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Problems on $\beta\mathbb{N}$

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A B S T R A C T

This is an update on, and expansion of, our paper *Open problems on $\beta\omega$* in the book *Open Problems in Topology*.

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0. Introduction

In 1990 we contributed a paper to the book *Open problems in Topology*, [107], titled *Open problems on $\beta\omega$* ([77]).

Through the years some of these problems were solved, some were shown to be related to other problems, and some are still unsolved. In the first years after the publication of the book there were regular updates on the problems in the journal *Topology and its Applications*; in 2004 these were collated and extended in a comprehensive status report, [112], by Elliott Pearl.

The COVID-19 pandemic provided a good opportunity to go through our original paper again and provide a new update of the status of the problems as well as to collect and formulate new questions on the fascinating object that is $\beta\mathbb{N}$.

Many of the comments below incorporate information from Elliott Pearl's update mentioned above, but there have, of course, been many developments in the years since.

The numbering of the problems is different from that in the first paper because we have moved some questions around and combined related questions into more comprehensive problems. We have made no attempt to separate the solved problems from the unsolved ones. We wanted to keep related problems together and even though we consider a problem solved the reader may disagree and be inspired to investigate variations or strengthenings of the answers.

We should also mention the book *Open problems in Topology II* [113], edited by Elliot Pearl, that contains many more problems in topology, and in particular a paper by Peter Nyikos, *Čech-Stone remainders of discrete spaces*, that, as the title indicates, deals with problems on $\beta\kappa$ for arbitrary infinite cardinals κ.

1. Preliminaries

The main objects of study in this paper are the space $\beta\mathbb{N}$ and its subspace $\mathbb{N}^*$.

For a quick overview of their properties we refer to Chapter D-18 of [80]; the paper [104] offers a more comprehensive treatment. Here we collect some of the basic facts about our spaces in order to fix some notation that will be used throughout the paper.

To begin: $\beta\mathbb{N}$ is the set of ultrafilters on the set $\mathbb{N}$ of natural numbers, endowed with the topology generated by the base $\{\overline{A} : A \subseteq \mathbb{N}\}$, where $\overline{A}$ denotes the set of ultrafilters that contain A. The readily established equality $\beta\mathbb{N} \setminus \overline{A} = \overline{\mathbb{N} \setminus A}$ confirms what the notation $\overline{A}$ suggests: the set $\overline{A}$ is open *and* closed, and also equal to the closure of A in $\beta\mathbb{N}$.

We identify an element n of $\mathbb{N}$ with the ultrafilter $\{A : n \in A\}$ and thus consider $\mathbb{N}$ to be a subset of $\beta\mathbb{N}$. The complement $\beta\mathbb{N} \setminus \mathbb{N}$ is the set of *free* ultrafilters on $\mathbb{N}$ and is denoted $\mathbb{N}^*$; we extend this notation to all subsets of $\mathbb{N}$ and write $A^* = \overline{A} \setminus A$ whenever $A \subseteq \mathbb{N}$.

A map φ from $\mathbb{N}$ to itself induces a map $\beta\varphi$ from $\beta\mathbb{N}$ to itself: $\beta\varphi(u)$ is the ultrafilter generated by $\{\varphi[A] : A \in u\}$.

2. Autohomeomorphisms

The autohomeomorphisms of $\beta\mathbb{N}$ correspond to the permutations of $\mathbb{N}$ and are, as such, not very interesting topologically. The autohomeomorphisms of $\mathbb{N}^*$ offer more challenges.

In what follows Aut denotes the autohomeomorphism group of $\mathbb{N}^*$, and Triv denotes the subgroup of trivial autohomeomorphisms. Here a *trivial* autohomeomorphism is one with an 'easy' description: an autohomeomorphism h of $\mathbb{N}^*$ is trivial if there are co-finite subsets A and B of $\mathbb{N}$ and a bijection $\varphi : A \to B$ such that $h = \varphi^*$, where φ^* denotes the restriction of $\beta\varphi$ to $\mathbb{N}^*$.

1. Can Triv be a proper normal subgroup of Aut, and if *yes* what is (or can be) the structure of the factor group $\mathsf{Aut}/\mathsf{Triv}$; and if *no* what is (or can be) $[\mathsf{Triv} : \mathsf{Aut}]$?
COMMENTS: A very concrete first step would be to investigate what can one say about an autohomeomorphism h that satisfies $h^{-1} \circ \mathsf{Triv} \circ h = \mathsf{Triv}$.

A related question: what is the minimum number of autohomeomorphisms necessary to add to Triv to get a generating set for Aut?

Of course that number is 0 when $\mathsf{Aut} = \mathsf{Triv}$, but can it be non-zero and finite?

If $h \in \mathsf{Aut}$ then $I(h)$ denotes the family of subsets of ω on which h is trivial, that is, $A \in I(h)$ iff there is a function $h' : A \to \omega$ such that $h(B^*) = h'[B]^*$ whenever $B \subseteq A$.

If $I(h)$ contains an infinite set then h is *somewhere trivial*, otherwise h is totally non-trivial.

The ideal $I(h)$ determines an open set O_h: the union $\bigcup\{A^* : A \in I(h)\}$; its complement F_h is closed and could be said to be the set of points of $\mathbb{N}^*$ where h is truly non-trivial.

2. Does the existence of a (totally) non-trivial automorphism imply that Aut is simple?
COMMENTS: This question asks more than the opposite of question 1; a yes answer here would imply a no answer there, but a negative answer there could go together with a negative answer here.

3. Is it consistent with $\mathsf{MA} + \neg\mathsf{CH}$ that a totally non-trivial automorphism exists?
COMMENTS: The answer yes. This was established by Shelah and Steprāns in [136]. Consistency is the best one can hope for: in [134] Shelah and Steprāns proved that PFA implies all autohomeomorphisms of $\mathbb{N}^*$ are trivial; they also indicated how the implicit large cardinal assumption can be avoided and use $\diamondsuit$ on ω_2 to capture and eliminate any potential non-trivial autohomeomorphisms in a countable support iteration of length ω_2. Though not stated explicitly by the authors it is clear that one can modify the iteration so as to obtain a model that satisfies $\mathsf{MA}_{\aleph_1}$ as well. In [145] Veličković showed that the conjunction of $\mathsf{MA}_{\aleph_1}$ and OCA implies that all autohomeomorphisms are trivial.

4. Is it consistent to have a non-trivial automorphism, while for every $h \in \mathsf{Aut}$ the ideal $I(h)$ is the intersection of finitely many prime ideals?
COMMENTS: In topological terms: can one have non-trivial autohomeomorphisms but only very mild ones; the set of points where an autohomeomorphism is truly non-trivial is always finite.

5. Is every ideal $I(h)$ a P-ideal?
COMMENTS: This was asked explicitly in [134, Question 2] in case every autohomeomorphism is somewhere trivial, after it was shown that PFA implies a yes answer. However, as mentioned above, PFA implies that all autohomeomorphisms are trivial, so that $I(h)$ is, in fact, always an improper ideal.

Of course this question only makes sense in case $I(h)$ is not equal to the ideal of finite sets. Also, if *every* autohomeomorphism is somewhere trivial then every $I(h)$ is a tall ideal and hence the set of points of non-triviality is nowhere dense.

Without the additional condition that every autohomeomorphism is somewhere trivial the answer is consistently negative. The Continuum Hypothesis lets one construct an autohomeomorphism h that is trivial, in fact the identity, on the members of a partition $\mathcal{A}$ of ω into infinite sets, and so that there is a point u on the boundary of $\bigcup\{A^* : A \in \mathcal{A}\}$ such that h is not trivial on each neighborhood of u. This implies there is no $B \in I(h)$ such that $A \subseteq B$ for all $A \in \mathcal{A}$.

6. If every automorphism is somewhere trivial, is then every automorphism trivial?
COMMENTS: This is undecidable.

Shelah proved the consistency of "all autohomeomorphisms are trivial", see [127]. Shelah and Steprāns proved the consistency with $\mathsf{MA}_{\aleph_1}$ of "every autohomeomorphism is somewhere trivial, yet there is a non-trivial autohomeomorphism" in [135]; as noted above they proved in [136] that MA does not imply that all autohomeomorphisms are somewhere trivial.

Given a cardinal κ call an autohomeomorphism h *weakly κ-trivial* if the set $\{p : p \not\equiv_{\mathbf{RK}} h(p)\}$ has cardinality less than κ. Here $p \equiv_{\mathbf{RK}} q$ means that p and q have the same type, i.e., $q = \pi^*(p)$ for some permutation π of $\mathbb{N}$.

7. For what cardinals κ is it consistent to have that all autohomeomorphisms are weakly κ-trivial?
Comments: Since a trivial autohomeomorphism is weakly 1-trivial we see that $\kappa = 1$ is a possibility. And of course the candidates are less than or equal to $2^{\mathfrak{c}}$.

8. If h is weakly 1-trivial is h then trivial?
Comments: This is a uniformization question: if for every $p \in \mathbb{N}^*$ there is a permutation π_p such that $h(p) = \pi_p^*(p)$ is there then one (almost) permutation π of $\mathbb{N}^*$ such that $h(p) = \pi^*(p)$ for all p?

9. ($\mathsf{MA} + \neg\mathsf{CH}$) if p and q are $P_{\mathfrak{c}}$-points is there an h in Aut such that $h(p) = q$?
Comments: This is undecidable.

Shelah and Steprāns proved the consistency of $\mathsf{MA} + \neg\mathsf{CH}$ with "all autohomeomorphisms are trivial" in [134]; in this model there are $\mathfrak{c}$ many autohomeomorphisms and $2^{\mathfrak{c}}$ many $P_{\mathfrak{c}}$-points.

Steprāns proved the consistency of a positive answer in [141].

In the investigations into the previous question the following equivalence relation was used: $p \equiv q$ means that there are two partitions $\{A_n : n \in \omega\}$ and $\{B_n : n \in \omega\}$ of $\mathbb{N}$ into finite sets such that

$$(\forall P \in p)(\exists Q \in q)(\forall n)\bigl(|P \cap A_n| = |Q \cap B_n|\bigr)$$

The following question was left open.

