top}{2}$ be positive definite over $\mathcal{A}^o$. With this idea, for the case that $n=3$ (i.e., the number of participants in the meetings is 3), the following theorem can be obtained.

Theorem 5: Consider the constrained social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ with $n=3$ and $\mathrm{W}_m = \mathbf{1c}_m^\top$ $\forall m \in [M]$. Let Assumption 1 hold. Assume that $K_i \geq \frac{M-1}{3}$ $\forall i \in \mathcal{V}$. If $3c_{i,m} > \sum_{h\neq i} c_{h,m}$ hold for all $i \in \mathcal{V}$ and $m \in [M]$, the game admits a unique NE.

The proof of Theorem 5 is given in Appendix F.

Recall that for general multiplayer (i.e., more than two players) constant-sum games, there is no simple rule to guarantee the existence and uniqueness of NE. For our three-player social power games over complete graphs, which is a special case of constant-sum game, Theorem 5 gives some sufficient conditions that are easy to verify, i.e., $K_i \geq \frac{M-1}{3}, 3c_{i,m} > \sum_{h\neq i} c_{h,m}\ \forall i \in \mathcal{V}, m \in [M]$.

B. Best-Response Dynamics

For a strategic game, another important problem is how the players will learn to play the game. If the game is repeated, a well-studied learning process is best-response dynamics, under which the NE can be achieved for some special games such as potential games [33], [49]. This section characterizes the best-response dynamics for the social power game.

Consider an agent $i \in \mathcal{V}$. Given the actions of the other agents, the best response of i is the solution of the optimization problem (5), or by taking the explicit form $\mathrm{W}_m = \frac{1}{n}\mathbf{1c}_m^\top$ into it, the following

$$\begin{aligned} &\underset{\boldsymbol{\theta}_i}{\text{Minimize}} \sum_{m=1}^{M} \frac{r_{i,m}(n - s_{i,m} - \theta_{i,m})}{c_{i,m}\theta_{i,m} + r_{i,m}} \\ &\text{s.t.} \quad \sum_{m=1}^{M} \theta_{i,m} = K_i; \quad 0 \leq \theta_{i,m} \leq 1 \quad \forall m \in [M] \end{aligned} \tag{9}$$

where the values of $r_{i,m} := \sum_{h\neq i} c_{h,m}\theta_{h,m}$ and $s_{i,m} := \sum_{h\neq i} \theta_{h,m}$ do not depend on $\theta_{i,m}$. Moreover, define $\ell_{i,m} := \sqrt{r_{i,m}^2 + c_{i,m}r_{i,m}(n-s_{i,m})}$ $\forall m \in [M]$. By considering the Laplacian function of (9) and exploiting the KKT conditions, we can obtain the following theorem, for which the detailed proof is omitted for the lack of space.

Theorem 6: Consider the social power optimization problem (9) with $s_{i,m} > 0$ for all $m \in [M]$. There exists a unique division of the set $[M]$, $[M] = \mathcal{M}_0 \cup \mathcal{M}_* \cup \mathcal{M}_1$, such that

1) if $m \in \mathcal{M}_0$, it holds $\frac{\ell_{i,m}}{r_{i,m}} \leq \zeta$,
2) if $m \in \mathcal{M}_*$, it holds $\frac{\ell_{i,m}}{c_{i,m}+r_{i,m}} < \zeta < \frac{\ell_{i,m}}{r_{i,m}}$, and
3) if $m \in \mathcal{M}_1$, it holds $\zeta \leq \frac{\ell_{i,m}}{c_{i,m}+r_{i,m}}$

where ζ is defined as $\zeta := \frac{\sum_{m\in\mathcal{M}^*}\frac{\ell_{i,m}}{c_{i,m}}}{K_i - M_1 + \sum_{m\in\mathcal{M}^*}\frac{r_{i,m}}{c_{i,m}}}$, with $M_1 = |\mathcal{M}_1|$. Moreover, the optimization problem (9) admits a unique solution $\boldsymbol{\theta}_i^* \in [0,1]^M$, with

$$\theta_{i,m}^* = \begin{cases} 0, & \text{if } m \in \mathcal{M}_0 \\ \frac{\ell_{i,m}}{c_{i,m}\zeta} - \frac{r_{i,m}}{c_{i,m}}, & \text{if } m \in \mathcal{M}_* \\ 1, & \text{if } m \in \mathcal{M}_1. \end{cases} \tag{10}$$

The meaning of the parameter ζ is that, when taking the best-response strategy, the marginal increase of social power on the meetings in which agent i is partially stubborn (i.e., $\theta_{i,m} \in (0,1)$) are all equal to ζ^2. In fact, by taking the partial derivative of the cost function in (9) w.r.t $\theta_{i,m}$, it can be seen that the marginal benefit of being more stubborn at the mth meeting is $\frac{\ell_{i,m}^2}{(c_{i,m}\theta_{i,m}+r_{i,m})^2}$. Therefore, Theorem 6 says that agent i's best response is to be nonstubborn (fully stubborn) when the marginal benefit is less (larger) than ζ^2.

Remark 8: Theorem 6 indicates that the parameter $r_{i,m}$, as the sum of stubbornness of the other agents weighted by their network centralities ($c_{j,m}, j \neq i$), plays an important role in determine the best response of agent i. In particular, if $r_{i,m}$ is small, $\ell_{i,m}$ becomes small and thereafter, $\frac{l_{i,m}}{c_{i,m}+r_{i,m}}$ becomes small. For this case, any optimal strategy of agent i will have $\theta_{i,m} \in (0,1)$ instead of $\theta_{i,m} = 1$. Intuitively, the reason is that agent i can obtain a high social power by being mildly stubborn at the meeting when the other agents have low network centralities or do not put much stubbornness resource onto it. Therefore, a better strategy for agent i is to save some resource in meeting m for the competition in the other meetings.

Remark 9: Theorem 6 does not specify a clear form of $\mathcal{M}_0, \mathcal{M}_1$, and $\mathcal{M}_*$. However, it gives an idea to solve the problem (9) in finite time. To do this, first sort the indexes of meetings along a decreasing order of $\frac{\ell_{i,m}}{c_{i,m}+r_{i,m}}$, say, $i_1, i_2, \ldots, i_M$, and sort the indexes of meetings along an increasing order of $\frac{\ell_{i,m}}{r_{i,m}}$, say, $i^1, i^2, \ldots, i^M$. Second, we go through the two sequences, construct $\mathcal{M}_1$ ($\mathcal{M}_0$) by adding each i_m (i^m) to it one by one, and check if the conditions 1)–3) in Theorem 6 are satisfied. Due to the existence of a solution for the problem (9), this conditions should be satisfied in finite time, and we can then calculate all $\theta_{i,m}^*$ in $\mathcal{M}_*$.

Example 2: Consider a social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ with 5 players, i.e., $\mathcal{V} = [5]$. Let $M = 5$. For each $m = 1, \ldots, 5$, let $W_m = \mathbf{1c}_m^\top$, with $\mathbf{c}_m = 0.5\mathbf{e}_m + 0.1\mathbf{1}$ being a simplex. The agents have the same stubbornness budget, i.e., $K_i = 1.2$ $\forall i = 1, \ldots, 5$. First, choose the initial stubbornness profile $\boldsymbol{\theta}$ (each column corresponds to an agent) as

$$\boldsymbol{\theta} = \begin{pmatrix} 0.3 & 0.37 & 0.01 & 0.25 & 0.25 \\ 0.23 & 0.15 & 0.28 & 0.15 & 0.13 \\ 0.35 & 0.46 & 0.05 & 0.46 & 0.15 \\ 0.19 & 0.18 & 0.48 & 0.03 & 0.41 \\ 0.13 & 0.04 & 0.38 & 0.31 & 0.26 \end{pmatrix}.$$

Then, let each agent follows the best-response dynamics. The trajectory of $\boldsymbol{\theta}_1$ is shown in Fig. 2(a),

from which it can be seen that $\boldsymbol{\theta}_1$ converges. In fact, in the simulation, $\boldsymbol{\theta}$ converges to $0.1966\mathrm{I} + 0.2007\mathbf{1}\mathbf{1}^\top$ (some remaining digits are omitted), which can be verified to be close to an NE. This also aligns with Remark 8, as in the equilibrium strategy,

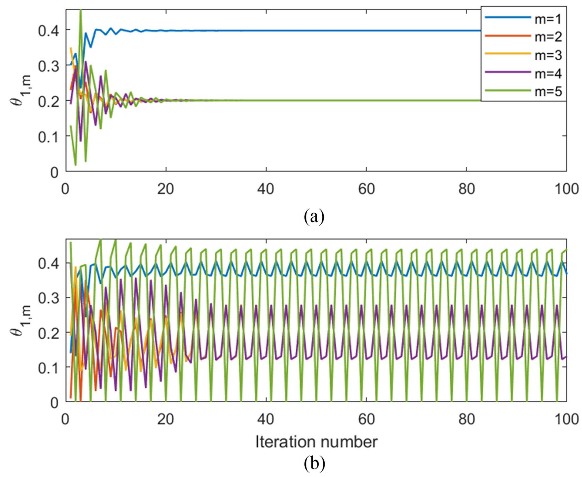

Fig. 2. Plot for Example 2. Each curve corresponds to a trajectory of $\theta_{1,m}$, generated by the best-response dynamics.

each agent i is not totally stubborn in the meeting that he/she has high network centrality (i.e., $m = i$, in which $c_{i,m} = 0.6$). However, the convergence not necessarily occurs. If we choose the initial stubbornness profile $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta} = \begin{pmatrix} 0.14 & 0.4 & 0.03 & 0.09 & 0.22 \\ 0.01 & 0.1 & 0.5 & 0.18 & 0.29 \\ 0.19 & 0.02 & 0.25 & 0.15 & 0.07 \\ 0.4 & 00.31 & 0.04 & 0.3 & 00.29 \\ 0.46 & 0.37 & 0.38 & 0.48 & 0.33 \end{pmatrix}$$

the best-response dynamics will oscillate, as shown in Fig. 2(b).

