A multivariate analogue of Ruijsenaars's generalised hypergeometric function

A quantum algebra approach

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Abstract

In this thesis, we derive a multivariate analogue of Ruijsenaars's ${}_2F_1$-generalisation R. We use Hopf algebra representation theory of the modular double of $\mathfrak{sl}(2)$, a Hopf algebra structure strongly related to quantum groups, to relate the function R to overlap coefficients of eigenfunctions. Using properties of the algebra and the representation, we derive an Askey-Wilson type difference equation. We moreover recover Ruijsenaars's unitary transformation kernel $\mathcal{E}$.

Expanding on the Hopf algebra structure, we extend our derivations to the multivariate version of R. Employing representation theory, we obtain multivariate difference equations. Furthermore, we demonstrate that the multivariate function enables the definition of a unitary transformation on multivariate functions in $L^2((0,\infty)^N)$.

Preface

This master thesis marks the culmination of my academic journey as a student. I am grateful for the opportunity I have had to pursue my studies. My sincere thanks go to my parents for their unwavering support, especially in my decision to pursue a second master's degree.

Throughout my educational journey, I have been fortunate to receive guidance and support from numerous teachers and mentors who have played a significant role in my personal and intellectual development. Their collective contributions have shaped me into the person I am today, and for that, I would like to thank them all.

A special word of gratitude addresses my thesis supervisor, Wolter Groenevelt, for supporting me in the process of writing this thesis and sharing his expertise. His guidance has been indispensable, providing the necessary direction and insights for this thesis.

I would like to acknowledge Jan van Neerven and Fernando de Oliveira Filho for their participation in the thesis committee. Additionally, I wish to thank Fokko van de Bult for sharing his helpful insights in the asymptotics of Ruijsenaars's R-function.

Lastly, I extend my gratitude to my roommate and friend Benjamin for his patience in hearing me out, and to all others who have supported me during the past months.

Writing this thesis has been a rewarding experience, and I hope that the reader will find enjoyment in exploring its contents.

Contents

Chapter 1

Introduction

This study aims to derive a multivariate analogue of Ruijsenaars's generalised hypergeometric function R. Ruijsenaars initially developed this function as a generalisation of the family of ${}_2F_1$ functions in the context of Calogero-Moser many-particle systems [40]. Several properties of the function R were proven in the subsequent works [36, 37, 38]. For this study, it is important to note that the function R is an eigenfunction of a system of Askey-Wilson type difference equations and can be used to construct the kernel of a unitary map. The main objective of this thesis is to extend these two properties to a multivariate version of the function R. We will demonstrate that the multivariate version of our function is an eigenfunction of a multivariate difference operator and also serves as the kernel of a unitary map on multivariate functions.

1.1 Background

The derivations in this thesis have a close relationship with the study of Krawtchouk polynomials. To provide some context, I will briefly highlight the parallels between our work and the study of these polynomials. It is important to note that no prior knowledge of Krawtchouk polynomials is necessary to understand this thesis; this section is only meant to provide additional background. The article [34] by Nomura and Terwilliger offers an excellent overview of the connection between Krawtchouk polynomials and the Lie algebra $\mathfrak{sl}(2)$. Many of the results presented in this thesis have corresponding counterparts in their work.

Krawtchouk polynomials form a set of polynomials that are orthogonal with respect to a discrete measure. These polynomials are proven to be equal to matrix elements of the natural representation on the Lie group $SU(2)$ [30]. Alternatively, they can be expressed as matrix elements of the symmetric power representation of the Lie algebra $\mathfrak{sl}(2)$ on polynomials of two variables. In this case, the polynomials in two variables are acted upon by $\mathfrak{sl}(2)$ through differentiation [34, lem 3.20].

In [34, thms 3.23 and 3.24], eigenfunctions of two elements h and h^* are used to derive difference equations for the Krawtchouk polynomials. The representation theory can also be employed to establish the orthogonality of these polynomials ([34, thm 3.22]). As a consequence of their orthogonality, the Krawtchouk polynomials can be used to construct an invertible transformation. Properties of this transformation have been studied in [10].

The study of Krawtchouk polynomials shares similarities with our research. In this study, we employ representations of a deformation of the Lie algebra $\mathfrak{sl}(2)$, which we refer to as the quantum group or quantum deformed algebra. We denote this deformation as $U_q(\mathfrak{sl}(2))$ or simply U_q. (Furthermore, we extend this deformation to a larger structure known as the modular double, discovered by Faddeev ([7]). This extension allows us to broaden the range of a pair of positive base parameters to complex conjugate parameters.)

We discover eigenfunctions of two classes of elements in the deformation and compute their overlap coefficients, a continuous generalisation of the matrix elements found in the case of Krawtchouk polynomials. The resulting overlap function, denoted as ψ, is found to be equivalent to Ruijsenaars's generalised hypergeometric function R, up to certain prefactors. This result is not surprising, as van de Bult ([4]) has previously derived a closely related outcome.

We employ the representation theory of U_q to derive difference equations for the function ψ, in a similar way as described in [34]. We demonstrate that these difference equations for ψ are equivalent to the Askey-Wilson difference equations, with appropriate prefactors.

Although the concept of orthogonality does not directly transfer from the Krawtchouk polynomials to ψ, we establish that the latter can still be regarded as the kernel of an invertible Hilbert space isomorphism. With some scaling, this isomorphism can be modified into a unitary transformation.

Having derived these results for ψ, we proceed to generalise the function to its multivariate version. This multivariate function shares similarities with a multivariate version of the Krawtchouk polynomials studied in [45]. Our derivation follows an algebraic approach similar to the derivations in the studies of multivariate Askey-Wilson functions and Askey-Wilson polynomials presented in [15, 16]. These articles provide a method for deriving a multivariate difference equation, which we also adopt.

1.2 Outline

The body of this thesis is divided into three parts. Part I can be considered a preliminary part, introducing the quantum group $U_q(\mathfrak{sl}(2))$, its modular double, and its representation on meromorphic functions. These concepts provide the necessary tools for the algebraic derivations later in this study. The structure of the quantum group U_q resembles that of a Hopf algebra, of which we use some properties. TTo keep this thesis self-contained, we will start the first part with a chapter on Hopf algebras (chapter 2). In principle, no general knowledge on the concept of Hopf algebras is needed, so that the contents of chapter 2 may be considered background knowledge. The reader may safely skip over it if they wish to do so. Chapter 3 introduces and discusses the quantum group U_q and its modular double, while we study its representations in chapter 4.

Part II focuses on specific elements of the modular double. In the case of Krawtchouk polynomials, the polynomials could be derived from eigenfunctions of certain elements h and h^* in the Lie algebra $\mathfrak{sl}(2)$. In our approach, we replace these elements h and h^* with a type of elements known as skew-primitive elements. We define those elements in chapter 5. It is followed by the introduction of the hyperbolic gamma function in chapter 6. This hyperbolic gamma, equivalent (up to reparametrisation) to the double sine function (cf. [31]) and the quantum dilogarithm (cf. [27, 9]) generalises the regular gamma function. We employ this hyperbolic gamma function in chapter 7 to express eigenfunctions of the skew-primitive elements.

In part III, we use the eigenfunctions derived in the previous part to construct the function ψ, which we do in chapter 8. In that same chapter, we establish the relationship between the function and Ruijsenaars's function R, as well as the related functions R_{ren} and $\mathcal{E}$. Additionally, we reproduce results concerning the analyticity of the function, and we extend Ruijsenaars's results regarding its asymptotics to complex parameters. In chapter 9, we derive a difference equation ψ and show how it relates to the Askey-Wilson difference equations. Chapter 10 demonstrates that ψ and $\mathcal{E}$ serve as kernels for Hilbert space isomorphism, which allows us to define a unitary transformation on $L^2(0, \infty)$ using these functions. The methods used in this chapter are related to those used in [24, 28, 17, 18, 23]. Its results extend the results of Ruijsenaars for different choices of the parameters.

In chapter 11 lastly, all the results converge. We use the Hopf algebra structure of the modular double to generalise the skew-primitive elements to a tensor product of Hopf algebras and derive their multivariate eigenfunctions. The latter will be used to define novel multivariate versions of the maps ψ and $\mathcal{E}$. We demonstrate that the multivariate ψ satisfies a multivariate difference equation, and that the multivariate $\mathcal{E}$ serves as the kernel for a unitary multivariate integral transform on $L^2((0, \infty)^N)$.

Part I

The quantum group $U_q(\mathfrak{sl}(2))$

This initial part can be considered preliminary. In chapter 2, we introduce the concept of coalgebras and Hopf algebras. Then, in chapter 3, we describe a special Hopf algebra $U_q(\mathfrak{sl}(2))$ known as the *quantum enveloping algebra of (the Lie algebra)* $\mathfrak{sl}(2)$ or the *quantum group of* $\mathfrak{sl}(2)$, as well as its modular double $\mathfrak{D}$. This Hopf algebra will be the underlying algebra, upon which we built the results of this thesis. In chapter 4, we introduce a class of representations on $U_q(\mathfrak{sl}(2))$ and $\mathfrak{D}$. The contents of these chapters are drawn from a variety of prior sources and do not include novel results. Readers who are already familiar with the ideas of Hopf algebras and the quantum group of $\mathfrak{sl}(2)$ may proceed directly to chapter 4.

Chapter 2

Hopf algebras

Hopf algebras were first introduced in the work of Heinz Hopf in the early 1940s. A formal definition of the concept was given in 1956 by Pierre Cartier. From the late 1980s onwards, Hopf algebras and its representation have been an object of study within many areas of mathematics and mathematical physics, such as category theory, combinatorics and conformal field theory ([1]).

In this chapter, we introduce the concept of Hopf algebras in the following steps. We recapitulate the definition of an algebra, as a preparation for introducing the dual concept of a coalgebra. Next, we introduce bialgebras, which combines the structures of algebra and coalgebra. Adding the aditional structure of an antipode, we arive at Hopf algebras. We complete our introduction of Hopf algebras by adding a star structure to them.

To illustrate the concepts we introduce, we will provide some simple examples. Some of these examples will be directly relevant to our main study. Other examples are included solely for the purpose of illustrating the concepts they exemplify. Their goal is to aid the reader in understanding the notions we have introduced.

In later chapters we will exclusively consider vector spaces over the ground field $\mathbb{C}$. The majority of the ideas covered in this chapter are applicable to arbitrary ground fields. Therefore, we use the symbol $\mathbb{K}$ to represent the ground field throughout this chapter. Except where other references are given, the information in this chapter is based on the introduction to Hopf algebras in [25].

2.1 Algebras

An *algebra* is a vector space A equipped with a bilinear map called the *product map*. The algebra is *unital* if it contains an element I such that for all $x \in A$, $Ix = xI = x$. Often, the product map is assumed to be associative, which we do from now on, unless explicitly stated otherwise. Moreover, again unless stated differently, we assume algebras to be unital.

To highlight the duality with the later definition of a coalgebra, we require a more abstract concept of an algebra. In this abstract context, we consider the unit as a mapping from the ground field into the algebra. Let's introduce our formal definition:

Definition 2.1 (Algebra)**.** Let A be a vector space over $\mathbb{K}$. Suppose $\mu : A \times A \to A$ is a bilinear and associative map. The pair (A, μ) forms an *algebra*, where μ is called the (algebra) product. An algebra (A, μ) is called a *unital algebra* if we extend it with a linear map $\eta : \mathbb{K} \to A$ that satisfies the condition of *unity*:

$$\mu(\eta(1), a) = \mu(a, \eta(1)) = a \quad \text{for all } a \in A.$$

We denote the unital algebra by the triplet (A, μ, η), and we refer to η as the *unit map* of the algebra.

The unit element I and the unit map η are related through the relationship $I = \eta(1)$, so that the definitions are equivalent. If we speak of 'an algebra A' from now on, the existence of the product μ and unit η is implicitly understood. We may occasionally add a subscript to their symbols (and e.g. write μ_A and η_A) to avoid confusion. At times we might also denote the product $\mu(a, b)$ of two matrix elements by the simpler notation ab or $a \cdot b$. Additionally, we use the symbol I, 1_A or just 1 to denote the element $\eta(1)$ in A. The use of the more formal notation is beneficial, however, for our study of the definition of a Hopf algebra.

Slightly abusing notation, we also write $\mu(a \otimes b) = \mu(a, b)$ for $a \otimes b \in A \otimes A$.[1] We then view μ to be a map from $A \otimes A$ to A. Using this updated notation, we define an algebra homomorphism as follows:

Definition 2.2 (Algebra homomorphism)**.** Let A, B be two algebras over $\mathbb{K}$. A linear map $f : A \to B$ is called an *algebra homomorphism* if it commutes with the algebra products and sends the unit of A to the unit of B, that is, if both

$$f \circ \mu_A = \mu_B \circ (f \otimes f) \quad \text{and} \quad f \circ \eta_A = \eta_B.$$

We say that an algebra homomorphism *preserves the algebra structure*.

Denote by $\mathrm{End}(V)$ the algebra of linear maps from V to itself (*endomorhpisms*).

Definition 2.3 (Representation)**.** Let A be an algebra and V a vector space. A *representation* is an algebra homomorphism from A to $\mathrm{End}(V)$.

Definition 2.4 (Subalgebra, ideal)**.** Let (A, μ) be an algebra, not neccesarily unital. A subspace B of A is called a *subalgebra* if it is closed under the multiplication operation μ, i.e., $\mu(a, b) \in B$ for all $a, b \in B$. A subalgebra I that satisfies $\mu(x, a), \mu(a, x) \in I$ for all $x \in I$ and $a \in A$, is called an *ideal* of A.

Example 2.5 (Ground algebra). The field $\mathbb{K}$ equipped with its regular product and the unit map id is an algebra. If (A, μ, η) is another algebra, then the map η serves as an algebra homomorphism from $\mathbb{K}$ to A. ∎

Example 2.6. Consider the vector space $M_2(\mathbb{K})$ of 2×2-matrices with elements in $\mathbb{K}$. By equipping $M_2(\mathbb{K})$ with the usual matrix product and the unit map $\lambda \mapsto \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda \end{smallmatrix}\right)$, we transform it into a unital algebra. ∎

[1] By the universal property of the tensor product, there is a unique map μ' from $A \otimes A$ to A such that $\mu'(a \otimes b) = \mu(a, b)$ for all $a, b \in A$, justifying the notation.

Example 2.7 (Tensor product of algebras). If A and B are algebras, we can give the tensor product space $A \otimes B$ an algebra structure by defining the product and unit maps as follows:

$$\mu_{A\otimes B}(a_1 \otimes b_1, a_2 \otimes b_2) := \mu_A(a_1, a_2) \otimes \mu_B(b_1, b_2) \quad \text{and} \quad \eta_{A\otimes B}(\lambda) = \lambda\eta_A(1) \otimes \eta_B(1).$$

As factors of the tensor product commute (i.e. $(a \otimes 1)(1 \otimes b) = a \otimes b = (1 \otimes b)(a \otimes 1)$), we can regard A and B as commuting subalgebras of $A \otimes B$. ∎

Example 2.8 (Tensor algebra). Let V be a vector space over $\mathbb{K}$. We define the n-th tensor power of V ($n \in \mathbb{N}$) as $T^n V = V^{\otimes n} = V \otimes \cdots \otimes V$, with $T^0 V = \mathbb{K}$. The *tensor algebra* $T(V)$ is the direct sum of all $T^n V$'s, i.e.,

$$T(V) = \bigoplus_{n\in\mathbb{N}} T^n V = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus \cdots .$$

We define the (associative) product μ on elements in $T^m V$ and $T^n V$ by

$$\mu(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_n) = u_1 \otimes \cdots \otimes u_m \otimes v_1 \otimes \cdots \otimes v_n \in T^{m+n}V$$

and extend it to a bilinear map on $T(V)$. The unit map is defined by mapping 1 to the element 1 in $T^0 V \subseteq T(V)$.

For notational convenience, we can omit the tensor product sign from our notation, using expressions like $v_1 \cdots v_n$ instead of $v_1 \otimes \cdots \otimes v_n$. We extend this convention to linear combinations as well. In the upcoming chapter on the quantum group $U_q(\mathfrak{sl}(2))$, we will make use of this tensor algebra, and of the quotient algebra introduced in the following example. ∎

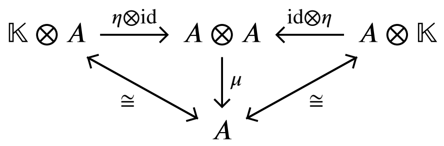
Example 2.9 (Quotient algebra). Let A be a vector space over $\mathbb{K}$ and I an ideal of A. The *quotient algebra* A/I is the space of equivalence classes of elements in A. We represent elements in A/I as $[a] := a + I$ for $a \in A$. We have $[a] = [b]$ whenever $a - b \in I$. The product on A/I is defined as $[a][b] = [ab]$, and addition is defined as $[a] + [b] = [a + b]$. The map $a \mapsto [a]$ preserves the product and the unit, so that it is an algebra homomorphism. We might ocassionally drop the square brackets from our notation. ∎

2.2 Coalgebras

The condition of associativity of μ in the definition of an algebra is equivalent to the commutativity of the following commutation diagram:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mathrm{id}\otimes\mu} & A \otimes A \\ {\scriptstyle \mu\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \mu} \\ A \otimes A & \xrightarrow[\mu]{} & A \end{array} \tag{2.1}$$

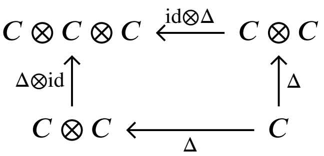
Similarly, the condition of unity of η is equivalent to the commutatitity of this diagram:[2]



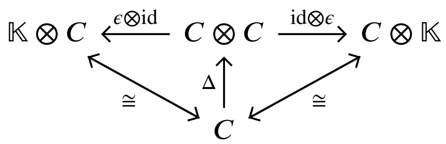
(2.2)

The triple of a vector space A, a product $\mu : A \times A \to A$, and a unit $\eta : \mathbb{K} \to A$ satisfying the above commutation relations define an algebra. Inverting all arrows in this expression, we arrive at the definition of a coalgebra:

Definition 2.10 (Coalgebra)**.** Let C be a vector space over $\mathbb{K}$ and let $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{K}$ be two linear maps such that the following two diagrams commute:



(2.3)



(2.4)

The triple (C, Δ, ϵ) is called a *coalgebra*. The maps Δ and ϵ are called the *coproduct* and *counit*, respectively. The condition that (2.3) commutes is called *coassociativity*; the commuting of (2.4) defines the notion of *counity*.

We define a coalgebra homomorphism by reversing directions in the definition of an algebra homomorphism (definition 2.2):

Definition 2.11 (Coalgebra homomorphism)**.** Let C and D be two coalgebras over $\mathbb{K}$. A linear map $f : C \to D$ is called a *coalgebra homomorphism* if both equations

$$(f \otimes f) \circ \Delta_C = \Delta_D \circ f \quad \text{and} \quad \epsilon_C = \epsilon_D \circ f$$

are satisfied.

Example 2.12 (Ground coalgebra). A very simple example of a coalgebra is the triple $(\mathbb{K}, \Delta_{\mathbb{K}}, \mathrm{id})$ with $\Delta_{\mathbb{K}} : \lambda \mapsto \lambda(1 \otimes 1)$. If $(C, \Delta_C, \epsilon_C)$ is another coalgebra, then ϵ_C is a coalgebra homomorphism from C to $\mathbb{K}$. ∎

Example 2.13. For a slightly more elaborate example of a coalgebra, let X be a set and let $C = \mathbb{K}X$ be the vector space spanned by elements of the set X, i.e., C is a vector space with basis X. Set $\Delta(x) := x \otimes x$ and $\epsilon(x) := 1$ for all basis vectors $x \in X$ and extend them to linear maps on C. Then (C, Δ, ϵ) is a coalgebra.

We can, for instance, take $X = \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$. We can identify the span of X with the vector space of 2×2-matrices. Defining the coproduct Δ and counit ϵ as indicated,

[2] Observe that for any vector space V over $\mathbb{K}$, the tensor product spaces $\mathbb{K} \otimes V$ and $V \otimes \mathbb{K}$ are isomorphic to V. The isomorphism from $\mathbb{K} \otimes V \to V$ is given by $\lambda \otimes v \mapsto \lambda v$, with inverse $v \mapsto 1 \otimes v$. The isomorphism $V \otimes \mathbb{K} \to V$ is defined similarly.

given a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we would have

$$\Delta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and $\epsilon_X\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + b + c + d$. ∎

Example 2.14 (Tensor product of coalgebras). The tensor product of two coalgebras C and D can be made into a coalgebra itself by in the following way: for $c \in C$ and $d \in D$, we define

$$\Delta_{C\otimes D}(c,d) := (\mathrm{id} \otimes \tau_{C,D} \otimes \mathrm{id})\circ(\Delta_C \otimes \Delta_D)(c,d) \quad \text{and} \quad \epsilon_{C\otimes D}(c,d) := \epsilon_C(c)\epsilon_D(d),$$

where $\tau_{C,D} : C \otimes D \to D \otimes C$ is defined as $\tau_{C,D}(x \otimes y) = y \otimes x$.[3] We extend these maps to linear maps on $C \otimes D$. ∎

2.3 Bialgebras

Before delving into the definition of a bialgebra, we present a lemma that establishes a relation between algebras and coalgebras:

Lemma 2.15. *Let (H, μ, η) be an algebra and (H, Δ, ϵ) a coalgebra. We equip $H \otimes H$ with the tensor product of algebras and coalgebras as defined in examples 2.7 and 2.14, and provide $\mathbb{K}$ with the ground algebra and coalgebra structures (examples 2.5 and 2.12). The following two statements are equivalent:*

1. *Both $\mu : H \otimes H \to H$ and $\eta : \mathbb{K} \to H$ are coalgebra homomorphisms.*

2. *Both $\Delta : H \to H \otimes H$ and $\epsilon : H \to \mathbb{K}$ are algebra homomorphisms.*

The proof of this lemma can be found in [25, thm. III.2.1]. It demonstrates that these two statements are equivalent by showing that they correspond to the same set of commutative diagrams.

Now, let's move on to the definition of a bialgebra:

Definition 2.16 (Bialgebra, bialgebra homomorhism)**.** Let (H, μ, η) be an algebra and (H, Δ, ϵ) a coalgebra such that the (equivalent) statements in lemma 2.15 are satisfied. The quintuple $(H, \mu, \eta, \Delta, \epsilon)$ is called a *bialgebra*.

A *bialgebra homomorphism* is a map that is simultaneously an algebra homomorphism and a coalgebra homomorphism and hence preserves algebraic and coalgebraic structures.

In simpler terms, a bialgebra combines the structures of both an algebra and a coalgebra. It requires that the product and unit maps “commute" with the coproduct and counit maps, respectively, as established by the equivalence presented in Lemma 2.15.

[3] Note that we need this map $\tau_{C,D}$ here, as $\Delta_C \otimes \Delta_D$ maps into $C \otimes C \otimes D \otimes D$. Note that we can write the product on $A \otimes B$ (example 2.7) using this map $\tau_{A,B}$ as well: $\mu_{A\otimes B} = (\mu_A \otimes \mu_B)\circ(\mathrm{id} \otimes \tau_{A,B} \otimes \mathrm{id})$.

Definition 2.17 (Star-bialgebra)**.** Let H be a bialgebra with ground field $\mathbb{C}$. Let $*: H \to H$ to be an antilinear involution, and write $x^* = *(x)$. Suppose that for all $x, y \in H$

$$(xy)^* = y^* x^* \quad \text{and} \quad \Delta(x^*) = (* \otimes *)\Delta(x).$$

Then we call $(H, \mu, \eta, \Delta, \epsilon, *)$ a *star-bialgebra*.

The notion of star-bialgebra extends the concept of a star-algebra, which we did not discuss. Two star structures $*$ and $\bullet$ on a bialgebra are called equivalent if there exists a bialgebra automorphism f such that $* \circ f = f \circ \bullet$.

Example 2.18. Recall the coalgebra of 2×2-matrices we discussed in example 2.13. It is not a bialgebra if we equip it with the standard matrix product, as

$$\epsilon\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \cdot \epsilon\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 2 \cdot 1 \neq 1 = \epsilon\left(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right),$$

so that ϵ is not an algebra homomorphism. ∎

Example 2.19 (Group bialgebra). Consider the group $G = \{1, e, f\}$ with $ef = fe = 1$, $ee = f$ and $ff = e$. We can define an algebra product on the linear span H of G by linearly extending the group product to H. We define the unit by $\eta(\lambda) = \lambda 1$. Writing the element $a_1 1 + a_e e + a_f f$ in H as a vector (a_1, a_e, a_f), we can write this product as

$$\mu\left(\begin{pmatrix} a_1 \\ a_e \\ a_f \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_e \\ b_f \end{pmatrix}\right) = \begin{pmatrix} a_1 b_1 + a_e b_f + a_f b_e \\ a_1 b_e + a_e b_1 + a_f b_f \\ a_1 b_f + a_e b_e + a_f b_1 \end{pmatrix}.$$

We can define a coalgebra on the basis elements by setting

$$\Delta(1) = 1 \otimes 1, \quad \Delta(e) = e \otimes e \quad \text{and} \quad \Delta(f) = f \otimes f,$$

and expand it to a linear map on H. Similarly, we define the counit on the basis by $\epsilon(1) = \epsilon(e) = \epsilon(f) = 1$ and extend it to a linear map, this defines a coalgebra structure (cf. example 2.13).

To see that $(H, \mu, \eta, \Delta, \epsilon)$ defines a bialgebra, we should check item 2 of lemma 2.15. We have e.g.

$$\Delta(\mu(e, f)) = \Delta(1) = 1 \otimes 1$$

and

$$\mu_{H\otimes H}(\Delta(e), \Delta(f)) = \mu_{H\otimes H}(e \otimes e, f \otimes f) = ef \otimes ef = 1 \otimes 1,$$

so that $\Delta(\mu(e, f)) = \mu_{H\otimes H}(\Delta(e), \Delta(f))$. We also have

$$\epsilon(\mu(e, f)) = \epsilon(1) = 1 = \epsilon(e) \cdot \epsilon(f).$$

In a similar way one checks that the above relations on other basis elements.

We can define a star structure on H by setting $1^* = 1$ and $e^* = f$ and extending it to an antilinear map on H. One could check that this defines a star-structure on the bialgebra H. ∎

Example 2.20 (Tensor product of bialgebras). Let A and B be bialgebras. We've seen already (examples 2.7 and 2.14) how to equip $A \otimes B$ with algebra and coalgebra structures. We can show that the resulting coproduct and counit are algebra homomorphisms.

Let a and c be elements of A and b and d elements of B. Write $\Delta_A(a) = \sum_j a'_j \otimes a''_j$ with the sum taken over some finite index set and a'_j, a''_j elements of A. Note that this is possible for any $a \in A$ as $\Delta_A(a) \in A \otimes A$. Similarly, we write $\Delta_A(c) = \sum_k c'_k \otimes c''_k$, $\Delta_B(b) = \sum_l b'_l \otimes b''_l$ and $\Delta_B(d) = \sum_m d'_m \otimes d''_m$. Then

$$\begin{aligned}
&\Delta_{A\otimes B}\mu_{A\otimes B}(a \otimes b \otimes c \otimes d) \\
&\qquad = (\mathrm{id} \otimes \tau_{A,B} \otimes \mathrm{id})\big(\Delta_A(ac) \otimes \Delta_B(bd)\big) \\
&\qquad = (\mathrm{id} \otimes \tau_{A,B} \otimes \mathrm{id})\big((\Delta_A(a)\Delta_A(c)) \otimes (\Delta_B(b)\Delta_B(d))\big) \\
&\qquad = (\mathrm{id} \otimes \tau_{A,B} \otimes \mathrm{id}) \sum_{j,k,l,m} a'_j c'_k \otimes a''_j c''_k \otimes b'_l d'_m \otimes b''_l d''_m \\
&\qquad = \sum_{j,k,l,m} a'_j c'_k \otimes b'_l d'_m \otimes a''_j c''_k \otimes b''_l d''_m \\
&\qquad = \mu_{A\otimes B\otimes A\otimes B}\left(\sum_{j,l} a'_j \otimes b'_l \otimes a''_j \otimes b''_l, \sum_{k,m} c'_k \otimes d'_m \otimes c''_k \otimes d''_m \right) \\
&\qquad = \mu_{A\otimes B\otimes A\otimes B}\big(\Delta_{A\otimes B}(a \otimes b), \Delta_{A\otimes B}(c \otimes d)\big) \\
&\qquad = \mu_{A\otimes B\otimes A\otimes B}\big(\Delta_{A\otimes B} \otimes \Delta_{A\otimes B}\big)(a \otimes b \otimes c \otimes d).
\end{aligned}$$

By linearity this result extends to $A \otimes B \otimes A \otimes B$, so that $\Delta_{A\otimes B}$ is an algebra homomorphism.

It is left to the reader to check that $\epsilon_{A\otimes B}$ is an algebra homomorphism as well. We conclude that $A \otimes B$, equiped with the algebra and coalgebra structures of the tensor products, is a bialgebra. ∎

2.4 Hopf algebras

If A is an algebra and C is a coalgebra, we can define a convolution on linear maps from A to C. For linear maps f and g, the convolution is defined as

$$f \star g = \mu \circ (f \otimes g) \circ \Delta.$$

Note that if we have $\Delta(x) = \sum_j x'_j \otimes x''_j$, we can write the convolution as $f \star g(x) = \sum_j f(x'_j) g(x''_j)$. Just as for the convolution of functions $\mathbb{R} \to \mathbb{R}$, it can be shown that the bialgebra convolution is associative. Unlike the convolution of functions on the real line, however, the bialgebra convolution admits an inverse: $\eta \circ \epsilon$. These properties were proved in [25, prop. III.3.1].

We use the convolution to define the antipode, which transforms a bialgebra into a Hopf algebra.

Definition 2.21 (Antipode, Hopf algebra)**.** Let $(H, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. An *antipode* is a linear map $S : H \to H$ that satisfies

$$S \star \mathrm{id} = \mathrm{id} \star S = \eta \circ \epsilon.$$

If S is an antipode, the sextuple $(H, \mu, \eta, \Delta, \epsilon, S)$ is called a *Hopf algebra*. A bialgebra homomorphism f between two Hopf algebras H_1 and H_2 that satisfies $f \circ S_{H_1} = S_{H_2} \circ f$ is called a *Hopf algebra homomorphism*. If a star-bialgebra admits an antipode, we call $(H, \mu, \eta, \Delta, \epsilon, S, *)$ a *Hopf star-algebra*.

Not every bialgebra admits an antipode. If an antipode exists, however, it is unique: suppose S and S' are both antipodes, then by associativity and unitality of the convolution,

$$S = S \star (\eta \circ \epsilon) = S \star (\mathrm{id} \star S') = (S \star \mathrm{id}) \star S' = (\eta \circ \epsilon) \star S' = S'.$$

Two Hopf star-involutions $*$ and $\bullet$ are called equivalent if there exists a Hopf algebra homomorphism f such that $* \circ f = f \circ \bullet$

Remark 2.22. Many authors, including Kassel ([25]), impose an additional requirement on a star-operatation $*$ in their definition of a Hopf star-algebra, demanding that it satisfies $* \circ S \circ * \circ S = \mathrm{id}$. However, it has been shown in [41] that this additional condition is a direct consequence of the conditions imposed in definition 2.17. As $* \circ S \circ * \circ S = \mathrm{id}$ on a Hopf star-algebra, it follows that the antipode of a Hopf star-algebra is invertible. Or, put differently, if the antipode of a Hopf-algebra is non-invertible, it cannot admit a star-structure. ∎

Example 2.23 (Group Hopf algebra). Consider the group bialgebra $(H, \mu, \eta, \Delta, \epsilon)$ from example 2.19. We define a linear map $S : H \to H$ by setting $S(g) = g^{-1}$ for all $g \in G = \{1, e, f\}$. Then for all $g \in G$,

$$S \star \mathrm{id}(g) = \mu(S \otimes \mathrm{id})\Delta(g) = \mu(S \otimes \mathrm{id})(g \otimes g) = S(g)g = e = \eta(\epsilon(g)),$$

and similarly $\mathrm{id} \star S(g) = \eta(\epsilon(g))$. By linearity, $S \star \mathrm{id} = \mathrm{id} \star S = \eta \circ \epsilon$, so S is an antipode and $(H, \mu, \eta, \Delta, \epsilon, S)$ is a Hopf algebra. ∎

Example 2.24 (Tensor product of Hopf algebras). We have seen in example 2.20 that the tensor product of two bialgebras is again a bialgebra. We will show that the sensor product of Hopf algebras is again a Hopf algebra. To do so, equip the vector spaces A and B with a Hopf algebra structure, with antipodes S_A and S_B. Set $S_{A \otimes B}(a \otimes b) = S_A(a) \otimes S_B(b)$ for $a \in A$ and $b \in B$ and extend it to a linear map.

Then, writing $\Delta_A(a) = \sum_j a'_j \otimes a''_j$ and $\Delta_B(b) = \sum_k b'_k \otimes b''_k$, we have

$$\begin{aligned} S_{A \otimes B} \star \mathrm{id}_{A \otimes B}(a \otimes b) &= \mu_{A \otimes B}\Big(\sum_{j,k} S(a'_j) \otimes S(b'_k) \otimes a''_j \otimes b''_k\Big) \\ &= \mu_A\Big(\sum_j S(a'_j) \otimes a''_j\Big) \otimes \mu_B\Big(\sum_k S(b'_k) \otimes b''_k\Big) \\ &= S_A \star \mathrm{id}_A(a) \otimes S_B \star \mathrm{id}_B(b) = \eta_A(\epsilon_A(a)) \otimes \eta_B(\epsilon_B(b)) \\ &= \eta_{A \otimes B}(\epsilon_{A \otimes B}(a \otimes b)). \end{aligned}$$

Similarly, $\mathrm{id}_{A \otimes B}(a \otimes b) \star S_{A \otimes B} = \eta_{A \otimes B}(\epsilon_{A \otimes B}(a \otimes b))$, and by linearity this result can be extended to $A \otimes B$, so that $S_{A \otimes B}$ is an antipode. ∎

If (A, μ, η) is an algebra, we define $\mu^{\mathrm{op}}(x, y) = \mu(y, x)$ to be *opposite algebra product*. The triple $(A, \mu^{\mathrm{op}}, \eta)$ defines an algebra again. We conclude this chapter with a lemma.

Lemma 2.25. *Let H be a bialgebra and $S : H \mapsto H$ be an algebra homomorphism from*

(H,μ,η) to (H,μ^{op},η). Let $X \subseteq H$ be a set that generates H as an algebra. If for all $x \in X$

$$S \star \mathrm{id}(x) = \mathrm{id} \star S(x) = \eta(\epsilon(x)), \tag{2.5}$$

then S is an antipode for H.

Proof. As S, Δ and η are algebra homomorphisms, it follows that eq. (2.5) holds on the unit element of H. As all elements in H can be written as a linear combination of iterated products of elements in X, and eq. (2.5) preserves linear combinations, it suffices to show that if eq. (2.5) is satisfied for $x = y$ and for $x = z$, then it is also satisfied for $x = yz$.

Write $\Delta y = \sum_j y'_j \otimes y''_j$ and $\Delta z = \sum_k z'_k \otimes z''_k$ with j and k running over some finite indexing sets. Then $\Delta(yz) = \Delta y \Delta z = \sum_{j,k} (y'_j z'_k) \otimes (y''_j z''_k)$, so that

$$\begin{aligned}
S \star \mathrm{id}(yz) &= \sum_{j,k} S(y'_j z'_k) y''_j z''_k \\
&= \sum_{j,k} S(z'_k) S(y'_j) y''_j z''_k && \text{(by our assumption on } S\text{)} \\
&= \sum_k S(z'_k) \left[\sum_j S(y'_j) y'' \right] z''_k \\
&= \sum_k S(z'_k) \eta(\epsilon(y)) z''_k && \text{(by our assumption on } y\text{)} \\
&= \eta(\epsilon(y)) \sum_k S(z'_k) z''_k && \text{(as } \eta \text{ commutes with all elements of } H\text{)} \\
&= \eta(\epsilon(y)) \eta(\epsilon(z)) = \eta(\epsilon(yz)),
\end{aligned}$$

and in a similar way we can show that

$$\mathrm{id} \star S(yz) = \eta(\epsilon(yz)).$$

□

Chapter 3

$U_q(\mathfrak{sl}(2))$ and its modular double $\mathfrak{D}$

Heisenberg introduced theory of representing physical observables in quantum theory by Hermitian matrices. He noted that, unlike the classical analogue, multiplication of those matrix observables was not commutative. Dirac then described how the observables formed a noncommutative algebra, with its noncommutativity quantised by $\hbar$, the reduced Plank constant. Dirac coined this deformed algebra the *quantum algebra* ([46]). The term 'quantum' has then since become a common prefix for referring to the study of noncommutative analogues of classical commutative objects such as groups and algebras.

A subclass of those quantum algebras is formed by the so-called *quantum groups*. Unlike the name suggests, they are not groups, but Hopf algebras. In particular, the name is used for deformations of algebras related to Lie groups and Lie algebras, formalised independently by Drinfeld ([5]) and Jimbo ([22]) around the year 1985. In the case of Lie groups, deformations are done on the commutative algebra of functions on the group manifold of the Lie group. In the case of Lie algebras, the deformation is done on the enveloping algebra (we will recapitulate the definitions), which is in fact not commutative, but is *cocommutative* as a Hopf algebra.[1] The underlying procedures in deforming these Hopf algebras were originally inspired by applications from mathematical physics, being quantum integrable models and statistical physics ([6]).

In this chapter, we introduce the quantum group that is obtained from deforming the enveloping algebra of the Lie algebra $\mathfrak{sl}(2)$, which we denote $U_q(\mathfrak{sl}(2))$ or just U_q, with q being the deformation parameter. This deformation was first introduced (as an algebra) in 1981 ([32]) and has been studied by many authors since. We start our discussion of U_q by recapitulating the definition of the Lie algebra $\mathfrak{sl}(2)$ and its enveloping algebra, denoted $U(\mathfrak{sl}(2))$ or U. Then we introduce U_q and show its relation to U. We introduce a parameter $\tilde{q}$, dual to q, and use it to introduce a larger Hopf algebra $\mathfrak{D}$, which is isomorphic to $U_q \otimes U_{\tilde{q}}$, and which we call the *modular double of* U_q. Lastly, we introduce and discuss some star-structures on U_q and $\mathfrak{D}$.

There exist some different ways of defining the Hopf algebra U_q, not all of which are equivalent. We decide to stick to the definition provided in [25] and match all our results to this definition.

[1] An algebra is said to be commutative if $\mu = \mu \circ \tau$, with τ the map that flips the factors of the tensor product. A coalgebra is cocommutative if $\Delta = \tau \circ \Delta$.

3.1 The Lie algebra $\mathfrak{sl}(2)$

To provide the necessary background for introducing $U_q(\mathfrak{sl}(2))$, we briefly recap the definition of a Lie algebra and especially of $\mathfrak{sl}(2)$, which lies at the root of U_q. For a more comprehensive treatment of Lie algebras, we recommend consulting e.g. [20], in which one will find more background on the concepts discussed in this section.

Definition 3.1 (Lie bracket, Lie algebra)**.** Let L be a vector space. A *Lie bracket* is a bilinear vector space map $[\cdot\,,\cdot]\colon L \times L \to L$ satisfying the two axioms $[x, x] = 0$ and $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in L$. The pair $(L, [\cdot\,,\cdot])$ defines a *Lie algebra*.

Lie algebras are in general non-associative non-unital algebras, although some nontrivial examples of associative Lie algebras exist.

Example 3.2 (L(A))*.* Let A be an associative algebra. One could check that $[a, b] := ab - ba$ defines a Lie bracket on A, so that $(A, [\cdot\,,\cdot])$ is a Lie algebra. We denote this Lie algebra by L(A). ∎

The Lie algebra $\mathfrak{sl}(2)$ is a non-associative algebra that we define as follows: its vector space is the linear span over $\mathbb{C}$ of three elements, which we call h, e and f. Its algebra product is the Lie bracket is given by

$$[h, e] = 2e, \quad [h, f] = -2f \quad \text{and} \quad [e, f] = h.$$

One can identify $\mathfrak{sl}(2)$ with the vector space of 2×2-matrices with zero trace, equipped with the product $[x, y] = xy - yx$. In this case, the basis elements h, e and f correspond to the matrices

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

3.2 Enveloping algebra

We will now introduce the concept of the enveloping algebra (sometimes referred to as the *universal* enveloping algebra) of a Lie algebra. The enveloping algebra $U(L)$ of a Lie algebra L is an associative algebra that preserves many of the properties of L, making it a valuable tool in the study of Lie algebras.

We define the enveloping algebra using the tensor and quotient algebras in examples 2.8 and 2.9. Let L be a Lie algebra, $T(L)$ the tensor algebra, and let $I(L)$ be the ideal of $T(L)$ generated by all elements in $T(L)$ of the form $xy - yx - [x, y]$, for $x, y \in L$.[2] We define the enveloping algebra of L as

$$U(L) := T(L)/I(L).$$

We now move over to the enveloping algebra of $\mathfrak{sl}(2)$. We will show that the algebra $U(\mathfrak{sl}(2))$ equals, up to isomorphism, the algebra U generated by the three elements E, F, H

[2] The ideal $I(L)$ thus is the space of all finite sums of the form $v(xy - yx - [x, y])w$ for $v, w \in T(L)$ and $x, y \in L$.

with relations

$$HE - EH = 2E, \quad HF - FH = -2F \quad \text{and} \quad EF - FE = H.$$

To see why U is isomorphic to $U(\mathfrak{sl}(2))$, define a map $\phi\colon T(\mathfrak{sl}(2)) \to U$ by sending the elements h, e and f in $T(\mathfrak{sl}(2))$ to H, E and F respectively in U and extending the map to a homomorphism. From the defining relations of U, it is clear that ϕ maps $I(\mathfrak{sl}(2))$ to 0. We can thus let ϕ descend to a well-defined map $\phi' : U(\mathfrak{sl}(2)) \to U$, setting $\phi'([x]) = \phi(x)$.

We now define the linear map $\psi : U \to U(\mathfrak{sl}(2))$ on the generators by $\psi(H) = [h]$ and $\psi(E) = [e]$ and $\psi(F) = [f]$. It is easy to check that the map ψ can be extended to a well defined algebra homomorphism, satisfying the defining relations of U. Note that ψ and ϕ' are eachothers inverses, so that indeed U is isomorphic to $U(\mathfrak{sl}(2))$.

We can define a Hopf algebra stucture on U by defining a coproduct and a counit. On the generators, set

$$\Delta H = 1 \otimes H + H \otimes 1, \quad \Delta E = 1 \otimes E + E \otimes 1 \quad \text{and} \quad \Delta F = 1 \otimes F + F \otimes 1,$$

and extend Δ to the span of the generators. One checks that such a map preserves the algebra structure, e.g. for the relation $EF - FE = H$ we find

$$\Delta E \Delta F = 1 \otimes (EF) + F \otimes E + E \otimes F + (EF) \otimes 1,$$

and

$$\Delta F \Delta E = 1 \otimes (FE) + E \otimes F + F \otimes E + (FE) \otimes 1,$$

so that

$$\Delta E \Delta F - \Delta F \Delta E = 1 \otimes (EF - FE) + (EF - FE) \otimes 1 = \Delta H.$$

Similarly, the other defining relations are preserved under Δ, so that we can extend it to an algebra homomorphism.

The counit of U can be by defining ϵ to map the span of H, E and F to 0 and η_U to $\eta_{\mathbb{K}} = \text{id}$. Preservation of the defining relations now is automatic, so that it extends to a homomorphism.

Lastly, for $n = 0, 1, 2, ...$ and $v_1, ..., v_n \in \text{span}\{H, E, F\}$, we set

$$S(v_1 v_2 \cdots v_n) = (-1)^n v_n v_{n-1} \cdots v_1$$

and extend it to a linear map on H. One checks that S preserves the defining relations (e.g. $S(EF - FE) = FE - EF = -H = S(H)$) and is an algebra homomorphism from (H, μ, η) into $(H, \mu^{\text{op}}, \eta)$, so that we can use lemma 2.25 to conclude that S is an antipode. It can be shown that S extends to an antipode of U, so that U is a Hopf algebra.

Several star structures exist on U, some of which are given on the generators by

$$E^* = E, \quad F^* = F \quad \text{and} \quad H^* = -H,$$

and

$$E^\bullet = F, \quad F^\bullet = E \quad \text{and} \quad H^\bullet = H.$$

3.3 The quantum group $U_q(\mathfrak{sl}(2))$

For complex q not equal to -1, 0 or 1, we define $U_q(\mathfrak{sl}(2))$ to be the algebra generated by the elements K, K^{-1}, E and F with relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}. \tag{3.1}$$

To simplify the notation, we will often write U_q instead of $U_q(\mathfrak{sl}(2))$. We denote the product by μ and the unit by η as usual. The parameter q quantifies the deformation of U_q with respect to U. (We will show in what sense U can be seen as a limiting case of U_q in the next section.)

We will extend U_q to a bialgebra. Let $\Delta : U_q \to U_q \otimes U_q$ be the algebra homomorphism defined by

$$\Delta K = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\Delta E = 1 \otimes E + E \otimes K \quad \text{and} \quad \Delta F = K^{-1} \otimes F + F \otimes 1.$$

It is straightforward to verify that Δ preserves the relations given in eq. (3.1), for example,

$$\Delta K \Delta E = K \otimes KE + KE \otimes K^2 = K \otimes (q^2EK) + (q^2EK) \otimes K^2 = q^2 \Delta E \Delta K.$$

Hence the extension to an algebra homomorphism is well defined.

Let the homomorphism $\epsilon : U_q \to \mathbb{C}$ be defined by

$$\epsilon(K) = \epsilon(K^{-1}) = 1 \quad \text{and} \quad \epsilon(E) = \epsilon(F) = 0.$$

(Compatibility with the relations eq. (3.1) follows almost immediately in this case.) The quintuple $(U_q, \mu, \eta, \Delta, \epsilon)$ is a bialgebra.