10. Is $\equiv$ different from $\equiv_{\mathbf{RK}}$ in ZFC?

11. Are the autohomeomorphisms of $\mathbb{N}^*$ induced by the shift map $\sigma : n \mapsto n+1$ and by its inverse conjugate?
Comments: Recently Will Brian showed that the answer to this question is affirmative assuming CH, see [20].

See [19,41] for earlier results. Under CH the autohomeomorphism group of $\mathbb{N}^*$ is simple, yet it has the maximum possible number of conjugacy classes: $2^{\mathfrak{c}}$. This suggests questions about the number and nature of conjugacy classes of this group, in ZFC or under various familiar extra set-theoretical assumptions, see also [79].

12. Does $\mathbb{N}^*$ have a universal autohomeomorphism?
Comments: This is a question with many possible variations. The definition of universality that we adopt here is as follows: $f : \mathbb{N}^* \to \mathbb{N}^*$ is universal if for every closed subspace F of $\mathbb{N}^*$ and every autohomeomorphism g of F there is an embedding $e : F \to \mathbb{N}^*$ such that $g = e^{-1} \circ (f \restriction e[F]) \circ e$.

One can ask whether there is a universal autohomeomorphism at all, whether the shift σ is universal (for autohomeomorphisms without fixed points), whether there is a universal autohomeomorphism just for autohomeomorphisms without fixed points.

The authors have shown that there is a universal autohomeomorphism of $\mathbb{N}^*$ under CH and that there is no trivial universal autohomeomorphism. See [78].

We also note that $\mathbb{N}$ has a universal permutation: take a permutation of $\mathbb{N}$ that has infinitely many n-cycles, for every n, and infinitely many infinite cycles (copies of $\mathbb{Z}$ with the shift). Every other permutation of $\mathbb{N}$ can be embedded into this one.

3. Subspaces

13. For what p are $\mathbb{N}^* \setminus \{p\}$ and $\beta\mathbb{N} \setminus \{p\}$ non-normal?
Comments: Originally this question had the word 'equivalently' after the 'and' (in parentheses). Since $\mathbb{N}^* \setminus \{p\}$ is closed in $\beta\mathbb{N} \setminus \{p\}$ there is an implication between the non-normality of these spaces but we do not know whether that implication is reversible. Thus this question may actually be two separate ones.

Under CH the answer is, in both cases, "for every point", see [32,105,118,147]. There are some results for some special types of points, see, e.g., Błaszczyk and Szymański [16], Gryzlov [73], and Logunev [96], but a general answer is wanting.

14. Is it consistent that there is a non-butterfly point in $\mathbb{N}^*$?
Comments: We call p a butterfly point if there are disjoint sets A and B such that p is the only common accumulation point of A and B, that is: $A^d \cap B^d = \{p\}$.

The points used by Błaszczyk and Szymański [16] in their (partial) answer to the previous question are easy-to-describe butterfly points. Let $X = \{x_n : n \in \omega\}$ be a discrete subset of $\mathbb{N}^*$, let $A = \operatorname{cl} X \setminus X$ and take $p \in A$. Let $\{B_n : n \in \omega\}$ be a partition of $\mathbb{N}$ such that $B_n \in x_n$ for all n and let q be the ultrafilter $\{Q : p \in \operatorname{cl}\{x_n : n \in Q\}\}$. The set $B = \bigcap_{Q\in q}\bigl(\bigcup\{B_n : n \in Q\}\bigr)^*$ is closed and $\{q\} = A^d \cap B^d$. Thus butterfly points exist.

By contrast, in [14] Bešlagić and Van Douwen showed that it is consistent with all consistent cardinal arithmetic that all points of $\mathbb{N}^*$ are butterfly points.

15. Is it consistent that $\mathbb{N}^* \setminus \{p\}$ is C^*-embedded in $\mathbb{N}^*$ for some but not all $p \in \mathbb{N}^*$?
Comments: The answer is yes: in [50] Alan Dow showed that in the Miller model $\mathbb{N}^* \setminus \{p\}$ is C^*-embedded iff p is not a P-point. There are P-points in the Miller model: every ground-model P-point generates a P-point in the extension.

16. What spaces can be embedded in $\beta\omega$?
Comments: This is a very general question and a definitive answer looks out of reach for now, even for closed subspaces.

The Continuum Hypothesis implies that the closed subspaces of $\beta\omega$ are exactly the compact zero-dimensional F-spaces; in fact, these are also exactly the closed P-sets in $\mathbb{N}^*$. The implication does not reverse: in [53] it is shown that every compact zero-dimensional F-space is a (closed) subspace of $\mathbb{N}^*$ in any model obtained by adding $\aleph_2$ many Cohen reals to a model of CH.

Dow and Vermeer proved in [60] that it is consistent that the σ-algebra of Borel sets of the unit interval is not the quotient of any complete Boolean algebra. By Stone duality, this yields a compact basically disconnected space, hence a compact zero-dimensional F-space, of weight $\mathfrak{c}$ that cannot be embedded into any extremally disconnected space, in particular not into $\beta\mathbb{N}$.

Some ZFC results are available. For instance: if X is a compact space of countable cellularity that is a continuous image of $\mathbb{N}^*$ then its projective cover $E(X)$ can be embedded in $\mathbb{N}^*$ as a $\mathfrak{c}$-OK set (a weakening of the notion of a P-set). This was proved by van Mill in [103] and applies to all separable compact extremally disconnected spaces as well as to the projective covers of Suslin lines and of Bell's ccc non-separable remainder [10].

Van Douwen proved in unpublished work that every P-space of weight $\mathfrak{c}$ (or less) can be embedded into $\beta\mathbb{N}$. In fact he proved that for every infinite cardinal κ every P-space of weight 2^κ can be embedded in $\beta\kappa$. The argument was sketched and extended in [58] and we summarize it here for the reader's convenience.

Let X be a P-space of weight 2^κ and embed it into the Cantor cube $C = 2^{2^\kappa}$ of weight 2^κ. Next consider the projective cover $\pi : E(C) \to C$ of this cube. The Cantor cube is a group under coordinatewise addition modulo 2, so for every $p \in C$ the map $\lambda_p : x \mapsto x + p$ is a homeomorphism; this homeomorphism lifts to a homeomorphism $\Lambda_p : E(C) \to E(C)$ with the property that $\pi \circ \Lambda_p = \lambda_p \circ \pi$. Now take one point $u_0 \in E(C)$ that maps to the neutral element 0 of C and consider the subspace $X' = \{\Lambda_p(u_0) : p \in X\}$ of $E(C)$. Using the fact that regular open sets in C are, up to permutation of the coordinates, of the form $U \times 2^I$ where U is regular open in the Cantor set 2^ω and $I = 2^\kappa \setminus \omega$, one shows that π is actually a homeomorphism from X' to X. Finally then, as π is irreducible and C has density κ, the density of $E(C)$ is equal to κ as well. Therefore there is a continuous surjection $f : \beta\kappa \to E(C)$ and one can take a closed subset F of $\beta\kappa$ such that $f \restriction F$ is irreducible and onto. As $E(C)$ is extremally disconnected this restriction is a homeomorphism and we find our copy of X in F.

The extension in [58] delivers more but at a cost: one embeds βX in a suitable Cantor cube, possibly of a larger weight than that of X itself. What this delivers is that the copy of X in $\beta\lambda$ (where λ may be larger than the κ above) is C^*-embedded.

Thus we get the general statement that every P-space can be C^*-embedded in a compact extremally disconnected space.

This argument also shows that $2^{\aleph_0} = 2^{\aleph_1}$ implies that $\beta\omega_1$ embeds into $\beta\mathbb{N}$. For $\beta\omega_1$ embeds into the Cantor cube $2^{2^{\omega_1}}$, which under our assumption is the same as $2^{\mathfrak{c}}$. The latter is a continuous image of $\beta\mathbb{N}$ and an irreducible preimage of $\beta\omega_1$ will be homeomorphic to $\beta\omega_1$. If $2^{\aleph_0} < 2^{\aleph_1}$ then $\beta\omega_1$ can not be embedded into $\mathbb{N}^*$ because its weight, which is $2^{\aleph_1}$, is larger than that of $\mathbb{N}^*$.

17. Describe the closed P-sets of $\mathbb{N}^*$.
COMMENTS: This has a quite definitive answer under CH: every compact zero-dimensional F-space of weight $\mathfrak{c}$ can be embedded in $\mathbb{N}^*$ as a P-set. What we are looking for are properties that can be established in ZFC, or provably can not. For example: one cannot prove in ZFC that there is a P-set homeomorphic to $\mathbb{N}^*$ itself, see [87], or that there is a P-set that satisfies the ccc, see [69].

One can ask if cellularity less than $\mathfrak{c}$ is at all possible.

There are various nowhere dense closed P-sets that one can write down explicitly. To give two familiar examples, among many, we consider the density ideal $\mathcal{I}_d$ and the summable ideal $\mathcal{I}_\Sigma$. The first is defined by

$$I \in \mathcal{I}_d \quad \text{iff} \quad \lim_{n\to\infty} \frac{1}{n}|A \cap n| = 0$$

and the second as

$$I \in \mathcal{I}_\Sigma \quad \text{iff} \quad \sum_{n \in A} \frac{1}{n} \text{ converges.}$$

These ideals have been studied widely but we would like to know: what are the topological properties of the nowhere dense closed P-sets

$$F_d = \mathbb{N}^* \setminus \bigcup\{A^* : A \in \mathcal{I}_d\} \quad \text{and} \quad F_\Sigma = \mathbb{N}^* \setminus \bigcup\{A^* : A \in \mathcal{I}_\Sigma\}$$

Rudin established in [120] that F_d contains no P-points and even that no countable set of P-points accumulates at a point of F_d. Indeed let u be a P-point and observe first that for every n there is an $i_n < n$ such that $U_n = \{m \in \mathbb{N} : m \equiv i_n \pmod{n}\}$ belongs to u. Because u is a P-point there is then a $U \in u$ such that $U \subseteq^* U_n$ for all n. But this implies $U \in \mathcal{I}_d$, so $u \notin F_d$. Because F_d is a P-set this implies that no countable set of P-points has accumulation points in F_d.

