

Nonmonotonic trajectories to a partially penetrating well in a semiconfined aquifer

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[1] The flow path to a partially penetrating well in a semiconfined aquifer with finite thickness can exhibit nonmonotonic behavior. Water particles entering a semiconfined aquifer far away from a well through the confining layer go downward, and closer to the well they move upward, while under certain circumstances they rise so high that they come down again to finally be captured by the well. An approximative problem is solved analytically under the assumptions that the aquifer is of infinite thickness and that the screen may be represented as a point. It is shown that this phenomenon will occur for particular values of parameters $Kc/a > 1.283$, where a is the position of the point extraction in the aquifer with respect to the top of the aquifer, K is the hydraulic conductivity of the aquifer, and c is the hydraulic resistance of the covering layer. Such upward bending groundwater path lines have ecological implications in the sense that water from far away will come close to the top of the aquifer in the neighborhood of the well.

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1. Introduction

[2] It was noted in 1984 at the National Institute for Public Health and Environmental Protection (now the National Institute for Public Health and the Environment, Bilthoven, Netherlands) that the flow of a water particle started far away at the top of a semiconfined homogeneous, isotropic aquifer, just underneath the semiconfining layer, to a partially penetrating well, showed a ripple (Figure 1).

[3] A particle can be seen to start at the top of the aquifer, far away from the well, initially go downward below the bottom elevation of the well screen, and then in the neighborhood of the well go upward. In fact, close to the well, the particle continues to rise such that it has to go downward again to be captured by the well. This trajectory exhibits a minimum and a local maximum as a function of the distance to the well. We show that for certain parameter values the nonmonotonic flow paths seen in Figure 1 are to be expected, which might be useful for the interpretation of water quality measurements in the field.

2. Mathematical Model

[4] Firstly, we describe the flow pattern by a mathematical model with the corresponding analytical solution for the drawdown $\phi = \phi(r, z)$. The governing differential equation and associated boundary conditions are:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad r > 0, \quad 0 < z < D, \quad (1)$$

$$\begin{aligned} \frac{\partial \phi}{\partial z}(r, 0) &= \frac{\phi(r, 0)}{Kc}, \quad r > 0, \\ \frac{\partial \phi}{\partial z}(r, D) &= 0, \quad r > 0, \end{aligned} \quad (2)$$

$$\begin{aligned} \lim_{r \rightarrow 0} r \frac{\partial \phi}{\partial r} &= -\frac{Q}{2\pi Kl}, \quad a - l/2 < z < a + l/2, \\ \lim_{r \rightarrow 0} r \frac{\partial \phi}{\partial r} &= 0, \quad 0 \leq z < a - l/2 \ \& \ a + l/2 < z \leq D, \end{aligned} \quad (3)$$

where (r, z) [L] are the spatial coordinates, with the z -axis pointing downward, ϕ [L] the drawdown, D [L] the thickness of the aquifer, K [LT^{-1}] the hydraulic conductivity, c [T] the hydraulic resistance of the aquitard at the top of the aquifer, a [L] the distance (assumed to be positive) between the center of the screen and the top of the aquifer, l [L] the length of the screen and Q [L^3T^{-1}] the pumping rate of the partially penetrating well. The analytical solution to the problem defined by (1), (2) and (3) is given by [Bruggeman, 1999, formula 532.06]:

$$\begin{aligned} \phi(r, z) &= \frac{2Q}{\pi Kl} \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \frac{\sin\left(\frac{\alpha_n l}{2D}\right) \cos\left(\frac{\alpha_n(D-a)}{D}\right)}{1 + \frac{D/(Kc)}{\alpha_n^2 + (D/(Kc))^2}} \\ &\times K_0\left(\frac{\alpha_n r}{D}\right) \cos\left(\frac{\alpha_n(D-z)}{D}\right), \end{aligned} \quad (4)$$

where α_n , $n = 0, \dots, \infty$ are the positive roots of the transcendental equation $\alpha \tan \alpha = D/(Kc)$, and K_0 the modified Bessel function of order zero. This analytical solution is implemented in FLOP3N [Veling, 1992], which

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calculates the drawdown in each of N coupled aquifers (in our case 2) based on the analytical solution for the full three-dimensional problem [Maas, 1987].