Define S to be the algebra homomorphism from (U_q, μ, η) to $(U_q, \mu^{\text{op}}, \eta)$ satisfying

$$S(K) = K^{-1}, \quad S(K^{-1}) = K, \quad S(E) = -EK^{-1} \quad \text{and} \quad S(F) = -KF.$$

One can check that the defining relations are preserved under S, e.g.

$$S(KE) = S(E)S(K) = -EK^{-2} = -q^2K^{-1}EK^{-1} = q^2S(K)S(E) = S(q^2EK),$$

so that the homomorphism is well-defined. By lemma 2.25, S is an antipode, showing that U_q is a Hopf algebra.

3.4 Relating U_q to U

In the previous section, I said that the parameter q quantifies the deformation of U_q with respect to U. We will see that in the limit $q = 1$, we retrieve, more or less, the algebra U.

The defining relations in eq. (3.1) do not allow us to set $q = 1$. Defining the element $H \in U_q$

as $H := \frac{K-K^{-1}}{q-q^{-1}}$, we have $EF - FE = H$ and we can calculate

$$\begin{aligned} HE - EH &= \frac{1}{q-q^{-1}}\big((KE - EK) - (K^{-1}E - EK^{-1})\big) \\ &= \frac{1}{q-q^{-1}}\big((q^2-1)EK - (q^{-2}-1)EK^{-1}\big) \\ &= qEK + q^{-1}EK^{-1}, \end{aligned}$$

and similarly

$$HF - FH = -qFK^{-1} - q^{-1}FK.$$

Hence, using H, we can rewrite the defining relations to

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \qquad KE = q^2EK, \qquad KF = q^{-2}FK, \\ &(q-q^{-1})H = K - K^{-1}, \qquad HE - EH = qEK + q^{-1}EK^{-1}, \\ &HF - FH = -qFK^{-1} - q^{-1}FK, \quad \text{and} \quad EF - FE = H. \end{aligned}$$

This more complicated set allows for setting $q = 1$. Simultaneously setting $K = 1$, we retieve the defining relations of U. If we denote by $\langle K-1\rangle_q$ the ideal of U_q generated by $K - 1$, we find that $U \cong U_1/\langle K-1\rangle_1$. (Setting $q = -1$, we can also retrieve U, but now with the roles of the generators E and F interchanged.)

3.5 The modular double $\mathfrak{D}$

The modular double is a structure first described by Faddeev ([7]) which unifies two quantum groups U_q and $U_{\tilde{q}}$ into one object. This new object is modular in the sense that it is compatible with the original structures.

For $q \in \mathbb{C} \setminus \{0\}$, set b in such a way that $q = e^{i\pi b^2}$. The dual parameter $\tilde{q}$ is defined as $\tilde{q} = e^{i\pi/b^2}$. We can define the modular double as

$$\mathfrak{D} := U_q \otimes U_{\tilde{q}}$$

By example 2.7, U_q and $U_{\tilde{q}}$ can be identified with commuting subalgebras. Hence we identify $\mathfrak{D}$ with the algebra generated by $K, K^{-1}, E, F, \tilde{K}, \tilde{K}^{-1}, \tilde{E}, \tilde{F}$ with relations

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK, \quad EF - FE = \frac{K-K^{-1}}{q-q^{-1}}, \\ &\tilde{K}\tilde{K}^{-1} = \tilde{K}^{-1}\tilde{K} = 1, \quad \tilde{K}\tilde{E} = \tilde{q}^2\tilde{E}\tilde{K}, \quad \tilde{K}\tilde{F} = \tilde{q}^{-2}\tilde{F}\tilde{K}, \quad \tilde{E}\tilde{F} - \tilde{F}\tilde{E} = \frac{\tilde{K}-\tilde{K}^{-1}}{\tilde{q}-\tilde{q}^{-1}}, \end{aligned}$$

and $MN = NM$ for $M \in \{K, K^{-1}, E, F\}$ and $N \in \{\tilde{K}, \tilde{K}^{-1}, \tilde{E}, \tilde{F}\}$. By example 2.24, the Hopf algebra structures of U_q and $U_{\tilde{q}}$ carry over to $\mathfrak{D}$. Note that q does not uniquely determine the values of b and $\tilde{q}$, and hence $\mathfrak{D}$ is not determined by the choice of q alone.

One could of course take the tensor product of U_q with U_r with any arbitrary r, and all of the above would still apply, but the two factors U_q and $U_{\tilde{q}}$ of the modular double turn out to be related in a more fundamental way: Faddeev showed how they can both be embedded in a natural way in an algebra generated by Heisenberg-type elements ([7]). We will present the main idea of this procedure in this chapter. It should be noted that while the derivation presented by Faddeev is not rigorous from a mathematical point of view, it serves as an illustration of the underlying thought process, and has proven to be useful e.g. in detemining representations of $\mathfrak{D}$, as will be shown in the next chapter.

First, let p_n, with $n \in \mathbb{Z}_4$ be elements with commutation relations $[p_n, p_{n+1}] = 2\pi i I$ (with I the unit element) and $[p_n, p_{n+2}] = 0$. Set $v_n = e^{bp_n}$ (assuming for now that this is well-defined as an element of the algebra generated by $\{p_1, p_2, p_3, p_4\}$). By the Baker-Campbell-Hausdorff formula, the elements v_n satisfy

$$v_n v_{n+1} = e^{b(p_n+p_{n+1})+\frac{b^2}{2}[p_n,p_{n+1}]} = q e^{b(p_n+p_{n+1})}$$

and

$$v_{n+1} v_n = e^{b(p_{n+1}+p_n)-\frac{b^2}{2}[p_n,p_{n+1}]} = q^{-1} e^{b(p_n+p_{n+1})},$$

so that

$$v_n v_{n+1} = q^2 v_{n+1} v_n \quad \text{and} \quad v_n v_{n+2} = v_{n+2} v_n \tag{3.2}$$

for $n \in \mathbb{Z}_4$. Denote the algebra generated by four elements v_1, v_2, v_3, v_4 and the relations in eq. (3.2) by C_q.

The algebra C_q has two central elements[3], $Z_1 = v_0 v_2$ and $Z_2 = v_1 v_3$. Consider the subalgebra generated by the elements

$$K = q v_0 v_3 = q^{-1} v_3 v_0, \quad K' = q^{-1} v_1 v_2, \quad E = i\frac{v_0 + v_1}{q - q^{-1}} \quad \text{and} \quad F = i\frac{v_2 + v_3}{q - q^{-1}}, \tag{3.3}$$

and their inherited relations. It is straightforward to check that these generators satisfy U_q-like commutation relations: $KK' = K'K$, $EF - FE = \frac{K-K'}{q-q^{-1}}$ and $KE = q^2 EK$. Note that $KK' = K'K = Z_1 Z_2$, so that it is in the centre. Taking the quotient over the ideal generated by $KK' - 1$, we retrieve an algebra with equivalent relations as U_q. Thus, U_q is (isomorphic to) a quotient algebra of a subalgebra of C_q.

In a similar way, setting $\tilde{v}_n = e^{p_n/b}$ for $n \in \mathbb{Z}_4$, we can find commutation relations on the elements $\tilde{v}_n$ to generate the algebra $C_{\tilde{q}}$. We define

$$\tilde{K} = \tilde{q} \tilde{v}_0 \tilde{v}_3 = \tilde{q}^{-1} \tilde{v}_3 \tilde{v}_0, \quad \tilde{K}' = \tilde{q}^{-1} \tilde{v}_1 \tilde{v}_2, \quad \tilde{E} = i\frac{\tilde{v}_0 + \tilde{v}_1}{\tilde{q} - \tilde{q}^{-1}} \quad \text{and} \quad \tilde{F} = i\frac{\tilde{v}_2 + \tilde{v}_3}{\tilde{q} - \tilde{q}^{-1}}, \tag{3.4}$$

and retrieve $U_{\tilde{q}}$ by taking the quotient over the ideal generated by $\tilde{K}\tilde{K}' - 1$.

Again by the Baker-Campbell-Hausdorff formula, we conclude that for all $m, n \in \mathbb{Z}_4$,

$$v_m \tilde{v}_n = e^{bp_m} e^{p_n/b} = e^{[p_m,p_n]} e^{p_n/b} e^{bp_m} = \tilde{v}_n v_m \tag{3.5}$$

in the algebra generated by the p_n's, as $[p_m, p_n]$ equals either $\pm 2\pi i$ or 0. Hence the elements K, K', E and F commute with $\tilde{K}$, $\tilde{K}'$, $\tilde{E}$ and $\tilde{F}$ in the algebra generated by the p_n's.

Although the algebras C_q and $C_{\tilde{q}}$ are well-defined, from a mathematical viewpoint, there seems to be some difficulty in viewing them as subalgebras of the algebra generated by p_0, p_1, p_2 and p_3. I see no evident direct way of how to define a notion of convergence such that $v_n = e^{bp_n}$

[3] An element is called central if it commutes with all elements of the algebra.

can be defined by a convergent power series and can be seen as an element of this algebra.[4]

3.6 The Hopf star-algebras U_q and $\mathfrak{D}$

It has been shown (cf. [25, 33]) that a Hopf star-structure exists on $U_q(\mathfrak{sl}(2))$ if either $q^2 \in \mathbb{R}$ or $|q| = 1$. In the former case (of real q^2), any Hopf star-involution on U_q is equivalent to an involution • of the form

$$K^\bullet = K, \quad (K^{-1})^\bullet = K^{-1}, \quad E^\bullet = \frac{q}{|q|} KF \quad \text{and} \quad F^\bullet = \frac{q}{|q|} EK^{-1}.$$

Using the definition of U_q in section 3.4, we see that in the limit $q \to 1$ over the real line this star-operation corresponds to the operation • we defined on U, as $H^\bullet = \left(\frac{K-K^{-1}}{q-q^{-1}}\right)^\bullet = \frac{K-K^{-1}}{q-q^{-1}} = H$ for real q.

In the case $|q| = 1$, any star-involution is equivalent to

$$K^* = K, \quad (K^{-1})^* = K^{-1}, \quad E^* = E, \quad \text{and} \quad F^* = F. \tag{3.6}$$

This structure corresponds to the operation $*$ we defined on U when we take the limit $q \to 1$ over the complex unit circle, as $H^* = \left(\frac{K-K^{-1}}{q-q^{-1}}\right)^* = \frac{K-K^{-1}}{q^{-1}-q} = -H$ for $|q| = 1$ and $q \neq \pm 1$.

Both star structures can be extended in a natural way to $\mathfrak{D}$, if the parameter b is chosen such that b^2 is imaginary for real q, so that $\tilde{q}$ is again real, or if b^2 is chosen real, such that q and $\tilde{q}$ are both contained in the unit circle.

Faddeev has noted ([7]) that a third type of star-involution exists on $\mathfrak{D}$, not corresponding to any star-involution on U_q. It corresponds to the case $\tilde{q} = \bar{q}^{-1}$, or $b = e^{i\theta}$ for some real θ. We denote this third type of involution by the symbol $\star$, and define it as

$$K^\star = \tilde{K}, \quad (K^{-1})^\star = \tilde{K}^{-1}, \quad E^\star = \tilde{E} \quad \text{and} \quad F^\star = \tilde{F}.$$

One can check that the defining relations of $\mathfrak{D}$ are preserved under $\star$, e.g.

$$(KE)^\star = E^\star K^\star = \tilde{E}\tilde{K} = \tilde{q}^{-2}\tilde{K}\tilde{E} = \tilde{q}^{-2}K^\star E^\star = (\bar{\tilde{q}}^{-2}EK)^\star = (q^2 EK)^\star.$$

The commutativity of the coproduct Δ with $\star$ follows almost automatically, and we conclude that $\star$ is a valid Hopf star-involution.

The existence of this latter star-structure on $\mathfrak{D}$ is another indicator that something special is going on with the modular double. The star structures $*$ and $\star$ correspond to the cases of weak and strong coupling, respectively, in certain applications in conformal field theory. Hence, the modular double is needed for studying the strong coupling case ([8]). In the rest of this thesis, we will be using the star-structures induced by $*$ and $\star$. The star structure $*$ agrees with the star given by $v_n^* = v_n$ and $\tilde{v}_n^* = \tilde{v}_n$ on C_q, for $n \in \mathbb{Z}_4$. The star $\star$ agrees with the star on C_q defined by $w_n^\star = \tilde{w}_n$ for $n \in \mathbb{Z}_4$, interchanging factors in the modular double.

[4]One could think to define a norm on the algebra and use it to define convergence. Note however that by the Wintner-Wielandt theorem, $[p_n, p_{n+1}] = 2\pi i I$ cannot hold for elements of a normed unital algebra.

Chapter 4

Representations on $\mathfrak{D}$

In this chapter, we will introduce representations of $\mathfrak{D}$ on the space $\mathcal{M}$ of meromorphic functions on $\mathbb{C}$. Additionally, we will discuss a sesquilinear form on $\mathcal{M}$ that establishes a connection between these representations and the star-structures on $\mathfrak{D}$.

The representations of U_q have been classified in [42]. We will utilise a specific representation defined in that work for $|q| = |\tilde{q}| = 1$. While this representation was originally defined for U_q, it can be extended to a representation on $\mathfrak{D}$, see e.g. [21]. Furthermore, it admits a generalisation that allows for the case $\tilde{q} = \bar{q}^{-1}$ as well. To derive these representations, we will refer back to our earlier discussion of Faddeev's mathematical derivation of $\mathfrak{D}$, as presented in Section section 3.5.

4.1 Position and momentum operators, and their exponentials

Before introducing the representation, we will shortly discuss the position operator X and the momentum operator P. They are defined on $\mathcal{M}$ by

$$Xf(x) = x \cdot f(x) \quad \text{and} \quad Pf(x) = \frac{1}{2\pi i}\frac{\mathrm{d}f}{\mathrm{d}x}(x).$$

Note that the derivative of a meromorhphic function is meromorhpic, so that this is well-defined.

The operators X and P satisfy the commutation relation $[X, P] = -\frac{1}{2\pi i}I$. We can define the exponentials of X and P as operators on $\mathcal{M}$ using power series expansions. For a in $\mathbb{C}$, the Taylor series of $f(x + \frac{a}{2\pi i})$ around x gives

$$f(x + \frac{a}{2\pi i}) = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{2\pi i}\right)^n \frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x) = \sum_{n=0}^{\infty} \frac{1}{n!}[(aP)^n f](x),$$

which converges for all $x \in \mathbb{C}$, except at the poles of f. Based on this, we define

$$e^{aP} f(x) := f(x + \frac{a}{2\pi i}).$$

Similarly,

$$e^{ax} f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(ax)^n f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[(aX)^n f\right](x),$$

which justifies setting

$$e^{aX} f(x) := e^{ax} f(x).$$

It is straightforward to check that $e^{aX}e^{bP} = e^{ab[X,P]}e^{bP}e^{aX}$, and we can use the Baker-Campbell-Hausdorff formula to express the exponentials of linear combinations of X and P:

$$e^{aX+bP} = e^{-\frac{1}{2}ab[X,P]}e^{aX+bP+\frac{1}{2}ab[X,P]} = e^{\frac{ab}{4\pi i}}e^{aX}e^{bP}.$$

4.2 The representation π_λ

We define the representation π_λ of $\mathfrak{D}$ using the following construction. Let $q = e^{i\pi b^2}$ either with a real positive value for b with $b^2 \notin \mathbb{Q}$, or with $b = e^{i\theta/2}$ for some $\theta \in (-\pi, 0) \cup (0, \pi)$. Pick w_1 and w_2 in the following way: if $b > 0$, select positive values for w_1 and w_2 and such that $w_1/w_2 = b^2$; if $b = e^{i\theta/2}$, choose w_1 and w_2 with positive real parts in such a way that $\bar{w}_1 = w_2$ and $w_1/w_2 = b^2 = e^{i\theta}$. Let $\beta := w_1/b = bw_2$.

Now set

$$\begin{aligned}
p_0 &= \frac{2\pi}{\beta}X - \frac{\pi\lambda}{\beta} - 2\pi\beta P,\\
p_1 &= \frac{2\pi}{\beta}X + \frac{\pi\lambda}{\beta},\\
p_2 &= -\frac{2\pi}{\beta}X - \frac{\pi\lambda}{\beta} + 2\pi\beta P,\\
p_3 &= -\frac{2\pi}{\beta}X + \frac{\pi\lambda}{\beta},
\end{aligned}$$

where X and P are the position and momentum operators defined in the previous section. It can be checked that $[p_n, p_{n+1}] = 2\pi i I$ and $[p_n, p_{n+2}] = 0$ for all $n \in \mathbb{Z}_4$.

We set $v_n = \exp\big(bp_n\big)$ and $\tilde{v}_n = \exp\big(p_n/b\big)$, and define a representation π_λ of U_q by setting

$$\pi_\lambda(K) = qv_0v_3, \quad \pi_\lambda(K^{-1}) = q^{-1}v_1v_2, \quad \pi_\lambda(E) = i\frac{v_0 + v_1}{q - q^{-1}} \quad \text{and} \quad \pi_\lambda(F) = i\frac{v_2 + v_3}{q - q^{-1}}.$$

It is straight-forward to see that the map π_λ extends to an algebra homomorphism: Note how the elements v_n satisfy the relations described in eq. (3.2). By comparing the definition of π_λ above with eq. (3.3), we see that π_λ preserves the product of U_q, modulo the ideal I generated by $KK' - 1 = v_0v_3v_1v_2 - 1$. (In section 3.5 we retrieved U_q by factoring out $KK' - 1$ from the algebra generated by the elements in eq. (3.3).) By the Baker-Campbell-Hausdorff formula applied to our current elements v_n, we have

$$\begin{aligned}
(v_0v_3)(v_1v_2) - 1 &= e^{bp_0+bp_3+\frac{1}{2}b^2[p_0,p_3]}e^{bp_1+bp_2+\frac{1}{2}b^2[p_1,p_2]} - 1\\
&= e^{-2\pi\beta P-4\pi^2b^2\beta^2[X,P]}e^{2\pi\beta P+4\pi^2b^2\beta^2[X,P]} - 1 = 1 - 1 = 0,
\end{aligned}$$

so that I equals $\{0\}$. It follows that π_λ is well defined as a representation.

We can use the elements $\tilde{v}_n$ to extend the representation π_λ to $\mathfrak{D}$, as $v_m\tilde{v}_n = \tilde{v}_nv_m$ for $m, n \in \mathbb{Z}_4$ (cf. eq. (3.5)), setting

$$\pi_\lambda(\tilde{K}) = \tilde{q}\tilde{v}_0\tilde{v}_3, \quad \pi_\lambda(\tilde{K}^{-1}) = \tilde{q}^{-1}\tilde{v}_1\tilde{v}_2, \quad \pi_\lambda(\tilde{E}) = i\frac{\tilde{v}_0 + \tilde{v}_1}{\tilde{q} - \tilde{q}^{-1}} \quad \text{and} \quad \pi_\lambda(\tilde{F}) = i\frac{\tilde{v}_2 + \tilde{v}_3}{\tilde{q} - \tilde{q}^{-1}}.$$

The elements v_n and $\tilde{v}_n$ might be useful when defining the representation, but the above notation is less convenient when we want to let the representation act on an element in $\mathcal{M}$. We therefore rewrite the action of π_λ on the generators of $\mathfrak{D}$ to the following equivalent form:

$$\begin{aligned}
&\pi_\lambda(K) = T^{iw_1}, \qquad \pi_\lambda(K^{-1}) = T^{-iw_1},\\
&\pi_\lambda(E) = \frac{iq^{\frac{1}{2}}}{q-q^{-1}} S^{1/iw_2}\Big(q^{-\frac{1}{2}}e^{\pi\lambda/w_2} + q^{\frac{1}{2}}e^{-\pi\lambda/w_2}T^{iw_1}\Big),\\
&\pi_\lambda(F) = \frac{iq^{\frac{1}{2}}}{q-q^{-1}} S^{-1/iw_2}\Big(q^{-\frac{1}{2}}e^{\pi\lambda/w_2} + q^{\frac{1}{2}}e^{-\pi\lambda/w_2}T^{-iw_1}\Big),\\
&\pi_\lambda(\tilde{K}) = T^{iw_2}, \qquad \pi_\lambda(\tilde{K}^{-1}) = T^{-iw_2},\\
&\pi_\lambda(\tilde{E}) = \frac{i\tilde{q}^{\frac{1}{2}}}{\tilde{q}-\tilde{q}^{-1}} S^{1/iw_1}\Big(\tilde{q}^{-\frac{1}{2}}e^{\pi\lambda/w_1} + \tilde{q}^{\frac{1}{2}}e^{-\pi\lambda/w_1}T^{iw_2}\Big), \quad \text{and}\\
&\pi_\lambda(\tilde{F}) = \frac{i\tilde{q}^{\frac{1}{2}}}{\tilde{q}-\tilde{q}^{-1}} S^{-1/iw_1}\Big(\tilde{q}^{-\frac{1}{2}}e^{\pi\lambda/w_1} + \tilde{q}^{\frac{1}{2}}e^{-\pi\lambda/w_1}T^{-iw_2}\Big),
\end{aligned} \tag{4.1}$$

where $T^y f(x) := f(x+y) = e^{2\pi i y P} f(x)$ and $S^y f(x) = e^{2\pi i x y} f(x) = e^{2\pi i y X} f(x)$, and w_1 and w_2 are defined as above. I leave it to the reader to verify that the representation thus defined coincides with the earlier definition. Note that the definitions on the generators of $U_{\tilde{q}}$ are now obtained by interchanging w_1 and w_2 in the relations that define π_λ in U_q. (Just as $\tilde{q} = e^{i\pi w_2/w_1}$ is obtained from interchanging w_1 and w_2 in $q = e^{i\pi w_1/w_2}$.)

4.3 A sesquilinear form on $\mathcal{M}$

In this section we introduce a sesquilinear form on a subspace of $\mathcal{M}$. Moreover, we relate it to the representation π_λ, showing that the representation is compatible with the star-structures we defined on the modular double. We start by defining some auxiliary concepts and notations.

Let C be a directed contour in $\mathbb{C}$ with $c : \mathbb{R} \to C$ a parametrisation of C. We call C a *deformation of the real line* if

- c is continuous, piecewise differentiable and injective;
- the imaginary part of c is bounded, i.e. $\sup_{x\in\mathbb{R}} |\mathrm{Im}\, c(x)| < \infty$; and
- $\mathrm{Re}\, c(x) \to \infty$ as $x \to \infty$ and $\mathrm{Re}\, c(x) \to -\infty$ if $x \to -\infty$.

For any $b \in \mathbb{C}$, $C + b$ is the deformation of $\mathbb{R}$ parametrised by $c_1(x) := c(x) + b$, the notation $\bar{C}$ refers to the deformation parametrised by $c_2(x) := \overline{c(x)}$ and $-C$ is parametrised by $c_3(x) := -c(-x)$.

Let $f \in \mathcal{M}$ be a meromorphic function. Let $r \in \mathbb{R}$. If for any compact interval $K \subset \mathbb{R}$ there exists some $R > 0$ such that $f(x+iy)$ has no poles in for $|x| \geq R$ and $y \in K$, and $|f(x+iy)| = O(e^{r|x|})$, we say that f has *exponential growth* with rate r.

For any function $f \in C$, we define the conjugate function $\bar{f}$ by $\bar{f}(z) := \overline{f(\bar{z})}$.

Now let C be a deformation of the real line and let $f, g \in \mathcal{M}$ be functions with exponential growths r_f and r_g, with f and $\bar{g}$ having no singularities on C. If $r_f + r_g < 0$, we define $\langle \cdot, \cdot \rangle_C$ by

$$\langle f, g\rangle_C := \int_C f(z)\bar{g}(z)\,\mathrm{d}z. \tag{4.2}$$

Note that the conditions imposed on f and g are sufficient to guarantee convergence of this integral. Restricting $\langle \cdot, \cdot \rangle_C$ to the subspace of $\mathcal{M}$ of functions f with negative exponential growth rates and with f and $\bar{f}$ both having no poles on C, it defines a sesquilinear form, i.e. linear in its first argument and antilinear in the second one.

We call two deformations C and C' *homologous* with respect to the poles of the function f if, intuitively speaking, a pole lies above C in the complex plane if and only if it lies above C'. A more formal definition would be to say that C and C' are homologous with respect to the poles of the function f if both curves have the same winding number with respect to all poles. (This requires e.g. a compactification of $\mathbb{C}$ to turn the deformations of $\mathbb{R}$ into closed contours. We will not go into detail, as we expect the concept of homology to be clear.)

If C and C' are homologous with respect to the poles of f, then $C + s$ and $C' + s$ are homologous with respect to the poles of $f(\cdot - s)$ for any $s \in \mathbb{C}$. If C and C' are homologous with respect to the poles of both f and $\bar{g}$, we have $\langle f, g\rangle_C = \langle f, g\rangle_{C'}$ by Cauchy's integral theorem, provided that both sides of the equation are well-defined.

We use this to relate the sesquilinear form to the star structures on $\mathfrak{D}$:

Lemma 4.1. *Let C be a deformation of the real line. Let $f, g \in \mathcal{M}$ be such that the sum of exponential growth rates of f and g is negative. Suppose that f and $\bar{g}$ have no poles on $C - iw_1$, C and $C + iw_1$, and, moreover, that these three curves are homologous with respect to both f and $\bar{g}$. Then*

$$\langle \pi_\lambda(M) f, g\rangle_C = \begin{cases} \langle f, \pi_{\bar{\lambda}}(M^*) g\rangle_C & \text{if } b > 0, \\ \langle f, \pi_{\bar{\lambda}}(M^\star) g\rangle_C & \text{if } |b| = 1. \end{cases} \tag{4.3}$$

for all $M \in \operatorname{span}\{1, K, K^{-1}, E, F, FK, K^{-1}E\}$.

The above justifies calling M^* (or $M^\star$) the *adjoint* of M.

Proof. Note that π_λ maps $\operatorname{span}\{1, K, K^{-1}, E, F, FK, K^{-1}E\}$ to $\operatorname{span}\{T^{i\xi_1 w_1} S^{\xi_2/iw_2} \mid \xi_1, \xi_2 \in \{-1, 0, 1\}\}$. Let us first relate the latter to the sesquilinear form.

We have

$$\begin{aligned} \langle T^{i\xi_1 w_1} S^{\xi_2/iw_2} f, g\rangle_C &= \int_C e^{2\pi\xi_2/w_2\cdot(z+i\xi_1 w_1)} f(z + i\xi_1 w_1)\overline{g(\bar{z})}\, \mathrm{d}z \\ &= \int_{C+i\xi_1 w_1} e^{2\pi\xi_2/w_2\cdot z} f(z) \overline{g(\overline{z - i\xi_1 w_1})}\, \mathrm{d}z. \end{aligned}$$

We now want to replace the integral domain by C again, by using the Cauchy integral theorem. We thus want to show that C and $C + i\xi_1 w_1$ are homologous with respect to the poles of the integrand. Note that the factor $e^{2\pi\xi_2/\overline{w_2}\cdot z}$ has no poles. By assumption C and $C + i\xi_1 w_1$ are homologous with respect to the poles of f, so it suffices to show that they are homologous with respect to the poles of $\overline{g(\overline{\cdot - i\xi_1 w_1})} = \bar{g}(z - i\xi_1 w_1)$. As we noted, this is true if $C - i\xi_1 w_1$ and

C are homologous with respect to the poles of $\bar{g}$, which is true by assumption. Hence, we have

$$
\begin{aligned}
\langle T^{i\xi_1 w_1} S^{\xi_2/iw_2} f, g\rangle_C &= \int_C e^{2\pi\xi_2/w_2 \cdot z} f(z)\overline{g(\overline{z - i\xi_1 w_1})}\, \mathrm{d}z \\
&= \int_C f(z)\overline{e^{2\pi\xi_2/\overline{w_2}\cdot\bar{z}} g(\bar{z} + i\xi_1 \overline{w_1})}\, \mathrm{d}z \\
&= \int_C f(z)\overline{S^{\xi_2/i\overline{w_2}} T^{i\xi_1\overline{w_1}} g(\bar{z})}\, \mathrm{d}z \\
&= \langle f, S^{\xi_2/i\overline{w_2}} T^{i\xi_1\overline{w_1}} g\rangle \\
&= \langle f, e^{-2\pi i\xi_1\xi_2\overline{w_1}/\overline{w_2}} T^{i\xi_1\overline{w_1}} S^{\xi_2/i\overline{w_2}} g\rangle \qquad \text{(using commutation relations for } S \text{ and } T) \\
&= e^{2\pi i\xi_1\xi_2 w_1/w_2}\langle f, T^{i\xi_1\overline{w_1}} S^{\xi_2/i\overline{w_2}} g\rangle
\end{aligned} \tag{4.4}
$$

For the case $b > 0$, remember that we chose w_1 and w_2 positive, so that the bars on w_1 and w_2 in eq. (4.4) have no effect. We can check that eq. (4.3) holds on the elements in $\{1, K, K^{-1}, E, F, FK, K^{-1}E\}$. For 1, K and K^{-1}, this follows rather directly from eq. (4.4) with $\xi_2 = 0$ and $\xi_1 = 0, 1, -1$, as $(K^{\pm 1})^* = K^{\pm 1}$. We check it for E. We had $E^* = E$, so we will verify whether

$$
\langle \tfrac{i}{q-q^{-1}}\left(e^{\pi\lambda/w_2} S^{1/iw_2} + q e^{-\pi\lambda/w_2} S^{1/iw_2} T^{iw_1}\right) f, g\rangle_C \stackrel{?}{=} \langle f, \tfrac{i}{q-q^{-1}}\left(e^{\pi\bar{\lambda}/w_2} S^{1/iw_2} + q e^{-\pi\bar{\lambda}/w_2} S^{1/iw_2} T^{iw_1}\right) g\rangle_C .
$$

Note that $\frac{i}{q-q^{-1}}$ is real for $|q| = 1$, and by sesquilinearity it suffices to check the two equalities

$$
\langle e^{\pi\lambda/w_2} S^{1/iw_2} f, g\rangle_C \stackrel{?}{=} \langle f, e^{\pi\bar{\lambda}/w_2} S^{1/iw_2} g\rangle_C
$$

and

$$
\langle q e^{-\pi\lambda/w_2} S^{1/iw_2} T^{iw_1} f, g\rangle_C \stackrel{?}{=} \langle f, q e^{-\pi\bar{\lambda}/w_2} S^{1/iw_2} T^{iw_1} g\rangle_C .
$$

The first one follows from eq. (4.4) with $\xi_1 = 0$ and $\xi_2 = 1$ after taking the exponentials out of the sesquilinear form. For the second equation, we can rewrite it to

$$
q e^{-\pi\lambda/w_2}\langle S^{1/iw_2} T^{iw_1} f, g\rangle_C \stackrel{?}{=} q^{-1} e^{-\pi\lambda/w_2}\langle f, S^{1/iw_2} T^{iw_1} g\rangle_C .
$$

Noting that $e^{2\pi i w_1 w_2} = q^2$, we see that this is true from the case $\xi_1 = \xi_2 = 1$ in eq. (4.4). Similar reasoning shows that eq. (4.3) is true on F, FK and $K^{-1}E$ in the case $b > 0$.

The case $|b| = 1$ of eq. (4.3) is checked by similar calculations. (Note $|b| = 1$ corresponds to complex conjugate parameters w_1 and $w_2 = \overline{w_1}$.) □

Replacing w_1 by w_2 in the conditions of lemma 4.1, the relation eq. (4.3) holds on the span of $\{1, \tilde{K}, \tilde{K}^{-1}, \tilde{E}, \tilde{F}, \tilde{F}\tilde{K}, \tilde{K}^{-1}\tilde{E}\}$.

Part II

Skew-primitive elements in $\mathfrak{D}$ and their eigenfunctions

So far, we have discussed properties of Hopf algebras, the quantum quantum group $U_q(\mathfrak{sl}(2))$ and its modular double $\mathfrak{D}$, and their representation π_λ. In this second part, we zoom in on certain elements of $\mathfrak{D}$, called the skew-primitive elements. We will define them and study some of their properties in chapter 5. In chapter 6 we will introduce the hyperbolic gamma function, which, in chapter 7 will be used to construct eigenfunctions of the skew-primitive elements under π_λ.

From this point onwards, unless stated otherwise, we assume all vector spaces to be over the field $\mathbb{C}$. We assume the deformation parameter q to be of the form $q = e^{i\pi w_1/w_2}$, with w_1 and w_2 either a pair of positive parameters with $w_1/w_2 \notin \mathbb{Q}$, or a pair of complex conjugate parameters with strictly positive real part and nonzero imaginary parts. Let $w = (w_1 + w_2)/2$ and $\tilde{q} = e^{i\pi w_2/w_1}$.

Chapter 5

Skew-primitive elements

This chapter introduces the notion of skew-primitive elements in the context of coalgebras and their properties in bialgebras. Skew-primitive elements are a generalisation of primitive elements, which play an important role in the representation theory of bialgebras and Hopf algebras. Many properties of the primitive elements carry over to skew-primitive elements, which receive considerable attention in the literature, cf. [26, 35]. We take over definitions and notation from the latter source.

After discussing primitive and skew-primitive elements in the context of general coalgebras and bialgebras, we will introduce two families of skew-primitive elements in $\mathfrak{D}$. We will study the effect of the star-operations on the skew primitive elements and we will relate the two families in a three-term relation.

We will use these specific elements later to derive properties of Ruijsenaars's R-function in part III. Especially, we use properties of skew-primitive elements for deriving a multivariate generalisation of the Ruijsenaars function transform in chapter 11.

5.1 Primitive elements

Let (C, Δ, ϵ) be a co-algebra. A *primitive element* is an element $x \in C$ that satisfies the relation $\Delta x = 1 \otimes x + x \otimes 1$. If we introduce use the inductive notation

$$\Delta^n := (\Delta \otimes \mathrm{id}^{\otimes(n-1)}) \circ \Delta^{n-1} \tag{5.1}$$

with $\Delta^0 = \mathrm{id}$, then for $n \in \mathbb{N}$, a primitive element x satisfies

$$\Delta^n x = 1^{\otimes n} \otimes x + 1^{\otimes(n-1)} \otimes x \otimes 1 + \cdots + x \otimes 1^{\otimes n}.$$

Note that due to coassociativity nothing changes if we apply the coproduct to another factor in eq. (5.1), so that we could equivalently write e.g.

$$\Delta^n = (\mathrm{id}^{\otimes(n-1)} \otimes \Delta) \circ \Delta^{n-1}.$$

In a bialgebra H, the primitive elements can be used to construct a basis for the Lie algebra of the bialgebra, with the Lie bracket defined by the commutator $[x, y] = xy - yx$. Indeed, if x

and y are primitive, then

$$\begin{aligned}\Delta(xy - yx) &= \Delta x\Delta y - \Delta y\Delta x \\ &= (1 \otimes xy + y \otimes x + x \otimes y + xy \otimes 1) - (1 \otimes yx + x \otimes y + y \otimes x + yx \otimes 1) \\ &= 1 \otimes (xy - yx) + (xy - yx) \otimes 1,\end{aligned}$$

so that the commutator of two primitive elements is primitive.

Let ρ_1 be a representation of H on V_1 and ρ_2 a representaion of H on V_2, and let x be a primitive element. If $v_1 \in V_1$ and $v_2 \in V_2$ are eigenvectors of $\rho_1(x)$ and $\rho_2(x)$ respectively, with eigenvalues μ_1 and μ_2 then $v_1 \otimes v_2$ is an eigenvector of $(\rho_1 \otimes \rho_2)(\Delta x)$, as

$$(\rho_1 \otimes \rho_2)(\Delta x) = \rho_1(1)v_1 \otimes \rho_2(x)v_2 + \rho_1(x)v_1 \otimes \rho_2(1)v_2 = (\mu_1 + \mu_2)v_1 \otimes v_2.$$

5.2 Skew-primitive elements

The concept of primitive elements can be generalised via group-like elements. A nonzero element $g \in C$ is called *group-like* if $\Delta g = g \otimes g$. From the counit's defining relations, one deduces that $\epsilon(g) = 1$ if g is group-like. It follows from the defintion of the antipode that group-like elements in a Hopf algebra are invertible. Hence, the group-like elements of a Hopf algebra form a group.

Suppose g and h are both group-like elements. We say that the element $x \in C$ is (g, h)-*primitive* or *skew-primitive* if it satisfies $\Delta x = g \otimes x + x \otimes h$. The coproduct of such skew-primitive elements mimics the convenient form of the coproduct of a (regular) primitive element. Iteratively applying the coproduct shows that

$$\Delta^n x = g^{\otimes n} \otimes x + g^{\otimes(n-1)} \otimes x \otimes h + \cdots + x \otimes h^{\otimes n}. \tag{5.2}$$

We distinguish some special cases of skew-primitive elements. Let $g \in C$ be group-like. We might call $x \in C$ *left g-primitive* or just *left primitive* if it is $(1, g)$-primitive, i.e. $\Delta x = 1 \otimes x + x \otimes g$. The element x is called *right g-primitive* or *right primitive* if it is $(g, 1)$-primitive. In a Hopf algebra, an element x is called a *twisted primitive* element if it is (g, g^{-1})-primitive. Sometimes left or right primitive elements are also referred to as twisted primitives. The concepts are closely related, since if x is (g, g^{-1})-primitive, then $g^{-1}x$ and xg^{-1} are $(1, g^{-2})$-primitive, while gx and xg are $(g^2, 1)$-primitive.

Suppose that the group-like elements form a commutative group. Then the set of all skew-primitive set is a Lie algebra with the Lie bracket defined by the commutator.

5.3 Left- and right-primitive elements in $\mathfrak{D}$

We will first define two families of primitive elements on U_q, each parametrised by a pair of variables. A second pair of families is given by their analogues in $U_{\tilde{q}}$. The skew primitive elements in this chapter are generalisations of the elements discussed in [4].

We start with a family of left K-primitives in U_q. Note that $\Delta E = 1 \otimes E + E \otimes K$, so that E is left K-primitive. Also, FK and $K - 1$ are left K-primitive, as

$$\Delta(FK) = (K^{-1} \otimes F + F \otimes 1)(K \otimes K) = 1 \otimes FK + FK \otimes K$$

and

$$\Delta(K-1) = K \otimes K - 1 \otimes 1 = (K-1) \otimes K + 1 \otimes (K-1).$$

By linearity of the coproduct, any linear combination of E, FK and $K-1$ is also left K-primitive. We set $v_\rho = 2\cosh\frac{2\pi\rho}{w_2}$ and define our family of left K-primitive elements by

$$Y_{u,\rho} := e^{2\pi u/w_2} E + e^{-2\pi u/w_2} q^{-1} FK + \frac{i v_\rho}{q - q^{-1}} (K-1), \tag{5.3}$$

with ρ and u complex parameters.

Similarly, F, $K^{-1}E$ and $K^{-1} - 1$ are right K^{-1}-primitives, so that

$$X_{v,\sigma} := e^{2\pi v/w_2} F + e^{-2\pi v/w_2} q K^{-1} E + \frac{i v_\sigma}{q - q^{-1}} (K^{-1} - 1), \tag{5.4}$$

with $v, \sigma \in \mathbb{C}$ gives a family of right primitive elements.

Both give rise to an equivalent family in $U_{\tilde{q}}$ by adding tildes and interchanging w_1 and w_2. If we set $\tilde{v}_\rho := 2\cosh\frac{2\pi\rho}{w_1}$, the resulting families are given by

$$\tilde{Y}_{u,\rho} := e^{2\pi u/w_1} \tilde{E} + e^{-2\pi u/w_1} \tilde{q}^{-1} \tilde{F}\tilde{K} + \frac{i \tilde{v}_\rho}{\tilde{q} - \tilde{q}^{-1}} (\tilde{K} - 1) \tag{5.5}$$

and

$$\tilde{X}_{v,\sigma} := e^{2\pi v/w_1} \tilde{F} + e^{-2\pi v/w_1} \tilde{q}^{-1} \tilde{K}^{-1} \tilde{E} + \frac{i \tilde{v}_\sigma}{\tilde{q} - \tilde{q}^{-1}} (\tilde{K}^{-1} - 1). \tag{5.6}$$

Let $\mu_\tau^\rho := i\frac{v_\tau - v_\rho}{q - q^{-1}}$ for $\rho, \tau \in \mathbb{C}$. It follows from a direct calculation that

$$Y_{u,\sigma} = Y_{u,\rho} - \mu_\rho^\sigma (K-1), \tag{5.7}$$

and

$$X_{v,\sigma} = X_{v,\rho} - \mu_\rho^\sigma (K^{-1} - 1). \tag{5.8}$$

Tilded versions of these equations hold as well, with $\tilde{\mu}_\tau^\rho := i\frac{\tilde{v}_\tau - \tilde{v}_\rho}{\tilde{q} - \tilde{q}^{-1}}$.

5.4 Adjoints of the skew-primitive elements

Note that the primitive elements $Y_{u,\rho}$ and $X_{v,\sigma}$ are contained in $\mathrm{span}\{1, K, K^{-1}, E, F, FK, K^{-1}E\}$, so that we can use lemma 4.1 to calculate the adjoints of $\pi_\lambda(Y_{u,\rho})$ and $\pi_\lambda(X_{v,\sigma})$ with respect to the sesquilinear form $\langle\cdot,\cdot\rangle_C$. In this section, we will calculate the adjoints of the primitive elements with respect to the star involutions $*$ and $\star$ on $\mathfrak{D}$.

The involution $*$

First consider the involution $*$, which applies if w_1 and w_2 are positive-valued. Recall the star-involution $*$ on $\mathfrak{D}$ defined in eq. (3.6). Let $w = (w_1 + w_2)/2$. As $i/(q - q^{-1})$ is real, we find that

$$\begin{aligned} Y_{u,\rho}^{*} &= e^{2\pi\bar{u}/w_2}E + e^{-2\pi\bar{u}/w_2}qKF + \frac{iv_{\bar{\rho}}}{q-q^{-1}}(K-1) \\ &= e^{2\pi\bar{u}/w_2}E + e^{-2\pi\bar{u}/w_2}q^{-1}FK + \frac{iv_{\bar{\rho}}}{q-q^{-1}}(K-1) \qquad (KF = q^{-2}FK) \\ &= Y_{\bar{u},\bar{\rho}}. \end{aligned}$$

Note that $v_\rho = 2\cosh\frac{2\pi\rho}{w_2}$, so that $v_\rho = v_{\bar{\rho}}$ whenever $\operatorname{Re}\rho = 0$ or $\operatorname{Im}\rho \in \frac{w_2}{2}\mathbb{Z}$. Hence, $Y_{u,\rho}^{*} = Y_{u,\rho}$ if $\rho \in i\mathbb{R} \cup (\mathbb{R} + i\frac{w_2}{2}\mathbb{Z})$ and $u \in \mathbb{R} + iw_2\mathbb{Z}$.

A similar calculation shows that

$$X_{v,\sigma}^{*} = X_{\bar{v},\bar{\sigma}},$$

and we have $X_{v,\sigma}^{*} = X_{v,\sigma}$ if $\sigma \in i\mathbb{R} \cup (\mathbb{R} + i\frac{w_2}{2}\mathbb{Z})$, $v \in \mathbb{R} + iw_2\mathbb{Z}$. Moreover,

$$\tilde{Y}_{u,\rho}^{*} = \tilde{Y}_{\bar{u},\bar{\rho}},$$

and

$$\tilde{X}_{v,\sigma}^{*} = \tilde{X}_{\bar{v},\bar{\sigma}},$$

which are self-adjoint if $\rho \in i\mathbb{R} \cup (\mathbb{R} + i\frac{w_1}{2}\mathbb{Z})$ and $u \in \mathbb{R} + i\frac{w}{2} + iw_1\mathbb{Z}$ and $\sigma \in i\mathbb{R} \cup (\mathbb{R} + i\frac{w_1}{2}\mathbb{Z})$, $v \in \mathbb{R} - i\frac{w}{2} + iw_1\mathbb{Z}$.

We find that $Y_{u,\rho}$ and $\tilde{Y}_{u,\rho}$ are simultaneously self-adjoint if $\rho \in i\mathbb{R} \cup \mathbb{R}$ and $u \in \mathbb{R}$, and similarly for $X_{v,\sigma}$ and $\tilde{X}_{v,\sigma}$ if $\sigma \in i\mathbb{R} \cup \mathbb{R}$, $v \in \mathbb{R}$.

The involution $\star$

For the involution $\star$, corresponding to a pair of complex conjugate parameters w_1 and w_2, similar calculations can be performed, resulting in

$$Y_{u,\rho}^{\star} = \tilde{Y}_{\bar{u},\bar{\rho}}$$

and

$$X_{v,\sigma}^{\star} = \tilde{X}_{\bar{v},\bar{\sigma}},$$

with $u, v, \rho, \sigma \in \mathbb{C}$.

We might construct self-adjoint elements with respect to $\star$, as

$$(Y_{u,\rho} + \tilde{Y}_{u,\rho})^{\star} = \tilde{Y}_{\bar{u},\bar{\rho}} + Y_{\bar{u},\bar{\rho}},$$

so that $Y_{u,\rho} + \tilde{Y}_{u,\rho}$ is self-adjoint if $\rho \in i\mathbb{R} \cup \mathbb{R}$ and $u \in \mathbb{R}$. We can do a similar thing for $X_{v,\sigma} + \tilde{X}_{v,\sigma}$.

