

Introduction to a depth-integrated model  
for suspended transport (Galappatti, 1983)

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Report no. 6-86

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## 1. INTRODUCTION

It is the aim of this note to give a simple introduction to the method as used by Galappatti (1983) to derive a depth-integrated model for suspended transport.

Contrary to other depth-integrated models (see e.g. Latteux and Teisson, 1985) this model is directly based on the 2D convection-diffusion equation (see e.g. de Vries, 1981) including the boundary condition near the bed. No extra empirical relation for deposition/pick-up rate near the bed is necessary. This note will concentrate on a schematic case in order to clarify the solution method as good as possible.

The following assumptions will be made:

1. A uniform but non-steady flow is considered, i.e.  $\partial/\partial x = 0$  and  $\partial/\partial t \neq 0$ .
2. The concentration of the suspended sediment is also uniform ( $\partial C/\partial x = 0$ ) but a function of time  $t$  and vertical coordinate  $z$ .
3. The diffusion coefficient for sediment  $\epsilon_s$  is constant in vertical direction and can be described by  $\epsilon_s = \beta u_* a$  ( $u_* =$  friction velocity,  $a =$  water depth and  $\beta = 0.067$ , i.e. the average value of a parabolic  $\epsilon_s(z)$ -distribution).

Under these assumptions the 2D convection-diffusion equation (in the vertical plane), as usually applied for solving the sediment concentration, can be written as follows:

$$\frac{\partial C}{\partial t} = w_s \frac{\partial C}{\partial z} + \frac{\partial C}{\partial z} \left( \epsilon_s \frac{\partial C}{\partial z} \right) = \frac{\partial \phi_z}{\partial z} \quad (1)$$

$$\epsilon_s = \beta u_* a$$

$w_s =$  fall velocity of the sediment

The vertical sediment flux  $\phi_z$  is zero at the water surface  $z = z_w$ , or:

$$\phi_z(z = z_w) = \left[ w_s \cdot c + \epsilon_s \frac{\partial C}{\partial z} \right]_{z=z_w} = 0 \quad (2)$$

Another boundary condition is applied at bed level  $z_b$  ; two possibilities will be treated:

$$1. \quad C(z = z_b) = C_e(z = z_b) \quad (3)$$

$$2. \quad \left. \frac{\partial C}{\partial z} \right]_{z=z_b} = \left. \frac{\partial C_e}{\partial z} \right]_{z=z_b} \quad (4)$$

in which subscript 'e' is an indication for equilibrium (i.e. steady, uniform) conditions.

*Remarks:*

- i. The first type of bed boundary condition (3) assumes that the concentration near the bed is determined by the instantaneous flow conditions and is thus not influenced by the concentration profile itself. The second type (4) assumes that the pick-up rate of sediment at the bed ( $= -\epsilon_s \partial C / \partial z$ ) is directly determined by the instantaneous flow conditions. The deposition rate ( $= w_s C$ ) then depends on the actual concentration near the bed.
- ii. For simplicity reasons the bed boundary is positioned at the level  $z = z_b$ . It should be noticed that in case  $\epsilon_s$  is a function of  $z$  with  $\epsilon_s(z_b) = 0$ , the bed boundary should be positioned at a small distance  $z_a$  above the bed.

In the next Chapter the depth-integrated model will be derived using an asymptotic method.

In Chapter 3 some important functions and a characteristic adaptation time (for the average concentration) will be derived in an analytical way.

In Chapter 4 some remarks will be made concerning more general cases (non-uniform conditions) and the validity of the model.

More information on this model is given by Galappatti and Vreugdenhil (1985), Wang (1984) and Wang and Ribberink (1986).

2. DERIVATION OF A DEPT-INTEGRATED MODEL FOR SUSPENDED TRANSPORT

Equation (1) can be made dimensionless using the following transformation of coordinates (see Fig. 1):

$$\zeta = (z - z_b)/a \quad (5)$$

$$\tau = t/T \quad (6)$$

in which  $T$  represents a characteristic time scale of the concentration variations.

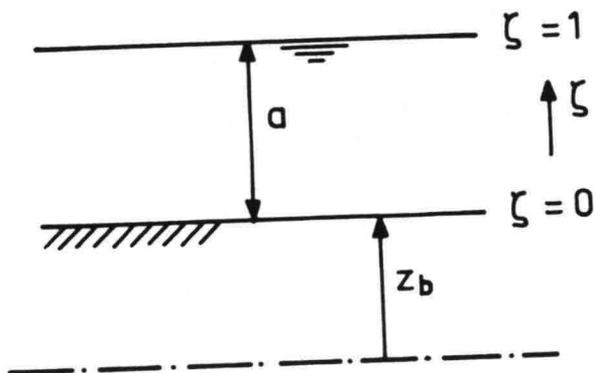


Fig. 1 Definition sketch

Equations (1) ... (4) can now be written as:

$$\frac{a}{w_s T} \frac{\partial C}{\partial \tau} = \frac{\partial C}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C}{\partial \zeta^2} \quad (7)$$

$$\epsilon^1 = \epsilon/w_s a$$

At  $\zeta = 1$  :

$$\left[ C + \epsilon^1 \frac{\partial C}{\partial \zeta} \right]_{\zeta=1} = 0 \quad (8)$$

At the bed ( $\zeta = 0$ ):

$$C(0) = C_e(0) \quad (9)$$

or:

$$\left. \frac{\partial C}{\partial \zeta} \right]_{\zeta=0} = \left. \frac{\partial C_e}{\partial \zeta} \right]_{\zeta=0} \quad (10)$$

It is assumed that the term on the left-hand side of (7) has a much smaller magnitude than both terms on the right-hand side. This includes that:

1. A small parameter  $\delta = a/w_s T \ll 1$  can be indicated.
2. The concentration profile  $C(\zeta)$  should not deviate too much from the equilibrium profile  $C_e(\zeta)$ .

