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The Elements of Deformation Analysis

Blending Geodetic Observations and Deformation Hypotheses

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PROEFSCHRIFT

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Preface

This study is the result of the PhD-research, which I started in 2011 at Delft University of Technology in the group of Prof. Hanssen. The incentive for this research was ten years of teaching geodesy at the HU University of Applied Sciences Utrecht (Hogeschool Utrecht). Several lecture notes had been written during this period (1991-2001), and software had been developed for educational purposes. In the course of these years I supervised many students in their thesis writing, which always took place in professional practice in the Netherlands.

These activities made it clear to me that the theoretical possibilities of the similarity transformation (1D, 2D and 3D) were not used to their full extent in professional practice in the Netherlands. The similarity transformation was, of course, well known to transform coordinates. But it was unusual to solve a transformation problem by considering it as an adjustment, where the coordinates are considered as "observations". As a consequence, the application to the similarity transformation of the achievements of the Delft School of Mathematical Geodesy (van Daalen, 1985) was poorly investigated. It seemed promising to me to test coordinate differences after transformation with statistical tests: one-dimensional w-tests and multidimensional point tests and even subfield tests. The concept of "minimal detectable biases" seemed useful to describe the minimal values of significant differences; significant in the sense that they are probably not caused by random noise.

The study of the application of these concepts to deformation monitoring was to be a focal point of my PhD-research. The concepts offer the possibility to improve communication about the results of deformation monitoring. The research started with interviews with professional practice. It taught me that principals, contractors, politicians and the general public want clear answers: "is something moving or not, and if it is moving, how will it move in future?". How can this be handled in an acceptable way, taking into account statistical considerations? The focus of the research subject was, therefore, put on the *communication* about deformation monitoring. The idea was that communication should not be hampered by statistical jargon, and yet a sound *mathematical analysis* and sound *statistical testing* should be the basis of good communication. A model based on the application of Delft School methods to the similarity transformation seemed a promising basis for clear communication.

Besides the mathematical and statistical deformation analysis, and communication, three other aspects are important in geodetic monitoring. To judge the reach and influence of statements about geodetic monitoring, it is important to know how geodetic

monitoring is organised in society. Put differently: how is the *governance* of geodetic monitoring. Moreover, it is important to notice that a statement may be scientifically justified (it is *credible*), but not accepted by non-specialists, i.e. by politicians or the general public (it is not *acceptable*) (van der Molen, 1999). Finally, to make statements about deformations comparable, *standardisation* of the models is important. Five models had, therefore, to be built: a geodetic deformation analysis model (with a mathematical and statistical basis), a model on governance, a model on acceptability and credibility, a model on standardisation, and finally, as the overarching model, a model on communication.

On three of the five models extensive research has been done: the model on governance, the model on standardisation and the geodetic deformation analysis model. The models on credibility/acceptability and communication have only slightly been elaborated upon. It turned out that the *geodetic deformation analysis model* needed much more attention and research than anticipated. So the focus of the research shifted to this model, which is the subject of this study.

The *standardisation model* has been worked out in a separate report (Velsink, 2016a, in Dutch). It is not part of this study, but it was crucial to formulate the requirements of the geodetic deformation analysis model. It was the result of the research project DefoGuide, which started in 2014, and had as subject the standardisation of geodetic deformation analysis. It was a project, funded by the research programme Maps4Society, with participation of several partners from professional practice in the Netherlands: government agency Rijkswaterstaat and engineering firms Fugro Geoservices, Grontmij, Antea Group, and Geomaat. During the project more partners joined the project group: gas and oil exploration and production company NAM, and engineering firms Brem Funderingsexpertise, and RPS Nederland. The discussions within the project group were very fruitful for linking experiences from professional practice and theoretical models. They led to two scientific publications, which are included as chapters in this thesis.

The research on the *governance of geodetic monitoring* led to a scientific publication in the proceedings of Geomonitoring 2012 (Velsink, 2012). It describes a set-up for a taxonomy of the governance of geodetic deformation analysis. It will serve as a basis for interviews to be held with professional practice, with the purpose to yield a governance taxonomy. The treatment of this research is not part of this study, but it was an important source for the formulation of the requirements for the geodetic deformation analysis model of this study.

Prof. Hanssen of Delft University of Technology, Prof. Niemeier of Technical University Braunschweig and Prof. Versendaal of HU University of Applied Sciences Utrecht were constant advisors during my research. I like to thank them for their guidance and help. Their support and positive feedback helped me to keep going in my research.

Many evenings and weekends I wandered in the realms of geodesy, its history and its outskirts, its neighbouring fields of science, and other sciences and humanities, to grasp the intricacies and possibilities of geodetic deformation analysis. I thank my wife for supporting me and letting me wander.

Hiddo Velsink, September 2018

Contents

Summary

The subject of this study is deformation analysis of the earth's surface (or part of it) and spatial objects on, above or below it. Such analyses are needed in many domains of society. Geodetic deformation analysis uses various types of geodetic measurements to substantiate statements about changes in geometric positions.

Professional practice, e.g. in the Netherlands, regularly applies methods for geodetic deformation analysis that have shortcomings, e.g. because the methods apply substandard analysis models or defective testing methods. These shortcomings hamper communication about the results of deformation analyses with the various parties involved. To improve communication solid analysis models and a common language have to be used, which requires standardisation.

Operational demands for geodetic deformation analysis are the reason to formulate in this study seven characteristic elements that a solid analysis model needs to possess. Such a model can handle *time series* of several epochs. It analyses only *size and form*, not position and orientation of the reference system; and datum points may be under influence of deformation. The *geodetic* and *physical* models are combined in one adjustment model. Full use is made of available *stochastic information*. Statistical testing and computation of *minimal detectable deformations* is incorporated. Solution methods can handle *rank deficient* matrices (both model matrix and cofactor matrix). And, finally, a search for the *best hypothesis/model* is implemented. Because a geodetic deformation analysis model with all seven elements does not exist, this study develops such a model.

For effective standardisation geodetic deformation analysis models need: practical *key performance indicators*; a *clear procedure* for using the model; and the possibility to *graphically visualise* the estimated deformations.

This study shows that *key performance indicators* can be derived from the method of hypothesis formulation and testing, and from rejection criteria. They can also stem from the description of the test quality by means of minimal detectable deformations. A *clear procedure* is possible, if an unambiguous way is provided to distinguish the observation noise, the deformation signal with zero mean in time, and the deformation trend from each other. The *graphical visualisation*, finally, demands clearly defined quantities that are sensitive only to the deformations of the object at hand and not to changes in, e.g., the reference system.

In this study I propose a geodetic deformation analysis model, which is built around a least-squares adjustment model. *Two* adjustment models are developed in this study: one model uses geodetic measurements in the observation vector. In the other model this vector holds pre-computed coordinates, which follow from separate adjustments per epoch. The parameter vector holds, for both models, the final coordinates. Both models yield the same adjustment results. The choice, which one to use, depends on the professional context in which the model is used.

The developed geodetic deformation analysis model is shown to be effective in several use cases. These use cases are geodetic networks in 1D, 2D and 3D that have been measured in several epochs, and which are analysed with one of the two adjustment models, mentioned above.

Moreover, the proposed analysis model not only possesses the seven necessary elements, mentioned before, it also has some additional advantageous characteristics. First, it is possible to define the S-basis of the geodetic network, used for deformation analysis, with points that are under influence of deformation. Secondly, there is no need for a separate analysis of reference and object points; they are analysed simultaneously. Thirdly, the deformation estimates of moving points are relative to all the other points of the same network (moving or not), not relative to an S-basis. These estimates are invariant for a change of S-basis, i.e. for an S-transformation. Finally, biases in geodetic measurements and deformation hypotheses can be tested simultaneously.

The availability of key performance indicators, based on the analysis model and its characteristic elements as described in this study, and the definition of a statistically significant deformation, provided in this study, make a standardised procedure for geodetic deformation analysis possible. Thus a tool is available for the improvement of communication about geodetic deformation analysis.

Resumo (in Esperanto)

La temo de ĉi tiu studo estas deformadanalizo de la surfaco de la tero (aŭ parto de ĝi) kaj spacaj objektoj sur, super aŭ sub ĝi. Tiaj analizoj estas bezonataj en multaj partoj de la socio. La geodezia deformadanalizo uzas diversajn specojn de geodeziaj mezuroj por apogi deklarojn pri ŝanĝoj en geometriaj pozicioj.

La profesia praktiko, ekzemple en Nederlando, regule aplikas metodojn por geodezia deformadanalizo, kiuj havas mankojn, ekzemple ĉar la metodoj aplikas subnivelajn analizmodelojn aŭ mankhavajn testmetodojn. Ĉi tiuj mankoj malhelpas komunikadon kun la diversaj partioj pri la rezultoj de deformadanalizoj. Por plibonigi komunikadon, oni devas uzi solidajn analizmodelojn kaj komunan lingvon, kio postulas normigon.

Praktikaj postuloj por geodeziaj deformadanalizoj estas la kialo por formuli en ĉi tiu studo sep karakterizajn elementojn, kiujn devas posedi solida analizmodelo. Tia modelo povu manipuli *temposeriojn* de pluraj mezurepokoj. Ĝi analizu nur *grandon kaj formon*, ne la pozicion kaj orientiĝon de la referenca sistemo; kaj datumaj punktoj povu esti sub influo de deformado. La modeloj *geodeziaj* kaj *fizikaj* kombiniĝu en unu kompensadmodelon. La kompletaj haveblaj *stokastaj informoj* estu uzataj. Statistikan testadon kaj komputadon de *minimumaj detekteblaj deformoj* la modelo enhavu. Solvometodoj povu manipuli *range mankajn* matricojn (kaj modelmatricon kaj matricon de kofaktoroj). Kaj, fine, la serĉado de la *plej bona hipotezo/modelo* estu realigita. Ĉar ne ekzistas analizmodelo de geodezia deformado kun ĉiuj sep elementoj, ĉi tiu studo disvolvas tian modelon.

Por efektiva normigo geodeziaj deformadanalizaj modeloj bezonas: praktikajn *ŝlosilajn plenumindikilojn*; *klaran proceduron* por uzi la modelon; kaj la eblecon *grafike bildigi* la stimitajn deformojn.

Ĉi tiu studo montras ke *ŝlosilaj plenumindikiloj* povas esti derivitaj de la metodo de hipotezo-formulado kaj testado, kaj de la kriterioj por malakcepto. Ili ankaŭ povas fonti el la priskribo de la testkvalito per minimumaj detekteblaj deformoj. *Klara proceduro* eblas, se unusenca maniero estas donita por distingi inter si la observan bruon, la deformajn signalojn kun nula mezo kaj la deforman tendencon. La *grafika bildigo* fine postulas klare difinitajn kvantojn, kiuj estas sentivaj nur al la deformoj de la objekto kaj ne al ŝanĝoj de ekzemple la referenca sistemo.

En ĉi tiu studo mi proponas geodezian deformadanalizmodelon, kiu estas konstruita ĉirkaŭ kompensadmodelo laŭ la metodo de minimumaj kvadratoj. *Du* kompensadmodelojn disvolvas ĉi tiu studo: unu modelo uzas geodeziajn mezurojn en la observa vektoro.

En la alia modelo ĉi tiu vektoro entenas pli frue kalkulitajn koordinatojn, kiuj fontas el apartaj kompensadoj por ĉiu mezurepoko. La parametra vektoro entenas, por ambaŭ modeloj, la finajn koordinatojn. Ambaŭ modeloj donas la samajn kompensadrezultojn. La elekto, kiun oni devas uzi, dependas de la profesia kunteksto, en kiu la modelo estas uzata.

La disvolvita geodezia deformadanalizmodelo montriĝas esti efika en pluraj uzkazoj. Ĉi tiuj uzkazoj estas geodeziaj retoj en 1D, 2D kaj 3D, kiuj estas mezuritaj en pluraj mezurepokoj, kaj kiuj analiziĝas per unu el la du kompensadmodeloj, menciitaj antaŭe.

Plie, la proponita analizmodelo ne nur posedas la sep necesajn elementojn, menciitajn antaŭe, ĝi ankaŭ havas kelkajn pliajn avantaĝajn karakterizaĵojn. Unue, ĝi povas difini la S-bazon de la geodezia reto, uzata por la deformadanalizo, pere de punktoj kiuj estas sub influo de deformado. Due, ne necesas aparta analizo de referencaj kaj objektaj punktoj; ili analiziĝas samtempe. Trie, la stimado de deformoj de moviĝantaj punktoj estas relativa al ĉiuj aliaj punktoj (moviĝantaj aŭ ne) de la sama reto, ne relativa al S-bazo. Ĉi tiuj stimoj estas invariantoj al ŝanĝo de la S-bazo, tio estas: al S-transformacio. Fine, eraroj en geodeziaj mezuroj kaj deformadhipotezoj povas esti testataj samtempe.

La disponebleco de ŝlosilaj plenumindikiloj, bazitaj sur la analizmodelo kaj ĝiaj karakterizaj elementoj kiel priskribitaj en ĉi tiu studo, kaj la difino de statistike signifa deformo, kiel donita en ĉi tiu studo, povas ebligi normigan proceduron por geodezia deformadanalizo. Tiel ilo estas havebla por plibonigo de komunikado pri geodezia deformadanalizo.

Samenvatting (in Dutch)

Het onderwerp van deze studie is deformatieanalyse van het aardoppervlak (of een deel ervan) en van ruimtelijke objecten erop, erboven of eronder. Dergelijke analyses zijn nodig in veel domeinen van de samenleving. Geodetische deformatieanalyse gebruikt verschillende soorten geodetische metingen om uitspraken over veranderingen in geometrische posities te onderbouwen.

De professionele praktijk, bijvoorbeeld in Nederland, past regelmatig methoden toe voor geodetische deformatieanalyse die tekortkomingen hebben, bijvoorbeeld omdat de methoden inferieure analysemodellen of gebrekkige testmethoden toepassen. Deze tekortkomingen bemoeilijken de communicatie over de resultaten van deformatieanalyses met de verschillende betrokken partijen. Om de communicatie te verbeteren, moeten solide analysemodellen en een gemeenschappelijke taal worden gebruikt, waarvoor standaardisatie vereist is.

De operationele eisen voor geodetische deformatieanalyse zijn de reden om in dit onderzoek zeven karakteristieke elementen te formuleren die een solide analysemodel moet bezitten. Zo'n model kan overweg met *tijdreeksen* van verschillende epochen. Het analyseert alleen *grootte en vorm*, niet de positie en oriëntatie van het referentiesysteem; en basispunten mogen aan deformatie onderhevig zijn. De *geodetische* en *fysische* modellen worden gecombineerd in één vereffeningsmodel. Er wordt gebruik gemaakt van de volledige beschikbare *stochastische informatie*. Statistische testen en de berekening van *minimaal detecteerbare deformaties* zijn onderdeel van het model. Oplossingsmethoden kunnen overweg met *rangdefecte* matrices (zowel modelmatrix als cofactorenmatrix). Ten slotte is een zoekmethode voor de *beste hypothese* (het *beste model*) geïmplementeerd. Omdat een geodetisch deformatieanalysemodel met alle zeven elementen niet bestaat, ontwikkelt deze studie een dergelijk model.

Voor effectieve standaardisatie hebben geodetische deformatieanalysemodellen nodig: praktische *key performance indicators*; een *duidelijke procedure* voor het gebruik van het model; en de mogelijkheid om de geschatte deformaties *grafisch te visualiseren*.

Deze studie laat zien dat *key performance indicators* kunnen worden afgeleid uit de methode voor hypotheseformulering en het testen daarvan, en van de criteria voor verwerping. Ze kunnen ook voortkomen uit de beschrijving van de testkwaliteit door middel van minimaal detecteerbare deformaties. Een *duidelijke procedure* is mogelijk als een eenduidige manier wordt geboden om de waarnemingsruis, het deformatiesignaal met een gemiddelde van nul en de deformatietrend van elkaar te onderscheiden. De

grafische visualisatie, ten slotte, vraagt om duidelijk gedefinieerde grootheden die alleen gevoelig zijn voor de deformaties van het object in kwestie en niet voor veranderingen in bijvoorbeeld het referentiesysteem.

In deze studie stel ik een geodetisch deformatieanalysemodel voor, dat is opgebouwd rond een kleinste-kwadratenvereffeningsmodel. *Twee* vereffeningsmodellen worden ontwikkeld in deze studie: het ene model gebruikt geodetische metingen in de waarnemingsvector. In het andere model bevat deze vector eerder berekende coördinaten, die volgen uit afzonderlijke vereffeningen per epoche. De parametervector bevat, voor beide modellen, de uiteindelijke coördinaten. Beide modellen leveren dezelfde vereffeningsresultaten op. Welke keuze men moet maken, hangt af van de professionele context waarin het model wordt gebruikt.

Het ontwikkelde geodetische deformatieanalysemodel blijkt effectief te zijn in verschillende gebruikssituaties. Deze gebruikssituaties zijn geodetische netwerken in 1D, 2D en 3D die in verschillende epochen zijn gemeten en die worden geanalyseerd met een van de twee hierboven genoemde vereffeningsmodellen.

Bovendien bezit het voorgestelde analysemodel niet alleen de zeven noodzakelijke elementen die eerder zijn genoemd, maar heeft het ook enkele aanvullende voordelige eigenschappen. Ten eerste is het mogelijk om de S-basis van het geodetische netwerk, dat wordt gebruikt voor deformatieanalyse, te definiëren met punten die aan deformatie onderhevig zijn. Ten tweede is er geen noodzaak voor een afzonderlijke analyse van referentie- en objectpunten; ze worden tegelijk geanalyseerd. Ten derde zijn deformatieschattingen van bewegende punten relatief ten opzichte van alle andere punten van hetzelfde netwerk (al dan niet bewegend), niet ten opzichte van een S-basis. Deze schattingen zijn invariant voor een verandering van S-basis, dat wil zeggen voor een S-transformatie. Ten slotte kunnen fouten in geodetische metingen en deformatiehypothesen gelijktijdig worden getest.

De beschikbaarheid van key performance indicators, gebaseerd op het analysemodel en de karakteristieke elementen zoals beschreven in deze studie, en de definitie van een statistisch significante deformatie, die in deze studie is beschreven, maken een gestandaardiseerde procedure voor geodetische deformatieanalyse mogelijk. Er is aldus een hulpmiddel beschikbaar voor het verbeteren van communicatie over geodetische deformatieanalyse.

1

Introduction

1.1 Motivation

Theoretical methodologies for geodetic deformation analysis are well developed, but are less well suited to be implemented for operational purposes. Here we propose a new geodetic deformation analysis model (as theoretical methodology) that is expected to answer operational demands appropriately.

1.2 Background

The subject of this study is geodetic deformation analysis of geo-objects. We use the term "geo-objects" to describe both the earth's surface (or part of it), and spatial objects on, above or below it. The geo-objects can be man-made structures or natural objects. Geodetic deformation analysis is about movements and deformations of geo-objects. They can be human-induced, e.g. because of construction engineering works, hydrological activities, ageing assets, and mining, or caused by natural forces, such as landslides and sinkholes. Geodetic deformation analysis is used to address a wide variety of issues in society, and is important for e.g. risk assessment and asset management (Caspary, 2000).

The analysis is based on monitoring points that represent the geo-objects. A wide variety of *techniques* is used to get relevant observations, e.g. spirit levelling, tacheometry, GPS, laser scanning (terrestrial, aerial, or from satellites), and InSAR (Interferometric Synthetic Aperture Radar). Traditionally a multitude of points is measured and analysed as a *geodetic network*. Modern developments, however, allow to measure large amounts of points, which are clustered in *point clouds*. Repeated measurements of point clouds do not necessarily give measurements of exactly the same points (e.g.: laser scanning, scanning tacheometers), or points that are unambiguously identifiable on the ground

(e.g.: InSAR (Interferometric Synthetic Aperture Radar)). These modern developments make the analysis as a geodetic network less obvious, but not impossible, if techniques are used to extract identifiable or repeatibly measurable points (real or virtual) from the point cloud.

The purpose of monitoring is the identification of changes in form and size, including relative position changes, of the point cloud or geodetic network. Ideally, the movements and deformations are analysed by the joint *adjustment* of the observations of the complete time sequence of the monitoring (in one step or sequentially), and the subsequent statistical *testing* of deformation hypotheses. The analysis method may be different depending on the monitoring being continuous or in discrete epochs. An epoch is a time period, in which a set of points is measured once. It is assumed that no movements or deformations occur during this time period, or that they are compensated for. For certain measuring techniques an epoch can be so short that it is considered a moment in time.

Thus, geodetic deformation analysis is determined by the *purpose* of analysis, the *representation of geo-objects*, which are the subject of analysis, the *measuring techniques* to get observations as input for the analysis, and finally the analysis itself by *adjustment and testing*. In the following sections we pay attention to the role of geodetic deformation analysis in society, and we dwell in more detail on the techniques used for monitoring, and the existing approaches for the analysis of monitoring results.

1.2.1 Geodetic deformation analysis in society

Geodetic deformation analysis has many applications in society. Interviews with several stakeholders in the Netherlands have made it clear that at least six domains of human activity, in which geodetic deformation analysis is of interest, can be distinguished (appendix A.3). Activities concerning *residential and non-residential buildings* define a domain, where primarily the private sector is active. Here risk assurance against unexpected movements and deformations of geo-objects is common practice. This makes insurance companies important stakeholders in this domain. In the domains of *infrastructure works* and *hydraulic engineering*, the public sector plays a much greater part. The standard guidelines, used in their invitations to tender for geodetic deformation analysis assignments, determine to a great extent the analysis methods used. In the *mining industry* (in the Netherlands: gas, oil, salt and coal) risks for society of movements and deformations are evident. Ample legal regulations have, therefore, been adopted, and have a large influence on the used analysis methods. In the domain of *industrial installations* regular monitoring for movements and deformations is required. The installations are mostly owned by the private sector, where interest groups and large companies have adopted regulations, e.g. for the monitoring of oil containers. The last domain to be mentioned here, is the domain of analysis of movements and deformations caused by *geophysical processes*. As examples can serve landslides, sinkholes, upheaval or subsidence of large regions or countries, and continental drift. This is almost exclusively the domain of public organisations, such as universities.