5.5 Relating the left- and right primitive elements

We will close this chapter with a relation that expresses $X_{-v,\sigma}$ in terms of products of $Y_{u,\rho}$ and K^{-1}, for arbitrary $u, v, \rho, \sigma \in \mathbb{C}$. We present it as a lemma:

Lemma 5.1. *Let $u, v, \rho, \sigma \in \mathbb{C}$, then*

$$X_{v,\sigma} = a_{u+v} K^{-1} Y_{u,\rho} + a_{-u-v} Y_{u,\rho} K^{-1} + b^{\rho,\sigma}_{u+v}(K^{-1} - 1),$$

with

$$a_x := \frac{q e^{2\pi x/w_2} - q^{-1} e^{-2\pi x/w_2}}{q^2 - q^{-2}} \quad \textit{and} \quad b^{\rho,\sigma}_x := \frac{i v_\rho (a_x + a_{-x})}{q - q^{-1}} + \frac{i v_\sigma}{q - q^{-1}}.$$

Proof. Using $EK^{-1} = q^2 K^{-1} E$, $K^{-1} F K = q^2 F$ and $K^{-1}(K - 1) = -(K^{-1} - 1)$, we find by applying the definition in eq. (5.3) that

$$\begin{aligned} a_{u+v} K^{-1} Y_{u,\rho} + a_{-u-v} Y_{u,\rho} K^{-1} + b^{\rho,\sigma}_{u+v}(K^{-1} - 1) \\ = e^{2\pi u/w_2}(a_{u+v} q^{-1} + a_{-u-v} q) q K^{-1} E \\ + e^{-2\pi u/w_2}(a_{u+v} q + a_{-u-v} q^{-1}) F \\ + \left(-\frac{i v_\rho (a_{u+v} + a_{-u-v})}{q - q^{-1}} + b^{\rho,\sigma}_{u+v} \right)(K^{-1} - 1). \end{aligned}$$

Now

$$\begin{aligned} a_{u+v} q^{-1} + a_{-u-v} q &= \frac{e^{2\pi(u+v)/w_2} - q^{-2} e^{2\pi(-u-v)/w_2} + q^2 e^{2\pi(-u-v)/w_2} - e^{2\pi(u+v)/w_2}}{q^2 - q^{-2}} \\ &= e^{2\pi(-u-v)/w_2} \end{aligned}$$

and flipping the signs of u and v,

$$a_{u+v} q + a_{-u-v} q^{-1} = e^{2\pi(u+v)/w_2}.$$

Moreover,

$$-\frac{i v_\rho (a_{u+v} + a_{-u-v})}{q - q^{-1}} + b^{\rho,\sigma}_{u+v} = \frac{i v_\sigma}{q - q^{-1}},$$

so that by substitution we find

$$\begin{aligned} a_{u+v} K^{-1} Y_{u,\rho} + a_{-u-v} Y_{u,\rho} K^{-1} + b^{\rho,\sigma}_{u+v}(K^{-1} - 1) \\ = e^{2\pi v/w_2} F + e^{-2\pi v/w_2} q K^{-1} E + \frac{i v_\sigma}{q - q^{-1}}(K^{-1} - 1) = X_{v,\sigma}. \end{aligned}$$

□

Chapter 6

The hyperbolic gamma function

The hyperbolic gamma function G is a generalisation of the regular gamma function. It was first introduced in this form by Ruijsenaars in [39] as a minimal solution to a set of three analytic difference equations. Up to a change of parameters, G is equivalent to the generalisation of Hölder's double sine function (cf. [19]) that was presented in [44]. In this latter form, the hyperbolic gamma function has been extensively studied within the field of multiple zeta functions (see [31] for a short overview). The hyperbolic gamma function is also equivalent to the quantum dilogarithm (see e.g. [27, 9]), which has applications in quantum conformal field theory.

In this chapter, we will introduce the hyperbolic gamma function in stages. We will begin by defining G on a strip in the complex plane, studying its analytic properties there. We will then explore its symmetries and derive a pair of difference equations. Using these difference equations, we will extend G to a meromorphic function on the complex plane and identify its zero and pole locations. We will close the chapter studying asymptotics of G and products of G.

Our derivation is mainly based on the derivation in [23]. Additional sources will be explicitly referenced when relevant.

6.1 The hyperbolic gamma function on a strip

We introduce the function G through an auxiliary function g. Consider two parameters w_1 and w_2 with positive real parts, and set $w := (w_1 + w_2)/2$. We define g as follows:

$$g(w_1, w_2; z) := \int_0^\infty \left(\frac{\sin(2yz)}{2\sinh(w_1 y)\sinh(w_2 y)} - \frac{z}{w_1 w_2 y} \right) \frac{\mathrm{d}y}{y}. \tag{6.1}$$

We verify that g is well-defined and analytic for $z \in \mathbb{R} \times i(-\operatorname{Re} w, \operatorname{Re} w)$: the integrand is continuous for $y > 0$. By applying l'Hôpital's rule, we can see that the singularity at $y = 0$ is removable, yielding a value of $-(z(w_1^2 + w_2^2 + 4z^2))/(6w_1 w_2)$ for the integrand at $y = 0$. Thus, the integral over $[0, 1]$ converges. As

$$\left| \frac{\sin(2yz)}{2\sinh(w_1 y)\sinh(w_2 y)} \right| = O(e^{2y(|\operatorname{Im} z| - \operatorname{Re} w)}),$$

one can deduce that the integral over $[1,\infty)$ converges as well, provided $\mathrm{Im}(z) \in (-\mathrm{Re}\, w, \mathrm{Re}\, w)$. Therefore, g is well defined on the said domain. By restricting z to a compact subset of the strip, we can uniformly bound the integrand. Noting that the derivative with respect to z of the integrand is also continuous and has a finite limit for $y \to 0$, we can apply Leibniz integral rule (as stated in [11, lem. II.3.3]), to see that the piecewise integrals

$$g_n(w_1, w_2; z) := \int_n^{n+1} \left(\frac{\sin(2yz)}{2\sinh(w_1 y)\sinh(w_2 y)} - \frac{z}{w_1 w_2 y} \right) \frac{\mathrm{d}y}{y},$$

with $n = 0, 1, 2, \ldots$ all converge to analytic functions of z. By the above, we can bound the g_n's locally uniformly in z, so that by the Weierstrass M-test, $\sum_{n=0}^{\infty} g_n$ converges locally uniformly to g. As the uniform limit of analytic functions is analytic, g is analytic as a function of z on the strip $\mathbb{R} \times i(-\mathrm{Re}\, w, \mathrm{Re}\, w)$.

For z in the same strip , we define the *hyperbolic gamma function* by

$$G(w_1, w_2; z) := e^{ig(w_1, w_2; z)}.$$

It is clear that G is also analytic on the same domain as g. From the definition of g, we can derive the symmetries $g(\bar{w}_1, \bar{w}_2; \bar{z}) = \overline{g(w_1, w_2; z)}$ and $g(w_1, w_2; -z) = -g(w_1, w_2; z)$, which translate to symmetries of G:

$$G(w_1, w_2; -z) = G(w_1, w_2; z)^{-1} = \overline{G(\bar{w}_1, \bar{w}_2; \bar{z})} \tag{6.2}$$

Moreover, $G(w_1, w_2; z) = G(w_2, w_1; z)$, so that whenever $w_1, w_2 > 0$ or $\bar{w}_1 = w_2$, (i.e. corresponding to the values of $q = e^{i\pi w_1/w_2}$ we consider), the latter term in eq. (6.2) equals $\overline{G(w_1, w_2; \bar{z})}$. As we consider w_1 and w_2 to be fixed parameters, we will often use the shorter notation $G(z) = G(w_1, w_2; z)$, and similarly for other functions that depend on w_1 and w_2.

6.2 Difference equations for G

By applying the identity $\sin(a + bi) = \sin(a - bi) + 2i \cos a \sinh b$, we may write

$$\begin{aligned} g(z + i\frac{w_1}{2}) &= \int_0^{\infty} \left(\frac{\sin\left(2y(z + i\frac{w_1}{2})\right)}{2\sinh(w_1 y)\sinh(w_2 y)} - \frac{z + i\frac{w_1}{2}}{w_1 w_2 y} \right) \frac{\mathrm{d}y}{y} \\ &= \int_0^{\infty} \left(\frac{\sin\left(2y(z - i\frac{w_1}{2})\right) + 2i\cos(2yz)\sinh(w_1 y)}{2\sinh(w_1 y)\sinh(w_2 y)} - \frac{(z - i\frac{w_1}{2}) + iw_1}{w_1 w_2 y} \right) \frac{\mathrm{d}y}{y} \\ &= g(z - i\frac{w_1}{2}) + i\int_0^{\infty} \left(\frac{\cos(2yz)}{\sinh(w_2 y)} - \frac{1}{w_2 y} \right) \frac{\mathrm{d}y}{y}, \end{aligned}$$

provided that the integral in the last step is convergent. As suggested in [23], we split it into

$$\int_0^{\infty} \left(\frac{\cos(2yz)}{\sinh(w_2 y)} - \frac{1}{w_2 y} \right) \frac{\mathrm{d}y}{y} = \int_0^{\infty} \left(\frac{1}{\sinh y} - \frac{1}{y} \right) \frac{\mathrm{d}y}{y} + \int_0^{\infty} \frac{\cos\left(\frac{2yz}{w_2}\right) - 1}{\sinh y} \frac{\mathrm{d}y}{y},$$

which equals $-\log\left(2\cosh\frac{\pi z}{w_2}\right)$ whenever $|z| < \left|\frac{w_2}{2}\right|$, cf. [14, nrs. 3.529.1-2], so that

$$g(z+i\frac{w_1}{2}) = g(z-i\frac{w_1}{2}) - i\log\left(2\cosh\frac{\pi z}{w_2}\right).$$

For $|z| < \left|\frac{w_2}{2}\right|$, we use the above g-difference equation to obtain

$$\frac{G(z+i\frac{w_1}{2})}{G(z-i\frac{w_1}{2})} = 2\cosh\frac{\pi z}{w_2}, \tag{6.3}$$

and by interchanging w_1 and w_2, we find for $|z| < \left|\frac{w_1}{2}\right|$

$$\frac{G(z+i\frac{w_2}{2})}{G(z-i\frac{w_2}{2})} = 2\cosh\frac{\pi z}{w_1}. \tag{6.4}$$

Remark 6.1. By eq. (6.2), the equations eqs. (6.3) and (6.4) can be rewritten into

$$G(-i\tfrac{w_1}{2}+z)G(-i\tfrac{w_1}{2}-z) = \frac{1}{2}\operatorname{sech}\frac{\pi z}{w_2}$$

and

$$G(-i\tfrac{w_2}{2}+z)G(-i\tfrac{w_2}{2}-z) = \frac{1}{2}\operatorname{sech}\frac{\pi z}{w_1}.$$

One could regard this pair of difference equations as the hyperbolic analogue of the Γ-difference equation

$$\Gamma(\tfrac{1}{2}+x)\Gamma(\tfrac{1}{2}-x) = \pi\sec\pi x$$

for $x+\frac{1}{2}\notin\mathbb{Z}$. ∎

6.3 Meromorphic extension to $\mathbb{C}$

The difference equations eqs. (6.3) and (6.4) allow us to extend the hyperbolic gamma function G to a meromorphic function on $\mathbb{C}$ through analytic continuation. By the continuity of g on $\mathbb{R}\times i(-\operatorname{Re} w, \operatorname{Re} w)$, the function G has no poles or zeros on this strip. From the difference equations, we can conclude that the extension, which will also call G, is analytic on the upper half-plane and nonzero on the lower half-plane.

The zeros of G are contained in the set

$$Z_+ = i\{w + kw_1 + lw_2 \mid k,l = 0,1,2,\ldots\}. \tag{6.5}$$

The poles of G are contained in the lower half-plane, in the set $Z_- = -Z_+$. The pole at $-iw$ is simple, and if $\frac{w_1}{w_2}\notin\mathbb{Q}$, all poles are simple, and their residues can be computed using the difference equations.

We will explicitly compute the residue at $-iw$: Using the difference equation eq. (6.3) to derive the second equality below, we have

$$G(z - iw) = G(z - i\frac{w_1}{2} - i\frac{w_2}{2}) = \frac{G(z + i\frac{w_1}{2} - i\frac{w_2}{2})}{2\cosh\frac{\pi(z - i\frac{w_2}{2})}{w_2}} = i\frac{G(z + i\frac{w_1}{2} - i\frac{w_2}{2})}{2\sinh\frac{\pi z}{w_2}},$$

so that

$$\mathrm{Res}(G; -iw) = \frac{iw_2}{2\pi} G(i\frac{w_1}{2} - i\frac{w_2}{2}).$$

Interchanging the roles of w_1 and w_2, we similarly find

$$\mathrm{Res}(G; -iw) = \frac{iw_1}{2\pi} G(i\frac{w_2}{2} - i\frac{w_1}{2}).$$

Equating the two, and noting that $G(i\frac{w_2}{2} - i\frac{w_1}{2}) = 1/G(i\frac{w_1}{2} - i\frac{w_2}{2})$ by eq. (6.2), we find

$$G(i\frac{w_1}{2} - i\frac{w_2}{2})^2 = \frac{w_1}{w_2},$$

which gives our explicit expression for the residue:

$$\mathrm{Res}(G; -iw) = \frac{i}{2\pi}\sqrt{w_1 w_2}. \tag{6.6}$$

The set of zeros Z_+ is contained in the sector $S_+ := \{iw + iz \mid \arg z \in [\arg w_1, \arg w_2]\}$, and the poles are contained in the sector $S_- := -S_+$. As $\mathbb{C} \setminus (S_+ \cup S_-)$ is simply connected and contains the strip $\mathbb{Z} \times i(-\mathrm{Re}\, w, \mathrm{Re}\, w)$, and G is nonzero and analytic on $\mathbb{C} \setminus (S_+ \cup S_-)$, we can extend g to an analytic function on this area by setting $g(z) = -i \log G(z)$, with its branch chosen to agree with the original definition of g.

6.4 Asymptotics of G

We will describe the limiting behaviour of $G(z)$ as $|\mathrm{Re}\, z| \to \infty$ using estimates due to Ruijsenaars. These asymptotics can be given for arbitrary w_1 and w_2 with positive real part. Restricting ourselves to a pair of either positive or complex conjugate parameters w_1 and w_2, notation simplifies considerably, and we limit our study to theses cases.

While discussing the asymptotics, we use the symbol C to denote any bounding constant. If the symbol appears in subsequent equations, it need not resemble the same constant, but just any positive finite number, chosen independently of the variable z.

Let $w = (w_1 + w_2)/2$. In case of positive parameters, let $w_0 := \min\{w_1, w_2\}$. In the conjugate case, let $w_0 := w = \mathrm{Re}\, w_1$. Let $\alpha := \frac{2\pi}{w_1 w_2}$ and fix $\varsigma \in (0, 1)$.

We define functions

$$\tilde{g}(z) := -\frac{\alpha}{4} z^2 - \frac{\alpha}{48}(w_1^2 + w_2^2).$$

and

$$f(z) := g(z) - \tilde{g}(z)$$

for $z \in \mathbb{C} \setminus (S_+ \cup S_-)$. Then

$$G(z) = e^{i\tilde{g}(z) + if(z)}.$$

As g is analytic on $\mathbb{C} \setminus (S_+ \cup S_-)$, it follows that f is analytic on that domain as well.

For $\epsilon > 0$, let S_+^ϵ denote the set of points in $\mathbb{C}$ with Euclidian distance less than ϵ to S_+ and let $S_-^\epsilon = -S_+^\epsilon$. See fig. 6.1 for an illustration of S_+^ϵ and S_-^ϵ.

The following lemma describes the asymptotics of G:

Lemma 6.2. *Let $K \subset \mathbb{R}$ be compact and $\epsilon > 0$. There exists a positive constant C such that*

$$\left|f(w_1, w_2; z)\right| < Ce^{-\alpha\varsigma w_0 \operatorname{Re} z}$$

for all $z \notin (S_+^\epsilon \cup S_-^\epsilon)$ with $\operatorname{Im} z \in K$ and $\operatorname{Re} z \geq 0$.

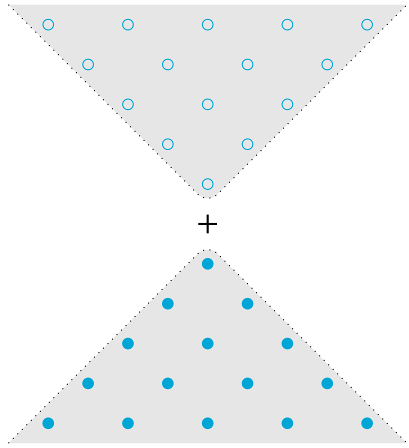


Figure 6.1: This figure illustrates the complex plane, showing the locations of the zeros and poles of G for conjugate parameters w_1 and w_2. The origin is marked with a black $+$. The open dots represent the positions of the zeros in Z_+ closest to the real line (corresponding to $k + l \leq 4$ in eq. (6.5)). The solid dots indicate the positions of the poles in $\mathbb{Z}_-$ closest to the real line. The shaded region represents $S_+^\epsilon \cup S_-^\epsilon$ for some small value of ϵ. In the case of positive w_1 and w_2, Z_+ and Z_- are contained in halflines on the imaginary axis.

The conditions imposed on z ensure that it stays away a distance of at least ϵ from the zeros and poles of G, which correspond to singularities of f. These conditions allow, for example, for z on the real halfline $[\epsilon, \infty)$. In general, if K is a compact interval, there exists some $R > 0$ such that the conditions also allow for z in the halfstrip $[R, \infty) \times iK$. Ruijsenaars proved a slightly stronger version of this lemma ([36, thm. A.1]). The proof of the lemma took him eight pages ([36, app. C]), and we are not going to reproduce it here. It should be noted that for positive parameters w_1 and w_2, our lemma equals [39, prop. III.4], which admits a considerably simpler proof. Such a simpler proof might also exist in the case of conjugate parameters.

By lemma 6.2, we can see that $|f(z)|$ approaches 0 as $\operatorname{Re} z \to \infty$. Since $e^x = 1 + O(|x|)$ for x around 0, we can derive the following asymptotic expression for the hyperbolic gamma function:

$$G(z) = e^{i\tilde{g}(z)}e^{if(z)} = e^{i\tilde{g}(z)}(1 + O(e^{-\alpha\varsigma w_0|\operatorname{Re} z|})) \tag{6.7}$$

for $\operatorname{Re} z \to \infty$ and $\operatorname{Im} z$ restricted to a compact interval. Since

$$G(-z) = 1/G(z) = e^{-i\tilde{g}(z)}e^{-if(z)},$$

similar bounds exist for $\operatorname{Re} z \to -\infty$ whenever $\operatorname{Im} z$ is restricted to a compact interval.

As we have $\left|e^{if(z)}\right| \in (e^{-C}, e^C)$ when z is restricted as in the lemma, we may write

$$e^{-C} < \left|G(z)/e^{i\tilde{g}(z)}\right| < e^C$$

for $z \in \mathbb{C} \setminus (S_+^\epsilon \cup S_-^\epsilon)$ with $\operatorname{Im} z$ contained in some compact interval and $\operatorname{Re} z$ positive. For $\operatorname{Re} z$ negative, we similarly have

$$e^{-C} < \left|G(z)/e^{-i\tilde{g}(-z)}\right| < e^C.$$

Using the definition of $\tilde{g}$, we can collect these results into

$$0 < e^{-C} < |G(z)| \cdot e^{-\alpha \operatorname{Im} z |\operatorname{Re} z|/2} < e^{C}. \tag{6.8}$$

If we restrict the imaginary part of z to a compact set, then for $a, b \in \mathbb{C}$ and some R large enough, we can use eq. (6.7) derive the following asymptotic expression for the ratio of the hyperbolic gamma function at two different arguments:

$$\frac{G(z-a)}{G(z-b)} = \exp\Big(-i\frac{\alpha}{2} z \cdot (b-a) + i\frac{\alpha}{4}(b^2 - a^2)\Big) \cdot (1 + O(e^{-\alpha\varsigma w_0 \operatorname{Re} z})), \tag{6.9}$$

and by analyticity of f, the remaining $O(e^{-\alpha\varsigma w_0 \operatorname{Re} z})$-term is uniform for a and b restricted to a compact domain.

We can write

$$\left|\frac{G(z-a)}{G(z-b)}\right| / \exp(-\alpha \operatorname{Im} z \operatorname{Re}(b-a)/2 - \alpha \operatorname{Im}(b-a) \operatorname{Re} z/2) = (1 + O(e^{-\alpha\varsigma w_0 \operatorname{Re} z}))$$

for $\operatorname{Re} z \geq R$. For $\operatorname{Re} z \to -\infty$, similar estimates can be made, so that we conclude

$$0 < C_- < \left|\frac{G(z-a)}{G(z-b)}\right| e^{-\alpha \operatorname{Im}(b-a)|\operatorname{Re} z|/2} < C_+, \tag{6.10}$$

for some positive constants C_-, C_+ and $|\operatorname{Re} z|$ large enough.

We introduce the notation $F(x \pm z) := F(x+z)F(x-z)$ for whenever a $\pm$-sign appears in the argument of a function. As a special case of eq. (6.9), let $a = -b = -iy$ for some $y \in \mathbb{C}$, then

$$G(iy \pm z) = \exp(\alpha y z) \cdot (1 + O(e^{-\alpha\varsigma w_0 |\operatorname{Re} z|})) \tag{6.11}$$

and

$$0 < C_- < |G(iy \pm z)| e^{-\alpha \operatorname{Re} y |\operatorname{Re} z|} < C_+ \tag{6.12}$$

if $\operatorname{Im} z$ is restricted to some compact set and $\operatorname{Re} z \to \pm\infty$, with all bounds uniform with respect to y if it is restricted to a compact set.

Chapter 7

Eigenfunctions of skew-primitives

We conclude part II with a brief chapter in which we calculate the eigenfunctions of $\pi_\lambda(Y_{u,\rho})$ and $\pi_\lambda(X_{v,\sigma})$.

7.1 Eigenfunctions of $\pi_\lambda(Y_{u,\rho})$

Suppose f is an eigenfunction of $\pi_\lambda(Y_{u,\rho})$ with eigenvalue $\mu_\tau^\rho = i\frac{v_\tau - v_\rho}{q-q^{-1}}$. We closely follow the calculation of eigenfunctions in [4]. We can calculate that

$$\begin{aligned}
\pi_\lambda(Y_{u,\rho})f(z) &= e^{2\pi u/w_2}\pi_\lambda(E)f(z) + e^{-2\pi u/w_2}q^{-1}\pi_\lambda(FK)f(z) + \frac{iv_\rho}{q-q^{-1}}\pi_\lambda((K-1))f(z) \\
&= \frac{iq^{\frac{1}{2}}e^{2\pi u/w_2}}{q-q^{-1}}e^{2\pi z/w_2}\left(q^{-\frac{1}{2}}e^{\pi\lambda/w_2}f(z) + q^{\frac{1}{2}}e^{-\pi\lambda/w_2}f(z+iw_1)\right) \\
&\quad + \frac{iq^{-\frac{1}{2}}e^{-2\pi u/w_2}}{q-q^{-1}}e^{-2\pi z/w_2}\left(q^{-\frac{1}{2}}e^{\pi\lambda/w_2}f(z+iw_1) + q^{\frac{1}{2}}e^{-\pi\lambda/w_2}f(z)\right) \\
&\qquad + \frac{iv_\rho}{q-q^{-1}}f(z+iw_1) - \frac{iv_\rho}{q-q^{-1}}f(z) \\
&= \frac{i}{q-q^{-1}}\Bigg[\left(-v_\rho + 2\cosh\left(2\pi u/w_2 + 2\pi z/w_2 + \pi\lambda/w_2\right)\right)f(z) \\
&\quad + \left(v_\rho + 2\cosh\left(2\pi u/w_2 + 2\pi z/w_2 + i\pi\frac{w_1}{w_2} - \pi\lambda/w_2\right)\right)f(z+iw_1)\Bigg],
\end{aligned}$$

If we now require $\pi_\lambda(Y_{u,\rho})f = \mu_\tau^\rho f$, we can rewrite the above to

$$\begin{aligned}
f(z+iw_1) &= \frac{v_\tau - 2\cosh\left(2\pi u/w_2 + 2\pi z/w_2 + \pi\lambda/w_2\right)}{v_\rho + 2\cosh\left(2\pi u/w_2 + 2\pi z/w_2 + i\pi\frac{w_1}{w_2} - \pi\lambda/w_2\right)}f(z) \\
&= \frac{v_\tau + 2\cosh\left(2\pi u/w_2 + 2\pi z/w_2 + \pi\lambda/w_2 - i\pi\right)}{v_\rho + 2\cosh\left(2\pi u/w_2 + 2\pi z/w_2 + i\pi\frac{w_1}{w_2} - \pi\lambda/w_2\right)}f(z).
\end{aligned}$$

Using $v_x = 2\cosh 2\pi x/\omega_2$, and the sum rule for hyperbolic cosines[1], we can rewrite this to

$$f(z+i\omega_1) = \frac{\cosh\frac{\pi}{\omega_2}\left(z+u+\frac{\lambda}{2}-i\frac{\omega_2}{2}\pm\tau\right)}{\cosh\frac{\pi}{\omega_2}\left(z+u-\frac{\lambda}{2}+i\frac{\omega_1}{2}\pm\rho\right)} f(z).$$

Replacing z by $z - i\frac{\omega_1}{2}$, we find

$$f(z+i\frac{\omega_1}{2}) = \frac{\cosh\frac{\pi}{\omega_2}\left(z+u+\frac{\lambda}{2}-i\omega\pm\tau\right)}{\cosh\frac{\pi}{\omega_2}\left(z+u-\frac{\lambda}{2}\pm\rho\right)} f(z-i\frac{\omega_1}{2}),$$

where we use convention to write $F(x \pm z) = F(x+z)F(x-z)$ for any function F.

From the difference equation for the hyperbolic gamma function, we see that this requirement is satisfied by

$$f(z) = \frac{G\left(z+u-i\omega+\frac{\lambda}{2}\pm\tau\right)}{G\left(z+u-\frac{\lambda}{2}\pm\rho\right)},$$

so that f is an eigenfunction of $\pi_\lambda(Y_{u,\rho})$ with eigenvalue μ_τ^ρ.

As G is invariant under interchanging ω_1 and ω_2, we see that

$$H_{\tau,\rho}^{\lambda,u}(z) := \frac{G\left(z+u-i\omega+\frac{\lambda}{2}\pm\tau\right)}{G\left(z+u-\frac{\lambda}{2}\pm\rho\right)} \tag{7.1}$$

is simultaneously an eigenfunction of $Y_{u,\rho}$ and $\tilde{Y}_{u,\rho}$, with $\pi_\lambda(\tilde{Y}_{u,\rho})H_{\tau,\rho}^{\lambda,u} = \tilde{\mu}_\tau^\rho H_{\tau,\rho}^{\lambda,u}$.

Let $H_{\tau,\rho}^{\lambda}(z) = H_{\tau,\rho}^{\lambda,0}(z)$, and observe that

$$H_{\tau,\rho}^{\lambda,u}(z) = T^u H_{\tau,\rho}^{\lambda}(z). \tag{7.2}$$

One could also check that $\pi_\lambda(Y_{u,\rho}) = T^u\pi_\lambda(Y_\rho)T^{-u}$.

7.2 The symmetry θ and the representation π_λ

We're planning to calculate eigenfunctions of our families of left and right primitive elements under the action of π_λ. Before doing so, we discuss a symmetry between the two families, and show how it relates to the representation. This will save us a lot of effort when calculating the eigenfunctions.

We can relate the two families of twisted primitive elements by introducing an involutive algebra automorphism θ, which we define on the generators of U_q by

$$\theta(K) = K^{-1} \quad \text{and} \quad \theta(E) = F.$$

Using the map θ, one easily calculates from eqs. (5.3) and (5.4) that

$$\theta(Y_{u,\rho}) = X_{u,\rho}. \tag{7.3}$$

Define the operator R on $\mathcal{M}$ to be the reflection operator $Rf(z) := f(i\omega - z)$ for $f \in \mathcal{M}$.

[1]The sum rule reads $\cosh a + \cosh b = 2\cosh\frac{a+b}{2}\cosh\frac{a-b}{2}$.

Then

$$RT^{iw_1}Rf(z) = R[z' \mapsto f(-(z'+iw_1))](z) = f(z-iw_1) = T^{-iw_1}f(z),$$

and

$$RS^{1/iw_2}Rf(z) = R[z' \mapsto e^{2\pi z'/w_2}f(-z')](z) = e^{-2\pi z/w_2}f(z) = S^{-1/iw_2}f(z).$$

Using these relations, one easily checks on eq. (4.1) that

$$\pi_\lambda(\theta(\,\cdot\,)) = R\circ \pi_\lambda(\,\cdot\,)\circ R \tag{7.4}$$

holds on the generators of U_q. Clearly eq. (7.4) extends to an algebra homomorphism and hence it holds on the algebra. In a similar way one checks that this relation also holds on $U_{\tilde q}$ and hence on D. We conclude that if f is an eigenfunction of $\pi_\lambda(Y)$ for some $Y \in \mathfrak{D}$, then Rf is an eigenfunction of $\pi_\lambda(\theta(Y))$ with the same eigenvalue.

7.3 Eigenfunctions of $\pi_\lambda(X_{\nu,\sigma})$

By using eqs. (7.3) and (7.4), we find that $RH^{\lambda,\nu}_{\sigma,\upsilon}$ is an eigenfunction of $\pi_\lambda(X_{\nu,\sigma})$ with eigenvalue μ^σ_υ, and of $\pi_\lambda(\tilde X_{\nu,\sigma})$ with eigenvalue $\tilde\mu^\sigma_\upsilon$. We define $F^{\lambda,\nu}_{\sigma,\upsilon} = RH^{\bar\lambda,\bar\nu}_{\bar\sigma,\bar\upsilon}$, so that

$$\pi_{\bar\lambda}(X^*_{\nu,\sigma})F^{\lambda,\nu}_{\sigma,\upsilon} = \mu^{\bar\sigma}_{\bar\upsilon}F^{\lambda,\nu}_{\sigma,\upsilon} = \overline{\mu^\sigma_\upsilon}F^{\lambda,\nu}_{\sigma,\upsilon} \tag{7.5a}$$

in the case of positive w_1 and w_2. For a conjugate pair w_1 and w_2, we similarly have

$$\pi_{\bar\lambda}(X^\star_{\nu,\sigma})F^{\lambda,\nu}_{\sigma,\upsilon} = \tilde\mu^{\bar\sigma}_{\bar\upsilon}F^{\lambda,\nu}_{\sigma,\upsilon} = \overline{\mu^\sigma_\upsilon}F^{\lambda,\nu}_{\sigma,\upsilon} \tag{7.5b}$$

We may explicitly write

$$F^{\lambda,\nu}_{\sigma,\upsilon}(z) = \frac{G(-z+\bar\nu-iw+\frac{\bar\lambda}{2}\pm\bar\upsilon)}{G(-z+\bar\nu-\frac{\bar\lambda}{2}\pm\bar\sigma)}.$$

For future reference, note that by eq. (6.2)

$$\overline{F^{\lambda,\nu}_{\sigma,\upsilon}(\bar z)} = \overline{\left(\frac{G(-\bar z+\bar\nu-iw+\frac{\bar\lambda}{2}\pm\bar\upsilon)}{G(-\bar z+\bar\nu-\frac{\bar\lambda}{2}\pm\bar\sigma)}\right)} = \frac{G(z-\nu-iw-\frac{\lambda}{2}\pm\upsilon)}{G(z-\nu+\frac{\lambda}{2}\pm\sigma)}. \tag{7.6}$$

This introduction of eigenfunctions of the skew-primitive elements closes this chapter and the second part of this thesis. At this point, we have laid the foundation for the rest of this text. In the next part, we will start the construction of our version of Ruijsenaars's generalised hypergeometric function.

Part III

The function ψ

In this third part, we use the eigenfunctions $H_{\rho,\tau}^{\lambda,u}$ and $F_{\sigma,\upsilon}^{\lambda,\upsilon}$ to define a function ψ in chapter 8, where we also study its symmetric and analytic properties, relating it to Ruijsenaars's functions R, R_{ren} and $\mathcal{E}$. In chapter 9 we show that the function solves a set of Askey-Wilson difference equations. In chapter 10 we show that ψ and $\mathcal{E}$ can be used as kernels of Hilbert space isomorphisms, the $\mathcal{E}$-based transformation being unitary. In Chapter 11, lastly, we derive the multivariate versions of ψ and $\mathcal{E}$ and extend the difference equations and the unitary transformation to these multivariate versions.

Chapter 8

Definition and basic properties of ψ

This chapter introduces the function ψ, defined as an integral over a product of the eigenfunctions $H^{\lambda,u}_{\rho,\tau}$ and $F^{\lambda,v}_{\sigma,\upsilon}$, which were introduced in the previous part. We relate this function ψ to the generalised hypergeometric function R, defined by Ruijsenaars in [40] and extensively studied in [36, 37, 38], as a generalisation of the ${}_2F_1$-functions. We will prove the analyticity of ψ and derive its asymptotics.

8.1 The function ψ

Recall the sesquilinear form we defined in section 4.3. By substituting $a \leftarrow -u + iw - \frac{\lambda}{2} \pm \tau$ and $b \leftarrow -u + \frac{\lambda}{2} \pm \rho$ into eq. (6.10), we find that the function $H^{\lambda,u}_{\rho,\tau}$ defined in eq. (7.1) exhibits exponential growth with a growth rate of $\alpha(\operatorname{Im}\lambda - w)$. Similarly, the function $F^{\lambda,v}_{\sigma,\upsilon}$ has a growth rate of $\alpha(-\operatorname{Im}\lambda - w)$. Consequently, the product $H^{\lambda,u}_{\rho,\tau}\overline{F^{\lambda,v}_{\sigma,\upsilon}}$ has exponential growth with rate $-2\alpha w$.

Let u, λ, ρ, σ, τ and υ be complex parameters. The product $H^{\lambda,u}_{\rho,\tau}\overline{F^{\lambda,0}_{\sigma,\upsilon}}$ possesses poles located in the sets

$$Z_- - u + iw - \frac{\lambda}{2} \pm \tau \quad \text{and} \quad Z_- + iw + \frac{\lambda}{2} \pm \upsilon, \tag{8.1}$$

which we refer to as the *downward pole sequences*. (Their elements have imaginary parts bounded from above but not from below.) Its other poles are contained in

$$Z_+ - u + \frac{\lambda}{2} \pm \rho \quad \text{and} \quad Z_+ - \frac{\lambda}{2} \pm \sigma, \tag{8.2}$$

which we call the *upward pole sequences*.

Let C denote a deformation of the real line that separates the upward and downward pole sequences of $H^{\lambda,u}_{\rho,\tau}\overline{F^{\lambda,0}_{\sigma,\upsilon}}$. We define

$$\psi^{\lambda,u}_{\rho,\sigma}(\tau,\upsilon) := \langle H^{\lambda,u}_{\tau,\rho}, F^{\lambda,0}_{\upsilon,\sigma}\rangle_C. \tag{8.3}$$

Such a curve C exists whenever the upward and downward pole sequences are disjoint as sets. This is true, for example, if $\rho,\sigma,\tau,\upsilon \in \mathbb{R}$ and $|\operatorname{Im} u|, |\operatorname{Im}\lambda| < \operatorname{Re} w$, as can be seen from eqs. (8.1) and (8.2). As a consequence of the Cauchy integral theorem, the value of $\psi^{\lambda,u}_{\rho,\sigma}(\tau,\upsilon)$ does not depend on the specific choice of C (as long as it separates the poles correctly).

We can express $\psi^{\lambda,u}_{\rho,\sigma}$ as an integral, using eq. (7.6):

$$\psi^{\lambda,u}_{\rho,\sigma}(\tau,\upsilon) = \int_C \frac{G(z+u-iw+\frac{\lambda}{2}\pm\tau)}{G(z+u-\frac{\lambda}{2}\pm\rho)} \frac{G(z-iw-\frac{\lambda}{2}\pm\upsilon)}{G(z+\frac{\lambda}{2}\pm\sigma)} \mathrm{d}z. \tag{8.4}$$

We can use this integral expression to deduce certain symmetries: Firstly, by substituting $z' = z + u$, we obtain from the integral that

$$\psi^{\lambda,u}_{\rho,\sigma}(\tau,\upsilon) = \psi^{-\lambda,-u}_{\sigma,\rho}(\upsilon,\tau). \tag{8.5}$$

Additionally, by performing the substitution $z' = z + \frac{u}{2} - \frac{\lambda}{2}$ in the integral, we find

$$\psi^{\lambda,u}_{\rho,\sigma}(\tau,\upsilon) = \psi^{u,\lambda}_{\sigma,\rho}(\tau,\upsilon). \tag{8.6}$$

Combining this with eq. (8.5), we obtain

$$\psi^{\lambda,u}_{\rho,\sigma}(\tau,\upsilon) = \psi^{-u,-\lambda}_{\rho,\sigma}(\upsilon,\tau). \tag{8.7}$$

Lastly, we consider the complex conjugate of ψ, and find

$$\begin{aligned}\overline{\psi^{\lambda,u}_{\rho,\sigma}(\tau,\upsilon)} &= \overline{\int_C H^{\lambda,0}_{\rho,\tau}(z+u)\overline{H^{\bar\lambda,0}_{\bar\sigma,\bar\upsilon}(-\bar z)}\,\mathrm{d}z} \\ &= \int_C \overline{H^{\lambda,0}_{\rho,\tau}(z+u)}H^{\bar\lambda,0}_{\bar\sigma,\bar\upsilon}(-\bar z)\,\mathrm{d}z.\end{aligned}$$

By performing the change of variables $z' = -\bar z - \bar u$, the above expression becomes

$$\begin{aligned} &= \int_{-\bar C-\bar u} \overline{H^{\lambda,0}_{\rho,\tau}(-\overline{z'})}H^{\bar\lambda,0}_{\bar\sigma,\bar\upsilon}(z'+\bar u)\,\mathrm{d}z' \\ &= \psi^{\bar\lambda,\bar u}_{\bar\sigma,\bar\rho}(\bar\upsilon,\bar\tau) = \psi^{-\bar\lambda,-\bar u}_{\bar\rho,\bar\sigma}(\bar\tau,\bar\upsilon), \end{aligned} \tag{8.8}$$

where the latter equality follows from eq. (8.5).

One should note that in each of the above symmetries, each shifted integral domain is again a curve separating the upward and downward pole sequences.

Remark 8.1. In eq. (8.3), we intentionally set the parameter v of $F^{\lambda,v}_{\sigma,\upsilon}$ to be equal to 0. This choice was made because $\langle H^{\lambda,u}_{\tau,\rho}, F^{\lambda,v}_{\upsilon,\sigma}\rangle_C$ would depend on u and v only in the combination $u+v$. ∎

Remark 8.2. Equation (8.6) highlights a duality between the parameters u and λ. Our derivation however does not allow for an algebraic interpretation for this duality. The parameter λ is associated to the representation, the parameter u is not related to any algebra operation.

The derivation of ψ as presented in [4] is highly similar to ours, but uses the notion of an extended modular double, defined as the crossed product of the modular double with an algebra of complex powers of K and $\tilde K$. We can regard the parameter u as the exponent of such a complex power of K. This might allow for an algebraic interpretation, but we did not pursue this. ∎

8.2 Ruijsenaars's generalised hypergeometric function

Our function ψ bears very close resemblance to three closely related functions studied by Ruijsenaars in his triptych on a generalised hypergeometric function ([36, 37, 38]). In this section, we will introduce the functions studied by Ruijsenaars and show how ψ relates to each of these functions.

Let $\gamma = (\gamma_0, \gamma_1, \gamma_2, \gamma_3) \in \mathbb{R}^4$, and let w_1 and w_2 have positive real parts, and let $w = (w_1 + w_2)/2$ as usual. We define the action of $\hat{}$ on $\mathbb{R}^4$ by

$$\hat{\gamma} := \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}\gamma.$$

We also introduce the map $\mathbf{c}$, defined by

$$\mathbf{c}(\gamma) := (\gamma_0 + w, \gamma_1 + \frac{w_1}{2}, \gamma_2 + \frac{w_2}{2}, \gamma_3). \tag{8.9}$$

Ruijsenaars's function R can be written in terms of those maps as

$$R(w_1, w_2, \mathbf{c}(\gamma); \tau, v) = \frac{1}{\sqrt{w_1 w_2}} \int_C \frac{1}{G(z + iw)} \frac{G(z \pm \tau + i\gamma_0)}{G(\pm\tau + i\gamma_0)} \frac{G(z \pm v + i\hat{\gamma}_0)}{G(\pm v + i\hat{\gamma}_0)} \prod_{j=1}^{3} \frac{G(i(w + \gamma_0 + \gamma_j))}{G(z + i(w + \gamma_0 + \gamma_j))} \mathrm{d}z, \tag{8.10}$$

with C again a deformation of the real line that separates the upward and downward pole sequences of the integrand. In Ruijsenaars's original definition of R in [40], the function depended on $c \in \mathbb{R}^4$. However, when studying the symmetries of R in [37] and its properties as the kernel of a unitary map in [38], Ruijsenaars started using the shifted parameter set γ, and the map $\mathbf{c}$ to relate the two. Thus the appearance of $\mathbf{c}(\gamma)$ in the arguments of R is a remnant of his original formulation.

Although originally defined for γ (or c) in $\mathbb{R}^4$, the function was extended by Ruijsenaars to a meromorphic function for $\gamma \in \mathbb{C}^4$ and w_1, w_2 complex with positive real parts. We can express R in terms of our function ψ by setting $\lambda, u, \rho, \sigma \in \mathbb{C}^4$ and considering the parameter relation

$$\gamma = -i(\rho + \lambda, \rho - \lambda, \sigma - u, -\sigma - u). \tag{8.11}$$

(This gives $\hat{\gamma} = -i(\rho - u, \rho + u, \sigma + \lambda, -\sigma + \lambda)$, so that $\hat{}$ acts on γ by interchanging λ with $-u$ and vice versa.) With this parameter relation for γ, we can write

$$R(w_1, w_2, \mathbf{c}(\gamma); \tau, v) = \frac{1}{\sqrt{w_1 w_2}} \frac{G(iw + 2\rho)G(iw + \rho + \lambda \pm \sigma - u)}{G(\pm\tau + \rho + \lambda)G(\pm v + \rho - u)} \times \int_C \frac{G(z \pm \tau + \rho + \lambda)G(z \pm v + \rho - u)}{G(z + iw)G(z + iw + 2\rho)G(z + iw + \rho + \lambda \pm \sigma - u)} \mathrm{d}z.$$

By substituting $z' = z + \rho + \frac{\lambda}{2} - u - iw$, we can rewrite the integral as

$$\int_{C + \rho + \frac{\lambda}{2} - u + iw} \frac{G(z' + u - iw + \frac{\lambda}{2} \pm \tau)G(z' - iw - \frac{\lambda}{2} \pm v)}{G(z + u - \frac{\lambda}{2} \pm \rho)G(z + \frac{\lambda}{2} \pm \sigma)} \mathrm{d}z = \psi^{\lambda,u}_{\rho,\sigma}(\tau, v).$$

Considering the relation in eq. (8.11) as a bijection from $\mathbb{C}^4$ to $\mathbb{C}^4$, we denote $\psi_\gamma := \psi_{\rho,\sigma}^{\lambda,\mu}$ with the parameters linked through eq. (8.11). We have

$$R(w_1, w_2, \mathbf{c}(\gamma); \tau, v) = \frac{1}{\sqrt{w_1 w_2}} \frac{\prod_{j=1}^{3} G(i(w+\gamma_0+\gamma_j))}{G(\pm\tau + i\gamma_0)G(\pm v + i\hat{\gamma}_0)} \times \psi_\gamma(\tau, v). \tag{8.12}$$

Ruijsenaars also introduced a renormalised version of R, denoted as R_{ren}, which is given by

$$\begin{aligned} R_{\text{ren}}(w_1, w_2, \mathbf{c}(\gamma); \tau, v) &:= \frac{R(w_1, w_2, \mathbf{c}(\gamma); \tau, v)}{\prod_{j=1}^{3} G(i(w+\gamma_0+\gamma_j))} \\ &= \frac{1}{\sqrt{w_1 w_2}} \frac{\psi_\gamma(\tau, v)}{G(\pm\tau + i\gamma_0)G(\pm v + i\hat{\gamma}_0)}. \end{aligned} \tag{8.13}$$

This renormalised version of the function has only τ- and v-related poles and zeros. Ruijsenaars showed that R_{ren} is meromorphic as a function of τ and v, γ and w_1 and w_2, with pole locations given by

$$\pm\tau = -i\gamma_j + Z_+ \quad \text{and} \quad \pm v = -i\hat{\gamma} + Z_+$$

for $j = 0, 1, 2, 3$.

From the above relations, one may readily observe a symmetry with regard to the triplet $(\gamma_1, \gamma_2, \gamma_3)$: any permutation of its elements leaves ψ_γ, R and R_{ren} invariant (note that $\hat{\gamma}_0$ does not change either under such permutations). Also, as G is invariant under interchanging w_1 and w_2, also ψ_γ, R and R_{ren} are invariant under this action. Noting that $\gamma_0 + \gamma_j = \hat{\gamma}_0 + \hat{\gamma}_j$, one observes $\psi_{\hat{\gamma}}(v, \tau) = \psi_\gamma(\tau, v)$, and similar relations hold for R and R_{ren}.

In [37], Ruijsenaars introduced a third version of his function, $\mathcal{E}$, which observes some additional symmetries. He set

$$c(\gamma; \tau) := \frac{\prod_{j=0}^{3} G(\tau - i\gamma_j)}{G(2\tau + iw)}$$

and

$$\chi(\gamma) := \exp\left(i\alpha \left[\sum_{j=0}^{4} \frac{\gamma_j^2}{4} - \frac{w_1^2 + w_2^2 + w_1 w_2}{8} \right] \right).$$

The function $\mathcal{E}$ then is defined as

$$\mathcal{E}(w_1, w_2, \gamma; \tau, v) := \frac{\chi(\gamma) R_{\text{ren}}(w_1, w_2, \mathbf{c}(\gamma); \tau, v)}{c(\gamma; \tau) c(\hat{\gamma}; v)} \tag{8.14}$$

$$= \frac{\chi(\gamma)}{\sqrt{w_1 w_2}} \frac{G(2\tau + iw)G(2v + iw)}{G(\tau + i\gamma_0)G(v + i\hat{\gamma}_0) \prod_{j=1}^{3} G(\tau - i\gamma_j)G(v - i\hat{\gamma}_j)} \psi_\gamma(\tau, v). \tag{8.15}$$

Note that $\chi(\hat{\gamma}) = \chi(\gamma)$, so that all symmetries of ψ_γ, R and R_{ren} that we described in the previous paragraph carry over to $\mathcal{E}$.