VI. Conclusion

This article studied a social power game for parallel FJ models, in which each FJ model describes an opinion evolution process disjoint and independent from the others, but involving the same agents. The scope of the game is to allocate a budget of stubbornness across the parallel FJ models so as to maximize the sum of the social powers achieved in the models. Under this formulation, the social power game becomes an n-player constant-sum game, with a cost function which is discontinuous on some boundary point of the action space. In spite of the lack of *continuity*, we were able to prove that an NE exists for the game by approximating the NEs with a sequence of *continuous convex* games. In addition, if the pseudogradient mapping inside the action space satisfies some strict monotonicity assumptions, the uniqueness of the NE is also guaranteed. These conditions were further applied to the case of rank-1 complete graphs, for which the best response dynamics of the social power game was characterized. Note that the game considered in this article is static essentially. An interesting future direction is to extend the model to an online scenario in which the agents need to take actions dynamically. Another interesting direction is the reverse problem—inferring network structures from observed actions using general network learning methods, such as those in [50] and [60], which could further broaden the applicability of the social power game framework.

Appendix A
Proof of Theorem 1

If $\theta_j = 0$ for all $j \neq i$, for any $\theta_i > 0$, it must be $\mathrm{sp}_i = 1$, which is the upper bound of any social power by definition. Therefore, any $\theta_i \in (0, K_i]$ is a solution of (5).

Consider the case that there exist $j \neq i$ such that $\theta_j > 0$. According to Lemmas 1 and 2, $\mathrm{Q} = \mathrm{I} - (\mathrm{I} - \Theta)\mathrm{W}$ and $\mathrm{P} = \mathrm{Q}^{-1}\Theta$ are well-defined. The social power of agent i is $\mathrm{sp}_i = \frac{1}{n}\mathbf{1}^\top \mathrm{P}\mathbf{e}_i$. To prove that the only solution to the problem (5) is $\theta_i = K_i$, it suffices to prove that $\mathrm{sp}_i(\theta_i, \boldsymbol{\theta}_{-i})$ is monotonically increasing with relation to θ_i. This obviously holds at the point $\theta_i = 0$, since $\mathrm{sp}_i(0, \boldsymbol{\theta}_{-i}) = 0 < \mathrm{sp}_i(\theta_i, \boldsymbol{\theta}_{-i})$ for any $\theta_i > 0$.

We now consider $\theta_i > 0$. From the derivative of matrix inverse, i.e., $\frac{\partial \mathrm{Q}^{-1}}{\partial \theta_i} = -\mathrm{Q}^{-1}\frac{\partial \mathrm{Q}}{\partial \theta_i}\mathrm{Q}^{-1}$, we have

$$\begin{aligned} \frac{\partial \mathrm{P}}{\partial \theta_i} &= \frac{\partial \mathrm{Q}^{-1}}{\partial \theta_i}\Theta + \mathrm{Q}^{-1}\frac{\partial \Theta}{\partial \theta_i} = -\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\Theta \\ &\quad + \mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top = -\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\Theta + \mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top . \end{aligned} \tag{11}$$

Let $\mathrm{W} = [w_{ij}] = [\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n]^\top$, $\mathrm{P} = [p_{ij}] = [\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n]$, and $\mathrm{Q}^{-1} = [q_{ij}] = [\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_n]$. Then

$$\begin{aligned} \frac{\partial \mathbf{p}_i}{\partial \theta_i} &= \left(-\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\Theta + \mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top\right)\mathbf{e}_i \\ &= -\mathbf{q}_i\mathbf{w}_i^\top \mathbf{p}_i + \mathbf{q}_i = \mathbf{q}_i(1 - \mathbf{w}_i^\top \mathbf{p}_i) \end{aligned} \tag{12}$$

and if $\theta_i > 0$

$$\frac{\partial \mathbf{p}_i}{\partial \theta_i} = \frac{1}{\theta_i}\mathbf{p}_i\left(1 - \mathbf{w}_i^\top \mathbf{p}_i\right). \tag{13}$$

Note that $(\mathrm{I} - \Theta)\mathrm{W}$ is strictly substochastic. From the proof of Lemma 2 in [43], $(\mathrm{I} - \Theta)\mathrm{W}$ is Schur stable. Accordingly

$$\mathrm{Q}^{-1} = \mathrm{I} + \sum_{k=1}^{\infty}\left[(\mathrm{I} - \Theta)\,\mathrm{W}\right]^k \tag{14}$$

and $q_{ii} > 1$ $\forall i \in \mathcal{V}$, which gives $\mathbf{1}^\top \mathbf{q}_i = \mathbf{1}^\top \mathrm{Q}^{-1}\mathbf{e}_i \geq \mathbf{1}^\top \mathbf{e}_i > 0$. From the strong connectivity of the graph, there exists a path $j_1 = j \to j_2 \to \cdots \to j_H \to i$. If $\theta_{j_H} = 1$, it holds $p_{j_H i} = 0$, and $\mathbf{w}_i^\top \mathbf{p}_i = \sum_{\ell \neq j_H} w_{i\ell}p_{\ell i} \leq \sum_{\ell \neq j_H} w_{i\ell} < 1$. If $\theta_H < 1$, there must exist $h \in [H-1]$ with $\theta_h > 0$ and $\theta_{h'} < 1$ $\forall h' > h$. From Lemma 2, $p_{j_H j_h} > 0$, which yields $p_{j_H i} < 1$ due to the stochasticity of P. Therefore, $\mathbf{w}_i^\top \mathbf{p}_i < w_{j_H i} + \sum_{\ell \neq j_H} w_{i\ell}p_{\ell i} \leq 1$. From (12), we then have $\frac{\partial \mathrm{sp}_i}{\partial \theta_i} = \frac{1}{n}\mathbf{1}^\top \mathbf{q}_i(1 - \mathbf{w}_i^\top \mathbf{p}_i) > 0$, i.e., sp_i is strictly monotonically increasing with the argument θ_i. The proof is then completed. □

Appendix B
Proofs for Propositions 1, 2, and Lemma 6

Proof of Proposition 1: According to the definition of social power, given $\boldsymbol{\theta}_{-i}$, $\mathrm{sp}_{i,m}$ is determined by $\theta_{i,m}$. Therefore, it suffices to prove that for all $m \in [M]$, $-\mathrm{sp}_{i,m}$ is (strictly) convex with relation to $\theta_{i,m} \in [0, 1]$.

We first consider the case that $\boldsymbol{\theta}_{-i,m} \neq \mathbf{0}$ holds for all $m \in [M]$. Given $m \in [M]$, it is easy to see that $\mathrm{sp}_{i,m}$ is continuously differentiable with relation to $\theta_{i,m}$. We only need to prove that the second order derivative of $\mathrm{sp}_{i,m}$ is negative for $\theta_{i,m} > 0$. For the simplicity, we omit the index m in the following proof when there is no confusion. According to (11), we have

$$\begin{aligned}\frac{\partial^2 \mathrm{P}}{(\partial\theta_i)^2} &= -\frac{\partial \mathrm{Q}^{-1}}{\partial\theta_i}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\Theta - \mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\frac{\partial \mathrm{Q}^{-1}}{\partial\theta_i}\Theta \\ &\quad - \mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\frac{\partial\Theta}{\partial\theta_i} + \frac{\partial \mathrm{Q}^{-1}}{\partial\theta_i}\mathbf{e}_i\mathbf{e}_i^\top \\ &= 2\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\Theta \\ &\quad - 2\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \mathrm{W}\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top \end{aligned} \tag{15}$$

where P and Q are defined as in (2) and (3) (with the subscript m omitted). Similar to the proof of Theorem 1, we have

$$\begin{aligned}\frac{n}{2}\frac{\partial^2 \mathrm{sp}_i}{(\partial\theta_i)^2} &= \mathbf{1}^\top\mathbf{q}_i\mathbf{w}_i^\top\mathbf{q}_i\mathbf{w}_i^\top\mathbf{p}_i - \mathbf{1}^\top\mathbf{q}_i\mathbf{w}_i^\top\mathbf{q}_i \\ &= -\mathbf{1}^\top\mathbf{q}_i\mathbf{w}_i^\top\mathbf{q}_i(1-\mathbf{w}_i^\top\mathbf{p}_i) < 0. \end{aligned} \tag{16}$$

Therefore, $-\mathrm{sp}_i$ is strictly convex with relation to θ_i.

Now we consider that for some $m \in [M]$, $\boldsymbol{\theta}_{-i,m} = \mathbf{0}$. For this case, the mth FJ model degenerates to a DeGroot model. It is easy to see that $-\mathrm{sp}_{i,m} = -\xi_{i,m}$ for $\theta_{i,m} = 0$, with $\xi_{i,m}$ as the mth entry of the normalized left eigenvector of W_m that corresponds to the eigenvalue 1, and $-\mathrm{sp}_{i,m} = -1$ if $\theta_{i,m} > 0$. Therefore, $-\mathrm{sp}_{i,m}$ is convex for $\theta_{i,m} \in [0,1]$. Combining all the arguments, the proof is completed. □

Proof of Proposition 2: The notation in the proof of Proposition 1 is followed. On one hand, we know that $\mathrm{sp}_{i,m_0} = \xi_{i,m_0} \in (0,1)$ hold for all $i \in \mathcal{V}$. On the other hand, consider $\boldsymbol{\theta}_i^\delta := \boldsymbol{\theta}_i^0 + \delta\boldsymbol{\zeta}$ for $\delta > 0$, where $\boldsymbol{\zeta} \in \mathbb{R}^M$ is chosen such that $\boldsymbol{\theta}_i^\delta \in \mathcal{A}_i, \zeta_{m_0} > 0$ and $\mathrm{sgn}(\zeta_m) = -\mathrm{sgn}(\theta_{i,m}^0)$ for all $m \neq m_0$. Then we have that $\mathrm{sp}_{i,m_0}(\boldsymbol{\theta}_i^\delta, \boldsymbol{\theta}_{-i}^0) = 1$ holds for any small $\delta > 0$. Therefore, the function $u_i(\cdot, \boldsymbol{\theta}_{-i}^0)$ is not continuous at $\boldsymbol{\theta}_i^0$. □

To prove Lemma 6, the following lemma is needed.