In each of the mentioned domains different legislation and regulations may apply. Different certified standards may be used, and different guidelines from industry or major

commissioning companies and public organisations may be present. They all have a large impact on the methods, quality and complexity of geodetic deformation monitoring and analysis. Some remarks can be made on the legislation, standards and guidelines in the Netherlands. The remarks concern especially the domain of the mining industry and the domain of infrastructure, where extensive geodetic guidelines for deformation measurements are available. The other domains have their own guidelines as well, but geodetic monitoring is less extensively treated.

In the Netherlands the mining law gives regulations regarding deformation measurements. The mining company has the obligation to do observations to monitor the consequences of mining activities. A government agency "State Supervision of Mines" supervises the measurements and analyses. Since 2014 there are guidelines to do the measurements (T.P.B., 2014).

In the domain of infrastructure the guidelines of the executive arm of the Dutch Ministry of Infrastructure and the Environment are used (Rijkswaterstaat, 2014). They are used not only by Rijkswaterstaat, but also by many other commissioning companies, also outside the domain of infrastructure.

The mathematical-geodetic models used in the mentioned guidelines of Rijkswaterstaat and T.P.B. have shortcomings. The guidelines of T.P.B. consider only two analysis models: "free network adjustment" and "space-time-analysis" (T.P.B., 2014, p. 23). In the first analysis the measurements of each epoch are adjusted as a free network, and the results are compared. For the comparison only one method (point-wise multi-epoch analysis) is described (T.P.B., 2014, p. 60), where many exist, cf. Holdahl (1975); Tengen (2010); Heunecke et al. (2013). For the second analysis method only reference is made to research from the Netherlands (Kenselaar and Quadvlieg, 2001; Houtenbos, 2004), where more research results are available (cf. Tengen, 2010; Heunecke et al., 2013).

The guidelines of Rijkswaterstaat give detailed instructions on the analysis of individual epochs of deformation measurements. For the combined analysis of all epochs, however, it merely states: "Information is provided to the client on the nature and extent of the deformation and the deformation process." (Rijkswaterstaat, 2014, pp. 38 & 48). No indication is provided on the analysis model and methods to be used, or on the requirements, the analysis has to fulfil.

Nowadays, positioning equipment, such as GPS or tacheometry, is widely available. This stimulates nonspecialised companies to get involved as contractors in geodetic deformation analysis. At the same time, for many commissioning companies and government institutions geodetic deformation analysis is a side issue, for which no specialised officer is available. Therefore, they lack know-how on geodetic deformation analysis, and they are unfamiliar with the mentioned guidelines.

The unfamiliarity with the guidelines and their shortcomings lead to many deformation analyses that are not statistically underpinned and do not fulfil basic geodetic requirements of precision and reliability (see the examples in appendix A.4). The result is that statements about deformations (or their absence) are formulated in ambiguous terms, often only accompanied by lists of coordinate or height differences (see appendix A.4).

Communication about deformations (or their absence) is hampered by such inadequate analysis results. This leads to calls from the surveying industry and their clients for a clear, unambiguous terminology, and the availability of statistically underpinned numerical indicators, preferably graphically visualised. Such a call was ushered in the Netherlands in 2014, and resulted in the wish to realise *standardisation* of geodetic deformation analysis. This, in its turn, led to a research project to build a standardisation model (appendix A.1). To substantiate such a standardisation, an underlying geodetic deformation analysis model is needed, and is provided in this study.

The above considerations suggest that legislation, standards and guidelines have a substantial influence on the extent and quality of geodetic deformation analysis. A closer study of the *governance* of geodetic deformation analysis in society is, therefore, important to get a good insight in this influence.

1.2.2 Monitoring techniques

Several geodetic monitoring techniques are available, and new ones are regularly introduced. Here we mention spirit levelling, tacheometry, GPS-positioning, satellite radar interferometry (InSAR), and laser scanning. In situ measuring techniques can be used as well for deformation analysis. We mention here inclinometers, crack meters, vibration meters and fibre optical sensors.

When discrete points are measured in discrete epochs, and points are *identifiable* as the same ones in different epochs, the determination of movements and deformations is based on comparing the point coordinates or point heights of different epochs as they follow from the measurements. Techniques like spirit levelling, tacheometry and GPS-positioning, fulfil these conditions, as points receive names and are often marked. With InSAR identifying identical points is possible, but more difficult is the identification of corresponding terrestrial points.

Modern techniques such as laser scanning produce point clouds. For them it is more involved to identify identical points through the epochs. These clouds can be reduced to smaller point sets by extracting virtual points (i.e. points derived from one or more measured points) that are identifiable, cf. van Goor (2011). This enables the use of analysis methods using points, identifiable as the same ones through the epochs. Deformation analysis of point clouds, using their full amount of information, needs research into correlation between measurements, specific surface parametrisations, and systematic effects of the acquisition method (Holst and Kuhlmann, 2016).

1.3 Approaches to geodetic deformation analysis

In this section a short overview is given of possible approaches in geodetic deformation analysis. A comprehensive overview is given by Heunecke et al. (2013). To put existing approaches in a general context, it is first described, how a deformation is principally a change of form (as the word itself says: "*de-form*-ation). Then, it is described what a geodetic point field is and how it can be tested for stability. *Stability* is used in this study in the sense of absence of deformation.

If a test for stability is rejected (or a deformation is known to be present, but type and size of deformation are unknown), the best deformation hypothesis has to be searched for[1]. This is described after the testing method has been treated.

If time series of coordinates are available, the analysis often consists of filtering the individual points and statistically testing the filtered results, which are then presented graphically per point.

1.3.1 Changes of form and size

The word "deformation" has its origin in the Latin verb "deformare", which means to deform, to disfigure. It consists of three parts: de-, forma and -are. De- is a Latin prefix with the meaning "down" (which is unfavourable), and "in the opposite sense"[2]. "Forma" is the Latin word for "form", and "-are" is the Latin suffix for a verb. "Deformare" thus means changing the form in an unfavourable way.

Hence, it follows naturally that the subject of geodetic deformation analysis is the analysis of changes of form of a collection of points. A collection of points that is measured by geodetic measurements is called in this study a *geodetic point field*. It can be a one-dimensional field, if only the point heights are determined. It is a two-dimensional or three-dimensional field, if two-dimensional or three-dimensional coordinates determine the point positions. It follows from the just given description of the subject of geodetic deformation analysis that we cannot talk about the deformation of a single point, because a point does not have a form.

If two point fields have the same form, they are *similar*. If, moreover, the size of both point fields is the same, they are *congruent*. Therefore, to test a point field for stability, i.e. to test whether it preserves its form ánd size, the congruence has to be checked by a *congruence analysis* (Niemeier, 1979). It is checked between the point field at a certain time epoch and the same point field at another time epoch. It may happen that, for some reason, the *changes of size* of the point field cannot be determined from the available measurements with the desired precision. In that case the deformation analysis is restricted to the analysis of form, which means that the *similarity* of the same point field in different time epochs has to be checked. An example of missing information about the size, is a triangulation network. Another example occurs, when monumented control points (base stations) of a national or international reference system are used as reference points, relative to which the deformation is monitored. In that case the unit of length of the measuring devices may differ from the unit of length of the reference system, in a way not known precisely enough.

[1] Here, the term 'best hypothesis' is considered to describe the hypothesis that is as close as possible to the observations ('closeness' defined by some norm, for example the Frobenius norm), and that can be considered a valid hypothesis; that is, a hypothesis that can stand up to severe tests (Popper, 2002, p. 249).

[2] Compare "destruction" as opposed to "construction", and words like "degenerate", "degrade".

1.3.2 Testing for stability

Testing for congruence and similarity To test a point field for stability, its form and size (or just its form, if size is not taken into account) are expected to stay the same between one epoch and a later epoch. This can be checked by performing a congruence or similarity transformation of the coordinates of the later epoch onto those of the first epoch. A congruence transformation preserves the form ánd size of a point field. A similarity transformation preserves only the form of a point field. Consequently, with the proper transformation parameters, the residuals in the coordinates after transformation should be zero, except for the measurement noise by which the coordinates are contaminated because they stem from geodetic measurements. The transformation parameters can be estimated by a least-squares adjustment. The residuals are then tested by a standard global test (F-test) (Welsch and Heunecke, 2001, p. 394). For a statistically optimal test, the covariance matrices of the coordinates have to be taken into account.

The congruence or similarity transformation can be omitted, if the information, contained in the coordinates and their covariance matrix, is only determined by the form and size (or only by the form) of the point field. This is only the case, if the origin and orientation (or origin, orientation and scale) of the reference system of the coordinates is determined by a deterministic function of the coordinates, or, put differently, if the coordinates and their covariance matrix are in an S-system (Teunissen, 1985a, p. 41). It means that the coordinates and their covariance matrix define themselves the origin, orientation and scale of the reference system. No external information is used for this definition, and thus no external inaccuracies, not related to form and size, can influence the coordinates and their covariance matrix. A covariance matrix defined in an S-system is singular with a rank deficiency equal to the freedom of rotations, translations and scale that the reference system posseses (7 for an S-system that defines only the form in 3D). In all other situations, the transformation should not be omitted, because the covariance matrix contains information on more than just form and size of the point field, cf. Teunissen (1985a, p. 65ff.). Using the coordinates and their covariance matrix for testing, would yield erroneous results, if the transformation is omitted.

If the covariance matrix of the coordinates is not known (or not available), approximate values for variances and covariances are often used. It is assumed, for example, that all coordinates are not correlated mutually, and have the same precision. This means that a scaled unit matrix is used as covariance matrix. If no correlation is assumed between the coordinates, and all coordinates have different variances, the covariance matrix is approximated by a diagonal matrix with nonzero elements on the main diagonal. Because a scaled unit matrix and a diagonal matrix with nonzero elements on the main diagonal are not singular, they are not defined in an S-system. Every reference system that is not an S-system is called an (a)-system (Baarda, 1973, p. 20). Because coordinates and their covariance matrix, when defined in an (a)-system, contain more information than just information on form and size, a congruence or similarity transformation has to be performed before the F-test can be executed. In general a congruence transformation (alternatively called in 3D a rigid-body-transformation) is used, unless information on the changes in scale cannot be extracted from the measurements with enough precision. If the transformation needed is just a differential one, the transformation is called an S-transformation (Baarda, 1973, p. 21).

Movement relative to reference points Conventionally, in geodetic deformation analysis object points are distinguished from reference points. *Object points* are points that are part of the object or earth's surface, whose deformation or movement is under consideration. *Reference points* are points that are assumed to be not under the influence of the forces that drive the deformation or movement under consideration. It is customary in geodetic deformation analysis to treat reference points (if they are present) differently from the object points (Caspary, 2000, p. 116). Consequently, before analysing the object points for deformations, the reference points need to be checked for stability. Subsequently the movements of object points relative to the reference points are analysed.

In this study, however, the reference points are considered part of the total point field and analysed in the same way as any other subset of points. In the total point field the movements of object points or points of the earth's surface relative to the reference points is treated as a change in form and size of the total point field.

Control points as reference points The coordinates of the points of a geodetic point field are defined relative to a reference system, which is determined by its origin, orientation, and scale. A reference *system* is operationally defined through a reference *frame*, which is defined by well-determined control points, for which the coordinates and velocities at a certain epoch are assumed to be known (Torge and Müller, 2012, p. 17). Such control points (called *first order control points*) or lower order control points, linked by densifications to the first order control points, can be used as reference points for a geodetic deformation analysis. If such reference points are assumed to have standard deviations and correlations of zero (or negligibly small), they define an S-system for the total point field under consideration.

In this case it may be justified to compare coordinates of different epochs directly, using the covariance matrices relative to the control/reference points, *without a previous transformation*. It is assumed, then, that the coordinates of all epochs and their covariance matrices are with certainty relative to this reference frame with the desired precision. It also implies that the control points are stable throughout all epochs. These assumptions, however, have to be tested for. Noncompliance to these assumptions of errorfree and stable control points will lead to erroneous deformation analysis results. Note that in this case the deformation analysis considers the deformation of the total point field of control points and object points together, and not the deformation of only the object points.

1.3.3 Formulating deformation hypotheses

If the global test (F-test) of stability leads to rejection, or if it follows from other considerations that a deformation is probable, it is in general desirable to get information on the type and size of the deformations. If the driving forces for the deformation are known, their description can be used to describe the deformations. If, however, the driving forces are not known, or not known sufficiently, the search for adequate formulations of deformations can be troublesome. If such formulations are to be tested statistically, we call them deformation hypotheses. We will now treat two common

methods to arrive at a formulation of deformations. The two methods are (a) the method that tries to find all unstable points successively, and (b) the method that analyses a time series of measured differences for one or more points by distinguishing trend, signal and noise in the time series.

(a) Search for unstable points A conventional method to find plausible deformation hypotheses is to test *two epochs* of measurements statistically for congruence or similarity. If the global test leads to rejection of the stability hypothesis, we search for individual points, of which the deformation is statistically significant.

Such a statistical testing of the congruence or similarity of a point field in two epochs, and the subsequent search for unstable points, was first described by Pelzer (1971). The approach of Pelzer has been elaborated upon in many further publications (Heunecke et al., 2013; Chen, 1983; Chrzanowski et al., 1986; Dong, 1993; Caspary, 2000). Because of the German city where Prof. Pelzer worked, these publications can be characterised as representing the "Hanover school".

Characteristic of the method is the analysis point-by-point: if one point is detected as moving most significantly, it is removed, and the remaining point field is analysed in the same way for a significantly moved point. Methods to analyse more than two epochs have been described as well (Niemeier, 1979, 1985; Caspary, 2000).

An improvement of the method aims at identification of a group of stable reference points, using the general concept of identification of the maximum subgroup of consistent data (Neitzel, 2004).

For testing and reliability concepts in geodesy the "Delft school of geodesy" is well-known (Staff LGR, 1982). The methods of the Hanover school have been extended by the testing and reliability concepts from the Delft school of geodesy (van Mierlo, 1978).

The resulting methods to arrive at a formulation of the best deformation hypothesis have been implemented in several commercial software systems (Heunecke et al., 2013, p. 521).

(b) Separation of trend, signal and noise; time series Besides the method mentioned under (a), the search for the best deformation hypothesis can be accomplished by interpreting the rejected F-test as caused by a deformation *trend* (the systematic or parametric part, i.e. with a lower frequency in time or space), a deformation *signal* (with a higher frequency in time or space) and remaining measurement *noise*, where the terminology of geodetic collocation is used (Moritz, 1980, p. 111ff.). This terminology is often used (Walwer et al., 2016; Didova et al., 2016; Ming et al., 2016). The trend is described by a deterministic function, while the signal is described stochastically with a temporal or spatial covariance function. To describe a trend or signal in the time domain with any reliability, a sufficiently high sampling rate in time is needed, i.e. more than two epochs are generally necessary. We then have a *time series* of measurements to determine the deformation hypothesis.

Because the measurements (or the coordinates per epoch, derived from them) constitute a time series, theories about stochastic processes, filtering, covariance functions

and spectral analysis come into play, cf. Papoulis (1984). The analysis of a time series is often restricted to individual points, e.g. Chang and Hanssen (2016); Ng et al. (2015). The analysis of a point field, where both the spatial correlation and the temporal correlation is considered, is less common.

If the formulation of the deformations is to be tested statistically, the trend and signal constitute the hypothesis. The estimated noise at times and points in space, for which measurements are available, can be used for testing the hypothesis. An unmotivated deterministic trend function (not motivated by driving forces), or an unmotivated temporal or spatial covariance function (and thus an unmotivated signal) may lead to wrong testing results.

Now that the approaches to geodetic deformation analysis have been described concisely, we can describe in the following section the link to professional practice, after which the problem area can be treated.

1.3.4 Link to professional practice

The topic of this study lies at the intersection of theoretical methodologies for geodetic deformation analysis and their implementations for operational purposes. Therefore, scientific research in geodetic deformation analysis is divided here in research into available theoretical models and development of new ones on the one hand, and research into the extent and maturity of their use in professional practice on the other hand. The theoretical models have been described concisely in the previous section. To illuminate the link with professional practice, the cooperation between scientific academia and professional practice in the Netherlands is described here. In section 1.2.1 six domains have been distinguished, where geodetic deformation analysis is of interest. For each domain the link between scientific research and theoretical methodologies on the one hand and their implementations for operational purposes on the other hand is described below. These links reveal that geodetic deformation analysis in professional practice in the Netherlands appears not to be performed according to the available theoretical methodologies, described in the previous section, in the majority of domains. This is confirmed by an analysis of the software that is used for geodetic deformation analysis.

Here we first confront the scientific research and the theoretical methodologies with professional practice. Then an overview of applied software is given. A picture is thus drawn that gives the foundation for the research question that will be derived afterwards.

Research and methodologies versus professional practice In the domain of analysis of movements and deformations caused by *geophysical processes*, the scientific academia itself constitutes the professional practice. This can be seen in the regular appearance of scientific papers on this subject, e.g. Chang and Hanssen (2016); Fuhrmann et al. (2015); Boyd et al. (2015); Ng et al. (2015); Xu et al. (2000).

In the domain of the mining industry, where there is ample legislation, close cooperation between professional practice and scientific research can be found. This follows, for example, from the settlement of conflicts by academic discussions, e.g. Blaauwendraad et al. (2009).

In the domains of industrial installations, the construction of residential and non-residential buildings, infrastructure, and also in hydraulic engineering, it is less common to use one of the approaches of section 1.3 in professional practice in the Netherlands. A usual method of working is to compute coordinates or heights for measured points for each epoch separately, and to determine the coordinate differences between the epochs. The analysis of the differences is not specifically elaborated upon, and is often restricted to visualisations of difference vectors. This is exemplified by the guidelines of the executive arm of the Dutch Ministry of Infrastructure and the Environment (Rijkswaterstaat, 2014). These guidelines are often used, also for assignments by commissioning companies and agencies outside the domain of activities of the Ministry.

Software as indicator for use of models To assess the dissemination of theoretical models in professional practice, the use of dedicated software, in which these models have been implemented, is a good indicator. For the analysis by means of similarity and congruence models, several commercial software packages that use methods of the Hanover school are available (Heunecke et al., 2013, p. 521). In the Netherlands this software is hardly used by professional practice and the academia (appendix A.5). Here it is described which software is used in the Netherlands.

Specific software has been made for deformation analysis in the mining industry by Delft University of Technology (Ketelaar, 2008; de Bruijne et al., 2001; de Heus et al., 1994b, and older). The software for the analysis of levelling networks, based on the models of de Heus et al. (1994b), aims at estimating linear trends per point in a least-squares adjustment, taking account of stochastic information of the measurements. The software DePSI can analyse InSAR-measurements (van Leijen, 2014). This software is based on time series analysis of individual points that have been identified as persistent scatterers.

Engineering firms have made specialised software, which is generally dedicated to specific application possibilities. An example is the computing tool made by Antea Group in Microsoft Excel for the deformation analysis of individual levelling points in an area of soil subsidence because of salt extraction (Velsink, 2016a, p. 75). It is based on the theory, treated by Verhoef (1994), which, in its turn, is based on the model of de Heus et al. (1994b). It estimates a linear trend per point.

MOVE3, a software package that is widely used in geodesy in the Netherlands, has some possibilities for deformation analysis (MOVE3, 2017). These possibilities have been extended recently, i.a. because of the research that has been conducted for this study. It is possible to use difference vectors in 1D, 2D or 3D as observations for the least-squares adjustment of MOVE3. Minimal detectable biases (see Teunissen (2006, p. 102)) can be computed for these observations. The deformation analysis option of MOVE3 is an implementation of the model described in chapter 6.

After this overview of the link between the approaches to geodetic deformation analysis and professional practice, we can direct ourselves to the research question and its subquestions.

1.4 Problem area and research question

Problem area The overview of conventional approaches to geodetic deformation analysis in section 1.3, and the link with professional practice in section 1.3.4, suggest that in many domains of professional practice in the Netherlands the comparison and analysis of point positions of different epochs are not handled with one of the conventional approaches to geodetic deformation analysis. In fact, the analysis is often restricted to visualisations of difference vectors. Moreover, the differences between epochs are not statistically analysed, and conclusions, therefore, do not in general meet any statistical optimality criterion. Hence, without a mathematical-statistical model as foundation for the geodetic deformation analysis, arguments arise in professional practice on how to draw conclusions, and on the attainable quality of analyses. The arguments are a sign of failing *communication*, and are caused by the absence of a clear, unambiguous terminology, and of statistically underpinned numerical indicators of the quality of analysis results. As a consequence, *standardisation* of terminology, processes and presentation of results is asked for. Such standardisation has to be based on a sound mathematical-statistical model. Models that follow from the approaches described in section 1.3 (the Hanover school, and time series analysis) fulfil these requirements, and implementations in software exist.

Hence, it is appealing to use one of these models, or a combination of them. There are, however, two drawbacks.

The first drawback concerns the lack of availability of suitable quantities for the assessment of the attainable quality of a deformation analysis. The Delft school of geodesy yields suitable quantities for such an assesment in the form of bounds to the values of errors that can be detected by statistical tests with a certain probability (Baarda, 1968b, p. 19). These bounds are called *boundary values* or *marginally detectable errors* (Staff LGR, 1982, p. 217), or *minimal detectable biases* (Teunissen, 2006, p. 102). van Mierlo (1978) gave an impulse to their application for deformation analysis. It has, however, not been extended for more intricate deformation hypotheses (e.g. when several points are moving, or when different subsets of points are moving relative to each other, or when a subset of points is subject to a rotation, or all of this together).