Let W be the group of all operations on $(\gamma_0, \gamma_1, \gamma_2, \gamma_3)$ that consist of permutations and an even number of sign flips. (This group W is isomorphic to the Weyl group of the Lie algebra

D_4.) By [37, thm 1.1], $\mathcal{E}$ satisfies

$$\mathcal{E}(w_1, w_2, p(\gamma); \tau, v) = \mathcal{E}(w_1, w_2, \gamma; \tau, v) \tag{8.16}$$

for all $p \in W$. As a consequence of this W-invariance, the function $\mathcal{E}$ is even in all four parameters λ, u, ρ, σ, and is invariant under the simultaneous interchange of any two pairs of these parameters.

Remark 8.3. For some elements in $p \in W$, the property in eq. (8.16) is immediately clear from our above analysis, as is the case for any permutation of the elements $\gamma_1, \gamma_2, \gamma_3$. One could deduce from *eq.* (8.4) that $\mathcal{E}$ is even in ρ and thus is also invariant under the map $p(\gamma) = (-\gamma_1, -\gamma_0, \gamma_2, \gamma_3)$ (which flips the sign of ρ). This is not sufficient to prove the equality eq. (8.16) for all $p \in W$. We will not provide a full prove for the claim in eq. (8.16), but rely on the proof of Ruijsenaars in [37]. Another proof, relying on a more general result, is found in [3, prop. 4.18]. ∎

8.3 Analyticity of ψ

In this section, we aim to prove the analyticity of the function $(\lambda, u, \rho, \sigma, \tau, v) \mapsto \psi^{\lambda,u}_{\rho,\sigma}(\tau, v)$ on the domain Ω defined by

$$\Omega = \mathbb{C}^6 \setminus \{(\lambda, u, \rho, \sigma, \tau, v) \mid \pm\tau \in Z_+ + \lambda \pm \rho \quad \text{or} \quad \pm\tau \in Z_+ + u \pm \sigma$$
$$\text{or} \quad \pm v \in Z_+ - u \pm \rho \quad \text{or} \quad \pm v \in Z_+ - \lambda \pm \sigma\}. \tag{8.17}$$

The domain Ω is defined in such a way that for $(\lambda, u, \rho, \sigma, \tau, v) \in \Omega$, the downward (eq. (8.1)) and upward (eq. (8.2)) pole sequences are disjoint as sets, so that a deformation C exists that separates those upward and downward pole sequences. The following lemma captures the analyticity of ψ:

Lemma 8.4 (Analyticity of ψ_γ)**.** *Let Ω be as defined in eq.* (8.17)*. The function $\psi^{\lambda,u}_{\rho,\sigma}(\tau, v)$ is analytic as a function from $\Omega \subset \mathbb{C}^6 \to \mathbb{C}$.*

Proof. Let $(\lambda, u, \rho, \sigma, \tau', v)$ be an arbitrary point in Ω. We will first show that $\psi^{\lambda,u}_{\rho,\sigma}(\tau, v)$ is analytic for τ around τ'.

Let C be a deformation of the real line that separates the upward and downward pole sequences determined by the choice of $(\lambda, u, \rho, \sigma, \tau', v)$. Let $\epsilon = \inf\{d(z, p) \mid z \in C, p \in Z_- - u + iw - \frac{\lambda}{2} \pm \tau'\}$, so that ϵ is the shortest distance between a point on C and a pole in $Z_- - u + iw - \frac{\lambda}{2} \pm \tau'$. The imaginary part of C is bounded (by definition of a deformation), and for each $R \in \mathbb{R}$ the set Z_- contains only finitely many points z with $|\mathrm{Im}\, z| \leq R$. It follows that $\epsilon > 0$. Given this ϵ, if $\tau \in \mathbb{C}$ is such that $|\tau - \tau'| < \epsilon$, we have $(\lambda, u, \rho, \sigma, \tau, v) \in \Omega$ and the same C can be used to separate the upward and downward pole sequences induced by $(\lambda, u, \rho, \sigma, \tau, v)$ for all τ in an open ball around τ'.

Let $c : \mathbb{R} \to C$ be a parametrisation of C. Define C_n as the curve $\{c(x) \mid x \in [-n, n]\}$ and define

$$\psi_n(\tau) := \int_{C_n} \frac{G(z + u - iw + \frac{\lambda}{2} \pm \tau)}{G(z + u - \frac{\lambda}{2} \pm \rho)} \frac{G(z - iw - \frac{\lambda}{2} \pm v)}{G(z + \frac{\lambda}{2} \pm \sigma)} \mathrm{d}z.$$

Note that the integrand is analytic for $z \in C_n$ and τ in the neighbourhood of τ'. We can apply the Leibniz integral rule ([11, lem. II.3.3]) to conclude that each ψ_n is analytic in the open neighbourhood of τ' with radius ϵ. As the integrand has exponential growth rate $-2\alpha w$, which is independent of τ, ψ_n converges uniformly to $\tau \mapsto \psi^{\lambda,u}_{\rho,\sigma}(\tau, v)$. As the uniform limit of analytic functions is analytic, it follows that the limit is an analytic function.

We can show analyticity with respect to the other variables in a similar way. By Hartogs's theorem it then follows that $(\lambda, u, \rho, \sigma, \tau, v) \mapsto \psi^{\lambda,u}_{\rho,\sigma}(\tau, v)$ is analytic on Ω. □

Remark 8.5. Ruijsenaars has shown ([36, thm. 2.2]) that R_{ren} is meromorphic with poles located at

$$\pm\tau = -i\gamma_j + Z_+ \quad \text{and} \quad \pm v - i\hat{\gamma}_j + Z_+, \quad j = 0, 1, 2, 3,$$

their orders agreeing with the orders of the zeros of $\prod_{j=0}^{3} G(\pm\tau + i\gamma_j)G(\pm v + i\hat{\gamma}_j)$.

It follows from eq. (8.13) that ψ_γ has poles at

$$\pm\tau = \pm\rho + \lambda + Z_+, \quad \pm\tau = \pm\sigma - u + Z_+, \quad \pm v = \pm\rho - u + Z_+ \quad \text{and} \quad \pm v = \pm\sigma - \lambda + Z_+,$$

so that ψ_γ is meromorphic. Orders of the poles agree with the zero orders of

$$G(\pm\tau \pm \rho - \lambda)G(\pm\tau \pm \sigma - u)G(\pm v \pm \rho + u)G(\pm v \pm \sigma + \lambda).$$

As we will not explicitly need this result, we will not provide a proof for it. ∎

8.4 Asymptotics of ψ

The asymptotics of $\mathcal{E}(w_1, w_2, \gamma; \tau, v)$ for $\operatorname{Re}\tau \to \infty$ were derived in [37] for real-valued w_1 and w_2, and for $\gamma \in \mathbb{R}^4$. However, for our study, we want to consider complex values for γ and we also want to allow for conjugate parameters w_1 and w_2, along with their induced asymptotics. Hence, we need to adapt Ruijsenaars results for our own case. We will derive and prove the asymptotics for ψ; the corresponding result for $\mathcal{E}$ can then be computed from the G-asymptotics and eq. (8.15). The proof follows a similar approach as presented in [37], with adjustments made to accommodate our complex parameters.

Recall that we consider the parameters w_1 and w_2 to form a fixed pair of either positive or complex conjugate parameters with positive real parts. We set $w = (w_1 + w_2)/2$. In the case of positive parameters, we define $w_0 := \min\{w_1, w_2\}$. For the conjugate case, we set $w_0 := w$, which equals $\operatorname{Re} w_1$. Let $\alpha := \frac{2\pi}{w_1 w_2}$ and fix $\varsigma \in (0, 1)$.

Our approach to finding the asymptotics consists of two steps. The first step involves deforming the curve C from the definition of ψ into a curve C_s. This deformation isolates two of the poles in the downward pole sequences. Since these specific poles are simple, we can compute their residues. By utilising the asymptotics of G (as given in equation 6.7), we can approximate these residues with exponential functions, and use the result as an approximation. The second step involves demonstrating that the integral along the shifted curve C_s vanishes as $\operatorname{Re}\tau \to \infty$, showing that our approximation is accurate.

Step 1: Approximation via residues

By performing a change of variables $z \leftarrow z - u + iw + \frac{\lambda}{2}$, we can write

$$\psi_\gamma(\tau, \upsilon) = \int_C I_\gamma(\tau, \upsilon; z)\mathrm{d}z, \tag{8.18}$$

where

$$I_\gamma(\tau, \upsilon; z) := \frac{G(z + \lambda \pm \tau)}{G(z + iw \pm \rho)} \frac{G(z - u \pm \upsilon)}{G(z + iw + \lambda - u \pm \sigma)}, \tag{8.19}$$

and C represents the shifted integral path.

The integrand I_γ has poles in upward pole sequences due to the denominator, located at

$$z = \pm\rho + ikw_1 + ilw_2 \quad \text{and} \quad z = \pm\sigma - (\lambda - u) + ikw_1 + ilw_2,$$

and downward pole sequences due to the numerator at

$$z = \pm\tau - (iw + \lambda) - ikw_1 - ilw_2 \quad \text{and} \quad z = \pm\upsilon - (iw - u) - ikw_1 - ilw_2,$$

for any combination of $k, l = 0, 1, 2, 3, ...$

Let u, λ, ρ, σ be such that $|\mathrm{Im}\,\rho| + \mathrm{Im}\,u < w$ and $|\mathrm{Im}\,\sigma| + \mathrm{Im}\,\lambda < w$. Note that these conditions ensure that the upward pole sequences are disjoint from the downward ones. Let r_1 and r_2 be such that $0 < r_1 < r_2$ and $\mathrm{Re}\,u, \mathrm{Re}\,\lambda, \mathrm{Re}\,\rho, \mathrm{Re}\,\sigma \in [-r_2, r_2]$. Choose $\upsilon \in [r_1, r_2]$.

Assume that $\mathrm{Im}\,\tau \in [-w_0, w_0]$ and $\mathrm{Re}\,\tau > R$, where R is some positive constant for which we will specify additional conditions shortly. Fix some $\eta \in (0, w_0)$.

We define the contour C_s by indenting the line $\mathrm{Im}\,z = -(w - \mathrm{Im}\,u) - \eta$ upwards around $\mathrm{Re}\,z = \pm\,\mathrm{Re}\,\tau$ along compact intervals such that the curve C_s stays a distance of at least $w_0 + 1$ away from the sectors $S_- \pm \mathrm{Re}\,\tau - \lambda$ (with S_- the sector containing all points in Z_-, see section 6.4). As $|\mathrm{Im}\,\tau| \leq w_0$, the distance of the indentation to the τ-dependent poles is at least 1.

We now pick the the lower bound R on $\mathrm{Re}\,\tau$ sufficiently large, in such a way that both indentations stay away a distance 1 from the sectors containing the other poles of I_γ. Let the shape of the indentations be independent of the value of the parameters ρ, σ, τ and υ, and let their positions depend linearly on $\mathrm{Re}\,\tau$ with coefficient ± 1. A visualisation of the poles and the $\mathbb{R}$-deformation C_s is provided in fig. 8.1.

Note that all upward pole sequences (black squares in fig. 8.1) lie above the curve C_s: the poles related to ρ at $\pm\rho + ikw_1 + ilw_2$ with $k, l = 0, 1, 2, ...$ have their imaginary parts bounded below by $-|\mathrm{Im}\,\rho|$. Hence, as $|\mathrm{Im}\,\rho| + \mathrm{Im}\,u < w$ by assumption, all these poles lie above the line $\mathrm{Im}\,z = -(w - \mathrm{Im}\,u) - \eta$. The σ-related poles lie above the line by a similar argument. By our conditions on R, the indentations in the curve also lay below the upward pole sequences.

The points in the downward pole sequences, on the other hand, lie below the curve, except for two of them. It is immediate that the poles at $\pm\tau - (iw + \lambda) - ikw_1 - ilw_2$ (solid blue bullets in fig. 8.1) lie below C_s.

The υ-related poles at $\pm\upsilon - (iw - u) - ikw_1 - ilw_2$ lie below the line $\mathrm{Im}\,z = -(w - \mathrm{Im}\,u) - \eta$ whenever at least k or l is nonzero (open blue bullets). The poles at $\pm\upsilon - (iw - u)$ (open pink bullets in the sketch) have imaginary parts equal to $-(w - \mathrm{Im}\,u)$, and as $\eta \in (0, w_0)$, these two

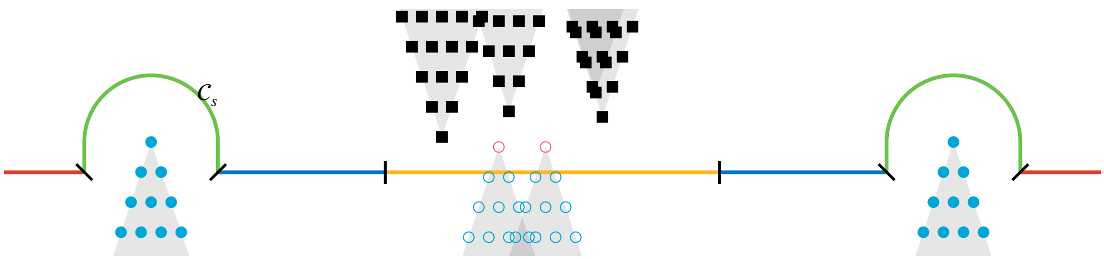


Figure 8.1: This figure shows the pole locations of the function I for a certain choice of u, λ, ρ, σ, τ and υ. The upward pole sequences are marked with black solid squares, whilst the downward pole sequences are marked with cyan and pink bullets. The open bullets correspond to poles centred around $\pm\upsilon$, whilst the closed bullets correspond to poles around $\pm\tau$. The shaded areas mark the sectors containing poles. The curve in the illustration represents a section of C_s and has been divided into coloured segments. The curve has indentations around the τ-related poles, which are green in the picture. The other coloured sections lie on the line $-iw - i\eta + i\mathrm{Im}\, u$. The colours of these sections will be explained in-text. The contour C_s separates the upward and downward pole sequences, except for the two poles at $\pm\upsilon - (iw - u)$ (pink), which deliberately lie above the curve.

poles lie above the line.

Let

$$L_\gamma(\tau, \upsilon) := \int_{C_s} I_\gamma(\tau, \upsilon; z)\, \mathrm{d}z. \tag{8.20}$$

By the residue theorem and eq. (6.6) we can write

$$\begin{aligned}\psi_\gamma(\tau,\upsilon) - L_\gamma(\tau,\upsilon) &= -2\pi i\big[\mathrm{Res}\big(I_\gamma(\tau,\upsilon;\cdot); -\upsilon-(iw-u)\big) \\ &\qquad + \mathrm{Res}\big(I_\gamma(\tau,\upsilon;\cdot); \upsilon-(iw-u)\big)\big] \\ &= \sqrt{w_1w_2}\left[\frac{G(-\upsilon - iw + u + \lambda \pm \tau)}{G(-\upsilon + u \pm \rho)}\,\frac{G(-2\upsilon - iw)}{G(-\upsilon + \lambda \pm \sigma)} + (\upsilon \leftrightarrow -\upsilon)\right] \\ &= \sqrt{w_1w_2}\big[c'(\hat{\gamma};\upsilon)G(-\upsilon - iw + u + \lambda \pm \tau) + (\upsilon \leftrightarrow -\upsilon)\big],\end{aligned} \tag{8.21, 8.22}$$

where c' is defined by

$$c'(\gamma;\tau) := \frac{G(\tau + i\gamma_0)\prod_{j=1}^{3} G(\tau - i\gamma_j)}{G(\tau + iw)}. \tag{8.23}$$

(We can retrieve $c'(\gamma;\tau)$ from $c(\gamma;\tau)$ by flipping the sign of γ_0.)

Using eq. (6.11), we can derive the asymptotic behavior of $\psi_\gamma(\tau,\upsilon) - L_\gamma(\tau,\upsilon)$:

$$\begin{aligned}&\psi_\gamma(\tau,\upsilon) - L_\gamma(\tau,\upsilon) \\ &\quad = \sqrt{w_1w_2}e^{-\alpha(w+iu+i\lambda)\tau}\big[c'(\hat{\gamma};\upsilon)e^{i\upsilon\tau} + (\upsilon \leftrightarrow -\upsilon)\big]\cdot(1 + O(e^{-\alpha\varsigma w_0|\mathrm{Re}\,\tau|})),\end{aligned} \tag{8.24}$$

where the term $O(e^{-\alpha\varsigma w_0|\mathrm{Re}\,\tau|})$ is uniformly bounded in γ and υ if their values are restricted to a compact set. Therefore, if we define

$$\psi_\gamma^{\mathrm{ass}}(\tau,\upsilon) := \sqrt{w_1w_2}e^{-\alpha(w+iu+i\lambda)\tau}\big[c'(\hat{\gamma};\upsilon)e^{i\upsilon\tau} + (\upsilon \leftrightarrow -\upsilon)\big], \tag{8.25}$$

we have

$$\psi_\gamma(\tau,\upsilon) = L_\gamma(\tau,\upsilon) + \psi_\gamma^{\mathrm{ass}}(\tau,\upsilon)\cdot(1 + O(e^{-\alpha\varsigma w_0|\mathrm{Re}\,\tau|})). \tag{8.26}$$

Step 2: The remainder vanishes

We will now demonstrate that $L_\gamma(\tau, \upsilon) \to 0$ as $\operatorname{Re}\tau \to \infty$, provided that $\operatorname{Im}\tau \in [-w_0, w_0]$:

Lemma 8.6. *Set r_1 and r_2 such that $0 < r_1 < r_2$ and fix some $\eta \in (0, w_0)$. Let u, λ, ρ, σ be such that $|\operatorname{Im}\rho| + \operatorname{Im} u < w$, $|\operatorname{Im}\sigma| + \operatorname{Im}\lambda < w$ and $\operatorname{Im}\lambda + \operatorname{Im} u < w+\eta$, with $\operatorname{Re} u, \operatorname{Re}\lambda, \operatorname{Re}\rho, \operatorname{Re}\sigma \in [-r_2, r_2]$, and let $\upsilon \in [r_1, r_2]$. Set $\kappa = \min\{w+\eta-\operatorname{Im}\lambda-\operatorname{Im} u, 2w-w_0\}$. There exists constants C and R independent of τ and υ, such that*

$$\left|L_\gamma(\tau, \upsilon)\right| < Ce^{-\alpha\kappa \operatorname{Re}\tau}$$

whenever $\operatorname{Im}\tau$ is restricted to $[-w_0, w_0]$ and $\operatorname{Re}\tau > R$. Furthermore, if we restrict the imaginary parts of ρ and σ to a compact set, the constants C and R can be chosen independently of ρ and σ as well.

Proof. Throughout this proof, any occurrence of the symbol C, possibly subscripted, refers to some finite positive constant independent of $\operatorname{Re}\rho$ and $\operatorname{Re}\sigma$, τ, and υ subject to the restrictions in the lemma. The value of this C may change every time the symbol is used. We consider the values of λ and u to be fixed.

We will establish upper bounds for $\int I_\gamma(\tau, \upsilon; z)\,\mathrm{d}z$ along different sections of the curve C_s, and utilise them to show that the contributions of these parts vanish as $\operatorname{Re}\tau \to \infty$. We analyse the sections of C_s corresponding to different colors in fig. 8.1 consecutively, starting from the red ones.

The red half-lines Let us begin by examining the part of the integral along the half-line to the right of the right indentation (which is red in fig. 8.1). Let t_+ be such that the real part of the half-line is given by $[\operatorname{Re}(\tau-\lambda)+t_+, \infty)$. The value of t_+ is independent of τ by our definition of C_s. Let R be so large that all instances of G in I_γ (eq. (8.19) have arguments with a positive real part along the half-line.

We use eq. (6.8) to estimate each of the hyperbolic gamma functions in I_γ on this half-line, giving

$$\begin{aligned}\left|I_\gamma(\tau, \upsilon; z)\right| &< C\frac{e^{\frac{\alpha}{2}\operatorname{Im}(z+\lambda+\tau)\operatorname{Re}(z+\lambda+\tau)}e^{\frac{\alpha}{2}\operatorname{Im}(z+\lambda-\tau)\operatorname{Re}(z+\lambda-\tau)}}{e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\rho)\operatorname{Re}(z+iw+\rho)}e^{\frac{\alpha}{2}\operatorname{Im}(z+iw-\rho)\operatorname{Re}(z+iw-\rho)}}\\ &\quad\times\frac{e^{\frac{\alpha}{2}\operatorname{Im}(z-u)\operatorname{Re}(z-u+\upsilon)}e^{\frac{\alpha}{2}\operatorname{Im}(z-u)\operatorname{Re}(z-u-\upsilon)}}{e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\lambda-u+\sigma)\operatorname{Re}(z+iw+\lambda-u+\sigma)}e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\lambda-u-\sigma)\operatorname{Re}(z+iw+\lambda-u-\sigma)}}\\ &= C\frac{e^{\alpha(\operatorname{Im}(z+\lambda)\operatorname{Re}(z+\lambda)+\operatorname{Im}\tau\operatorname{Re}\tau)}}{e^{\alpha((w+\operatorname{Im}z)\operatorname{Re}z+\operatorname{Im}\rho\operatorname{Re}\rho)}}\frac{e^{\alpha(\operatorname{Im}(z-u)\operatorname{Re}(z-u))}}{e^{\alpha((w+\operatorname{Im}(z+\lambda-u))\operatorname{Re}(z+\lambda-u)+\operatorname{Im}\sigma\operatorname{Re}\sigma)}}\\ &= Ce^{\alpha(-2w\operatorname{Re}z+\operatorname{Im}\tau\operatorname{Re}\tau-\operatorname{Im}\rho\operatorname{Re}\rho-\operatorname{Im}\sigma\operatorname{Re}\sigma-w\operatorname{Re}(\lambda-u))}.\end{aligned}$$

By exploiting the compactness of the domains of ρ, σ, and υ, we can find a weaker bound that is independent of those parameters (but might depend on u and λ):

$$< Ce^{\alpha(-2w\operatorname{Re}z+\operatorname{Im}\tau\operatorname{Re}\tau)},$$

which, by $\operatorname{Im}\tau \le w_0$, simplifies to

$$< Ce^{\alpha\left(-2w\operatorname{Re}z+w_0\operatorname{Re}\tau\right)}.$$

Integrating this along the half-line with real part $[\operatorname{Re}(\tau-\lambda)+t_+,\infty)$, we find that its contribution to L is dominated by $Ce^{\alpha(w_0-2w)\operatorname{Re}\tau}$. For the other tail of the integral (to the left of the left indentation) we obtain the same bound by a similar calculation.

The green indentations Next we turn to the indentation around τ (the right indentation in fig. 8.1, which has been given a green colour). The length of this indentation is bounded and, by definition of C_s, independent of ρ, σ, τ and υ. By continuity we can therefore bound the value of $|G(z+\lambda-\tau)|$ along this indentation, independently of ρ, σ, τ and υ.

Similarly to before, we can estimate the remaining terms in I_γ using eq. (6.8), which gives

$$\left|I_\gamma(\tau,\upsilon;z)\right| < C\frac{e^{\frac{\alpha}{2}\operatorname{Im}(z+\lambda+\tau)\operatorname{Re}(z+\lambda+\tau)}}{e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\rho)\operatorname{Re}(z+iw+\rho)}e^{\frac{\alpha}{2}\operatorname{Im}(z+iw-\rho)\operatorname{Re}(z+iw-\rho)}} \times\frac{e^{\frac{\alpha}{2}\operatorname{Im}(z-u)\operatorname{Re}(z-u+\upsilon)}e^{\frac{\alpha}{2}\operatorname{Im}(z-u)\operatorname{Re}(z-u-\upsilon)}}{e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\lambda-u+\sigma)\operatorname{Re}(z+iw+\lambda-u+\sigma)}e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\lambda-u-\sigma)\operatorname{Re}(z+iw+\lambda-u-\sigma)}}$$

which by similar reasoning as above simplifies to

$$< Ce^{\alpha(-2w\operatorname{Re}z+w_0\operatorname{Re}\tau)}e^{-\frac{\alpha}{2}\operatorname{Im}(z+\lambda-\tau)\operatorname{Re}(z+\lambda-\tau))}.$$

Since $z-\tau$ is bounded (independently of τ) along the indentation, the second exponential term is bounded, and we can simplify the first one to

$$< Ce^{\alpha\left(w_0-2w\right)\operatorname{Re}\tau}.$$

As the length of the indentation is finite, its contribution to L_γ is also bounded by $Ce^{\alpha(w_0-2w)\operatorname{Re}\tau}$. The same bound holds for the indentation around $-\operatorname{Re}\tau$, as can be seen from a similar calculation.

The yellow middle part Thirdly, we turn to estimating the contribution along the line segment with $|\operatorname{Re}z|\le 3r_2+1$, which is yellow in fig. 8.1.[1] Since this line segment has a finite length, independent of our parameters, we can estimate the contribution of the hyperbolic gamma functions involving ρ, σ and υ by their maximum value along the line segment. As we take the parameters in a compact subspace, we can make this bound independent of ρ, σ and υ. By continuity of $\frac{G(z-u\pm\upsilon)}{G(z+iw\pm\rho)G(z+iw+\lambda-u\pm\sigma)}$, this bound is finite.

We can thus bound the contribution of I_γ on this line segment by focusing on the two τ-dependent hyperbolic gamma functions. Let R be at least so large that $|\operatorname{Re}(z+\lambda-\tau)| = \operatorname{Re}(\tau-z-\lambda)$ and $\operatorname{Re}(z+\lambda-\tau)>0$ whenever z is in the segment and $\operatorname{Re}\tau>R$. Then we find using eq. (6.8) that

$$\begin{aligned}\left|I_\gamma(\tau,\upsilon;z)\right| &< Ce^{\frac{\alpha}{2}(\operatorname{Im}(z+\lambda+\tau)\operatorname{Re}(z+\lambda+\tau)+\operatorname{Im}(z+\lambda-\tau)\operatorname{Re}(\tau-\lambda-z))}\\ &= Ce^{\alpha(\operatorname{Im}\tau\operatorname{Re}(z+\lambda)+\operatorname{Im}(z+\lambda)\operatorname{Re}\tau)}.\end{aligned}$$

Since $\operatorname{Re}z$ is bounded along this curve, and $\operatorname{Im}z=-w-\eta+\operatorname{Im}u$, we can bound this by

$$< Ce^{-\alpha(w+\eta-\operatorname{Im}\lambda-\operatorname{Im}u)\operatorname{Re}\tau}.$$

As the line segment has finite length, its contribution to L is thus bounded by $Ce^{-\alpha(w+\eta-\operatorname{Im}\lambda-\operatorname{Im}u)\operatorname{Re}\tau}$.

[1] The poles at $\pm\sigma-(\lambda-u)$ have their real part bounded absolutely by $3r_2$. The additional $+1$ ensures that the remaining (blue in fig. 8.1) parts of the integral stay away sufficiently far from all poles.

The blue remaining sections Lastly, we turn to estimate the contribution of the two remaining line segments. Let t_- be such that the right one of the remaining line segments is given by $\operatorname{Re} z \in (3r_2 + 1, \operatorname{Re}(\tau - \lambda) - t_-)$ and $\operatorname{Im} z = -w - \eta + \operatorname{Im} u$. This t_- is independent of ρ, σ, τ and v by definition of $\mathcal{C}_s$.

We estimate an upper bound for the contribution to L_γ along this right line segment with positive real part. The calculation is highly similar to the one for the integral tails, note however that along this segment $|\operatorname{Re}(z + \lambda - \tau)| = \operatorname{Re}(\tau - z - \lambda)$.

We find

$$\left|I_\gamma(\tau, v; z)\right| < C \frac{e^{\frac{\alpha}{2}\operatorname{Im}(z+\lambda+\tau)\operatorname{Re}(z+\lambda+\tau)} e^{\frac{\alpha}{2}\operatorname{Im}(z+\lambda-\tau)\operatorname{Re}(\tau-z-\lambda)}}{e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\rho)\operatorname{Re}(z+iw+\rho)} e^{\frac{\alpha}{2}\operatorname{Im}(z+iw-\rho)\operatorname{Re}(z+iw-\rho)}} \times \frac{e^{\frac{\alpha}{2}\operatorname{Im}(z-u)\operatorname{Re}(z-u+v)} e^{\frac{\alpha}{2}\operatorname{Im}(z-u)\operatorname{Re}(z-u-v)}}{e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\lambda-u+\sigma)\operatorname{Re}(z+iw+\lambda-u+\sigma)} e^{\frac{\alpha}{2}\operatorname{Im}(z+iw+\lambda-u-\sigma)\operatorname{Re}(z+iw+\lambda-u-\sigma)}}$$

which, using boundedness of parameters and $\operatorname{Im} z = -w - \eta + \operatorname{Im} u$, we can simplify to

$$< C e^{-\alpha(w+\eta-\operatorname{Im}\lambda-\operatorname{Im} u)\operatorname{Re}\tau} e^{-\alpha(w+\operatorname{Im}\lambda+\operatorname{Im} u-\eta-\operatorname{Im}\tau)\operatorname{Re} z}.$$

We integrate this along the line-segment to find the following bound for its contribution to L_γ:

$$C_1 e^{-\alpha(w+\eta-\operatorname{Im}\lambda-\operatorname{Im} u)\operatorname{Re}\tau} \times (C_2 e^{-\alpha(w+\operatorname{Im}\lambda+\operatorname{Im} u-\eta-w_0)\operatorname{Re}\tau} + C_3).$$

If $w + \operatorname{Im}\lambda + \operatorname{Im} u - \eta - w_0 \geq 0$, the C_3-term dominates the one involving C_2 as $\operatorname{Re}\tau \to \infty$, and the contribution is bounded by

$$C e^{-\alpha(w+\eta-\operatorname{Im}\lambda-\operatorname{Im} u)\operatorname{Re}\tau}.$$

If $w + \operatorname{Im}\lambda + \operatorname{Im} u - \eta - w_0 < 0$, the C_2-term is dominant and the estimate is

$$C e^{-\alpha(w+\eta-\operatorname{Im}\lambda-\operatorname{Im} u+w+\operatorname{Im}\lambda+\operatorname{Im} u-\eta-w_0)\operatorname{Re}\tau} = C e^{\alpha(w_0-2w)\operatorname{Re}\tau}.$$

In both cases the estimate equals

$$C e^{-\alpha\kappa\operatorname{Re}\tau}.$$

Similarly, we bound the part of the integral along other remaining line segment by the same expression.

Now we have bounded all contributions to L_γ by either $C e^{-\alpha(w+\eta-\operatorname{Im}\lambda-\operatorname{Im} u)\operatorname{Re}\tau}$ or $C e^{\alpha(w_0-2w)\operatorname{Re}\tau}$, concluding our proof. □

We now easily derive the following theorem:

Theorem 8.7 (Asymptotic behaviour of ψ)**.** *Set r_1 and r_2 such that $0 < r_1 < r_2$ and fix some $\eta \in (0, w_0)$. Let u, λ, ρ, σ be such that $|\operatorname{Im}\rho| + \operatorname{Im} u < w$, $|\operatorname{Im}\sigma| + \operatorname{Im}\lambda < w$ and $\operatorname{Im}\lambda + \operatorname{Im} u < w + \eta$, with $\operatorname{Re} u, \operatorname{Re}\lambda, \operatorname{Re}\rho, \operatorname{Re}\sigma \in [-r_2, r_2]$, and let $v \in [r_1, r_2]$. Set $\kappa = \min\{w + \eta - \operatorname{Im}\lambda - \operatorname{Im} u, 2w - w_0\}$. There exists constants C and R independent of τ and v, such that*

$$\left|\psi_\gamma(\tau, v) - \psi_\gamma^{\text{ass}}(\tau, v)\right| < C e^{-\alpha\kappa\operatorname{Re}\tau} \tag{8.27}$$

whenever $\operatorname{Im}\tau$ is restricted to $[-w_0, w_0]$ and $\operatorname{Re}\tau > R$. Furthermore, if we restrict the imaginary *parts of ρ and σ to a compact set, the constants $\boldsymbol{C}$ and $\boldsymbol{R}$ can be chosen independently of ρ and σ as well.*

Proof. If we set $\varsigma = \eta / w_0$, the theorem follows immediately from combining eqs. (8.25) and (8.26) with lemma 8.6 and the triangle inequality. □

Chapter 9

Solution to the Askey-Wilson difference equation

In 1985, Askey and Wilson published a paper ([2]) introducing a special family of polynomials, now known as the Askey-Wilson polynomials. This family of polynomials depends on four parameters and a coefficient q, and encompasses all classical families of orthogonal polynomials, either as a special case or as a limiting case ([29]).

A fundamental result in the theory of special functions is that if (P_n) is a family of orthogonal polynomials with P_n of degree n and $P_0 \equiv 1$, then there exist coefficients a_n and b_n so that

$$xP_n(x) = a_{n+1}P_{n+1}(x) + b_nP_n(x) + a_nP_{n-1}(x), \tag{9.1}$$

with $P_{-1} \equiv 0$, see e.g. [12]. The coefficients a_n and b_n can be expressed in terms of inner products, with $a_n = \langle XP_n, P_{n-1}\rangle / \langle P_{n-1}, P_{n-1}\rangle$ and $b_n = \langle XP_n, P_n\rangle / \langle P_n, P_n\rangle$, where $XP_n(x) = xP_n(x)$.

We can see eq. (9.1) as difference equation in the discrete parameter n. Some systems of classical orthogonal polynomials also satisfy a difference equation for the continuous variable, of the form

$$\lambda_nP_n(x) = A(x)P_n(x+1) + B(x)P_n(x) + C(x)P_n(x-1),$$

with coefficients independent of n and an eigenvalue λ_n independent of x. An example of such polynomials are the Krawtchouk polynomials ([34]).

The Askey-Wilson difference equations generalise this second form of difference equation, and have the orthogonal family of Askey-Wilson polynomials as their eigenfunctions. However, nonpolynomial eigenfunctions to the corresponding difference operator exist, and Ruijsenaars has demonstrated that his generalised hypergeometric function R is an eigenfunction of this operator ([36]). In this chapter, we will use properties of the quantum group $\mathfrak{D}$ and the representation π_λ to provide a novel proof for this established result.

Our approach in this proof is similar to that used by Groenevelt in [15, 16] on other solutions of the Askey-Wilson difference equations. We will first present the difference equation in the upcoming section, and subsequently outline the steps we will undertake in constructing the proof.

9.1 The Askey-Wilson difference equation

This section defines the Askey-Wilson difference equation in the form we will use. Let $\gamma \in \mathbb{C}^4$ and let w_1 and w_2 be either positive or complex conjugate parameters with positive real parts as before. Set

$$\mathcal{A}_{\gamma}^{w_1,w_2}(x) = \frac{\prod_{j=0}^{3} \cosh \frac{\pi}{w_2}\left(x + i\frac{w_1}{2} + i\gamma_j\right)}{\sinh \frac{2\pi x}{w_2} \sinh \frac{2\pi(x+iw)}{w_2}}. \tag{9.2}$$

Then[1]

$$\begin{aligned}\mathcal{L}_{\gamma,x}^{w_1,w_2} f(x,y) = \mathcal{A}_{\gamma}^{w_1,w_2}(x)\big[f(x+iw_1,y) - f(x,y)\big] \\ + \mathcal{A}_{\gamma}^{w_1,w_2}(-x)\big[f(x-iw_1,y) - f(x,y)\big]\end{aligned} \tag{9.3}$$

defines the second-order Askey-Wilson difference operator. In this relation, the continuous variable y has replaced the discrete variable n of eq. (9.1). The x in the subscript indicates that it acts on the x-variable. The operator $\mathcal{L}_{\gamma,y}^{w_1,w_2}$ (with a y in the subscript) acts in a similar manner, with the coefficients taken as a function of y, and the shifts applied in the y-variable:

$$\begin{aligned}\mathcal{L}_{\gamma,y}^{w_1,w_2} f(x,y) = \mathcal{A}_{\gamma}^{w_1,w_2}(y)\big[f(x,y+iw_1) - f(x,y)\big] \\ + \mathcal{A}_{\gamma}^{w_1,w_2}(-y)\big[f(x,y-iw_1) - f(x,y)\big].\end{aligned}$$

It can been shown that R is an eigenfunction of $\mathcal{L}_{\gamma,x}^{w_1,w_2}$, with eigenvalue

$$v(w_1,w_2,\gamma;y) = \frac{1}{2}\left[\cosh \frac{2\pi y}{w_2} + \cosh \frac{\pi i}{w_2}\left(w_1 + 2\hat{\gamma}_0\right)\right]. \tag{9.4}$$

In section 9.2, we will prove that the function $H_{\tau,\rho}^{\lambda,u}$ we defined in chapter 7 satisfies a three-term relation

$$H_{\tau,\rho}^{\lambda,u}(z - iw_1) = B_{\tau,\rho}^{\lambda} H_{\tau+iw_1,\rho}^{\lambda,u}(z) + C_{\tau,\rho}^{\lambda} H_{\tau,\rho}^{\lambda,u}(z) + B_{-\tau,\rho}^{\lambda} H_{\tau-iw_1,\rho}^{\lambda,u}(z),$$

with coefficients $B_{\tau,\rho}^{\lambda}$ and $C_{\tau,\rho}^{\lambda}$ independent of z. This relation shows that a shift of $-iw_1$ in the variable z can be translated to shifts in τ. By identifying the left-hand side of this expression as $\pi_\lambda(K^{-1})H_{\tau,\rho}^{\lambda,u}$, we can then use lemma 5.1 to express $\pi_\lambda(X_{0,\sigma})H_{\tau,\rho}^{\lambda,u}$ in terms of $H_{\tau+iw_1,\rho}^{\lambda,u}$, $H_{\tau,\rho}^{\lambda,u}$ and $H_{\tau-iw_1,\rho}^{\lambda,u}$. Furthermore, using lemma 4.1, and the fact that $F_{v,\sigma}^{\lambda,0}$ is an eigenfunction of $\pi_{\bar{\lambda}}(X_{0,\sigma}^{*})$ (or of $\pi_{\bar{\lambda}}(X_{0,\sigma}^{\star})$, in the case of conjugate parameters), we can express

$$\mu_v^{\sigma}\psi_\gamma = \langle H_{\tau,\rho}^{\lambda,u}, \pi_{\bar{\lambda}}(X_{0,\sigma}^{*/\star})F_{v,\sigma}^{\lambda}\rangle_C$$

in terms of $\psi_\gamma(\tau + iw_1, v)$, $\psi_\gamma(\tau, v)$ and $\psi_\gamma(\tau - iw_1, v)$. We discuss this in section 9.3. By rewriting this result using Ruijsenaars's function R, we derive a three-term relation for R that is equivalent to

$$\mathcal{L}_{\gamma,\tau}^{w_1,w_2} R(w_1,w_2,\mathbf{c}(\gamma);\tau,v) = v(w_1,w_2,\gamma;v)R(w_1,w_2,\mathbf{c}(\gamma);\tau,v),$$

[1] The difference operator as we present it here resembles the form used in [36], with the map in eq. (8.9) used to retrieve $\mathbf{c}$ from γ. Up to a reparametrisation, the coefficients in this operator are equivalent to the coefficients used by Askey and Wilson in [2].

which will be shown in section 9.4.

After we have shown this, we will use the symmetries of R to show that it satisfies three other Askey-Wilson-type difference equations, and we will define a difference operator for which ψ_γ is an eigenfunction.

9.2 Calculating $\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho}$

The calculation of $\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho}$ follows from straightforward applications of the difference equations for G (eqs. (6.3) and (6.4)). We will first find an expression for $H^{\lambda,u}_{\tau,\rho}(z-i\frac{w_1}{2})$ and then apply it twice to find the desired expression for $\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho}(z) = H^{\lambda,u}_{\tau,\rho}(z-iw_1)$.

Lemma 9.1. *Let $\tau \notin i\frac{w_2}{2}\cdot\mathbb{Z}$. Set $A^\lambda_{\tau,\rho} := i\frac{\cosh\frac{\pi}{w_2}\left(\tau+\rho+i\frac{w_1}{2}-\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}}$. Then*

$$\begin{aligned} H^{\lambda,u}_{\tau,\rho}(z-i\frac{w_1}{2}) &= A^\lambda_{\tau,\rho}H^{\lambda,u}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z) + A^\lambda_{-\tau,\rho}H^{\lambda,u}_{\tau-i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z) \qquad (9.5)\\ &= A^\lambda_{\tau,-\rho}H^{\lambda,u}_{\tau+i\frac{w_1}{2},\rho-i\frac{w_1}{2}}(z) + A^\lambda_{-\tau,-\rho}H^{\lambda,u}_{\tau-i\frac{w_1}{2},\rho-i\frac{w_1}{2}}(z). \qquad (9.6)\end{aligned}$$

Proof. Note that the coefficients in the lemma are independent of u. As $H^{\lambda,u}_{\tau,\rho} = H^{\lambda,0}_{\tau,\rho}(\cdot+u)$, it suffices to prove the lemma for the case $u=0$, which simplifies the notation. As $H^{\lambda,u}_{\tau,\rho}$ is even in ρ, the second line of the equality follows from the first by flipping the sign of ρ. It thus remains to prove the first equality for $u=0$.

By the definition of $H^{\lambda,u}_{\tau,\rho}$ in eq. (7.1), we have

$$H^{\lambda,0}_{\tau,\rho}(z-i\frac{w_1}{2}) = \frac{G(z-iw+\frac{\lambda}{2}\pm\tau-i\frac{w_1}{2})}{G(z-\frac{\lambda}{2}\pm\rho-i\frac{w_1}{2})}$$

which by simple arithmetic can be written as

$$= \frac{G(z-\frac{\lambda}{2}+(\rho+i\frac{w_1}{2}))}{G(z-\frac{\lambda}{2}+(\rho-i\frac{w_1}{2}))}\frac{G(z-iw+\frac{\lambda}{2}+(\tau-i\frac{w_1}{2}))}{G(z-iw+\frac{\lambda}{2}+(\tau+i\frac{w_1}{2}))} \times\frac{G(z-iw+\frac{\lambda}{2}\pm(\tau+i\frac{w_1}{2}))}{G(z-\frac{\lambda}{2}\pm(\rho+i\frac{w_1}{2}))}.$$

We can apply the difference equation eq. (6.3) to both the first and the second fraction, rewriting them in terms of hyperbolic cosines. The third fraction we recognise as a shifted version of $H^{\lambda,0}_{\tau,\rho}$, so that we can write

$$H^{\lambda,0}_{\tau,\rho}(z-i\frac{w_1}{2}) = \frac{\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)}{\cosh\frac{\pi}{w_2}\left(z-iw+\frac{\lambda}{2}+\tau\right)}\times H^{\lambda,0}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z).$$

Dividing both sides by the fraction of hyperbolic cosines, we have

$$H^{\lambda,0}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z) = \frac{\cosh\frac{\pi}{w_2}\left(z-iw+\frac{\lambda}{2}+\tau\right)}{\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)}H^{\lambda,0}_{\tau,\rho}(z-i\frac{w_1}{2}). \qquad (9.7)$$

Multiplying this latter expression by $A^{\lambda}_{\tau,\rho}$, we find, using the product rule $\cosh a \cosh b = \frac{1}{2}(\cosh(a+b)+\cosh(a-b))$, that

$$\begin{aligned}
&A^{\lambda}_{\tau,\rho} H^{\lambda,0}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z) \\
&= i\frac{\cosh\frac{\pi}{w_2}\left(\tau+\rho+i\frac{w_1}{2}-\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}} \frac{\cosh\frac{\pi}{w_2}\left(z-iw+\frac{\lambda}{2}+\tau\right)}{\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)} H^{\lambda,0}_{\tau,\rho}(z-i\frac{w_1}{2}) \\
&= i\frac{\cosh\frac{\pi}{w_2}\left(z-iw-\frac{\lambda}{2}+2\tau+\rho+i\frac{w_1}{2}\right)+\cosh\frac{\pi}{w_2}\left(z-iw+\frac{3\lambda}{2}-\rho-i\frac{w_1}{2}\right)}{2\sinh\frac{2\pi\tau}{w_2}\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)} \\
&\qquad \times H^{\lambda,0}_{\tau,\rho}(z-i\frac{w_1}{2}).
\end{aligned}$$

As $H^{\lambda,0}_{\tau,\rho}$ is even in τ, we can flip the sign of τ in the latter expression to find

$$\begin{aligned}
&A^{\lambda}_{-\tau,\rho} H^{\lambda,0}_{\tau-i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z) \\
&= -i\frac{\cosh\frac{\pi}{w_2}\left(z-iw-\frac{\lambda}{2}-2\tau+\rho+i\frac{w_1}{2}\right)+\cosh\frac{\pi}{w_2}\left(z-iw+\frac{3\lambda}{2}-\rho-i\frac{w_1}{2}\right)}{2\sinh\frac{2\pi\tau}{w_2}\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)} \\
&\qquad \times H^{\lambda,0}_{\tau,\rho}(z-i\frac{w_1}{2}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&A^{\lambda}_{\tau,\rho} H^{\lambda,0}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z) + A^{\lambda}_{-\tau,\rho} H^{\lambda,0}_{\tau-i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z) \\
&= i\frac{\cosh\frac{\pi}{w_2}\left(z-iw-\frac{\lambda}{2}+2\tau+\rho+i\frac{w_1}{2}\right)-\cosh\frac{\pi}{w_2}\left(z-iw-\frac{\lambda}{2}-2\tau+\rho+i\frac{w_1}{2}\right)}{2\sinh\frac{2\pi\tau}{w_2}\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)} \\
&\qquad \times H^{\lambda,0}_{\tau,\rho}(z-i\frac{w_1}{2}).
\end{aligned}$$

Using $\frac{1}{2}(\cosh(a+b)-\cosh(a-b)) = \sinh a \sinh b$, we can rewrite the fraction in this expression to

$$\frac{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{\pi}{w_2}\left(z-iw-\frac{\lambda}{2}+\rho+i\frac{w_1}{2}\right)}{\sinh\frac{2\pi\tau}{w_2}\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)} = \frac{-i\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)}{\cosh\frac{\pi}{w_2}\left(z-\frac{\lambda}{2}+\rho\right)} = -i,$$

concluding our proof. □

If $H^{\lambda,u}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z)$ has no pole for $\tau \to 0$, the singularities of the right-hand sides of eqs. (9.5) and (9.6) at $\tau = 0$ are removable.