Lemma 11: Consider the FJ model (1) with $\mathcal{G}_\mathrm{W}$ as a strongly connected graph. Assume that there exists $i \in [n]$ such that $\theta_i > 0$. Let $\mathrm{Q}^{-1} := (\mathrm{I} - (\mathrm{I}-\Theta)\mathrm{W})^{-1} = [q_{ij}]$. For any $N > 0$, there exist $\epsilon > 0$ such that if $\max_{i\in[n]}\theta_i < \epsilon$, it holds that $q_{ji} > N$ for all $i, j \in [n]$.

Proof: Due to the strong connectivity, it holds $\lim_{k\to\infty}\mathrm{W}^k = \mathbf{1}\boldsymbol{\xi}^\top$ for some $\boldsymbol{\xi} > 0$, and $\boldsymbol{\xi}^\top\mathbf{1} = 1$. There exists $\bar{k} > 0$ such that $|[\mathrm{W}^k]_{ij} - \xi_j| < \frac{\min_{i\in[n]}\xi_i}{2}$ for all $k \geq \bar{k}$. Moreover, from (14), it is

$$\begin{aligned}\mathrm{Q}^{-1} &= \sum_{k=0}^{\infty}((\mathrm{I}-\Theta)\mathrm{W})^k \geq \sum_{k=0}^{\infty}(1-\max_{i\in[n]}\theta_i)^k\mathrm{W}^k \\ &\geq \sum_{k=\bar{k}}^{\infty}(1-\max_{i\in[n]}\theta_i)^k\mathrm{W}^k > \frac{1}{2}\sum_{k=\bar{k}}^{\infty}(1-\max_{i\in[n]}\theta_i)^k\mathbf{1}\boldsymbol{\xi}^\top \\ &= \frac{(1-\max_{i\in[n]}\theta_i)^{\bar{k}}}{2\max_{i\in[n]}\theta_i}\mathbf{1}\boldsymbol{\xi}^\top\end{aligned}$$

where the third inequality is from $\mathrm{W}^k \geq \mathbf{1}\boldsymbol{\xi}^\top - \frac{\min_{i\in[n]}\xi_i}{2}\mathbf{1}\mathbf{1}^\top \geq \frac{1}{2}\mathbf{1}\boldsymbol{\xi}^\top$. Therefore, for $\forall N > 0$, if we choose $\epsilon = \min\{\frac{1}{2}, \frac{\min_{i\in[n]}\xi_i}{2^{\bar{k}+1}N}\}$, the desired conclusion is obtained. □

Proof of Lemma 6: We use the proof of contradiction. Suppose that for any $\epsilon > 0$, there exists $\delta > 0$ such that $\max_{i\in\mathcal{V}}\theta_{i,m}^{\delta*} < \epsilon$ for some $m \in [M]$. For each $\epsilon > 0$, choose (any) one of such pairs (δ, m), denoted as $(\delta(\epsilon), m(\epsilon))$ to indicate their dependence on ϵ. Without loss of generality, suppose $\epsilon < \min_{i\in\mathcal{V}}\frac{K_i}{M}$.

Let

$$i(\epsilon) = \arg\min_{i\in\mathcal{V}}\ \mathrm{sp}_{i,m(\epsilon)}(\boldsymbol{\theta}^{\delta(\epsilon)*}).$$

It must be $\mathrm{sp}_{i(\epsilon),m(\epsilon)}(\boldsymbol{\theta}^{\delta(\epsilon)*}) \leq \frac{1}{n}$, that is, $\sum_{j\in\mathcal{V}}p_{ji(\epsilon),m(\epsilon)} \leq 1$. This, combined with (13), yields

$$\begin{aligned}\left.\frac{\partial \mathrm{sp}_{i,m}}{\partial\theta_{i,m}}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m(\epsilon)}} &= \left.\frac{(1-\mathbf{w}_{i,m}^\top\mathbf{p}_{i,m})}{n}\mathbf{1}^\top\mathbf{q}_{i,m}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m(\epsilon)}} \\ &\geq \left.\frac{1-\bar{w}}{n}\mathbf{1}^\top\mathbf{q}_{i,m}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m(\epsilon)}} \quad \forall i \in \mathcal{V}, m \in [M].\end{aligned} \tag{17}$$

On the other hand, as $\epsilon < \min_{i\in\mathcal{V}}\frac{K_i}{M}$, we can find $m'(\epsilon) \in [M]\backslash m(\epsilon)$ such that $\theta_{i(\epsilon),m'(\epsilon)}^{\delta(\epsilon)*} \geq \frac{K_{i(\epsilon)}}{M}$, for which (13) leads to

$$\begin{aligned}\left.\frac{\partial \mathrm{sp}_{i,m}}{\partial\theta_{i,m}}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m'(\epsilon)}} &\leq \left.\frac{1}{\theta_{i(\epsilon),m'(\epsilon)}^{\delta(\epsilon)*}}\mathrm{sp}_{i,m}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m'(\epsilon)}} \\ &\leq \frac{M}{K_{i(\epsilon)}} \leq \frac{M}{\min_{i\in\mathcal{V}}K_i}.\end{aligned} \tag{18}$$

According to Lemma 11, there exists $\bar{\epsilon} > 0$ such that

$$\left.\frac{1-\bar{w}}{n}\mathbf{1}^\top\mathbf{q}_{i,m}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m(\epsilon)}} > \frac{M}{\min_{i\in\mathcal{V}}K_i}$$

holds if $\max_{i\in\mathcal{V}}\theta_{i,m(\epsilon)}^{\delta(\epsilon)*} < \bar{\epsilon}\ \forall i \in \mathcal{V}$. Therefore, if $\epsilon < \bar{\epsilon}$, (17) combined with (18) lead to

$$\begin{aligned}\nabla_{\mathbf{e}_{m(\epsilon)}-\mathbf{e}_{m'(\epsilon)}}^{i(\epsilon)} &:= \nabla_{\mathbf{e}_{m(\epsilon)}-\mathbf{e}_{m'(\epsilon)}}u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}^{\delta(\epsilon)*})\Big|_{\substack{i=i(\epsilon)\\ \boldsymbol{\theta}_i=\boldsymbol{\theta}_i^{\delta(\epsilon)*}}} \\ &= \left.\frac{\partial \mathrm{sp}_{i,m}}{\partial\theta_{i,m}}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m(\epsilon)}} - \left.\frac{\partial \mathrm{sp}_{i,m}}{\partial\theta_{i,m}}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m'(\epsilon)}} \\ &\geq \left.\frac{1-\bar{w}}{n}\mathbf{1}^\top\mathbf{q}_{i,m}\right|_{\substack{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta(\epsilon)*}\\ i=i(\epsilon)\\ m=m(\epsilon)}} - \frac{M}{\min_{i\in\mathcal{V}}K_i} > 0\end{aligned} \tag{19}$$

where $\nabla_{\mathbf{e}_{m(\epsilon)}-\mathbf{e}_{m'(\epsilon)}}$ represents the derivative operator along the direction of $\mathbf{e}_{m(\epsilon)} - \mathbf{e}_{m'(\epsilon)}$. This means that the directional derivative $\nabla_{\mathbf{e}_{m(\epsilon)}-\mathbf{e}_{m'(\epsilon)}}^{i(\epsilon)}$ is strictly larger than 0. It is easy to see that $\boldsymbol{\theta}_{i(\epsilon)}^{\delta(\epsilon)*} + \sigma(\mathbf{e}_{m(\epsilon)} - \mathbf{e}_{m'(\epsilon)})$ lies in the $\mathcal{A}_{i(\epsilon)}^{\delta(\epsilon)}$ if $\sigma > 0$ is small enough. Therefore, the best response action of agent $i(\epsilon)$ [i.e., solution of the optimization problem (5)] cannot be $\boldsymbol{\theta}_{i(\epsilon)}^{\delta(\epsilon)*}$, when the profile of the other agents is $\boldsymbol{\theta}_{-i(\epsilon)}^{\delta(\epsilon)*}$.

To summarize, if $\epsilon < \bar{\epsilon}$, $\boldsymbol{\theta}^{\delta(\epsilon)*}$ cannot be an NE, which leads to a contradiction. Thus, Lemma 6 must hold. □

Appendix C
Proofs for Theorem 2 and Proposition 4

Proof of Theorem 2: For any $\delta > 0$, let $\boldsymbol{\theta}^{\delta *}$ be an NE of the constrained social power game $\langle \mathcal{V}, (\mathcal{A}_i^\delta), (u_i)\rangle$. We are going to prove that any $\boldsymbol{\theta}^* = (\boldsymbol{\theta}_i^*) \in \limsup_{\delta\to 0^+} \boldsymbol{\theta}^{\delta *}$ is an NE of the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$. Note that the set $\limsup_{\delta\to 0^+} \boldsymbol{\theta}^{\delta *}$ is nonempty, since the action profile set $(\mathcal{A}_i)$ is compact and $\boldsymbol{\theta}^{\delta *} \in (\mathcal{A}_i)$ for all $\delta \in (0, \frac{1}{n})$. Choose $\{\delta_k : k \in \mathbb{N}_+\}$ such that $\lim_{k\to\infty} \boldsymbol{\theta}^{\delta_k *} = \boldsymbol{\theta}^*$. From Lemma 6, there exists a constant $\bar{\epsilon} > 0$ such that for all $k \in \mathbb{N}_+, m \in [M]$, for at least one agent $i \in \mathcal{V}$, it holds $\theta_{i,m}^{\delta_k *} \geq \bar{\epsilon}$. Accordingly, for all $m \in [M]$, there exists $i \in \mathcal{V}$ such that $\theta_{i,m}^* \geq \bar{\epsilon}$.

If for some $m_1 \in [M]$, there exists $i_1 \in \mathcal{V}$ such that $\theta_{i_1,m_1}^* \geq \bar{\epsilon}$ and $\theta_{i,m_1}^* = 0$ for all $i \neq i_1$, the following Claim 1 holds, for which the proof is given below.