There are certain similarities between the two sets and $\mathbb{N}^*$ itself. Consider the map $f : \mathbb{N} \to \mathbb{N}$, defined by $f(n) = k$ iff $k! < n \le (k+1)!$. It is an elementary exercise to show that

$$\limsup_{n\to\infty} \frac{1}{n}|f^{\leftarrow}[X] \cap n| = 1 \quad \text{en} \quad \sum_{n \in f^{\leftarrow}[X]} \frac{1}{n} = \infty$$

whenever X is an infinite subset of $\mathbb{N}$. This implies that βf maps both F_d and F_Σ onto $\mathbb{N}^*$ and it allows for the lifting of many combinatorial structures on $\mathbb{N}^*$ to these sets. It is clear that the restriction of βf to $\mathbb{N}^*$ is an open map onto $\mathbb{N}^*$ itself, whether its restrictions to F_d and F_Σ are open as well is less clear.

18. Which compact zero-dimensional F-spaces admit an open map onto $\mathbb{N}^*$?
Comments: This question is related to Van Douwen's paper [40], where open maps are used to transfer information from $\mathbb{N}^*$ to other remainders. As a special case one can investigate whether the sets F_d and F_Σ from the Question 17 admit open maps onto $\mathbb{N}^*$ (if the map βf given there does not already give open maps).

19. Is there a nowhere dense copy of $\mathbb{N}^*$ in $\mathbb{N}^*$ that is a $\mathfrak{c}$-OK-set in $\mathbb{N}^*$?
Comments: Alan Dow showed in [52] that there a nowhere dense copy of $\mathbb{N}^*$ that is not of the form $\operatorname{cl} D \setminus D$ for some countable and discrete subset D of $\mathbb{N}^*$. This was later improved by Dow and van Mill in [59] to a nowhere dense copy that is a weak P-set. In light of the comments for question 17 the present question asks for the best that we can get in ZFC. Most likely the answer to this question will require a new idea as the constructions in the papers cited above produce sets that are definitely not $\mathfrak{c}$-OK in $\mathbb{N}^*$.

20. Is every subspace of $\mathbb{N}^*$ strongly zero-dimensional?
Comments: It is clear that every subspace is zero-dimensional and that closed subspaces are even strongly zero-dimensional, but for general subspaces this question is quite open. Until recently it was not even known whether there was an example of a zero-dimensional F-space that is not strongly zero-dimensional, see [56].

If the answer is negative then a secondary question suggests itself immediately: is there an upper bound to the covering dimension of subspaces of $\mathbb{N}^*$?

21. Is every nowhere dense subset of $\mathbb{N}^*$ a $\mathfrak{c}$-set?
Comments: In general a set A is called a κ-set if there is a pairwise disjoint family $\mathcal{O}$ of open sets of cardinality κ and such that $A \subseteq \bigcap\{\operatorname{cl} O : O \in \mathcal{O}\}$.

That the answer is positive is called by some "The $\mathfrak{c}$-set conjecture". In [139] Simon proved that this question is the same as "Is there a maximal nowhere dense subset in $\mathbb{N}^*$?". The questions are the same in that the answer "no" to one is equivalent to the answer "yes" to the other: Every nowhere dense set in $\mathbb{N}^*$ is a $\mathfrak{c}$-set if and only if every nowhere dense set in $\mathbb{N}^*$ is a nowhere dense subset of another nowhere dense set (this is the order that we are considering).

There is a purely combinatorial reformulation of this question, denoted $\mathrm{RPC}(\omega)$ in [4]: if $\mathcal{A}$ is an infinite maximal almost disjoint family then $\mathcal{I}^+(\mathcal{A})$ has an almost disjoint refinement. Here, $\mathcal{I}^+(\mathcal{A})$ is the family of sets not in the ideal $\mathcal{I}(\mathcal{A})$ generated by $\mathcal{A}$ and the finite sets and an almost disjoint refinement is an almost disjoint family $\mathcal{B}$ with a map $X \mapsto B_X$ from $\mathcal{I}^+(\mathcal{A})$ to $\mathcal{B}$ such that $B_X \subseteq^* X$ for all X.

Finally, we should mention that the answer is positive for one-point sets: all points of $\mathbb{N}^*$ are $\mathfrak{c}$-points, see [5].

22. Does there exist a completely separable maximal almost disjoint family?
Comments: This question is related to Question 21 because by [4, Theorem 4.19] a positive answer to that question implies the existence of an abundance of completely separable maximal almost disjoint families; where a maximal almost disjoint family $\mathcal{A}$ is *completely separable* if it is itself an almost disjoint refinement of $\mathcal{I}^+(\mathcal{A})$.

Whether completely separable maximal almost disjoint families exist is a problem first raised by Erdős and Shelah in [64].

Currently the best result is due to Shelah who showed in [130] that the answer is positive if $\mathfrak{c} < \aleph_\omega$ and that a negative solution would imply consistency of the existence of large cardinals.

It is not (yet) clear whether this question and Question 21 are equivalent. Thus far *constructions* of completely separable maximal almost disjoint families (in some model or another) could always be adapted to prove RPC(ω), but there is currently no proof of RPC(ω) from the *mere existence* of such a family.

23. Describe the retracts of $\beta\mathbb{N}$ and $\mathbb{N}^*$, as well as their *absolute* retracts.
Comments: A retract of $\beta\mathbb{N}$ is necessarily a closed separable extremally disconnected subspace. It is known that a compact separable extremally disconnected can be embedded as a retract of $\beta\mathbb{N}$. If X is such a space then there is a continuous surjection $f : \beta\mathbb{N} \to X$ and if K is such that $f \restriction K$ is irreducible then $f \restriction K$ is a homeomorphism to X and $(f \restriction K)^{-1} \circ f$ is a retraction of $\beta\mathbb{N}$ onto K.

Shapiro [123] and Simon [138] have shown independently and by quite different means that not every closed separable subset of $\beta\mathbb{N}$ is a retract. This gives rise to the notion of an absolute retract of $\beta\mathbb{N}$: a (sub)space that is a retract irrespective of how it is embedded.

Bella, Błaszczyk and Szymański proved in [13] that if X is compact, extremally disconnected, without isolated points and of π-weight $\aleph_1$ or less then X is an absolute retract for extremally disconnected spaces iff X is the absolute of one of the following three spaces: the Cantor set, the Cantor cube ${}^{\omega_1}2$, or the sum of these two spaces. This shows that under CH there are very few absolute retracts of $\beta\mathbb{N}$.

We have less information about the retracts of $\mathbb{N}^*$, absolute or not. Of course if a subset of $\mathbb{N}^*$ is a retract of $\beta\mathbb{N}$ then it is a retract of $\mathbb{N}^*$ as well. We do not know whether the converse is true, for separable closed subsets of course.

We do know that non-trivial zero-sets are not retracts. Such a set is of the form $Z = \mathbb{N}^* \setminus \bigcup_{n\in\omega} A_n^*$, where the A_n are infinite and pairwise disjoint subsets of $\mathbb{N}$. We write $C = \bigcup_{n\in\omega} A_n^*$. Now the closure of C is a P-set in $\mathbb{N}^*$, it is the union of C and the boundary of Z, and if we take one point $u_n \in A_n^*$ for each n then $K = \mathrm{cl}\{x_n : n \in \omega\}$ is a copy of $\beta\mathbb{N}$ and $K^* = K \setminus \{x_n : n \in \omega\}$ is a P-set in the boundary of Z and hence in Z. If we now take assume $r : \mathbb{N}^* \to Z$ is a retraction then $r \restriction K^*$ is the identity and for all but finitely many n we must have $r(x_n) \in K^*$. But this would imply that K^* is separable, a contradiction.

In addition the closure of a non-trivial (not itself closed) cozero-set may, under CH ([110]), or may not, in the $\aleph_2$ Cohen model ([47, Theorem 4.5]), be a retract of $\mathbb{N}^*$.

4. Individual ultrafilters

24. Is there a model in which there are no P-points and no Q-points?
Comments: The Continuum Hypothesis implies that both kinds of points exist. If $\mathfrak{c} = \aleph_2$ then at least one kind exists; this depends on the value of $\mathfrak{d}$. If $\mathfrak{d} = \mathfrak{c}$ then P-points exist, in fact Ketonen showed in [90] that then every filter of cardinality less than $\mathfrak{c}$ can be extended to a P-point. In the present case, if $\mathfrak{d} < \mathfrak{c}$ then $\mathfrak{d} = \aleph_1$ and then the result of Coplakova and Vojtáš from [33] applies to show that there are Q-points; this relies on the fact that the Novák number of $\mathbb{N}^*$ is at least $\aleph_2$, see [3].

The current methods for creating models without P-points involve iterations with countable supports and these invariably produce models where $\mathfrak{c} = \aleph_2$, and hence these will contain Q-points. A recent exception is [30], where models without P-points and arbitrarily large continuum are constructed. However $\mathfrak{d} = \aleph_1$ in these models, hence these contain Q-points as well.

25. Is there a model in which there is a rapid ultrafilter but in which there is no Q-point?
Comments: In [119] it was shown that the existence of a countable non-discrete extremally disconnected group implies the existence of rapid ultrafilters.

26. What are the possible compactifications of spaces of the form $\mathbb{N} \cup \{p\}$ for $p \in \mathbb{N}^*$?
Comments: Of course for every p we have $\beta(\mathbb{N} \cup \{p\}) = \beta\mathbb{N}$. There are points where this phenomenon persists: Dow and Zhou showed that is $f : \beta\mathbb{N} \to {}^{\mathfrak{c}}2$ is continuous and onto and $K \subset \mathbb{N}^*$ is a closed set such

that $f \restriction K$ is irreducible and onto then for *every* point in K *every* compactification of $\mathbb{N} \cup \{p\}$ contains a copy of $\beta\mathbb{N}$, see [62].

Other examples of spaces of the form $\mathbb{N} \cup \{x\}$, where x is the only non-isolated point, for which every compactification contains $\beta\mathbb{N}$ were constructed by Van Douwen and Przymusiński in [45].

The case of scattered compactifications has received considerable interest.

In [122] Semadeni asked whether $\mathbb{N} \cup \{p\}$ always has a scattered compactification.