[5] Although one may start with finding the zeros of the first derivative of (4), the terms in the resulting series exhibit a rather erratic behavior of the sign depending on the relative magnitudes of α_n , l and a . To avoid this we will solve an approximative problem with a more convenient solution.

3. Approximative Problem

[6] Since we know that the strange behavior of interest occurs only then when the screen is situated rather high in the aquifer, the first approximation is to ignore the lower aquifer boundary, hence take the aquifer to be infinitely deep. Moreover, in such a case it is reasonable to approximate the well screen by a point extraction. This means that we address the mathematical problem:

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad r > 0, \quad 0 < z < \infty, \quad (5)$$

$$\frac{\partial \phi}{\partial z}(r, 0) = \frac{\phi(r, 0)}{Kc}, \quad r > 0, \quad (6)$$

$$\lim_{\rho \rightarrow 0} \rho^2 \frac{\partial \phi}{\partial \rho} = -\frac{Q}{4\pi K}, \quad \rho = \sqrt{r^2 + (z-a)^2},$$

$$\lim_{r \rightarrow 0} \frac{\partial \phi}{\partial r} = 0, \quad r = 0, \quad z \neq a, \quad (7)$$

$$\lim_{r \rightarrow \infty} \phi(r, z) = 0, \quad 0 < z < \infty,$$

$$\lim_{z \rightarrow \infty} \phi(r, z) = 0, \quad 0 < r < \infty.$$

Here we assume $a > 0$ because otherwise there can be no ripple at all. The solution to (5), with (6) and (7), is given by [see Bruggeman, 1999, formula 522.24]

$$\begin{aligned} \phi(r, z) &= \frac{Q}{4\pi K} \left[\frac{1}{\sqrt{r^2 + (z-a)^2}} + \frac{1}{\sqrt{r^2 + (z+a)^2}} \right. \\ &\quad \left. - 2 \int_0^\infty \frac{1}{1 + Kc\alpha} J_0(r\alpha) e^{-(z+a)\alpha} d\alpha \right]. \end{aligned} \quad (8)$$

Herein is J_0 the Bessel function of the first kind and order zero. In this representation use has been made of [Abramowitz and Stegun, 1964, formula 29.3.55]

$$\int_0^\infty J_0(r\alpha) e^{-(z+a)\alpha} d\alpha = \frac{1}{\sqrt{r^2 + (z+a)^2}}. \quad (9)$$

4. Estimates

[7] Our purpose is to show that there are values for r , a , K and c such that $\frac{\partial \phi(r, a)}{\partial z} < 0$, because then for that value of r a trajectory passing the elevation a of the extraction will rise but since it will be captured by the well afterward, it will show a ripple. For this to happen the velocity component at

(r, a) must be negative. The vertical velocity component is proportional to

$$\begin{aligned} \frac{\partial \phi}{\partial z}(r, z) &= -\frac{Q}{4\pi K} \left[\frac{z-a}{\left(\sqrt{r^2 + (z-a)^2}\right)^3} + \frac{z+a}{\left(\sqrt{r^2 + (z+a)^2}\right)^3} \right. \\ &\quad \left. - \frac{2}{Kc} \left\{ \frac{1}{\sqrt{r^2 + (z+a)^2}} \right. \right. \\ &\quad \left. \left. - \int_0^\infty \frac{1}{1 + Kc\alpha} J_0(r\alpha) e^{-(z+a)\alpha} d\alpha \right\} \right]. \end{aligned} \quad (10)$$

At the derivation of (10) use has been made of (9). If one evaluates (10) at the level of the well, thus for $z = a$, then it follows that

$$\begin{aligned} \frac{\partial \phi}{\partial z}(r, a) &= -\frac{Q}{4\pi K} \left[\frac{2a}{\left(\sqrt{r^2 + 4a^2}\right)^3} - \frac{2}{Kc} \left\{ \frac{1}{\sqrt{r^2 + 4a^2}} \right. \right. \\ &\quad \left. \left. - \int_0^\infty \frac{1}{1 + Kc\alpha} J_0(r\alpha) e^{-(2a)\alpha} d\alpha \right\} \right]. \end{aligned} \quad (11)$$