It is now assumed that the solution  $C(\zeta, \tau)$  can be written according to the following asymptotic expansion:

$$C = C_0 + \delta C_1 + \delta^2 C_2 + \dots + \delta^i \cdot C_i + \dots \quad (11)$$

*Remark:* Because  $\delta \ll 1$  every higher-order part of the solution should be much smaller than a lower order part, thus:  $\delta^i C_i \ll \delta^{i-1} C_{i-1}$ .

Substitution of this solution in (7) gives:

$$\delta \frac{\partial C_0}{\partial \tau} + \delta^2 \frac{\partial C_1}{\partial \tau} + \dots + \delta^i \cdot \frac{\partial C_i}{\partial \tau} + \dots =$$

$$\frac{\partial C_0}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C_0}{\partial \zeta^2} + \delta \left[ \frac{\partial C_1}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C_1}{\partial \zeta^2} \right] + \dots + \delta^i \cdot \left[ \frac{\partial C_i}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C_i}{\partial \zeta^2} \right] + \dots \quad (12)$$

A zero-order equation is formed by the largest terms in (12) :

$$0\text{-order:} \quad \frac{\partial C_0}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C_0}{\partial \zeta^2} = 0 \quad (13)$$

First order and higher order equations can be derived from (12) by gathering terms containing  $\delta$ ,  $\dots$ ,  $\delta^i$  etc.:

$$\begin{aligned}
 1^{\text{st}} \text{ order : } & \frac{\partial C_0}{\partial \tau} = \frac{\partial C_1}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C_1}{\partial \zeta^2} \\
 2^{\text{nd}} \text{ order : } & \frac{\partial C_1}{\partial \tau} = \frac{\partial C_2}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C_2}{\partial \zeta^2} \\
 i^{\text{th}} \text{ order : } & \frac{\partial C_{i-1}}{\partial \tau} = \frac{\partial C_i}{\partial \zeta} + \epsilon^1 \frac{\partial^2 C_i}{\partial \zeta^2}
 \end{aligned}
 \tag{14}$$

Equation (13) represents the convection-diffusion equation in equilibrium conditions and therefore  $C_0(\zeta, \tau)$  will have the same shape as  $C_e(\zeta, \tau)$  (i.e.  $C_0/\bar{C}_0 = C_e/\bar{C}_e$ ).

By using the operator  $D [ ] = \frac{\partial}{\partial \zeta} + \epsilon^1 \frac{\partial^2}{\partial \zeta^2}$ , the general  $i^{\text{th}}$  order equation (14) can be written as:

$$\frac{\partial C_{i-1}}{\partial \tau} = D [ C_i ]$$

or with the inverse operator  $D^{-1}$  :

$$C_i = D^{-1} \left[ \frac{\partial C_{i-1}}{\partial \tau} \right]
 \tag{16}$$

An important assumption, as made by Galappatti (1983) is that only the zero-order part of the solution contributes to the dept-averaged concentration  $\bar{C}(\tau)$ , thus:

$$\begin{aligned}
 \bar{C}(\tau) &= \int_0^1 C(\zeta, \tau) d \zeta = \int_0^1 \left\{ C_0 + \delta C_1 + \delta^2 C_2 + \dots \right\} d \zeta = \\
 &= \int_0^1 C_0 d \zeta = \bar{C}_0(\tau)
 \end{aligned}
 \tag{17}$$

By introducing a shape function  $\phi_0(\zeta, \tau) (= C_0(\zeta, \tau)/\bar{C}_0(\tau))$  the zero-order solution can be written as:

$$C_0(\zeta, \tau) = \phi_0(\zeta, \tau) \cdot \bar{C}(\tau)
 \tag{18}$$

Application of (16) yields the first-order part of the solution:

$$\begin{aligned}
 C_1(\zeta, \tau) &= D^{-1} \left[ \frac{\partial C_0}{\partial \tau} \right] \\
 &= D^{-1} \left[ \frac{\partial}{\partial \tau} (\phi_0 \cdot \bar{C}) \right] \\
 &= D^{-1} \left[ \bar{C} \cdot \frac{\partial \phi_0}{\partial \tau} + \phi_0 \frac{\partial \bar{C}}{\partial \tau} \right]
 \end{aligned} \tag{19}$$

By neglecting shape variations ( $\partial \phi_0 / \partial \tau \approx 0$ ) and by realising that the application of operator  $D^{-1}$  to a function of  $\tau$  equals zero,  $C_1(\zeta, \tau)$  can be written as:

$$C_1(\zeta, \tau) = D^{-1} [\phi_0] \cdot \frac{\partial \bar{C}}{\partial \tau} = \phi_1(\zeta, \tau) \cdot \frac{\partial \bar{C}}{\partial \tau} \tag{20}$$

with  $\phi_1(\zeta, \tau)$  is the first-order shape function.