In [121] Ryll-Nardzewski and Telgarsky proved that the answer is yes if p is a P-point and the Continuum Hypothesis holds; the compactification is a version of the compactification $\gamma\mathbb{N}$ of Franklin-Rajagopalan from [70], where $\gamma\mathbb{N} \setminus \mathbb{N}$ is a copy of the ordinal $\omega_1 + 1$ and p corresponds to the point ω_1.

In [86] Jayachandran and Rajagopalan constructed a scattered compactification of $\mathbb{N} \cup \{p\}$, where p is a P-point limit of a sequence of P-points.

Solomon, Telgarski, and Malykhin, in [140], [143], and [97], respectively, exhibited points p in $\mathbb{N}^*$ such that $\mathbb{N} \cup \{p\}$ has no scattered compactification.

Malykhin's paper and the paper [144] by Telgarsky contain investigations of the structure of the (complementary) sets S and NS of points for which $\mathbb{N} \cup \{p\}$ does and does not have a scattered compactification respectively. The set NS is quite rich: it contains the closures of all of its countable subsets and it is upward closed in the Rudin-Frolík order.

This richness foreshadowed a later result of Malykhin's from [98,99]: in the Cohen model it is the case that for *every* point $p \in \mathbb{N}^*$ *every* compactification of $\mathbb{N} \cup \{p\}$ contains a copy of $\beta\mathbb{N}$; in particular $\mathrm{NS} = \mathbb{N}^*$ in this model.

27. Is there $p \in \mathbb{Q}_d^*$ such that $\mathcal{B} = \{A \in p : A$ is closed and nowhere dense in $\mathbb{Q}$ and without isolated points$\}$ is a base for p?

Comments: To eliminate possible confusion: we wrote $p \in \mathbb{Q}_d^*$ to emphasize that we are asking for an ultrafilter on the countable *set of rationals* (with the discrete topology), and $\mathbb{Q}$ in the description of $\mathcal{B}$ to emphasize that we want a base for the ultrafilter that is closely connected to the topological structure of the *space of rationals*.

One could ask the question in the opposite direction: is there a point x in $\beta\mathbb{Q} \setminus \mathbb{Q}$ (the *space* of rationals) that, when considered as an ultrafilter of closed sets has a base consisting of closed nowhere dense copies of $\mathbb{Q}$ and that also generates a real ultrafilter on the set $\mathbb{Q}$.

A third way of looking at this question is to consider $\beta\,\mathrm{Id} : \beta\mathbb{Q}_d \to \beta\mathbb{Q}$, where Id is the identity map and look for points in $\beta\mathbb{Q} \setminus \mathbb{Q}$ with one-point preimages. Such points are easily found in the closure of $\mathbb{N}$ for example, but we want a point whose elements are topologically as rich as possible.

These ultrafilters were dubbed 'gruff ultrafilters' by Van Douwen. This question is still open but there are many consistent positive answers:

- Van Douwen [43]: from $\mathsf{MA}_{\text{countable}}$,
- Coplakova and Hart [34]: from $\mathfrak{b} = \mathfrak{c}$,
- Ciesielski and Pawlikowski [31]: from a version of the Covering Property Axiom (hence in the Sacks model),
- Millán [109]: from the same assumption a Q-point with this property,
- Fernández-Bretón and Hrušák [66]: from a parametrized $\diamondsuit$-principle, from $\mathfrak{d} = \mathfrak{c}$, and in the random real model; a correction in [67] points out that in the third case one needs to add $\aleph_1$ many Cohen reals first.

28. Is there a $p \in \mathbb{N}^*$ such that whenever $\langle x_n : n \in \omega\rangle$ is a sequence in $\mathbb{Q}$ there is an $A \in p$ such that $\{\, x_n : n \in A \,\}$ is nowhere dense?

Comments: Such ultrafilters are called *nowhere dense*. A P-point is nowhere dense: it will have a member A such that $\{x_n : n \in A\}$ converges to a point or is closed and discrete. On the other hand, in [126] Shelah showed that it is consistent that there are no nowhere dense ultrafilters. In [15] it is shown that a nowhere dense ultrafilter exists iff there is a σ-centered partial order that does *not* add a Cohen real.

Research into this type of problem was initiated by Baumgartner in [8]: the general situation involves a set S and a notion of smallness on S, usually expressed in terms of ideals. One then calls an ultrafilter u on $\mathbb{N}$ small if for every map $f:\mathbb{N}\to S$ there is a member of u whose image under f is small.

29. Is there an ultrafilter u such that for every map $f:\mathbb{N}\to\mathbb{N}$ there is a member U of u such that $f[U]$ has density zero?
Comments: This is a special case of the general problem mentioned in the comments above. We mention it here because it is related to some special cases of Problem 34, which deals with permutations, rather than arbitrary maps.

30. Is there in ZFC an ultrafilter that is Sacks-indestructible?
Comments: This question is inspired by the many proofs that ultrafilters of small character may exist. Sacks forcing preserves selective ultrafilters, P-points and many ultrafilters constructed from these. Those ultrafilters need not exist of course, so the question becomes if there are ultrafilters that are preserved by this partial order.

5. Dynamics, algebra, and number theory

31. Is there a point in $\mathbb{N}^*$ that is not an element of any maximal orbit closure?
Comments: In this problem we consider the integers $\mathbb{Z}$ rather than $\mathbb{N}$ and the shift map σ, defined by $\sigma(n)=n+1$. The orbit of $u\in\mathbb{N}^*$ is the set $\{\sigma(u):n\in\mathbb{Z}\}$ and its closure C_u is the *orbit closure* of u.

32. Is there an infinite strictly increasing sequence of orbit closures?
Comments: This problem is related to the previous problem: if there is no increasing sequence of orbit closures then the family of orbit closures is well-founded under reverse inclusion and every point is in some maximal orbit closure. A negative answer to this question, and hence to Question 31, was given recently by Zelenyuk in [150].

33. Is there a $p\in\mathbb{N}^*$ such that for every pair of commuting continuous maps $f,g:{}^{\omega}2\to{}^{\omega}2$ there is an $x\in{}^{\omega}2$ such that $p\text{-}\lim f^n(x)=p\text{-}\lim g^n(x)=x$?
Comments: This question is related in two ways to Birkhoff's multiple recurrence theorem, which states that commuting continuous self-maps of the Cantor set have common recurrent points. Using ultrafilters one can state this theorem as: for every pair of commuting continuous maps $f,g:{}^{\omega}2\to{}^{\omega}2$ there are $p\in\mathbb{N}^*$ and $x\in{}^{\omega}2$ such that $p\text{-}\lim f^n(x)=p\text{-}\lim g^n(x)=x$.

So the first connection to our question is clear: is there one single ultrafilter that works for all pairs.

The second connection is the question whether the theorem holds for the Cantor cube ${}^{\mathfrak{c}}2$?

If it does then the answer to our question is positive. To see this note first that there are $\mathfrak{c}$ many pairs of commuting self-maps of ${}^{\omega}2$, enumerated these as $\{\langle f_\alpha,g_\alpha\rangle:\alpha<\mathfrak{c}\}$. These determine one pair $\langle f,g\rangle$ of commuting self maps of ${}^{\mathfrak{c}}2$: write ${}^{\mathfrak{c}}2$ as ${}^{\mathfrak{c}\times\omega}2$, and let $f=\prod_{\alpha<\mathfrak{c}}f_\alpha$ and $g=\prod_{\alpha<\mathfrak{c}}g_\alpha$. The maps f and g commute and if $x\in{}^{\mathfrak{c}\times\omega}2$ is a common recurrent point then $p\text{-}\lim f^n(x)=p\text{-}\lim g^n(x)=x$ for some $p\in\mathbb{N}^*$. But then also $p\text{-}\lim f^n_\alpha(x_\alpha)=p\text{-}\lim g^n_\alpha(x_\alpha)=x_\alpha$ for all α.

34. For what nowhere dense sets $A\subseteq\mathbb{N}^*$ do we have $\bigcup_{\pi\in S_{\mathbb{N}}}\pi^*[A]\neq\mathbb{N}^*$?
Comments: Here $S_{\mathbb{N}}$ denotes the permutation group of $\mathbb{N}$.

It is consistent to assume that this happens for all nowhere dense sets. In [3] Balcar, Pelant and Simon studied $\mathfrak{n}$, the Novák number of $\mathbb{N}^*$, defined as the smallest number of nowhere dense sets needed to cover $\mathbb{N}^*$. The inequality $\mathfrak{c} < \mathfrak{n}$ is consistent and yields the consistency of "for all nowhere dense sets"; it follows from CH (because $\mathfrak{n} \geq \aleph_2$), but is also consistent with other values of $\mathfrak{c}$.

The inequality $\mathfrak{n} \leq \mathfrak{c}$ is also consistent and that case there is not such an easy way out and it becomes an interesting project to investigate whether the permutations of individual nowhere sets do, or do not, cover $\mathbb{N}^*$ in ZFC.

Permuting a singleton will not yield a cover, as $|\mathbb{N}^*| = 2^{\mathfrak{c}}$.

Less obvious is Gryzlov's result from [74] that the permutations of the set F_d from Question 17 do not form a cover. This was improved by Flašková in [65]: the permutations of the larger set F_Σ do not cover $\mathbb{N}^*$ either.

There is another natural nowhere dense subset of $\mathbb{N}^*$ the permutations of which may, or may not, cover $\mathbb{N}^*$. Identify $\mathbb{N}$ with $\mathbb{N} \times \mathbb{N}$ and for $k \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{N}$ write $U(f,k) = \{\langle m,n\rangle : m \geq k \text{ and } n \geq f(m)\}$. The set $B = \bigcap_{f,k} U(f,k)^*$ is nowhere dense and it is well known then $\bigcup_{\pi \in S_\mathbb{N}} \pi[B]$ consists of all non P-points of $\mathbb{N}^*$. Hence the permutations of B cover $\mathbb{N}^*$ iff there are no P-points.

35. For what nowhere dense sets $A \subseteq \mathbb{N}^*$ do we have $\bigcup\{h[A] : h \in \mathsf{Aut}\} \neq \mathbb{N}^*$?
Comments: This question is more difficult than the previous one.

For example, singleton sets still do not provide covers in ZFC, but the easy counting argument is replaced by the non-trivial fact that $\mathbb{N}^*$ is not homogeneous.