In Appendix A we prove that the integral in this expression is always positive, so $\frac{\partial \phi(r, a)}{\partial z}$ can be estimated to

$$\frac{\partial \phi}{\partial z}(r, a) < -\frac{Q}{4\pi K} \left[\frac{2a}{\left(\sqrt{r^2 + 4a^2}\right)^3} - \frac{2}{Kc} \frac{1}{\sqrt{r^2 + 4a^2}} \right]. \quad (12)$$

So, $\frac{\partial \phi(r, a)}{\partial z}$ is negative if

$$\frac{2a}{\left(\sqrt{r^2 + 4a^2}\right)^3} - \frac{2}{Kc} \frac{1}{\sqrt{r^2 + 4a^2}} > 0. \quad (13)$$

Condition (13) is equivalent with

$$r^2 < a(Kc - 4a), \quad (14)$$

and it is only possible to satisfy this condition if

$$Kc/a > 4. \quad (15)$$

Because (15) is a sufficient condition, it does not mean that the phenomenon is absent in cases where $Kc/a \leq 4$. In Appendix A we derive a more general sufficient condition

$$Kc/a > 1.282371 \dots \quad (16)$$

[8] In the example presented in Figure 1, condition (15) has been satisfied. In Figures 2, 3, 4, and 5 the quotient Kc/a takes on values of 24.0, 8.89, 3.33, and 1.0, respectively. Condition (15) is satisfied only by the first and second cases, which exhibit the phenomenon (Figures 2 and 3). Condition (15) has not been satisfied by the third case, but (16) has been satisfied: the phenomenon is present but hardly visible (Figure 4). When (16) is not satisfied, the