This proces of successive approximations can be repeated for all higher-order parts, leading to:

$$C_i(\zeta, \tau) = \phi_i(\zeta, \tau) \cdot \frac{\partial^i \bar{C}}{\partial \tau^i} \tag{21}$$

with the general  $i$ -th order shape function:

$$\phi_i(\zeta, \tau) = D^{-1} \left[ \phi_{i-1}(\zeta, \tau) \right] \tag{22}$$

The complete asymptotic solution  $C(\zeta, \tau)$  is obtained by substitution of all parts in (11):

$$C(\zeta, \tau) = \phi_0 \cdot \bar{C} + \delta \cdot \phi_1 \cdot \frac{\partial \bar{C}}{\partial \tau} + \dots + \delta^i \cdot \phi_i \cdot \frac{\partial^i \bar{C}}{\partial \tau^i} + \dots \tag{23}$$

*The depth-integrated model*

By applying a boundary condition at the bed, (23) can be transformed into an ordinary differential equation with only the depth-averaged concentration  $\bar{C}$  as unknown. For example, application of (9) gives:

$$\begin{aligned}
 C(o, \tau) = C_e(o, \tau) &= \phi_0(o, \tau) \cdot \bar{C}(\tau) + \delta \cdot \phi_1(o, \tau) \cdot \frac{d\bar{C}}{d\tau} + \dots \\
 &\dots + \delta^i \cdot \phi_i(o, \tau) \cdot \frac{d^i \bar{C}}{d\tau^i} + \dots
 \end{aligned} \tag{24}$$

Because  $C_o(\zeta, \tau)$  and  $C_e(\zeta, \tau)$  have the same (equilibrium) shape  $\phi_o(\zeta, \tau)$ , and thus  $C_e(\zeta, \tau) = \phi_o \cdot \bar{C}_e(\tau)$ , (24) can be written as:

$$\bar{C}_e(\tau) = \bar{C}(\tau) + \delta \frac{\phi_1(o, \tau)}{\phi_o(o, \tau)} \cdot \frac{d\bar{C}}{d\tau} + \delta^2 \frac{\phi_2(o, \tau)}{\phi_o(o, \tau)} \cdot \frac{d^2\bar{C}}{d\tau^2} + \dots \quad (25)$$

By defining the shape factors  $\gamma_i = \phi_i(o, \tau)$  and returning to the original time coordinate  $t$ , (25) can be written as:

$$\bar{C}_e(t) = \bar{C}(t) + \frac{\gamma_1}{\gamma_o} \frac{a}{w_s} \frac{d\bar{C}}{dt} + \frac{\gamma_2}{\gamma_o} \left( \frac{a}{w_s} \right)^2 \frac{d^2\bar{C}}{dt^2} + \dots \quad (26)$$

It is shown in Appendix 1 that in case of the gradient type of bed boundary condition (10), in stead of (26), the following equation results:

$$\bar{C}_e(t) = \bar{C}(t) + \frac{\gamma_1 + 1}{\gamma_o} \frac{a}{w_s} \frac{d\bar{C}}{dt} + \frac{\gamma_2}{\gamma_o} \left( \frac{a}{w_s} \right)^2 \frac{d^2\bar{C}}{dt^2} + \dots \quad (27)$$

The equations (26) and (27) can be solved if the flow conditions (depth  $a$ , discharge  $q$ ), sediment properties ( $w_s$ ), shape factors  $\gamma_i$  and the equilibrium concentration  $\bar{C}_e$  are known. The flow conditions should be obtained with a separate flow model (e.g. long-wave equations). Analytical expressions for  $\bar{C}_e$  and  $\gamma_i$  will be derived in Chapter 3.

### *Morphological computations*

Bed-level changes can be calculated using the complete sediment continuity equation:

$$(1 - \epsilon_o) \frac{\partial z_b}{\partial t} = - \frac{\partial \bar{C} a}{\partial t} - \frac{\partial s_b}{\partial x} - \frac{\partial s_{susp}}{\partial x} \quad (28)$$

- $s_b$  = bed-load transport rate
- $s_{susp}$  = suspended transport rate
- $\epsilon_o$  = porosity of a sand bed

For the schematic case of uniform conditions, as assumed in this note, the  $\partial/\partial x$ -terms are zero, and thus once  $\bar{C}(t)$  is solved from (26) or (27) the bed-level change can be calculated using:

$$\frac{\partial z_b}{\partial t} = - \frac{\partial \bar{C} a}{\partial t} \quad (29)$$

### 3. EQUILIBRIUM CONCENTRATION, THE SHAPE FACTORS AND A CHARACTERISTIC ADAPTATION TIME

#### 3.1 Equilibrium concentration $\bar{C}_e$

The equilibrium value of the depth-averaged concentration  $\bar{C}_e$  should be determined with a transport formula for suspended sediment transport. For example, assume a power law to be valid:

$$s_{\text{susp}} = m \bar{u}^n \quad (30)$$

$\bar{u}$  = depth-averaged flow velocity

The suspended transport rate  $s_{\text{susp}}$  (per unit width, including pores) can also be calculated with:

$$s_{\text{susp}} = \frac{1}{1 - \epsilon_0} \int_{z_b}^{z_b+a} C_e(z) \cdot u(z) dz \quad (31)$$

which, by applying a distribution factor  $\alpha$ , can be written as:

$$s_{\text{susp}} = \frac{\alpha}{1 - \epsilon_0} \bar{C}_e \bar{u} a = \frac{\alpha}{1 - \epsilon_0} \bar{C}_e q \quad (32)$$

Combination of (30) and (32) leads to an expression for  $\bar{C}_e$ :

$$\bar{C}_e = \frac{(1 - \epsilon_0) \cdot m \bar{u}^n}{\alpha q} \quad (33)$$

Herein  $\bar{C}_e$  is completely determined by the local flow conditions and sediment properties (e.g. grain size is included in  $m$ ).