The desired expression for $\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho}$ can now be derived easily:

Corollary 9.2. *For* $\tau \notin i\frac{w_2}{2}\cdot\mathbb{Z} \cup (\pm i\frac{w_1}{2} + i\frac{w_2}{2}\cdot\mathbb{Z})$,

$$\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho} = B^{\lambda}_{\tau,\rho}H^{\lambda,u}_{\tau+iw_1,\rho} + C^{\lambda}_{\tau,\rho}H^{\lambda,u}_{\tau,\rho} + B^{\lambda}_{-\tau,\rho}H^{\lambda,u}_{\tau-iw_1,\rho}.$$

with

$$B^{\lambda}_{\tau,\rho} := \frac{\cosh \frac{\pi}{w_2}(\tau \pm \rho - \lambda + i\frac{w_1}{2})}{\sinh \frac{2\pi\tau}{w_2} \sinh \frac{2\pi(\tau+iw)}{w_2}}$$

and

$$C^{\lambda}_{\tau,\rho} := -\frac{\cosh \frac{\pi}{w_2}(\tau + \rho \pm \lambda + i\frac{w_1}{2})}{\sinh \frac{2\pi\tau}{w_2} \sinh \frac{2\pi(\tau+iw)}{w_2}} - \frac{\cosh \frac{\pi}{w_2}(\tau - \rho \pm \lambda - i\frac{w_1}{2})}{\sinh \frac{2\pi\tau}{w_2} \sinh \frac{2\pi(\tau-iw)}{w_2}}.$$

Proof. By applying the relation eq. (9.5) of lemma 9.1 in the second equality below, we have

$$\begin{aligned}\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho}(z) &= H^{\lambda,u}_{\tau,\rho}((z - i\frac{w_1}{2}) - i\frac{w_1}{2}) \\ &= A^{\lambda}_{\tau,\rho} H^{\lambda,u}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z - i\frac{w_1}{2}) + A^{\lambda}_{-\tau,\rho} H^{\lambda,u}_{\tau-i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z - i\frac{w_1}{2}).\end{aligned} \tag{9.8}$$

Setting $\hat{\tau} = \tau + i\frac{w_1}{2}$, $\hat{\rho} = \rho + i\frac{w_1}{2}$, applying lemma 9.1 now in the form of eq. (9.6) to derive the second equality below gives

$$\begin{aligned}H^{\lambda,u}_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z - i\frac{w_1}{2}) &= H^{\lambda,u}_{\hat{\tau},\hat{\rho}}(z - i\frac{w_1}{2}) \\ &= A^{\lambda}_{\hat{\tau},-\hat{\rho}} H^{\lambda,u}_{\hat{\tau}+i\frac{w_1}{2},\hat{\rho}-i\frac{w_1}{2}}(z) + A^{\lambda}_{-\hat{\tau},-\hat{\rho}} H^{\lambda,u}_{\hat{\tau}-i\frac{w_1}{2},\hat{\rho}-i\frac{w_1}{2}}(z) \\ &= A^{\lambda}_{\tau+i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} H^{\lambda,u}_{\tau+iw_1,\rho}(z) + A^{\lambda}_{-\tau-i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} H^{\lambda,u}_{\tau,\rho}(z).\end{aligned} \tag{9.9}$$

Flipping the sign of τ and using the fact that $H^{\lambda,u}_{\tau,\rho}$ is even in τ, we derive

$$H^{\lambda,u}_{\tau-i\frac{w_1}{2},\rho+i\frac{w_1}{2}}(z - i\frac{w_1}{2}) = A^{\lambda}_{-\tau+i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} H^{\lambda,u}_{\tau-iw_1,\rho}(z) + A^{\lambda}_{\tau-i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} H^{\lambda,u}_{\tau,\rho}(z). \tag{9.10}$$

Substituting eqs. (9.9) and (9.10) into eq. (9.8) gives

$$\begin{aligned}\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho} = A^{\lambda}_{\tau,\rho} A^{\lambda}_{\tau+i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} H^{\lambda,u}_{\tau+iw_1,\rho} \\ + \left(A^{\lambda}_{\tau,\rho} A^{\lambda}_{-\tau-i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} + A^{\lambda}_{-\tau,\rho} A^{\lambda}_{\tau-i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} \right) H^{\lambda,u}_{\tau,\rho} \\ + A^{\lambda}_{-\tau,\rho} A^{\lambda}_{-\tau+i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} H^{\lambda,u}_{\tau-iw_1,\rho}.\end{aligned}$$

By direct calculation we find

$$A^{\lambda}_{\tau,\rho} A^{\lambda}_{\tau+i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} = -\frac{\cosh \frac{\pi}{w_2}(\tau \pm \rho - \lambda + i\frac{w_1}{2})}{\sinh \frac{2\pi\tau}{w_2} \sinh \frac{2\pi(\tau+i\frac{w_1}{2})}{w_2}} = \frac{\cosh \frac{\pi}{w_2}(\tau \pm \rho - \lambda + i\frac{w_1}{2})}{\sinh \frac{2\pi\tau}{w_2} \sinh \frac{2\pi(\tau+iw)}{w_2}} = B^{\lambda}_{\tau,\rho}$$

and similarly

$$\begin{aligned}&A^{\lambda}_{\tau,\rho} A^{\lambda}_{-\tau-i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} + A^{\lambda}_{-\tau,\rho} A^{\lambda}_{\tau-i\frac{w_1}{2},-\rho-i\frac{w_1}{2}} \\ &\quad = -\frac{\cosh \frac{\pi}{w_2}(\tau + \rho \pm \lambda + i\frac{w_1}{2})}{\sinh \frac{2\pi\tau}{w_2} \sinh \frac{2\pi(\tau+iw)}{w_2}} - \frac{\cosh \frac{\pi}{w_2}(\tau - \rho \pm \lambda - i\frac{w_1}{2})}{\sinh \frac{2\pi\tau}{w_2} \sinh \frac{2\pi(\tau-iw)}{w_2}} = C^{\lambda}_{\tau,\rho},\end{aligned}$$

finishing the proof. □

The results in this section remain true after interchanging the roles of w_1 and w_2. Additionally, we have

$$\pi_{\bar{\lambda}}(K)F^{\lambda,v}_{v,\sigma} = B^{\bar{\lambda}}_{\bar{v},\bar{\sigma}}F^{\lambda,v}_{v+iw_1,\sigma} + C^{\bar{\lambda}}_{\bar{v},\bar{\sigma}}F^{\lambda,v}_{v,\sigma} + B^{\bar{\lambda}}_{-\bar{v},\bar{\sigma}}F^{\lambda,v}_{v-iw_1,\sigma}.$$

To see this, recall that $F^{\lambda,v}_{v,\sigma}(z) = H^{\bar{\lambda},\bar{v}}_{\bar{v},\bar{\sigma}}(-z)$, so that

$$\pi_{\bar{\lambda}}(K)F^{\lambda,v}_{v,\sigma}(z) = F^{\lambda,v}_{v,\sigma}(z+iw_1) = H^{\bar{\lambda},\bar{v}}_{\bar{v},\bar{\sigma}}(-z-iw_1) = \pi_{\bar{\lambda}}(K^{-1})H^{\bar{\lambda},\bar{v}}_{\bar{v},\bar{\sigma}}(-z).$$

The result now follows from applying corollary 9.2 to the right-hand side.

The following result will be relevant in our discussion of the multivariate hypergeometric function in chapter 11. We state it here because of its likeness to corollary 9.2.

Corollary 9.3. *For $\tau \notin i\frac{w_2}{2}\cdot\mathbb{Z} \cup (i\frac{w_1}{2} + i\frac{w_2}{2}\cdot\mathbb{Z})$,*

$$\pi_\lambda(K^{-1})H^{\lambda,u}_{\tau,\rho} = \widehat{B}^\lambda_{\tau,\rho}H^{\lambda,u}_{\tau+iw_1,\rho+iw_1} + \widehat{C}^\lambda_{\tau,\rho}H^{\lambda,u}_{\tau,\rho+iw_1} + \widehat{B}^\lambda_{-\tau,\rho}H^{\lambda,u}_{\tau-iw_1,\rho+iw_1} \tag{9.11}$$

$$= \widehat{B}^\lambda_{\tau,-\rho}H^{\lambda,u}_{\tau+iw_1,\rho-iw_1} + \widehat{C}^\lambda_{\tau,-\rho}H^{\lambda,u}_{\tau,\rho-iw_1} + \widehat{B}^\lambda_{-\tau,-\rho}H^{\lambda,u}_{\tau-iw_1,\rho-iw_1}, \tag{9.12}$$

with

$$\widehat{B}^\lambda_{\tau,\rho} = A^\lambda_{\tau,\rho}A^\lambda_{\tau+i\frac{w_1}{2},\rho+i\frac{w_1}{2}}, \quad \textit{and} \quad \widehat{C}^\lambda_{\tau,\rho} = A^\lambda_{\tau,\rho}A^\lambda_{-\tau-i\frac{w_1}{2},\rho+i\frac{w_1}{2}} + A^\lambda_{-\tau,\rho}A^\lambda_{\tau-i\frac{w_1}{2},\rho+i\frac{w_1}{2}}.$$

Proof. The proof of this corollary is highly similar to the proof of corollary 9.2. The first equality follows from applying eq. (9.5) instead of eq. (9.6) in the step after eq. (9.8). The second equality follows from flipping the sign of ρ. □

9.3 A three-term relation for ψ_γ

In this section we will combine previous results to derive a three-term relation for ψ_γ.

Recall the coefficient

$$b^{\rho,\sigma}_x = \frac{iv_\rho(a_x + a_{-x})}{q - q^{-1}} + \frac{iv_\sigma}{q - q^{-1}}.$$

of lemma 5.1, and let

$$\hat{b}^{\rho,\sigma}_x = \frac{iv_\rho a_x + iv_{\rho+iw_1}a_{-x}}{q - q^{-1}} + \frac{iv_\sigma}{q - q^{-1}}. \tag{9.13}$$

Using these coefficients, we state a difference equation for ψ_γ:

Proposition 9.4. *Let $u, \lambda, \rho, \sigma, \tau, v \in \mathbb{C}$, related to $\gamma \in \mathbb{C}^4$ by eq. (8.11)). The following difference equation holds:*

$$\mu^\sigma_v\psi_\gamma(\tau, v) = \mathcal{B}^{\lambda,u}_{\rho,\sigma}(\tau)\psi_\gamma(\tau + iw_1, v) + \mathcal{C}^{\lambda,u}_{\rho,\sigma}(\tau)\psi_\gamma(\tau, v) + \mathcal{B}^{\lambda,u}_{\rho,\sigma}(-\tau)\psi_\gamma(\tau - iw_1, v),$$

with $\mathcal{B}^{\lambda,u}_{\rho,\sigma}(\tau) := \hat{b}^{\tau,\sigma}_u B^\lambda_{\tau,\rho}$, and $\mathcal{C}^{\lambda,u}_{\rho,\sigma}(\tau) := b^{\tau,\sigma}_u C^\lambda_{\tau,\rho} - b^{\rho,\sigma}_u$.

Proof. The verification of this relation is quite straightforward from results we have proved before. We will give the proof for the case of positive w_1 and w_2. The proof for conjugate parameters follows from a similar reasoning, and in fact only requires replacing the $*$ by a $\star$ in the equation below. We have

$$\begin{aligned}
\mu_v^\sigma \psi_\gamma(\tau,v) &= \mu_v^\sigma \langle H^{\lambda,u}_{\tau,\rho}, F^{\lambda,0}_{v,\sigma}\rangle_C = \langle H^{\lambda,u}_{\tau,\rho}, \pi_{\bar\lambda}(X^*_{0,\sigma}) F^{\lambda,0}_{v,\sigma}\rangle_C && \text{(by eq. (7.5))}\\
&= \langle \pi_\lambda(X_\sigma) H^{\lambda,u}_{\tau,\rho}, F^{\lambda,0}_{v,\sigma}\rangle_C && \text{(by lemma 4.1)}\\
&= a_u \langle \pi_\lambda(K^{-1}Y_{u,\rho}) H^{\lambda,u}_{\tau,\rho}, F^{\lambda,0}_{v,\sigma}\rangle_C\\
&\qquad + a_{-u}\langle \pi_\lambda(Y_{u,\rho}K^{-1}) H^{\lambda,u}_{\tau,\rho}, F^{\lambda,0}_{v,\sigma}\rangle_C && \text{(by lemma 5.1)}\\
&\qquad\qquad + b_u^{\rho,\sigma}\langle \pi_\lambda(K^{-1}-1) H^{\lambda,u}_{\tau,\rho}, F^{\lambda,0}_{v,\sigma}\rangle_C .
\end{aligned}$$

We can use corollary 9.2 to translate all instances of $\pi_\lambda(K^{-1})$ into τ-shifts. We can leverage the fact that $\pi_\lambda(Y_{u,\rho})H^{\lambda,u}_{\tau,\rho} = \mu^\rho_\tau H^{\lambda,u}_{\tau,\rho}$ for any value of τ to get rid of all $\pi_\lambda(Y_{u,\rho})$'s. We arrive at

$$\begin{aligned}
\mu_v^\sigma \psi_\gamma(\tau,v) &= a_u\mu^\rho_\tau\Big(B^\lambda_{\tau,\rho}\psi_\gamma(\tau+iw_1,v) + C^\lambda_{\tau,\rho}\psi_\gamma(\tau,v) + B^\lambda_{-\tau,\rho}\psi_\gamma(\tau-iw_1,v)\Big)\\
&\quad + a_{-u}\Big(B^\lambda_{\tau,\rho}\mu^\rho_{\tau+iw_1}\psi_\gamma(\tau+iw_1,v) + C^\lambda_{\tau,\rho}\mu^\rho_\tau\psi_\gamma(\tau,v)\\
&\qquad + B^\lambda_{-\tau,\rho}\mu^\rho_{\tau-iw_1}\psi_\gamma(\tau-iw_1,v)\Big)\\
&\qquad\quad + b_u^{\rho,\sigma}\Big(\Big(B^\lambda_{\tau,\rho}\psi_\gamma(\tau+iw_1,v) + C^\lambda_{\tau,\rho}\psi_\gamma(\tau,v)\\
&\qquad\qquad + B^\lambda_{-\tau,\rho}\psi_\gamma(\tau-iw_1,v)\Big) - \psi_\gamma(\tau,v)\Big). \qquad (9.14)
\end{aligned}$$

By regrouping coefficients, we find

$$\begin{aligned}
\mu_v^\sigma \psi_\gamma(\tau,v) &= \Big(a_u\mu^\rho_\tau + a_{-u}\mu^\rho_{\tau+iw_1} + b_u^{\rho,\sigma}\Big)B^\lambda_{\tau,\rho}\psi_\gamma(\tau+iw_1,v)\\
&\quad + \Big[\Big((a_u+a_{-u})\mu^\rho_\tau + b_u^{\rho,\sigma}\Big)C^\lambda_{\tau,\rho} - b_u^{\rho,\sigma}\Big]\psi_\gamma(\tau,v)\\
&\qquad + \Big(a_u\mu^\rho_\tau + a_{-u}\mu^\rho_{\tau-iw_1} + b_u^{\rho,\sigma}\Big)B^\lambda_{-\tau,\rho}\psi_\gamma(\tau-iw_1,v). \qquad (9.15)
\end{aligned}$$

We conclude the proof by observing that the coefficients of lemma 5.1 satisfy

$$\begin{aligned}
a_u\mu^\rho_\tau + a_{-u}\mu^\rho_{\tau+iw_1} + b_u^{\rho,\sigma} &= a_u\frac{iv_\tau - iv_\rho}{q-q^{-1}} + a_{-u}\frac{iv_{\tau+iw_1} - iv_\rho}{q-q^{-1}} + \frac{iv_\rho(a_u+a_{-u})}{q-q^{-1}} + \frac{iv_\sigma}{q-q^{-1}}\\
&= \frac{iv_\tau a_u + iv_{\tau+iw_1}a_{-u}}{q-q^{-1}} + \frac{iv_\sigma}{q-q^{-1}} = \hat b_u^{\tau,\sigma}, \qquad (9.16)
\end{aligned}$$

and similarly

$$(a_u+a_{-u})\mu^\rho_\tau + b_u^{\rho,\sigma} = \frac{iv_\tau(a_u+a_{-u})}{q-q^{-1}} + \frac{iv_\sigma}{q-q^{-1}} = b_u^{\tau,\sigma} \qquad (9.17)$$

and

$$a_u\mu^\rho_\tau + a_{-u}\mu^\rho_{\tau-iw_1} + b_u^{\rho,\sigma} = \hat b_u^{-\tau,\sigma}, \qquad (9.18)$$

so that filling in these values into eq. (9.15) gives the desired result. ☐

9.4 Ruijsenaars's function R solving the Askey-Wilson difference equations

By now, we have collected most ingredients for proving that Ruijsenaars's function R is an eigenfunction of the Askey-Wilson operator, which we will do in this section. The proposition below shows that it is actually an eigenfunction of four related Askey-Wilson difference operators:

Proposition 9.5. *Let $\gamma \in \mathbb{C}^4$ and let w_1 and w_2 be either both positive, or a conjugate pair with positive real part. Let the operator $\mathcal{L}^{w_1,w_2}_{\gamma,x}$, the functions υ and R, and the vector $\hat{\gamma}$ be defined as before. Then*

$$\mathcal{L}^{w_1,w_2}_{\gamma,\tau} R(w_1, w_2, \gamma; \tau, \upsilon) = \upsilon(w_1, w_2, \gamma; \upsilon) R(w_1, w_2, \gamma; \tau, \upsilon),$$

and

$$\mathcal{L}^{w_1,w_2}_{\hat{\gamma},\upsilon} R(w_1, w_2, \gamma; \tau, \upsilon) = \upsilon(w_1, w_2, \hat{\gamma}; \tau) R(w_1, w_2, \gamma; \tau, \upsilon).$$

Furthermore, we can interchange the roles of w_1 and w_2 in these relations, so that also

$$\mathcal{L}^{w_2,w_1}_{\gamma,\tau} R(w_1, w_2, \gamma; \tau, \upsilon) = \upsilon(w_2, w_1, \gamma; \upsilon) R(w_1, w_2, \gamma; \tau, \upsilon),$$

and

$$\mathcal{L}^{w_2,w_1}_{\hat{\gamma},\upsilon} R(w_1, w_2, \gamma; \tau, \upsilon) = \upsilon(w_2, w_1, \hat{\gamma}; \tau) R(w_1, w_2, \gamma; \tau, \upsilon).$$

Proof. The second equality follows from the first one if we recalling from section 8.2 that

$$R(w_1, w_2, \gamma; \tau, \upsilon) = R(w_1, w_2, \hat{\gamma}; \upsilon, \tau).$$

The latter two equalities follow directly from the former two by observing that R is invariant under the interchange of w_1 and w_2. Thus, it suffices to show that the first equality holds.

Recall from eq. (8.12) that

$$R(w_1, w_2, \mathbf{c}(\gamma); \tau, \upsilon) = \frac{1}{\sqrt{a_1 a_2}} \frac{\prod_{j=1}^{3} G(i(w + \gamma_0 + \gamma_j))}{G(\pm\tau + i\gamma_0) G(\pm\upsilon + i\hat{\gamma}_0)} \times \psi_\gamma(\tau, \upsilon).$$

Since the operator $\mathcal{L}^{w_1,w_2}_{\gamma,\tau}$ only acts on the variable τ, the proposition holds if and only if

$$\Psi(\tau) := \frac{1}{G(\pm\tau + i\gamma_0)} \times \psi_\gamma(\tau, \upsilon) = G(\pm\tau - i\gamma_0)\psi_\gamma(\tau) \tag{9.19}$$

is an eigenfunction with eigenvalue $\upsilon(w_1, w_2, \gamma; \upsilon)$.[2]

First, note that

$$\Psi(\tau + iw_1) = \frac{\cosh \frac{\pi}{w_2}\left(\tau + i\frac{w_1}{2} - i\gamma_0\right)}{\cosh \frac{\pi}{w_2}\left(\tau + i\frac{w_1}{2} + i\gamma_0\right)} G(\pm\tau - i\gamma_0)\psi_\gamma(\tau + iw_1, \upsilon) \tag{9.20}$$

by the difference equation eq. (6.3) of G.

[2] Although the function Ψ implicitly depends on υ and γ, we do not add them as arguments in order to simplify the notation.

Writing out $\mathcal{L}^{w_1,w_2}_{\gamma,\tau}\Psi(\tau)$ as a difference equation, we have

$$\mathcal{L}^{w_1,w_2}_{\gamma,\tau}\Psi(\tau) = \mathcal{A}^{w_1,w_2}_{\gamma}(\tau)\big[\Psi(\tau+iw_1)-\Psi(\tau)\big] + \mathcal{A}^{w_1,w_2}_{\gamma}(-\tau)\big[\Psi(\tau-iw_1)-\Psi(\tau)\big].$$

Using eq. (9.20), we can rewrite the right-hand side as

$$\begin{aligned} G(\pm\tau - i\gamma_0)\Bigg[\mathcal{A}^{w_1,w_2}_{\gamma}(\tau)&\left(\frac{\cosh\frac{\pi}{w_2}\big(\tau+i\frac{w_1}{2}-i\gamma_0\big)}{\cosh\frac{\pi}{w_2}\big(\tau+i\frac{w_1}{2}+i\gamma_0\big)}\psi_\gamma(\tau+iw_1,v)-\psi_\gamma(\tau,v)\right)\\ +\mathcal{A}^{w_1,w_2}_{\gamma}(-\tau)&\left(\frac{\cosh\frac{\pi}{w_2}\big(-\tau+i\frac{w_1}{2}-i\gamma_0\big)}{\cosh\frac{\pi}{w_2}\big(-\tau+i\frac{w_1}{2}+i\gamma_0\big)}\psi_\gamma(\tau-iw_1,v)-\psi_\gamma(\tau,v)\right)\Bigg]. \end{aligned} \tag{9.21}$$

We will focus our attention on the part of the expression between the square brackets. In appendix A, we show (eq. (A.2)) that the coefficient of $\psi_\gamma(\tau+iw_1,v)$ in this part equals

$$-i\frac{q-q^{-1}}{4}\mathcal{B}^{\lambda,u}_{\rho,\sigma}(\tau).$$

Flipping the sign of τ, the coefficient of $\psi_\gamma(\tau-iw_1,v)$ equals $-i\frac{q-q^{-1}}{4}\mathcal{B}^{\lambda,u}_{\rho,\sigma}(-\tau)$. The coefficient of $\psi_\gamma(\tau,v)$ within the square brackets of eq. (9.21) is equal to

$$-i\frac{q-q^{-1}}{4}\mathcal{C}^{\lambda,u}_{\rho,\sigma}(\tau)+\cosh\frac{\pi}{w_2}\big(\rho+i\frac{w_1}{2}-u\pm\sigma\big),$$

which follows from eq. (A.6) in appendix A.

By proposition 9.4, we find that the contents of the pair of square brackets in eq. (9.21) equal

$$\begin{aligned} &\left[-i\frac{q-q^{-1}}{4}\mu^\sigma_v+\cosh\frac{\pi}{w_2}\big(\rho+i\frac{w_1}{2}-u\pm\sigma\big)\right]\psi_\gamma(\tau,v)\\ &\qquad=\frac{1}{2}\left[\cosh\frac{2\pi v}{w_2}-\cosh\frac{2\pi\sigma}{w_2}+\cosh\frac{\pi}{w_2}(iw_1+2\rho-2u)+\cosh\frac{2\pi\sigma}{w_2}\right]\psi_\gamma(\tau,v)\\ &\qquad=\frac{1}{2}\left[\cosh\frac{2\pi v}{w_2}+\cosh\frac{\pi}{w_2}(iw_1+2i\hat{\gamma}_0)\right]\psi_\gamma(\tau,v). \end{aligned}$$

We conclude that

$$\mathcal{L}^{w_1,w_2}_{\gamma,\tau}\Psi(\tau)=v(w_1,w_2,\gamma;v)\Psi(\tau) \tag{9.22}$$

and hence that

$$\mathcal{L}^{w_1,w_2}_{\gamma,\tau}R(w_1,w_2,\gamma;\tau,v)=v(w_1,w_2,\gamma;v)R(w_1,w_2,\gamma;\tau,v).$$ □

We define $L^{w_1,w_2}_{\gamma,x}$ to be the operator acting on a function f by

$$L^{w_1,w_2}_{\gamma,x}f(x):=\frac{\mathcal{L}^{w_1,w_2}_{\gamma,x}\big[G(\pm(\,\cdot\,)-i\gamma_0)\times f\big](x)}{G(\pm x-i\gamma_0)}. \tag{9.23}$$

Then the following corollary is immediate from eqs. (9.19) and (9.22) in the proof of proposition 9.5:

Corollary 9.6. *The function ψ_γ is an eigenfunction of $L^{w_1,w_2}_{\gamma,x}$:*

$$L^{w_1,w_2}_{\gamma,\tau}\psi_\gamma(\tau,v)=v(w_1,w_2,\gamma;v)\psi_\gamma(\tau,v),$$

and similarly

$$L_{\widehat{\gamma},\upsilon}^{w_1,w_2}\psi_\gamma(\tau,\upsilon) = \upsilon(w_1,w_2,\widehat{\gamma};\tau)\psi_\gamma(\tau,\upsilon),$$
$$L_{\gamma,\tau}^{w_2,w_1}\psi_\gamma(\tau,\upsilon) = \upsilon(w_2,w_1,\gamma;\upsilon)\psi_\gamma(\tau,\upsilon) \quad \textit{and}$$
$$L_{\widehat{\gamma},\upsilon}^{w_2,w_1}\psi_\gamma(\tau,\upsilon) = \upsilon(w_2,w_1,\widehat{\gamma};\tau)\psi_\gamma(\tau,\upsilon).$$

Remark 9.7. Our derivation of the difference equations explicitly employs the star structure on the modular double. As we defined that structure only for w_1 and w_2 either positive of complex conjugates, we cannot straightforwardly extend the results to arbitrary w_1 and w_2. Ruijsenaars has shown ([36, thm. 3.1]) that the results can be extended to all w_1 and w_2 in the complex plane with positive real parts. ∎

Chapter 10

Kernel of a unitary map on $L^2(0, \infty)$

In [38], Ruijsenaars demonstrated that the function R_{ren} discussed in section 8.2 can be viewed as the kernel of a Hilbert space isomorphism[1] on weighted L^2 function spaces. The function $\mathcal{E}$ moreover is the kernel of a unitary map on $L^2(0, \infty)$. Ruijsenaars proved these results for real parameters $\gamma \in \mathbb{R}^4$ and positive ω_1 and ω_2. By these results, we can straightforwardly conclude that, under the same conditions on γ, ω_1, and ω_2, ψ_γ is also the kernel of a Hilbert space isomorphism between weighted L^2 spaces.

The real values of γ for which ψ_γ forms the kernel of a Hilbert space isomorphism correspond to purely imaginary values of u, λ, ρ and σ, as follows from eq. (8.11). In our multivariate generalisation of this kernel in chapter 11 of this thesis, we need to choose ρ and σ on the real axis. Additionally, we desire our results to hold for a pair of conjugate parameters ω_1 and ω_2. Therefore, we must adapt Ruijsenaars's results to our specific requirements, which we will accomplish in this chapter.

Our approach does not mimic Ruijsenaars's proof. We used a method similar to methods used for other quantum group-related transformations in e.g. [24, 28, 17, 18]. It has been employed in [23] to recover Ruijsenaars's results on R_{ren} and $\mathcal{E}$ as kernels of Hilbert space isomorphisms.

The function transformation we study will be of the form

$$\mathcal{R}_\gamma f(v) := \int f(\tau)\psi_\gamma(\tau, v)W_\gamma(\tau)\,\mathrm{d}\tau$$

for $f \in L^2((0, \infty), W_\gamma)$, and the image will be in $L^2((0, \infty), W_{\hat{\gamma}})$. We will specify the weight functions shortly in section 10.1. Our objective is to establish conditions on γ ensure the convergence of $\mathcal{R}_\gamma$ for all $f \in L^2(0, \infty)$ and preserves the inner product. To achieve this, we will define a truncated inner product in section 10.2, using the weight function. We then introduce an integral form related to the concept of a Wronski determinant. This form allows us to compute the truncated inner product of two copies of ψ_γ. By utilising the asymptotics of ψ_γ, we demonstrate in section 10.3 that this truncated inner product approximates the Dirac delta function (in a weak sense and up to a certain weight function). We use the latter result in section 10.4 to show that our function transform preserves inner products. By explicit construction of the inverse transformation, we show that $\mathcal{R}$ is a Hilbert space isomorphism. In

[1] A surjective map $U : H_1 \to H_2$ between Hilbert spaces is a *Hilbert space isomorphism* if it preserves the inner product: $\langle f, g\rangle_{H_1} = \langle Uf, Ug\rangle_{H_2}$. If $H_1 = H_2$, so that U is an endomorphism, we call U a *unitary* map.

section 10.4, we employ the aforementioned result to show that our function transform preserves inner products. Furthermore, through explicit construction of the inverse transformation, we establish that $\mathcal{R}$ is a Hilbert space isomorphism. Likewise, we show that $\mathcal{E}$ is the kernel of a unitary operator, and we identify its inverse.

Throughout this chapter, we assume again that ω_1 and ω_2 are either positive or a pair of conjugate parameters with positive real part. In the case of positive parameters, assume $\omega_1 < \omega_2$, else interchange their roles. Let $\omega = (\omega_1 + \omega_2)/2$. Let $\omega_0 = \omega$ when dealing with conjugate parameters, and let $\omega_0 = \omega_1$ in the positive case. In the latter case, we have $\omega_0 - \omega = \frac{\omega_1 - \omega_2}{2}$; in the conjugate case, $\omega_0 - \omega = 0$ holds.

10.1 The weight function W_γ

In this chapter, we will work with the weight function defined as follows:

$$W_\gamma(\tau) := \frac{1}{\omega_1\omega_2}\frac{1}{c'(\gamma;\pm\tau)} = \frac{1}{\omega_1\omega_2}G(i\omega \pm 2\tau)G(-i\gamma_0 \pm \tau)\prod_{j=1}^{3}G(i\gamma_j \pm \tau)$$
$$= \frac{1}{\omega_1\omega_2}G(i\omega \pm 2\tau)G(\pm\tau \pm \rho - \lambda)G(\pm\tau \pm \sigma - u). \qquad (10.1)$$

By using the relation $\overline{G(a+bi)} = G(-\overline{a+bi}) = G(-a+bi)$ for real a and b, we observe that $G(\pm a + bi)$ is real and positive (non-negative). Therefore, for $\tau \in \mathbb{R}$, the weight $W_\gamma(\tau)$ is a positive (non-negative) weight function whenever λ and u are purely imaginary, and ρ and σ are either real or imaginary.

We denote $\mathcal{H}_\gamma := L^2((0,\infty), W_\gamma)$. Its inner product is given by

$$\langle f, g\rangle_\gamma = \int_0^\infty f(x)\overline{g(x)}W_\gamma(x)\,\mathrm{d}x.$$

We use the convention to regard the functions in $\mathcal{H}_\gamma$ as even functions on the real line, extending them the obvious way.

10.2 The truncated inner product and the Wronskian

We will now define the truncated inner product and our version of the Wronskian. In the case of conjugate parameters ω_1 and ω_2, we will need a slightly more complicated notion of the truncated inner product, to avoid integration over a singularity in a later stage.

We will start with this complicated version of the truncated inner product for the case $\omega_1 = \overline{\omega_2}$. For $N > 2|\operatorname{Im}\omega_1|$ and $\epsilon \in [0, |\operatorname{Im}\omega_1|)$, we define the curve $C_{N,\epsilon}$ by modifying the directed line segment $[-N, N]$ as follows: we remove line segments of length 2ϵ around the points $\pm\operatorname{Im}\omega_1$, replacing them by an upward-facing half-circle of radius ϵ at $-\operatorname{Im}\omega_1$ and a downward-facing one around $\operatorname{Im}\omega_1$, as sketched in fig. 10.1, so that the curve

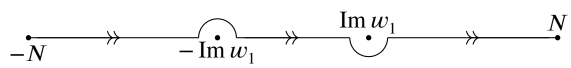


Figure 10.1: An illustration of the curve $C_{N,\epsilon}$ in the case $\operatorname{Im}\omega_1 > 0$. The curve is a deformation of the line-segment $[-N, N]$ with oppositely facing indentations at $\pm\operatorname{Im}\omega_1$.

passes above the point $-\operatorname{Im} w_1$ and below the point $\operatorname{Im} w_1$. We introduce those circular arcs to avoid the upcoming singularities I mentioned.

Let f and g be functions on $\mathbb{C}$, let $N > 2\operatorname{Im} w_1$ and $\epsilon \in [0, |\operatorname{Im} w_1|)$. We define

$$\langle f, g\rangle_{N,\epsilon} := \int_{C_{N,\epsilon}} f(x)\bar{g}(x)W_\gamma(x)\,\mathrm{d}x,$$

whenever the integral converges, with $\bar{g}(x) = \overline{g(\bar{x})}$ as before. If the restrictions of f and g to $\mathbb{R}$ are in $\mathcal{H}_\gamma$, and f and g are continuous around $\pm\operatorname{Im} w$, taking the limits $N \to \infty, \epsilon \to 0$ gives twice the (W_γ-weighted) inner product of f and g.

In the general case, so for either positive or conjugate parameters w_1 and w_2, and for functions f and g functions defined on $\mathbb{R}$ (not necessarily in $\mathcal{H}_\gamma$) and $N > 0$, we define the ϵ-independent truncated inner product $\langle f, g\rangle_N$ by

$$\langle f, g\rangle_N := \int_{-N}^{N} f(x)\bar{g}(x)W_\gamma(x)\,\mathrm{d}x.$$

If f and g are analytic on the strip $\mathbb{R} \times i[-w_0, w_0]$, we define their Wronskian $[f, g]$ by

$$[f, g](z) := \int_{z-iw_1}^{z} \Big(f(x + iw_1)\bar{g}(x) - f(x)\bar{g}(x + iw_1)\Big) \times t_\gamma(x)\mathcal{A}_\gamma^{w_1,w_2}(x)W_\gamma(x)\,\mathrm{d}x, \tag{10.2}$$

where $\mathcal{A}_\gamma^{w_1,w_2}$ is the Askey-Wilson coefficient defined in eq. (9.2), t_γ is defined as

$$t_\gamma(x) := \frac{\cosh\frac{\pi}{w_2}\left(x + i\frac{w_1}{2} - i\gamma_0\right)}{\cosh\frac{\pi}{w_2}\left(x + i\frac{w_1}{2} + i\gamma_0\right)} = G(\pm x + i\gamma_0)G(\pm(x + iw_1) - i\gamma_0),$$

and the integration is performed along the line segment from $z - iw_1$ to z.

Note that the product $t_\gamma(x)\mathcal{A}_\gamma^{w_1,w_2}(x)W_\gamma(x)$ is invariant under interchanging x by $-x - iw_1$, as we derive in appendix B. Abbreviating $V_\gamma(x) := t_\gamma(x)\mathcal{A}_\gamma^{w_1,w_2}(x)W_\gamma(x)$, and assuming f and g to be even, we can write

$$\begin{aligned}
[f, g](-z) &= \int_{-z-iw_1}^{-z} \Big(f(x + iw_1)\bar{g}(x) - f(x)\bar{g}(x + iw_1)\Big)V_\gamma(x)\,\mathrm{d}x \\
&= -\int_{z}^{z-iw_1} \Big(f(-x')\bar{g}(-x' - iw_1) - f(-x' - iw_1)\bar{g}(-x')\Big)V_\gamma(-x' - iw_1)\,\mathrm{d}x' \\
&\qquad\qquad\text{(by substituting } x' = -x - iw_1) \\
&= \int_{z-iw_1}^{z} \Big(f(x')\bar{g}(x' + iw_1) - f(x' + iw_1)\bar{g}(x')\Big)V_\gamma(x')\,\mathrm{d}x' \\
&\qquad\qquad\text{(as } f, g \text{ are even and using the symmetry of } V_\gamma) \\
&= -\int_{z-iw_1}^{z} \Big(f(x' + iw_1)\bar{g}(x') - f(x')\bar{g}(x' + iw_1)\Big)V_\gamma(x')\,\mathrm{d}x' \\
&= -[f, g](z),
\end{aligned}$$

so that $[f, g]$ is an odd function for even f and g. We will use this property in proving the following lemma, which relates the truncated inner product, the difference operator $L_\gamma^{w_1,w_2}$ and the Wronskian. The lemma comes in two versions, one for the case of a conjugate pair w_1 and

w_2, one for the positive case. We will first state and prove the lemma for conjugate parameters.

Lemma 10.1 (Conjugate w_j's)**.** *Suppose that* $w_1 = \overline{w_2}$ *with* $\operatorname{Im} w_1 \neq 0$ *and* $\operatorname{Re} w_1 > 0$*, and that* $\operatorname{Im}\lambda$ *and* $\operatorname{Im} u$ *are both smaller than* w*. Let* f, g *be even functions that are analytic on the strip* $\mathbb{R}\times i[-w,w]$*. Then for* N *sufficiently large, and* $\epsilon > 0$ *sufficiently small, the following relation holds:*

$$\langle L_\gamma^{w_1,w_2} f, g\rangle_{N,\epsilon} - \langle f, L_{\bar\gamma}^{w_2,w_1} g\rangle_{N,\epsilon} = 2[f,g](N).$$

Proof. Let f and g be as in the statement of the lemma. From the definition of $\mathcal{A}_\gamma^{w_1,w_2}$ in eq. (9.2), we can directly derive that

$$\overline{\mathcal{A}_{\bar\gamma}^{w_2,w_1}(\bar x)} = \mathcal{A}_\gamma^{\bar w_2,\bar w_1}(-x) = \mathcal{A}_\gamma^{w_1,w_2}(-x).$$

Additionally,

$$\overline{G(-i\bar\gamma_0 \pm \bar x)} = G(-i\gamma_0 \pm x).$$

Hence,

$$\begin{aligned}
&\overline{L_{\bar\gamma}^{w_2,w_1} g(\bar x)}\\
&\quad = \operatorname{Conj}\Bigg[\frac{1}{G(-i\bar\gamma_0 \pm \bar x)}\\
&\qquad\quad \times\Big(\mathcal{A}_{\bar\gamma}^{w_2,w_1}(\bar x)\big(G(-i\bar\gamma_0 \pm (\bar x + i\bar w_1))g(\bar x + i\bar w_1) - G(-i\bar\gamma_0 \pm \bar x)g(\bar x)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (x \leftrightarrow -x)\Big)\Bigg]\\
&\quad = \operatorname{Conj}\Bigg[\frac{1}{G(-i\bar\gamma_0 \pm \bar x)}\\
&\qquad\quad \times\Big(\mathcal{A}_{\bar\gamma}^{w_2,w_1}(\bar x)\big(G(-i\bar\gamma_0 \pm \overline{(x - iw_1)})g(\overline{x - iw_1}) - G(-i\bar\gamma_0 \pm \bar x)g(\bar x)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (x \leftrightarrow -x)\Big)\Bigg]\\
&\quad = \frac{1}{G(-i\gamma_0 \pm x)}\Big(\mathcal{A}_\gamma^{w_1,w_2}(-x)\big(G(-i\gamma_0 \pm (x - iw_1))\bar g(x - iw_1) - G(-i\gamma_0 \pm x)\bar g(x)\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (x \leftrightarrow -x)\Big)\\
&\quad = L_\gamma^{w_1,w_2}\bar g(x).
\end{aligned} \tag{10.3}$$

Note moreover that we can use the function t_γ to express

$$L_\gamma^{w_1,w_2} f(x) = \mathcal{A}_\gamma^{w_1,w_2}(x)\big(t_\gamma(x) f(x + iw_1) - f(x)\big) + (x \leftrightarrow -x). \tag{10.4}$$

Applying both eqs. (10.3) and (10.4), for $N > 2\left|\operatorname{Im} w_1\right|$ and $\epsilon > 0$ sufficiently small,[2] we can

[2] By the assumptions on f and g, $f(\cdot + iw_1)$ is analytic around at the point $\operatorname{Im} w_1$. Analyticity at a point implies analyticity in some open neighbourhood of that point by definition, hence by picking $\epsilon > 0$ sufficiently small, $f(\cdot + iw_1)$ is analytic on $C_{N,\epsilon}$. The same extends to $\bar g(x + iw_1)$, possibly after picking ϵ even smaller.

write

$$\langle L_\gamma^{w_1,w_2} f, g\rangle_{N,\epsilon} - \langle f, L_{\bar\gamma}^{w_2,w_1} g\rangle_{N,\epsilon}$$
$$= \int_{C_{N,\epsilon}} \Big[A(x)\big(t_\gamma(x) f(x+iw_1) - f(x)\big) + (x \leftrightarrow -x)\Big] \bar g(x) W_\gamma(x)\,\mathrm{d}x$$
$$- \int_{C_{N,\epsilon}} f(x)\Big[A(x)\big(t_\gamma(x)\bar g(x+iw_1) - \bar g(x)\big) + (x \leftrightarrow -x)\Big] W_\gamma(x)\,\mathrm{d}x.$$

Reordering terms, we write this as

$$= \int_{C_{N,\epsilon}} A(x) t_\gamma(x)\big[f(x+iw_1)\bar g(x) - f(x)\bar g(x+iw_1)\big] W_\gamma(x)\,\mathrm{d}x$$
$$+ \int_{C_{N,\epsilon}} A(-x) t_\gamma(-x)\big[f(x-iw_1)\bar g(x) - f(x)\bar g(x-iw_1)\big] W_\gamma(x)\,\mathrm{d}x.$$

Recall that $t_\gamma A W_\gamma$ is invariant under replacing x by $-x-iw_1$, and that W_γ is even in x, as shown in appendix B. We can utilise this to rewrite the latter integral to

$$\int_{C_{N,\epsilon}} A(x-iw_1) t_\gamma(x-iw_1)\big[f(x-iw_1)\bar g(x) - f(x)\bar g(x-iw_1)\big] W_\gamma(x-iw_1)\,\mathrm{d}x.$$

By a shift of the integral domain this equals

$$\int_{C_{N,\epsilon}-iw_1} A(x) t_\gamma(x)\big[f(x)\bar g(x+iw_1) - f(x+iw_1)\bar g(x)\big] W_\gamma(x)\,\mathrm{d}x.$$

With a slight abuse of notation,[3] we can thus write

$$\langle L_\gamma^{w_1,w_2} f, g\rangle_{N,\epsilon} - \langle f, L_{\bar\gamma}^{w_2,w_1} g\rangle_{N,\epsilon}$$
$$= \left[\int_{C_{N,\epsilon}} - \int_{C_{N,\epsilon}-iw_1}\right] A(x) t_\gamma(x)\big[f(x+iw_1)\bar g(x) - f(x)\bar g(x+iw_1)\big] W_\gamma(x)\,\mathrm{d}x. \tag{10.5}$$

Assumed that the integrand in analytic in the area enclosed by $C_{N,\epsilon}$, $C_{N,\epsilon} - iw_1$ and the line segments connecting the endpoints at $-N$ and $-N-iw_1$, respectively at N and $N-iw_1$, by Cauchy's integral theorem this equals

$$= \left[\int_{N-iw_1}^{N} - \int_{-N-iw_1}^{-N}\right] A(x) t_\gamma(x)\big[f(x+iw_1)\bar g(x) - f(x)\bar g(x+iw_1)\big] W_\gamma(x)\,\mathrm{d}x \tag{10.6}$$
$$= [f,g](N) - [f,g](-N) = 2[f,g](N),$$

the last equality holding as $[f,g]$ is odd.

It remains to check that the integrand is indeed analytic in the given region. By the premise of the lemma, the product $f(x+iw_1)\bar g(x) - f(x)\bar g(x+iw_1)$ is analytic in the given region. We may rewrite eq. (B.4) as follows:

$$t_\gamma(x) A(x) W_\gamma(x) = \frac{1}{4w_1w_2} \frac{\sinh\frac{2\pi x}{w_1}}{\sinh\frac{2\pi(x+iw)}{w_2}} \frac{G(x+iw_1-i\gamma_0)}{G(x+i\gamma_0)} \prod_{j=1}^{3} \frac{G(x+iw_1+i\gamma_j)}{G(x-i\gamma_j)}. \tag{10.7}$$

[3] We use the notation $\left[\int_A - \int_B\right] f(x)\,\mathrm{d}x$ to abbreviate $\int_A f(x)\,\mathrm{d}x - \int_B f(x)\,\mathrm{d}x$.

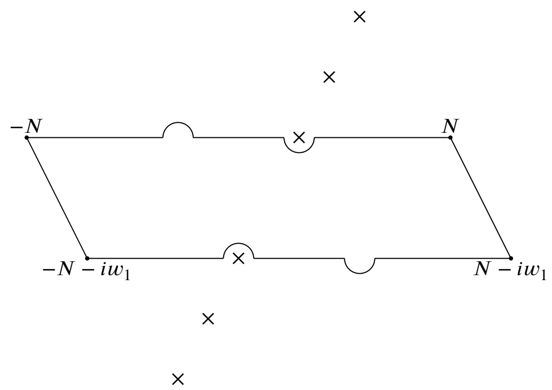


Figure 10.2: An illustration of the area enclosed by the paths of integration in eqs. (10.5) and (10.6) in the case $\operatorname{Im} w_1 > 0$. The marks "×" indicate the positions of the poles at $\pm m = 1, 2, 3$ given by eq. (10.9).

We study its analytic properties.

The hyperbolic gamma functions in eq. (10.7) have poles at

$$x = \pm\rho + \lambda - iw - iw_1 - ikw_1 - ilw_2, \quad x = \pm\rho - \lambda + iw + ikw_1 + ilw_2,$$
$$x = \pm\sigma + u - iw - iw_1 - ikw_1 - ilw_2 \quad \text{and} \quad x = \pm\sigma - u + iw + ikw_1 + ilw_2, \tag{10.8}$$

with $j = 1, 2, 3$ and $k, l = 0, 1, 2, 3, \ldots$. Clearly, if $\operatorname{Im}\lambda, \operatorname{Im} u < w$, then for $\epsilon > 0$ sufficiently small, all of these poles are located outside the enclosed region.