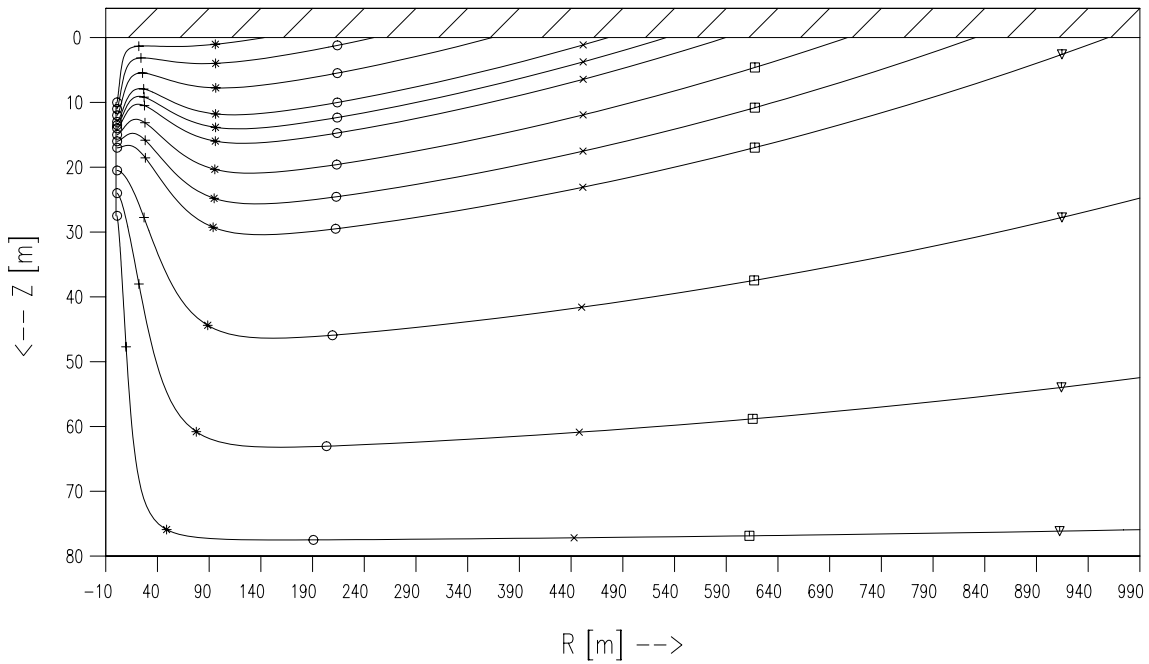


Figure 1. Example of the upward flow near a well. The horizontal axis shows the radial distance to the well; the vertical axis shows the height of the aquifer covered by an aquitard. The aquifer is bounded on the bottom by an impervious layer. The z axis is pointed downward. The screen is located at $r = 0$, from $z = 10$ m to $z = 27.5$ m. The six trajectories starting at $r = 475$ m, . . . , 959 m at the top of the aquifer exhibit the upward flow behavior the most obviously. Parameter values are $K = 25$ m/d, $c = 450$ days, $D = 80$ m, and $a = 18.75$ m, $l = 17.5$ m. Horizontal dimension is 1000 m, vertical dimension is 84.5 m. The small circles at the screen indicate the location where the water particles reach the screen. The other symbols indicate the time before the water particles will be kept by the screen (3, 60, 365, 1825, 3650, and 9125 days, respectively, for a pumping rate of 9240 m³/d and a porosity of 0.3).

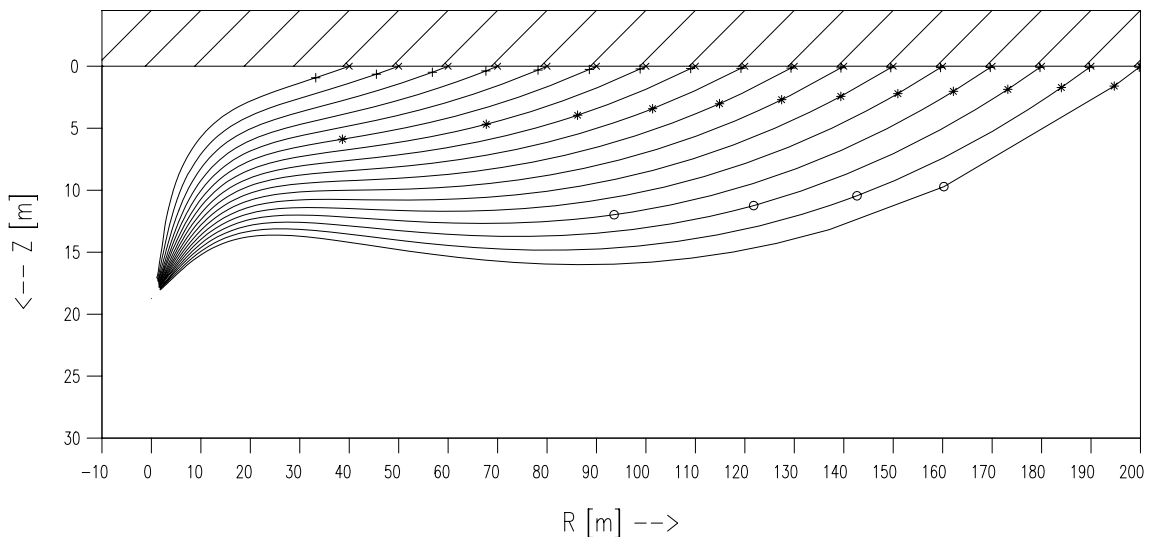


Figure 2. First of a series of four figures with variation of the hydraulic resistance parameter c . The horizontal axis shows the radial distance to the well; the vertical axis shows the height of the aquifer covered by an aquitard. The aquifer is unbounded from below. The z axis is pointed downward. The point extraction is located at $r = 0$, $z = a = 18.75$ m. Horizontal dimension is 210 m, and vertical dimension is 34.5 m. The symbols indicate the time the water particles starting at the top of the aquifer have traveled into the direction of the screen (3, 60, and 365 days, respectively, for a pumping rate of 9240 m³/d and a porosity of 0.3). Parameter values $K = 25$ m/d and $c = 18$ days. Condition (15) has been satisfied.