The distribution factor  $\alpha$  is determined by the shape of the equilibrium concentration profile and the shape of the velocity profile (e.g. a logarithmic profile) and generally has a value close to unity.

#### 3.2 The shape functions and shape factors

For the schematic case as treated in this note the shape functions  $\phi_i(\zeta)$  can be determined in an analytical way.

The zero-order shape function  $\phi_0(\zeta)$  (also representing the shape of the equilibrium concentration profile  $C_e(\zeta)$ ) can be obtained by solving (13) through integration:

$$\frac{\partial}{\partial \zeta} \left\{ C_0 + \varepsilon^1 \frac{\partial C_0}{\partial \zeta} \right\} = 0 \quad (34)$$

or: 
$$C_0 + \varepsilon^1 \frac{\partial C_0}{\partial \zeta} = \text{constant} = k \quad (35)$$

Because at the water-surface ( $\zeta = 1$ ) this expression, representing the vertical sediment flux, equals zero,  $k = 0$ . The remaining equation can be solved directly because  $\varepsilon^1 (= \beta u_* / w_s)$  is no function of  $\zeta$ . A general solution is:

$$C_0(\zeta) = k_1 \exp(-\zeta/\varepsilon^1) \quad (36)$$

The average value of  $C_0(\zeta)$  is:

$$\bar{C}_0 = \int_0^1 C_0(\zeta) d\zeta = k_1 \varepsilon^1 (1 - \exp(-1/\varepsilon^1)) \quad (37)$$

By dividing (36) and (37),  $\phi_0(\zeta)$  is obtained:

$$\phi_0(\zeta) = \frac{C_0(\zeta)}{\bar{C}_0} = \frac{\exp(-\zeta/\varepsilon^1)}{\varepsilon^1 (1 - \exp(-1/\varepsilon^1))} \quad (38)$$

The first-order shape function  $\phi_1(\zeta)$  can be derived using (21) :

$$\phi_1(\zeta) = D^{-1} \left[ \phi_0(\zeta) \right] \quad (39)$$

The following mathematical property of the inverse operator  $D^{-1}$  applied to a shape function  $\phi_i(\zeta)$  is proven by Galappatti (see also Appendix 2):

$$D^{-1} \left[ \phi_i(\zeta) \right] = - \int_{\zeta}^1 \phi_i d\zeta + \phi_0 \int_{\zeta}^1 \frac{\phi_i}{\phi_0} d\zeta + B \phi_0 \quad (40)$$

By substituting  $\phi_i = \phi_0$  and (38), the following expression for  $\phi_1(\zeta)$  can be derived after carrying out the integrations in (40):

$$\phi_1(\zeta) = \frac{\exp(-1/\epsilon^1)}{1 - \exp(-1/\epsilon^1)} + \frac{\exp(-\zeta/\epsilon^1)}{\epsilon^1 (1 - \exp(-\zeta/\epsilon^1))} \left\{ \epsilon^1 - \zeta - \frac{2 \exp(-1/\epsilon^1)}{1 - \exp(-1/\epsilon^1)} \right\} \quad (41)$$

*Remark:* The constant B in (40) is determined by making an additional assumption, viz. that each individual higher order part of the solution ( $\delta^i C_i$ ,  $i \geq 1$ ) does not contribute to the depth-averaged concentration, or:

$$\bar{\phi}_i = \int_0^1 \phi_i(\zeta) d\zeta = 0$$

Higher-order shape function  $\phi_2(\zeta)$ , ... are not derived because for practical applications generally only  $\phi_0(\zeta)$  and  $\phi_1(\zeta)$  are used. Figure 2 shows the derived shape functions  $\phi_0(\zeta)$  and  $\phi_1(\zeta)$  for the case that  $w_s/u_* = 0.2$  ( $\beta = 0.067$ ).

How the concentration profile is influenced by net erosion or sedimentation from/to the bed can be explained with Fig. 2 and the first-order form of (23):

$$C(\zeta, t) = \phi_0(\zeta) \bar{C}(t) + \phi_1(\zeta) \frac{a}{w_s} \cdot \frac{d\bar{C}(t)}{dt} \quad (43)$$

In case of net erosion and thus  $d\bar{C}/dt > 0$  (see (29)), the third term will lead to a relatively large concentration near the bed and a relatively small concentration near the water surface (i.e. compared with the equilibrium profile). The opposite occurs in case of net-sedimentation ( $d\bar{C}/dt < 0$ ). These corrections of the equilibrium profile describe the time-dependent redistribution process of sediment in the vertical.