Claim 1: For all $m \neq m_1$, if there exists $i_2 \neq i_1$ such that $\theta_{i_2,m}^* > 0$, it must be $\theta_{i_1,m}^* = 1$.

According to Claim 1, for each $m \in [M]$, either $\theta_{i_1,m}^* = 1$ or $\theta_{i_1,m}^* \geq \bar{\epsilon}$ while $\theta_{i,m}^* = 0 \ \forall i \neq i_1$. Obviously, for the agent i_1, her social power has reached the maximum (given the actions of the other agents) and will not increase when changing her action unilaterally.

We then consider an arbitrary $i \neq i_1$. For the simplicity of symbols, let $\Omega = \mathcal{A}_i$ and $\Omega_k = \mathcal{A}_i^{\delta_k} \ \forall k \in \mathbb{N}_+$. Define the cost functions $c^k : \Omega_k \to \mathbb{R}$ and $c^\infty : \Omega \to \mathbb{R}$ as

$$c^k(\cdot) = -u_i(\cdot, \boldsymbol{\theta}_{-i}^{\delta_k *}),\ c^\infty(\cdot) \quad = -u_i(\cdot, \boldsymbol{\theta}_{-i}^*).$$

Here, with some abuse of notation, we omit the index i for c^k, c^∞ and use the superscripts k, ∞ to indicate their dependence on $\boldsymbol{\theta}_{-i}^{\delta_k *}, \boldsymbol{\theta}_{-i}^*$. According to Proposition 1, $c^k(\cdot)\ \forall k \in \mathbb{N}_+$ and $c^\infty(\cdot)$ are all convex and continuously differentiable. We can then define continuous operators $T_k : \Omega_k \to \mathbb{R}^M, T : \Omega \to \mathbb{R}^M$ as the derivative operators of $c^k(\cdot)$ and $c^\infty(\cdot)$, respectively, that is

$$T_k \boldsymbol{\theta}_i = \nabla c^k(\boldsymbol{\theta}_i)\ \forall \boldsymbol{\theta}_i \in \Omega_k;\ T\boldsymbol{\theta}_i \quad = \nabla c^\infty(\boldsymbol{\theta}_i) \quad \forall \boldsymbol{\theta}_i \in \Omega.$$

Observe that Ω_k and Ω are compact convex sets, and $\Omega_k \uparrow \Omega$. Henceforth, T_k can be continuously extended to Ω in the way that $\tilde{T}_k \boldsymbol{\theta}_i = T_k(\mathrm{Pj}_{\Omega_k}(\boldsymbol{\theta}_i))\ \forall \boldsymbol{\theta}_i \in \Omega$, where $\tilde{T}_k$ is the extension and Pj_{Ω_k} is the projection operator to Ω_k. Note that $\frac{\partial \mathrm{sp}_{i,m}}{\partial \theta_{i,m}}|_{\boldsymbol{\theta}_{-i}=\boldsymbol{\theta}_{-i}^{\delta_k *}}$ and $\frac{\partial \mathrm{sp}_{i,m}}{\partial \theta_{i,m}}|_{\boldsymbol{\theta}_{-i}=\boldsymbol{\theta}_{-i}^*}$ are continuous on Ω_k and Ω, respectively, w.r.t $\theta_{i,m}$ for all $i \neq i_1$ and $m \in [M]$, and

$$T_k \boldsymbol{\theta}_i = -\left(\frac{\partial \mathrm{sp}_{i,1}}{\partial \theta_{i,1}}, \ldots, \frac{\partial \mathrm{sp}_{i,M}}{\partial \theta_{i,M}}\right)^\top |_{\boldsymbol{\theta}_{-i}=\boldsymbol{\theta}_{-i}^{\delta_k *}}$$

$$T\boldsymbol{\theta}_i = -\left(\frac{\partial \mathrm{sp}_{i,1}}{\partial \theta_{i,1}}, \ldots, \frac{\partial \mathrm{sp}_{i,M}}{\partial \theta_{i,M}}\right)^\top |_{\boldsymbol{\theta}_{-i}=\boldsymbol{\theta}_{-i}^*}.$$

Therefore, it holds that $\lim_{k\to\infty} \mathcal{G}(T_k) = \mathcal{G}(T)$. Moreover

$$\mathbf{1}^\top \mathbf{q}_{i,m}\big|_{\boldsymbol{\theta}_{-i}=\boldsymbol{\theta}_{-i}^{\delta_k *}} \leq \max_{\Theta_m' \in \mathcal{C}^{\bar{\epsilon}}} \mathbf{1}^\top \mathbf{q}_{i,m}\big|_{\Theta_m=\Theta_m'} := \bar{q}_i$$

$$\mathbf{1}^\top \mathbf{q}_{i,m}\big|_{\boldsymbol{\theta}_{-i}=\boldsymbol{\theta}_{-i}^*} \leq \bar{q}_i$$

where

$$\mathcal{C}^{\bar{\epsilon}} := \{\mathrm{diag}(\theta_1, \ldots, \theta_n) \in [0,1]^{n\times n} : \exists j \in \mathcal{V} s.t. \theta_j \geq \bar{\epsilon}\}.$$

As $\mathcal{C}^{\bar{\epsilon}}$ is a compact set and $\bar{\epsilon}$ is a constant that only depends on $\{\mathrm{W}_m : m \in [M]\}$ and $\{K_i : i \in \mathcal{V}\}$, it must be $\bar{q}_i < \infty$. This, combined with (13), further gives that T_k and T are uniformly bounded. So far, all assumptions in Lemma 3 have been verified. We then have $\lim_{k\to\infty} \mathcal{S}(\Omega_k, T_k) = \mathcal{S}(\Omega, T)$. According to the equality between convex optimization and VI problems [52], given $\boldsymbol{\theta}_{-i}^{\delta_k *}$ (or $\boldsymbol{\theta}_{-i}^*$), the solution of the optimization problem (5) is exactly $\mathcal{S}(\Omega_k, T_k)$ (or $\mathcal{S}(\Omega, T)$). Note that $\mathcal{S}(\Omega_k, T_k) = \{\boldsymbol{\theta}_i^{\delta_k *}\}$. Therefore, it holds $\mathcal{S}(\Omega, T) = \{\boldsymbol{\theta}_i^*\}$, which is the solution of (5) given $\boldsymbol{\theta}_{-i}^*$. From the arbitrariness of i, it is proved that $\boldsymbol{\theta}^*$ is a NE.

If for all $m \in [M]$, there exists j_1, j_2 such that $\theta_{j_1,m}^*, \theta_{j_2,m}^* > 0$, there exists $N > 0$ such that for all $k > N$, it hold $\theta_{j_1,m}^{\delta_k *} \geq \frac{\theta_{j_1,m}^*}{2} > 0$ and $\theta_{j_2,m}^{\delta_k *} \geq \frac{\theta_{j_2,m}^*}{2} > 0$. Therefore, for all $i \in \mathcal{V}$, if $k > N$, $u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}^{\delta_k *})$ is continuously differentiable on on $\mathcal{A}_i^{\delta_k}$ w.r.t $\boldsymbol{\theta}_i$. Following the same arguments as above (for $i \neq i_1$), it can be proved that $\boldsymbol{\theta}^*$ is a NE.

Combining all the arguments, the proof is completed. □

Proof of Claim 1: We use the proof of contradiction. Suppose that for a $m_2 \neq m_1$, there exists $i_2 \neq i_1$ such that $\theta_{i_2,m_2}^* > 0$ and $\theta_{i_1,m_2}^* < 1$. According to the definition of $\boldsymbol{\theta}^*$, for any $\epsilon > 0$, there exists $N > 0$ such that for all $k > N$, it hold

$$\theta_{i_1,m_1}^{\delta_k *} \geq \frac{\theta_{i_1,m_1}^*}{2} > 0; \quad \theta_{i,m_1}^{\delta_k *} \leq \epsilon \quad \forall i \neq i_1$$
$$\theta_{i_1,m_2}^{\delta_k *} \leq \frac{\theta_{i_1,m_2}^* + 1}{2} < 1; \quad \theta_{i_2,m_2}^{\delta_k *} \geq \frac{\theta_{i_2,m_2}^*}{2} > 0. \qquad (20)$$

Let

$$\mathcal{C}_{m_2} = \Bigg\{\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_n) : \theta_i \in [0,1], i \neq i_1, i_2;$$
$$\theta_{i_1} \in \left[0, \frac{\theta_{i_1,m_2}^* + 1}{2}\right]; \theta_{i_2} \in \left[\frac{\theta_{i_2,m_2}^*}{2}, 1\right]\Bigg\}.$$

For each $\Theta \in \mathcal{C}_{m_2}$, according to Lemma 2, it must be $\mathbf{p}_{i_1,m_2}|_{\Theta_{m_2}=\Theta} < \mathbf{1}$. As $\mathcal{C}_{m_2}$ is a compact set, we have $\max_{\Theta_{m_2}\in\mathcal{C}_{m_2}} \mathbf{p}_{i_1,m_2} < \mathbf{1}$, which further gives $\max_{\Theta_{m_2}\in\mathcal{C}_{m_2}} \mathbf{w}_{i_1,m_2}^\top \mathbf{p}_{i_1,m_2} < 1$. From (13) and $\mathbf{1}^\top \mathbf{q}_{i,m} > 1$, for each $k > N$, we obtain

$$\frac{\partial \mathrm{sp}_{i_1,m_2}}{\partial \theta_{i_1,m_2}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta_k *}} \geq \frac{1 - \max_{\Theta_{m_2}\in\mathcal{C}_{m_2}} \mathbf{w}_{i_1,m_2}^\top \mathbf{p}_{i_1,m_2}}{n} := b_{m_2}.$$

According to Lemma 2 and the continuity of matrix inverse, for any $\sigma > 0$, there exists $\epsilon > 0$ such that for $\boldsymbol{\theta}^{\delta_k *}$ satisfying (20), it is $\mathbf{p}_{i_1,m_1}|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta_k *}} \geq (1-\sigma)\mathbf{1}$, correspondingly, if $k > N$ (N is chosen such that (20) holds), from (13) and (20)

$$\frac{\partial \mathrm{sp}_{i_1,m_1}}{\partial \theta_{i_1,m_1}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{\delta_k *}} \leq \frac{2}{n\theta_{i_1,m_1}^*}(1 - \mathbf{w}_{i_1,m_1}^\top \mathbf{p}_{i_1,m_1}) \leq \frac{2\sigma}{n\theta_{i_1,m_1}^*}.$$

Therefore, if we choose σ such that $\frac{2\sigma}{n\theta_{i_1,m_1}^*} < b_{m_2}$, for all $k > N$, the directional derivative

$$\nabla_{\mathbf{e}_{m_2}-\mathbf{e}_{m_1}} u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}_{-i}^{\delta_k *})\Big|_{\substack{i=i_1 \\ \boldsymbol{\theta}_i=\boldsymbol{\theta}_i^{\delta_k *}}} \geq b_{m_2} - \frac{2\sigma}{n\theta_{i_1,m_1}^*} > 0.$$

This means that $\boldsymbol{\theta}^{\delta_k *}$ is not a NE, which generates the contradiction. Therefore, Claim 1 holds.