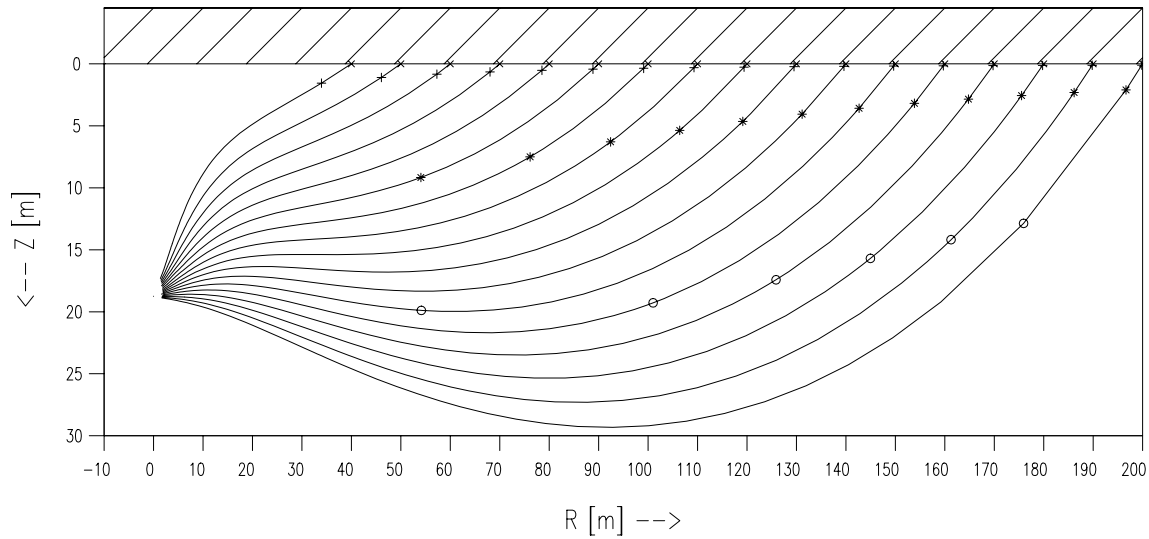


Figure 3. Parameter value $c = 6.67$ days. See Figure 2 for further explanation. Condition (15) has been satisfied.

phenomena is not present (Figure 5). From (14) one can get an estimate at which distance from the well this particular behavior will occur. For the first three cases we find: $r < 89.4$ m, $r < 51.7$ m and $r < 22.1$ m, respectively, so it will occur at distances easily measurable in the field. In the fourth case (16) is not satisfied.

[9] In the limit for $c \rightarrow \infty$, thus in a confined aquifer, it is clear that the local maximum occurs always, since (15) is always satisfied. The locus of the points of that local maximum is given by

$$r^2 = (a^2 - z^2)^{2/3} \left((z + a)^{2/3} + (a - z)^{2/3} \right), \quad (17)$$

as can be found by putting $\frac{\partial \phi(r, z)}{\partial z} = 0$, where $\frac{\partial \phi}{\partial z}$ is given by the first two terms in (10). So, the character of the

flow field changes dramatically by changing the condition $\frac{\partial \phi(r, 0)}{\partial z} = \frac{\phi(r, 0)}{Kc}$ into $\frac{\partial \phi(r, 0)}{\partial z} = 0$.

[10] The dimensionless parameter combination Kc/a is the only relevant parameter for this problem. This can be seen by scaling the axes as $r = r'a$ and $z = z'a$; in that case the extraction is located at $(r', z') = (0, 1)$, and the equation (6) becomes $\frac{\partial \phi(r', 0)}{\partial z'} = \frac{a\phi(r', 0)}{Kc}$.

5. Discussion

[11] We have shown that particles which enter an aquifer far away from an extraction well may follow a path through the aquifer in such a way that close to the well they move upward before they are captured by the