The fraction of hyperbolic sines in eq. (10.7) has a removable singularity at $x = -i\frac{w_1}{2}$, and has its remaining poles located at

$$x = -i\frac{w_1}{2} + im\frac{w_2}{2}, \tag{10.9}$$

with $m \in \mathbb{Z} \setminus \{0\}$. Note that the poles with $m = \pm 1$ are located at $-i\frac{w_1}{2} + i\frac{w_2}{2} = \operatorname{Im} w_1$ and $-i\frac{w_1}{2} - i\frac{w_2}{2} = -iw$. Our choice of indentations of $C_{N,\epsilon}$ was done in such a way that for $\epsilon > 0$, these poles remain outside the enclosed region; see fig. 10.2 for an illustration. Thus, if we choose $\epsilon > 0$ sufficiently small and $N > 2\left|\operatorname{Im} w_1\right|$, indeed the equality eq. (10.6) holds. □

For the case of positive coefficients w_1 and w_2, a similar result holds. We state it for the case $0 < w_1 < w2$. In case $w_1 > w_2$, the result holds by interchanging the roles of w_1 and w_2.

Lemma 10.2 (Positive w_j's)**.** *Suppose that* $0 < w_1 < w_2$, *and that* $\operatorname{Im}\lambda$ *and* $\operatorname{Im} u$ *are both smaller than* w. *Let* f, g *be even functions that are analytic on the strip* $\mathbb{R} \times i[-w_1, w_1]$. *For N sufficiently large, the following relation holds:*

$$\langle L_\gamma^{w_1,w_2} f, g\rangle_N - \langle f, L_{\bar\gamma}^{w_1,w_2} g\rangle_N = 2[f,g](N).$$

Proof. The proof of this lemma is highly similar to the proof of lemma 10.1, but slightly simpler. We provide only the general outline.

In a similar way as in the proof of the conjugate case, one checks that $\overline{L_{\bar\gamma}^{w_1,w_2} g(\bar x)} = L_\gamma^{w_1,w_2}\bar g(x)$. Following our earlier reasoning we observe that

$$\begin{aligned}
&\langle L_\gamma^{w_1,w_2} f, g\rangle_N - \langle f, L_{\bar\gamma}^{w_1,w_2} g\rangle_N \\
&\quad = \left[\int_{-N}^{N} - \int_{-N-iw_1}^{N-iw_1}\right] A(x) t_\gamma(x) \big[f(x+iw_1)\bar g(x) - f(x)\bar g(x+iw_1)\big] W_\gamma(x)\, \mathrm{d}x.
\end{aligned}$$

As the pole locations of $At_\gamma W_\gamma$ are again adequately described by eqs. (10.8) and (10.9) with $j = 1,2,3$, $k,l = 0,1,2,3,\dots$ and $m \in \mathbb{Z}\setminus\{0\}$, and since we assumed $\operatorname{Im}\lambda, \operatorname{Im} u < w$ and $w_2 > w_1$, the integrand is analytic in the rectangle with corners $\{-N, N, N - iw_1, -N - iw_1\}$. Therefore, by Cauchy's integral theorem, the right-hand side of above expression equals

$$\left[\int_{N-iw_1}^{N} - \int_{-N-iw_1}^{-N}\right] A(x)t_\gamma(x)\big[f(x+iw_1)\bar{g}(x) - f(x)\bar{g}(x+iw_1)\big]W_\gamma(x)\,\mathrm{d}x$$
$$= [f,g](N) - [f,g](-N) = 2[f,g](N). \qquad \square$$

The following result holds in both the conjugate case as in the real case and follows from lemmas 10.1 and 10.2 respectively.

Corollary 10.3. *Let $\rho,\sigma \in \mathbb{R}$ and $u,\lambda \in i(w_0 - w, w)$. Let $y, y' \in \mathbb{R}$ with $y \neq \pm y'$. The following equality holds:*

$$\langle \psi_\gamma(\,\cdot\,,y), \psi_\gamma(\,\cdot\,,y')\rangle_N = \frac{2[\psi_\gamma(\,\cdot\,,y), \psi_\gamma(\,\cdot\,,y')](N)}{\sinh \frac{\pi}{w_2}(y \pm y')}.$$

Proof. Our plan is to use the fact that $\psi_\gamma(\,\cdot\,,y)$ is an eigenfunction of L_γ^{w1,w_2}, together with the appropriate choice from the previous two lemmas.

Let us introduce the shortened notation $\psi_\gamma^y := \psi_\gamma(\,\cdot\,,y)$.

In the case of conjugate parameters w_1, w_2, note that by corollary 9.6,

$$\langle L_\gamma^{w_1,w_2}\psi_\gamma^y, \psi_\gamma^{y'}\rangle_{N,\epsilon} = v(w_1,w_2,\gamma;y)\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_{N,\epsilon}.$$

Considering $\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_{N,\epsilon}$ as an integral, we observe that its integrand is analytic and has no singularities around $\pm \operatorname{Im} w_1$. Thus, for $\epsilon \geq 0$ sufficiently small, $\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_{N,\epsilon}$ is independent of ϵ and equals $\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_N$ by Cauchy's integral theorem. Hence, for $\epsilon > 0$ sufficiently small,

$$\langle L_\gamma^{w_1,w_2}\psi_\gamma^y, \psi_\gamma^{y'}\rangle_{N,\epsilon} = v(w_1,w_2,\gamma;y)\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_N. \tag{10.10}$$

As $\gamma = -i(\rho+\lambda, \rho-\lambda, \sigma-u, -\sigma-u)$, for ρ and σ real, and u and λ on the imaginary axis, we have $\bar\gamma = -i(-\rho+\lambda, -\rho-\lambda, -\sigma-u, \sigma-u)$, i.e. the signs of ρ and σ have flipped. Since ψ_γ is even in ρ and σ, this flip of signs is irrelevant, and we conclude that $\psi_{\bar\gamma} = \psi_\gamma$. By a similar reasoning as the one proceeding eq. (10.10), we find

$$\langle \psi_\gamma^y, L_{\bar\gamma}^{w_2,w_1}\psi_\gamma^{y'}\rangle_{N,\epsilon} = \langle \psi_\gamma^y, L_{\bar\gamma}^{w_2,w_1}\psi_{\bar\gamma}^{y'}\rangle_{N,\epsilon} = \overline{v(w_2,w_1,\bar\gamma;y')}\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_N$$

for small values of ϵ.

By lemma 8.4, when $\rho,\sigma \in \mathbb{R}$ and $u,\lambda \in i(w_0 - w, w)$, the function ψ_γ^y has no singularities on the strip $\mathbb{R}\times[-iw_0, iw_0]$. Therefore, we can apply either lemma 10.1 or lemma 10.2, depending on the values of w_1 and w_2.

In both the conjugate case and the real case, utilising either lemma 10.1 or lemma 10.2, we obtain that (in the conjugate case: for ϵ sufficiently small):

$$\begin{aligned}
2[\psi_\gamma^y,\psi_\gamma^{y'}](N) &= \langle L_\gamma^{w_1,w_2}\psi_\gamma^y,\psi_\gamma^{y'}\rangle_{N(,\epsilon)} - \langle \psi_\gamma^y, L_{\bar\gamma}^{\bar w_1,\bar w_2}\psi_\gamma^{y'}\rangle_{N(,\epsilon)}\\
&= \tfrac{1}{2}\Big[\cosh\tfrac{2\pi y}{w_2} + \cosh\tfrac{\pi i}{w_2}(w_1+2\hat\gamma_0)\Big]\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N\\
&\quad - \tfrac{1}{2}\overline{\Big[\cosh\tfrac{2\pi y'}{\bar w_2} + \cosh\tfrac{\pi i}{\bar w_2}\Big(\bar w_1+2\hat{\bar\gamma}_0\Big)\Big]}\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N\\
&= \frac{1}{2}\left(\cosh\frac{2\pi y}{w_2} - \cosh\frac{2\pi y'}{w_2}\right)\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N \qquad (\text{as } \bar{\hat{\bar\gamma}}_0 = \hat\gamma_0)\\
&= \sinh\frac{\pi(y\pm y')}{w_2}\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N. \qquad \square
\end{aligned}$$

10.3 Convergence to Dirac delta

In this section, we aim to demonstrate that $\frac{1}{2}\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N$ weakly converges to the Dirac delta function, up to some weight factor. More precisely, we want to show that

$$\lim_{N\to\infty}\frac{1}{2}\int_0^\infty f(y)\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N W_{\hat\gamma}(y)\,\mathrm{d}y = f(y')$$

for suitable functions f.

To prove this result, we will first study the asymptotic behaviour of $\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N$ using corollary 10.3. For the rest of this chapter, we assume ρ and σ to be real, and λ and u to be purely imaginary.

We begin by considering the expression

$$[\psi_\gamma^y,\psi_\gamma^{y'}](N) = \int_{N-iw_1}^{N}\left(\psi_\gamma^y(x+iw_1)\overline{\psi_\gamma^{y'}}(\bar x) - \psi_\gamma^y(x)\overline{\psi_\gamma^{y'}(\overline{x+iw_1})}\right)t_\gamma(x)A(x)W_\gamma(x)\,\mathrm{d}x$$

which, for real ρ,σ,y' and purely imaginary u and λ by eq. (8.8) equals

$$= \int_{N-iw_1}^{N}\left(\psi_\gamma^y(x+iw_1)\psi_\gamma^{y'}(x) - (y\leftrightarrow y')\right)t_\gamma(x)A(x)W_\gamma(x)\,\mathrm{d}x. \qquad (10.11)$$

We examine the asymptotics of the factor $t_\gamma(x)A(x)W_\gamma(x)$ that appears in the Wronskian. Recall from eq. (10.7) that we can express it as

$$t_\gamma(x)A(x)W_\gamma(x) = \frac{1}{4w_1w_2}\frac{\sinh\frac{2\pi x}{w_1}}{\sinh\frac{2\pi(x+iw)}{w_2}}\frac{G(x+iw_1-i\gamma_0)}{G(x+i\gamma_0)}\prod_{j=1}^{3}\frac{G(x+iw_1+i\gamma_j)}{G(x-i\gamma_j)}.$$

For $\operatorname{Re} x\to\infty$ with the imaginary part of x bounded, we can write

$$\begin{aligned}
\frac{\sinh\frac{2\pi x}{w_1}}{\sinh\frac{2\pi(x+iw)}{w_2}} &= e^{\frac{2\pi x}{w_1}-\frac{2\pi(x+iw)}{w_2}}\cdot\frac{e^{\frac{2\pi(x+iw)}{w_2}} - e^{\frac{2\pi(x+iw)}{w_2}-2\frac{2\pi x}{w_1}}}{e^{\frac{2\pi(x+iw)}{w_2}} - e^{-\frac{2\pi(x+iw)}{w_2}}}\\
&= e^{\frac{2\pi x}{w_1}-\frac{2\pi(x+iw)}{w_2}}\cdot\left(1+\frac{e^{\frac{-2\pi(x+iw)}{w_2}} - e^{\frac{2\pi(x+iw)}{w_2}-2\frac{2\pi x}{w_1}}}{e^{\frac{2\pi(x+iw)}{w_2}} - e^{-\frac{2\pi(x+iw)}{w_2}}}\right)\\
&= e^{\frac{\pi i(w_2-w_1)}{w_2}}e^{\frac{2\pi x(w_2-w_1)}{w_1w_2}}\cdot(1+O(e^{-2\operatorname{Re}\frac{2\pi}{w_1}\operatorname{Re}x})).
\end{aligned}$$

The term $\operatorname{Re}\frac{2\pi}{w_1}$ in the exponent equals $\alpha\operatorname{Re}w_2$. In the real case, $\operatorname{Re}w_2 = w_2 > w_1 = w_0$, in the conjugate case, $\operatorname{Re} w_2 = \operatorname{Re} w_1 = w_0$, so that we have (possibly with a weaker bound)

$$\frac{\sinh \frac{2\pi x}{w_1}}{\sinh \frac{2\pi(x+iw)}{w_2}} = e^{\frac{\pi i(w_2-w_1)}{w_2}} e^{\frac{2\pi x(w_2-w_1)}{w_1 w_2}} \cdot (1 + O(e^{-2\alpha w_0 \operatorname{Re} x})). \tag{10.12}$$

Using eq. (6.9), we can express

$$\frac{G(x + iw_1 - i\gamma_0)}{G(x + i\gamma_0)} = e^{\alpha(w_1/2-\gamma_0)x} e^{i\alpha(w_1^2/4-\gamma_0 w_1/2)} (1 + O(e^{-\alpha\varsigma w_0 \operatorname{Re}_x})),$$

for $\varsigma \in (0, 1)$ chosen arbitrarily. Similar expressions hold for the other fractions of hyperbolic gamma functions, so that we may write

$$t_\gamma(x)A(x)W_\gamma(x) = \frac{1}{4w_1 w_2} e^{\frac{\pi i(w_2-w_1)}{w_2}} e^{\frac{2\pi x(w_2-w_1)}{w_1 w_2}} e^{\alpha(2w_1-(\gamma_0-\gamma_1-\gamma_2-\gamma_3))x}$$
$$\times e^{i\alpha(w_1^2-(\gamma_0-\gamma_1-\gamma_2-\gamma_3)w_1/2)} (1 + O(e^{-\alpha\varsigma w_0 \operatorname{Re}_x})),$$

which, noting $\gamma_0 - \gamma_1 - \gamma_2 - \gamma_3 = -2i\lambda - 2iu$, we rewrite as

$$= \frac{1}{4w_1 w_2} e^{\alpha(2x+iw_1)(w+iu+i\lambda)} \cdot (1 + O(e^{-\alpha\varsigma w_0 \operatorname{Re} x})). \tag{10.13}$$

Using theorem 8.7, we can estimate

$$\psi_\gamma^y(x + iw_1)\psi_\gamma^{y'}(x) = w_1 w_2 \cdot e^{-\alpha(2x+iw_1)\cdot(w+iu+i\lambda)}$$
$$\times \sum_{\xi_1,\xi_2\in\{-1,1\}} c'_{\hat\gamma}(\xi_1 y)c'_{\hat\gamma}(\xi_2 y')e^{i\alpha x(\xi_1 y+\xi_2 y')}e^{-\alpha w_1\xi_1 y}$$
$$+ O(e^{-\alpha \operatorname{Re}(w+iu+i\lambda)\operatorname{Re} x} e^{-\alpha\kappa \operatorname{Re} x}),$$

and, as $O(e^{-\alpha \operatorname{Re}(w+iu+i\lambda)\operatorname{Re} x} e^{-\alpha\kappa \operatorname{Re} x}) = O(e^{\alpha(-iu-i\lambda-w-\kappa)\operatorname{Re} x}$ for $u, \lambda \in i\mathbb{R}$, we have

$$\left(\psi_\gamma^y(x + iw_1)\psi_\gamma^{y'}(x) - (y \leftrightarrow y')\right) =$$
$$w_1 w_2 \cdot e^{-\alpha(2x+iw_1)\cdot(w+iu+i\lambda)}$$
$$\times \sum_{\xi_1,\xi_2\in\{-1,1\}} c'_{\hat\gamma}(\xi_1 y)c'_{\hat\gamma}(\xi_2 y')e^{i\alpha x(\xi_1 y+\xi_2 y')}\left(e^{-\alpha w_1\xi_1 y} - e^{-\alpha w_1 \xi_2 y'}\right)$$
$$+ O(e^{-\alpha(i\lambda+iu+w+\kappa)\operatorname{Re} x}). \tag{10.14}$$

Substituting $x = N + iz$ into eq. (10.11) gives

$$[\psi_\gamma^y, \psi_\gamma^{y'}](N) = i \int_{-w_1}^{0} \left(\psi_\gamma^y(N + i(z + w_1))\psi_\gamma^{y'}(N + iz) - (y \leftrightarrow y')\right)$$
$$\times t_\gamma(N + iz)A(N + iz)W_\gamma(N + iz)\,\mathrm{d}z,$$

and using our estimates in eqs. (10.13) and (10.14), we rewrite this as

$$= \frac{i}{4} \sum_{\xi_1,\xi_2\in\{-1,1\}} c'_{\hat\gamma}(\xi_1 y)c'_{\hat\gamma}(\xi_2 y')\left(e^{-\alpha w_1\xi_1 y} - e^{-\alpha w_1\xi_2 y'}\right)$$
$$\times \int_{-w_1}^{0} e^{i\alpha(N+iz)(\xi_1 y+\xi_2 y')}\,\mathrm{d}z$$
$$+ O(e^{2\alpha(w+iu+i\lambda)N} e^{-\alpha(i\lambda+iu+w+\kappa)N}).$$

Working out the integral gives

$$= \frac{i}{4} \sum_{\xi_1,\xi_2\in\{-1,1\}} c'_{\hat{\gamma}}(\xi_1 y) c'_{\hat{\gamma}}(\xi_2 y') \Big(e^{-\alpha w_1 \xi_1 y} - e^{-\alpha w_1 \xi_2 y'} \Big) \\ \times \frac{e^{i\alpha(N+iz)(\xi_1 y+\xi_2 y')}\Big|_{-w_1}^{0}}{-\alpha(\xi_1 y + \xi_2 y')} + O(e^{\alpha(w+i\lambda+iu-\kappa)N}),$$

which we can rewrite as

$$= -\frac{i}{4\alpha} \sum_{\xi_1,\xi_2\in\{-1,1\}} c'_{\hat{\gamma}}(\xi_1 y) c'_{\hat{\gamma}}(\xi_2 y') \frac{e^{i\alpha N(\xi_1 y+\xi_2 y')}}{\xi_1 y + \xi_2 y'} \\ \times \underbrace{\Big(1 - e^{\alpha w_1(\xi_1 y+\xi_2 y')}\Big)\Big(e^{-\alpha w_1 \xi_1 y} - e^{-\alpha w_1 \xi_2 y'}\Big)}_{=2(\cosh \frac{2\pi y}{w_2} - \cosh \frac{2\pi y'}{w_2}) = 4 \sinh \frac{\pi}{w_2}(y \pm y')} \\ + O(e^{\alpha(w+i\lambda+iu-\kappa)N}).$$

Hence, according to corollary 10.3 and considering that $\alpha = \frac{2\pi}{w_1 w_2}$, we obtain

$$\langle \psi_\gamma(\,\cdot\,, y), \psi_\gamma(\,\cdot\,, y')\rangle_N = -i\frac{w_1 w_2}{\pi} \sum_{\xi_1,\xi_2\in\{-1,1\}} c'_{\hat{\gamma}}(\xi_1 y) c'_{\hat{\gamma}}(\xi_2 y') \frac{e^{i\alpha N(\xi_1 y+\xi_2 y')}}{\xi_1 y + \xi_2 y'} \\ + O(e^{\alpha(w+i\lambda+iu-\kappa)N}).$$

Using Euler's formula, we may rewrite this as

$$\langle \psi_\gamma(\,\cdot\,, y), \psi_\gamma(\,\cdot\,, y')\rangle_N = \phi_1(y, y') \cos \alpha N(y + y') + \phi_2(y, y') \sin \alpha N(y + y') \\ + \phi_3(y, y') \cos \alpha N(y - y') + \phi_4(y, y') \frac{\sin \alpha N(y - y')}{y - y'} + O(e^{\alpha(w+i\lambda+iu-\kappa)N}), \quad (10.15)$$

with

$$\phi_1(y, y') := -i\frac{w_1 w_2}{\pi} \times \frac{c'_{\hat{\gamma}}(y)c'_{\hat{\gamma}}(y') - c'_{\hat{\gamma}}(-y)c'_{\hat{\gamma}}(-y')}{y + y'},$$

$$\phi_2(y, y') := \frac{w_1 w_2}{\pi} \times \frac{c'_{\hat{\gamma}}(y)c'_{\hat{\gamma}}(y') + c'_{\hat{\gamma}}(-y)c'_{\hat{\gamma}}(-y')}{y + y'},$$

$$\phi_3(y, y') := -i\frac{w_1 w_2}{\pi} \times \frac{c'_{\hat{\gamma}}(y)c'_{\hat{\gamma}}(-y') - c'_{\hat{\gamma}}(-y)c'_{\hat{\gamma}}(y')}{y - y'}$$

and

$$\phi_4(y, y') := \frac{w_1 w_2}{\pi} \times \Big(c'_{\hat{\gamma}}(y)c'_{\hat{\gamma}}(-y') + c'_{\hat{\gamma}}(-y)c'_{\hat{\gamma}}(y')\Big).$$

If we fix $y' > 0$ and consider the ϕ_j's solely as functions of y, we observe that all of them have a simple pole at $y = 0$ due to the terms $c'_{\hat{\gamma}}(y)$ and $c'_{\hat{\gamma}}(-y)$. Moreover, since $c'_{\hat{\gamma}}(y)c'_{\hat{\gamma}}(-y') - c'_{\hat{\gamma}}(-y)c'_{\hat{\gamma}}(y')$ is zero for $y = y'$ (and analytic), ϕ_3 has a removable singularity at $y = y'$. The other ϕ_j's (with $j = 1, 2, 4$) do not have any singularities for $y \in (0, \infty)$.

Remark 10.4. Let us consider the convergence of the remaining term in eq. (10.15). Recalling the definition of κ from theorem 8.7, note that

$$w + i\lambda + iu - \kappa = w - \mathrm{Im}(\lambda + u) - \kappa = \max\{w_0 - w - \mathrm{Im}(\lambda + u), -w - \eta\}.$$

For this factor to be negative, so that the $O(e^{\alpha(w+i\lambda+iu-\kappa)N})$ converges to 0 as $N \to \infty$, we thus need to impose the condition $\operatorname{Im}(\lambda+u) > w_0 - w$.

If we restrict y and y' to some closed interval $[r_1, r_2]$ (with $0 < r_1 < r_2$), by continuity of $c'_{\hat{\gamma}}$ on $\mathbb{R} \setminus \{0\}$ and by theorem 8.7, this remaining term converges to 0 uniformly in y and y', i.e.

$$\begin{aligned}\Big|\langle \psi_\gamma(\,\cdot\,,y), \psi_\gamma(\,\cdot\,,y')\rangle_N - \Big(&\phi_1(y,y')\cos\alpha N(y+y') + \phi_2(y,y')\sin\alpha N(y+y') \\ &+ \phi_3(y,y')\cos\alpha N(y-y') + \phi_4(y,y')\frac{\sin\alpha N(y-y')}{y-y'}\Big)\Big| < Ce^{\alpha(w+i\lambda+iu-\kappa)N}, \end{aligned} \tag{10.16}$$

for sufficiently large N, where C is independent of N, y and y'. ∎

We can use the above approximations to establish the following result:

Lemma 10.5. *Let $\lambda, u \in i(w_0 - w, w)$ with $\operatorname{Im}(\lambda+u) > w_0 - w$ and let ρ and σ be real. Let $f \in C_c^\infty(0,\infty)$. The following limit holds:*

$$\lim_{N\to\infty} \int_0^\infty f(y)\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_N \,\mathrm{d}y = 2\frac{f(y')}{W_{\hat{\gamma}}(y')}.$$

Proof. Since f is bounded with compact support, using eq. (10.15), we can apply the dominated convergence to the integral over the remaining term, and we have

$$\begin{aligned}\lim_{N\to\infty} &\int_0^\infty f(y)\langle \psi_\gamma^y, \psi_\gamma^{y'}\rangle_N \,\mathrm{d}y \\ &= \lim_{N\to\infty}\int_0^\infty f(y)\Big(\phi_1(y,y')\cos\alpha N(y+y') + \phi_2(y,y')\sin\alpha N(y+y') \\ &\qquad + \phi_3(y,y')\cos\alpha N(y-y') + \phi_4(y,y')\frac{\sin\alpha N(y-y')}{y-y'}\Big)\,\mathrm{d}y,\end{aligned}$$

provided that the limit on the right-hand side exists.

By the conditions imposed on f, we have $f\phi_j(\,\cdot\,,y') \in L^1(0,\infty)$ for $j = 1,2,3$. Consequently, we can employ the Riemann-Lebesgue lemma to the terms involving ϕ_1, ϕ_2, and ϕ_3 in the integrand. This lemma allows us to conclude that their contributions vanish as N tends to infinity.

A well-known result from the theory of Fourier analysis (see e.g., [47, sec. 9.7]) states that if g is a continuous function on $(0,\infty)$ with bounded variation[4] that is differentiable around $x' > 0$, then

$$\lim_{t\to\infty}\int_0^\infty g(x)\frac{\sin t(x-x')}{x-x'}\,\mathrm{d}x = \pi g(x').$$

Since $f\phi_4$ is compactly supported and smooth on $(0,\infty)$ by our assumptions, we can employ

[4] We say that $f : (0,\infty) \to \mathbb{C}$ has *bounded variation* if there exists a finite number M such that for any choice of $x_0 < x_1 < \ldots < x_n$ in $(0,\infty)$, we have $\sum_{i=1}^n |f(x_i) - f(x_{i-1})| \le M$. Compactly supported smooth functions have bounded variation.

the above result to compute the limit

$$\lim_{N\to\infty}\int_0^\infty f(y)\phi_4(y,y')\frac{\sin\alpha N(y-y')}{y-y'}\,\mathrm{d}y$$
$$= \pi f(y')\phi_4(y',y') = 2\pi\frac{w_1 w_2}{\pi}c'_{\hat{\gamma}}(\pm y')f(y') = 2\frac{f(y')}{W_{\hat{\gamma}}(y')}. \qquad \square$$

Suppose that g is a compactly supported smooth function. It follows that $gW_{\hat{\gamma}}/2$ is also compactly supported and smooth. Therefore, we have

$$\lim_{N\to\infty}\frac{1}{2}\int_0^\infty g(y)\langle\psi_\gamma^y,\psi_\gamma^{y'}\rangle_N W_{\hat{\gamma}}(y)\,\mathrm{d}y = g(y')$$

for such functions g.

Remark 10.6. It is important to note that lemma 10.5 is not invariant under interchanging γ with the dual parameters $\hat{\gamma}$: This interchange corresponds to replacing λ with $-u$ and vice versa, as shown in the paragraph containing eq. (8.11). If we perform this interchange in the lemma, the conditions for λ, u, and $\lambda+u$ would undergo a sign change. For positive parameters w_1 and w_2, we can define a smaller domain for λ and u where the lemma holds true simultaneously for a fixed γ and its dual parameter set $\hat{\gamma}$.

In the case of conjugate parameters, the lower bound $w_0 - w$ of the open interval is zero. Consequently, it is impossible to find a range that accommodates both λ and $-\lambda$, as well as u and $-u$ simultaneously. As a result, we will encounter difficulties in obtaining an explicit formula for the inverse of the Ruijsenaars function transform with kernel $\psi_{\hat{\gamma}}$ in this case of conjugate parameters. However, we can overcome this obstacle by utilising the kernel $\mathcal{E}$ instead, taking advantage of its symmetries. ∎

10.4 The Ruijsenaars function transform

In this final section of the chapter, we introduce a function transform on a dense subspace of $\mathcal{H}_\gamma$ that has ψ_γ as its kernel. We use the results of the previous section to show that for certain values of γ this transformation extends to a Hilbert space isomorphism, and we try to identify its inverse.

We define the Ruijsenaars function transform $\mathcal{R}_\gamma$ with kernel ψ_γ for $f \in C_c^\infty(0,\infty)$ as

$$\mathcal{R}_\gamma f(y) := \int_0^\infty f(x)\psi_\gamma(x,y)W_\gamma(x)\,\mathrm{d}x.$$

For $f \in \mathcal{H}_{\gamma,0}$, the integral converges pointwise for any $y \in (0,\infty)$ since the integrand has compact support and is continuous. In the following theorem we extend the operator to a Hilbert space isomorphism.

Theorem 10.7. *Let $\lambda, u \in i(w_0 - w, w)$ satisfy* $\mathrm{Im}(\lambda+u) > w_0 - w$*, and let $\rho,\sigma\in\mathbb{R}$. Suppose $f \in C_c^\infty(k\pi,(k+1)\pi)$ for some $k\in\mathbb{N}$ and let $g \in C_c^\infty(0,\infty)$. Then, we have*

$$\langle\mathcal{R}_{\hat{\gamma}}f,\mathcal{R}_{\hat{\gamma}}g\rangle_\gamma = \langle f,g\rangle_{\hat{\gamma}}.$$

As a consequence, $\mathcal{R}_{\hat\gamma}$ uniquely extends to a Hilbert space isomorphism from $\mathcal{H}_{\hat\gamma}$ onto its image, which is a subspace of $\mathcal{H}_\gamma$.

Proof. Using on our previous results, we can calculate

$$\begin{aligned}\langle \mathcal{R}_{\hat\gamma} f, \mathcal{R}_{\hat\gamma} g\rangle_\gamma &= \int_0^\infty \mathcal{R}_{\hat\gamma} f(x)\overline{\mathcal{R}_{\hat\gamma} g(x)} W_\gamma(x)\,\mathrm{d}x \\ &= \lim_{N\to\infty}\int_0^N\int_0^\infty\int_0^\infty f(y)\underbrace{\psi_{\hat\gamma}(y,x)}_{=\psi_\gamma(x,y)} W_{\hat\gamma}(y)\bar g(z)\underbrace{\overline{\psi}_{\hat\gamma}(z,x)}_{=\overline{\psi}_\gamma(x,z)} W_{\hat\gamma}(z) W_\gamma(x)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Note that the integrand of the triple integral is continuous and has compact support on $[0,N]\times(0,\infty)\times(0,\infty)$, so that we can apply Fubini's theorem to interchange the order of the integrations. By eq. (8.7), we can rewrite $\psi_{\hat\gamma}(y,x) = \psi_\gamma(x,y)$. Using the truncated inner product notation, we can rewrite the above integral to

$$= \lim_{N\to\infty}\frac{1}{2}\int_0^\infty \bar g(z) W_{\hat\gamma}(z)\int_0^\infty f(y) W_{\hat\gamma}(y)\langle \psi_\gamma^y, \psi_\gamma^z\rangle_N\,\mathrm{d}y\,\mathrm{d}z.$$

If we could interchange the limit and the outer integration, the first claim of the theorem would follow from lemma 10.5. Our goal now is to show that we can indeed do this using the dominated convergence theorem.

We need to bound $\int_0^\infty f(y)W_{\hat\gamma}(y)\langle\psi_\gamma^y,\psi_\gamma^z\rangle_N\,\mathrm{d}y$ independently of N, for $z\in\operatorname{supp} g$. Using eq. (10.15) and recalling remark 10.4, we observe that

$$\int_0^\infty f(y)W_{\hat\gamma}(y)\left(\langle\psi_\gamma^y,\psi_\gamma^z\rangle_N - \phi_4(y,z)\frac{\sin\alpha N(y-z)}{y-z}\right)\mathrm{d}y$$

can be bounded independently of N and z. By our reasoning in appendix C, we can bound

$$\int_0^\infty f(y)W_{\hat\gamma}(y)\phi_4(y,z)\frac{\sin\alpha N(y-z)}{y-z}\,\mathrm{d}y$$

independently of N, so that indeed we may apply the dominated convergence theorem to conclude

$$\begin{aligned}\langle \mathcal{R}_{\hat\gamma} f, \mathcal{R}_{\hat\gamma} g\rangle_\gamma &= \frac{1}{2}\int_0^\infty \bar g(z)W_{\hat\gamma}(z)\lim_{N\to\infty}\int_0^\infty f(y)W_{\hat\gamma}(y)\langle\psi_\gamma^y,\psi_\gamma^z\rangle_N\,\mathrm{d}y\,\mathrm{d}z \\ &= \int_0^\infty f(z)\bar g(z)W_{\hat\gamma}(z)\,\mathrm{d}z \qquad\qquad \text{(by lemma 10.5)} \\ &= \langle f, g\rangle_{\hat\gamma}.\end{aligned}$$

Now we turn to extend the transformation to $\mathcal{H}_{\hat\gamma}$. By linearity, the latter result still holds for $f\in C_c^\infty((0,\infty)\setminus\pi\mathbb{N})$, and then obviously also for g in the latter space. Note the following properties:

1. If $f\in C_c^\infty((0,\infty)\setminus\pi\mathbb{N})$, then $f\sqrt{W_{\hat\gamma}}\in C_c^\infty((0,\infty)\setminus\pi\mathbb{N})$ and vice versa, as $W_{\hat\gamma}$ is a strictly positive analytic function on compact subsets of $(0,\infty)$.
2. We have $f\in\mathcal{H}_{\hat\gamma}$ if and only if $f\sqrt{W_{\hat\gamma}}\in L^2(0,\infty)$, with $\|f\|_{\mathcal{H}_{\hat\gamma}} = \left\|f\sqrt{W_{\hat\gamma}}\right\|_{L^2}$.
3. The space $C_c^\infty((0,\infty)\setminus\pi\mathbb{N})$ is densely contained in $L^2(0,\infty)$, as e.g. the indicator functions on finite intervals can be approximated with elements in the former set, and those span a dense subspace of $L^2(0,\infty)$.

Hence, if $f \in \mathcal{H}_{\hat\gamma}$, there exist functions $f_n \in C_c^\infty((0,\infty) \setminus \pi\mathbb{N})$ such that $f_n\sqrt{W_{\hat\gamma}}$ converges to $f\sqrt{W_{\hat\gamma}}$ with respect to the L^2 norm, and therefore $f_n \to f$ with respect to the norm of $\mathcal{H}_{\hat\gamma}$. Thus, $C_c^\infty((0,\infty) \setminus \pi\mathbb{N})$ is dense in $\mathcal{H}_{\hat\gamma}$.

For arbitrary $f \in \mathcal{H}_{\hat\gamma}$, let $(f_n) \subset C_c^\infty((0,\infty) \setminus \pi\mathbb{N})$ be a sequence converging to f. Since $\mathcal{H}_{\hat\gamma}$ is complete, (f_n) is a Cauchy sequence. Note that

$$\left\|\mathcal{R}_{\hat\gamma} f_n - \mathcal{R}_{\hat\gamma} f_m\right\|_{\mathcal{H}_\gamma} = \left\|\mathcal{R}_{\hat\gamma}(f_n - f_m)\right\|_{\mathcal{H}_\gamma} = \|f_n - f_m\|_{\mathcal{H}_{\hat\gamma}},$$

so $(\mathcal{R}_{\hat\gamma} f_n)$ is a Cauchy sequence as well, and therefore converges in $\mathcal{H}_\gamma$. Denoting the limit by $\mathcal{R}_{\hat\gamma} f$, we have extended $\mathcal{R}_{\hat\gamma}$ to a Hilbert space isomorphism from $\mathcal{H}_{\hat\gamma}$ to its image. □

Recall remark 10.6; its reasoning also applies to theorem 10.7. Moving to dual parameters, if λ and u are in $-i(w_0 - w, w) = i(-w, w - w_0)$ with $\operatorname{Im}(\lambda + u) < w - w_0$, with $\rho, \sigma \in \mathbb{R}$, then by theorem 10.7, we conclude that $\mathcal{R}_\gamma$ is a Hilbert space isomorhpism from $\mathcal{H}_\gamma$ onto a subspace of $\mathcal{H}_{\hat\gamma}$. For the case of positive w_1 and w_2 we thus can conclude:

Corollary 10.8. *Suppose $0 < w_1 < w_2$ with $\frac{w_1}{w_2} \notin \mathbb{Q}$. Let $\lambda, u \in i(w_0 - w, w - w_0)$ with $\lambda + u \in i(w_0 - w, w - w_0)$ as well, and let $\rho, \sigma \in \mathbb{R}$. Then $\mathcal{R}_{\hat\gamma}$ is a Hilbert space isomorphism onto $\mathcal{H}_\gamma$ with inverse $\mathcal{R}_\gamma$.*

Proof. The conditions on u, λ, ρ, σ imply, by theorem 10.7, that $\mathcal{R}_{\hat\gamma}$ and $\mathcal{R}_\gamma$ are Hilbert space isomorphisms onto their ranges. For $f \in C_c^\infty(0,\infty) \subseteq \mathcal{H}_{\hat\gamma}$, we have $\mathcal{R}_{\hat\gamma} f \in \mathcal{H}_\gamma$, so that $\mathcal{R}_\gamma[\mathcal{R}_{\hat\gamma} f]$ is defined.

We have

$$\mathcal{R}_\gamma[\mathcal{R}_{\hat\gamma} f](z) = \lim_{N\to\infty} \int_0^N \int_0^\infty f(x) W_{\hat\gamma}(x) \psi_{\hat\gamma}(x,y) W_\gamma(y) \psi_\gamma(y,z)\,\mathrm{d}x\,\mathrm{d}y$$

with an absolutely integrable integrand, so that we can apply Fubini's theorem to find

$$\begin{aligned} &= \lim_{N\to\infty} \int_0^\infty f(x) W_{\hat\gamma}(x) \int_0^N \psi_\gamma(y,x)\psi_\gamma(y,z) W_\gamma(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \lim_{N\to\infty} \frac{1}{2} \int_0^\infty f(x) W_{\hat\gamma}(x) \langle \psi_\gamma^x, \psi_\gamma^z \rangle_N\,\mathrm{d}x \\ &= f(z) \end{aligned}$$

by lemma 10.5. Therefore, $\mathcal{R}_\gamma \circ \mathcal{R}_{\hat\gamma} = \mathrm{id}$ holds on a dense subspace of $\mathcal{H}_{\hat\gamma}$ and hence on $\mathcal{H}_{\hat\gamma}$. Replacing γ by $\hat\gamma$ and vice versa in lemma 10.5, we find that $\mathcal{R}_{\hat\gamma} \circ \mathcal{R}_\gamma = \mathrm{id}$ on $\mathcal{H}_\gamma$. We conclude that $\mathcal{R}_\gamma = \mathcal{R}_{\hat\gamma}^{-1}$. □

The result of corollary 10.8 does not carry over to the case where w_1 and w_2 are a conjugate pair, as $w_0 - w = 0$ in that case, making the interval $(w_0 - w, w - w_0)$ empty.

Abbreviate $\mathcal{E}(\gamma;\tau,\upsilon) = \mathcal{E}(w_1, w_2, \gamma; \tau, \upsilon)$. By eq. (8.15), we can write

$$\mathcal{E}(\gamma;\tau,\upsilon) := \frac{\chi(\gamma)}{\sqrt{w_1 w_2}} \frac{\psi_\gamma(\tau,\upsilon)}{c'(\gamma;\tau)c'(\hat\gamma;\upsilon)}.$$

We define, for $f \in C_c^\infty(0,\infty)$,

$$\tilde{\mathcal{R}}_\gamma f(y) := \frac{1}{\sqrt{w_1 w_2}} \int_0^\infty f(x)\mathcal{E}(\gamma;x,y)\,\mathrm{d}x.$$

We can use the results on $\mathcal{R}_{\hat{\gamma}}$ to prove the following corollary:

Corollary 10.9. *Let $\lambda, u \in i(w_0 - w, w)$ satisfy $\mathrm{Im}(\lambda + u) > w_0 - w$ and let $\rho, \sigma \in \mathbb{R}$. The operator $\tilde{\mathcal{R}}_{\hat{\gamma}}$ extends to a Hilbert space isomorphism from $L^2(0,\infty)$ to its range.*

Proof. Recall that χ is defined as

$$\chi(\gamma) = \exp\left(i\alpha \left[\sum_{j=0}^{4} \frac{\gamma_j^2}{4} - \frac{w_1^2 + w_2^2 + w_1 w_2}{8} \right] \right).$$

Note that

$$\sum_{j=0}^{4} \gamma_j^2 = -(\rho+\lambda)^2 - (\rho-\lambda)^2 - (\sigma-u)^2 - (-\sigma-u)^2 = -2(\rho^2 + \lambda^2 + u^2 + \sigma^2)$$

is real for our choice of parameters. Moreover, $w_1^2 + w_2^2 + w_1 w_2 = w_1^2 + \overline{w_1^2} + w_1 \overline{w}_2$ is real, so $|\chi(\gamma)| = 1$, and the same holds when γ is replaced by $\hat{\gamma}$.

Note from the definition of c' (eq. (8.23)) and the symmetries of G (eq. (6.2)) that $c'(\gamma; -\tau) = \overline{c'(\gamma;\tau)}$. Hence

$$\sqrt{W_\gamma(\tau)} = \frac{1}{\sqrt{w_1 w_2}} \left| \frac{1}{c'(\gamma;\tau)} \right|,$$

so that

$$\phi_\gamma(\tau) = \sqrt{W_\gamma(\tau)} \sqrt{w_1 w_2}\, c'(\gamma;\tau)$$

satisfies $\left|\phi_\gamma(\tau)\right| = 1$.

Now let $f \in C_c^\infty((0,\infty) \setminus \pi\mathbb{N})$ and $g \in C_c^\infty(0,\infty)$. We have

$$\langle \tilde{\mathcal{R}}_{\hat{\gamma}} f, \tilde{\mathcal{R}}_{\hat{\gamma}} g \rangle_{L^2} = \left(\frac{1}{w_1 w_2} \right)^2 \int_0^\infty \int_0^\infty \int_0^\infty \frac{\chi(\hat{\gamma}) \overline{\chi(\hat{\gamma})}}{c'(\hat{\gamma}; x) \overline{c'(\hat{\gamma}; y)} c'(\gamma; z) \overline{c'(\gamma; z)}} \\ \times \psi_{\hat{\gamma}}(x,z) \overline{\psi}_{\hat{\gamma}}(y,z) f(x) \bar{g}(y) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z,$$

which we can rewrite as

$$\int_0^\infty \int_0^\infty \int_0^\infty [f(x) c'(\hat{\gamma}; -x) \sqrt{w_1 w_2}] \overline{[g(y) c'(\hat{\gamma}; -y) \sqrt{w_1 w_2}]} \\ \times \psi_{\hat{\gamma}}(x,z) \overline{\psi}_{\hat{\gamma}}(y,z) W_{\hat{\gamma}}(x) W_{\hat{\gamma}}(y) W_\gamma(z) \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

Identifying $f(x) c'(\hat{\gamma}; -x) \sqrt{w_1 w_2} = \frac{f(x)}{\sqrt{W_{\hat{\gamma}}(x)}} \phi_{\hat{\gamma}}(x)$ and similarly for g, we recognise that the above equals

$$\langle \mathcal{R}_{\hat{\gamma}}[f\phi_{\hat{\gamma}}/\sqrt{W_{\hat{\gamma}}}], \mathcal{R}_{\hat{\gamma}}[g\phi_{\hat{\gamma}}/\sqrt{W_{\hat{\gamma}}}] \rangle_\gamma = \langle f\phi_{\hat{\gamma}}/\sqrt{W_{\hat{\gamma}}}, g\phi_{\hat{\gamma}}/\sqrt{W_{\hat{\gamma}}} \rangle_{\hat{\gamma}}$$

by theorem 10.7. The right-hand side can be recognised as

$$\langle f\phi_{\hat{\gamma}}, g\phi_{\hat{\gamma}} \rangle_{L^2} = \langle f, g \rangle_{L^2},$$

so that $\tilde{\mathcal{R}}_{\hat{\gamma}}$ preserves the inner product. By a similar reasoning as in the proof of theorem 10.7, we extend it to a Hilbert space isomorphism on $L^2(0,\infty)$. □

We can extend the result of corollary 10.9 to $\tilde{\mathcal{R}}_\gamma$ (without a hat on γ) using the D_4-symmetry of $\mathcal{E}$. Initially, it may seem that we need to flip the signs of λ and u in the corollary's conditions to ensure $-u$ and $-\lambda$ are in $i(\omega_0 - \omega, \omega)$ with $\mathrm{Im}(-u-\lambda) > \omega_0 - \omega$. Recall that we did this previously in corollary 10.8 for the transformation with kernel ψ_γ.

However, by the D_4-symmetry with respect to γ (eq. (8.16)), the function $\mathcal{E}(\gamma;\tau,v)$ is even in both u and λ. We find that $\tilde{\mathcal{R}}_\gamma = \tilde{\mathcal{R}}_{\gamma'}$, where the prime acts on γ by flipping the signs of both u and λ. Therefore, we can apply corollary 10.9 to derive the desired properties via $\tilde{\mathcal{R}}_{\gamma'}$, without changing the domain of λ and u.

We will now present the main result of this chapter.

Theorem 10.10 (Unitarity of the Ruijsenaars transform)**.** *Let $\lambda, u \in i(\omega_0 - \omega, \omega)$ satisfy* $\mathrm{Im}(\lambda + u) > \omega_0 - \omega$ *and let $\rho, \sigma \in \mathbb{R}$. The operator $\tilde{\mathcal{R}}_{\hat{\gamma}}$ is a unitary map on $L^2(0,\infty)$ with inverse*

$$\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1} f(y) := \frac{1}{\sqrt{\omega_1\omega_2}} \int_0^\infty f(x)\overline{\mathcal{E}(\gamma;x,y)}\,\mathrm{d}x.$$

Proof. We observe that $\overline{\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1} f(y)} = \tilde{\mathcal{R}}_\gamma \bar{f}(y)$. Thus, based on the discussion above the theorem, $\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1}$ is a Hilbert space isomorphism from $L^2(0,\infty)$ onto its range. As in the proof of corollary 10.8, for $f \in C_c(0,\infty)$, we calculate

$$\begin{aligned}
\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1}\tilde{\mathcal{R}}_{\hat{\gamma}} f(z) &= \frac{1}{\omega_1\omega_2}\lim_{N\to\infty}\int_0^N\int_0^\infty f(x)\mathcal{E}(\hat{\gamma};x,y)\overline{\mathcal{E}(\gamma;y,z)}\,\mathrm{d}y\,\mathrm{d}x \\
&= \left(\frac{1}{\omega_1\omega_2}\right)^2 \lim_{N\to\infty}\int_0^\infty\int_0^N f(x)\frac{\chi(\hat{\gamma})\overline{\chi(\gamma)}}{c'(\hat{\gamma};x)c'(\gamma;y)\overline{c'(\gamma;y)c'(\hat{\gamma};z)}} \\
&\qquad\qquad \times \psi_{\hat{\gamma}}(x,y)\overline{\psi}_\gamma(y,z)\,\mathrm{d}x\,\mathrm{d}y \\
&= \frac{1}{2\omega_1\omega_2}\lim_{N\to\infty}\int_0^\infty \frac{f(x)}{c'(\hat{\gamma};x)c'(\hat{\gamma};-z)}\langle\psi_\gamma^x,\psi_\gamma^z\rangle_N\,\mathrm{d}x \\
&= \frac{1}{\omega_1\omega_2}\frac{f(z)}{c'(\hat{\gamma};z)c'(\hat{\gamma};-z)}/W_{\hat{\gamma}}(z) = f(z),
\end{aligned}$$

where we deployed lemma 10.5 once more. This result extends to $f \in L^2(0,\infty)$ by density, thus establishing $\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1}\circ\tilde{\mathcal{R}}_{\hat{\gamma}} = \mathrm{id}$ on $L^2(0,\infty)$.