Also the methods of the Hanover school do not in general provide information on the attainable quality of deformation analyses. This is caused by the fact that the methods of the Hanover school, and also many time series analyses, are based on determining point-by-point, whether one point has moved. Neitzel (2004) gives an extension to the methods of the Hanover school to test for blocks of stable points. However, methods to test more intricate deformation hypotheses, such as given above, in a straightforward, uniform, procedure, have not been published.

The second drawback is the absence of ways to incorporate physical models, describing the deformations, into the models of the Hanover school. A physical model is a description of driving forces, from which a hypothesis can be deduced about the movements of one or more points. Incorporation of a physical model means that the hypothesis is part of the adjustment model, and can be tested simultaneously with the testing of the geodetic observations.

In conclusion, a suitable geodetic deformation analysis model that enables standardisation, uses physical and geometric models in a combined model, and enables intricate deformation hypotheses to be tested, is not yet available.

Such a model is a crucial prerequisite for effective communication between all stakeholders concerned with geodetic deformation analyses (legislators and government agencies that ensure protection against deformation risks, commissioning companies and agencies, companies that perform geodetic deformation analyses, and the general public, which may or may not be subject to the deformation risks).

Thus it is the purpose of this study is to build such an optimal geodetic deformation analysis model on the basis of the body of knowledge of the Delft school of geodesy, and, in this way, to have a foundation for standardisation and communication.

Research question The research question has been formulated in view of the described problem area:

> How can a generic mathematical-geodetic model be formulated that is:
> (i) usable for geodetic deformation analysis, (ii) enables standardisation of terminology, processes and presentation of results for geodetic deformation analysis, and (iii) is usable as a basis for communication about goals, possibilities and analysis results of geodetic deformation measurements?

The model is intended to use geodetic observables on the one hand and information on deformations in physical reality on the other hand, to test intricate deformation hypotheses. The standardisation and the communication have to be based on statistically valid methods.

Subquestions

The construction of a model for geodetic deformation analysis, and this model's use for standardisation and communication, is elaborated upon along the following lines. The method of least squares with its optimality characteristics, and the statistical methods that can easily be coupled to the method, forms the basis of model construction. To describe deformation hypotheses deterministic formulations have to be possible, for which constraints on model parameters provide an effective apparatus. To weigh hypotheses against each other, a search mechanism for finding the best hypothesis has to be present. Because the model is concerned with deformations, analysis of form and size is pivotal. For adequate analyses the (physical) causes of deformations have to be taken into account. Finally, time is of importance in deformation analysis and, thus, time series analysis is crucial. Formulation of standards and communication, based on these standards, is made possible, if a geodetic deformation analysis model has been built along the sketched lines.

In accordance with these lines five subquestions are formulated into which the research question is broken down.

1. How can a model be built in such a way that the full *existing body of knowledge of least-squares theory* is applied for deformation analysis? The focus is here on

the achievements of the Delft school of geodesy regarding testing and quality description (Baarda, 1968b, 1973, 1995; Staff LGR, 1982; Teunissen, 2007), and their application for testing deformation hypotheses. This includes the concept that the deformation of a point field should be modelled in such a way that *only form and size* of the point field are analysed for changes and not changes in other geometric quantities, such as origin and orientation of a reference system (the point field is considered here the discretised representation of the monitored object), cf. Baarda (1995, p. 6).

2. How can a *physical model* with its functional relations and parameters be incorporated in the geodetic deformation analysis model, thus enabling the analysis of the fit of a *time series* of observations within both the geodetic and physical model?

3. When measuring the same geodetic observables during subsequent time epochs in a deforming point field, the values of the measured observations will change because of the deformations (and because of changes caused by measurement noise and other error sources). This is expressed mathematically by the fact that the mathematical expectations of the observables will comply with the deformation hypothesis (if this hypothesis is valid). Put differently: a deformation hypothesis constrains the observables. The measured observations will, of course, not comply with the constraints, because of measurement noise and observation errors. They will also not comply with the constraints of a deformation hypothesis, if the hypothesis is not a valid one.

   From this follows the subquestion, how *constraints* can be used effectively to describe deformation hypotheses, and how these constraints can be tested and provided with a quality description, expressing minimal detectable deformations.

   In the process of handling constraints in an adjustment model, *rank deficient* model matrices and covariance matrices can appear. Hence the question follows, how least-squares solutions of the geodetic deformation analysis model can be formulated in a way that permits rank deficiency of those matrices.

4. What kind of *search method* helps finding the hypothesis that best describes the deformation?

5. What are the requirements that a geodetic deformation analysis model has to fulfil to be usable for *standardisation* and, as a derivative, for effective and efficient *communication*.

1.5 Research method and outline

Research method To answer the main research question an analysis model will be built around a mathematical-geodetic adjustment model. The adjustment model takes geodetic measurements as input and delivers the coordinates of all points under consideration in all epochs as output. The adjustment can be done in one step. The corresponding adjustment model will be elaborated upon in this study. It is called here the *measurements model*. The deformation analysis will be based on the adjustment

results, especially on the computed coordinates of all points in all epochs and the computed deformation parameters, and on the results of statistical tests, performed on the adjustment results.

In scientific research and in professional practice the adjustment is often split in two steps. The splitting up into two steps is often necessary, because the original geodetic measurements are not available during all epochs, or they are considered too numerous. The first step comprises the adjustments of the observations for each epoch separately, resulting in coordinates per epoch. There may be a consirable time (days, months, or even years) between the adjustments of the first and last epoch. This type of adjustment is well-known and often used (and in professional practice many software packages are available). The corresponding adjustment model will, therefore, not be treated in this study. The coordinates per epoch are used as input for the second step, in which all epochs are taken together and the coordinates are analysed for deformations. Customarily, the second step is not realised as a least-squares adjustment. The present research method, nevertheless, will be based on the construction of an adjustment model for the second step, called the *coordinates model*, that is suitable for least-squares adjustment. The advantages, which are closely linked to the optimality properties of the least-squares method, will be presented. The model takes coordinates both as input and as output.

To formulate the two adjustment models (measurements model and coordinates model), several problems will have to be solved. To keep their treatment manageable, chapters 3 and 4 will treat some of the problems in the framework of simpler models (e.g. only two-dimensional, or only two epochs). Chapter 5 treats some necessary mathematical tools, after which the measurements model and the coordinates model are treated in their full extent in chapters 6 and 7.
The following problems will be dealt with. First it will be studied how similarity and congruence transformations can be incorporated in an adjustment model for deformation analysis, how a singular covariance matrix of the input coordinates can be handled, and how a best deformation hypothesis can be found. This will be studied in two-dimensional Euclidian space, and with two epochs of observations. Then the problem will be extended to 3D, where the inclusion of transformations into the adjustment model is more complicated, because of the rotation parameters.
To extend the model to more than two epochs, i.e. to a time series of observations, with the incorporation of deformation hypotheses by means of constraints, the handling of constraints has to be based on a more generic basis than literature provides. This more generic basis will allow for testing constraints in a model with a singular covariance matrix and a rank deficient model matrix.
The intermediate models and solutions that will solve the mentioned problems will be tested by designing algorithms and applying them to simulated test situations. Finally the geodetic deformation analysis for a time series of observations (measurements model), and for a time series of coordinates (coordinates model) will be developed.

To answer the last subquestion the results of the research for the set-up of a taxonomy of governance of geodetic deformation analysis (Velsink, 2012), and of the results of research project *DefoGuide* (see the description in appendix A.1), will be used.

Most results have been published before (Velsink, 2015a,b, 2016b, 2017, 2018).

Outline The outline of this study is described below and summarised in table 1.1.

After this introductory chapter, chapter 2 gives a description of the *proposed analysis model* and its essential elements. This analysis model is the foundation for the subsequent chapters. It includes an adjustment model, which will be developed in two variants, the measurements model and the coordinates model. The requirements are treated that the analysis model has to fulfil to be usable for *standardisation* and for effective and efficient *communication*.

Chapter	3	4	5	6	7
Epochs	2	2	–	> 2	> 2
Dimension	2D	3D	–	any	3D
Computation methods	–	–	yes	–	–
Nonstochastic observations	–	–	yes	yes	yes
Transformations	yes	yes	–	–	yes
Search for best hypothesis	yes	yes	yes	yes	yes
Testing & test quality	yes	yes	yes	yes	yes

Table 1.1: Adjustment model characteristics in chapters 3 – 7.

In chapter 3 the situation is treated that two epochs of deformation measurements in the plane (two-dimensional Euclidean space) are available. The adjustment model takes coordinates and their covariance matrix as input, and treats, therefore, the *coordinates model for two epochs and in 2D*. The measurements are assumed to have been adjusted already prior to the use of the model. This prior adjustment has yielded estimates for the coordinates of the network under consideration. Also the covariance matrix of these coordinates is used. The prior adjustment is often the adjustment of a free network, which results necessarily in a singular covariance matrix of the coordinates. Incorporating a transformation into the adjustment model is shown to have important advantages. Handling a singular covariance matrix for the observations is treated. Subsequently testing and quality description of deformation hypotheses are described. It is shown how the best alternative hypothesis can be found by testing a large amount of possible hypotheses.

Chapter 4 is about deformation analysis with coordinates in three-dimensional Euclidean space, where the introduction of transformation parameters, especially the rotation parameters, needs special care. Two epochs of coordinates of the same physical points are assumed to be available. This chapter, therefore, treats the *coordinates model in the case of two epochs and in 3D*. The coordinates may originate from previous adjustments, where geodetic observations were used to estimate the coordinates. This implies that the covariance matrices of the coordinates can be full and singular, e.g. when a free network was adjusted in the previous adjustment. To guarantee that only the change of form and size of the point field is analysed, a three-dimensional transformation is included in the adjustment model, as explained in section 1.3.2. The transformation can be a similarity or congruence transformation. The transformation is set up in such a way that no approximate values of the transformation parameters are needed, and that gimbal lock[3] cannot occur.

[3]Gimbal lock can occur, if a 3D-rotation is realised by three successive rotations around three axes

Chapter 5 is devoted to the adjustment and testing methods that are needed when *constraints* are present in the adjustment model. This happens in the models of chapters 6 and 7, so chapter 5 is a preparatory chapter. To enable a quality description of the testing of constraints (by means of minimal detectable biases), here the constraints are handled as nonstochastic observations (constants, treated as observations with a standard deviation of zero and with no correlation to other observations). They can be seen as pseudo-observations. An advantage of formulating constraints as nonstochastic observations, is that testing and test quality description are realised by an identical and simple procedure for both stochastic observations and constraints. A consequence is, however, that the covariance matrix of the observations will always be singular. An overview of several methods is given to accomplish adjustment, testing and test quality description in case the covariance matrix is singular and the model matrix rank deficient.

In chapter 6 an adjustment model is presented to adjust and test a time series of geodetic observations. This chapter, therefore, treats the *measurements model*, for a *time series of observations in 1D, 2D, or 3D*. Functional relations, originating from a physical deformation model, are incorporated in the model by means of constraints and extra parameters. The significance of these extra parameters can be tested by the usual testing of an alternative hypothesis against a null hypothesis. The validity of the physical deformation model is, therefore, tested as an alternative hypothesis. Because the constraints, which describe the physical model, are introduced in the model as nonstochastic observations, testing and test quality description of deformation hypotheses on the one hand, and of measurement errors on the other hand, is accomplished with the same procedure.

Chapter 7 handles time series of three-dimensional coordinates. This chapter, therefore, treats the *coordinates model*, for a *time series of coordinates in 3D*. The coordinates can originate from previous free network adjustments, which have singular covariance matrices. Each epoch of coordinates is linked to a following epoch by means of a transformation (similarity or congruence transformation). This guarantees that only changes in form and size are analysed. Deformation hypotheses can relate to several points and to several epochs (e.g. a set of points, belonging to a certain building, that is subsiding during a number of epochs; or a set of points belonging to one side of a fault, moving to the points on the other side of the fault during a certain amount of epochs). The functional relationships, describing the deformation, can contain additional parameters, and can be, for example, polynomial functions. The deformation hypotheses are, again, described by a set of nonstochastic observations.

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that stand orthogonally to each other. For certain values of the successive rotations two rotation planes can coincide, and a degree of (rotation) freedom is lost. In this situation the transformation parameters cannot be determined unambiguously.

2

Analysis model and its essential elements

2.1 Introduction

The following chapters treat various elements of geodetic deformation analysis. Each chapter is self-contained and can be read separately. Most chapters have been published before as scientific papers. This chapter provides a framework for the following chapters. This framework I provide in the form of an *analysis model* for geodetic deformation analysis. The model distinguishes *three analysis stages* (design, implementation and realisation), and seven *essential model elements* (listed in table 2.1). The next section elaborates upon the components of each of the three analysis stages. Figure 2.1 presents a graphical illustration. The seven essential model elements follow from the requirements as formulated in the research question and its subquestions, which follow, in their turn, from the intended use of the model. Therefore, I first treat the *use* of the analysis model in section 2.3. Subsequently, section 2.4 gives a description of the seven essential model elements. Models are available in literature, in which one or more of the essential model elements are fully or partly incorporated (van Mierlo, 1978; de Heus et al., 1994b; Caspary, 2000; Heunecke et al., 2013; Chang and Hanssen, 2016). I will treat them thoroughly in this and the following chapters. My purpose, however, is to present a model that contains *all* elements, and enables a full answer to the research question.

2.2 Analysis stages

This section describes components of the three stages of the analysis model: design stage, implementation stage and realisation stage, visualised in figure 2.1.

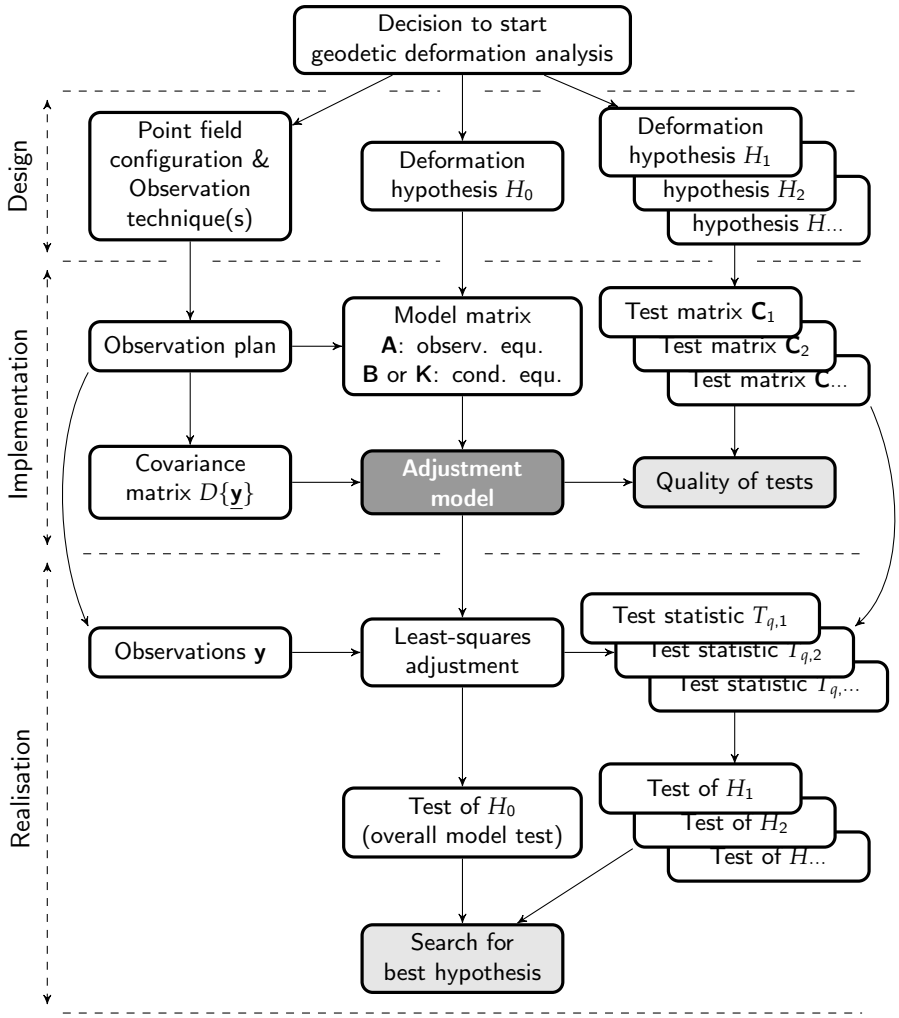

Figure 2.1: *Analysis model*: Geodetic deformation analysis, divided in three stages: design, implementation and realisation.
$\underline{\mathbf{y}}$, $D\{\underline{\mathbf{y}}\}$, **A**, **B**, $\mathbf{C}_1$, etc.: variable names used in this study.

2.2.1 Design stage

The analysis model describes a process that starts with the decision that a geodetic deformation analysis is necessary. The subject of the analysis is the deformation of objects on, above or below the earth's surface, or of parts of the surface itself. I call these objects or parts of the earth's surface "geo-objects".

The *design* of the analysis is the first stage of the analysis process. In the design stage one makes the decision, advised by the geodesist, how a mathematical-geodetic model will describe the geo-object or geo-objects. An option is to make a *discretisation* of

the geo-objects with several or many *points*. The coordinates of the points (1D, 2D or 3D) are determined. They constitute a mathematical representation of the geo-objects. Other options to make a mathematical representation are e.g. raster elements (pixels), volume elements (voxels), and Delaunay triangles. To limit the scope of my research, I only consider mathematical-geodetic models that handle 1D, 2D or 3D points.

The determination of deformations of geo-objects now becomes the determination of movements of representative points. All points together constitute a point field, and I call it a *geodetic point field*, if geodetic techniques are used to determine its deformations. Geodetic techniques are for example spirit levelling, GPS differential carrier phase positioning, tacheometry, and interferometric synthetic aperture radar (InSAR). Form, size and nature of the geo-objects, but also the geodetic observation techniques, used to measure the deformations, determine the optimal *configuration* of the geodetic point field. Questions that arise are: What are suitable point locations? How dense should the distribution of points be? Are monumentations of the points needed, and what do they look like?

The formulation of deformation hypotheses is part of the design stage: which deformations can be expected, considering the forces exerted on the geo-objects? To assess the validity of a presumed deformation, a test of the deformation hypothesis is necessary. Figure 2.1 distinguishes the null hypothesis (H_0) from alternative hypotheses (H_1, H_2, ...), in which I follow the Neyman-Pearson testing theory (Neyman and Pearson, 1933; Teunissen, 2006).

2.2.2 Implementation stage

The second stage of geodetic deformation analysis is the *implementation* stage, where the geodesist takes the lead. He implements the chosen geodetic observation techniques by deciding which physical instruments or sensors to use. This determines the precision of the observations, and is the basis of the covariance matrix. The geodesist also implements the point field configuration by making an observation plan. For techniques such as tacheometry and GPS it means choosing the instrument points and the object points to be measured. For a technique such as InSAR it means choosing the necessary images and checking the availability of usable coherent scatterers.

In the centre of the implementation stage stands the *adjustment model*[1], which determines the way in which the measured observations are adjusted with the least-squares method.

The least-squares theory gives two fundamental models to adjust observations (Helmert, 1907; Tienstra, 1956; Teunissen, 2000). The first model, and the most used one, is the model of observation equations (Gauss-Markov model), first published in 1805 (Legendre, 1805)[2]. The second model is the model of condition equations, first published in 1828 (Gauss, 1828). Both models are *dual* to each other, which means that the adjustment results are the same, and that each model can be transferred into the other

[1] The adjustment model is equal to the mathematical-geodetic model of the research question.

[2] Gauss claimed to have used the least-squares method before 1805; the Gauss-Markov theorem (proved by Gauss, recovered by Markov) gave a statistical foundation for the method.

(Teunissen and Amiri-Simkooei, 2008, eq. (3)). The model of observation equations contains both observables and parameters. The model of condition equations contains only observables. I will use both models in this and the following chapters, but I will always take the *model of observation equations as starting point*, because, in general, the construction of the adjustment model is easier with observation equations for the problems I intend to solve.

Three extensions of the two fundamental adjustment models exist (Tienstra, 1956; Teunissen, 2000): (i) the model of observation equations with constraints on the parameters, (ii) the model of observation equations, where the observations, used in the adjustment, are functions of the measured observations, (iii) the two previous models combined. I will not treat these extensions separately here, but I will use them in this study, and derive the necessary equations when they are needed.

The adjustment model for geodetic deformation analysis describes the relations between the mathematical expectations of the observables according to some deformation hypothesis.[3] If the model of observation equations is used, the introduction of parameters into the model is necessary. The parameters that describe the deformations of the geo-objects constitute an important part of the parameters. The choice which parameters are used, is free, as long as they describe the deformations adequately. In this study I choose to use coordinates for these parameters. The reason is that the analysis model presents deformations of geo-objects as position changes of points. To describe positions and position changes, coordinates and coordinate changes are convenient. However, the determination of form and size changes of a geodetic point field demands special precautions, if coordinates are used as parameters (problem of the geodetic datum). I will treat this in section 2.4.2 in more detail.

With the observation plan, the covariance matrix, the choice of parameters and the null hypothesis, the observation equations can now be formulated. They are in general nonlinear. My approach in least-squares adjustment is to linearise all nonlinear relations and to solve the adjustment models in their linearised form. This makes it necessary to have adequate initial values for the parameters (model of observation equations), and initial values for the observations that are consistent with the initial parameter values and comply with the adjustment conditions. Iteration of the solution process is necessary as well. The linearised observation equations (or, for the model of condition equations, the linearised condition equations) yield the *model matrix*[4], see figure 2.1.