The shape factors  $\gamma_i (= \phi_i(0))$  can be derived from (38) and (41):

$$\gamma_0 = \phi_0(0) = \frac{1}{\epsilon^1 (1 - p)} \quad (44)$$

$$\gamma_1 = \phi_1(0) = \frac{\epsilon^1 (1 - p^2) - 2p}{\epsilon^1 (1 - p)^2} \quad (45)$$

with  $p = \exp(-1/\epsilon^1)$

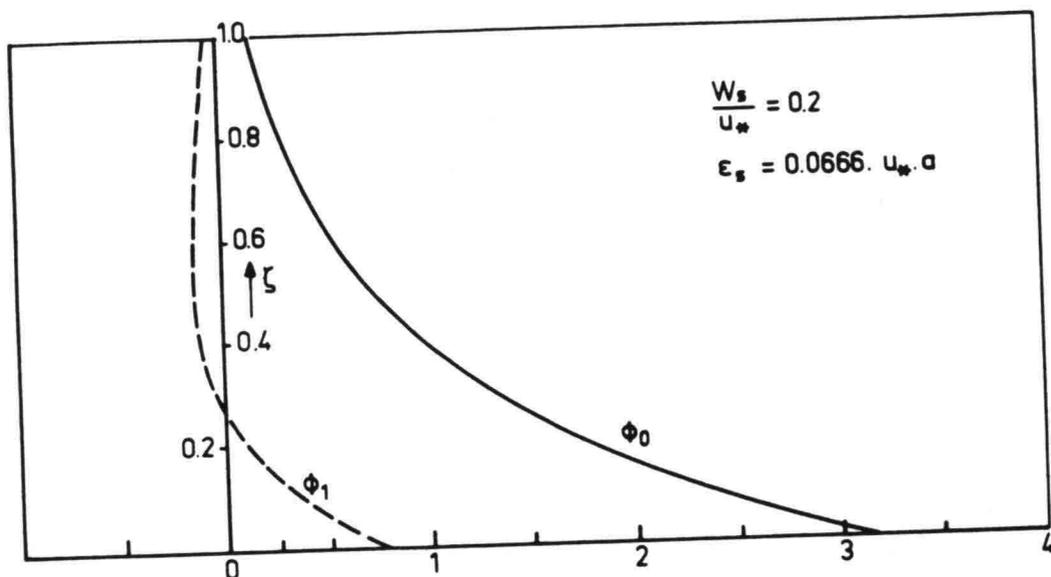


Fig. 2 Shape functions

### 3.3 A characteristic adaptation time

The redistribution process of the concentration in case of net erosion or sedimentation takes a certain time which can be characterized by the so-called adaptation time  $T_A$ . The first-order form of the equation for the depth-averaged concentration  $\bar{C}(t)$  ((26) and (27)) can be written as:

$$\bar{C}_e(t) = \bar{C}(t) + T_A \cdot \frac{d\bar{C}}{dt} \quad (46)$$

in which  $T_A$  represents the adaptation time which equals:

$$T_A = \frac{\gamma_1}{\gamma_0} \cdot \frac{a}{w_s} \quad (\text{imposed concentration at } z = z_b) \quad (47)$$

$$T_A = \frac{\gamma_1 + 1}{\gamma_0} \cdot \frac{a}{w_s} \quad (\text{imposed vertical concentration gradient at } z = z_b) \quad (48)$$

*Remark:* That  $T_A$  indeed represents a characteristic time scale can be seen when (46) is solved analytically in case of constant flow conditions ( $T_A$  and  $\bar{C}_e$  are constant).  
The analytical solution is:

$$\bar{C}(t) = \bar{C}_e - (\bar{C}_e - \bar{C}(0)) \exp(-t/T_A)$$

in which  $\bar{C}(0)$  is the imposed initial value of the concentration.  
The solution is depicted in Fig. 3.

The dependence of  $T_A$  of  $w_s/u_*$  and the type of bed-boundary condition can be determined after substitution of (44) and (45) in (47) and (48) (see also Fig. 4). The gradient type of bed-boundary condition always causes a larger adaptation time. For  $w_s/u_* \rightarrow 0$  the influence of the type of bed boundary condition becomes very pronounced.

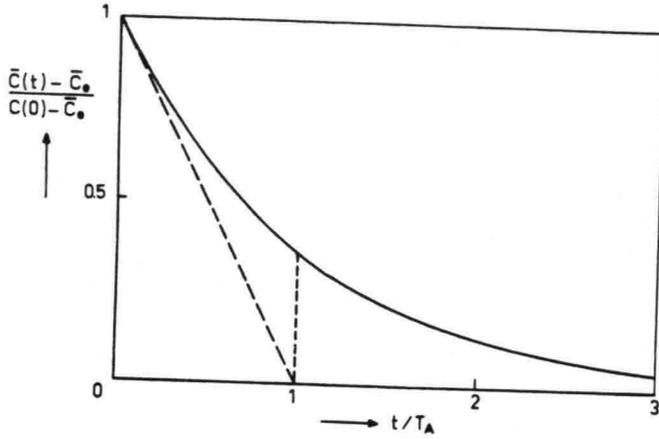


Fig. 3 Adaptation of concentration  $\bar{C}(t)$  to equilibrium conditions (uniform, steady flow).