We have no information about the sets F_d and F_Σ in this context, except for the general fact that under CH the space $\mathbb{N}^*$ cannot be covered by nowhere dense P-sets, see [93]. Also, in [2] it was shown that it is consistent that $\mathbb{N}^*$ can be covered by nowhere dense P-sets, and the principle NCF (Near Coherence of Filters) implies that $\mathbb{N}^*$ is even the union of a *chain* of nowhere dense P-sets, see [151], but the sets in these covers are unrelated to the sets F_d and F_Σ. It is also unclear whether any one of the individual sets in these families will produce a cover when moved around by the members of Aut.

The answer for the set B remains the same because the union $\bigcup\{h[B] : h \in \mathsf{Aut}\}$ consists of all non-P-points.

6. Other

36. Are ω_0^* and ω_1^* ever homeomorphic?
Comments: This is known as the Katowice Problem, or rather the last remaining case of this problem. It was posed in full by Marian Turzański, when he was in Katowice (hence the name of the problem). The general question is: if κ and λ are infinite cardinals, endowed with the discrete topology, and the remainders κ^* and λ^* are homeomorphic must the cardinals κ and λ be equal?

Since the weight of κ^* is equal to 2^κ it is immediate that the Generalized Continuum Hypothesis implies a yes answer. In joint work Balcar and Frankiewicz established that the answer is actually positive without any additional assumptions, *except possibly for the first two infinite cardinals.* More precisely, see [1,68]: If $\langle\kappa,\lambda\rangle \neq \langle\aleph_0,\aleph_1\rangle$ and $\kappa < \lambda$ then the remainders κ^* and λ^* are not homeomorphic.

The paper [29] contains a list of the current known of consequences of ω_0^* and ω_1^* being homeomorphic; all but one of these can be made to hold in a single model of ZFC.

By Stone-duality the Katowice problem can be formulated algebraically: are the quotient (Boolean) algebras $\mathcal{P}(\omega_0)/\mathrm{fin}$ and $\mathcal{P}(\omega_1)/\mathrm{fin}$ ever isomorphic? In this form the question even makes sense in ZF: in models without non-trivial ultrafilters the spaces ω_0^* and ω_1^* are empty (and so trivially homeomorphic) but the structures of the algebras may still differ.

37. Is there consistently an uncountable cardinal κ such that ω^* and $U(\kappa)$ are homeomorphic?
Comments: This problem is part of the uniform version of the Katowice problem, Question 36. The full question asks whether for distinct infinite cardinals κ and λ spaces $U(\kappa)$ and $U(\lambda)$ of *uniform* ultrafilters

can be homeomorphic, or algebraically whether the quotient algebras $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ and $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ can be isomorphic. This is Question 47 in [44], where we also find the information that, in general, the algebra $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ has cardinality 2^κ and is μ-complete for $\mu < \operatorname{cf}\kappa$ but not $\operatorname{cf}\kappa$-complete. Therefore we can concentrate on cases where $2^\kappa = 2^\lambda$ and $\operatorname{cf}\kappa = \operatorname{cf}\lambda$.

In [42] Van Douwen investigated the statements S_n:

$$\text{if } \kappa \neq \aleph_n \text{ then } \mathcal{P}(\kappa)/[\kappa]^{<\kappa} \text{ and } \mathcal{P}(\omega_n)/[\omega_n]^{<\aleph_n} \text{ are not isomorphic.}$$

Thus, our question is whether it is consistent that S_0 is false. Van Douwen showed that there is at most one n for which S_n is false, but the proof offers no information on the location of that n (if any) as it simply establishes the implication “if $m < n$ and S_m is false then S_n holds”.

38. What is the structure of the sequences $\langle n((\mathbb{N}^*)^n) : n \in \mathbb{N}\rangle$ and $\langle wn((\mathbb{N}^*)^n) : n \in \mathbb{N}\rangle$?
Comments: Here n and wn denote the Novák and weak Novák numbers, defined as the minimum cardinality of a family of nowhere dense sets that covers the space, or has a dense union, respectively.

It is clear that if N is nowhere dense in a space X then $N \times Y$ is nowhere dense in the product $X \times Y$. This shows that, in general, $n(X \times Y) \leq \min\{n(X), n(Y)\}$ and likewise for wn. It follows that both sequences in our question are non-increasing and hence must become constant eventually.

One could ask when they do become constant. For wn this is undetermined: in [132] Shelah and Spinas showed that for every n there is a model in which $wn((\mathbb{N}^*)^n) > wn((\mathbb{N}^*)^{n+1})$. In particular $wn(\mathbb{N}^*) > wn(\mathbb{N}^* \times \mathbb{N}^*)$ is possible, in [133] the latter inequality was shown to hold in the Mathias model.

For the Novák numbers of the finite powers nothing is known as yet.

39. What is the status of the statement that all Parovichenko spaces are co-absolute (with $\mathbb{N}^*$)?
Comments: This question is related to Parovichenko’s theorem from [111], which states that under CH all Parovichenko spaces are homeomorphic to $\mathbb{N}^*$. Of course Parovichenko spaces were named after this theorem was proved: they are compact, zero-dimensional F-spaces of weight $\mathfrak{c}$ without isolated points in which every non-empty G_δ-set has non-empty interior. For the nonce we say that a space is of *Parovichenko type* if it satisfies the conditions above, except for possibly the weight restriction.

In [21] Broverman and Weiss proved that under CH all spaces of Parovichenko type of π-weight $\mathfrak{c}$ are co-absolute (with $\mathbb{N}^*$). They also established that if CH fails and $\mathfrak{c} = 2^{<\mathfrak{c}}$ then there is a Parovichenko space that is not co-absolute with $\mathbb{N}^*$. They also proved that ω_0^* and ω_1^* are co-absolute or, in algebraic terms that the Boolean algebras $\mathcal{P}(\omega_0)/\mathrm{fin}$ and $\mathcal{P}(\omega_1)/\mathrm{fin}$ have isomorphic completions, which shows that completions do not have a direct effect on Question 36.

In [148] Williams also established the π-weight result and showed that $\mathbb{N}^*$ is co-absolute with a linearly ordered space.

In [108] Van Mill and Williams improved the negative result of Broverman and Weiss: if our statement holds then not only do we have $\mathfrak{c} < 2^{<\mathfrak{c}}$, but even $\mathfrak{c} < 2^{\aleph_1}$.

In [46] Dow proved that the equality $\operatorname{cf}\mathfrak{c} = \aleph_1$ already implies that all Parovichenko spaces are co-absolute.

The definition of the absolute as the Stone space of the Boolean algebra of regular open sets makes sense for any compact space, so one may also seek co-absolutes of $\mathbb{N}^*$ among spaces that are not zero-dimensional. In [32] Comfort and Negrepontis showed that under CH if X is locally compact and σ-compact, but not compact, and if $\left|C(X)\right| = \mathfrak{c}$ then the set of P-points in X^* is homeomorphic to the G_δ-modification of the ordered space 2^{ω_1}; Parovichenko had already established this fact for $\mathbb{N}^*$ in [111]. This implies that for such spaces the remainders share a homeomorphic dense subspace and hence that all such remainders are co-absolute with $\mathbb{N}^*$, still under CH of course. So, for example, under CH the spaces $\mathbb{N}^*$ and $\mathbb{H}^*$ are co-absolute.

In [51] Dow showed that in the Mathias model $\mathbb{N}^*$ and $\mathbb{H}^*$ are not co-absolute.

40. Let X be a compact space that can be mapped onto $\mathbb{N}^*$. Is X non-homogeneous?
COMMENTS: Since $\mathbb{N}^*$ maps onto $\beta\mathbb{N}$, a space as in the question will also map onto $\beta\mathbb{N}$. If the weight of X is at most $\mathfrak{c}$ then Theorem 4.1 (c) of [37] applies and we find that X is indeed non-homogeneous.

41. Is it consistent that every compact space contains either a convergent sequence or a copy of $\beta\mathbb{N}$?
COMMENTS: Efimov asked in [63] whether every compact space contains either a convergent sequence or a copy of $\beta\mathbb{N}$ and a counterexample is now called a *Efimov space*. In [76] one finds a survey of the status of the problem in 2007; it lists various consistent Efimov spaces, which explains why the present formulation asks for a consistency result. We mention here some of the additional results that have been obtained in the meantime.

To begin there is a positive answer in [61] to Question 1 from [76]: Martin's Axiom, or even the equality $\mathfrak{b} = \mathfrak{c}$, implies that there is a Efimov space.

In addition there has been progress on two related questions due to Juhász and Hušek. The latter asked whether every compact Hausdorff space contains either a convergent ω-sequence or a convergent ω_1-sequence; Juhász' question is stronger: must a compact Hausdorff space that does not contain a convergent ω_1-sequence be first-countable? A counterexample to Hušek's question would be a Efimov space because $\beta\mathbb{N}$ contains a convergent ω_1-sequence. In [54] one finds a result that provides many models in which Juhász' question, and hence that of Hušek's, have a positive answer. One of these models satisfies $\mathfrak{b} = \mathfrak{c}$, hence Efimov's question is strictly stronger than that of Hušek's.

42. Is there a locally connected continuum such that every proper subcontinuum contains a copy of $\beta\mathbb{N}$?
COMMENTS: There are various continua that have the property that every proper subcontinuum contains a copy of $\beta\mathbb{N}$: the remainders $\beta\mathbb{R}^n \setminus \mathbb{R}^n$ all have this property for example. The reason is that they are F-spaces, hence the closure of every countable relatively discrete subset is a copy of $\beta\mathbb{N}$. However, these remainders are not locally connected; indeed if a space X is not pseudocompact then one can use an unbounded continuous function to exhibit points in X^* at which neither βX nor X^* is locally connected, see [81].

In [106] we find a construction, from CH, of a locally connected continuum without non-trivial convergent sequences. This construction, an inverse limit in which all potential convergent sequences are destroyed, can be modified with some extra bookkeeping to yield a locally connected continuum in which every infinite subset contains a countable discrete subset whose closure is homeomorphic to $\beta\mathbb{N}$, still under CH of course.