The geodesist uses the null hypothesis to construct the model matrix. An overall model test (F-test) gives information about the validity of the null hypothesis (Teunissen, 2006). If it is rejected (in the realisation stage), the alternative hypotheses come into play. An alternative hypothesis is the null hypothesis, complemented with additional parameters. Such an additional parameter can be, e.g., the subsidence coefficient of a linear subsidence, or the coefficients of a truncated Fourier series to describe some periodic deformation. The additional parameters can also describe errors in the observations (e.g. point identification errors, atmospheric errors, instrument calibration errors). The relation between the additional parameters and the rest of the adjustment model is, in general, nonlinear. It is linearised and yields a *test matrix*. The geodesist uses the

[3] The hypothesis can be that no deformation is present.
[4] Also called: coefficient matrix, design matrix.

test matrix to compute a test statistic in the realisation stage, but also to determine the *quality of the test* of the alternative hypothesis.

The geodesist ends the implementation stage before the start of the measurements. The result is an overview of the efforts that will be necessary in the realisation stage, as well as an overview of the quality of the tests. This means that before the start of the measurements the geodesist can make a statement about the types and sizes of deformations that will be detectable.

2.2.3 Realisation stage

The geodesist collects observations according to the observation plan, and adjusts them according to the adjustment model. He checks the estimated least-squares residuals with an overall model test. If the test does not lead to rejection of the null hypothesis, he is ready. If not, the alternative hypotheses are tested one after the other. For this purpose *test statistics* are computed for each hypothesis. The question is, how to determine which the best hypothesis is. The *search for the best hypothesis* is complicated and will be treated in section 2.4.7.

2.3 Model use

The geodetic deformation analysis model, visualised in figure 2.1, and its essential elements, listed in table 2.1, are intended for use in professional practice. This section is dedicated to the *use* of the analysis model. It describes the operational demands for an analysis model (the requirements that professional practice asks for), and the communication and standardisation that should be possible with the analysis model.

2.4.1:	A *time series* of several epochs can be handled
2.4.2:	Only *size and form* are analysed, not position and orientation of the reference system
2.4.3:	*Geodetic and physical* model are combined
2.4.4:	Full use of *stochastic information*
2.4.5:	Statistical testing and computation of *minimal detectable biases*
2.4.6:	Solution methods can handle *rank deficient* matrices (both model matrix and cofactor matrix)
2.4.7:	Search for *best hypothesis/model* is possible

Table 2.1: Essential model elements and sections in which they are described.

2.3.1 Operational demands

The operational demands have been derived from interviews with several stakeholders in the Netherlands (Velsink, 2012). The interviewees were from several companies and public bodies, active in the commissioning and implementation of geodetic deformation measurements. The operational demands have been derived as well from the results of a research project on guidelines for geodetic deformation monitoring (see appendix A).

From the interviews it follows that tasks, legislation and regulations, and operational practice regarding geodetic deformation measurements can differ considerably for the domains of activity where they are used, such as: residential and non-residential buildings, infrastructure works and hydraulic engineering, mining industry, industrial installations, and research into deformations of large areas, like countries and continents.

Below I look, succesively, at specific operational demands, originating from Velsink (2016a), and present their connections to the required model elements, summarised in table 2.1.

First, it occurs in professional practice that companies, involved in geodetic deformation measurements for a certain assignment, experience a lack of knowledge in geodetic deformation analysis, and, if their knowledge is sufficient, lack adequate geodetic deformation analysis models. The result is that they cannot agree on the attainable precision of deformation estimates, using a certain network configuration. The requirement is, therefore, that an unambiguous procedure should exist to quantify the attainable precisions of deformation estimates. Directly related is the need for unambiguous testing procedures to determine, whether a tolerance has been exceeded. The requirement can be restated that a quantification of minimal detectable deformations, linked to an unambiguous testing procedure, should be available.

Secondly, it occurs that the supplier of geodetic deformation measurements delivers coordinates or heights of the points of a geodetic network at different epochs as a product to the client, without a clarification what they mean and how they should be interpreted. The client can be e.g. a geophysict, a construction engineer, or a hydraulics engineer. He may have made his own calculations of which deformations are to be expected. It is often not clear how to link the coordinates and heights to the own calculations. More generally phrased: how should the geodetic model and the physical model be linked? Is it possible to give an unambiguous procedure to test the physical model by means of the geodetic model?

A third operational demand follows from the experience that spreadsheets with coordinate or height differences between epochs, or listings of computer programmes are not acceptable for clients. They want a graphical overview of the deformation analysis results, and additionally a few numbers or statements (such as "accepted") as indicators for stability or for the deformations detected. A visualisation is, for example, a map, on which the deformation is visible, or a film or filmlike experience of the changes in time. Such a requirement demands a complete deformation analysis model, where all measurements are treated simultaneously, a testing procedure is incorporated, and testing procedures are integrated. Hence, the demand encompasses all elements of table 2.1.

The fourth operational demand is related to insufficient knowledge of covariance matrices and their dependence on S-systems in professional practice in the Netherlands. This shows itself in the mixing up of local, regional and national precision descriptions, resulting in suboptimal deformation analyses. It happens, for example, that twice the value of a one-dimensional standard deviation is interpreted as a one- or more-dimensional deformation tolerance. This means that the geodetic deformation analysis model must be capable of handling, without much interference of the user, the full stochastic information available. Also, the analysis must be restricted to size and form, without mixing it with information on position, orientation and scale of the reference system, or of points, not related and not of interest for the deformation. S-systems must be handled adequately by the model.

The fifth and final operational demand follows from the often encountered lack of tools (partly because they do not exist, partly because the specialist does not know them, or has no access to them) to adequately perform statistical tests on geodetic deformation measurements. In such a situation, the conclusion what the best deformation hypothesis is, is based much more on practical expertise than on statistical considerations. It may be expected that this leads to difficult considerations and consultations with other specialists to reach a conclusion. This can, indeed, be seen in professional practice in the Netherlands. Hence, the demand is for an overall statistical analysis of all epochs and all measurements (time series analysis), and for a procedure to find the best deformation hypothesis, based on statistical considerations.

2.3.2 Communication and standardisation

The analysis model, presented in this chapter, is intended to improve geodetic deformation analysis, and, by using the results that follow from the model, to improve *communication* on results. Good communication needs a common terminology, because words need to have an unambiguous meaning for mutual understanding. This is especially important, when we use words that describe statistical conclusions of measurements. However, even if we use a common terminology, it is inevitable that different specialists, stakeholders, journalists and laymen draw different conclusions from the same observation material and model assumptions. This is caused by the stochastic character of the observations, which causes many different statistical models to be applicable. It is also caused by different interpretations of the words in which statistical conclusions are phrased. Different conclusions from the observational material can also be caused by unknown parameters of applied physical models, which can be approximated by different values or mathematical functions.

A different physical model means that the geodetic model uses other or even wrong functional relationships to which the observations are adjusted. This implies that geodetic observations cannot give a description of deformations without the use of some physical model.

Hence, to improve communication on the results of geodetic deformation analysis, a common terminology and a clear perception of the used models, are necessary. *Standardisation* can help here. Standardisation means that terms are defined unambiguously, models to be used are listed, standard processing options are available, and the interpre-

tation of analysis results is performed in a uniform way. Standardisation, however, does not solve the multi-interpretability of observations. It only prevents misconceptions, promotes mutual understanding, and stimulates research to improve analysis methods (Blind, 2013).

Standardisation facilitates communication on statistical analysis results. The degree of knowledge and expertise of persons involved in the communication influences the possible intricacy of terminology and models. Consequently, we distinguish *four categories* of persons involved in deformation analysis. The first category is the *expert*, who is well-educated in the field of such analyses and has sufficient expertise in it. The second category is the *technically educated stakeholder*, who is sufficiently educated and has enough expertise to understand the technical language of the expert, but is not an expert himself. The third category is the *non-technically educated stakeholder*. It may be the principal, for example the representative of a tendering authority, it may also be a person, who is responsible for the consequences of failures to handle risks (of deformations) adequately. The fourth and last category is the *general public*, who is informed by the media and by specific information material, and may be or become subject to negative consequences of failures regarding deformations. Communication with each category has to be done in a terminology and a way of reasoning that is adequate for that category.

2.4 Model elements

The model elements (see table 2.1) are expatiated upon in this section in general terms.

To be able to describe the elements, I assume that the geodetic observables can be clustered into epochs. An epoch is a moment in time or a time interval in which all necessary observations of the objects under deformation are collected and, if the epoch is a time interval, during which interval the deformations are negligible relative to the possible deformations in between epochs (or they are compensated for). The objects under deformation are represented by points, and I assume every point to be identifiable as the same point in each epoch (unless there are physical hindrances).

2.4.1 Time series

The adjustment model, which is used in the geodetic deformation analysis, should be capable of handling several epochs of observations of the objects under deformation. This means that *time series* of 1D, 2D or 3D coordinates have to be determined from the geodetic observables. Testing such time series is the basis of the geodetic deformation analysis.

Two different adjustment models are presented in this study. The first one (figure 2.2) takes geodetic observations in the observation vector and coordinates in the parameter vector. The second adjustment model (figure 2.3) distinguishes two phases. In the first phase the geodetic observations are adjusted for each epoch separately. In the second phase the coordinates, which result from all epoch adjustments, are taken as observations, and coordinates appear in the parameter vector as well.

The second adjustment model is added to the straightforward first one, because it is common practice to base geodetic deformation analysis on the analysis of the coordinates, obtained from the separate epochs (see section 1.3). The use of the second model is a necessity in case the original measurements are not available, only the coordinates per epoch.

In the first model (figure 2.2), hereafter called the *measurements model*, the vector of observations contains all geodetic measurements, clustered in subvectors: one subvector for each epoch. The parameters are the coordinates of all points. Each point has a different set of coordinates for each epoch. The epochs are linked together by constraints, which describe the hypothesised deformation pattern[5].

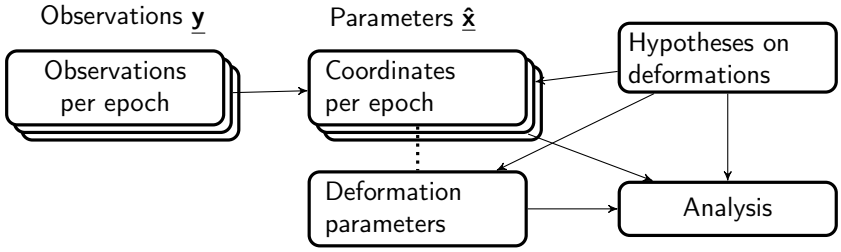

Figure 2.2: *Measurements model*: adjustment in one phase.
Dotted line links elements that constitute the parameter vector.
Arrows mean "determine(s), are (is) the basis for".
$\underline{\mathbf{y}}$, $\underline{\hat{\mathbf{x}}}$: variable names used in this study.

In the second model (figure 2.3) the analysis is split in two phases. First it is assumed that the geodetic observations of each epoch have been adjusted separately, resulting in coordinates per epoch, including their covariance matrix. In the second phase, a model is built that takes the epoch coordinates both as input (observations), and as output (parameters). Because the analysis is only concerned with form and size of the point field, the second phase includes transformations of all epochs to a reference epoch. The necessity of this inclusion is treated more deeply in section 2.4.2[6].

The results of both models are the same, if, in the second model, the covariance matrices of the epoch adjustments of the first phase, are taken care of in the second phase. This follows from the *principal property* of least-squares adjustment (Tienstra, 1956, p. 154):

> "Every problem of adjustment may be divided into an arbitrary number of phases, provided that in each following phase the cofactors resulting from preceding phase(s) are used."

2.4.2 Change in size and form[7]

In geodetic deformation analysis the form and size, and their changes, of a geodetic point field are studied. The analysis should be invariant for the scale, position and orientation

[5] Chapter 6 treats the resulting model, and its adjustment, testing and the description of test quality.
[6] The resulting model is the subject of chapter 7.
[7] Large parts of the text in this section are translated and adapted from (Velsink, 2016a).

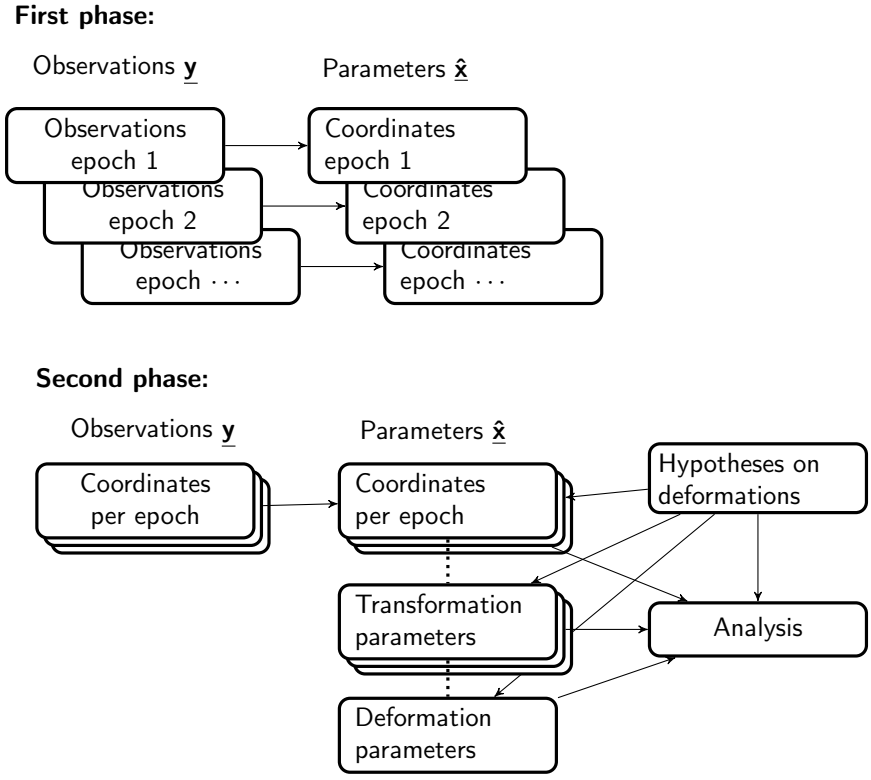

Figure 2.3: *Coordinates model*: adjustment in two phases.
Dotted lines link elements that constitute the parameter vector.
Arrows mean "determine(s), are (is) the basis for".
$\underline{\mathbf{y}}$, $\underline{\hat{\mathbf{x}}}$: variable names used in this study.

of the reference system, used to fix the positions of the points. In this section first the concepts "form" and "size" are defined for the 1D, 2D and 3D Euclidian space. From these definitions the concepts "size and form elements" and "form quantities" follow. Deformation analysis is about changes in size elements and form quantities. Subsequently, the concept of "coordinates" is introduced, where it will be shown that they are merely intermediate quantities in the analysis of changes of form and size. Then, the customary approach to geodetic deformation analysis and its disadvantages are treated. The choice for a model with coordinates as parameters is discussed, and finally the conclusion regarding the requirements for the two adjustment models is drawn.

Size and form elements

The word "deformation" stems from the Latin word "deformatio", which means "disfiguration", "deformation". Its kernel is the word "forma": "form". This indicates that deformation is about *change of form*, and, as an extension, about *change of form and*

size. A deformation analysis model must therefore be capable of giving information about the existence, size and process of this change of form and size.

The mathematical concept "form" follows from the concept "similarity" as defined by the Greek mathematician Euclid (third century B.C.). When two objects are similar, they have the same form. Euclid starts his book "The Elements" with 23 definitions (Fitzpatrick and Heiberg, 2007). Among them are the definitions of the *planar angle* and the *distance*. Using the concept of planar angles, Euclid introduces the concept of *similarity* in book 6 of his Elements. The first definition of book 6 states: "Similar rectilinear figures are those (which) have (their) angles separately equal and the (corresponding) sides about the equal angles proportional." A more recent definition is: "Two figures are said to be similar when all corresponding angles are equal and all distances are increased (or decreased) in the same ratio, called the ratio of magnification (Coxeter and Greitzer, 1967, p. 94)." A transformation that takes figures to similar figures is called a similarity transformation. The angles stay the same under such a transformation and are called the *form elements*. Distances convey information on the size and are, therefore, called *size elements*.

If no increase or decrease in the distances is required (the ratio of magnification is 1) the objects are *congruent*: they not only have the same form, but also the same size. The congruence transformation (also called "direct isometry" or "rigid body transformation") transforms an object to another one without changing the form or size. The congruence transformation is normally used in deformation analysis, unless changes in size are not determinable from the measurement with enough precision, in which case a similarity transformation is used.

The following three paragraphs treat the definitions of size and form elements in, first, the two-dimensinal plan, then in three-dimensional Euclidian space, and finally in one-dimensional Euclidian space.

Two-dimensional plane In the two-dimensional plane (Euclidean two-dimensional space) the smallest set of points, by which an angle can be defined, and for which it is thus sensible to speak of its form, contains three points. The three points constitute a triangle. The form of the triangle is defined, if two angles are known (figure 2.4(a)). The size of the triangle is fixed, if additionally one of the distances is known. The form and size of the triangle are thus defined, if two angles and one distance are known. They are, however, also defined, if the three distances (AB, BC, AC) are known. Baarda (1966) pointed out, that the form, without the size, can be defined by two length ratios, for example:

$$v_{CAB} = \frac{AB}{AC} \text{ en } v_{ABC} = \frac{AB}{BC}. \tag{2.1}$$

The combination of an angle and a length ratio determines the form as well, for example α_{ABC} en v_{ABC}, see figure 2.4(b). An angle can be considered to be a length ratio, if it is seen as the central angle of circle: it is the arc length on the circle, divided by the radius (Baarda, 1966, blz. 27). Both the length ratio and the angle are taken together in one variable, the Π-quantity, and written as a complex number:

$$\Pi_{ABC} = \ln v_{ABC} + i\,\alpha_{ABC}, \tag{2.2}$$

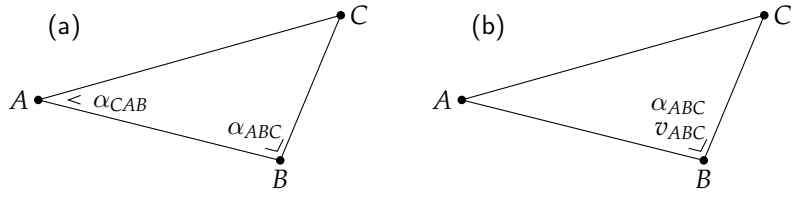

Figure 2.4: Form of a triangle: α_{CAB}, α_{ABC}: angles; v_{ABC}: length ratio.

where "ln" denotes the natural logarithm, and i the imaginary unit, for which holds $i^2 = -1$. The advantage of taking the natural logarithm is the analogy between the length ratio and the angle:

$$\begin{aligned} \ln v_{ABC} &= \ln BC - \ln BA; \\ \alpha_{ABC} &= r_{BC} - r_{BA}, \end{aligned} \tag{2.3}$$

with r_{BC} and r_{BA} the directions from B to respectively C and A. The Π-quantity allows to capture the form of three points in a single complex quantity. It is immediately clear that there are three Π-quantities that can capture the form of a triangle. Each of the three can be be determined from each of the other two. By increasing the number of points, the form of, say, five points is determined from three Π-quantities. In general, the form of a point field of n points in the plane is defined by $(n-2)$ Π-quantities.

The Π-quantities are determined in geodetic practice by carrying out measurements of distances and directions. If more measurements are carried out than are needed to determine all the required Π-quantities, adjustment and thus also testing and quality description of the measurements are possible. Conventionally, the method of least squares is used to do that.

The Π-quantity is the basic *form quantity* in Euclidean two-dimensional space. Deformation analysis is concerned with form *change* and thus with changes in the Π-quantities.

Three-dimensional Euclidian space In three-dimensional Euclidean space, four is the smallest number of points, for which the form can be defined, provided that all points do not lie in the same two-dimensional subspace. The four points form a tetrahedron, as shown in figure 2.5(a). Five form elements are necessary to fix the form of the tetrahedron (van Mierlo, 1976, p. 5). This can be seen in figure 2.5(a): the form of the triangle ABD is defined by a Π-quantity: α_{ABD} and v_{ABD}. The same applies to the triangle DBC. If the dihedral angle θ of ABD relative to DBC is known, the form of the entire tetrahedron is fixed.

With five points (figure 2.5(b)) eight form elements are required. For the form of the triangle BCE one Π-quantity (two form elements) is needed. To determine the position of the triangle relative to the other triangles additionally a third form element is needed. With each additional point three additional form elements are required. The form of a point field of n points is thus defined by $5 + (n-4) * 3 = (3n-7)$ form elements.

In two-dimensional Euclidean space a point field may be linked by triangles (Delaunay triangulation), after which the form of each triangle is fixed by one Π-quantity. The

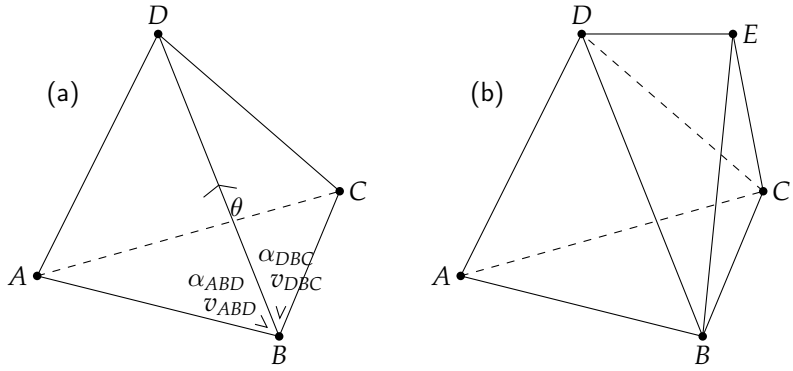

Figure 2.5: Four and five points; α: angle; v: length ratio; θ: dihedral angle.

entire point field is fixed by a collection of Π-quantities. In three-dimensional space the set-up is more complex. The Π-quantities of a subset of the triangles between the points is fixed. Then the dihedral angles between these triangles are fixed in order to obtain the total of $(3n - 7)$form elements.