Proof of Proposition 4: Consider any agent i. Let $f_i(\theta)$ be her social power given that the underlying graph is $\mathcal{G}$ and any other agent j's stubbornness is $\frac{K_j}{M}$ for all $m \in [M]$. According to Proposition 1, the function $-f_i$ is convex, which gives

$$-\sum_{m=1}^{M} \mathrm{sp}_{i,m}(\boldsymbol{\theta}_i, \boldsymbol{\theta}^*_{-i}) = M\sum_{m=1}^{M}\left(-\frac{1}{M}f_i(\theta_{i,m})\right)$$
$$\geq -Mf_i\left(\frac{1}{M}\sum_{m=1}^{M}\theta_{i,m}\right) = -Mf_i\left(\frac{K_i}{M}\right).$$

The equality holds iff $\theta_{i,m} = \frac{K_i}{M}$ for all $m \in [M]$. Therefore, agent i has no incentive to deviate from θ^*_i. As the agent i is arbitrarily chosen, the desired conclusion is obtained. □

Appendix D
Proofs for Theorem 3

Proof of Lemma 7: Note that $\mathcal{A}^\delta$ is a compact set, thus $\sigma_\delta > 0$. Let $\boldsymbol{\theta}^1, \boldsymbol{\theta}^2 \in \mathbb{R}^{nM} \in \mathcal{A}^\delta$ and $\boldsymbol{\theta}(\alpha) = \alpha\boldsymbol{\theta}^2 + (1-\alpha)\boldsymbol{\theta}^1$ for any $\alpha \in [0,1]$. Define a function $f(\alpha) := -(\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1)^\top \nabla_{\boldsymbol{\theta}}\mathbf{u}(\boldsymbol{\theta}(\alpha))$. Obviously, $f(\alpha)$ is continuously differentiable on [0,1]. According to the mean-value theorem, there exists α^* such that

$$f(1) - f(0) = f(\alpha^*)' = (\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1)^\top \mathrm{J}(\boldsymbol{\theta}(\alpha^*))(\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1)$$
$$= (\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1)^\top \frac{\mathrm{J}(\boldsymbol{\theta}(\alpha^*)) + \mathrm{J}(\boldsymbol{\theta}(\alpha^*))^\top}{2}(\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1)$$
$$\geq \sigma_\delta\|\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1\|^2$$

that is, $(\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1)^\top(\nabla_{\boldsymbol{\theta}}\mathbf{u}(\boldsymbol{\theta}^1) - \nabla_{\boldsymbol{\theta}}\mathbf{u}(\boldsymbol{\theta}^2)) \geq \sigma_\delta\|\boldsymbol{\theta}^2 - \boldsymbol{\theta}^1\|^2$. The arbitrariness of $\boldsymbol{\theta}^1, \boldsymbol{\theta}^2$ then gives desired conclusion. □

Proof of Theorem 3: Let $\boldsymbol{\theta}^*$ be an NE of the game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$. According to Assumption 2, for all $m \in [M]$, there exists $i_1, i_2 \in \mathcal{V}$ such that $\theta^*_{i_1,m}, \theta^*_{i_2,m} > 0$. Let $\epsilon = \min_{i\in\mathcal{V}, m\in[M]} \theta^*_{i,m}$, and $\mathcal{B}_{\frac{\epsilon}{2}}$ be the closed ball centered at $\boldsymbol{\theta}^*$ with radius $\frac{\epsilon}{2}$, i.e.,

$$\mathcal{B}_{\frac{\epsilon}{2}} := \left\{\boldsymbol{\theta} \in \mathbb{R}^{nM} : \|\boldsymbol{\theta} - \boldsymbol{\theta}^*\| \leq \frac{\epsilon}{2}\right\}.$$

It is easy to see that $(\mathcal{B}_{\frac{\epsilon}{2}} \cap \mathcal{A}) \subset \mathcal{A}^o$. From a similar proof to that of Lemma 7, the pseudogradient mapping $-\nabla_{\boldsymbol{\theta}}\mathbf{u}$ is $\bar{\sigma}$-strongly monotone over $\mathcal{B}_{\frac{\epsilon}{2}} \cap \mathcal{A}$, with $\bar{\sigma} := \min_{\boldsymbol{\theta}\in\mathcal{B}_{\frac{\epsilon}{2}}\cap\mathcal{A}} \sigma(\boldsymbol{\theta}) > 0$. Moreover, it is easy to see that $-\nabla_{\boldsymbol{\theta}}u(\boldsymbol{\theta})$ is continuous on all $\boldsymbol{\theta} \in \mathcal{A}^o$. Since $\mathcal{B}_{\frac{\epsilon}{2}} \cap \mathcal{A}$ is compact, $-\nabla_{\boldsymbol{\theta}}u(\boldsymbol{\theta})$ is Lipschitz continuous over $\mathcal{B}_{\frac{\epsilon}{2}} \cap \mathcal{A}$. As $\boldsymbol{\theta}^*$ is an NE, it must be $\boldsymbol{\theta}^* \in \mathcal{S}(\mathcal{A}, -\nabla_{\boldsymbol{\theta}}u)$. Note that here the values of $-\nabla_{\boldsymbol{\theta}}u(\boldsymbol{\theta})$ on $\mathcal{A}\backslash\mathcal{A}^o$ does not matter, thus can be arbitrarily defined. According to Lemma 4, there exists $\bar{\delta} > 0$ such that for any $\delta < \bar{\delta}$, $\mathrm{VI}(\mathcal{A}^\delta, -\nabla_{\boldsymbol{\theta}}u)$ admits a unique solution $\boldsymbol{\theta}^{\delta*}$ in $\mathcal{B}_{\frac{\epsilon}{2}} \cap \mathcal{A}$, such that $\lim_{\delta\to 0}\boldsymbol{\theta}^{\delta*} = \boldsymbol{\theta}^*$. On the other hand, from Lemma 5, $\boldsymbol{\theta}^{\delta*}$ is an NE of the constrained social power game $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i)\rangle$. According to Proposition 5, $\langle \mathcal{V}, \mathcal{A}^\delta, (u_i)\rangle$ admits a unique NE. Therefore, $\boldsymbol{\theta}^*$ must be unique. □

Appendix E
Proofs for the Lemmas in Section IV-D

Proof of Lemma 8: Let $\boldsymbol{\theta}^*$ be any NE of the game. According to Remark 6, for any $m \in [M]$, there exists $\max_{i\in\mathcal{V}} \theta^*_{i,m} > 0$. To prove Lemma 8, it suffices to exclude the possibility that there exist $i' \in \mathcal{V}$ and $m' \in [M]$ such that $\theta^*_{i',m'} > 0$ and $\theta^*_{i,m'} = 0\,\forall i \neq i'$. This is obvious if the condition 1) holds, which implies for the agents i_1 and i_2, it must be $\theta^*_{i_1,m}, \theta^*_{i_2,m} > 0$ for all $m \in [M]$. For the condition 2), we first note that if such m', i' exist, then for all $m \neq m'$, if $\theta^*_{i',m} < 1$, it must be $\theta^*_{i,m} = 0\;\forall i \neq i'$, otherwise the social power of i' can be increased by replacing some stubbornness from the meeting m' to m. This also indicates that $K_{i'} > K_i$ for all $i \neq i'$, or being more accurate, according to Remark 6, it holds $K_{i'} - K_i \geq \bar{\epsilon}$. On the other hand, if the condition 2) is satisfied, there must be at least one other agent i'' such that $\theta^*_{i'',m} > \theta^*_{i',m} - \bar{\epsilon}$. Therefore, when either the condition 1) or 2) hold, for each $m \in [M]$, there are at least two agents i_1, i_2 with $\theta^*_{i_1,m}, \theta^*_{i_2,m} > 0$. The proof is then finished. □

Proof of Lemma 9: As in the proof of Lemma 8, it suffices to exclude the case that for the agent i' with $K_{i'} = \max_{i\in\mathcal{V}} K_i$, it holds $\theta^*_{i',m'} > 0$ and $\theta_{i,m'} = 0\;\forall i \neq i'$ for some $m' \in [M]$.