To see that also $\tilde{\mathcal{R}}_{\hat{\gamma}}\circ\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1} = \mathrm{id}$, again take note of remark 10.6. We can use the D_4-symmetry to flip the signs of u and λ, leaving $\tilde{\mathcal{R}}_{\hat{\gamma}}$ and $\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1}$ invariant. On the result, we can use lemma 10.5 in a similar fashion as above to show that $\tilde{\mathcal{R}}_{\hat{\gamma}}\circ\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1} = \mathrm{id}$. We conclude that $\tilde{\mathcal{R}}_{\hat{\gamma}}$ is unitary with inverse $\tilde{\mathcal{R}}_{\hat{\gamma}}^{-1}$. □

Remark 10.11. By the D_4-symmetry on γ, the statement of the theorem is still true if $\lambda, u \in i(\omega_0 - \omega, \omega)$ and $\mathrm{Im}(\lambda + u) > \omega_0 - \omega$ hold only after flipping signs on λ and/or u. Moreover, the D_4-symmetry on γ allows us to interchange the roles of any two pairs of parameters in $(u, \lambda, \rho, \sigma)$ simultaneously, e.g. we can interchange $(u,\rho) \leftrightarrow (\sigma,\lambda)$ or $(u,\lambda) \leftrightarrow (\rho,\sigma)$, as can be seen from eq. (8.11). These symmetries imply that we could take λ and u to be real parameters and ρ and σ to be imaginary parameters with certain restrictions.

Ruijsenaars has established unitarity for the transformation with ω_1 and ω_2 positive and all parameters u, λ, ρ and σ restricted to the imaginary axis, without any constraints on their size ([38]). This result cannot be deduced from the D_4-symmetry. However, it does raise an intriguing question regarding the possibility of relaxing the restrictions imposed on λ and u. For now, it remains an open question if this is possible. ■

Chapter 11

The multivariate generalisation of ψ

In this chapter, we derive a multivariat version of the hypergemoemetric funcitons ψ_γ and $\mathcal{E}$. We will motivate our derivation based on properties of elements of the Hopf algebra $\mathfrak{D}$ and its representations. The generalisation we perform is closely related to the methods used on Askey-Wilson polynomials and Askey-Wilson functions, respectively, in [15, 16].

11.1 Coproducts of skew-primitives and their eigenfunctions

In this section, we generalise results of part II: we study coproducts of skew-primitive elements and their eigenfunctions, which are multivariate versions of the eigenfunctions we derived in chapter 7. We will use these multivariate eigenfunctions later to define multivariate hypergeometric functions.

Throughout this chapter, fix $N \in \mathbb{N}_{>0}$. As before, we assume that the parameters w_1 and w_2 are either positive, with $0 < w_1 < w_2$ and $w_1/w_2 \notin \mathbb{Q}$, or a pair of conjugate parameters with positive real part. Taking $N = 1$, we retrieve the original functions and their properties as discussed in the previous chapters.

For $j \in \{1, ..., N\}$, we set

$$\boldsymbol{Y}^{(j)}_{u,\rho} = 1^{\otimes(N-j)} \otimes \Delta^{j-1} Y_{u,\rho} \quad \text{and} \quad \boldsymbol{X}^{(j)}_{\sigma} = 1^{\otimes(N-j)} \otimes \Delta^{j-1} X_{0,\sigma}. \tag{11.1}$$

These are elements in $\mathfrak{D}^{\otimes N}$. We construct them by using of iterated coproducts of the skew-primitive elements, tensored from the left by the algebra unit.

By induction, we show that for $n \in \mathbb{N}_{>0}$

$$\Delta^n Y_{u,\rho} = 1^{\otimes n} \otimes Y_{u,\rho} + \Delta^{n-1} Y_{u,\rho} \otimes K. \tag{11.2}$$

For $n = 1$ this follows from our original discussion of $Y_{u,\rho}$. Assuming that eq. (11.2) holds for some $n \geq 1$, we find

$$\begin{aligned}\Delta^{n+1} Y_{u,\rho} &= \left(\Delta \otimes \mathrm{id}^{\otimes n}\right)\Delta^n Y_{u,\rho} \\ &= 1^{\otimes(n+1)} \otimes Y_{u,\rho} + (\Delta \otimes \mathrm{id}^{\otimes n-1})\Delta^{n-1} Y_{u,\rho} \otimes \mathrm{id}(K) \\ &= 1^{\otimes(n+1)} \otimes Y_{u,\rho} + \Delta^n Y_{u,\rho} \otimes K.\end{aligned}$$

This property can also be derived in a more direct way from eq. (5.2).

To be able to discuss eigenfunctions of these elements, we first need to extend our representation to $\mathfrak{D}^{\otimes N}$. A natural generalisation of the representation π_λ is to consider the representation $\pi_\lambda^{\otimes N}$. However, from a mathematical perspective, it is not necessary to have copies of the same representation on each factor of the tensor product. Letting $\boldsymbol{\lambda} = (\lambda_1, ..., \lambda_N) \in \mathbb{C}^N$, we define $\pi_{\boldsymbol{\lambda}}$ to be a representation of $\mathfrak{D}^{\otimes N}$ acting on the space $\boldsymbol{\mathcal{M}_N}$ of meromorphic functions on $\mathbb{C}^N$, given by $\pi_{\boldsymbol{\lambda}} = \pi_{\lambda_1} \otimes \cdots \otimes \pi_{\lambda_N}$. The element in the kth factor of the representation acts on the kth variable of the function in $\boldsymbol{\mathcal{M}_N}$.

Let $\rho \in \mathbb{C}$ and $\boldsymbol{u}, \boldsymbol{\lambda}, \boldsymbol{\tau}, \boldsymbol{z} \in \mathbb{C}^N$. We define a meromorphic function $\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho} \in \boldsymbol{\mathcal{M}_N} : \mathbb{C}^N \to \mathbb{C}^{\otimes N} \cong \mathbb{C}$ by

$$\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}(\boldsymbol{z}) = \bigotimes_{j=1}^{N} H^{\lambda_j,u_j}_{\tau_j,\tau_{j-1}}(z_j), \tag{11.3}$$

where $\tau_0 = \rho$. Since $\mathbb{C}^{\otimes N} \cong \mathbb{C}$, we can replace the symbol $\bigotimes$ by $\prod$. For $\boldsymbol{u} = (u, u, ..., u)$ with $u \in \mathbb{C}$, we use the notation $\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}(\boldsymbol{z})$, where the symbol u in the superscript now has regular weight.

Using induction, we can show that

$$\pi_{\boldsymbol{\lambda}}(\Delta^{N-1} Y_{u,\rho})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \mu^{\rho}_{\tau_N} \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}.$$

For $N = 1$ this was clear already from chapter 7. For $N > 1$ let $\boldsymbol{\lambda}, \boldsymbol{\tau} \in \mathbb{C}^{N+1}$, we define $\vec{\boldsymbol{\lambda}} \in \mathbb{C}^N$ to be a version of $\boldsymbol{\lambda}$ with its last entry chopped off, i.e. $\vec{\boldsymbol{\lambda}} = (\lambda_1, ..., \lambda_N)$. We similarly define $\vec{\boldsymbol{\tau}} = (\tau_1, ..., \tau_N)$. We assume for the sake of induction that

$$\pi_{\vec{\boldsymbol{\lambda}}}(\Delta^{N-1} Y_{u,\rho})\boldsymbol{H}^{\vec{\boldsymbol{\lambda}},u}_{\vec{\boldsymbol{\tau}},\rho} = \mu^{\rho}_{\tau_N} \boldsymbol{H}^{\vec{\boldsymbol{\lambda}},u}_{\vec{\boldsymbol{\tau}},\rho}.$$

Observe that $\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \boldsymbol{H}^{\vec{\boldsymbol{\lambda}},u}_{\vec{\boldsymbol{\tau}},\rho} \otimes H^{\lambda_{N+1},u}_{\tau_{N+1},\tau_N}$. By eq. (11.2), we can express

$$\pi_{\boldsymbol{\lambda}}(\Delta^{N} Y_{u,\rho})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \left(1^{\otimes N} \otimes \pi_{\lambda_{N+1}}(Y_{u,\rho}) + \pi_{\vec{\boldsymbol{\lambda}}}(\Delta^{N-1} Y_{u,\rho}) \otimes \pi_{\lambda_{N+1}}(K)\right)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}.$$

Using eq. (5.7), we can write

$$\begin{aligned}\left(1^{\otimes N} \otimes \pi_{\lambda_{N+1}}(Y_{u,\rho})\right)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} &= \boldsymbol{H}^{\vec{\boldsymbol{\lambda}},u}_{\vec{\boldsymbol{\tau}},\rho} \otimes \pi_{\lambda_{N+1}}(Y_{u,\rho}) H^{\lambda_{N+1},u}_{\tau_{N+1},\tau_N} \\ &= \boldsymbol{H}^{\vec{\boldsymbol{\lambda}},u}_{\vec{\boldsymbol{\tau}},\rho} \otimes \pi_{\lambda_{N+1}}\left(Y_{u,\tau_N} - \mu^{\rho}_{\tau_N}(K-1)\right) H^{\lambda_{N+1},u}_{\tau_{N+1},\tau_N} \\ &= \left(\mu^{\tau_N}_{\tau_{N+1}} + \mu^{\rho}_{\tau_N}\right)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} - \mu^{\rho}_{\tau_N} \boldsymbol{H}^{\vec{\boldsymbol{\lambda}},u}_{\vec{\boldsymbol{\tau}},\rho} \otimes \pi_{\lambda_{N+1}}(K) H^{\lambda_{N+1},u}_{\tau_{N+1},\tau_N},\end{aligned}$$

where we used $\pi_{\lambda_{N+1}}(Y_{u,\tau_N}) H^{\lambda_{N+1},u}_{\tau_{N+1},\tau_N} = \mu^{\tau_N}_{\tau_{N+1}}$ in the last line.

Using the induction hypothesis, we also compute

$$\left(\pi_{\vec{\boldsymbol{\lambda}}}(\Delta^{N-1} Y_{u,\rho}) \otimes \pi_{\lambda_{N+1}}(K)\right)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \mu^{\rho}_{\tau_N} \boldsymbol{H}^{\vec{\boldsymbol{\lambda}},u}_{\vec{\boldsymbol{\tau}},\rho} \otimes \pi_{\lambda_{N+1}}(K) H^{\lambda_{N+1},u}_{\tau_{N+1},\tau_N}.$$

Adding the latter two results, and observing that $\mu^{\tau_N}_{\tau_{N+1}} + \mu^{\rho}_{\tau_N} = \mu^{\rho}_{\tau_{N+1}}$, completing the induction.

Moving back to $\boldsymbol{\tau}, \boldsymbol{\lambda} \in \mathbb{C}^N$, we conclude that the element $\boldsymbol{Y}^{(j)}_{u,\tau_{N-j}}$ acts similarly under the representation, but only effects the latter j factors of the tensor product. In that case we have

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{Y}^{(j)}_{u,\tau_{N-j}})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \mu^{\tau_{N-j}}_{\tau_N} \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}. \tag{11.4}$$

By eqs. (5.2) and (11.1) we can write

$$\boldsymbol{Y}^{(j)}_{u,\rho} = 1^{\otimes(N-j)} \otimes \Delta^{j-1} Y_{u,\rho} = \sum_{k=N-j+1}^{N} 1^{\otimes(k-1)} \otimes Y_{u,\rho} \otimes K^{N-k}.$$

For $\boldsymbol{u} \in \mathbb{C}^N$, we modify the right-hand side of the above to define define the element $\boldsymbol{Y}^{(j)}_{\boldsymbol{u},\rho}$ as

$$\boldsymbol{Y}^{(j)}_{\boldsymbol{u},\rho} := \sum_{k=N-j+1}^{N} 1^{\otimes(k-1)} \otimes Y_{u_k,\rho} \otimes K^{N-k}. \tag{11.5}$$

There is no direct way to express this newly defined element by means of the coproduct. By eq. (7.2) and note below it, we can however write

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{Y}^{(j)}_{\boldsymbol{u},\rho}) = (T^{u_1} \otimes \cdots \otimes T^{u_N}) \pi_{\boldsymbol{\lambda}}(1^{\otimes(N-j)} \otimes \Delta^j Y_\rho)(T^{-u_1} \otimes \cdots \otimes T^{-u_N}),$$

with each shift operator $T^{\pm u_k}$ acting on the kth variable for $k = 1, ..., N$. Since

$$\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho} = (T^{u_1} \otimes \cdots \otimes T^{u_N}) \boldsymbol{H}^{\boldsymbol{\lambda},0}_{\boldsymbol{\tau},\rho},$$

we can use the above and eq. (11.4) to conclude that $\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}$ is an eigenfunction of $\pi_{\boldsymbol{\lambda}}(\boldsymbol{Y}^{(j)}_{\boldsymbol{u},\rho})$ with eigenvalue $\mu^{\tau_{N-j}}_{\tau_N}$.

We set

$$\boldsymbol{F}^{\boldsymbol{\lambda}}_{\boldsymbol{v},\sigma}(\boldsymbol{z}) = \bigotimes_{j=1}^{N} F^{\lambda_j,0}_{v_j,v_{j+1}}(z_j), \tag{11.6}$$

with $v_{N+1} = \sigma$. (We have set the parameter v in the superscripts of F equal to zero, as we will not need different values for it.) We extend the star involution of $\mathfrak{D}$ to a star involution on $\mathfrak{D}^{\otimes N}$ by applying the star to each factor of the tensor product simultaneously. We can apply a similar procedure as above. In the case of real parameters w_1 and w_2, we compute

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{X}^{(j)*}_{\sigma}) \boldsymbol{F}^{\boldsymbol{\lambda}}_{\boldsymbol{v},\sigma} = \overline{\mu^{\sigma}_{v_{N-j+1}}} \boldsymbol{F}^{\boldsymbol{\lambda}}_{\boldsymbol{v},\sigma}.$$

In the case of conjugate parameters w_1 and w_2 we have

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{X}^{(j)\star}_{\sigma}) \boldsymbol{F}^{\boldsymbol{\lambda}}_{\boldsymbol{v},\sigma} = \overline{\mu^{\sigma}_{v_{N-j+1}}} \boldsymbol{F}^{\boldsymbol{\lambda}}_{\boldsymbol{v},\sigma}.$$

11.2 Multivariate hypergeometric functions

We use the eigenfunctions derived in the previous section to define our multivariate version of ψ. We define it by

$$\boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau}, \boldsymbol{v}) := \prod_{j=1}^{N} \langle H^{\lambda_j,u_j}_{\tau_j,\tau_{j-1}}, F^{\bar{\lambda}_j}_{v_j,v_{j+1}} \rangle_{C_j}, \tag{11.7}$$

where we define each curve C_j in such a way that it separates the upward and downward pole sequences of $\psi^{\lambda_j,u_j}_{\tau_{j-1},v_{j+1}}(\tau_j, v_j)$. Note that the terms appearing in the sesquilinear form are exactly the factors of the tensor products eqs. (11.3) and (11.6). If $\boldsymbol{u} = (u, u, ..., u)$, we use the notation $\boldsymbol{\psi}^{\boldsymbol{\lambda},u}_{\rho,\sigma}$, where the symbol u in the superscript has regular weight. From the definition we observe that

$$\boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v}) = \prod_{j=1}^{N} \psi^{\lambda_j,u_j}_{\tau_{j-1},v_{j+1}}(\tau_j, v_j).$$

Many properties, such as the analyticity and the positions of the poles, carry over from the original function ψ: the function $\boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}$ is analytic if all its factors are analytic. We can also deduce analogues of some symmetries of the original function. Let us introduce the notation $\overline{\boldsymbol{x}} := (x_N, x_{N-1}, ..., x_1)$ to denote the reversely ordered version of a vector $\boldsymbol{x} = (x_1, ..., x_N) \in \mathbb{C}^N$, so that $\overline{x}_k = x_{N+1-k}$ (we set $\overline{\tau}_{N+1} = \rho$ and $\overline{v}_0 = \sigma$). We use this notation to deduce analogues of eq. (8.5), by applying the symmetry to each factor of the product. It results in the following relation:

$$\boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v}) = \prod_{j=1}^{N} \psi^{-\lambda_j,-u_j}_{v_{j+1},\tau_{j-1}}(v_j, \tau_j) = \prod_{j=1}^{N} \psi^{-\overline{\lambda}_j,-\overline{u}_j}_{\overline{v}_{j-1},\overline{\tau}_{j+1}}(\overline{v}_j, \overline{\tau}_j) = \boldsymbol{\psi}^{-\overline{\boldsymbol{\lambda}},-\overline{\boldsymbol{u}}}_{\sigma,\rho}(\overline{\boldsymbol{v}}, \overline{\boldsymbol{\tau}}). \tag{11.8}$$

Using eq. (8.8) we derive in a similar way that

$$\overline{\boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v})} = \boldsymbol{\psi}^{\overline{\bar{\boldsymbol{\lambda}}},\overline{\bar{\boldsymbol{u}}}}_{\bar{\sigma},\bar{\rho}}(\overline{\bar{\boldsymbol{v}}}, \overline{\bar{\boldsymbol{\tau}}}) = \boldsymbol{\psi}^{-\bar{\boldsymbol{\lambda}},-\bar{\boldsymbol{u}}}_{\bar{\rho},\bar{\sigma}}(\bar{\boldsymbol{\tau}}, \bar{\boldsymbol{v}}).$$

Setting

$$\gamma^j = -i(\tau_{j-1} + \lambda_j, \tau_{j-1} - \lambda_j, -v_{j+1} - u_j, v_{j+1} - u_j),$$

we may write

$$\boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v}) = \prod_{j=1}^{N} \psi_{\gamma^j}(\tau_j, v_j),$$

and define

$$\widehat{\boldsymbol{\psi}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{v},\boldsymbol{\tau}) := \prod_{j=1}^{N} \psi_{\widehat{\gamma}^j}(v_j, \tau_j) \tag{11.9}$$

with

$$\widehat{\gamma}^j = -i(\tau_{j-1} - u_j, \tau_{j-1} + u_j, -v_{j+1} + \lambda_j.v_{j+1} + \lambda_j).$$

(By observing $\psi_{\widehat{\gamma}^j}(v_j, \tau_j) = \psi_{\gamma^j}(\tau_j, v_j)$, we deduce from eq. (11.8) that $\widehat{\boldsymbol{\psi}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{v},\boldsymbol{\tau}) = \boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v})$.)

We define the multivariate version of $\mathcal{E}$ as

$$\boldsymbol{\mathcal{E}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v}) := \prod_{j=1}^{N} \mathcal{E}(\gamma^j; \tau_j, v_j) = \left[\prod_{j=1}^{N} \frac{\chi(\gamma^j)}{\sqrt{w_1 w_2}} \frac{1}{c'(\gamma^j;\tau_j)c'(\widehat{\gamma}^j;v_j)}\right] \boldsymbol{\psi}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v}). \tag{11.10}$$

Its dual version we define as

$$\widehat{\boldsymbol{\mathcal{E}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{v},\boldsymbol{\tau}) := \prod_{j=1}^{N} \mathcal{E}(\widehat{\gamma}^j; v_j, \tau_j) = \left[\prod_{j=1}^{N} \frac{\chi(\widehat{\gamma}^j)}{\sqrt{w_1 w_2}} \frac{1}{c'(\widehat{\gamma}^j;\tau_j)c'(\gamma^j;v_j)}\right] \widehat{\boldsymbol{\psi}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{v},\boldsymbol{\tau}). \tag{11.11}$$

One could similarly define multivariate generalisations of the functions R and R_{ren}. We will not state these generalisations, but the procedure should be clear from the above two cases. We will use the multivariate $\boldsymbol{\psi}$ in the upcoming section to show that is satisfies a difference equation. The function $\boldsymbol{\mathcal{E}}$ will later be used to define a unitary transformation on multivariate functions.

Remark 11.1. Observe that the jth factor of the product defining $\boldsymbol{\psi}$ depends on the variables τ_j and υ_j, but also on variables τ_{j-1} and υ_{j+1}. Multivariate versions of the Krawtchouk polynomials have be derived in a similar way, cf. [45]. ∎

11.3 A multivariate difference equation

Now that we have defined the multivariate versions of ψ and $\mathcal{E}$, we move to discuss their applications. In this section, we will present and prove a multivariate version of proposition 9.4, where we replace each of the variables λ, τ, υ with vectors in $\mathbb{C}^N$. Similar equations for multivariate Askey-Wilson functions have been derived in [13]. The methods used in this section have previously been applied on multivariate Askey-Wilson polynomials and multivariate Askey-Wilson functions in [15, 16].

The original difference equation in proposition 9.4 expresses ψ_γ in terms of a three-term relation. The multivariate version we discuss here will involve (up to) 3^N terms. The derivation of the multivariate version will mainly consist of repeatedly applying previous results and employing some bookkeeping skills.

We understand that the derivations in this section can be tedious to read. Due to the exponential growth in the number of coefficients, some notational inconvenience seems inevitable. To manage, and hopefully slightly ease, the bookkeeping process, we introduce new notation before presenting the results. Additionally, we provide some examples throughout this section to illustrate the meaning of the notation and demonstrate the workings of the proofs.

We begin by introducing some notation. We define difference operators D_k, where $k \in \{1, ..., N\}$, that act on the $\boldsymbol{\tau}$ variable, applying a shift of $-iw_1$ to the kth entry of $\boldsymbol{\tau}$:

$$D_k \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \boldsymbol{H}^{\boldsymbol{\lambda},u}_{(\tau_1,...,\tau_{k-1},\tau_k-iw_1,\tau_{k+1},...,\tau_N),\rho}.$$

If we express $\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}$ in the form of a tensor product we did as in eq. (11.3), then D_k modifies the kth and $(k+1)$th factors, replacing them with $H^{\lambda_k,u_k}_{\tau_k-iw_1,\tau_{k-1}}$ and $H^{\lambda_{k+1},u_{k+1}}_{\tau_{k+1},\tau_k-iw_1}$, respectively.

We define $\overleftarrow{D}_k \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}$ as a copy of $\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}$ with τ_k replaced by $\tau_k - iw_1$ only in the kth factor of the tensor product, while leaving the $(k+1)$th factor untouched. Note that $\overleftarrow{D}_N \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = D_N \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}$, as there is no $(N+1)$th factor.

Example 11.2. For $N = 3$, the function $\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}$ can be written as

$$\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = H^{\lambda_1,u_1}_{\tau_1,\rho} \otimes H^{\lambda_2,u_2}_{\tau_2,\tau_1} \otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2}.$$

We have

$$D_2 \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = H^{\lambda_1,u_1}_{\tau_1,\rho} \otimes H^{\lambda_2,u_2}_{\tau_2-iw_1,\tau_1} \otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2-iw_1},$$

and

$$\overleftarrow{D}_2 \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = H^{\lambda_1,u_1}_{\tau_1,\rho} \otimes H^{\lambda_2,u_2}_{\tau_2-iw_1,\tau_1} \otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2}.$$

∎

For $\boldsymbol{\xi} = (\xi_1, ..., \xi_N) \in \{-1, 0, 1\}^N$, we set $D^{\boldsymbol{\xi}} = D_1^{\xi_1} \cdots D_N^{\xi_N}$. If we regard $iw_1\boldsymbol{\xi}$ as a vector in $\mathbb{C}^N$, we may also write $D^{\boldsymbol{\xi}} \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau}-iw_1\boldsymbol{\xi},\rho}$.

We further define

$$\boldsymbol{K}^{-1,(j)} := 1^{\otimes(N-j)} \otimes (K^{-1})^{\otimes j} = 1^{\otimes(N-j)} \otimes \Delta^{j-1}(K^{-1})$$

and

$$\boldsymbol{K}^{-1,(j,k)} = 1^{\otimes(N-j)} \otimes (K^{-1})^{\otimes k} \otimes 1^{\otimes(j-k)},$$

for $k = 0, 1, ..., j$. It should be noted that

$$\boldsymbol{K}^{-1,(j)} = \boldsymbol{K}^{-1,(j-k)} \cdot \boldsymbol{K}^{-1,(j,k)},$$

where $\cdot$ denotes the (Hopf) algebra product.

For $k = 0, 1, ..., j$, we set

$$\Xi^{(j,k)} := \{0\}^{N-j} \times \{-1, 0, 1\}^k \times \{0\}^{j-k}$$

and

$$\Xi^{(j)} := \Xi^{(j,j)}.$$

Using the coefficients defined in corollaries 9.2 and 9.3, we set

$$B^{\lambda,\xi}_{\tau,\rho} := \begin{cases} \widehat{B}^{\lambda}_{\tau,\rho} & \text{if } \xi = -1, \\ B^{\lambda}_{\tau,\rho} & \text{if } \xi = 0, \\ \widehat{B}^{\lambda}_{\tau,-\rho} & \text{if } \xi = 1, \end{cases} \quad \text{and} \quad C^{\lambda,\xi}_{\tau,\rho} := \begin{cases} \widehat{C}^{\lambda}_{\tau,\rho} & \text{if } \xi = -1, \\ C^{\lambda}_{\tau,\rho} & \text{if } \xi = 0, \\ \widehat{C}^{\lambda}_{\tau,-\rho} & \text{if } \xi = 1. \end{cases}$$

Furthermore, we set

$$E^{(k)}_{\boldsymbol{\xi},\lambda}(\boldsymbol{\tau}) := \begin{cases} B^{\lambda,\xi_{k-1}}_{\tau_k,\tau_{k-1}} & \textit{if } \xi_k = -1, \\ C^{\lambda,\xi_{k-1}}_{\tau_k,\tau_{k-1}} & \textit{if } \xi_k = 0, \\ B^{\lambda,\xi_{k-1}}_{-\tau_k,\tau_{k-1}} & \textit{if } \xi_k = 1, \end{cases}$$

where $\xi_0 := 0$, and

$$\boldsymbol{E}^{(j)}_{\boldsymbol{\xi},\boldsymbol{\lambda}}(\boldsymbol{\tau}) := \prod_{k=N-j+1}^{N} E^{(k)}_{\boldsymbol{\xi},\lambda_k}(\boldsymbol{\tau}).$$

Lastly, we set

$$\mathfrak{b}^{\rho,\sigma}_{u,\xi} := \begin{cases} \hat{b}^{\rho,\sigma}_u = -\frac{a_x v_\rho + a_{-x} v_{\rho+iw_1} + v_\sigma}{q-q^{-1}} & \text{if } \xi = -1, \\ b^{\rho,\sigma}_u = -\frac{a_x v_\rho + a_{-x} v_\rho + v_\sigma}{q-q^{-1}} & \text{if } \xi = 0, \\ \hat{b}^{-\rho,\sigma}_u = -\frac{a_x v_\rho + a_{-x} v_{\rho-iw_1} + v_\sigma}{q-q^{-1}} & \text{if } \xi = 1. \end{cases} \tag{11.12}$$

We will prove the following multivariate generalisation of corollary 9.2:

Lemma 11.3. *Let $j \in \{1, ..., N\}$. For $\tau_k \notin i\frac{w_2}{2} \cdot \mathbb{Z} \cup (\pm i\frac{w_1}{2} + i\frac{w_2}{2} \cdot \mathbb{Z})$, where $k = N - j +$*

$1, N-j+2, ..., N$, *we have*

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)})\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho} = \sum_{\xi\in\Xi^{(j)}} \boldsymbol{E}^{(j)}_{\xi,\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\xi}\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}. \tag{11.13}$$

Before proving the general result, I will show that the lemma holds true for a simple case in the following example.

Example 11.4. We show that lemma 11.3 holds in the case $N=3$, $j=2$. We have

$$\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = H^{\lambda_1,u_1}_{\tau_1,\rho} \otimes H^{\lambda_2,u_2}_{\tau_2,\tau_1} \otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2}$$

and

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(2)}) = 1\otimes\pi_{\lambda_2}(K^{-1})\otimes\pi_{\lambda_3}(K^{-1}) = \Big(1\otimes 1\otimes\pi_{\lambda_3}(K^{-1})\Big)\cdot\Big(1\otimes\pi_{\lambda_2}(K^{-1})\otimes 1\Big).$$

We first calculate

$$\begin{aligned}\Big(1\otimes\pi_{\lambda_2}(K^{-1})\otimes 1\Big)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} &= H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes\Big(\pi_{\lambda_2}(K^{-1})H^{\lambda_2,u_2}_{\tau_2,\tau_1}\Big)\otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2} \\ &= H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes\Big(\pi_{\lambda_2}(K^{-1})H^{\lambda_2,u_2}_{\tau_2,\tau_1}\Big)\otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2}.\end{aligned}$$

By corollary 9.2, we can rewrite this to

$$\begin{aligned}= H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes\Big(B^{\lambda}_{\tau_2,\tau_1}H^{\lambda_2,u_2}_{\tau_2+iw_1,\tau_1} + C^{\lambda}_{\tau_2,\tau_1}H^{\lambda_2,u_2}_{\tau_2,\tau_1} \\ + B^{\lambda}_{-\tau_2,\tau_1}H^{\lambda_2,u_2}_{\tau_2-iw_1,\tau_1}\Big)\otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2}.\end{aligned} \tag{11.14}$$

For the sake of the general proof of the lemma, note that we can write this as

$$\begin{aligned}&= \Big(B^{\lambda}_{\tau_2,\tau_1}\overleftarrow{D}^1_2 + C^{\lambda}_{\tau_2,\tau_1}\overleftarrow{D}^0_2 + B^{\lambda}_{-\tau_2,\tau_1}\overleftarrow{D}^{-1}_2\Big)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} \\ &= \sum_{\xi\in\Xi^{(2,1)}} E^{(2)}_{\xi,\lambda_2}(\boldsymbol{\tau})\overleftarrow{D}^{\xi_2}_2\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}.\end{aligned}$$

Now we apply $\Big(1\otimes 1\otimes\pi_{\lambda_3}(K^{-1})\Big)$ each term in the sum to find $\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(2)})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}$. We do this in the following way: Note that the third factor of the tensor product has been left untouched so far, whereas a shift has been applied to the τ_2-variable in the middle factor of some terms in the sum. We use corollaries 9.2 and 9.3 to apply K^{-1} to the terms in the third factor: For each term in the sum, whenever a shift has been applied to τ_2 in the second factor of $\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}$ in eq. (11.14), i.e. we're talking about the terms $H^{\lambda_2,u_2}_{\tau_2\pm iw_1,\tau_1}$ that appear in the sum, we want to apply the suitable form of corollary 9.3 to apply the same shift in τ_2 in the third factor. If no shift has been applied, i.e. we have a term $H^{\lambda_2,u_2}_{\tau_2,\tau_1}$, we apply corollary 9.2.

We use eq. (9.11) to write

$$\begin{aligned}&B^{\lambda}_{\tau_2,\tau_1}\Big(1\otimes 1\otimes\pi_{\lambda_3}(K^{-1})\Big)H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes H^{\lambda_2,u_2}_{\tau_2+iw_1,\tau_1}\otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2} \\ &\quad= B^{\lambda}_{\tau_2,\tau_1}H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes H^{\lambda_2,u_2}_{\tau_2+iw_1,\tau_1}\otimes\Big(\widehat{B}^{\lambda_3}_{\tau_3,\tau_2}H^{\lambda_3,u_3}_{\tau_3+iw_1,\tau_2+iw_1} \\ &\qquad + \widehat{C}^{\lambda_3}_{\tau_3,\tau_2}H^{\lambda_3,u_3}_{\tau_3,\tau_2+iw_1} + \widehat{B}^{\lambda_3}_{-\tau_3,\tau_2}H^{\lambda_3,u_3}_{\tau_3-iw_1,\tau_2+iw_1}\Big),\end{aligned}$$

so that all instances of τ_2 in the third factor of this term have now been replaced by τ_2+iw_1, as they are in the second term.

Similarly, by eq. (9.12),

$$\begin{aligned}B^{\lambda}_{-\tau_2,\tau_1}&\Big(1\otimes 1\otimes \pi_{\lambda_3}(K^{-1})\Big)H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes H^{\lambda_2,u_2}_{\tau_2-iw_1,\tau_1}\otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2}\\ &= B^{\lambda}_{-\tau_2,\tau_1}H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes H^{\lambda_2,u_2}_{\tau_2-iw_1,\tau_1}\otimes\Big(\widehat{B}^{\lambda_3}_{\tau_3,-\tau_2}H^{\lambda_3,u_3}_{\tau_3+iw_1,\tau_2-iw_1}\\ &\qquad+\widehat{C}^{\lambda_3}_{\tau_3,-\tau_2}H^{\lambda_3,u_3}_{\tau_3,\tau_2-iw_1}+\widehat{B}^{\lambda_3}_{-\tau_3,-\tau_2}H^{\lambda_3,u_3}_{\tau_3-iw_1,\tau_2-iw_1}\Big),\end{aligned}$$

and corollary 9.2 gives

$$\begin{aligned}C^{\lambda}_{\tau_2,\tau_1}&\Big(1\otimes 1\otimes \pi_{\lambda_3}(K^{-1})\Big)H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes H^{\lambda_2,u_2}_{\tau_2,\tau_1}\otimes H^{\lambda_3,u_3}_{\tau_3,\tau_2}\\ &= C^{\lambda}_{\tau_2,\tau_1}H^{\lambda_1,u_1}_{\tau_1,\rho}\otimes H^{\lambda_2,u_2}_{\tau_2-iw_1,\tau_1}\otimes\Big(B^{\lambda_3}_{\tau_3,\tau_2}H^{\lambda_3,u_3}_{\tau_3+iw_1,\tau_2}\\ &\qquad+C^{\lambda_3}_{\tau_3,\tau_2}H^{\lambda_3,u_3}_{\tau_3,\tau_2}+B^{\lambda_3}_{-\tau_3,\tau_2}H^{\lambda_3,u_3}_{\tau_3-iw_1,\tau_2}\Big).\end{aligned}$$

Taking the sum over these last three expressions and factoring out the brackets, we find

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(2)})\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}=\sum_{\boldsymbol{\xi}\in\Xi^{(2)}}\boldsymbol{E}^{(2)}_{\boldsymbol{\xi},\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\boldsymbol{\xi}}\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}$$

as we wanted to show. ∎

We now generalise this procedure in the following proof:

Proof of lemma 11.3. For $k=1,...,j$, we set $k' := N-j+k$. We can write $\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)}) = \pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-1)})\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j,1)})$, where the latter term $\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j,1)})$ is just K^{-1} applied in the $(N-j+1)$th variable (so the $1'$th factor). By applying corollary 9.2 to $1'$th factor, we obtain that

$$\begin{aligned}\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)})\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}&=\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-1)})\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j,1)})\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}\\ &=\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-1)})\Big(B^{\lambda}_{\tau_{1'},\tau_{0'}}\overleftarrow{D}^{-1}_{1'}+C^{\lambda}_{\tau_{1'},\tau_{0'}}\overleftarrow{D}^{0}_{1'}+B^{\lambda}_{-\tau_{1'},\tau_{0'}}\overleftarrow{D}^{1}_{1'}\Big)\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}\\ &=\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-1)})\sum_{\boldsymbol{\xi}\in\Xi^{(j,1)}}E^{(1')}_{\boldsymbol{\xi},\lambda_{1'}}(\boldsymbol{\tau})\overleftarrow{D}^{\xi_{1'}}_{1'}\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}.\end{aligned}$$

Next, if $j>1$, we factor $\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-1)}) = \pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-2)})\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-1,1)})$ and apply it to $\sum_{\boldsymbol{\xi}\in\Xi^{(j,1)}}E^{(1')}_{\boldsymbol{\xi},\lambda_{1'}}(\boldsymbol{\tau})\overleftarrow{D}^{\xi_{1'}}_{1'}\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}$. For each term in the sum we apply one of the following steps: for the term with $\xi_{1'}=0$ we apply corollary 9.2 to the $2'$th factor of the tensor product, for the terms with $\xi_{1'}=-1$ and $\xi_{1'}=1$ we apply eqs. (9.11) and (9.12) from corollary 9.3 respectively. As a result, for $\xi_{1'}=\pm1$, we add or subtract iw_1 in the $\tau_{1'}$ that appears in the $2'$th factor of $\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}$, according to the action of $\overleftarrow{D}^{\xi_{1'}}_{1'}$. We may therefore remove the $\leftarrow$ from $D_{1'}$. The result can be written as

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)})\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}=\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-2)})\sum_{\boldsymbol{\xi}\in\Xi^{(j,2)}}E^{(1')}_{\boldsymbol{\xi},\lambda_{1'}}(\boldsymbol{\tau})E^{(2')}_{\boldsymbol{\xi},\lambda_{2'}}(\boldsymbol{\tau})D^{\xi_{1'}}_{1'}\overleftarrow{D}^{\xi_{2'}}_{2'}\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}.$$

In our example with $j=2$, we were finished at this point. If $j>2$, we keep iterating this procedure for $k=3,...,j$, writing $\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-k)}) = \pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-k-1)})\cdot\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j-k,1)})$. We apply this to $\sum_{\boldsymbol{\xi}\in\Xi^{(j,k-1)}}\Big(\prod_{l=1}^{k-1}E^{(l')}_{\boldsymbol{\xi},\lambda_{l'}}(\boldsymbol{\tau})\Big)\Big(\prod_{l=1}^{k-2}D^{\xi_{l'}}_{l'}\Big)\overleftarrow{D}^{\xi_{(k-1)'}}_{(k-1)'}\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}$ in the following way: for terms of the sum with $\xi_{(k-1)'}=0$ we apply corollary 9.2, and for terms with $\xi_{(k-1)'}=-1$ and $\xi_{(k-1)'}=1$

we apply eqs. (9.11) and (9.12) respectively to the k'th factor and variable. We find

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-\mathbf{1},(j)})\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho} = \pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-\mathbf{1},(j-k)}) \sum_{\boldsymbol{\xi}\in\Xi^{(j,k)}} \left(\prod_{l=1}^{k} E^{(l')}_{\boldsymbol{\xi},\lambda_{l'}}(\boldsymbol{\tau})\right)\left(\prod_{l=1}^{k-1} D^{\xi_{l'}}_{l'}\right) \overleftarrow{D}^{\xi_{k'}}_{k'} \boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}.$$

The procedure terminates after handling the case $j = k$. At that point, we have derived the relation stated in eq. (11.13). □

The above lemma allows $\boldsymbol{u}$ to be an arbitrary vector in $\mathbb{C}^N$, whereas the coefficients are $\boldsymbol{u}$-independent. Since $\boldsymbol{H}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\boldsymbol{\tau},\rho}(\boldsymbol{z}) = \boldsymbol{H}^{\boldsymbol{\lambda},0}_{\boldsymbol{\tau},\rho}(\boldsymbol{z}+\boldsymbol{u})$), and this $\boldsymbol{u}$-shift commutes with $\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-\mathbf{1},(j)})$, the $\boldsymbol{u}$-independence of the coefficients is quite immediate. For the difference equation below, we need $\boldsymbol{u} = (u, u, ..., u)$ however.

Proposition 11.5. *Let $j \in \{1, ..., N\}$. For $\tau_k \notin i\frac{w_2}{2} \cdot \mathbb{Z} \cup (\pm i\frac{w_1}{2} + i\frac{w_2}{2} \cdot \mathbb{Z})$, where $k = N - j + 1, N - j + 2, ..., N$, we have*

$$\mu^{\sigma}_{v_{N-j+1}} \psi^{\boldsymbol{\lambda},u}_{\rho,\sigma}(\boldsymbol{\tau}, \boldsymbol{v}) = \sum_{\boldsymbol{\xi}\in\Xi^{(j)}} \left[\mathfrak{b}^{\tau_N,\sigma}_{u,\xi_N} \boldsymbol{E}^{(j)}_{\boldsymbol{\xi},\boldsymbol{\lambda}}(\boldsymbol{\tau}) - \delta_{\boldsymbol{\xi},0} b^{\tau_{N-j},\sigma}_{u} \right] D^{\boldsymbol{\xi}} \psi^{\boldsymbol{\lambda},u}_{\rho,\sigma}(\boldsymbol{\tau}, \boldsymbol{v}).$$

Note that the coefficients in the sum depend on $u, \boldsymbol{\lambda}, \rho, \sigma, \boldsymbol{\tau}$, but not on $\boldsymbol{v}$ (except for $v_{N+1} = \sigma$). The eigenvalue on the left-hand side, $\mu^{\sigma}_{v_{N-j+1}}$, is independent of $u, \boldsymbol{\lambda}, \rho$ and $\boldsymbol{\tau}$.

Proof. We are going to apply variations of lemmas 4.1 and 5.1 to mimic the proof of proposition 9.4. We first define a sesquilinear form on $\boldsymbol{\mathcal{M}_N}$ by setting

$$\langle f, g\rangle_{C_1,...,C_N} = \int_{C_1} \cdots \int_{C_N} f(\boldsymbol{z})\bar{g}(\boldsymbol{z})\, \mathrm{d}z_N \cdots \mathrm{d}z_1,$$

with $\bar{g}(\boldsymbol{z}) := \overline{g(\overline{\boldsymbol{z}})}$ and $C_1, ..., C_N$ being N deformations of the real line. We may use this notation for whenever the integral converges.

Let $f, g \in \boldsymbol{\mathcal{M}}_N$ be of the form $f(\boldsymbol{z}) = f_1(z_1) \cdots f_N(z_N)$, and similarly $g(\boldsymbol{z}) = g_1(z_1) \cdots g_N(z_N)$. Let $\boldsymbol{M} = M_1 \otimes \cdots \otimes M_N \in \{1, K, K^{-1}, E, F, FK, EK^{-1}\}^{\otimes N}$. For such functions and such elements in $\mathfrak{D}^{\otimes N}$ we can write

$$\langle \pi_{\boldsymbol{\lambda}}(\boldsymbol{M}) f, g\rangle_{C_1,...,C_N} = \prod_{j=1}^{N} \langle \pi_{\lambda_j}(M_j) f_j, g_j\rangle_{C_j}.$$

If each pair f_j, g_j satisfies the conditions of lemma 4.1, we can apply that lemma to each term, resulting in either

$$\langle \pi_{\boldsymbol{\lambda}}(\boldsymbol{M}) f, g\rangle_{C_1,...,C_N} = \langle f, \pi_{\bar{\boldsymbol{\lambda}}}(\boldsymbol{M}^*) g\rangle_{C_1,...,C_N}$$

or

$$\langle \pi_{\boldsymbol{\lambda}}(\boldsymbol{M}) f, g\rangle_{C_1,...,C_N} = \langle f, \pi_{\bar{\boldsymbol{\lambda}}}(\boldsymbol{M}^\star) g\rangle_{C_1,...,C_N},$$

depending on whether w_1 and w_2 are positive or complex conjugates. By linearity, this result carries over to elements in the span of $\{1, K, K^{-1}, E, F, FK, EK^{-1}\}^{\otimes N}$, such as $\boldsymbol{X}^{(j)}_{\sigma}$.

As the coproduct is an algebra homomorphism, we can apply lemma 5.1 to write this element

$\boldsymbol{X}_\sigma^{(j)}$ as

$$\begin{aligned}\boldsymbol{X}_\sigma^{(j)} &= 1^{\otimes(N-j)} \otimes \Delta^{j-1} X_{0,\sigma} \\ &= 1^{\otimes(N-j)} \otimes \Delta^{j-1}\Big(a_u K^{-1} Y_{u,\tau_{N-j}} + a_{-u} Y_{u,\tau_{N-j}} K^{-1} + b_u^{\tau_{N-j},\sigma}(K^{-1}-1)\Big) \\ &= a_u \boldsymbol{K}^{-1,(j)} \boldsymbol{Y}^{(j)}_{u,\tau_{N-j}} + a_{-u} \boldsymbol{Y}^{(j)}_{u,\tau_{N-j}} \boldsymbol{K}^{-1,(j)} + b_u^{\tau_{N-j},\sigma}(\boldsymbol{K}^{-1,(j)} - 1). \end{aligned} \tag{11.15}$$

Note that we have

$$\psi^{\boldsymbol{\lambda},u}_{\rho,\sigma}(\boldsymbol{\tau},\boldsymbol{v}) = \langle \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}, \boldsymbol{F}^{\boldsymbol{\lambda},0}_{\boldsymbol{v},\sigma}\rangle_{C_1,\dots,C_N},$$

so that for real w_1 and w_2,

$$\mu^\sigma_{v_{N-j+1}} \psi(\boldsymbol{\tau},\boldsymbol{v}) = \langle \boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}, \pi_{\bar{\boldsymbol{\lambda}}}(\boldsymbol{X}_\sigma^{(j)*})\boldsymbol{F}^{\boldsymbol{\lambda},0}_{\boldsymbol{v},\sigma}\rangle_{C_1,\dots,C_N} = \langle \pi_{\boldsymbol{\lambda}}(\boldsymbol{X}_\sigma^{(j)})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}, \boldsymbol{F}^{\boldsymbol{\lambda},0}_{\boldsymbol{v},\sigma}\rangle_{C_1,\dots,C_N}. \tag{11.16}$$

For a conjugate pair w_1, w_2, the above relation holds after replacing the $*$ by $\star$.