This leaves the question for a ZFC-example open but also suggest some further variations. The example has the property that *some* countable relatively discrete subsets have $\beta\mathbb{N}$ as their closures. One can ask whether one can ensure this for *all* countable relatively discrete subsets, or whether one can even make all countable subsets C^*-embedded. The reason for this is that a compact F-space cannot be locally connected, hence we would like to know how close to an F-space a locally connected continuum can be.

We would also like to know whether there is a natural example that answers our question; natural in the sense that one can simply write it down, as in "$\beta\mathbb{N}$ is a compact space without convergent sequences" and "$\mathbb{H}^*$ is a continuum in which every proper subcontinuum contains a copy of $\beta\mathbb{N}$".

43. Is there an extremally disconnected normal locally compact space that is not paracompact?
COMMENTS: The ordinal space ω_1 is locally compact and normal, but not paracompact. There are, however, various additional assumptions that when added to local compactness and normality will ensure paracompactness. Extremal disconnectedness may or may not be such an assumption: Kunen and Parsons showed in [94] that if κ is weakly compact then $\beta\kappa \setminus U(\kappa)$ is normal and locally compact but not paracompact. As weak compactness is a large cardinal property the answer to this question can go many ways: a consistent counterexample, a real counterexample, or even an equiconsistency result involving a large cardinal.

The weaker property of basic disconnectedness does not work, as shown by Van Douwen's example in [38]. In this paper Van Douwen attributes the present question to Grant Woods.

44. Is every compact hereditarily paracompact space of weight at most $\mathfrak{c}$ a continuous image of $\mathbb{N}^*$? Is every hereditarily c.c.c. compact space a continuous image of $\mathbb{N}^*$?

Comments: These questions are part of the general problem of identifying the continuous images of $\mathbb{N}^*$. Przymusiński proved in [116] that all perfectly normal compact spaces are continuous images of $\mathbb{N}^*$. One can therefore look for weakenings of perfect normality that still make the space an image of $\mathbb{N}^*$. The present two properties are such weakenings and they have not been ruled out yet.

Another weakening, first-countability, was ruled out by Bell in [11]: the $\aleph_2$-Cohen model contains a first-countable compact space that is not a continuous image of $\mathbb{N}^*$; this space is also hereditarily metacompact. In the same paper Bell showed that the compact ordered space 2^{ω_1} (with the lexicographic order) is an image of $\mathbb{N}^*$. Theorems 15 and 17 in Chapter 1 of [100] imply that every compact ordered space that is first-countable is a continuous image of the latter space, hence also of $\mathbb{N}^*$.

In connection with the latter result we note that it is consistent with the negation of CH that all linear orders of cardinality $\mathfrak{c}$ are embeddable into the Boolean algebra $\mathcal{P}(\mathbb{N})/\mathrm{fin}$, see [95]. By a combination of the Stone and Wallman dualities this implies that it is consistent with $\neg\mathsf{CH}$ that every compact ordered space of weight $\mathfrak{c}$ is a continuous image of $\mathbb{N}^*$.

This was later generalized in [9] to the consistency of Martin's Axiom for σ-linked partial orders, the negation of CH, and the statement that all compact spaces of weight $\mathfrak{c}$ are continuous images of $\mathbb{N}^*$.

In both cases the proof constructs an embedding of a universal linear order or a universal Boolean algebra of cardinality $\mathfrak{c}$ into $\mathcal{P}(\mathbb{N})/\mathrm{fin}$. This raises the question whether there is a universal compact space of weight $\mathfrak{c}$; one that maps onto all such spaces. The answer is negative, see [54, Section 6].

45. Is every compact space of weight at most $\aleph_1$ a 1-soft remainder of ω?

Comments: A compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ is 1-soft if for every subset A of $\mathbb{N}$ with $\mathrm{cl}\,A \cap \mathrm{cl}(\mathbb{N} \setminus A) \neq \emptyset$ there is an autohomeomorphism h of $\gamma\mathbb{N}$ that is the identity on $\gamma\mathbb{N} \setminus \mathbb{N}$ and is such that $\{n \in A : h(n) \notin A\}$ is infinite.

See Question 351527 on MathOverFlow, [6], and also the papers [7] and [55] for related information.

46. Is there a universal compact space of weight $\aleph_1$?

Comments: We mean universal in the mapping-onto sense; the dual question has the well-known answer $[0,1]^{\omega_1}$ and Parovichenko's theorem suggests that the answer might be positive. The answer is negative in the $\aleph_2$-Cohen model but a good reference is hard to find. There are references to the result, [12,102,124,125], but no concrete proof.

However, the argument in [54, Section 6] can readily be adapted to provide an accessible proof. We apply Stone duality and show that in the model there is no Boolean algebra of cardinality $\aleph_1$ in which every Boolean algebra of that cardinality can be embedded. Let $\mathrm{Fn}(\omega_2 \times \omega_0, 2)$ denote the Cohen partial order and let G be a generic filter.

The main steps are: we can assume that the Boolean algebra is determined by a partial order $\prec$ on a subset of ω_1. By the ccc of $\mathrm{Fn}(\omega_2 \times \omega_0, 2)$ the order $\prec$ is a member of $V[G \restriction \alpha]$ for some $\alpha < \omega_2$. Take the next $\aleph_1$ many Cohen reals $\langle c_\beta : \beta < \omega_1\rangle$, defined by $c_\beta(n) = \bigcup G(\alpha + \beta, n)$. The union, T, of the binary tree $2^{<\omega}$ and the set $\{c_\beta : \beta < \omega_1\}$ is a partially ordered set which, when turned upside-down generates a Boolean algebra B. Assume $\varphi : T \to \omega_1$ is the restriction of an embedding of B into $\langle \omega_1, \prec\rangle$. There is a countable subset C of ω_2 such that the restriction of φ to $2^{<\omega}$ belongs to $V[G \restriction (\alpha \cup C)]$. Now take $\beta \in \omega_1$ such that $\alpha + \beta \notin C$. Then c_β does not belong to $V[G \restriction (\alpha \cup C)]$, yet it can be defined from the elements $\gamma = \varphi(c_\beta)$ and $\varphi \restriction 2^{<\omega}$ by the formula $\bigcup\{s : \gamma \prec \varphi(s)\}$.

47. Investigate ultrafilters as topological spaces.

Comments: This is a very general question, so let us discuss some specific ones that may be investigated.

An ultrafilter can be viewed as a subspace of the Cantor set ${}^{\omega}2$, if one identifies a subset of ω with its characteristic function.

Of course this makes ultrafilters separable metric spaces, and hence relatively well-behaved. But not too well-behaved: free ultrafilters are non-measurable and do not have the property of Baire.

To begin one can repeat many of the investigations into the Rudin-Keisler order using more general kinds of maps. We know $p \leq_{\mathbf{RK}} q$ means that there is a map $\varphi : \mathbb{N} \to \mathbb{N}$ such that $\beta\varphi(q) = p$. The map φ determines a continuous map from ${}^{\omega}2$ to itself, so the following definition suggests itself at once: say $p \leq_c q$ if there is a continuous map $f : {}^{\omega}2 \to {}^{\omega}2$ such that $f[q] = p$.

One can ask whether $p \leq_c q$ and $q \leq_c p$ together imply that $p \equiv_c q$, which means that there is a homeomorphism of ${}^{\omega}2$ that maps p to q. The structure of the partial order $\leq_c$, minimal elements, incomparable elements, etc., would warrant investigation as well.

There is no reason to stop there of course: one can ask the same questions about Borel maps of any specific order, or of maps of arbitrary Baire classes.

One need not work with maps on ${}^{\omega}2$, though that may make life easier, one can investigate what it means for two ultrafilters to be homeomorphic, or what it means that one is a continuous image of the other. The methods of [49] may be of use in determining the possible sizes of sets of ultrafilters that are incomparable in this sense.

We note that an ultrafilter can be homeomorphic to at most $\mathfrak{c}$ many other ultrafilters: if $f : p \to q$ is a homeomorphism then Lavrentieff's theorem implies that f can be extended to a homeomorphism of G_δ-subsets of ${}^{\omega}2$, and the number of such homeomorphisms is equal to $\mathfrak{c}$.

The paper [101] contains many results on the topology of ultrafilters.

48. Is it consistent that all free ultrafilters have the same Tukey type?
COMMENTS: Isbell [85] raised the question of the number of Tukey types of ultrafilters on $\mathbb{N}$ and gave the obvious bounds 2 (trivial or not) and $2^{\mathfrak{c}}$. Tukey types of free ultrafilters were investigated by Dobrinen and Todorčević in [36] who gave a combinatorial characterization of ultrafilters that are Tukey-equivalent to the partial order of finite subsets of $\mathfrak{c}$: the ultrafilter $\mathcal{U}$ should contain a subfamily $\mathcal{X}$ of cardinality $\mathfrak{c}$ such that for every infinite subfamily $\mathcal{Y}$ of $\mathcal{X}$ the intersection $\bigcap\mathcal{Y}$ does not belong to $\mathcal{U}$.

Such ultrafilters exist see [85, Theorem 5.4]; they are the ultrafilters of character $\mathfrak{c}$ constructed from an independent family of cardinality $\mathfrak{c}$, see also [114,115].

In [30, Announcement 9] Chodounský and Guzmán announce a result that comes close to the statement that all free ultrafilters have this property.

Added in proof: in [28] Cancino-Manríquez and Zapletal construct models where all free ultrafilters are Tukey equivalent to the partial order of finite subsets of $\mathfrak{c}$.

49. Is the space of minimal prime ideals of $C(\mathbb{N}^*)$ *not* basically disconnected?
COMMENTS: For a commutative ring R we let mR denote the set of minimal prime ideals endowed with the hull-kernel topology. In [82,83] Henriksen and Jerison studied this space and asked whether $mC(\mathbb{N}^*)$ is basically disconnected.

In the papers [57] and [48] various conditions were found that imply $mC(\mathbb{N}^*)$ is *not* basically disconnected. For example, MA implies that $mC(\mathbb{N}^*)$ is not even an F-space ([57]). In [48] it was shown that the equality $\mathrm{cf}[\mathfrak{d}]^{\aleph_0} = \mathfrak{d}$ suffices to show that $mC(\mathbb{N}^*)$ is not basically disconnected. Failure of this equality entails the existence of inner models with measurable cardinals. The actual consequence, called **Mel**, of this equality that was used in the proof identifies $\mathbb{N}$ with $\mathbb{Q}$ and asks for a P-filter $\mathcal{F}$ on $\mathbb{Q}$, and two countable disjoint dense subsets A and B of $\mathbb{R} \setminus \mathbb{Q}$ such that the closure in $\mathbb{R}$ of every member of $\mathcal{F}$ meets both A and B.