We use the proof of contradiction: assume that such i', m' exist. Then, for each $i \neq i'$, there exists $m \in [M]$ such that $\theta^*_{i,m} \geq \frac{K_i}{M-1}$. As proved in Lemma 8, it also holds $\theta^*_{i',m} = 1$. According to the definition of NE, the derivative of $u_i(\cdot, \boldsymbol{\theta}^*_{-i})$ along the direction of $\mathbf{e}_{m'} - \mathbf{e}_m$ should be nonpositive, i.e.,

$$\nabla_{\mathbf{e}_{m'}-\mathbf{e}_m} u_i(\boldsymbol{\theta}_i, \boldsymbol{\theta}^*_{-i})\big|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i}$$
$$= \left.\frac{\partial \mathrm{sp}_{i,m'}}{\partial\theta_{i,m'}}\right|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} - \left.\frac{\partial \mathrm{sp}_{i,m}}{\partial\theta_{i,m}}\right|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i}$$
$$= \mathbf{1}^\top\mathbf{q}_{i,m'}\big|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} - \mathbf{1}^\top\mathbf{q}_{i,m}(1 - \mathbf{w}_i^\top\mathbf{p}_{i,m})\big|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} \leq 0 \quad (21)$$

where the second equality is from the fact that $\mathbf{p}_{i,m'} = \mathbf{0}$ (since only the agent i' has positive stubbornness for the meeting m'). On the other hand, from (14), it is easy to see that $\mathbf{1}^\top\mathbf{q}_{i,m'}$ (or $\mathbf{1}^\top\mathbf{q}_{i,m}$) is decreasing w.r.t $\theta^*_{j,m'}$ (or $\theta^*_{j,m}$) for all $j \in [M]$. Therefore, it holds

$$\mathbf{1}^\top\mathbf{q}_{i,m'}\big|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} \geq \mathbf{1}^\top\mathbf{q}_{i,m'}\Big|_{\substack{\theta_{i',m'}=1\\ \theta_{j,m'}=0\;\forall j\neq i}}$$
$$\mathbf{1}^\top\mathbf{q}_{i,m}\big|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} \leq \mathbf{1}^\top\mathbf{q}_{i,m'}\Big|_{\substack{\theta_{i',m}=1\\ \theta_{i,m}=\frac{K_i}{M-1}\\ \theta_{j,m}=0\;\forall j\neq i,i'}}.$$

According to Assumption 5, we then have $\mathbf{1}^\top\mathbf{q}_{i,m'}|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} \geq \underline{q}_i > \bar{q}_i \geq \mathbf{1}^\top\mathbf{q}_{i,m}|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i}$, which leads to $\mathbf{1}^\top\mathbf{q}_{i,m'}|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} - \mathbf{1}^\top\mathbf{q}_{i,m}(1 - \mathbf{w}_i^\top\mathbf{p}_{i,m})|_{\boldsymbol{\theta}_i=\boldsymbol{\theta}^*_i} > 0$. This contradicts (21). Therefore, the contradiction hypothesis does not hold. The proof is then finished. □

Appendix F
Proofs for Lemma 10 and Theorem 5

The following lemma is needed to prove Lemma 10.

Lemma 12: Given $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n_{>0}$ with $\mathbf{b}^\top\mathbf{1} = 1, \mathbf{c}^\top\mathbf{1} = 1$, consider the matrix $F \in \mathbb{R}^{n\times n}$ defined as $\mathrm{F} := \frac{\mathrm{CB}+\mathrm{BC}}{2}$, with $\mathrm{C} :=$

$\mathrm{diag}(\mathbf{c})$ being a diagonal matrix and

$$\mathrm{B} = \mathrm{I} + \mathbf{1}\mathbf{1}^\top - \mathbf{1}\mathbf{b}^\top - \mathbf{b}\mathbf{1}^\top.$$

Suppose that for some $\bar{b} \in (0, 1]$ it holds $b_i \geq \frac{\bar{b}}{n}\ \forall i \in [n]$. If $\mathbf{c} > \frac{\alpha}{n}\mathbf{1}$ for some $\alpha > 0$ with $\alpha > \frac{1}{1+\frac{(n-1)(2\bar{b}-\bar{b}^2)}{n^3}}$, it holds $\mathrm{F} \succeq \beta(\alpha)\mathrm{I}$, where

$$\beta(\alpha) := \left[n + \frac{1}{n}\left(1 - \frac{1}{n}\right)\left(2\bar{b} - \bar{b}^2\right)\right]\alpha - n. \tag{22}$$

Proof: Assume that $\mathbf{c} \geq \frac{\alpha}{n}\mathbf{1}$ for some $\alpha \in (0, 1]$. We then have $\mathbf{c} = \frac{\alpha}{n}\mathbf{1} + (1-\alpha)\mathbf{d}$, with $\mathbf{d} = [d_1, \ldots, d_n]' \geq \mathbf{0}$ and $\mathbf{d}^\top\mathbf{1} = 1$. The matrix F can be written as $\mathrm{F} = \frac{\alpha}{n}\mathrm{B} + \frac{(1-\alpha)}{2}(\mathrm{DB} + \mathrm{BD})$, with $\mathrm{D} = \mathrm{diag}(\mathbf{d})$. It can be verified that $\|\mathrm{B}\| = \rho(\mathrm{B}) \leq n$, since for any $\mathbf{x} \in \mathbb{R}^n$, it holds

$$\begin{aligned}\mathbf{x}^\top\mathrm{B}\mathbf{x} &= \|\mathbf{x}\|^2 + \|(\mathbf{1}-\mathbf{b})^\top\mathbf{x}\|^2 - \|\mathbf{b}^\top\mathbf{x}\|^2 \\ &\leq \|\mathbf{x}\|^2 + \|(\mathbf{1}-\mathbf{b})^\top\mathbf{x}\|^2 \leq n\|\mathbf{x}\|^2.\end{aligned} \tag{23}$$

Therefore, we obtain $\|\mathrm{DB} + \mathrm{BD}\| \leq 2\|\mathrm{D}\|\|\mathrm{B}\| \leq 2n$. On the other hand, from (23), it holds

$$\mathbf{x}^\top\mathrm{B}\mathbf{x} \geq \|\mathbf{x}\|^2 - \|\mathbf{b}^\top\mathbf{x}\|^2 \geq (1 - \|\mathbf{b}\|^2)\|\mathbf{x}\|^2.$$

As $\mathbf{b} \geq \frac{\bar{b}}{n}\mathbf{1}$, it should be $\mathbf{b} = \frac{\bar{b}}{n}\mathbf{1} + (1-\bar{b})\tilde{\mathbf{b}}$ for some $\tilde{\mathbf{b}} \in \mathbb{R}^n_{\geq 0}$ with $\tilde{\mathbf{b}}^\top\mathbf{1} = 1$. Therefore

$$\|\mathbf{b}\|^2 = -\frac{\bar{b}^2}{n} + \frac{2\bar{b}}{n} + (1-\bar{b})^2\|\tilde{\mathbf{b}}\|^2 \leq 1 - \left(1 - \frac{1}{n}\right)(2\bar{b} - \bar{b}^2).$$

As a result, for any $\mathbf{x} \in \mathbb{R}^n$, it holds

$$\mathbf{x}^\top\mathrm{B}\mathbf{x} \geq \left(1 - \frac{1}{n}\right)\left(2\bar{b} - \bar{b}^2\right)\|\mathbf{x}\|^2. \tag{24}$$

That is, $\lambda_{\min}(\mathrm{B}) \geq (1 - \frac{1}{n})(2\bar{b} - \bar{b}^2)$. According to the Weyl's inequality [29], we have

$$\lambda_{\min}(\mathrm{F}) \geq \frac{\alpha}{n}\left(1 - \frac{1}{n}\right)\left(2\bar{b} - \bar{b}^2\right) - n(1-\alpha) = \beta(\alpha).$$

Therefore, if $\alpha > \frac{1}{1+\frac{(n-1)(2\bar{b}-\bar{b}^2)}{n^3}}$, it must be $\lambda_{\min}(\mathrm{F}) \geq \beta(\alpha) > 0$. The proof is then completed. □

Proof of Lemma 10: From Lemma 7, it suffices to prove that $\frac{\mathrm{J}(\boldsymbol{\theta})+\mathrm{J}(\boldsymbol{\theta})^\top}{2} \succeq \frac{\beta(\bar{\alpha})}{n}\mathrm{I}\ \forall\boldsymbol{\theta} \in (\mathcal{A}_i^\delta)$. Note that $\mathrm{J}(\boldsymbol{\theta}) = \sum_{m=1}^M \mathrm{J}_m(\boldsymbol{\theta}) \otimes (\mathbf{e}_m\mathbf{e}_m^\top)$, where $\mathrm{J}_m(\boldsymbol{\theta}) \in \mathbb{R}^{n\times n}$ is defined as $[\mathrm{J}_m(\boldsymbol{\theta})]_{ij} := -\frac{\partial^2 \mathrm{sp}_{i,m}(\boldsymbol{\theta})}{\partial\theta_{i,m}\partial\theta_{j,m}}\quad \forall i, j \in \mathcal{V}$. Therefore, we have $\frac{\mathrm{J}(\boldsymbol{\theta})+\mathrm{J}(\boldsymbol{\theta})^\top}{2} = \sum_{m=1}^M(\frac{\mathrm{J}_m(\boldsymbol{\theta})+\mathrm{J}_m(\boldsymbol{\theta})^\top}{2}) \otimes (\mathbf{e}_m\mathbf{e}_m^\top)$, and it suffices to prove $\frac{\mathrm{J}_m+\mathrm{J}_m^\top}{2} \succeq \frac{\beta(\bar{\alpha})}{n}\mathrm{I}$ for each $m \in [M]$. Hereafter in the proof, when clear, we omit the index m for the simplicity of notation.