The form elements can be grouped as follows.

- Start with one of the triangles, for instance triangle ABD in figure 2.5. Take the Π-quantity of this triangle.
- Take a connecting triangle and its Π-quantity, plus an extra angle in order to fix the position of the connecting triangle with respect to the start triangle (a dihedral angle). Take these three form elements into one quantity, which I call a *P-quantity*. The P-quantity contains one form element more than a Π-quantity[8]. We write the P-quantity as a vector:

$$P_{DBCA} = \begin{pmatrix} \alpha_{DBC} \\ v_{DBC} \\ \alpha_{ABC} \end{pmatrix}. \tag{2.4}$$

The addition of an extra point, for example point E in figure 2.5(b), requires one additional P-quantity in order to fix the form of the entire point field. A point field of n points is thus fixed by one Π-quantity and $(n - 3)$ P-quantities. The Π-quantity is the fundamental form quantity in two-dimensional Euclidean space. Likewise the P-quantity is fundamental in three-dimensional Euclidean space. If more measurements are carried out than are needed to determine all necessary form elements, adjustment of the measurements, and also testing and quality description are possible.

Deformation analysis is about analysing *changes* in the form elements (if necessary grouped together in form quantities).

[8] I call it "P-quantity", because the letter following the letter Π in the Greek alphabet is the letter P (capital letter rho). This letter is identical in shape to the Roman capital letter P, which is equal in pronunciation to Π.

One-dimensional Euclidian space After the examination of the situation in 2D and 3D, we now consider the 1D situation. Here it seems that the concepts of form and similarity are not available, because the concept of an angle is not defined. We have seen, however, that angles can be defined by ratio's of distances, and similarity is defined by equality of angles. Applying this to heights, the equivalent form quantity for the 1D situation is the *height difference ratio*. It is defined as the ratio of two height differences that are expressed in the same unit of length.

Using coordinates to describe size and form

In the previous sections it has been argued that deformation analysis is concerned with the study of changes in size and form quantities. It should be possible to determine the precision of these changes and to derive a testing method from it. It should also be possible to derive quantities that provide information about the deformations that are detectable by the testing method: the minimal detectable deformations. The geometric problem of deformation analysis is tackled by formulating it with Cartesian coordinates as an intermediary (figure 2.6).

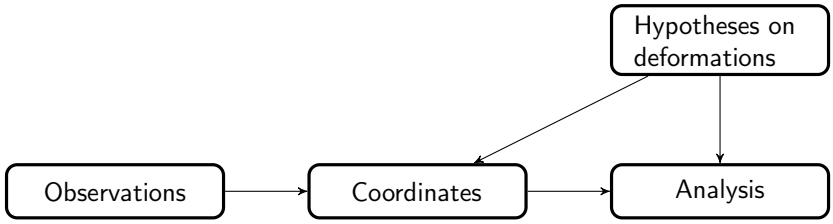

Figure 2.6: Coordinates as an intermediary for analysis.

Fixing the form and size of a point field with Cartesian coordinates (or other coordinates, such as geographic) has advantages. There is no need to make a spatial configuration of the point field, such as a triangulation or a collection of polyhedrons. The coordinates are linked directly to the points that have been measured. Disadvantages also stick to the method. Coordinates fix more than just the form and size. They contain additional information on the location and orientation of the coordinate system (or coordinate axes, and reference ellipsoid when geographical coordinates are used).

This study deals with deformation analysis by using coordinates. The premise remains, however, that deformation analysis is about analysing changes in size and form quantities. The tests and test quality descriptions should only depend on form changes and possibly size changes, and should be invariant for modification of other information, such as the location and orientation of the reference system.

Form, size and coordinates in geodesy

In the previous sections geodetic deformation analysis was linked to changes in form elements, taken together in form quantities. To define form quantities we need at

least 3 points (1D, 2D) or 4 points (3D). A deformation analysis, therefore, assumes a minimum of these numbers of points.

In geodesy it is customary, however, to analyse deformations by computing quantities for individual points. First coordinate differences between epochs are determined. Subsequently a *strain tensor* (strain matrix) for each point is computed (Welsch, 1982). The coordinates, coordinate differences, and also the strain tensor are not invariant for the location, orientation and scale of the reference system, in which the coordinates are expressed. Other quantities are needed that are invariant for the location, orientation and scale of the reference system, both in 2D, and in 3D (Vaníček et al., 2008). The quantities mentioned by Vaníček are tied to individual points. Also further studies into strain and strain rate parameters confine themselves to point-oriented quantities (Berber et al., 2009; Dermanis, 2010, 2011).

Because deformation analysis is about change of form and size, it makes sense to use quantities that are defined by several points together. This has been done in an analysis of deformations of the Tokai area, Japan (Xu et al., 2000). In the analysis a triangulation of 1883 and a trilateration of 1994 of the same network were compared. The network was divided in 53 blocks, each consisting of four network points. For each block strain quantities were computed that are invariant for the location, orientation and scale of the reference system. The analysis was done by visualising the strain quantities with ellipses for each block of four points, and showed deformations in an area near Shizuoka, not detected by previous conventional analyses (figure 2.7, the area near Shizuoka is indicated by a grey line) (Xu et al., 2000, p. 600).

This Japanese example shows the advantage of form and size analysis by invariant quantities of groups of points over the analysis per point.

Model with coordinates as parameters

Coordinates Two adjustment models (the *measurements model* and the *coordinates model*) are developed in this study, which use coordinates as intermediary quantities: the parameters in the adjustment model that describe form and size and that are to be estimated, are coordinates. This is reasonable, because otherwise an adjustment model has to be built that uses other parameters to describe form and size, such as the Π- and P-quantities. This would make it necessary to analyse each network to find the necessary amount of these quantities. Using coordinates makes such an analysis superfluous: each point has one, two or three coordinates, and that is all that is needed. Moreover, adjustment models that use coordinates as elements of the parameter vector, are well known in geodesy, and are based on a large body of knowledge. A further advantage of using coordinates is that formulating a deformation hypothesis using coordinates is much easier and transparant, than formulating such an hypothesis by using Π- and P-quantities. A disadvantage is, however, that S-systems and S-transformations have to be used to apply coordinates for the description of form and size.

S-system and transformations Geodetic deformation analysis is only concerned with the changes in size and form of monitored geo-objects, i.e. in changes in their size and

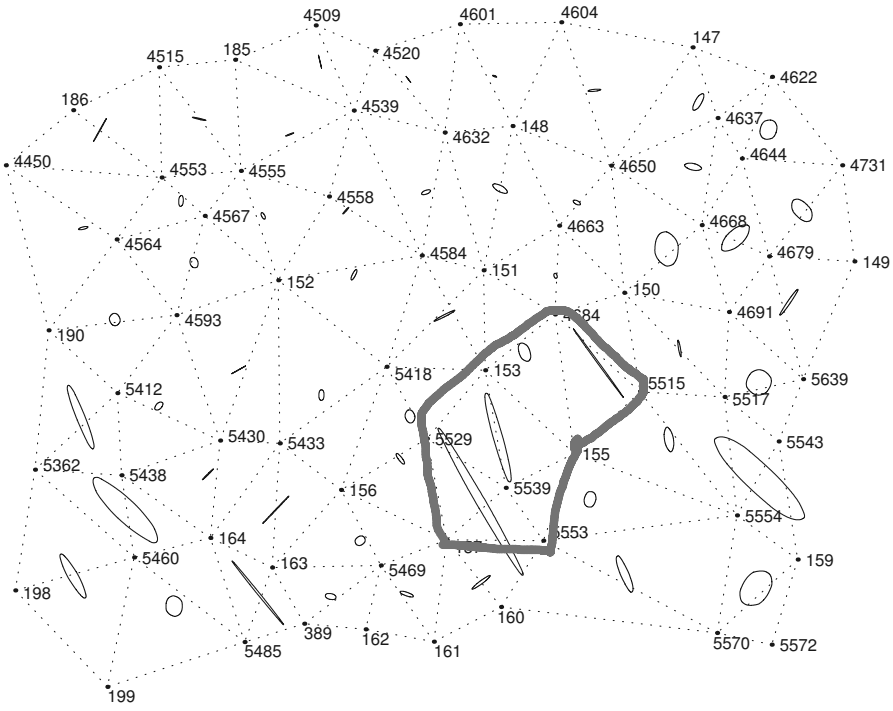

Figure 2.7: Strain quantities: invariant for reference system and visualised by ellipses. Picture from Xu et al. (2000).

form quantities. This means that the analysis has to be invariant for changes in position, orientation and scale of the reference system. Therefore, the degrees of freedom in the adjustment model concerning the position, orientation and scale, are to be fixed at some constant values. This can be done by fixing functions of the parameter vector at some constant values. The number of functions to be fixed is determined by the difference between the amount of coordinates and the minimal amount of size and form elements, needed to describe the size and form of the point field. For example, we saw before that in 3D the minimal amount of form elements to describe the form of a point field is $3n - 7$, with n the number of points. As there are $3n$ coordinates, the number of degrees of freedom to be fixed is 7. A reference system, where the position, orientation and scale are fixed in such a way, is called an *S-system* (Teunissen, 1985a, p. 41). Choosing functions of the parameter vector that have fixed values, by which the S-system is defined, is called here *fixing the S-system*. The elements of the parameter vector that are constrained by the functions constitute the *S-basis*.

Two different adjustment models are developed in this study. The *measurements model* (figure 2.2) takes geodetic observations in the observation vector and coordinates in the parameter vector. To fix the S-system, constraints on the parameters are formulated. Hence, the requirement for each of the two adjustment models is that it is extendable with constraints on the parameters.

The *coordinates model* (figure 2.3) takes coordinates as observations. The coordinates have been determined for each epoch separately. They can be defined in different reference systems, and in different S-systems[9]. One of the epochs is taken as reference epoch, and the coordinates of all epochs are transformed to the reference epoch by means of similarity or congruence transformations. This is done in the deformation analysis models by putting the transformation parameters into the parameter vector of the adjustment model. This has to be done no matter whether we have a 1D, 2D or 3D situation. The S-system of the reference epoch is fixed by adding constraints to the model, or by the singularity of the submatrix of the covariance matrix that relates to this epoch. The equivalence of using constraints and using a singular covariance matrix is treated later on (section 2.4.6). The requirement for each of the two adjustment models is, consequently, that it is extendable with constraints on the parameters, that it can handle singular covariance matrices, and that transformation parameters are included in the parameter vector.

In conventional geodetic deformation analysis the transformation to a common S-system is done separately and before the final adjustment (Setan and Singh, 2001; Heunecke et al., 2013, p. 492ff.). My choice in this study is to do it simultaneously, thereby making the procedure simpler, because no separate S-transformations are necessary to transform coordinates and their covariance matrices, and because transformations and deformation analysis are evaluated simultaneously.

To fix the S-system it is possible *not* to use constraints, but to leave the degrees of freedom that have to be fixed, unfixed. The result is that the parameter vector is not estimable, or, equivalently, that the model matrix is *rank deficient*. Methods exist to solve an adjustment model with a rank deficient model matrix, e.g. with generalised inverses (Rao and Mitra, 1971). It is worth noting, that the use of constraints to fix the S-system can be seen as a method to solve a rank deficient adjustment model. The requirement for the two adjustment models is, consequently, that an adjustment model with a rank deficient model matrix can be handled.

Conclusion

The two adjustment models to be developed in this study as part of the geodetic deformation analysis model, are required to be able to fix an S-system. This means that a rank deficient model matrix can be handled. As a way to do this, constraints on the parameters can be used. This is, therefore, a more specific requirement for the two adjustment models. For the model of figure 2.3 it is, additionally, necessary that transformation parameters are included in the parameter vector of the adjustment model. The transformation parameters are those of a similarity or congruence transformation, depending on the analysis of just the form (similarity transformation), or of form and size (congruence transformation). In the 1D situation the transformation is limited to just a shift, or, in rare exceptions, a shift and a scale change.

[9] Different S-systems (different S-bases) can be defined within one reference system.

2.4.3 Combine geodetic and physical model

A requirement of the two adjustment models that will be developed in the following chapters, is the ability to combine geodetic and physical models into one model. First the importance of modelling the *driving forces* for the deformations, and using them in the adjustment model by formulating constraints, is treated. Then the advantages of using *nonstochastic observations* to formulate constraints are mentioned. Finally it is shown how *reference and object points* should be handled in the proposed models.

Driving forces Knowledge of the driving forces of a deformation is valuable for a successful deformation analysis. Without it, only geometric changes in time can be viewed from the measurements, and it is more troublesome to separate the measurement noise and the deformation signal with zero mean from the deformation trend. Knowledge of the driving forces can be used to formulate a (forward) model, describing the deformations, and to test the model(s) subsequently (cf. Xu et al., 2000; Boyd et al., 2015). Such an approach is often chosen in geophysics, and the amount of models to be tested is generally limited.

If knowledge of the driving forces is not readily available, one can try to fit a deterministic mathematical function in time to a time series of coordinates. The time series is considered a random process and analysed accordingly (Papoulis, 1984; Chang and Hanssen, 2016; Ng et al., 2015).

It may be that it is difficult to determine which deterministic function should be used. This happens for example when it is assumed that there is no deformation, but this assumption is rejected by the measurements. A heuristic can be used in this case, where the congruence of the same point field in different epochs is compared to find a satisfactory description (Heunecke et al., 2013; Caspary, 2000; Nowel, 2016). The heuristic uses the stochastic information of the observations, and it tests hypotheses that assume deformations per single point and per pair of two epochs. If necessary, the heuristic successively removes points to arrive at an acceptable hypothesis. Then it tries to add previously rejected points.

A requirement for the two adjustment models is that information about the driving forces can be incorporated easily. This is realised by formulating constraints on the coordinates that are part of the parameter vector of the adjustment model. To make it possible that such constraints are easily and clearly formulated, each epoch is completely separated from each other epoch, in the sense that the same point has a different set of coordinates in each epoch. A constraint can in that case be formulated, for example, that a point has the same coordinate set in epoch i and in epoch j. Or, the constraint is that a point has different coordinate sets in epochs i, j and k, to be described by a linear subsidence with a certain subsidence parameter α. These constraints are part of the adjustment model. This means that after adjustment a least-squares estimate is obtained for, for example, the subsidence parameter α. It also means that the adjusted coordinates comply with the constraints. Testing a deformation hypothesis boils down to statistically testing whether it was justified to add the constraints to the adjustment model.

The use of constraints on coordinates to test geodetic deformation hypotheses is elaborated upon in chapters 6 and 7.

Nonstochastic observations Several methods exist to incorporate constraints into an adjustment model. In this study the choice has been made (motivated below) to use nonstochastic observations for this purpose. Nonstochastic observations are constants that are treated as if they were observations (also known as pseudo-observations). They have a standard deviation of zero and are not correlated to any other observation. There are several advantages to treat constraints as observations.

First it is possible to use the same, simple, procedure to test both stochastic ("normal") observations and nonstochastic ("pseudo") observations.

Secondly, the computation of minimal detectable biases (MDBs) for constraints is the same as the computation of MDBs for stochastic observations. In the context of deformation analysis the MDBs describe the minimal detectable deformations. They give information about the deformations that can be detected by testing the adjustment model under consideration. Because the MDBs can be computed *before* the measurements are performed, they enable to assess a deformation analysis network already in the design phase regarding its fitness for use.

Finally stochastic and nonstochastic observations can be tested simultaneously. For a successful deformation analysis, biases (errors) in the measurements should be distinguishable form deformations. Hence, it is advisable to design the deformation analysis in such a way that the testing procedure can distinguish biases in the measurements from deformations (e.g. by checking each epoch separately for measurement biases before testing several epochs for deformations). Nevertheless, there is always a chance that measurement biases are not discovered and deteriorate the deformation analysis. Being able to test hypotheses that include both measurement errors and deformations, is, then, an advantage.

Introducing observations with a standard deviation of zero and no correlation to other observations, necessarily means that the cofactor matrix of the observations is singular. It may be that also, because of other reasons, the partial cofactor matrix of the stochastic ("normal") observations is singular. The determination of the least-squares solution of the adjustment model has, therefore, to be capable to handle singular cofactor matrices.

In chapter 5 the use and processing of nonstochastic observations is treated extensively.

Reference and object points In geodetic deformation analysis it is customary to first identify stable reference points, to keep these fixed, and then to analyse the points of the object or part of the earth's surface under consideration (Nowel, 2016). The motivation for this two-step approach is that the coordinates of all epochs have to be defined in the same reference system (Caspary, 2000, p. 112). If, however, as discussed in section 2.4.2, we incorporate the transformations into the adjustment model, there is no need for such a two-step procedure. The reference points can be considered to be a similar set of points as the object points. They can be analysed simultaneously

with the object points. If some reference points are not stable, it will follow from the analysis in the same way as it follows for the unstable object points.

We can, therefore, require that the adjustment model enables an equivalent, simultaneous treatment of both reference and object points. An elaboration is given in chapter 3.

2.4.4 Full use of stochastic information

In the two adjustment models (measurements model and coordinates model) the elements of the observation vector do not have, in general, an equal standard deviation. In the coordinates model it is even the normal situation that the cofactor matrix of the coordinates, as they follow from the first phase, is a full, but singular matrix.

The requirement for the two adjustment models is, therefore, that they must be capable of handling full, singular cofactor matrices. The requirement also includes the possibility to handle correlation between observations (or coordinates) from different epochs. The necessity to be capable of handling singular cofactor matrices was already present, because of the nonstochastic observations, see the previous section 2.4.3. In all chapters from chapter 3 up to 7 the possibility to use full, singular cofactor matrices is a premise.

2.4.5 Testing and minimal detectable biases

My approach to *testing* deformation hypotheses is, first, to formulate constraints that describe a deformation, and then to test these constraints. This approach will be elaborated upon in chapters 5, 6 and 7. The *quality* of such tests is conveniently described by *minimal detectable deformations*. Their use to describe the quality of tests of deformation hypotheses, is introduced in chapter 3, and used in chapters 5 up to 7. Minimal detectable deformations are an important tool for standardisation.

In this section a short overview is given of the approach of testing deformation hypotheses, of the description of the test quality, and of its use for standardisation.

Testing deformation hypotheses The purpose of geodetic deformation analysis is formulating a valid deformation hypothesis that will stand the tests against other hypotheses. Hence a requirement for the two adjustment models is the availability of testing methods. Testing (stochastic) observations is common practice in geodetic applications (Teunissen, 2006). To test deformation hypotheses, a similar procedure is possible. The way to proceed is to first formulate a null hypothesis. This hypothesis is, for example, that deformation is absent, or that the deformation is a simple one (like a linear subsidence of one or more points). An alternative hypothesis is formulated as well. In case of a overall model test, the alternative hypothesis states only that the null hypothesis is not valid. If the overall model test is rejected, a more specific and intricate deformation hypothesis is formulated. The reasoning is that we want to describe reality with a model that is as simple as possible (law of parsimony, Occam's razor). Only when this simple model (the null hypothesis) does not satisfy, we extend it with extra parameters (the alternative hypothesis) to let the model better fit the observations, cf. Popper (2002, p. 248ff.).

A convenient way to test an alternative hypothesis is by defining the null hypothesis as the alternative hypothesis with additional constraints on the parameters. These constraints define the deformation hypothesis, for example that a point is stable. Testing the validity of the constraints gives information on the validity of the null hypothesis. It means that a reasonable way to formulate a geodetic deformation hypothesis is to consider each epoch as a separate geodetic network. To formulate the null hypothesis for each point, it is determined whether it is assumed to be stable, or to have some deformation pattern. This stability or deformation pattern of a point is made explicit by formulating constraints on the parameters of the adjustment model. If the null hypothesis is rejected by an overall model test, an alternative hypothesis is formulated. It can mean that a constraint on a point is removed, or that some parameter is added to the constraint (e.g. a parameter that describes a linear subsidence of a point).

Test quality description For testing deformation hypotheses, as for any testing procedure, it is important to know, what might stay undetected by the tests. The question is, what errors, biases and unspecified deformations are still present in the observations, although the test accepts the hypothesis. We are, therefore, interested in a good description of the test quality. Valuable quantities for this purpose are the minimal detectable biases (MDBs) (Baarda, 1968b; Teunissen, 2006). If these MDBs are computed for tests of specific deformation hypotheses, they indicate minimal detectable *deformations*.

MDBs can fulfil an important role for *standardisation* of geodetic deformation analysis. By standardising testing procedures according to the principles of this study, the MDBs that belong to specific testing procedures can be used as criteria in assignments, call for tenders, and similar documents, to which criteria the geodetic deformation analysis tasks have to comply.

In section 2.4.3 it was substantiated why constraints are introduced in the two adjustment models of this study as nonstochastic observations. For testing and test quality description the use of nonstochastic observations means that both the geodetic observations and the deformation pattern constraints are tested in the same way. Also the MDBs are computed in the same way.

2.4.6 Solution methods

The basic adjustment model of observation equations as described in section 2.2.2 has to be refined to fulfil all requirements for the two adjustment models, formulated in the previous sections. In this section the focus is on the availability of solutions for such a refined adjustment model. With "solutions" I refer to the least-squares solutions: the least-squares estimates of the parameters in the adjustment model, the adjusted observations, and the covariance matrices of both the estimated parameters and the adjusted observations. Additionally to the least-squares solutions, it is necessary for geodetic deformation analysis to have test results of testing alternative hypotheses against the null hypothesis by means of uniformly most powerful invariant tests.

From the requirements for the two adjustment models, listed in the previous sections, the most demanding are (i) the addition of constraints on the parameters, (ii) the related demand that a singular covariance matrix can be handled, and (iii) the demand that the model matrix can be rank deficient.