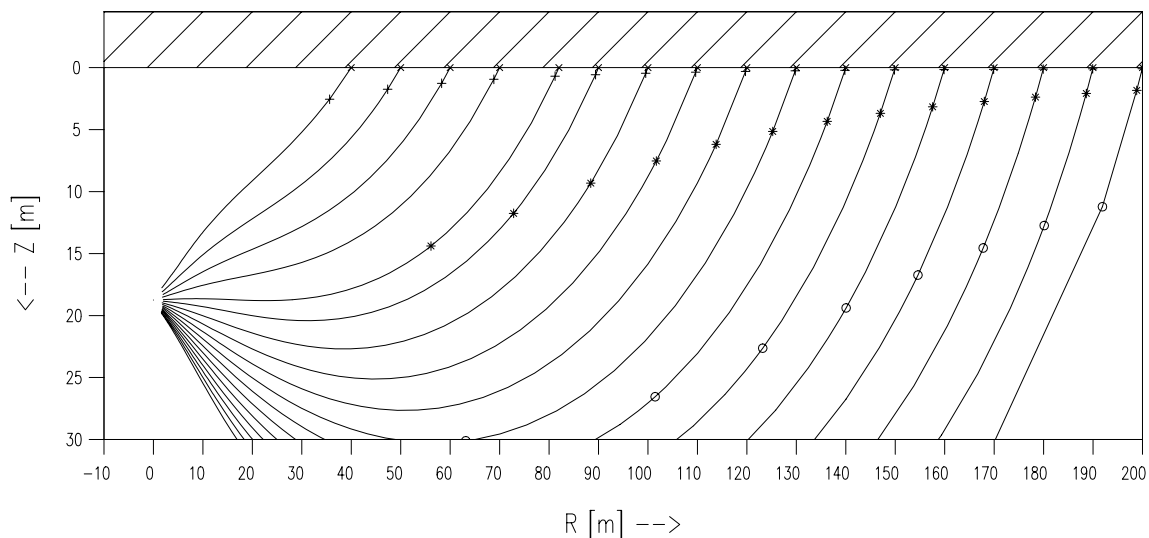


Figure 4. Parameter value $c = 2$ days. See Figure 2 for further explanation. Condition (15) has not been satisfied, but (16) has. However, the phenomenon is hardly visible at the fifth streamline from the top left, starting at $r = 82$ m.

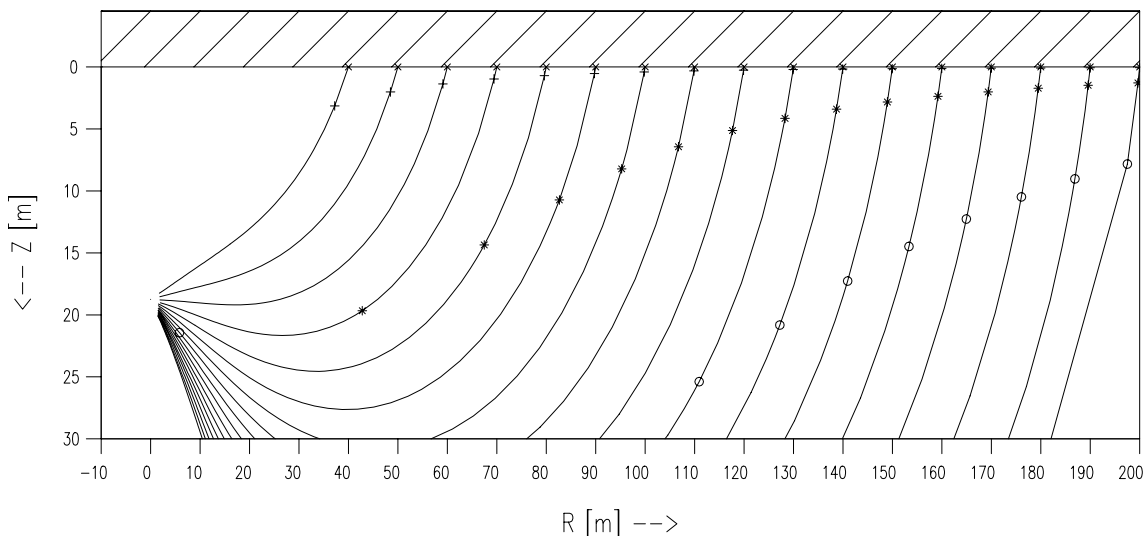


Figure 5. Parameter value $c = 0.75$ days. See Figure 2 for further explanation. Condition (16) has not been satisfied.

well. This has implications regarding the assessments of the shallow groundwater quality close to the extraction point.

[12] It is useful to try to understand why this phenomenon occurs. It certainly is a balance between the infiltration through the confining layer close to the extraction point and the amount of water coming from far. It can be proved that all the extracted water comes through that confining layer, as long as $c > 0$. If Kc/a becomes larger (see equation (6), after the scaling discussed above) the infiltration through the confining layer for the same r becomes less, giving more room for trajectories from below to overshoot the level of the extraction. In the limit for $c \rightarrow \infty$ this will occur always, because the aquifer is confined then, and the water has to be extracted from the aquifer itself.