Thus, to show that $mC(\mathbb{N}^*)$ is not basically disconnected it suffices to show that **Mel** holds, or the following stronger, but possibly more manageable, statement: the ideal of nowhere dense subsets of $\mathbb{Q}$ can be extended to a P-ideal.

50. Is there a c.c.c. forcing extension of L in which there are no P-points?
COMMENTS: The consistency of the nonexistence of P-points was proven by Shelah, see [149] and also [127, VI §4].

After this there have been various attempts to (dis)prove the existence of P-points in various standard models. Quite often the outcome was that ground model P-points remained ultrafilters and P-points in the extension.

A notable exception is the Silver model: in [30] we find a proof that iterating Silver forcing ω_2 times with countable supports produces a model without P-points; the same holds for the countable support product of arbitrarily many copies of the partial order. This establishes the consistency of the nonexistence of P-points with arbitrarily large values of $\mathfrak{c}$.

A question that is still open is whether P-points exist in the random real model. If not then this would answer the present question positively. If there are P-points in this model then our question gains interest as it is as yet unknown whether c.c.c. forcing can be used to kill P-points.

51. What is the relationship between ultrafilters of small character (less than $\mathfrak{c}$) and P-points?
COMMENTS: One of the first ultrafilters of small character can be found in [92, Exercise VII.A10]; it is a simple $P_{\aleph_1}$-point constructed by iterated forcing over a model of $\neg\mathsf{CH}$. There are many more examples of ultrafilters of small character but their constructions seem to involve P-points in some form or another. A common method is to start with a model of CH and enlarge the continuum while preserving some ultrafilters; these will then have character $\aleph_1$, which is smaller than $\mathfrak{c}$. Almost always these 'indestructible' ultrafilters are P-points (or stronger) and remain P-points in the extension. There are a few exceptions, see [75] for instance, but there the ultrafilters are built using P-points and these are preserved as well.

52. We let Sp_χ denote the set of characters of ultrafilters on $\mathbb{N}$, the *character spectrum* of $\mathbb{N}$. The general question is what one can say about this set.
COMMENTS: We know that $\mathfrak{c} \in \mathrm{Sp}_\chi$, and that $\mathrm{Sp}_\chi = \{\mathfrak{c}\}$ is possible.

In [128] Shelah showed the consistency of there being three cardinals κ, λ, and μ such that $\kappa < \lambda < \mu$, and $\kappa, \mu \in \mathrm{Sp}_\chi$ and $\lambda \notin \mathrm{Sp}_\chi$. The construction uses a c.c.c. forcing over a ground model in which the three cardinals are regular, λ is measurable, and there is another measurable cardinal below κ. In [129] he extended this result by showing how to build, given two disjoint sets Θ_1 and Θ_2 of regular cardinals, a cardinal-preserving partial order that forces Θ_1 to be a subset of Sp_χ and Θ_2 to be disjoint from it; the construction requires Θ_2 to consist of measurable cardinals. The same paper also contains models in which $\{n : \aleph_n \in \mathrm{Sp}_\chi\}$ can be any subset of $\mathbb{N}$, starting from infinitely many compact cardinals. This answers a question from [18], namely whether if there are ultrafilters of character $\aleph_1$ and $\aleph_3$ there must be one of character $\aleph_2$, but at the cost of large cardinals.

This leaves open the question whether the conjunction of $\aleph_1, \aleph_3 \in \mathrm{Sp}_\chi$ and $\aleph_2 \notin \mathrm{Sp}_\chi$ can be proven consistent from the consistency of just ZFC. To be very specific we ask whether there is an ultrafilter of character $\aleph_2$ in the model(s) of [92, Exercise VII.A10], where one starts with a model of $\mathfrak{c} = \aleph_3$, and in the side-by-side Sacks model where $\mathfrak{c} = \aleph_3$.

53. Is there consistently a point in $\mathbb{N}^*$ whose π-character has countable cofinality?
COMMENTS: The paper [18] contains a wealth of material on π-characters of ultrafilters, including a model with an ultrafilter of π-character $\aleph_\omega$.

Unlike the results on the character spectrum the results on the π-character spectrum do not require large cardinals.

54. Is it consistent that $t(p, \mathbb{N}^*) < \chi(p)$ for some $p \in \mathbb{N}^*$?
COMMENTS: There are plenty of compact spaces with points where the tightness is smaller than the charac-

ter; the one-point compactification of the any uncountable discrete space will do: the tightness at the point at infinity is countable, the character of the point is not.

Let us remark that no point of $\mathbb{N}^*$ has countable tightness: certainly at P-points the tightness is uncountable; if p is not a P-point then it lies on the boundary of a zero-set C and in the closure of its interior, but the closure of every countable subset of that interior is a subset of that interior. This implies that $t(p, \mathbb{N}^*) = \chi(p) = \mathfrak{c}$ if CH holds, hence the question for a consistency result.

As an aside we mention that there are consistent examples of regular extremally disconnected spaces of countable tightness: in [146] and [72] one finds constructions of extremally disconnected S-spaces. The constructions use $\clubsuit$ and that some extra assumption is necessary is shown in [142]: there are no extremally disconnected S-spaces if $\mathsf{MA} + \neg\mathsf{CH}$ holds. Both [72] and [142] contain constructions of extremally disconnected S-spaces in $\beta\mathbb{N}$.

55. If $C(\omega+1, \mathbb{C})$ admits an incomplete norm then does $C(\beta\mathbb{N}, \mathbb{C})$ admit one too?
Comments: This question is related to a conjecture/question of Kaplansky's about algebra norms on the spaces $C(X, \mathbb{C})$, with X compact. The question is whether every algebra norm is equivalent to the sup-norm $\|\cdot\|_\infty$. The answer is positive if the norm is complete, hence the question became whether every algebra norm on $C(X, \mathbb{C})$ is complete.

The book [35] surveys the solution to this problem: under CH every $C(X, \mathbb{C})$ carries an incomplete algebra norm (Dales and Esterlé) and it is consistent that every algebra norm on every $C(X, \mathbb{C})$ is complete (Solovay and Woodin).

The present question comes from the results that if $C(\beta\mathbb{N}, \mathbb{C})$ admits an incomplete norm then so does every $C(X, \mathbb{C})$, and if *some* $C(X, \mathbb{C})$ carries an incomplete norm then so does $C(\omega+1, \mathbb{C})$. In short it asks whether all compact spaces are equivalent for Kaplansky's conjecture.

The question can be translated into terms of individual ultrafilters and this leads to some interesting subquestions. A seminorm on an algebra is a function that satisfies all conditions of an algebra norm except for the condition that non-zero elements should have non-zero norm. An algebra is semi-normable if it carries a non-trivial seminorm.

For a point p of $\beta\mathbb{N}$ we let A_p denote the quotient algebra M_p/I_p, where $M_p = \{f \in C(\beta\mathbb{N}, \mathbb{C}) : f(p) = 0\}$, and $I_p = \{f \in C(\beta\mathbb{N}, \mathbb{C}) : (\exists P \in p)(f \restriction P = 0)\}$. We also let c_0 be the subalgebra of $C(\beta\mathbb{N}, \mathbb{C})$ of functions that vanish on $\mathbb{N}^*$ and we let c_0/p denote the quotient algebra $c_0/(c_0 \cap I_p)$.

Theorem 2.21 in [35] shows why we should be interested in these algebras: The algebra $C(\beta\mathbb{N}, C)$ admits an incomplete norm iff for some p the algebra A_p is seminormable, and $C(\omega+1, \mathbb{C})$ admits an incomplete norm iff for some q the algebra c_0/q is seminormable.

We see that *if* there is a p such that A_p is seminormable *then* there is a q such that c_0/q is seminormable. The present question ask whether this implication can be reversed.

Further questions regarding these algebras suggest themselves: is it the case that the seminormability of A_p implies that of c_0/p? In other words can we get $q = p$ in the previous paragraph?

Also, what is the answer to the stronger version of our question: if c_0/p is seminormable is A_p seminormable too?

We recommend [35, Chapters 1, 2 and 3] for more detailed information on this question.

56. ($\mathsf{MA} + \neg\mathsf{CH}$) Are there $\mathbb{G}$ and p (P-point, selective) such that $p \subseteq I_{\mathbb{G}}^+$?
Comments: Here $\mathbb{G}$ denotes a Hausdorff-gap in ${}^\omega\omega$ and $I_{\mathbb{G}}$ is the ideal of sets over which $\mathbb{G}$ is filled.

S. Kamo [88] proved that if V is obtained from a model of CH by adding Cohen reals then in V an ideal is a gap-ideal iff it is $\le \aleph_1$-generated. Also, CH implies that any nontrivial ideal is a gap-ideal.

The commentary in [112] mentions a further preprint by Kamo, [89], where it is shown that, under $\mathsf{MA} + \neg\mathsf{CH}$, for every Hausdorff gap $\mathbb{G}$ there are both selective ultrafilters and non-P-points consisting of positive sets (with respect to the gap-ideal $I_{\mathbb{G}}$). Also under $\mathsf{MA} + \neg\mathsf{CH}$ there is a selective non-$P_{\aleph_2}$-point that meets every gap-ideal.

Unfortunately we were unable to locate this preprint and verify these statements.

7. Orders

57. Is there for every $p \in \mathbb{N}^*$ a $q \in \mathbb{N}^*$ such that p and q are $\leq_{\mathbf{RK}}$-incomparable?
Comments: This question has a long history; it is as old as the Rudin-Keisler order itself. In [91] Kunen constructed two points that are $\leq_{\mathbf{RK}}$-incomparable. In [131] Shelah and Rudin proved that there is a set of $2^{\mathfrak{c}}$ incomparable points. In [137] Simon proved that these points may be taken to be $\aleph_1$-OK. In [49] Dow showed that there are many more situations where such sets may be constructed.