In the following paragraphs first the available adjustment methods are treated. Then special attention is paid to the dual model of condition equations and its capability of easily fulfilling the requirements for the two adjustment models. Subsequently the consequences of using nonstochastic observations are treated, and the close relation is hightlighted between constraints, nonstochastic observations and singular covariance matrices. Finally the methods are treated to test adjustment models with singular covariance matrices.

Adjustment The basic model to adjust observations is well-known, and its least-squares solution poses no problems and can be found in many textbooks, e.g. Tienstra (1956); Rao and Mitra (1971); Bjerhammar (1973); Vaníček and Krakiwsky (1986); Teunissen (2000); Koch (2013), and dates back to the publications of Legendre and Gauss in the beginning of the nineteenth century (Legendre, 1805; Gauss, 1809). Neither Legendre nor Gauss did yet consider the option of a full covariance matrix (first published by Aitken (1936)). A *singular* covariance matrix was treated first by Anderson (1948), according to Goldman and Zelen (1964, p. 165), who give themselves a more general solution for an adjustment model with singular covariance matrix, however under a certain restriction concerning the model matrix. This restriction is lifted by the solution given by Rao and Mitra (1971, p. 147ff.). That solution thus poses no special restrictions on the model matrix, and it may, therefore, be rank deficient.

Model of condition equations All the mentioned publications treat the model of observation equations (Gauss-Markov model) to arrive at a least-squares solution. The dual model is the model of condition equations. This model does not contain parameters, and can thus be arrived at by eliminating the parameters from the model of observations equations (for example by Gaussian elimination). It gives exactly the same least-squares solution, but does not formulate observation equations, but condition equations between the observations. Important advantages of this model are the fact that its algorithm does not need the inversion of the cofactor matrix, and that the size (dimensions) of its normal matrix is different from the size of the normal matrix of the model of observation equations. If the number of conditions is smaller than the number of parameters, the normal matrix is smaller. The model of condition equations was first described by Gauss in 1828, who specifically mentions higher geodesy (geodaesia sublimior) as the science, where this model finds frequent and satisfactory application (Gauss, 1828, p. 30).

The model of condition equations has a central position in the approach of the so-called Delft school, shaped by Tienstra and Baarda (van Daalen, 1985, p. 260). They viewed the model of condition equations as the formulation of the "laws of nature", to which observations will obey on average. They called this model the first standard problem, and the model of observation equations the second standard problem (Tienstra, 1956,

p. 140ff.). The first standard problem is primary, from which the second one follows (Staff LGR, 1982, p. 206).

In this study I show (in chapter 5) that for an example of deformation analysis taken from professional practice, the computation of the solution for the adjustment model of that analysis is fastest, if use is made of the model of condition equations. It is, therefore, remarkable that the use of the model of condition equations has found so little attention in geodetic literature. It can be seen from the textbooks, mentioned before. Rao and Mitra (1971) do not treat the model. Vaníček and Krakiwsky (1986, p. 179) only mention the "condition model", but do not mention its dual character to the model of observation equations. Koch (2013) treats the model of condition equations concisely, where the model of observation equations is treated extensively.

If the covariance matrix is singular, the same algorithm for the model of condition equations can be used as the one that is used to get a solution with a nonsingular covariance matrix. Because the covariance matrix does not need to be inverted, the computation poses no problems. The computed solution is a least-squares solution.

Many authors have put effort in the search for a solution for the model of *observation* equations with a singular covariance matrix (Anderson, 1948; Goldman and Zelen, 1964; Zyskind and Martin, 1969; Pringle and Rayner, 1971; Rao and Mitra, 1971). This solution requires two changes in the algorithm, as it is known for a nonsingular covariance matrix. These two changes are (i) what I call in this study the *amplification* of the covariance matrix (treated in chapter 5), and (ii) the use of generalised inverses instead of regular (Cayley) inverses (Rao and Mitra, 1971). In contrast, the algorithm for the model of condition equations does not need any changes. Moreover, it is straightforward to switch from the model of observation equations to the model of condition equations by a numerical procedure. The switch back, to get estimates for the parameters, can be performed by using a generalised inverse. This is shown in chapter 5. It seems, therefore, obvious to follow this path to arrive at a solution, but it has not been published before.

It is a requirement for the two adjustment models that they can handle a singular covariance matrix. The two models are derived as models of observation equations. Because the switch to and from the model of condition equations is easy, the solution of this last model with a singular covariance matrix is of interest. The derivation of the (unchanged) algorithm in case of a singular covariance matrix, is therefore given in this study. Note that this derivation is different from the one of Bjerhammar (1973).

Nonstochastic observations Constraints can be added to a model of observation equations. They constrain the estimated values of these parameters. A model of observation equations with constraints can be reduced to a model without constraints and with a scaled unit matrix as covariance matrix, by changing the observation vector and the model matrix (Rao and Mitra, 1971, p. 144). Subsequently, this reduced model can be solved by the standard method. A different way to rewrite the model of observation equations with constraints is by formulating the constraints as observations with a standard deviation of zero and without correlation to any other observation (Rao and Mitra, 1971, p. 148ff.). This means that an observation with a standard deviation of zero (and no correlation to other observations), called a nonstochastic observation

after Rao and Mitra (1971, p. 149), is equivalent to a constraint on the parameters. Let the covariance matrix of a model of observation equations have a rank deficiency d. It follows from the derivations of Rao and Mitra (1971) that such a model can be rewritten as a model with d constraints and a nonsingular covariance matrix.

The conclusion is that (i) constraints on parameters, (ii) rank deficiency of the covariance matrix, and (iii) nonstochastic observations, are three different ways to express the same phenomenon.

The approach chosen in this study is to formulate constraints as nonstochastic observations, see the rationale in section 2.4.3. The consequence is that the rows and columns in the covariance matrix that pertain to these nonstochastic observations, contain only zeros. The covariance matrix is therefore singular. Also the submatrix of the covariance matrix that pertains to the other, stochastic observations, can be singular. This can happen, for example, if the observations are coordinates that have resulted from a previous adjustment. Therefore, the computation of solutions in this study needs to assume that the covariance matrix is singular. A possible way to get a solution is the solution via the model of condition equations, shown above. Other methods, however, exist, and are treated in chapter 5.

We notice that methods to compute an exact, rigorous least-squares solution for a model of observation equations with a singular covariance matrix, exist already more than forty years (Rao and Mitra, 1971). Nevertheless, it happens that only approximate solutions are presented (Koch, 2013, section 3.2.7). Some publications state that only an approximate solution is possible (Lehmann and Neitzel, 2013; Shi et al., 2017). The assumption is that the model of observation equations has to be used and that infinitely large weights have to be approximated by finite weights. It is, however, possible to avoid the need for infinite weights, as will be shown in chapter 5.

Testing An important component of geodetic deformation analysis is testing the deformation hypotheses. In this study the choice has been made, see section 2.4.3, to formulate deformation hypotheses as constraints in the adjustment model. Testing a deformation hypothesis thus means testing the pertaining constraints. Because these constraints are formulated as nonstochastic observations, it is an obvious choice to test the constraints in exactly the same way as the other, stochastic observations. The methods and characteristics to test observations are well known (Teunissen, 2006), but they are valid only for adjustment models with nonsingular covariance matrices. The requirement for the two adjustment models is, therefore, that the equations are derived for the situation with singular covariance matrices. This is evaluated in a generic way in chapter 5.

For testing whether just one observation is biased, the conventional w-test has been developed (Baarda, 1968b, p. 15). Let us assume that the covariance matrix is a diagonal matrix, i.e. there is no correlation between the observations, and all observations have a finite standard deviation. For such a covariance matrix the test quantity of the conventional w-test (the w-quantity) of an observation is equal to the ratio between the estimated least-squares residual of that observation, and the standard deviation of that residual. This could give the impression that the estimated least-squares residuals are

essential for testing observations, and that, because for constraints these residuals are zero, constraints cannot be tested. In chapter 5, however, I will show that not the ordinary least-squares residuals are the essential quantities, but the *reciprocal* residuals[10]. They are introduced in chapter 3 and defined in a generic way in chapter 5 for use in the subsequent chapters. They are not zero for constraints, and neither are the conventional w-quantities of constraints, and thus constraints can be tested by conventional w-tests. Also more intricate hypotheses, concerning one or more constraints, or even constraints and stochastic observations simultaneously, can be tested.

Two adjustment models are developed in this study. The requirement is that these models should be capable of being tested for biases in both stochastic and nonstochastic observations in the same way (see section 2.4.3). It follows from the preceding that such a requirement can be fulfilled. Also the *test quality description* by means of minimal detectable biases can be realised for stochastic and nonstochastic observations in the same way.

2.4.7 Search for best hypothesis/model

Testing an adjustment model is done by acquiring the observations, subsequently adjusting them by the method of least-squares, and finally performing an *overall model test* (Teunissen, 2006, p. 93). If this test leads to rejection of the model, an alternative model has to be found. Comparing the null hypothesis and alternative hypotheses is done in pairs (Teunissen, 2006, p. 71). Each pair consists of the null hypothesis and one alternative hypothesis. In general, several, or even many, reasonable alternative hypotheses can be formulated. The problem to be addressed in this section, is how to select the *best alternative hypothesis*. Two situations, how the concept "best" can be interpreted, will be treated in the following, a simple and a more complicated situation.

Simple situation A relatively simple situation is treated first. Suppose that an alternative hypothesis has more parameters in its parameter vector than the null hypothesis. This is a reasonable assumption if we apply Occam's razor (see section 2.4.5). Suppose further that we want the testing procedure to use Uniformly Most Powerful Invariant tests (UMPI-tests) (Teunissen, 2006, p. 62ff.). Finally suppose that all observables have a normal distribution. In that case the test statistic for testing the alternative hypothesis against the null hypothesis is a quadratic form, which has a χ^2-distribution with q degrees of freedom (Teunissen, 2006, p. 78). q is the extra number of parameters that the alternative hypothesis has compared to the null hypothesis. Suppose then that the parameter vectors of all alternative hypotheses have the same number of extra parameters, i.e. q is equal for them. In that case, all test statistics to test each of them against the null hypothesis have the same χ^2-distribution, if the null hypothesis is valid. It is now possible to state that the alternative hypothesis that yields the largest test statistic is most probable, and to define the concept "best" in terms of probability.

[10] Reciprocal residuals can be viewed as the contravariant components relative to the reciprocal basis in a multidimensional Euclidian space of the difference between the observation vector and the expectation of the observation vector, where the metric is defined by a weight matrix, derived from the covariance matrix. Hence the adjective "reciprocal".

This situation can be found in the procedure of *data snooping* (Baarda, 1968b, p. 33). This procedure tries to find biases in observations by formulating as many alternative hypotheses, as there are observations. Each alternative hypothesis assumes one observation to be biased, and all other observations to be without bias. All these alternative hypotheses are mutually exclusive, i.e. any two of these hypotheses cannot be both valid. For these hypotheses $q = 1$. The procedure states that most probably thát observation is biased, for which the alternative hypothesis yields the largest test statistic.

A similar situation we find in geodetic deformation analysis for the *heuristic* described in both chapter 1 (section 1.3.3), and this chapter (section 2.4.3). The alternative hypothesis is that one point (in 1D, 2D or 3D) is biased (has moved because of deformation), and all others are not. For all points such a hypothesis is formulated and tested statistically. The alternative hypothesis, for which the test statistic is largest, is sustained, and the point concerned is assumed to have moved.

For both situations described above two problems can arise.

The first problem arises, when not one observation is biased, but more than one. Then, no one of the tested alternative hypotheses is valid. Both in data snooping and in the heuristic for geodetic deformation analysis, the following procedure is followed. The observation (data snooping) or point (heuristic) that gave the largest test statistic, is considered biased, or under influence of deformation. To detect other biased observations or points under influence of deformation, the observation or point is removed from the observation vector. The adjustment is repeated and also the testing of all observations/points. The observation/point that has now the largest test statistic is considered biased, or under influence of deformation, and removed to detect still more biases or deformations. This process is repeated until the overall model test of the remaining observations/points leads to acceptance of the null hypothesis.

It may be that observations/points have been removed mistakenly. Therefore, after the null hypothesis is not rejected any more, observations or points that have been removed are again added to the observation vector, and it is tested whether the null hypothesis is still not rejected. This process of removing and again adding is called a *forward* and *backward* search (Welsch et al. (2000, pp. 395-397), Niemeier (2008, pp. 446-450)).

The disadvantage of this procedure is that a valid alternative hypothesis, e.g. that two observations/points are biased, is not directly compared with other similar alternative hypotheses. There is, therefore, no statistical evidence that the accepted hypothesis that the heuristic has arrived at, is really better than other possible hypotheses.

The second problem of data snooping or the heuristic of geodetic deformation analysis is the *multiple testing problem*. If many similar hypotheses are tested, the chance of acceptance of an erroneous hypothesis increases (Lehmann, 2012). A solution is to adapt the critical values of the tests.

More complicated situation A more complicated situation arises, when q is different for different alternative hypotheses, i.e. they have a different number of additional parameters in relation to the null hypothesis. For geodetic deformation analysis this is a common situation. It arises, for example, if both one moved point and two moved

points are quite reasonable options. It arises as well, if, for example, a point is subsiding, and both a linear subsidence (one subsidence parameter) and an exponential decay of subsidence (described by, e.g., two parameters) are realistic options. In this situation the test statistics of different alternative hypotheses will not have the same distribution under the null hypothesis. Consequently, the critical values for testing the alternative hypotheses will be different, if the same level of signifance is taken. Because the distributions and the critical values are different, we cannot state that the largest of the computed values for the test statistics of both alternative hypotheses represents the most probable hypothesis.

Best hypothesis/model Both the simple situation and the complicated one show that finding the "best alternative hypothesis", when it is defined as the hypothesis, under which the vector of observations is most likely, is not straightforward. Here it is noticed here that finding the best alternative hypothesis is a *model selection problem* (cf. Lehmann and Lösler (2017)). To accomplish model selection, use can be made of a procedure such as data-snooping or the heuristic, described above for geodetic deformation analysis. They are examples of methods for *multiple comparison* (cf. Imparato, 2016, p. 42ff.) and *subset selection* (selection of a suitable set of model parameters, cf. Imparato, 2016, p. 189ff.).

From the given references it is clear that an optimal method for model selection in all situations is not available, as yet. Here, the choice has been made to use an *information criterion* for model selection. Examples of information criteria are the Akaike Information Criterion (Akaike, 1974) and the Bayesian Information Criterion (Schwarz, 1978). This study uses a different information criterion (treated more fully in chapter 6), based on the B-method of testing (cf. Baarda (1968b, p. 33)). The criterion has been introduced for the purpose of determining deformations in the gas fields in the Northern part of the Netherlands (de Heus et al. (1994b,a)). The criterion is the ratio between the value of the test statistic of a hypothesis and its critical value. This test ratio is computed for different hypotheses, for which the critical values are linked by means of the B-method of testing. It considers the best hypothesis the one that has the largest test ratio.

Whereas the method of test ratios, linked with the B-method of testing, has been justified by de Heus et al. (1994b,a), it has not been shown to perform better than methods that use different information criteria. Therefore, the justification is first expatiated upon in the next paragraph, and then the desirable characteristics of a method to find the best hypothesis, are mentioned in the subsequent paragraph.

At the centre of the justification is the linkage by B-method of testing, which means that a certain deformation (the so-called reference minimal detectable bias or deformation) will be detected by valid hypotheses of any dimension with the same power[11]. This power is a monotonic increasing function of the size of the deformation (Teunissen, 2006, p. 97). The function is well-behaved. Therefore, it may be expected that for deformations that are moderately larger or smaller, the power of valid hypotheses of different dimensions, although not the same, are still quite similar. The test ratios of valid hypotheses will increase with increasing deformations. Nonvalid hypotheses,

[11] The concepts "reference minimal detectable bias" and "valid model" are defined in chapter 6.

however, will in general lead to smaller test ratios (de Heus et al., 1994a). Moreover, the power is a monotonic decreasing function of the dimension of the test (Teunissen, 2006, p. 97). This will enhance the tendency to get larger test ratios for valid hypotheses with less parameters. Hence, it may be expected that a large test ratio is a strong indication of a valid hypothesis with a small amount of parameters.

It is often considered a desirable characteristic of a testing method that it is a uniformly most powerful test (Teunissen, 2006). If two alternative hypotheses are to be compared with each other, it is not to be expected that a simple function of the test ratios of both hypotheses can be used as the test statistic of an UMPI-test. Further research is, therefore, necessary into the optimality characteristics of the method with test ratios, coupled via the B-method of testing. It may be that other information criteria are better suited for specific deformation situations. An example will be encountered in chapter 5, in which both the criterion based on test ratios, and Akaike's Information Criterion are used in a verification example. The latter performs better in that situation.

Conclusion In many situations where geodetic deformation analysis is applied, it is not clear, which alternative hypotheses to consider and how to arrive at the best one. The described disadvantages of the method used by data snooping and the heuristic mentioned above, leads to the requirement for the two adjustment models that an information criterion is needed to discriminate between competing alternative hypotheses.

In chapter 3 the search for the best alternative hypothesis is elaborated upon by considering a large amount of alternative hypotheses, each considering one, two, or more points to be subject to deformation. Each alternative hypothesis is evaluated with the method that is based on the B-method of testing, combined with test ratios. In the other chapters the same information criterion is used to discriminate between deformation hypotheses. Additionally, in chapter 5 the results of this information criterion are compared to the results when the Akaike Information Criterion is used.

2.5 Findings for standardisation and communication

To finish this chapter, I draw the attention to the last subquestion of the research question (see section 1.4). It asks for the requirements that a geodetic deformation analysis model has to fulfil to be usable for *standardisation* and, as a derivative, for effective and efficient *communication*. The requirements are given below in the form of an operational definition of a statistically significant deformation and of key performance indicators. Both the definition and the key performance indicators make use of the analysis model and its essential elements, as they have been treated in this chapter.

For standardisation of geodetic deformation analysis, it is necessary to have a clear idea of what a deformation is. Therefore, an operational definition of a statistically significant deformation is needed. My formulation is as follows:

> "A *statistically significant deformation* is a deformation that fits a deformation hypothesis, which is described by constraints on the parameters of an adjustment model of geodetic observations (or of coordinates, derived from

> them); the deformation hypothesis is described by physically interpretable parameters (they have been derived from a physical model); the hypothesis has been tested by means of a statistical test, which has a known level of significance, if the hypothesis (either the null hypothesis, or the alternative hypothesis) is a valid one; and the deformation hypothesis has been shown to be the best among its competitors, where "best" is defined by an information criterion."

For a geodetic *stability* analysis ("no deformation" is assumed as starting point) it is generally necessary to know the minimal deformation that can be detected with the analysis method. The *minimal detectable bias* (MDB) is the tool that is needed to describe this quantity. The MDB is connected to a specific *deformation test* and to a specific *power* of the test to detect a deformation, so this information has to be provided if an MDB is mentioned. The MDBs needed are the MDBs of the constraints that define the deformation hypothesis. Because they give information about deformations, they may be called *minimal detectable deformations*. A certain *threshold* for the deformations is commonly given to the geodesist (a deformation larger than this threshold will lead to actions to stop the deformation): if this threshold is exceeded, the geodesist has to give a signal. The threshold is generally not formulated in statistical terms. The MDBs should be smaller than this threshold. In this way the threshold is given a statistical interpretation.

Also if not a stability analysis is asked for, but a *deformation* analysis, that gives information on size and form of the deformation, the minimal detectable bias is a fundamental concept in the communication between the geodesist and the client/non-expert. It provides a means to convey information on the size and form of the minimal detectable deviations from the assumed deformation hypothesis.

The reference power that is used to compute the MDB has to be chosen in accordance with the demands of the client. It may be that the probability of a type I error has to stay low to avoid possibly superfluous work, and the MDB should give the impression to be small enough (resulting in a low reference power, for example 70%), or it may be that the probability of a type II error has to stay low (resulting in a high reference power, for example 99%).

Besides the statistically significant deformation, the availability of *key performance indicators* is essential for the assessment of the quality of a geodetic deformation analysis. They can be used to compare the quality of different analyses, and thus to improve communication about the analyses. The key performance indicators should give information about the presence of a deformation, about its size and form (if a deformation is present), and about the test quality (i.e. how good is the test in detecting deformations).
The key performance indicators are realised in this study as follows. The *presence of a deformation* is determined with a statistical test of the constraints that define the deformation. Its size and form is determined with *least-squares estimates of the deformation parameters*. The *test quality*, finally, is determined with the MDBs.

The availability of the mentioned key performance indicators, based on the analysis model and its characteristic elements as described in this chapter, and the definition of a statistically significant deformation are the requirements that an analysis model has

to fulfil to make a standardised procedure for geodetic deformation analysis possible. With this standardised procedure a tool is available for improvement of communication about geodetic deformation analysis.

3

Analysis model in 2D (2 epochs)[1]

3.1 Introduction

Deformation analysis is done in many fields of human activity, e.g. the production of gas and oil, civil and construction engineering, water management, industrial installations, and because of threats of natural phenomena like land slides. An often applied method is to determine coordinates (one-, two- or three-dimensional) of points that are representative of the earth's surface or the object that is or may be subject to deformation. This object can be a civil engineering work, a building, a dam, an industrial storage tank, part of the earths' surface, etcetera. The object is represented by point coordinates. If the coordinates are acquired by geodetic means, the point field is called in this chapter a geodetic network. The deformation analysis looks at the changes of the coordinates in the course of time.

Modern technology offers many possibilities to produce coordinates, e.g. total station measurements, levelling, GPS, terrestrial photogrammetry and 3D laser scanning (terrestrial or air-borne). Also hydrological techniques, like networks of transponders on the seafloor, can be considered here.

The approach to compare coordinates is appropriate, as it is generally natural to describe deformations in their terms. Where in the past the amount of acquired coordinates was often limited to a few tens or hundreds, modern techniques can deliver streams of coordinates almost continuously, both in time and in space.