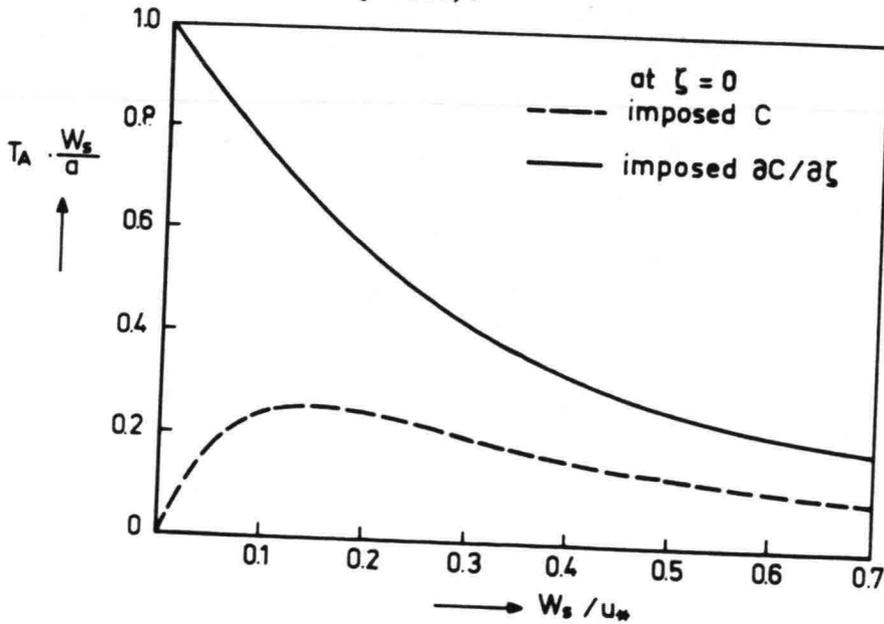


Fig. 4 Characteristic adaptation time as a function of  $w_s/u_*$  and the type of boundary condition near the bed.

4. SOME REMARKS ON GENERAL CASES AND THE VALIDITY OF THE ASYMPTOTIC MODEL

Some remarks will be made with respect to more general cases, i.e. non-uniform flow and more realistic vertical distributions of  $\epsilon_s(z)$  (see also Galappatti, 1983). In general cases the convection-diffusion equation including a horizontal convection term is considered:

$$\frac{\partial C}{\partial t} + u \frac{\partial C}{\partial x} = w_s \frac{\partial C}{\partial z} + \frac{\partial}{\partial z} \left( \epsilon_s \frac{\partial C}{\partial z} \right) \quad (49)$$

Using the same small parameter  $\delta$  for the convection term also, in stead of (23), the following asymptotic solution is obtained (first-order):

$$C(\xi, \zeta, \tau) = \phi_0 \bar{C} + \delta \phi_{21} \frac{\partial \bar{C}}{\partial \tau} + \delta \phi_{22} \frac{\partial \bar{C}}{\partial \xi} \quad (50)$$

with

$$\frac{\partial}{\partial \xi} = \frac{\bar{u}}{a w_s} \cdot \frac{\partial}{\partial x} \quad (51)$$

In stead of 26:

$$\bar{C}_e(x, t) = \bar{C}(x, t) + \underbrace{\frac{\gamma_{21}}{\gamma_0} \frac{a}{w_s} \frac{\partial \bar{C}}{\partial t}}_{T_A} + \underbrace{\frac{\gamma_{22}}{\gamma_0} \frac{\bar{u} a}{w_s} \frac{\partial \bar{C}}{\partial x}}_{L_A} \quad (52)$$

Herein not only an adaptation time  $T_A$  but also an adaptation length  $L_A$  can be distinguished. Because  $L_A \approx \bar{u} T_A$ ,  $T_A$  (e.g. from Fig. 3) can be transformed into  $L_A$ .

In the preceding Chapters  $\epsilon_s$  was assumed to be constant in vertical direction (i.e. the average value of a parabolic distribution). For practical applications it is advised to apply a more realistic distribution, e.g. the parabolic distribution as used by Rouse:

$$\epsilon_s(z) = \kappa u_* \cdot z (1 - z/a) \quad (53)$$

$$\kappa = 0.4$$

or probably an even more realistic parabolic-constant distribution as proposed by DHL (1980).

It should be realised that for this type of distributions  $\epsilon_s(z)$ , the shape functions/factors and thus adaptation time/length cannot be calculated analytically but should be determined numerically.

The influence of the type of distribution on  $T_A$  can be considerable (see Fig. 5 and Fig. 6).

*Remark:* Because of the introduction of the convective term  $u \partial C / \partial x$  and the application of more general  $\epsilon_s(z)$ -distributions, 2 extra parameters, besides  $w_s/u_*$ , will affect the shape functions, i.e.  $\beta (= z_a/a)$  and  $\bar{u}/u_*$  (assumption: logarithmic velocity profile!)

The validity of the asymptotic model is studied by Wang (1984) and Wang and Ribberink (1986). Because of the asymptotic character of the model, large deviations of the concentration profile compared with the equilibrium profile are not allowed. Three requirements are obtained:

1. The factor  $w_s/u_*$  should be much smaller than unity (e.g.  $w_s/u_* < 0.3$ ).
2. The time scale of the flow variations should be much larger than  $a/u_*$ .
3. The length scale of the flow variations should be much larger than  $\bar{u}a/u_*$ .

Further, it should be noticed that in case of flow variations with a time/length scale which largely exceed  $T_A$  respectively  $L_A$ , the asymptotic model is unnecessary complicated and in stead a transportformula can be applied (local, instantaneous equilibrium conditions).