Appendix A

[13] In this appendix we show that the integral in (11) is always positive.

$$\begin{aligned} I &= \int_0^\infty \frac{1}{1 + Kc\alpha} J_0(r\alpha) e^{-(2a)\alpha} d\alpha \\ &= s \int_0^\infty \frac{1}{s + \alpha} J_0(r\alpha) e^{-(2a)\alpha} d\alpha, \quad \text{where } s = 1/(Kc). \\ I &= s \int_0^\infty \left(\int_0^\infty e^{-(\alpha+s)t} dt \right) J_0(r\alpha) e^{-(2a)\alpha} d\alpha, \end{aligned}$$

using $\frac{1}{s + \alpha}$ as the Laplace transform for $e^{-\alpha t}$.

$$\begin{aligned} I &= s \int_0^\infty \left(\int_0^\infty e^{-st} dt \right) J_0(r\alpha) e^{-(2a+t)\alpha} d\alpha \\ &= s \int_0^\infty e^{-st} \left(\int_0^\infty J_0(r\alpha) e^{-(2a+t)\alpha} d\alpha \right) dt \\ &= s \int_0^\infty e^{-st} \frac{1}{\sqrt{r^2 + (2a + t)^2}} dt > 0. \end{aligned}$$

Substitution of $t = 2a(\tau - 1)$ leads to (remark that we had assumed that $a > 0$):

$$\begin{aligned} I &= \frac{2a}{Kc} e^{2a/(Kc)} \int_1^\infty e^{-2a\tau/(Kc)} \frac{1}{\sqrt{r^2 + 4a^2\tau^2}} d\tau. \\ I &> \frac{2a}{Kc} e^{2a/(Kc)} \int_1^\infty e^{-2a\tau/(Kc)} \frac{1}{\sqrt{r^2\tau^2 + 4a^2\tau^2}} d\tau \\ &= \frac{1}{\sqrt{r^2 + 4a^2}} \frac{2a}{Kc} e^{2a/(Kc)} E_1\left(\frac{2a}{Kc}\right), \end{aligned}$$

with $E_1(y) = \int_y^\infty e^{-t}/t dt$ the exponential integral. If one introduces $p = 2a/(Kc)$ then (11) transforms into

$$\frac{\partial \phi}{\partial z}(r, a) < -\frac{Q}{4\pi Ka\sqrt{r^2 + 4a^2}} \left[\frac{2a^2}{r^2 + 4a^2} - p + p^2 e^p E_1(p) \right]. \quad (\text{A1})$$

From here, it follows that if

$$r^2 < \frac{2a^2}{p - p^2 e^p E_1(p)} - 4a^2, \quad (\text{A2})$$

$\frac{\partial \phi}{\partial z}(r, a)$ is negative. Condition (A2) is only relevant if the right-hand side of (A2) is positive, so

$$p - p^2 e^p E_1(p) < 1/2. \quad (\text{A3})$$

The function $f(y) = y - y^2 e^y E_1(y)$ is monotonously increasing with $f(0) = 0$, so, if $y = 2a/(Kc) < y_0 = 1.559611 \dots$, with $y_0 - y_0^2 e^{y_0} E_1(y_0) = 1/2$, then condition (A3) is fulfilled and the derivation of (16) is complete.

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References

- Abramowitz, M., and I. A. Stegun (Eds.) (1964), *Handbook of Mathematical Functions*, Natl. Bur. of Stand., Washington, D. C.
- Bruggeman, G. A. (1999), *Analytical Solutions of Geohydrological Problems*, Dev. Water Sci., vol. 46, Elsevier, New York.
- Maas, C. (1987), Groundwater flow to a well in a layered porous medium: 1. Steady flow, *Water Resour. Res.*, 23(8), 1675–1681.
- Veling, E. J. M. (1992), Three-dimensional groundwaterflow modelling for the calculation of capture zones around extraction sites, in *Computational Fluid Dynamics*, vol. 2, *Proceedings of the First European Computational Fluid Dynamics Conference, September 7–11, 1992, Brussels*,

edited by C. Hirsch, J. Périaux, and W. Kordulla, pp. 1013–1020, Elsevier, New York.

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