An approach towards the geometric analysis of the deformation of a point field is presented that differs notably from conventional methods as described in section 3.2.

[1] This chapter has been published before in Journal of Geodesy (Velsink, 2015b). To fit the publication into this study minor changes have been made. In contrast to the following chapters, in this chapter stochastic variables are not underlined, and vectors and matrices are not printed in bold face.

The proposed method The method, proposed in this chapter, is not based on an analysis of displacement vectors, but on testing the results of connection adjustments and can test for more deformed points simultaneously. It is invariant under changes of the chosen S-systems (the reference systems of both the coordinates and their covariance matrix), also known as the geodetic datums. The concept of an S-system is introduced by Baarda (1973) and generalised by Teunissen (1985a, p. 41).

The proposed method elaborates on the theory in Teunissen (2006); Teunissen et al. (1987a); Velsink (1998a) and applies it to the geometric analysis of a geodetic network that has been measured in two epochs. The least squares adjustment of the coordinates, resulting from the measurements at each epoch, and the detection, specification and quantification of existing deformations are treated. The application of the theory of testing multidimensional alternative hypotheses of Teunissen (2006, p. 71ff.), an extension of the testing of one-dimensional alternative hypotheses of Baarda (1968b), is shown. The need to perform S-transformations during the testing process is avoided by inserting transformation parameters into the adjustment model.

The proposed method follows an approach of formulating alternative hypotheses that allow for complex hypotheses. By testing large amounts of multidimensional tests it is possible to find the points that have been deformed most likely, without the need to have prior information about the deformations. The method is capable of giving a least squares estimates of the deformations. Moreover it can compute the minimal detectable deformations, i.e. the size of the deformations, specified by an alternative hypothesis, that can be found with a specified probability by testing the hypothesis. It is an important tool for designing a geodetic network for deformation analysis.

Overview of this chapter After a description of the conventional approaches in section 3.2 this chapter gives in section 3.3 a review of the adjustment model and its solution for the connection of two epochs of coordinates of a geodetic network. Section 3.4 describes the theory of formulating one- and multidimensional alternative hypotheses and the way to test them. The least squares estimation of the deformation and the concept of a minimal detectable deformation are treated. Section 3.5 shows how an alternative hypothesis can be specified that describes the deformation of several points: the deformation of a partial point field. Also the case of two or more partial point fields, each of which can have a different deformation, is treated. The connection adjustment of section 3.3 and the test strategy in section 3.5.2, based on the testing theory of section 3.4, lie at the heart of the method proposed in this chapter. An algorithm, designed to test the method, is described in section 3.6. The results of the application of this algorithm to a simulated network are given. They show that the method is capable of detecting deformed partial point fields and estimate the size of the deformation. The method of this chapter gives rise to a reconsideration of a few aspects of geodetic deformation analysis. They are considered in section 3.7. Section 3.8 gives the conclusions of this chapter.

3.2 Conventional approaches

If deformations are suspected in a geodetic network, but the exact points that have been affected are not known, the conventional analysis method is to determine coordinates of object points in two epochs in the same reference system (S-system) and to compute the displacement vectors from these coordinates in order to test them (Kamiński and Nowel, 2013; Setan and Singh, 2001; Welsch et al., 2000; Caspary, 2000). The covariance matrix of the displacement vectors is computed as well. Each object point is consecutively tested by determining the 95%-confidence ellipse in 2D or ellipsoid in 3D and determining if the displacement vector is outside this ellips(oid) (Koch, 1985; Cederholm, 2003). Such a test is in general not invariant under a change of S-system, as will be shown in section 3.7.3. A solution can be sought in performing an S-transformation towards such an S-system, in which the lengths of the displacement vectors are minimised. The idea is that deformed points will show most clearly large displacement vectors in such an S-system. A possible S-system is the inner constraint solution (Baarda, 1960; Pope, 1971). Chen (1983) and Caspary and Borutta (1987) use so-called "robust" methods, e.g. by minimising the L1-norm of the displacement vector lengths, to find an optimal S-system.

Welsch et al. (2000) describe also a different approach. They build an adjustment model, in which the observations of two epochs are combined and constraints are imposed on the point coordinates. The constraints state that coordinates of common points should coincide, if no deformation has occurred. The quadratic form of the weighted estimated least squares residuals that result from the adjustment, is tested. If this test fails, the quadratic form is analysed to determine which points cause the failure. To this end a decomposition of the quadratic form is performed by using Gauss, Cholesky or spectral decomposition.

Typical for all these methods is the search for deformed points one-by-one. Because more than one point can be subject to deformation, Welsch et al. (2000, pp. 395-397) and Niemeier (2008, pp. 446-450) describe a strategy of "backward" and "forward" searching for deformed points. "Backward" means removing points that are suspected of being deformed and "forward" means (again) adding points that were formerly removed. Koch (1985) describes a similar approach.

The method proposed in this chapter provides a test for a deformation pattern of several deformed points that is more powerful (in the sense of a "most powerful test" as defined by Teunissen (2006, p. 62)) than the mentioned conventional test strategies. It is invariant under a change of the S-systems, in which the point coordinates are defined. The need for such an invariance is mentioned in literature (Koch, 1985), but is not present in the described conventional methods.

3.3 Review of the connection adjustment of two epochs of a geodetic network

The method proposed in this chapter performs a connection adjustment of the adjustment results of two epochs of geodetic measurements. The deformation analysis is done

with the results of this connection adjustment. In this section the adjustment model of the connection adjustment is reviewed. The theory can also be found in Teunissen (1985a); Teunissen et al. (1987b); Velsink (1998a).

3.3.1 Three partial point fields and two reference systems

Consider a point field of several or even many points, where geodetic measurements have been made with the intention to determine one-, two- or three-dimensional coordinates of all points. It is assumed that a least squares adjustment of the measurements has been performed, resulting in a set of coordinates with its covariance matrix of the point field. At a later moment, the second epoch, the measurements are repeated, and also the adjustment, resulting again in coordinates and their covariance matrix. Let us take together all coordinates of epoch 1 in a vector a. In a two-dimensional plane, for instance, the column vector a of Cartesian coordinates is written as:

$$a = \begin{pmatrix} x_1^a, & y_1^a, & x_2^a, & y_2^a, & \dots & x_{n_a}^a, & y_{n_a}^a \end{pmatrix}^* \tag{3.1}$$

where x_1^a and y_1^a are the x- and y-coordinates of point 1, etc, and n_a is the number of points in vector a. The asterisk * indicates the transpose of the vector. In the same way all coordinates of epoch 2 are taken together in a vector b.

Now vector a is partitioned in two parts: part 1 contains all coordinates of points that have no coordinates in vector b (point field 1) and part 2 contains the coordinates of all connection points (point field 2), i.e. the coordinates of those points that also have coordinates in vector b. Vector b is divided in the same way in two subvectors b_2 and b_3. Subvector b_2 contains the coordinates of the connection points, subvector b_3 the coordinates of the points that have no coordinates in vector a (point field 3), see figure 3.1.

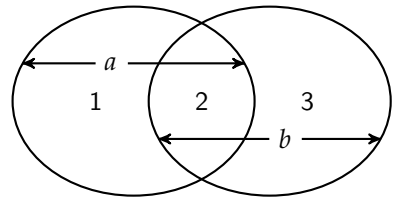

Figure 3.1: Point fields a and b and partial point fields 1, 2 and 3. Partial point field 2 contains the connection points.

Only the coordinates in a_2 and b_2, the coordinates of the connection points, give us information by which we can perform a connection adjustment. Influenced by the adjustment are all coordinates: a_1, a_2, b_2 and b_3.

Vectors a and b are supposed to be random vectors with a normal distribution, described by covariance matrices. The covariance matrix of a is indicated by D_a. It can be divided in a scalar variance factor σ^2 and a cofactor matrix Q_a as $D_a = \sigma^2 Q_a$. In the same way we have $D_b = \sigma^2 Q_b$. The variance factor is seen as a convenient way to get cofactor matrices with elements that are neither too large nor too small. The same variance factor is taken for a and b.

A vector c contains the coordinates of all points as unknown parameters, to be estimated in the least squares adjustment. Vector c is divided into three subvectors c_1, c_2 and c_3, in accordance with the three point fields 1, 2 and 3.

It is assumed that c is defined in the same reference system (S-system) as a. Assume that b is defined in a reference system (S-system) that differs from the reference system of a by a similarity transformation or a congruence transformation. In 2D a similarity transformation has four degrees of freedom: two translations in the directions of the x- and y-axis, a rotation and a change of scale. In 3D there are seven degrees of freedom: three translations, three rotations and a change of scale. In 1D there are two degrees of freedom: a translation and a change of scale. A congruence transformation has in all dimensions one degree of freedom less: the change of scale is missing.

To determine the coordinates in vector a geodetic measurements have been performed that yield more, less or an equal amount of degrees of freedom in the resulting coordinates as the measurements that resulted in vector b. It should be noted that only the information that is common to both vectors can influence the adjustment. The degrees of freedom of the transformation should encompass all degrees of freedom that both a and b have (Teunissen, 1985a, p. 70).

3.3.2 Linearised adjustment model and its solution

Non-linear model and its linearisation The functional relationship between a, b and c is given by the following equation:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 - e_{a_1} \\ a_2 - e_{a_2} \\ t(b_2 - e_{b_2}, f) \\ t(b_3 - e_{b_3}, f) \end{pmatrix} \tag{3.2}$$

where $e_{a_1}, e_{a_2}, e_{b_2}, e_{b_3}$ are random errors with a mathematical expectation of zero, f is the vector of transformation parameters from the reference system of b to that of a and $t(.)$ is the non-linear function describing the transformation.

Applying a non-linear adjustment to this model is generally not easy, so it is linearised.

To get simpler equations vector b is first loosely transformed to vector b' in such a way that the elements of b_2' approximate those of a_2:

$$\begin{pmatrix} b_2' \\ b_3' \end{pmatrix} = \begin{pmatrix} t'(b_2, f') \\ t'(b_3, f') \end{pmatrix} \tag{3.3}$$

where $b_2' \approx a_2$.

Eq. (3.2) is now written as:

$$\begin{pmatrix} c_1 \\ c_2 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} a_1 - e_{a_1} \\ a_2 - e_{a_2} \\ t(b_2' - e_{b_2'}, f) \\ t(b_3' - e_{b_3'}, f) \end{pmatrix} \tag{3.4}$$

where the transformation t and the transformation parameters f now cause only small changes in the coordinates.

Linearisation of function t yields the following linearised equation:

$$\begin{pmatrix} \Delta a_1 \\ \Delta a_2 \\ \Delta b_2' \\ \Delta b_3' \end{pmatrix} = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & I & 0 & -E_2 \\ 0 & 0 & I & -E_3 \end{pmatrix} \begin{pmatrix} \Delta c_1 \\ \Delta c_2 \\ \Delta c_3 \\ \Delta f \end{pmatrix} + \begin{pmatrix} e_{a_1} \\ e_{a_2} \\ e_{b_2'} \\ e_{b_3'} \end{pmatrix} \tag{3.5}$$

where I is the unit matrix, 0 is the matrix of zeros, E_2, E_3 are matrices to be looked closer at in the sequel, Δ is the difference of a vector and its vector of approximate values, and $e_{a_1}, \cdots, e_{b_3'}$ are random errors. For $a_1, a_2, b_2', \cdots, c_2, c_3, f$ see section 3.3.1.

As approximate values are taken:

for a_1 and c_1 : a_1;
for a_2, b_2' and c_2 : a_2;
for b_3' and c_3 : b_3';

For f approximate values are taken that leave the coordinates unchanged.

Matrix of coefficients of the transformation The matrices E_2 and E_3 are linearised coefficient matrices of the transformation from the reference system of b to that of a. As described in section 3.3.1 it is assumed that the transformation is a similarity or a congruence transformation, which means that E_2 and E_3 result from the linearisation of these transformations.

If for instance the transformation is a four parameter similarity transformation in a two-dimensional plane (change of scale, change of orientation, translation along the x-axis, translation along the y-axis), the matrices E_2 and E_3 have the following structure (Velsink, 1998a, p. 60):

$$E = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ x_i^0 & -y_i^0 & 1 & 0 \\ y_i^0 & x_i^0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \tag{3.6}$$

where x_i^0 and y_i^0 are approximate coordinates of point i. The first column of E pertains to the change of scale, the second to the change of orientation, the third and fourth to the translation along respectively the x- and y-axis. For each point i the first row concerns the x-coordinate and the second row the y-coordinate.

Reduced linearised adjustment model In the following the deltas will be dropped and also the primes of b', therefore e.g. Δa_1, $\Delta b_2'$ or Δc_3 will be indicated as a_1, b_2 and c_3 respectively. From eq. (3.5) the rows concerning vector a_1 and vector b_3 are omitted, because they give no redundant information and do not influence the adjustment results. Eq. (3.5) becomes:

$$\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ I & -E_2 \end{pmatrix} \begin{pmatrix} c_2 \\ f \end{pmatrix} + \begin{pmatrix} e_{a_2} \\ e_{b_2} \end{pmatrix} \tag{3.7}$$

Subtracting the second row from the first and putting $d = a_2 - b_2$ and its random error vector as e_d with its cofactor matrix Q_d gives:

$$d = E_2 f + e_d \tag{3.8}$$

$$Q_d = Q_{a_2} + Q_{b_2} \tag{3.9}$$

where Q_{a_2} and Q_{b_2} are the cofactor matrices of a_2 and b_2 respectively. The stochastic vectors a and b (and therefore also a_2 and b_2) are supposed to be stochastically not correlated mutually. Q_{a_2} and Q_{b_2} however can be full matrices.

The vector d contains $2n_{a_2}$ elements. The vector of transformation parameters f contains p elements, where p is 2, 4 or 7 in 1D, 2D and 3D respectively in case of a similarity transformation. For a congruence transformation p is 1, 3 or 6. The redundancy is therefore $2n_{a_2} - p$.

Adjustment The model consisting of equations (3.8) and (3.9) can be adjusted according to the method of least squares. The result is:

$$\hat{f} = (E_2^* Q_d^{-1} E_2)^{-1} E_2^* Q_d^{-1} d \tag{3.10}$$

$$\hat{d} = E_2 \hat{f} \tag{3.11}$$

$$\hat{e}_d = d - \hat{d} \tag{3.12}$$

where $\hat{f}$ contains the adjusted transformation parameters; $\hat{d}$ the adjusted values of d, and $\hat{e}_d$ the estimated values of e_d.

Because the model is a linearised one, iteration of the computation is necessary until a certain iteration criterion is met.

Adjusted coordinates for all partial point fields From the adjusted vector $\hat{d}$ the adjusted vectors $\hat{c}_1$, $\hat{c}_2$ and $\hat{c}_3$ can be calculated. The equations to use follow from the equations to estimate the random errors of free variates (Teunissen, 2000, p. 76, equation (14)) and can be written as (Velsink, 1998a, p. 79):

$$\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_2 \\ \hat{c}_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ b_2 \\ b_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ E_2 \\ E_3 \end{pmatrix} \hat{f} + \begin{pmatrix} -Q_{a_1,a_2} \\ -Q_{a_2} \\ Q_{b_2} \\ Q_{b3,b2} \end{pmatrix} Q_d^{-1} \hat{e}_d \tag{3.13}$$

As can be seen $\hat{c}_2$ can be calculated along two paths. In practical calculations one path is used. The other one may serve as a check.

3.3.3 The solution of the datum problem

In the previous section it is shown that a solution for vector c in eq. (3.2) can be found via the following steps:

1. Perform an approximate transformation on b in such a way that the transformed vector b_2' is almost the same as vector a_2.
2. Calculate the adjusted coordinates $\hat{c}_1$, $\hat{c}_2$ and $\hat{c}_3$ with equations (3.8) to (3.13).

In performing these steps a problem occurs, if the matrix Q_d of eq. (3.9) is singular and the inverse of Q_d in eq. (3.10) and (3.13) cannot be calculated. This may occur for instance, if both vectors a_2 and b_2 are defined in the same S-system and Q_{a_2} and Q_{b_2} are both singular matrices. The singularity of Q_d is related to the S-systems, in which a_2 and b_2 are defined and can therefore be solved by performing S-transformations, i.e. by changing the geodetic datums. Generally this is done by having the datum defined by some or all stable points, making it possible in that way to test the other points for deformation. This means that a change of datum is necessary if a datum point is detected as being influenced by deformation. This is done by S-transformations (van Mierlo, 1978) or generalised inverses (Koch, 1985). Another solution is however possible.

Matrix Q_d is calculated from eq. (3.9). In Teunissen et al. (1987b, p. 231) it is proven that the same solution in eq. (3.13) is arrived at, if Q_d of eq. (3.9) is replaced by the regular matrix $Q_{d'}$:

$$Q_{d'} = Q_d + E_2 Q_t E_2^* \tag{3.14}$$

where Q_t is any positive definite matrix with the right dimensions, for example the unit matrix. Changing matrix Q_d into the regular matrix $Q_{d'}$ is called the *regularisation* of Q_d. Almost the same equation gives Teunissen (1985a, eq. (3.2.14.a)), where the product $E_2 E_2^*$ is used.

Schaffrin (1975, p. 27) shows that for any adjustment problem, formulated with observation equations, any symmetric positive semi-definite generalised inverse of $Q_d + k^2 E_2 E_2^*$, with $k \neq 0$ an arbitrary real scalar, can be used as weight matrix of the observations to arrive at the least squares solution. Because in our case $Q_d + k^2 E_2 E_2^*$ is a regular matrix, the ordinary (Cayley) inverse can be used. From the proof of Schaffrin (1975, p. 28) it is clear that instead of $Q_d + k^2 E_2 E_2^*$ also $Q_{d'} = Q_d + E_2 Q_t E_2^*$ can be taken (the proof requires that $R(E_2) \subset R(Q_{d'})$; this is true, because $Q_{d'}$ has full rank and so $R(Q_{d'}) = \mathbb{R}^m$).

If Q_d has a rank defect, i.e. $\text{rank}(Q_d) < n_{a_2}$, with n_{a_2} the dimension of Q_d, the rank of $Q_{d'}$ is larger than that of Q_d, because $Q_{d'}$ has full rank n_{a_2}. The regularisation of Q_d can therefore be interpreted as moving the datum outside the point field under consideration and it is unnecessary to perform datum transformations: the datum problem is solved by the transformation that is implicit in the adjustment model (Teunissen, 1985a, p. 75).

3.4 Testing theory applied to deformation analysis

The way to determine the deformation of an object is to represent the object by a point field and to determine the changes in position, size and form of this point field. When looking at just one point it is tempting to calculate the differences in coordinates of this one point at two different epochs and to see these differences as the deformation. This is common practice in many deformation analyses, see for example the product

specification for performing and analysing deformation measurements of civil engineering works in the Netherlands (Rijkswaterstaat, 2014).

It is however better to perform a connection between the coordinate set at the first epoch and that of the second one. In section 3.3 the equations are given to fuse both coordinate sets into one vector c by means of the method of least squares. In that case it is assumed that both coordinate sets describe the same point field, without any deformation. If a deformation has taken place it should be tested by one or more appropriate statistical tests. If a statistical test leads to rejection, i.e. a deformation is present, the corresponding equations for estimating errors give estimates for the size of the deformation. These equations are given in section 3.4.2.

Before any deformation measurement is done it is possible to assess the smallest deformations that can be discovered with a certain probability by means of the designed deformation network. Section 3.4.3 treats the necessary statistical quantities.

3.4.1 Detection and specification of a deformation

Performing a statistical test on the connection of two coordinate sets and concluding that both coordinate sets describe the same point field differently and that a deformation has occurred, is the *detection* phase of the analysis. Closely related to the detection is the *specification* of the deformation, described by an alternative hypothesis.

3.4.1.1 Detection of blunders

Before any detection of deformations can be done, both a and b, and therefore d, have to be free of blunders. If checking for blunders has not been done well, the remaining blunders will lead to wrong conclusions regarding the deformations. A careful analysis of external reliability (Baarda (1968a, p. 68), van Mierlo (1978, p. 19)) of the models by which a and b were acquired, is necessary to assess the influence of possible remaining blunders.

3.4.1.2 Null and alternative hypothesis

The detection of a deformation can be done by performing χ^2-tests on the results of the least squares adjustment. The equations given here are based on Velsink (1998c), which in their turn are based on the first student edition of Teunissen (2006).

The adjustment model for the connection of both coordinate sets is given by equations (3.8) and (3.9). It is written as a linear model of observation equations, where d is the vector of observations and Q_d its cofactor matrix. The matrix E_2 is the matrix of coefficients, f the vector of unknowns and e_d the vector of random errors.

Solving the model of observation equations by means of the method of least squares gives equations (3.10), (3.11) and (3.12). Testing this solution by means of χ^2-tests is done (Teunissen, 2006, p. 78) by considering the model of equations (3.8) and (3.9) as

a null hypothesis and opposing it to an alternative hypothesis, defined by a specification matrix C and a vector of unknowns ∇ (pronounced as "nabla"):

$$\begin{array}{lcl|lcl} \multicolumn{3}{c|}{\text{Null hypothesis}} & \multicolumn{3}{c}{\text{Alternative hypothesis}} \\ d & = & E_2 f + e_d & d & = & E_2 f + C\nabla + e_d \\ Q_d & = & Q_{a_2} + Q_{b_2} & Q_d & = & Q_{a_2} + Q_{b_2} \end{array} \tag{3.15}$$

where the random errors are supposed to be normally distributed, described by the cofactor matrix Q_d, and to have an expected value of zero, i.e.: $E\{e_d\} = 0$.