Using eq. (11.15), we can write

$$\begin{aligned}\pi_{\boldsymbol{\lambda}}(\boldsymbol{X}_\sigma^{(j)})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = a_u \pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)}\boldsymbol{Y}^{(j)}_{u,\tau_{N-j}})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} + a_{-u}\pi_{\boldsymbol{\lambda}}(\boldsymbol{Y}^{(j)}_{u,\tau_{N-j}}\boldsymbol{K}^{-1,(j)})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} \\ + b_u^{\tau_{N-j},\sigma}\big(\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)}) - 1\big)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}.\end{aligned}$$

By eq. (11.4) and lemma 11.3, we have

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)}\boldsymbol{Y}^{(j)}_{u,\tau_{N-j}})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \pi_{\boldsymbol{\lambda}}(\boldsymbol{K}^{-1,(j)})\mu^{\tau_{N-j}}_{\tau_N}\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \mu^{\tau_{N-j}}_{\tau_N}\sum_{\xi\in\Xi^{(j)}}\boldsymbol{E}^{(j)}_{\xi,\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\boldsymbol{\xi}}\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho},$$

and

$$\begin{aligned}\pi_{\boldsymbol{\lambda}}(\boldsymbol{Y}^{(j)}_{u,\tau_{N-j}}\boldsymbol{K}^{-1,(j)})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} &= \pi_{\boldsymbol{\lambda}}(\boldsymbol{Y}^{(j)}_{u,\tau_{N-j}})\sum_{\xi\in\Xi^{(j)}}\boldsymbol{E}^{(j)}_{\xi,\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\boldsymbol{\xi}}\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} \\ &= \sum_{\xi\in\Xi^{(j)}}\mu^{\tau_{N-j}}_{\tau_N - i w_1 \xi_N}\boldsymbol{E}^{(j)}_{\xi,\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\boldsymbol{\xi}}\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}.\end{aligned}$$

Hence,

$$\pi_{\boldsymbol{\lambda}}(\boldsymbol{X}_\sigma^{(j)})\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} = \Big(-b_u^{\tau_{N-j},\sigma} + \sum_{\xi\in\Xi^{(j)}}\Big[a_u\mu^{\tau_{N-j}}_{\tau_N} + a_{-u}\mu^{\tau_{N-j}}_{\tau_N - i w_1\xi_N} + b_u^{\tau_{N-j},\sigma}\Big]\boldsymbol{E}^{(j)}_{\xi,\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\boldsymbol{\xi}}\Big)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho},$$

which, using eqs. (9.16) to (9.18), rewrites to

$$\begin{aligned}&= \Big(-b_u^{\tau_{N-j},\sigma} + \sum_{\xi\in\Xi^{(j)}}\frac{a_u v_{\tau_N} + a_{-u}v_{\tau_N - i w_1\xi_N} + v_\sigma}{q-q^{-1}}\boldsymbol{E}^{(j)}_{\xi,\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\boldsymbol{\xi}}\Big)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho} \\ &= \Big(-b_u^{\tau_{N-j},\sigma} + \sum_{\xi\in\Xi^{(j)}}\mathfrak{b}^{\tau_N,\sigma}_{u,\xi_N}\boldsymbol{E}^{(j)}_{\xi,\boldsymbol{\lambda}}(\boldsymbol{\tau})D^{\boldsymbol{\xi}}\Big)\boldsymbol{H}^{\boldsymbol{\lambda},u}_{\boldsymbol{\tau},\rho}.\end{aligned}$$

We obtain the expression in the statement of the proposition by substituting this back into the right-hand side of eq. (11.16). □

The proposition gives a system of N difference equations (i.e. for $j = 1, \dots, N$, we get N independent difference equations). Interchanging the roles of w_1 and w_2 in these equations, we find another N difference equations. Moreover, we can move to the multivariate version of the dual function, as given in eq. (11.8), and find two more systems of N difference equations.

Remark 11.6. In the original function ψ_γ, we discovered a duality between the parameters λ and u, see eq. (8.6). We did not present an algebraic interpretation of this duality; it merely followed from a shift of the domain in the integral form of ψ_γ, remark 8.2.

In the above multivariate difference equation, the parameter λ was generalised into a vector. Naturally, the question rises whether we can still write down a difference equation if we also vectorise u. Our attempts to extend the above equations for a vector $\boldsymbol{u}$ have failed thus far. In the next section we will see, however, that the results of chapter 10 extend to a transformation with kernel $\boldsymbol{\mathcal{E}}$, even for vectors $\boldsymbol{u} \in \mathbb{C}^N$. ∎

11.4 A unitary transformation on $L^2((0,\infty)^N)$

We conclude this chapter, as well as this thesis, by extending the unitary transformation $\tilde{\mathcal{R}}_\gamma$ from chapter 10 to a unitary transformation on $L^2((0,\infty)^N)$ and identifying its inverse.

Let $N \in \mathbb{N}_{>0}$ and let f be a bounded function with compact support in $L^2((0,N)^N)$. We define

$$\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f(\boldsymbol{y}) := \big(w_1 w_2\big)^{-\frac{N}{2}} \int_0^\infty \cdots \int_0^\infty f(\boldsymbol{x}) \boldsymbol{\mathcal{E}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{x},\boldsymbol{y})\, \mathrm{d}x_N \cdots \mathrm{d}x_1 \tag{11.17}$$

and

$$\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1}_{\rho,\sigma} f(\boldsymbol{x}) := \big(w_1 w_2\big)^{-\frac{N}{2}} \int_0^\infty \cdots \int_0^\infty f(\boldsymbol{y}) \overline{\hat{\boldsymbol{\mathcal{E}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(\boldsymbol{y},\boldsymbol{x})}\, \mathrm{d}y_1 \cdots \mathrm{d}y_N. \tag{11.18}$$

Remark 11.7. Note that the order of the integration is different in eq. (11.17) as compared to eq. (11.18). Although the integrands are bounded with compact support, so that we may apply Fubini's theorem to reorder the integrations at will, our choice of ordering them was not arbitrary:

Considering the integral kernels as a product as in eqs. (11.10) and (11.11), we note that in eq. (11.17) the variable x_j appears in both the jth and $(j+1)$th factor of the kernel. Integrating over x_N first (which appears only in the very last factor), the inner integral is just an application of the transformation $\tilde{\mathcal{R}}_\gamma$ that we studied in chapter 10. Next, each subsequent integration, over x_{N-1}, x_{N-2}, etc., is another application of the transformation $\tilde{\mathcal{R}}_\gamma$, with different parameters.

The kernel factors in eq. (11.18) have a reversed cross-dependence: the variable y_j appears in the jth and in the $(j-1)$th factor. This explains the reversed order of integration. ∎

We will show that, under appropriate conditions on $\boldsymbol{\lambda}$ and $\boldsymbol{u}$, the above transformations define unitary maps and are each other's inverses. We start with the following lemma.

Lemma 11.8. *Let $\lambda_j, u_j \in i(w_0 - w, w)$ with $\mathrm{Im}(\lambda_j + u_j) \in (w_0 - w, w)$ for $j = 1, 2, ..., N$. Let $\rho, \sigma \in \mathbb{R}$, let $R = [a_1, b_1] \times \cdots [a_N, b_N]$ be a closed N-rectangle in $(0,\infty)^N$ and let $f := \mathbb{1}_R$ be the indicator function on R. The maps $\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}$ and $\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1}_{\rho,\sigma}$ preserve the norm of f and*

$$\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1}_{\rho,\sigma} \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f = \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1}_{\rho,\sigma} f = f. \tag{11.19}$$

Proof. For $j = 1, ..., N$, let $f_j = \mathbb{1}_{[a_j,b_j]}$, so that $f(\boldsymbol{x}) = f_1(x_1) \times \cdots \times f_N(x_N)$. We will recursively write the action of $\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}$ on f. Set $\hat{f}^{(N)}(y_N; -) = 1$. For $j = N-1, N-2, ..., 1, 0,$

in decreasing order, set

$$\widehat{f}^{(j)}(x_j; y_{j+1}, \ldots, y_N) := \tilde{\mathcal{R}}_{\gamma^{j+1}}\Big[x_{j+1} \mapsto f_{j+1}(x_{j+1}) \times \widehat{f}^{(j+1)}(x_{j+1}; y_{j+2}, \ldots, y_N)\Big],$$

with $\tilde{\mathcal{R}}_\gamma$ the transformation with kernel $\mathcal{E}(\gamma;\,\cdot\,,\,\cdot\,)$ as defined in chapter 10. Note that $\widehat{f}^{(j)}$ depends on x_j, as γ_{j+1} depends on x_j.

We rewrite eq. (11.17) as an iterated integral over kernels, i.e.

$$\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f(\boldsymbol{y}) = \big(w_1 w_2\big)^{-\frac{N}{2}} \int_0^\infty f_1(x_1)\mathcal{E}(\gamma^1; x_1, y_1) \int_0^\infty f_2(x_2)\mathcal{E}(\gamma^2; x_2, y_2) \cdots \int_0^\infty f_N(x_N)\mathcal{E}(\gamma^N; x_N, y_N)\,\mathrm{d}x_N \cdots \mathrm{d}x_1. \tag{11.20}$$

We observe that we can write the innermost integral as $\sqrt{w_1 w_2}\tilde{\mathcal{R}}_{\gamma^N} f_N = \sqrt{w_1 w_2}\widehat{f}^{(N-1)}$. (Note that $\tilde{\mathcal{R}}_{\gamma^N} f_N$ is a function of the transformation parameter y_N, but it also implicitly depends on x_{N-1} due to its appearance in γ^N.)

The function f_N is continuous and differentiable on its support, and $\mathcal{E}(\gamma^N; x_N, y_N)$ is analytic as a function of (x_{N-1}, x_N, y_N) for x_{N-1} in the support of f_{N-1}, x_N in the support of f_N and $y_N \in (0, \infty)$. Thus, by Leibniz rule, $\widehat{f}^{(N-1)}(x_{N-1}; y_N)$ is analytic for x_{N-1} on $\operatorname{supp} f_{N-1}$ and $y_N \in (0, \infty)$. It follows that $f_{N-1}(x_{N-1}) \times \widehat{f}^{(N-1)}(x_{N-1}; y_N)$, as a function of x_{N-1}, is continuous and differentiable on its (compact) support, so that it is in $L^2(0, \infty)$. Therefore, we can apply the transformation $\tilde{\mathcal{R}}_{\gamma^{N-1}}$ on it to compute $\widehat{f}^{(N-2)}(x_{N-2}; y_{N-1}, y_N)$. In doing so, we obtain the result of the two innermost integrals in eq. (11.20) (up to a factor $w_1 w_2$). By iterating this process, we eventually arrive at $\widehat{f}^{(0)}(-; y_1, \ldots, y_N) = \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f(\boldsymbol{y})$. Moreover, we can write

$$\widehat{f}^{(j)}(x_j; y_{j+1}, \ldots, y_N) = \int_0^\infty \cdots \int_0^\infty f_{j+1}(x_{j+1}) \times \cdots \times f_N(x_N) \prod_{k=j+1}^{N} \mathcal{E}(\gamma^k; x_k, y_k)\,\mathrm{d}x_{j+1} \cdots \mathrm{d}x_N. \tag{11.21}$$

Let us compute $\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}\,-1}_{\rho,\sigma}\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f$ now. We can write this as

$$\begin{aligned}
&\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}\,-1}_{\rho,\sigma}\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f(\boldsymbol{z}) \\
&= \big(w_1 w_2\big)^{-\frac{N}{2}} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^N; y_N, z_N)} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^{N-1}; y_{N-1}, z_{N-1})} \cdots \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^1; y_1, z_1)}\widehat{f}^{(0)}(-; \boldsymbol{y})\,\mathrm{d}y_1 \cdots \mathrm{d}y_N \\
&= \big(w_1 w_2\big)^{-\frac{N-1}{2}} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^N; y_N, z_N)} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^{N-1}; y_{N-1}, z_{N-1})} \\
&\qquad \cdots \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^2; y_2, z_2)}\tilde{\mathcal{R}}^{-1}_{\widehat{\gamma}^1}\Big[\tilde{\mathcal{R}}_{\widehat{\gamma}^1}\Big[f_1(\,\cdot\,)\widehat{f}^{(1)}(\,\cdot\,; y_2, \ldots, y_N)\Big]\Big](z_1)\,\mathrm{d}y_2 \cdots \mathrm{d}y_N.
\end{aligned}$$

By the conditions on u_1 and λ_1, we have $\tilde{\mathcal{R}}^{-1}_{\widehat{\gamma}^1}\tilde{\mathcal{R}}_{\widehat{\gamma}^1} = \mathrm{id}$ (this follows from theorem 10.10), so that the above expression can be rewritten to

$$= \big(w_1 w_2\big)^{-\frac{N-1}{2}} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^N; y_N, z_N)} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^{N-1}; y_{N-1}, z_{N-1})} \cdots \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^2; y_2, z_2)} f_1(z_1)\widehat{f}^{(1)}(z_1; y_2, \ldots, y_N)\,\mathrm{d}y_2 \cdots \mathrm{d}y_N.$$

We can iterate these steps to rewrite the expression to

$$
\begin{aligned}
&= (w_1 w_2)^{-\frac{N-2}{2}} f_1(z_1) \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^N; y_N, z_N)} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^{N-1}; y_{N-1}, z_{N-1})} \\
&\qquad \cdots \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^3; y_3, z_3)} \tilde{\mathcal{R}}_{\widehat{\gamma}^2}^{-1} \left[\tilde{\mathcal{R}}_{\widehat{\gamma}^2} \left[f_2(\,\cdot\,) \hat{f}^{(2)}(\,\cdot\,; y_3, \ldots, y_N) \right] \right] (z_2) \,\mathrm{d}y_3 \cdots \mathrm{d}y_N \\
&= (w_1 w_2)^{-\frac{N-2}{2}} f_1(z_1) f_2(z_2) \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^N; y_N, z_N)} \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^{N-1}; y_{N-1}, z_{N-1})} \\
&\qquad \cdots \int_0^\infty \overline{\mathcal{E}(\widehat{\gamma}^3; y_3, z_3)} \hat{f}^{(2)}(z_2; y_3, \ldots, y_N) \,\mathrm{d}y_3 \cdots \mathrm{d}y_N.
\end{aligned}
$$

After a total of N iterations, we retrieve $f(\boldsymbol{z})$, so that $\tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1} \tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}} f = f$ on indicator functions of closed N-rectangles. A similar approach may be applied to show that $\tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}} \tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1} f = f$.

It remains to show the norm preservation. We will compute $\|f\|_{L^2}^2 = \langle \tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}} f, \tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}} f \rangle_{L^2}$, which we can write as

$$
\begin{aligned}
\langle \tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}} f, \tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}} f \rangle_{L^2} &= \langle \hat{f}^{(0)}, \hat{f}^{(0)} \rangle_{L^2} \\
&= \int_0^\infty \cdots \int_0^\infty \hat{f}^{(0)}(\boldsymbol{y}) \overline{\hat{f}^{(0)}(\boldsymbol{y})} \,\mathrm{d}y_1 \ldots \mathrm{d}y_N \\
&= \int_0^\infty \cdots \int_0^\infty \tilde{\mathcal{R}}_{\gamma^1} \left[f_1(\,\cdot\,) \hat{f}^{(1)}(\,\cdot\,; y_2, \ldots, y_N) \right] (y_1) \\
&\qquad \times \overline{\tilde{\mathcal{R}}_{\gamma^1} \left[f_1(\,\cdot\,) \hat{f}^{(1)}(\,\cdot\,; y_2, \ldots, y_N) \right] (y_1)} \,\mathrm{d}y_1 \ldots \mathrm{d}y_N.
\end{aligned}
$$

Viewing the inner integral as an inner product, and noting that $\tilde{\mathcal{R}}_{\gamma^1}$ is unitary by theorem 10.10, we rewrite the above to

$$
\begin{aligned}
&= \int_0^\infty \cdots \int_0^\infty f_1(x_1) \hat{f}^{(1)}(x_1; y_2, \ldots, y_N) \\
&\qquad \times \overline{f_1(x_1) \hat{f}^{(1)}(x_1; y_2, \ldots, y_N)](y_1)} \,\mathrm{d}x_1 \mathrm{d}y_2 \ldots \mathrm{d}y_N.
\end{aligned}
$$

Since the integrand is nonnegative, we can apply Tonelli's theorem to change the order of integration:

$$
\begin{aligned}
&= \int_0^\infty f_1(x_1) \overline{f_1(x_1)} \int_0^\infty \cdots \int_0^\infty \tilde{\mathcal{R}}_{\gamma^2} \left[f_2(\,\cdot\,) \hat{f}^{(2)}(\,\cdot\,; y_3, \ldots, y_N) \right] (y_2) \\
&\qquad \times \overline{\tilde{\mathcal{R}}_{\gamma^2} \left[f_2(\,\cdot\,) \hat{f}^{(2)}(\,\cdot\,; y_3, \ldots, y_N) \right] (y_2)} \,\mathrm{d}y_2 \ldots \mathrm{d}y_N \mathrm{d}x_1.
\end{aligned}
$$

Proceeding this way, we are eventually left with

$$
\begin{aligned}
&= \int_0^\infty \cdots \int_0^\infty f_1(x_1) \cdots f_N(x_N) \overline{f_1(x_1) \cdots f_N(x_N)} \,\mathrm{d}x_1 \cdots \mathrm{d}x_N \\
&= \langle f, f \rangle_{L^2} = \|f\|_{L^2}^2.
\end{aligned}
$$

In a similar way, it can be shown that $\tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1}$ preserves the norm of f. ☐

We use the above result to prove the main theorem of this section:

Theorem 11.9 (Unitarity of the multivariate Ruijsenaars transform)**.** *Let $\lambda_j, u_j \in i(w_0 - w, w)$ with $\mathrm{Im}(\lambda_j + u_j) \in (w_0 - w, w)$ for $j = 1, 2, \ldots, N$. Let $\rho, \sigma \in \mathbb{R}$. The map $\tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}}$ extend to a unitary operator on $L^2((0,\infty))$, and $\tilde{\boldsymbol{\mathcal{R}}}_{\rho,\sigma}^{\boldsymbol{\lambda},\boldsymbol{u}\;-1}$ extends to its inverse.*

Proof. One easily deduces that the results of lemma 11.8 extend to scalar multiples of indicator functions. The equality eq. (11.19) immediately extends to linear combinations of indicator functions as well. It will take a little more work to show that norm-preservation also extends to linear combinations of indicator functions:

If f and g are scalar multiples of indicator functions of N-rectangles with disjoint interiors, then by lemma D.1 we have $\langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f, \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} g\rangle_{L^2} = 0$. Hence,

$$\begin{aligned}\langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(f+g), \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(f+g)\rangle_{L^2} &= \langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f, \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f\rangle_{L^2} + \langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f, \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} g\rangle_{L^2} \\ &\quad + \langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} g, \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f\rangle_{L^2} + \langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} g, \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} g\rangle_{L^2} \\ &= \langle f, f\rangle_{L^2} + \langle g, g\rangle_{L^2} = \langle f+g, f+g\rangle_{L^2}.\end{aligned}$$

This result generalises to finite sums of scalar multiples of indicator functions of rectangles that have mutually disjoint interiors.

Now suppose that two such functions f and g have supports with non-disjoint interiors. We can find a finite set of (scalar multiples of) indicator functions $h_1, ..., h_n$ of which the supports do have mutually disjoint interiors, and which satisfy $f + g = h_1 + \cdots + h_n$ almost everywhere. Hence we can use them to compute

$$\begin{aligned}\langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(f+g), \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(f+g)\rangle_{L^2} &= \langle \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(h_1+\cdots+h_n), \tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(h_1+\cdots+h_n)\rangle_{L^2} \\ &= \langle h_1+\cdots+h_n, h_1+\cdots+h_n\rangle_{L^2} = \langle f+g, f+g\rangle_{L^2}.\end{aligned}$$

This, we extend the norm preservation to the linear span of the indicator functions of N-rectangles. By the polarisation identity, also the inner product is preserved under $\tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}$ on this space. As the span of indicator functions of N-rectangles is a dense subspace of $L^2((0,\infty)^N)$, we can now extend the transformation $\tilde{\mathcal{R}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}$ to a unitary transformation on $L^2((0,\infty)^N)$. □

Appendix A

The coefficients of proposition 9.5

In this appendix, we perform manipulations on the coefficients in eq. (9.21) to help prove proposition 9.5.

First, we aim to demonstrate that

$$\mathcal{A}_{\gamma}^{w_1,w_2}(\tau)\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-i\gamma_0\right)}{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+i\gamma_0\right)}=-i\frac{q-q^{-1}}{4}\mathcal{B}_{\rho,\sigma}^{\lambda,u}(\tau).$$

Using the definition of $\mathcal{A}_{\gamma}^{w_1,w_2}$ (eq. (9.2)), we can express the left-hand side as

$$\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-i\gamma_0\right)\prod_{j=1}^{3}\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+i\gamma_j\right)}{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi(\tau+iw)}{w_2}}$$
$$=\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-u\pm\sigma\right)\times\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-\lambda\pm\rho\right)}{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi(\tau+iw)}{w_2}},$$

where we utilised eq. (8.11) to relate γ to (u,λ,ρ,σ). The term in the fraction is recognisable as $B_{\tau,\rho}^{\lambda}$. Thus, we need to establish

$$\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-u\pm\sigma\right)=-i\frac{q-q^{-1}}{4}\hat{b}_u^{\tau,\sigma}.$$

We begin from the right-hand side. Using the definition of $\hat{b}_u^{\tau,\sigma}$ (eq. (9.13)), we have

$$-i\frac{q-q^{-1}}{4}\hat{b}_u^{\tau,\sigma}=\frac{v_\tau a_u+v_{\tau+iw_1}a_{-u}}{4}+\frac{v_\sigma}{4}.$$

Since $a_u=\frac{2}{q^2-q^{-2}}\sinh\frac{2\pi}{w_2}\left(u+i\frac{w_1}{2}\right)$, we can rewrite the right-hand side as

$$\frac{\sinh\frac{2\pi}{w_2}\left(u+i\frac{w_1}{2}\right)\cosh\frac{2\pi\tau}{w_2}-\sinh\frac{2\pi}{w_2}\left(u-i\frac{w_1}{2}\right)\cosh\frac{2\pi(\tau+iw_1)}{w_2}}{q^2-q^{-2}}+\frac{v_\sigma}{4}.$$

By applying the product rules for hyperbolic sines and cosines, we can further simplify it to

$$\frac{\sinh\frac{2\pi}{w_2}\left(u+i\frac{w_1}{2}+\tau\right)+\sinh\frac{2\pi}{w_2}\left(u+i\frac{w_1}{2}-\tau\right)-\left(\sinh\frac{2\pi}{w_2}\left(u+i\frac{w_1}{2}+\tau\right)+\sinh\frac{2\pi}{w_2}\left(u-3i\frac{w_1}{2}-\tau\right)\right)}{2(q^2-q^{-2})}+\frac{v_\sigma}{4}$$

$$=\frac{\sinh\frac{2\pi}{w_2}\left(u+i\frac{w_1}{2}-\tau\right)-\sinh\frac{2\pi}{w_2}\left(u-\frac{3}{2}iw_1-\tau\right)}{2(q^2-q^{-2})}+\frac{v_\sigma}{4}$$

$$=\frac{\sinh\frac{2\pi i w_1}{w_2}\cosh\frac{2\pi}{w_2}\left(u-\tau-i\frac{w_1}{2}\right)}{q^2-q^{-2}}+\frac{v_\sigma}{4}$$

and using $\sinh\frac{2\pi i w_1}{w_2}=\frac{1}{2}(q^2-q^{-2})$, this simplifies to

$$=\frac{1}{2}\cosh\frac{2\pi}{w_2}\left(u-\tau-i\frac{w_1}{2}\right)+\frac{1}{2}\cosh\frac{2\pi\sigma}{w_2}$$

$$=\cosh\frac{\pi}{w_2}\left(u-\tau-i\frac{w_1}{2}\pm\sigma\right)=\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-u\pm\sigma\right),$$

which shows that indeed

$$-i\frac{q-q^{-1}}{4}\hat{b}_u^{\tau,\sigma}=\frac{v_\tau a_u+v_{\tau+iw_1}a_{-u}}{4}+\frac{v_\sigma}{4}=\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-u\pm\sigma\right). \tag{A.1}$$

Thus, we indeed have

$$\mathcal{A}_\gamma^{w_1,w_2}(\tau)\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-i\gamma_0\right)}{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+i\gamma_0\right)}=-i\frac{q-q^{-1}}{4}\mathcal{B}_{\rho,\sigma}^{\lambda,u}(\tau). \tag{A.2}$$

We moreover want to show that

$$-\mathcal{A}_\gamma^{w_1,w_2}(\tau)-\mathcal{A}_\gamma^{w_1,w_2}(-\tau)=-i\frac{q-q^{-1}}{4}C_{\rho,\sigma}^{\lambda,u}(\tau)+\cosh\frac{\pi}{w_2}\left(\rho+i\frac{w_1}{2}-u\pm\sigma\right).$$

We begin by writing the left-hand side of this expression as

$$-\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\pm\lambda\right)\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-u\pm\sigma\right)}{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi(\tau+iw)}{w_2}}+(\tau\leftrightarrow-\tau)$$

$$=-\left[\frac{v_\tau a_u+v_{\tau+iw_1}a_{-u}}{4}+\frac{v_\sigma}{4}\right]\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\pm\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi(\tau+iw)}{w_2}}+(\tau\leftrightarrow-\tau),$$

using eq. (A.1) to replace the term $\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}-u\pm\sigma\right)$. We can rewrite the right-hand side to

$$i\left[\frac{iv_\tau(a_u+a_{-u})-i(v_\tau-v_{\tau+iw_1})a_{-u}}{4}+\frac{iv_\sigma}{4}\right]\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\pm\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi(\tau+iw)}{w_2}}+(\tau\leftrightarrow-\tau)$$

and we recognise this as

$$i\frac{q-q^{-1}}{4}b_u^{\tau,\sigma}\left[\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\pm\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi(\tau+iw)}{w_2}}+(\tau\leftrightarrow-\tau)\right]$$
$$-i\left[\frac{i(v_\tau-v_{\tau+iw_1})a_{-u}}{4}\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\pm\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi(\tau+iw)}{w_2}}+(\tau\leftrightarrow-\tau)\right] \quad \text{(A.3)}$$

We recognise the coefficient $C^\lambda_{\tau,\rho}$ of corollary 9.2 in the term in the first pair of square brackets. Moreover, we can write

$$v_\tau-v_{\tau+iw_1}=2\cosh\frac{2\pi\tau}{w_2}-2\cosh\frac{2\pi(\tau+iw_1)}{w_2}=4\sinh\frac{2\pi(\tau+i\frac{w_1}{2})}{w_2}\sinh\frac{-i\pi w_1}{w_2}$$
$$=-2(q-q^{-1})\sinh\frac{2\pi(\tau+i\frac{w_1}{2})}{w_2}=2(q-q^{-1})\sinh\frac{2\pi(\tau+iw)}{w_2} \quad \text{(A.4)}$$

so that eq. (A.3) rewrites to

$$-i\frac{q-q^{-1}}{4}b_u^{\tau,\sigma}C^\lambda_{\tau,\rho}-i\left[\frac{i(q-q^{-1})a_{-u}}{2}\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\pm\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}}+(\tau\leftrightarrow-\tau)\right]. \quad \text{(A.5)}$$

Now we write out the term within the square brackets, and simplify it using sum and product rules for hyperbolic sines and cosines. We arrive at

$$\frac{i(q-q^{-1})a_{-u}}{2}\frac{\cosh\frac{\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\pm\lambda\right)-\cosh\frac{\pi}{w_2}\left(\tau-i\frac{w_1}{2}-\rho\pm\lambda\right)}{\sinh\frac{2\pi\tau}{w_2}}$$
$$=\frac{i(q-q^{-1})a_{-u}}{2}\frac{\cosh\frac{2\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\right)+\cosh\frac{2\pi\lambda}{w_2}-\cosh\frac{2\pi}{w_2}\left(\tau-i\frac{w_1}{2}-\rho\right)-\cosh\frac{2\pi\lambda}{w_2}}{\sinh\frac{2\pi\tau}{w_2}}$$
$$=\frac{i(q-q^{-1})a_{-u}}{2}\frac{\cosh\frac{2\pi}{w_2}\left(\tau+i\frac{w_1}{2}+\rho\right)-\cosh\frac{2\pi}{w_2}\left(\tau-i\frac{w_1}{2}-\rho\right)}{\sinh\frac{2\pi\tau}{w_2}}$$
$$=\frac{i(q-q^{-1})a_{-u}}{2}\frac{\sinh\frac{2\pi\tau}{w_2}\sinh\frac{2\pi}{w_2}\left(\rho+i\frac{w_1}{2}\right)}{\sinh\frac{2\pi\tau}{w_2}}$$
$$=\frac{i(q-q^{-1})a_{-u}}{2}\sinh\frac{2\pi}{w_2}\left(\rho+i\frac{w_1}{2}\right)$$
$$=-\frac{i(q-q^{-1})a_{-u}}{2}\sinh\frac{2\pi}{w_2}\left(\rho+iw\right)$$
$$=-\frac{i(v_\rho-v_{\rho+iw_1})a_{-u}}{4},$$

where we used eq. (A.4) in the last step, with τ replaced by ρ. Substituting this back into eq. (A.5), we get

$$
\begin{aligned}
-i\frac{q-q^{-1}}{4}b_u^{\tau,\sigma}C^{\lambda}_{\tau,\rho}+i\frac{i(\nu_\rho-\nu_{\rho+iw_1})a_{-u}}{4}&\\
&=-i\frac{q-q^{-1}}{4}\left[b_u^{\tau,\sigma}C^{\lambda}_{\tau,\rho}-\frac{i(\nu_\rho-\nu_{\rho+iw_1})a_{-u}}{q-q^{-1}}\right]\\
&=-i\frac{q-q^{-1}}{4}\left[b_u^{\tau,\sigma}C^{\lambda}_{\tau,\rho}-\frac{i\nu_\rho(a_u+a_{-u})}{q-q^{-1}}-\frac{i\nu_\sigma}{q-q^{-1}}\right.\\
&\qquad\qquad\left.+\frac{i\nu_\rho a_u+i\nu_{\rho+iw_1}a_{-u}}{q-q^{-1}}+\frac{i\nu_\sigma}{q-q^{-1}}\right]\\
&=-i\frac{q-q^{-1}}{4}\left[b_u^{\tau,\sigma}C^{\lambda}_{\tau,\rho}-b_u^{\rho,\sigma}+\frac{i\nu_\rho a_u+i\nu_{\rho+iw_1}a_{-u}}{q-q^{-1}}+\frac{i\nu_\sigma}{q-q^{-1}}\right]\\
&=-i\frac{q-q^{-1}}{4}\left[b_u^{\tau,\sigma}C^{\lambda}_{\tau,\rho}-b_u^{\rho,\sigma}+\frac{i\nu_\rho a_u+i\nu_{\rho+iw_1}a_{-u}}{q-q^{-1}}+\frac{i\nu_\sigma}{q-q^{-1}}\right]\\
&=-i\frac{q-q^{-1}}{4}C^{\lambda,u}_{\rho,\sigma}(\tau)+\cosh\frac{\pi}{w_2}\big(\rho+i\frac{w_1}{2}-u\pm\sigma\big),
\end{aligned}
$$

where we apply the definition of $C^{\lambda,u}_{\rho,\sigma}$ in proposition 9.4 and the relation eq. (A.1) in the last step. We conclude that

$$
-\mathcal{A}^{w_1,w_2}_{\gamma}(\tau)-\mathcal{A}^{w_1,w_2}_{\gamma}(-\tau)=-i\frac{q-q^{-1}}{4}C^{\lambda,u}_{\rho,\sigma}(\tau)+\cosh\frac{\pi}{w_2}\big(\rho+i\frac{w_1}{2}-u\pm\sigma\big), \tag{A.6}
$$

as we desired to show.

Appendix B

Invariance of the product $t_\gamma \mathcal{A}_\gamma^{w_1,w_2} W_\gamma$

We want to show that the product $t_\gamma(x)\mathcal{A}_\gamma^{w_1,w_2}(x)W_\gamma(x)$ is invariant under replacing x by $-x-iw_1$.

First, note that we can write this product as

$$t_\gamma(x)\mathcal{A}_\gamma^{w_1,w_2}(x)W_\gamma(x) = G(\pm x - i\gamma_0)G(\pm(x+iw_1) - i\gamma_0) \\ \times \frac{G(iw \pm x)}{w_1 w_2} \frac{\prod_{j=0}^3 G(\pm x + i\gamma_j)\cosh\frac{\pi}{w_2}\left(x + i\frac{w_1}{2} + i\gamma_j\right)}{\sinh\frac{2\pi x}{w_2}\sinh\frac{2\pi(x+iw)}{w_2}}. \tag{B.1}$$

We immediately see that the factor $G(\pm x - i\gamma_0)G(\pm(x+iw_1) - i\gamma_0)$ satisfies this invariance, so that it remains to check that the invariance holds on

$$G(iw \pm x)\frac{\prod_{j=0}^3 G(\pm x + i\gamma_j)\cosh\frac{\pi}{w_2}\left(x + i\frac{w_1}{2} + i\gamma_j\right)}{\sinh\frac{2\pi x}{w_2}\sinh\frac{2\pi(x+iw)}{w_2}} \tag{B.2}$$

By using the difference equations eqs. (6.3) and (6.4), we deduce that

$$\begin{aligned} G(iw \pm 2x) = \frac{G(2x+iw)}{G(2x-iw)} &= 2\cosh\frac{\pi}{w_2}\left(2x + i\frac{w_2}{2}\right)\frac{G(2x + i\frac{w_2}{2} - i\frac{w_1}{2})}{G(2x-iw)} \\ &= 4\cosh\frac{\pi}{w_2}\left(2x + i\frac{w_2}{2}\right)\cosh\frac{\pi}{w_1}\left(2x - i\frac{w_1}{2}\right)\frac{G(2x-iw)}{G(2x-iw)} \\ &= 4\sinh\frac{2\pi x}{w_1}\sinh\frac{2\pi x}{w_2}. \end{aligned}$$

Thus, eq. (B.2) rewrites to

$$4\frac{\sinh\frac{2\pi x}{w_1}}{\sinh\frac{2\pi(x+iw)}{w_2}}\prod_{j=0}^{3} G(\pm x + i\gamma_j)\cosh\frac{\pi}{w_2}\left(x + i\frac{w_1}{2} + i\gamma_j\right). \tag{B.3}$$

It is straightforward to check that the fraction of hyperbolic sines is invariant under replacing x by $-x - iw_1$, as

$$\sinh\frac{2\pi(-x-iw_1)}{w_1} = -\sinh\frac{2\pi(x+iw_1)}{w_1} = -\sinh\frac{2\pi x}{w_1},$$

and

$$\sinh\frac{2\pi(-x-iw_1+iw)}{w_2} = -\sinh\frac{2\pi(x+iw_1-iw)}{w_2} = -\sinh\frac{2\pi(x+iw-iw_2)}{w_2} = -\sinh\frac{2\pi(x+iw)}{w_2}.$$

We're left with checking that

$$\prod_{j=0}^{3} G(\pm x+i\gamma_j)\cosh\frac{\pi}{w_2}\big(x+i\frac{w_1}{2}+i\gamma_j\big)$$

is invariant. To see this, note that

$$2\cosh\frac{\pi}{w_2}\big(x+i\frac{w_1}{2}+i\gamma_j\big) = G(x+i\gamma_j+iw_1)/G(x+i\gamma_j),$$

so that

$$\prod_{j=0}^{3} G(\pm x+i\gamma_j)\cosh\frac{\pi}{w_2}\big(x+i\frac{w_1}{2}+i\gamma_j\big) = \frac{1}{16}\prod_{j=0}^{3} G(-x+i\gamma_j)G(x+i\gamma_j+iw_1),$$

and the right-hand side is indeed invariant under replacing x by $-x-iw_1$.

Note that we have shown also that

$$t_\gamma(x)\mathcal{A}_\gamma^{w_1,w_2}(x)W_\gamma(x) = \frac{1}{4w_1w_2}\frac{\sinh\frac{2\pi x}{w_1}}{\sinh\frac{2\pi(x+iw)}{w_2}}G(-x-i\gamma_0)G(x-i\gamma_0+iw_1) \times\prod_{j=1}^{3} G(-x+i\gamma_j)G(x+i\gamma_j+iw_1). \quad \text{(B.4)}$$

Appendix C

Dominating the Dirichlet integral

Let $k \in \mathbb{N}$ and $f \in C_c^\infty(k\pi, (k+1)\pi)$. Let K be a compact set. Our objective in this appendix is to demonstrate that the integral

$$\int_0^\infty f(x) \frac{\sin\Big((N+\frac{1}{2})(x-y)\Big)}{x-y} \, \mathrm{d}x \tag{C.1}$$

can be bounded independently of N for $y \in K$.

By a general result from Fourier theory, the integral

$$\frac{1}{2} \int_0^\infty f(x) \frac{\sin\Big((N+\frac{1}{2})(x-y)\Big)}{\sin\frac{1}{2}(x-y)} \, \mathrm{d}x \tag{C.2}$$

converges to $2\pi f(y)$ uniformly for $y \in [k\pi, (k+1)\pi]$. This result, that can be derived from e.g. [43, cor. 10.4], allows us to bound the integral independently of N on $[k\pi, (k+1)\pi]$.

Note that $\left|\frac{2}{x-y} - \frac{1}{\sin(x-y)}\right|$ remains bounded for $y \in [k\pi, (k+1)\pi]$ and $x \in \operatorname{supp} f$. As a consequence, the integral

$$\frac{1}{2} \int_0^\infty f(x) \sin\Big((N+\frac{1}{2})(x-y)\Big)\left(\frac{2}{x-y} - \frac{1}{\sin\frac{1}{2}(x-y)}\right) \mathrm{d}x \tag{C.3}$$

can be bounded independently of N for $y \in [k\pi, (k+1)\pi]$. By adding eq. (C.2) and eq. (C.3), we can now bound the integral eq. (C.1) independently of N for y within $[k\pi, (k+1)\pi]$.

It remains to establish a bound for eq. (C.1) for $y \in K \setminus (k\pi, (k+1)\pi)$. As the support of f is compact, there exists a positive δ such that $|x-y| \geq \delta$ for all $x \in \operatorname{supp} f$ and $y \in K \setminus (k\pi, (k+1)\pi)$. Due to the smoothness of f, the term $\left|\frac{f(x)}{x-y}\right|$ is bounded for such x and y, and as the sine function is also absolutely bounded, we find that eq. (C.1) can be bounded independently of N for $y \in K \setminus (k\pi, (k+1)\pi)$.

Appendix D

Lemma for the multivariate transformation

We require the following lemma to establish the unitarity of $\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}$:

Lemma D.1. *Let $N \in \mathbb{N}_{>0}$ and let $\tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}$ denote the transformation defined in eq. (11.17). Suppose λ_j and u_j ($j = 1, 2, ..., N$) are in $i(w_0 - w, w)$ with $\mathrm{Im}(\lambda_j + u_j) \in (w_0 - w, w)$. Let $\rho, \sigma \in \mathbb{R}$ and let f and g be indicator functions of the N-rectangles R_f and R_g with disjoint interiors. We have*

$$\langle \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} f, \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma} g \rangle_{L^2} = 0. \tag{D.1}$$

Proof. We will prove this lemma using induction on N. For $N = 1$, the lemma holds true based on theorem 10.10. Now, let us assume that the lemma holds for $N = 1, 2, ..., N' - 1$, where $N' > 1$. We express $f(x_1, ..., x_{N'}) = f_1(x_1) \times \cdots \times f_{N'}(x_{N'})$ and $g(x_1, ..., x_{N'}) = g_1(x_1) \times \cdots \times g_{N'}(x_{N'})$. Let $\overleftarrow{f}(x_2, ..., x_{N'}) = f_2(x_2) \times \cdots \times f_{N'}(x_{N'})$ and similarly for g. As the interiors of R_f and R_g are disjoint, at least one of the equalities $f_1 \overline{g}_1 = 0$ and $\overleftarrow{f}\overline{\overleftarrow{g}} = 0$ must hold almost everywhere.

Using the notation with hats as introduced in the proof of lemma 11.8, we calculate

$$\begin{aligned}
&\langle \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(f+g), \tilde{\boldsymbol{\mathcal{R}}}^{\boldsymbol{\lambda},\boldsymbol{u}}_{\rho,\sigma}(f+g) \rangle_{L^2} \\
&\quad = \langle \hat{f}^{(0)} + \hat{g}^{(0)}, \hat{f}^{(0)} + \hat{g}^{(0)} \rangle_{L^2} \\
&\quad = \int_0^\infty \cdots \int_0^\infty \tilde{\mathcal{R}}_{\gamma^1}\left[f_1(\,\cdot\,)\hat{f}^{(1)}(\,\cdot\,; y_2, ..., y_{N'}) + g_1(\,\cdot\,)\hat{g}^{(1)}(\,\cdot\,; y_2, ..., y_{N'}) \right](y_1) \\
&\qquad\qquad \times \overline{\tilde{\mathcal{R}}_{\gamma^1}\left[f_1(\,\cdot\,)\hat{f}^{(1)}(\,\cdot\,; y_2, ..., y_{N'}) + g_1(\,\cdot\,)\hat{g}^{(1)}(\,\cdot\,; y_2, ..., y_{N'}) \right](y_1)}\, \mathrm{d}y_1 ... \mathrm{d}y_{N'}.
\end{aligned}$$

Viewing the inner integral as an inner product and noting that $\tilde{\mathcal{R}}_{\gamma^1}$ is unitary according to theorem 10.10, we can rewrite the above expression as

$$\begin{aligned}
&= \int_0^\infty \cdots \int_0^\infty \left[f_1(x_1)\hat{f}^{(1)}(x_1; y_2, ..., y_{N'}) + g_1(x_1)\hat{g}^{(1)}(x_1; y_2, ..., y_{N'}) \right] \\
&\qquad \times \overline{\left[f_1(x_1)\hat{f}^{(1)}(x_1; y_2, ..., y_{N'}) + g_1(x_1)\hat{g}^{(1)}(x_1; y_2, ..., y_{N'}) \right]}\, \mathrm{d}x_1 \mathrm{d}y_2 ... \mathrm{d}y_{N'}.
\end{aligned}$$

Since the integrand is nonnegative, we can apply Tonelli's theorem to change the order of integration, allowing us to write

$$= \int_0^\infty \cdots \int_0^\infty \left[f_1(x_1) \widehat{f}^{(1)}(x_1; y_2, ..., y_{N'}) + g_1(x_1) \widehat{g}^{(1)}(x_1; y_2, ..., y_{N'}) \right] \\ \times \overline{\left[f_1(x_1) \widehat{f}^{(1)}(x_1; y_2, ..., y_{N'}) + g_1(x_1) \widehat{g}^{(1)}(x_1; y_2, ..., y_{N'}) \right]} \, \mathrm{d}y_2 ... \mathrm{d}y_{N'} \mathrm{d}x_1 .$$

Expanding the brackets, we can rewrite this as

$$\begin{aligned} &= \int_0^\infty \cdots \int_0^\infty f_1(x_1) \widehat{f}^{(1)}(x_1; y_2, ..., y_{N'}) \overline{f_1(x_1) \widehat{f}^{(1)}(x_1; y_2, ..., y_{N'})} \, \mathrm{d}y_2 ... \mathrm{d}y_{N'} \mathrm{d}x_1 \\ &+ \int_0^\infty \cdots \int_0^\infty g_1(x_1) \widehat{g}^{(1)}(x_1; y_2, ..., y_{N'}) \overline{g_1(x_1) \widehat{g}^{(1)}(x_1; y_2, ..., y_{N'})} \, \mathrm{d}y_2 ... \mathrm{d}y_{N'} \mathrm{d}x_1 \\ &+ \int_0^\infty f_1(x_1) \overline{g_1(x_1)} \int_0^\infty \cdots \int_0^\infty \widehat{f}^{(1)}(x_1; y_2, ..., y_{N'}) \overline{\widehat{g}^{(1)}(x_1; y_2, ..., y_{N'})} \, \mathrm{d}y_2 ... \mathrm{d}y_{N'} \mathrm{d}x_1 \\ &+ \int_0^\infty g_1(x_1) \overline{f_1(x_1)} \int_0^\infty \cdots \int_0^\infty \widehat{g}^{(1)}(x_1; y_2, ..., y_{N'}) \overline{\widehat{f}^{(1)}(x_1; y_2, ..., y_{N'})} \, \mathrm{d}y_2 ... \mathrm{d}y_{N'} \mathrm{d}x_1 . \end{aligned}$$

The first two integrals in the latter expression can be recognised $\langle f, f \rangle_{L^2}$ and $\langle g, g \rangle_{L^2}$ respectively. The remaining two integrals can be shown to be zero by the following observations:

- If $f_1 \overline{g}_1 = 0$, then the latter two integrals are trivially zero.
- If $\overleftarrow{f} \overline{\overleftarrow{g}} = 0$, the indicator functions $\overleftarrow{f}$ and $\overleftarrow{g}$ have supports with disjoint interiors. Consider the integral

  $$\int_0^\infty \cdots \int_0^\infty \widehat{f}^{(1)}(x_1; y_2, ..., y_{N'}) \overline{\widehat{g}^{(1)}(x_1; y_2, ..., y_{N'})} \, \mathrm{d}y_2 ... \mathrm{d}y_{N'}$$

  that appears in the third line of the expression. It represents an instance of eq. (D.1) with a transformation for the case $N = N' - 1$, where f and g are replaced by the functions $\overleftarrow{f}$ and $\overleftarrow{g}$. Since $\overleftarrow{f}$ and $\overleftarrow{g}$ are indicator functions of sets with disjoint interiors, the integral in the third line of the expression vanishes. A similar reasoning applies to the integral in the fourth line.

Therefore, we have

$$\langle \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma}(f+g), \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma}(f+g) \rangle_{L^2} = \langle f, f \rangle_{L^2} + \langle g, g \rangle_{L^2} .$$

Expanding the left-hand side, we find

$$\langle \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} f, \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} g \rangle_{L^2} + \langle \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} g, \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} f \rangle_{L^2} = 0.$$

By replacing g with ig, we obtain

$$\langle \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} f, i\tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} g \rangle_{L^2} + \langle i\tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} g, \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} f \rangle_{L^2} = -i \langle \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} f, \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} g \rangle_{L^2} + i \langle \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} g, \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} f \rangle_{L^2} = 0.$$

Combining these two results, we conclude that $\langle \tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} f, i\tilde{\mathcal{R}}^{\lambda,u}_{\rho,\sigma} g \rangle_{L^2} = 0$. □

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