However, none of these results shed light on the present question. Some partial results are available: in [84] Hindman proved: if p is such that $\chi(r) = \mathfrak{c}$ whenever $r \leq_{\mathbf{RK}} p$ then there is a point that is incomparable with p, so the answer to the present question is positive if all ultrafilters have character $\mathfrak{c}$. Furthermore if $\mathfrak{c}$ is singular and $\chi(p) = \mathfrak{c}$ then again there is a point that is incomparable with p. The latter result was extended by Butkovičová in [27]: if $\kappa < \mathfrak{c}$ is such that $\mathfrak{c} < 2^\kappa$ then for every ultrafilter of character $\mathfrak{c}$ there are 2^κ many ultrafilters incomparable with it. Note that these results all impose conditions on individual ultrafilters in order to find an incomparable point; only the condition "all ultrafilters have character $\mathfrak{c}$" answers this question directly.

In [127, XVIII §4] Shelah proved that it is consistent that up to permutation there is one P-point.

We recall the definition of the Rudin-Frolík order: we say $p \leq_{\mathbf{RF}} q$ if there is an embedding $f : \beta\mathbb{N} \to \beta\mathbb{N}$ such that $f(p) = q$. This is a preorder that induces a partial order on the types of ultrafilters. To see this note that $p \leq_{\mathbf{RF}} q$ implies $p \leq_{\mathbf{RK}} q$: given f take a partition $\{A_n : n \in \mathbb{N}\}$ of N such that $A_n \in f(n)$ for all n. The map $g = \bigcup_n (A_n \times \{n\})$ satisfies $p = g(q)$ and shows $p \leq_{\mathbf{RK}} q$.

As usual $p <_{\mathbf{RF}} q$ will mean $p \leq_{\mathbf{RF}} q$ plus not-$q \leq_{\mathbf{RF}} p$, and this is readily seen to be equivalent to there being an embedding $f : \beta\mathbb{N} \to \mathbb{N}^*$ such that $f(p) = q$.

The Rudin-Frolík order is tree-like: if $p, q \leq_{\mathbf{RF}} r$ then $p \leq_{\mathbf{RF}} q$ or $q \leq_{\mathbf{RF}} p$. And due to the relation with $\leq_{\mathbf{RK}}$ we see at once that $\{p : p \leq_{\mathbf{RK}} q\}$ always has cardinality at most $\mathfrak{c}$.

In many papers on the Rudin-Frolík order Frolík's original notation is employed where one writes $X = f[\mathbb{N}]$, and $q = \Sigma(X, p)$ as well as $p = \Omega(X, q)$.

58. For what cardinals κ is there a strictly decreasing chain of copies of $\beta\mathbb{N}$ in $\mathbb{N}^*$ with a one-point intersection?
Comments: This question is related to decreasing chains in $\leq_{\mathbf{RF}}$. A decreasing sequence of copies of $\beta\mathbb{N}$ determines and is determined by a sequence $\langle X_\alpha : \alpha \in \delta \rangle$ of countable discrete subsets of $\mathbb{N}^*$ with the property that $X_\alpha \subseteq \operatorname{cl} X_\beta \setminus X_\beta$ whenever $\beta \in \alpha$. Take a point p in the intersection of the sequence; then $\langle \Omega(X_\alpha, p) : \alpha \in \delta \rangle$ is a decreasing $\leq_{\mathbf{RF}}$-chain.

To ensure that this chain does not have a lower bound one should make sure that p is not an accumulation point of a countable discrete subset of the intersection. Having a one-point intersection is certainly sufficient for this. In [39] Van Douwen showed that it is possible to have a chain of length $\mathfrak{c}$ with a one-point intersection. In [22] and [26] we find constructions of decreasing $\leq_{\mathbf{RF}}$-chains of type ω and of type μ for uncountable $\mu < \mathfrak{c}$ respectively. The latter two constructions provide a point in the intersection of a suitable chain of copies of $\beta\mathbb{N}$ that is not an accumulation point of a countable discrete subset of that intersection.

We want to know when in these cases the intersection can be made to be a one-point set.

59. If $\kappa \leq \mathfrak{c}$ has uncountable cofinality and if $\langle X_\alpha : \alpha < \kappa \rangle$ is a strictly decreasing sequence of copies of $\beta\mathbb{N}$ with intersection K, is there a point p in K that is not an accumulation point of any countable discrete

subset of K?
Comments: This is related to Question 58: the chains of copies of $\beta\mathbb{N}$ in the positive results were chosen with care. We want to know if that care is necessary.

60. What are the possible lengths of unbounded **RF**-chains?
Comments: Since every point has at most $\mathfrak{c}$ predecessors every chain has cardinality at most $\mathfrak{c}^+$. In [22] we find a point with exactly $\aleph_0$ many predecessors, with the order type of the set of negative integers.

Every unbounded chain will have cardinality at least $\mathfrak{c}$ (this follows from results in [17, Theorem 2.9] or [23]), so the cardinality of an unbounded chain is equal to either $\mathfrak{c}$ or $\mathfrak{c}^+$. In [24] and [25] Butkovičová constructed unbounded chains of order-type $\mathfrak{c}^+$ and ω_1 respectively.

What other order-type are possible? Can one prove that a chain or order-type $\mathfrak{c}$ (or its cofinality) exists, irrespective of CH?

61. Is every finite partial order embeddable in the Rudin-Keisler order?
Comments: See MathOverFlow `https://mathoverflow.net/questions/375365`. To get a positive answer it suffices to embed every finite power set into this order. It is relatively easy to adapt the construction of two incomparable ultrafilters to yield an embedding of the power set of $\{0, 1\}$ (see [152]), but an embedding of the power set of $\{0, 1, 2\}$ already poses unexpected difficulties.

The analogous question for the Rudin-Frolík order has an easier answer. This order is tree-like in that the predecessors of a point are linearly ordered, and every point has $2^{\mathfrak{c}}$ many successors. This implies that every finite rooted tree, and only those among the finite partial orders, can be embedded into this order.

8. Uncountable cardinals

62. Is there consistently an uncountable cardinal κ with $p \in U(\kappa)$ such that $\chi(p) < 2^\kappa$?
Comments: It is well known that if κ is an infinite cardinal then there are 2^{2^κ} many uniform ultrafilters on κ with character equal to 2^κ, see [115].

It is also well known, and referred to in other questions, that it is consistent that there are ultrafilters on $\mathbb{N}$ of character less than $\mathfrak{c}$.

Of course the Generalized Continuum Hypothesis implies that every uniform ultrafilter on every κ has character 2^κ, but we are not aware of any consistency result the other way for uncountable cardinals.

We formulate two special cases of our question:

- Is it consistent to have a uniform ultrafilter on ω_1 of character $\aleph_2$ (with $\aleph_2 < 2^{\aleph_1}$ of course)?
- Is it consistent to have a measurable cardinal κ with a $p \in U(\kappa)$ such that $\chi(p) < 2^\kappa$?

The first question simply looks at the smallest possible case and the second question asks, implicitly, if having a uniform ultrafilter of small character is actually a large-cardinal property of $\aleph_0$.

There has been recent activity in this area; the paper [71] deals with the character spectrum of uncountable cardinals of countable cofinality, and in [117] one finds models with $\mathfrak{u}_\kappa < 2^\kappa$ for $\kappa = \mathfrak{c}$ and for $\kappa = \aleph_{\omega+1}$. These results use large cardinals in the ground model: the spectrum result uses a supercompact and many measurables; the results for $\mathfrak{c}$ and $\aleph_{\omega+1}$ use a measurable and supercompact cardinal respectively.

63. Is it consistent to have cardinals $\kappa < \lambda$ with points $p \in U(\kappa)$ and $q \in U(\lambda)$ such that $\chi(p) > \chi(q)$?
Comments: This is a follow-up question to Question 62: if uniform ultrafilters of small character are at all possible, how much variation can we achieve among various cardinals?

64. If κ is regular and uncountable, $\mathcal{F}$ is a countably complete uniform filter on κ then what is the cardinality of the closed set $U_{\mathcal{F}} = \{u \in U(\kappa) : \mathcal{F} \subseteq u\}$?

Comments: In case κ is measurable one can use a measure ultrafilter to create filters $\mathcal{F}$ such that $U_{\mathcal{F}}$ is finite or a copy of $\beta\lambda$, for any $\lambda < \kappa$.

For other cardinals the set $U_{\mathcal{F}}$ will always be at least infinite and given the nature of $\beta\kappa$ the cardinality will be closely related to numbers of the form 2^{2^λ} for $\lambda \le \kappa$.

For the closed unbounded filter the answer is 2^{2^κ}: using a family of κ many pairwise disjoint stationary sets and an independent family on κ of cardinality 2^κ one can produce a map from $U_{\mathcal{F}}$ onto the Cantor cube of weight 2^κ.

65. Assume that κ is regular, that $\kappa \subseteq X \subseteq \beta\kappa$ is such that $[X]_{<\kappa} = X$ and $\beta_X\kappa = X$. Now if Y is a closed subspace of a power of X, is then also X a closed subspace of a power of Y?
Comments: Some notation: $[X]_{<\kappa}$ denotes $\bigcup\{\operatorname{cl} B : B \in [X]^{<\kappa}\}$, and if $\kappa \subseteq X \subseteq \beta\kappa$ then $\beta_X\kappa$ is the maximal subset of $\beta\kappa$ such that every function from κ to X has a continuous extension from $\beta_X\kappa$ to X.

66. Are there κ and $p \in U(\kappa)$ such that $|\mathbb{R}_p| > |\mathbb{R}_p/\equiv| = \mathfrak{c}$?
Comments: Here $\mathbb{R}_p$ denotes the ultrapower of $\mathbb{R}$ by the ultrafilter p. The relation $\equiv$ is that of Archimedean equivalence: $a \equiv b$ means that there is an $n \in \mathbb{N}$ such that both $|a| < |nb|$ and $|b| < |na|$.

67. Is there a C^*-embedded bi-Bernstein set in $U(\omega_1)$?

68. Are there open sets G_1 and G_2 in $U(\omega_1)$such that $\operatorname{cl} G_1 \cap \operatorname{cl} G_2$ consists of exactly one point?

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