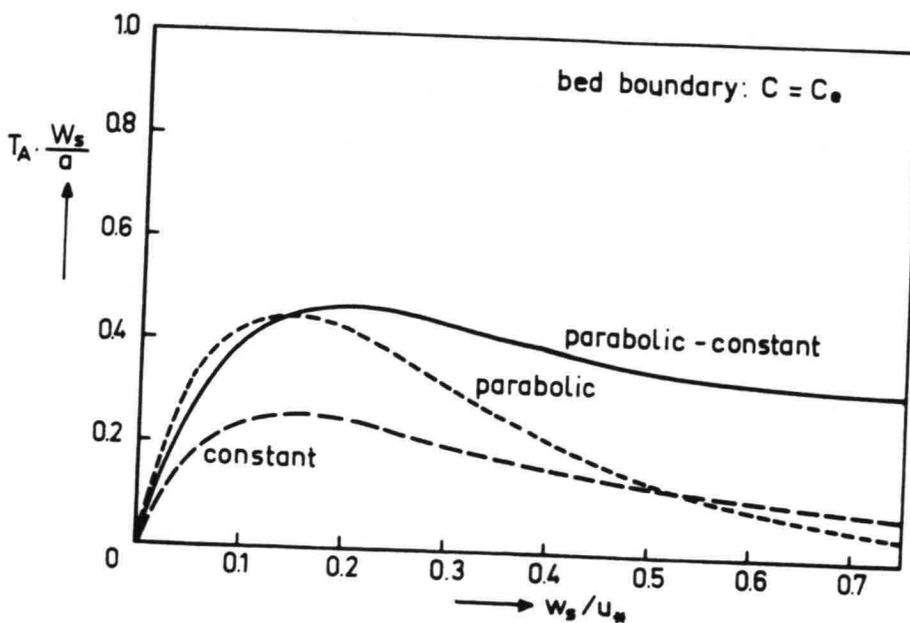


Fig. 5 Influence of the vertical distribution of  $\epsilon_s(z)$  on the adaptation time in case of an imposed concentration at the bed boundary.

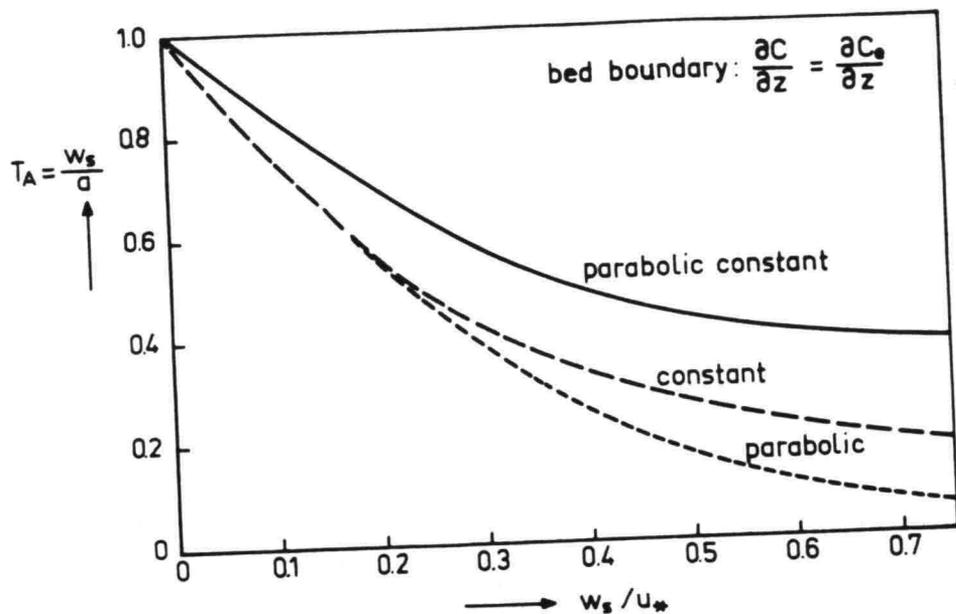


Fig. 6 Influence of the vertical distribution of  $\epsilon_s(z)$  on the adaptation time in case of an imposed concentration gradient at the bed boundary.

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Appendix 1

THE DEPTH-INTEGRATED MODEL IN CASE OF THE GRADIENT TYPE OF BOUNDARY CONDITION NEAR THE BED.

Starting with the asymptotic solution for the concentration profile (23):

$$C(\zeta, \tau) = \phi_0 \bar{C} + \delta \phi_1 \frac{\partial \bar{C}}{\partial \tau} + \delta^2 \phi_2 \frac{\partial^2 \bar{C}}{\partial \tau^2} + \dots \quad (1.1)$$

the following boundary condition is applied near the bed ( $\zeta = 0$ ):

$$\epsilon^1 \left[ \frac{\partial C}{\partial \zeta} \right]_{\zeta=0} = \epsilon^1 \left[ \frac{\partial C_e}{\partial \zeta} \right]_{\zeta=0} = -C_e(0) \quad (1.2)$$

Application of (1.2) to (1.1) gives:

$$\epsilon^1 \left[ \frac{\partial C}{\partial \zeta} \right]_{\zeta=0} = -C_e(0) = \epsilon^1 \left[ \frac{\partial \phi_0}{\partial \zeta} \right]_{\zeta=0} \bar{C} + \epsilon^1 \left[ \frac{\partial \phi_1}{\partial \zeta} \right]_{\zeta=0} \delta \frac{\partial \bar{C}}{\partial \tau} + \epsilon^1 \left[ \frac{\partial \phi_2}{\partial \zeta} \right]_{\zeta=0} \delta^2 \frac{\partial^2 \bar{C}}{\partial \tau^2} + \dots \quad (1.3)$$

Because  $\phi_0(\zeta)$  describes the shape of the equilibrium profile:

$$\epsilon^1 \left[ \frac{\partial \phi_0}{\partial \zeta} \right]_{\zeta=0} = -\phi_0(0) = -\gamma_0 \quad (1.4)$$