According to (11), for $j \neq i$, we have

$$\begin{aligned}\frac{\partial^2\mathrm{P}}{\partial\theta_i\partial\theta_j} &= \mathrm{Q}^{-1}\mathbf{e}_j\mathbf{e}_j^\top\mathrm{W}\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top\mathrm{W}\mathrm{Q}^{-1}\Theta \\ &\quad + \mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top\mathrm{W}\mathrm{Q}^{-1}\mathbf{e}_j\mathbf{e}_j^\top\mathrm{W}\mathrm{Q}^{-1}\Theta \\ &\quad - \mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top\mathrm{W}\mathrm{Q}^{-1}\mathbf{e}_j\mathbf{e}_j^\top - \mathrm{Q}^{-1}\mathbf{e}_j\mathbf{e}_j^\top\mathrm{W}\mathrm{Q}^{-1}\mathbf{e}_i\mathbf{e}_i^\top\end{aligned}$$

which gives

$$\begin{aligned}n\frac{\partial^2\mathrm{sp}_i}{\partial\theta_i\partial\theta_j} &= \mathbf{1}^\top\frac{\partial^2\mathrm{P}}{\partial\theta_i\partial\theta_j}\mathbf{e}_i \\ &= \mathbf{1}^\top\mathbf{q}_j\mathbf{w}_j^\top\mathbf{q}_i\mathbf{w}_i^\top\mathbf{p}_i + \mathbf{1}^\top\mathbf{q}_i\mathbf{w}_i^\top\mathbf{q}_j\mathbf{w}_j^\top\mathbf{p}_i \\ &\quad - \mathbf{1}^\top\mathbf{q}_j\mathbf{w}_j^\top\mathbf{q}_i.\end{aligned} \tag{25}$$

Applying (16) and (25) to (7), we have

$$\begin{aligned}-\frac{\partial^2\mathrm{sp}_i}{(\partial\theta_i)^2} &= \left[2c_i + \frac{2\sum_{h\in\mathcal{V}}(1-\theta_h)c_i^2}{\sum_{h\in\mathcal{V}}c_h\theta_h}\right]\frac{\sum_{h\neq i}c_h\theta_h}{n(\sum_{h\in\mathcal{V}}c_h\theta_h)^2} \\ -\frac{\partial^2\mathrm{sp}_i}{\partial\theta_i\partial\theta_j} &= \left[c_i + c_j + \frac{2\sum_{h\in\mathcal{V}}(1-\theta_h)c_ic_j}{\sum_{h\in\mathcal{V}}c_h\theta_h}\right] \\ &\quad\cdot\frac{c_i\theta_i}{n(\sum_{h\in\mathcal{V}}c_h\theta_h)^2} - \left[c_i + \frac{\sum_{h\in\mathcal{V}}(1-\theta_h)c_ic_j}{\sum_{h\in\mathcal{V}}c_h\theta_h}\right] \\ &\quad\cdot\frac{1}{n\sum_{h\in\mathcal{V}}c_h\theta_h}.\end{aligned} \tag{26}$$

This further gives that

$$\begin{aligned}\left[\frac{\mathrm{J}_m+\mathrm{J}'_m}{2}\right]_{ii} &= \left[2c_i + \frac{2\sum_{h\in\mathcal{V}}(1-\theta_h)c_i^2}{\sum_{h\in\mathcal{V}}c_h\theta_h}\right] \\ &\cdot\frac{\sum_{h\neq i}c_h\theta_h}{n(\sum_{h\in\mathcal{V}}c_h\theta_h)^2},\quad \left[\frac{\mathrm{J}_m+\mathrm{J}'_m}{2}\right]_{ij} = \frac{\sum_{h\neq i,j}c_h\theta_h}{n(\sum_{h\in\mathcal{V}}c_h\theta_h)^2} \\ &\cdot\left[\frac{c_i+c_j}{2} + \frac{\sum_{h\in\mathcal{V}}(1-\theta_h)c_ic_j}{\sum_{h\in\mathcal{V}}c_h\theta_h}\right]\end{aligned} \tag{27}$$

where the subscript m is omitted in the right-hand side terms. Let $b_i = \frac{c_i\theta_i}{\sum_{h\in\mathcal{V}}c_h\theta_h}$ for all $i \in \mathcal{V}$ and $\mathbf{b} = [b_1, \ldots, b_n]'$. Note that $b_i \geq c_i\theta_i \geq \frac{\bar{\alpha}\delta}{n}$ and $\mathbf{b}^\top\mathbf{1} = 1$. The equations (27) become

$$\begin{aligned}&n\left(\sum_{h\in\mathcal{V}}c_h\theta_h\right)^2\left[\frac{\mathrm{J}_m+\mathrm{J}'_m}{2}\right]_{ii} \\ &= \left[2c_i\sum_{h\in\mathcal{V}}c_h\theta_h + 2c_i^2\sum_{h\in\mathcal{V}}(1-\theta_h)\right]\cdot(1-b_i), \\ &n\left(\sum_{h\in\mathcal{V}}c_h\theta_h\right)^2\left[\frac{\mathrm{J}_m+\mathrm{J}'_m}{2}\right]_{ij} \\ &= (1-b_i-b_j)\cdot\left[\frac{c_i+c_j}{2}\sum_{h\in\mathcal{V}}c_h\theta_h + c_ic_j\sum_{h\in\mathcal{V}}(1-\theta_h)\right].\end{aligned} \tag{28}$$

Define $\mathrm{B} := \mathrm{I} + \mathbf{1}\mathbf{1}^\top - \mathbf{1}\mathbf{b}^\top - \mathbf{b}\mathbf{1}^\top$. The (28) can then be written in the following compact form:

$$\begin{aligned}&n\left(\sum_{h\in\mathcal{V}}c_h\theta_h\right)^2\frac{\mathrm{J}_m+\mathrm{J}'_m}{2} \\ &= \left(\sum_{h\in\mathcal{V}}c_h\theta_h\right)\left(\frac{\mathrm{CB}+\mathrm{BC}}{2}\right) + \sum_{h\in\mathcal{V}}(1-\theta_h)\mathrm{CBC}\end{aligned} \tag{29}$$

where $\mathrm{C} = \mathrm{diag}(c_1, \ldots, c_n)$. From (24) in the appendix, B is positive definite, so as the matrix CBC. On the other hand, according to Lemma 12, $\frac{\mathrm{CB}+\mathrm{BC}}{2} \succeq \beta(\bar{\alpha})$, which further gives $\frac{\mathrm{J}_m+\mathrm{J}_m^\top}{2} \succeq \frac{\beta(\bar{\alpha})}{n}\mathrm{I}$. The proof is then completed. □

Proof of Theorem 5: With the previous arguments, Assumption 2 holds. In the following we follow the symbols in the proof of Lemma 10 (the subscript m is also omitted when there is no confusion). It suffices to prove that the matrix $\frac{\mathrm{BC}+\mathrm{CB}}{2}$ is positive definite for all $\boldsymbol{\theta} \in \mathcal{A}^o$. For $n = 3$, according to (27), we have

$$\frac{\mathrm{BC}+\mathrm{CB}}{2} = \begin{pmatrix} 2c_1(b_2+b_3) & \frac{c_1+c_2}{2}b_3 & \frac{c_1+c_3}{2}b_2 \\ \frac{c_1+c_2}{2}b_3 & 2c_2(b_1+b_3) & \frac{c_2+c_3}{2}b_1 \\ \frac{c_1+c_3}{2}b_2 & \frac{c_2+c_3}{2}b_1 & 2c_3(b_1+b_2) \end{pmatrix}.$$

For the first row, we have

$$2c_1(b_2+b_3) - \frac{c_1+c_2}{2}b_3 - \frac{c_1+c_3}{2}b_2 = \frac{3}{2}c_1(b_2+b_3) - \frac{c_2}{2}b_3 - \frac{c_3}{2}b_2 > \frac{b_2+b_3}{2}(3c_1 - c_2 - c_3) > 0$$

where the inequalities are due to $\boldsymbol{\theta} \in \mathcal{A}^o$, which yields that either b_2 or b_3 be positive. Similarly, it can be proved that the inequality above holds for the second and third rows. This means that $\frac{\mathrm{BC}+\mathrm{CB}}{2}$ is strictly diagonally dominant matrix and the diagonal elements are greater than 0. Therefore, $\frac{\mathrm{BC}+\mathrm{CB}}{2}$ is positive definite for all $\boldsymbol{\theta} \in \mathcal{A}^o$, which further gives that $\frac{\mathrm{J}(\boldsymbol{\theta})+\mathrm{J}(\boldsymbol{\theta})^\top}{2}$ is positive definite. According to Theorem 3, the social power game $\langle \mathcal{V}, (\mathcal{A}_i), (u_i)\rangle$ admits a unique NE. □

References

[1] R. Abebe et al., “Opinion dynamics optimization by varying susceptibility to persuasion via non-convex local search,” *ACM Trans. Knowl. Discov. from Data (TKDD)*, vol. 16, no. 2, pp. 1–34, 2021.

[2] B. D. Anderson and M. Ye, “Recent advances in the modelling and analysis of opinion dynamics on influence networks,” *Int. J. Automat. Comput.*, vol. 16, no. 2, pp. 129–149, 2019.

[3] Y. Ao and Y. Jia, “Agents attraction competition in an extended Friedkin-Johnsen social network,” *IEEE Trans. Control Netw. Syst.*, vol. 10, no. 3, pp. 1100–1112, Sep. 2023.

[4] T. Basar, “On the uniqueness of the Nash solution in linear-quadratic differential games,” *Int. J. Game Theory*, vol. 5, pp. 65–90, 1976.

[5] C. Bernardo, L. Wang, M. Fridahl, and C. Altafini, “Quantifying leadership in climate negotiations: A social power game,” *PNAS Nexus*, vol. 2, no. 11, 2023, Art. no. 365.

[6] C. Bernardo, L. Wang, F. Vasca, Y. Hong, G. Shi, and C. Altafini, “Achieving consensus in multilateral international negotiations: The case study of the 2015 paris agreement on climate change,” *Sci. Adv.*, vol. 7, no. 51, 2021, Art. no. eabg 8068.

[7] E. Boix-Adserà, B. L. Edelman, and S. Jayanti, “The multiplayer colonel blotto game,” in *Proc. 21st ACM Conf. Econ. Computation*, 2020, pp. 47–48.

[8] W. Chen, Y. Wang, and S. Yang, “Efficient influence maximization in social networks,” in *Proc. 15th ACM SIGKDD Int. Conf. Knowl. Discov. Data Mining*, 2009, pp. 199–208.

[9] L. C. Corchón, “The theory of contests: A survey,” *Rev. Econ. Des.*, vol. 11, pp. 69–100, 2007.

[10] S. Cui, U. V. Shanbhag, and M. Staudigl, “A regularized variance-reduced modified extragradient method for stochastic hierarchical games,” 2023, *arXiv:2302.06497*.

[11] S. Dafermos, “Sensitivity analysis in variational inequalities,” *Math. Operations Res.*, vol. 13, no. 3, pp. 421–434, 1988.