The other derivatives of  $\phi_i$  ( $\partial \phi_i / \partial \zeta$ ) can be eliminated using the following mathematical property (see Appendix 2):

$$\frac{\partial \phi_i}{\partial \zeta} = -\frac{1}{\epsilon^i(\zeta)} \left( \phi_i(\zeta) + \int_{\zeta}^1 \phi_{i-1} d\zeta \right) \quad (1.5)$$

With this rule:

$$\begin{aligned} \left[ \frac{\partial \phi_1}{\partial \zeta} \right]_{\zeta=0} &= -\frac{1}{\epsilon^1} \left( \phi_1(0) + \int_0^1 \phi_0 d\zeta \right) \\ &= -\frac{1}{\epsilon^1} (\gamma_1 + 1) \end{aligned} \quad (1.6)$$

$$\left. \frac{\partial \phi_2}{\partial \zeta} \right]_{\zeta=0} = -\frac{1}{\epsilon^1} \left( \phi_2(0) + \int_0^1 \phi_1 d\zeta \right)$$

$$= -\frac{1}{\epsilon^1} (\gamma_2)$$
(1.7)

Substitution of (1.4), (1.6) and (1.7) in (1.3) yields:

$$-c_e(0) = -\gamma_0 \bar{c} - \delta (\gamma_1 + 1) \cdot \frac{\partial \bar{c}}{\partial \tau} - \delta^2 \gamma_2 \frac{\partial^2 \bar{c}}{\partial \tau^2} + \dots$$
(1.8)

Because  $c_e(0) = \gamma_0 \cdot \bar{c}_e$  (1.8) can be written as:

$$\bar{c}_e = \bar{c} + \delta \frac{\gamma_1 + 1}{\gamma_0} \cdot \frac{\partial \bar{c}}{\partial \tau} + \delta^2 \frac{\gamma_2}{\gamma_0} \cdot \frac{\partial^2 \bar{c}}{\partial \tau^2} + \dots$$
(1.9)

or with the original coordinates:

$$\bar{c}_e = \bar{c} + \frac{\gamma_1 + 1}{\gamma_0} \cdot \frac{a}{w_s} \cdot \frac{\partial \bar{c}}{\partial t} + \frac{\gamma_2}{\gamma_0} \cdot \left( \frac{a}{w_s} \right)^2 \frac{\partial^2 \bar{c}}{\partial t^2} + \dots$$
(1.10)

Appendix 2

THE  $D^{-1} [ ]$  OPERATOR (After Wang (1984)).

The operator  $D^{-1}$  is defined as:

$$D^{-1} [ g(\zeta) ] = f(\zeta) \quad (2.1)$$

if

$$D [ f(\zeta) ] = \frac{\partial f}{\partial \zeta} + \frac{\partial}{\partial \zeta} \left( \epsilon' \frac{\partial f}{\partial \zeta} \right) = g(\zeta) \quad (2.2)$$

$$\left[ f + \epsilon' \frac{\partial f}{\partial \zeta} \right]_{\zeta=1} = 0 \quad (2.3)$$

and

$$\int_0^1 f(\zeta) d\zeta = 0 \quad (2.4)$$

The expression  $f(\zeta)$  found by Galappatti (1983) is:

$$f(\zeta) = \phi_0 \int_{\zeta}^1 \frac{g}{\phi_0} d\zeta - \int_{\zeta}^1 g(\zeta) d\zeta + B \phi_0 \quad (2.5)$$

where  $B$  is a constant satisfying (2.4).

Important qualities of this operator are:

(i) The operator is a linear operator, this means:

$$D^{-1} [ \alpha g + \beta h ] = \alpha D^{-1} [g] + \beta D^{-1} [h] \quad (2.6)$$

if  $\alpha$  and  $\beta$  are constants.

(ii) The derivative with respect to  $\zeta$  can be expressed as:

$$\frac{\partial D^{-1} [ g(\zeta) ]}{\partial \zeta} = - \frac{1}{\epsilon^1(\zeta)} \left( D^{-1} [g] + \int_{\zeta}^1 g d\zeta \right) \quad (2.7)$$

This can be shown as follows:

Differentiation of (2.5) gives:

$$\begin{aligned}\frac{\partial f(\zeta)}{\partial \zeta} &= -\phi_0 \frac{g}{\phi_0} + \frac{\partial \phi_0}{\partial \zeta} \int_{\zeta}^1 \frac{g}{\phi_0} d\zeta + g(\zeta) + B \frac{\partial \phi_0}{\partial \zeta} \\ &= \frac{\partial \phi_0}{\partial \zeta} \left[ \int_{\zeta}^1 \frac{g}{\phi_0} d\zeta + B \right]\end{aligned}$$

From (35) it follows that:

$$\frac{\partial \phi_0}{\partial \zeta} = -\frac{\phi_0(\zeta)}{\epsilon^1(\zeta)}$$

thus

$$\begin{aligned}\frac{\partial f}{\partial \zeta} &= -\frac{1}{\epsilon^1(\zeta)} \phi_0(\zeta) \left( \int_{\zeta}^1 \frac{g}{\phi_0} d\zeta + B \phi_0 \right) \\ &= -\frac{1}{\epsilon^1(\zeta)} \left( f + \int_{\zeta}^1 g d\zeta \right)\end{aligned}$$