[12] P. Dasgupta and E. Maskin. *The Existence of Equilibrium in Discontinuous Economic Games, I: Theory. The Review of Economic Studies*, vol. 53, no. 1, pp. 1–26, 1986.

[13] M. H. DeGroot, “Reaching a consensus,” *J. Amer. Stat. Assoc.*, vol. 69, no. 345, pp. 118–121, 1974.

[14] S. Dhamal, W. Ben-Ameur, T. Chahed, and E. Altman, “A two phase investment game for competitive opinion dynamics in social networks,” *Inf. Process. Manage.*, vol. 57, no. 2, 2020, Art. no. 102064.

[15] M. E. Dickison, M. Magnani, and L. Rossi, *Multilayer Social Networks*. Cambridge, MA, USA: Cambridge Univ. Press, 2016.

[16] S. R. Etesami, “Open-loop equilibrium strategies for dynamic influence maximization game over social networks,” *IEEE Control Syst. Lett.*, vol. 6, pp. 1496–1500, 2021.

[17] S. R. Etesami and T. Başar, “Complexity of equilibrium in competitive diffusion games on social networks,” *Automatica*, vol. 68, pp. 100–110, 2016.

[18] C. L. Evans, *Leadership in Committee: A Comparative Analysis of Leadership Behavior in the US Senate*. Ann Arbor, MI, USA; Univ. Michigan Press, 2001.

[19] F. Facchinei and J.-S. Pang, “Nash equilibria: The variational approach,” in *Convex Optim. Signal Process. Commun.*, Cambridge, U.K.: Cambridge Univ. Press, 2010, pp. 443–493.

[20] A. Fazeli, A. Ajorlou, and A. Jadbabaie, “Competitive diffusion in social networks: Quality or seeding?,” *IEEE Trans. Control Netw. Syst.*, vol. 4, no. 3, pp. 665–675, Mar. 2016.

[21] S. T. Fiske and J. Berdahl, *Social Power.*, in A. W. Kruglanski & E. T. Higgins, Eds., 2007, The Guilford Press.

[22] A. Fontan and C. Altafini, “A signed network perspective on the government formation process in parliamentary democracies,” *Sci. Rep.*, vol. 11, no. 1, 2021, Art. no. 5134.

[23] N. E. Friedkin and E. C. Johnsen, “Influence networks and opinion change,” *Adv. Group Processes*, vol. 16, no. 1, pp. 1–29, 1999.

[24] J. Ghaderi and R. Srikant, “Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate,” *Automatica*, vol. 50, no. 12, pp. 3209–3215, 2014.

[25] A. Gionis, E. Terzi, and P. Tsaparas, “Opinion maximization in social networks,” in *Proc. SIAM Int. Conf. Data Mining*, 2013, pp. 387–395.

[26] M. Grabisch, A. Mandel, A. Rusinowska, and E. Tanimura, “Strategic influence in social networks,” *Math. Operations Res.*, vol. 43, no. 1, pp. 29–50, 2018.

[27] N. Groot, G. Zaccour, and B. De Schutter, “Hierarchical game theory for system-optimal control: Applications of reverse Stackelberg games in regulating marketing channels and traffic routing,” *IEEE Control Syst. Mag.*, vol. 37, no. 2, pp. 129–152, Feb. 2017.

[28] R. Hegselmann and U. Krause, “Opinion dynamics and bounded confidence models, analysis, and simulation,” *J. Artif. Societies Social Simul.*, vol. 5 no. 3, pp. 1–33, 2002.

[29] R. A. Horn and C. R. Johnson. *Matrix Analysis*. Cambridge, MA, USA: Cambridge Univ. Press, 2012.

[30] M. Hu and M. Fukushima, “Multi-leader-follower games: Models, methods and applications,” *J. Operations Res. Soc. Jpn.*, vol. 58, no. 1, pp. 1–23, 2015.

[31] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo, “Opinion dynamics and the evolution of social power in influence networks,” *SIAM Rev.*, vol. 57, no. 3, pp. 367–397, 2015.

[32] D. Kempe, J. Kleinberg, and É. Tardos, “Maximizing the spread of influence through a social network,” in *Proc. 9th ACM SIGKDD Int. Conf. Knowl. Discov. data mining*, 2003, pp. 137–146.

[33] J. R. Marden and J. S. Shamma, “Game theory and control,” *Annu. Rev. Control, Robot., Auton. Syst.*, vol. 1, pp. 105–134, 2018.

[34] A. Matakos, E. Terzi, and P. Tsaparas, “Measuring and moderating opinion polarization in social networks,” *Data Mining Knowl. Discov.*, vol. 31, pp. 1480–1505, 2017.

[35] A. Mirtabatabaei, P. Jia, N. E. Friedkin, and F. Bullo, “On the reflected appraisals dynamics of influence networks with stubborn agents,” in *Proc. Amer. Control Conf.*, 2014, pp. 3978–3983.

[36] U. Mosco, “Convergence of convex sets and of solutions of variational inequalities,” *Adv. Math.*, vol. 3, no. 4, pp. 510–585, 1969.

[37] C. Musco, C. Musco, and C. E. Tsourakakis, “Minimizing polarization and disagreement in social networks,” in *Proc. World Wide Web Conf.*, 2018, pp. 369–378.

[38] A. Nedić and B. Touri, “Multi-dimensional Hegselmann-Krause dynamics,” in *Proc. IEEE 51st Conf. Decis. control (CDC)*, 2012, pp. 68–73.

[39] M. J. Osborne and A. Rubinstein. *A Course in Game Theory*. Cambridge, MA, USA: MIT Press, 1994.

[40] K. Paarporn, R. Chandan, M. Alizadeh, and J. R. Marden, “Characterizing the interplay between information and strength in blotto games,” in *Proc. IEEE 58th Conf. Decis. Control (CDC)*, 2019, pp. 5977–5982.

[41] K. Paarporn, R. Chandan, D. Kovenock, M. Alizadeh, and J. R. Marden, “Strategically revealing intentions in general lotto games,” *IEEE Trans. Autom. Control*, vol. 69, no. 8, pp. 5396–5407, Aug. 2024.

[42] N. Pagan and F. Dörfler, “Game theoretical inference of human behavior in social networks,” *Nature Commun.*, vol. 10, no. 1, 2019, Art. no. 5507.

[43] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, “Novel multidimensional models of opinion dynamics in social networks,” *IEEE Trans. Autom. Control*, vol. 62, no. 5, pp. 2270–2285, May 2016.

[44] A. V. Proskurnikov and R. Tempo, “A tutorial on modeling and analysis of dynamic social networks. part I,” *Annu. Rev. Control*, vol. 43, pp. 65–79, 2017.

[45] A. V. Proskurnikov and R. Tempo, “A tutorial on modeling and analysis of dynamic social networks. part II,” *Annu. Rev. Control*, vol. 45, pp. 166–190, 2018.

[46] P. J. Reny, “On the existence of pure and mixed strategy Nash equilibria in discontinuous games,” *Econometrica*, vol. 67, no. 5, pp. 1029–1056, 1999.

[47] P. J. Reny, “Nash equilibrium in discontinuous games,” *Annu. Rev. Econ.*, vol. 12, no. 1, pp. 439–470, 2020.

[48] F. A. Rodrigues. *Network Centrality: An Introduction. A Mathematical Modeling Approach From Nonlinear Dynamics to Complex Systems*, pp. 177–196, 2019.

[49] J. B. Rosen, “Existence and uniqueness of equilibrium points for concave n-person games,” *Econometrica: J. Econometric Soc.*, pp. 520–534, 1965.

[50] E. Rossi, F. Monti, Y. Leng, M. Bronstein, and X. Dong, “Learning to infer structures of network games,” in *Proc. Int. Conf. Mach. Learn.*, 2022, pp. 18809–18827.

[51] J. Scott. *What is Social Network Analysis?*, London, U.K.: Bloomsbury Academic, 2012.

[52] G. Scutari, D. P. Palomar, F. Facchinei, and J.-S. Pang, “Convex optimization, game theory, and variational inequality theory,” *IEEE Signal Process. Mag.*, vol. 27, no. 3, pp. 35–49, Mar. 2010.

[53] J. Sherman and W. J. Morrison, “Adjustment of an inverse matrix corresponding to a change in one element of a given matrix,” *Ann. Math. Statist.*, vol. 21, no. 1, pp. 124–127, 1950.

[54] M. Sion, “On general minimax theorems,” *Pacific J. Math.*, vol. 8, no. 1, 1958.

[55] H. Sun and Z. Zhang, “Opinion optimization in directed social networks,” in *Proc. AAAI Conf. Artif. Intell.*, 2023, pp. 4623–4632.

[56] Y. Tian, P. Jia, A. Mirtabatabaei, L. Wang, N. E. Friedkin, and F. Bullo, “Social power evolution in influence networks with stubborn individuals,” *IEEE Trans. Autom. Control*, vol. 67, no. 2, pp. 574–588, Feb. 2021.

[57] Y. Tian and L. Wang, “Opinion dynamics in social networks with stubborn agents: An issue-based perspective,” *Automatica*, vol. 96, pp. 213–223, 2018.

[58] L. Wang, G. Chen, C. Bernardo, Y. Hong, G. Shi, and C. Altafini, “Social power games in concatenated opinion dynamics,” *IEEE Trans. Autom. Control*, vol. 69, no. 11, pp. 7614–7629, Nov. 2024.

[59] L. Wang, Y. Xing, C. Altafini, and K. H. Johansson, “Maximizing social power in multiple independent Friedkin-Johnsen models,” in *Proc. Eur. Control Conf. (ECC)*, 2024, pp. 3422–3427.

[60] T.-Z. Wang, T. Qin, and Z.-H. Zhou, “Sound and complete causal identification with latent variables given local background knowledge,” *Artif. Intell.*, vol. 322, 2023, Art. no. 103964.

[61] M. Ye, J. Liu, L. Wang, B. D. Anderson, and M. Cao, “Consensus and disagreement of heterogeneous belief systems in influence networks,” *IEEE Trans. Autom. Control*, vol. 65, no. 11, pp. 4679–4694, Nov. 2019.

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