Examples of the specification matrix C As an example take a two-dimensional plane point field, of which x- and y-coordinates have been determined in two epochs. Suppose that all points except points 4 and 5 stayed at the same position. Points 4 and 5 however have shifted away together in an equal, but unknown direction with an equal, but unknown amount. The vector ∇ now consists of two elements, a shift in the x-direction and a shift in the y-direction. The matrix C has two columns and twice as many rows as there are points:

$$\begin{pmatrix} \vdots \\ \nabla d_{x_4} \\ \nabla d_{y_4} \\ \nabla d_{x_5} \\ \nabla d_{y_5} \\ \vdots \end{pmatrix} = C\nabla = \begin{pmatrix} \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} \nabla_x \\ \nabla_y \end{pmatrix} \tag{3.16}$$

where ∇d_{x_i} and ∇d_{y_i} are the shifts in the x- and y-direction of point i and ∇_x and ∇_y the size of the shift in both directions. The elements of C that are represented by dots are all zero.

If point 4 and 5 have shifted independently from each other, the vector ∇ has four elements and we have:

$$\begin{pmatrix} \vdots \\ \nabla d_{x_4} \\ \nabla d_{y_4} \\ \nabla d_{x_5} \\ \nabla d_{y_5} \\ \vdots \end{pmatrix} = C\nabla = \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} \nabla_{x_4} \\ \nabla_{y_4} \\ \nabla_{x_5} \\ \nabla_{y_5} \end{pmatrix} \tag{3.17}$$

3.4.1.3 Testing the alternative hypothesis

To test the alternative hypothesis against the null hypothesis the test statistic V_q is calculated, where q indicates the number of elements of ∇ (and the rank of C) (Velsink, 1998c, p. (3)32):

$$V_q = \hat{r}^* C(C^* Q_{\hat{r}} C)^{-1} C^* \hat{r} \tag{3.18}$$

where the reciprocal random errors $\hat{r}$ and their cofactor matrix $Q_{\hat{r}}$ are calculated as:

$$\hat{r} = Q_d^{-1}\hat{e}_d \tag{3.19}$$

$$Q_{\hat{r}} = Q_d^{-1}Q_{\hat{e}_d}Q_d^{-1} \tag{3.20}$$

where $\hat{e}_d$ is calculated with eq. (3.12) and Q_d with eq. (3.9). The cofactor matrix $Q_{\hat{e}_d}$ is calculated as follows:

$$\begin{aligned} Q_{\hat{e}_d} &= Q_d - Q_{\hat{d}} \\ &= Q_d - E_2(E_2^* Q_d^{-1} E_2)^{-1} E_2^* \end{aligned} \tag{3.21}$$

with E_2 from equation (3.5) and $Q_{\hat{d}}$ the cofactor matrix of $\hat{d}$.

The test statistic V_q is statistically distributed according to the χ^2-distribution and can therefore be tested by comparing it with a critical value, calculated from a chosen level of significance. It is the uniformly most powerful invariant test (Teunissen, 2006, pp. 69, 78) to test the alternative hypothesis against the null hypothesis according to (3.15). We get the same result if V_q is divided by q and compared with a critical value, computed from the F-distribution. The test is as follows:

$$\text{If } F_{q,\infty} = \frac{V_q}{q\sigma^2} > F_{\text{critical}} \quad \text{reject the null hypothesis.} \tag{3.22}$$

with σ^2 the variance factor (variance of unit weight). It is assumed here that the variance factor is known and is not estimated from the adjustment results.

In section 3.5.2 this test is adapted to take account of the situation where alternative hypotheses of different dimension q are to be compared with each other.

3.4.1.4 S-system invariance of the test

Test (3.22) is invariant under changes of the S-systems relative to which vectors a and b are defined. It is proven as follows. In Teunissen et al. (1987b, pp. 231, 232) it is proven that $\hat{r}$ is invariant under changes of the S-systems, in which a_2 and b_2 are defined (hereafter called: $\hat{r}$ is S-system invariant). Because the proof in Teunissen et al. (1987b) is in Dutch, it is repeated here in English, adapted to the reasoning and formulation of this chapter. Subsequently it is proven that also $Q_{\hat{r}}$ is S-system invariant. Because $\hat{r}$ and $Q_{\hat{r}}$ are S-system invariant, also V_q of eq. (3.18) and therefore test (3.22) are S-system invariant.

Lemma 3.4.1. *$\hat{r}$ is S-system invariant*

Proof. Model (3.8) includes a transformation f that describes, according to sections 3.3.1 and 3.3.2, the transformation from the reference system of b to the reference system of a by a similarity or congruence transformation. Suppose that a or b or both are defined in different S-systems, i.e. a and b are replaced by:

$$a' = a + E_a \Delta f_a \tag{3.23}$$

$$b' = b + E_b \Delta f_b \tag{3.24}$$

where E_a and E_b are the linearised coefficient matrices of the similarity or congruence transformation as described in section 3.3.2 and Δf_a and Δf_b are the vectors of transformation parameters. Considering only the coordinates a_2 and b_2 of the connection points and switching to vector d of eq. (3.8) we have:

$$d' = a'_2 - b'_2 = d + E_2 \Delta f \tag{3.25}$$

with $\Delta f = \Delta f_a - \Delta f_b$. For the cofactor matrix $Q_{d'}$ of d' we get:

$$Q_{d'} = Q_d + Q_{d,f} E_2^* + E_2 Q_{f,d} + E_2 Q_f E_2^* \tag{3.26}$$

where $Q_{d,f}$ is the cofactor matrix between d and Δf, $Q_{f,d}$ its transpose and Q_f the cofactor matrix of Δf.

Model (3.8), (3.9) is formulated as a system of observation equations. It is now reformulated as the associated system of condition equations (that yields the same least squares solution). First the full rank $(2n_{a_2} \times (2n_{a_2} - p))$-matrix $E_2^{\perp}$ is introduced as the matrix defined by:

$$E_2^{\perp *} E_2 = 0 \tag{3.27}$$

Premultiplying eq. (3.8) on both sides with $E_2^{\perp *}$ we get

$$E_2^{\perp *} d = t \tag{3.28}$$

where $t = E_2^{\perp *} e_d$ is the vector of misclosures with $E\{t\} = 0$. Eq. (3.28) is a system of condition equations. Solving it by the method of least squares yields the following equation:

$$\hat{r} = E_2^{\perp} (E_2^{\perp *} Q_d E_2^{\perp})^{-1} E_2^{\perp *} d \tag{3.29}$$

where it should be noted that in solving the model of condition equations first $\hat{r}$ is calculated and then $\hat{e}_d$ with:

$$\hat{e}_d = Q_d \hat{r} \tag{3.30}$$

which means that $\hat{r}$ can be calculated without using the inverse of Q_d.

If d is replaced in eq. (3.29) by d' and Q_d by $Q_{d'}$ we get the changed $\hat{r}'$, caused by the transition of a and b to other S-systems:

$$\hat{r}' = E_2^{\perp} (E_2^{\perp *} Q_{d'} E_2^{\perp})^{-1} E_2^{\perp *} d' \tag{3.31}$$

Substituting equations (3.25) and (3.26) into eq. (3.31), we see that the second term of (3.25) and the last three terms of (3.26) lead to terms in (3.31) that are zero, because $E_2^* E_2^{\perp} = 0$ and also $E_2^{\perp *} E_2 = 0$ and therefore:

$$\hat{r}' = \hat{r} \tag{3.32}$$

This means that $\hat{r}$ remains unchanged and is S-system invariant. □

Lemma 3.4.2. *$Q_{\hat{r}}$ is S-system invariant*

Proof. The cofactor matrix $Q_{\hat{r}}$ is obtained by applying the law of propagation of cofactors to eq. (3.29):

$$Q_{\hat{r}} = E_2^{\perp}(E_2^{\perp *} Q_d E_2^{\perp})^{-1} E_2^{\perp *} \tag{3.33}$$

Switching to other S-systems for a and b means replacing Q_d in eq. (3.33) with $Q_{d'}$ of eq. (3.26). Elaborating this shows that the last three terms of eq. (3.26) lead to terms in eq. (3.33) that are zero, because $E_2^* E_2^{\perp} = 0$ and also $E_2^{\perp *} E_2 = 0$. Therefore $Q_{\hat{r}}$ remains unchanged and is S-system invariant. □

3.4.1.5 Invariance under regularisation of Q_d

In section 3.3.3 it is shown how regularisation of Q_d can solve the datum problem. Therefore it is important to prove the following lemma.
Lemma 3.4.3. *Test* (3.22) *is invariant under regularisation of* Q_d.

Proof. The regularisation of Q_d is done with eq. (3.14), which is repeated here:

$$Q_{d'} = Q_d + E_2 Q_t E_2^* \tag{3.34}$$

Substituting (3.34) into eq. (3.29) shows that the second term of (3.34) is postmultiplied with $E_2^{\perp}$, which yields a term of zero, because $E_2^* E_2^{\perp} = 0$. Therefore $\hat{r}$ is invariant under the replacement of Q_d with $Q_{d'}$.

Eq. (3.33) yields the cofactor matrix $Q_{\hat{r}}$. Replacing Q_d in this equation with $Q_{d'}$ doesn't change $Q_{\hat{r}}$ for the same reason it didn't change $\hat{r}$. Therefore also $Q_{\hat{r}}$ is invariant under the replacement of Q_d with $Q_{d'}$.

The conclusion is that both $\hat{r}$ and $Q_{\hat{r}}$ are invariant under regularisation of Q_d, which means that also V_q of eq. (3.18) and therefore test (3.22) are invariant under regularisation of Q_d. □

3.4.1.6 Types of alternative hypotheses

There are many alternative hypotheses, each defined by its own specification matrix C, that can be tested by the test statistic V_q. The number of elements of vector ∇ (and so the number of independent errors that can be tested for) is q and it is limited by the redundancy of the adjustment model (equations (3.8), (3.9)). In this adjustment model 2, 4 or 7 transformation parameters (in a 1-, 2- or 3-dimensional connection problem respectively) are determined by at least as many coordinate differences. The excess coordinate differences determine the redundancy. So the redundancy ρ is equal to the difference of the number of rows of the matrix E_2 and its number of columns.

The number of elements of vector ∇ is q. It cannot exceed the redundancy ρ, so:

$$1 \leq q \leq \rho \tag{3.35}$$

Overall model test If $q = \rho$ the test statistic V_q, now indicated as V_ρ, can be simplified to (Velsink, 1998c, p. (3)44):

$$V_\rho = \hat{e}_d^* Q_d^{-1} \hat{e}_d \tag{3.36}$$

Because the matrix C is eliminated from the equation, any matrix C of full rank and with ρ columns, of which the range space is complementary to the range space of E_2, will give the same test result when used in eq. (3.18). This test is therefore a general test on the correctness of the null hypothesis and is called the overall model test.

w-test If $q = 1$ the test statistic V_q, written as V_1 can be transformed into the well-known w test statistic (Baarda, 1968b, p. 14). The matrix C is now a vector, indicated by a lower-case letter c. Eq. (3.18) can in this case be written as:

$$V_1 = \frac{(c^* \hat{r})^2}{c^* Q_{\hat{r}} c} \tag{3.37}$$

The test statistic w is defined by:

$$w = \frac{c^* \hat{r}}{\sigma \sqrt{c^* Q_{\hat{r}} c}} \tag{3.38}$$

Therefore the relation between V_1 and w is:

$$w^2 = \sigma^2 V_1 \tag{3.39}$$

The test statistic w has a standard normal distribution and therefore its expectation is 0 and its standard deviation 1.

Data-snooping and point test If in performing the w-test the vector c in eq. (3.38) is defined as:

$$c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \tag{3.40}$$

the test is, if one point has changed in one coordinate direction only, while all other points have not changed position. Testing successively all coordinates with (3.22), (3.37) and (3.40) is called data-snooping (Baarda, 1968b, p. 30). In a 1D-connection of point fields this is a realistic test. In a 2D- or 3D-connection it can be used to check for example for an input error. To test for a deformation in 2D and 3D however, it is less useful, because a deformation affects in general two or three coordinates of one

point simultaneously. A more logical test is then a test with $q = 2$ or $q = 3$ and a C-matrix that looks like, for example in 3D, with the dots indicating zeros:

$$C = \begin{pmatrix} \vdots & \vdots & \vdots \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \end{pmatrix} \tag{3.41}$$

Such a test is called a point test.

3.4.1.7 Tests with $1 < q < \rho$

The overall model test with V_ρ of eq. (3.36) is called a ρ-dimensional test and the w-test with the test statistic of eq. (3.38), where $q = 1$, is called a 1-dimensional test. Between these two extremes of $q = 1$ and $q = \rho$ many tests can be devised where $1 < q < \rho$. The point tests of the previous section 3.4.1.6, where $q = 2$ or $q = 3$, are just one example. The case, where a deformation pattern of many points is tested by specifying an appropriate C-matrix with q a value between 1 and ρ will be treated in section 3.5.

3.4.2 Quantification of a deformation (its least squares estimation)

As shown in the previous subsections many alternative hypotheses can be formulated. In section 3.6 it will be shown that it is worthwhile to automate the process and to test for thousands of alternative hypotheses to find the best one in the case that the overall model test of section 3.4.1.6 has rejected the null hypothesis.

If the best alternative hypothesis has been found, the associated C-matrix is known and it is possible to estimate the size of the deformation by estimating $\hat{\nabla}$:

$$\hat{\nabla} = (C^* Q_{\hat{r}} C)^{-1} C^* \hat{r} \tag{3.42}$$

The estimated deformation in each coordinate is in e.g. 2D:

$$\begin{pmatrix} \hat{\nabla} d_{x_1} \\ \hat{\nabla} d_{y_1} \\ \vdots \\ \hat{\nabla} d_{x_{n_{a_2}}} \\ \hat{\nabla} d_{y_{n_{a_2}}} \end{pmatrix} = C\hat{\nabla} \tag{3.43}$$

It is worth noting that this estimation of deformations for each coordinate direction of a point is in general different from the coordinate differences between the first and second epoch of that point: the least squares estimator of the deformation is a best

linear unbiased estimator (Teunissen, 2000) and is in that sense better than the in practice often applied method of assessing a deformation by computing differences and at the most presenting a graphical representation of the differences. Much however depends upon the correctness of the alternative hypothesis. Finding the right alternative hypothesis is the subject of sections 3.5 and 3.6.

3.4.3 Minimal Detectable Deformation

Designing a geodetic network for the purpose of deformation analysis involves consideration of the type and size of deformation that can be detected by the analysis. Deformation measurements are subject to stochasticity that is caused by the measuring instruments, the observer (human or not) and the idealisation precision (the precision by which an object in reality is represented in a mathematical model, i.e. the precision of the "linking-up" of the model (Baarda, 1967, p. 6)). It is difficult to distinguish a deformation from this stochastical variation of measurement values. The way to go is shown by Baarda (1968b) and extended by Teunissen (2000).

Using the analysis procedure as described in the preceding sections it is natural to use the concept of the minimal detectable bias (Teunissen, 2006, p. 102) to evaluate the type and size of deformation that can be detected by the analysis. In the context of deformation analysis we will talk about the minimal detectable deformation ∇_0, defined by:

$$\sigma^2 \lambda_0 = \nabla_0^* C^* Q_{\hat{r}} C \nabla_0 \tag{3.44}$$

where σ^2 is the variance factor (variance of unit weight), λ_0 is the reference noncentrality parameter (its computation is explained below), and C and $Q_{\hat{r}}$ are as defined above.

The reference noncentrality parameter λ_0 is dependent on the power γ of test (3.22), the size α of this test and the dimension q, symbolically written as:

$$\lambda_0 = \lambda(\gamma, \alpha, q) \tag{3.45}$$

If a 1-dimensional test is performed ($q = 1$) and the size is chosen as $\alpha = 0.1\%$ and the power as $\gamma = 80\%$, a value of $\lambda_0 = 17.075$ results.

In general it is desirable that different tests (3.22) with different dimensions q have the same power γ (indicated as γ_0) and the same noncentrality parameter λ_0 (the reference noncentrality parameter). That means that for different dimensions different sizes α are used. Usually the value for $q = 1$, indicated by α_0, is fixed at a certain value, often $\alpha_0 = 0.1\%$. This procedure is called the B-method of testing (Baarda, 1968b, p. 34), see also Velsink (1998c, p. (3)54) and Niemeier (2008, p. 303).

The minimal detectable deformation ∇_0 of eq. (3.44) describes the deformation that, if present, will be detected by test (3.22) with a probability of γ_0 (e.g. 80%), if the critical value of the test is computed with the chosen reference noncentrality parameter λ_0 (e.g. 17.075), using the B-method of testing.
Deformations that are smaller will be detected with a smaller probability than γ_0.

The minimal detectable deformation ∇_0 (MDD) is defined in eq. (3.44). In a 1-dimensional test ($q = 1$) it is a scalar, that can be readily derived from eq. (3.44), except for its sign:

$$|\nabla_0| = \sigma \sqrt{\frac{\lambda_0}{c^* Q_{\hat{r}} c}} \tag{3.46}$$

where c is written with a lower-case letter, because it is a vector in case $q = 1$.

If $q = 2$ eq. (3.44) describes an ellipse, if $q = 3$ an ellipsoid and for $q > 3$ a hyperellipsoid.

The principal axes of the hyperellipsoid are determined by computing the eigenvectors of $C^* Q_{\hat{r}} C$. The lengths of ∇_0 in the direction of these axes are determined by the eigenvalues of the said matrix. The following equation gives the relation (Teunissen, 2006, p. 105):

$$\nabla_{0_k} = \sigma \sqrt{\frac{\lambda_0}{\lambda_k}} d_k \ , \quad k = 1, 2, \ldots, q \tag{3.47}$$

where ∇_{0_k} is the vector that gives the MDD in the direction of the k-th eigenvector, σ is the square root of the variance factor, λ_0 is the reference noncentrality parameter, λ_k is the k-th eigenvalue of $C^* Q_{\hat{r}} C$, and d_k is the k-th normalised eigenvector of $C^* Q_{\hat{r}} C$.

The hyperellipsoid of eq. (3.44) gives information about the whole point field, not just about one or a few points. The equation to give information about individual coordinates is:

$$\begin{pmatrix} \nabla_0 d_{x_1} \\ \nabla_0 d_{y_1} \\ \vdots \\ \nabla_0 d_{x_{n_{a_2}}} \\ \nabla_0 d_{y_{n_{a_2}}} \end{pmatrix} = C \nabla_0 \tag{3.48}$$

In the case of data snooping the matrix C is a vector with only one element different from zero and eq. (3.46) can be used. The MDD can then be uniquely attributed to one coordinate. For higher dimensional alternative hypotheses the quantity ∇_0 in eq. (3.48) can be chosen to be ∇_{0_k} of eq. (3.47) belonging to the largest eigenvalue of $C^* Q_{\hat{r}} C$.

Designing a geodetic network for deformation analysis is strongly supported by computing the MDD's for those alternative hypotheses that describe the deformation situations that might occur. An advantage of the equations given is that the MDD's can be computed before the first epoch is measured, i.e. in the design stage of the deformation network.

3.5 Testing the deformation of partial point fields

3.5.1 Data snooping strategy

If coordinates of a point field have been determined at two or more epochs a deformation analysis can be performed. What is an optimal strategy to perform such an analysis? When no specific indication is present about the points that have been deformed, it seems appropriate to start with data snooping, if the overall model test rejects the null hypothesis. That means that each coordinate is tested by means of the test statistic w of eq. (3.38) with the c-vector from eq. (3.40). The idea is that by checking each coordinate successively for a deformation one will effectively find all deformed coordinates. In the following these test statistics w are called *conventional w-quantities*.

Baarda (1968b) introduced the idea of data snooping and he warned immediately for the limitations of it. He writes on page 12: "These "possible" model errors are now described by a number of alternative hypotheses, *which in principle do not have to occur simultaneously*" (emphasis from Baarda). The alternative hypothesis that checks coordinate i tests if that coordinate is deformed and *all other coordinates are not*. This alternative hypothesis cannot be true simultaneously with the alternative hypothesis that coordinate $j \neq i$ is deformed and all other coordinates are not.

The conventional strategy of data snooping is to compute the conventional w-quantities and consider the coordinate with the largest w-quantity deformed, if its absolute value is larger than the critical value of the normal distribution (3.29 if $\alpha = 0.1\%$). After removing this coordinate from the data set and repeating the adjustment and testing the then largest w-quantity, its coordinate is removed if the critical value is exceeded. This process is repeated until no critical value is exceeded anymore. This strategy does not provide the uniformly most powerful invariant test as defined in Teunissen (2006, p. 62), if the deformation concerns more than one coordinate. The uniformly most powerful invariant test is test (3.22) with a matrix C that describes all coordinates affected by the deformation.

In a 2D or 3D point field testing for deformations is done more logically not by testing individual coordinates, but individual points. The alternative hypothesis is in this case, that two or three coordinates (for 2D and 3D respectively) are deformed and all other coordinates are not. Here we call a strategy that tests every point successively with such an alternative hypothesis *point data snooping*.

Point data snooping does not provide the uniformly most powerful invariant test if more than one point is subject to deformation. Such a test is test (3.22) with a matrix C that describes all points affected by the deformation.

3.5.2 Formulating alternative hypotheses to test for a deformation

Usually a deformation affects not just one point, but several points of a geodetic network. Testing for the occurrence of such a deformation can be done by choosing an appropriate C-matrix and using test (3.22). But how to know what an appropriate C-matrix is? If information is available about the processes that underlie the deformation, these

processes may dictate the C-matrix to use. In many cases however the underlying processes are not known well enough or even not known at all. Then one can simply try several different C-matrices and perform test (3.22). The C-matrix that delivers the largest value for the test statistic indicates the best alternative hypothesis.

Dimension q differs for different C-matrices A problem arises if the different C-matrices that are tested, have different dimensions, i.e. q is different. This means that test (3.22) is executed with different sizes α (if the B-method of testing is used). As a consequence different critical values are used. In that case the fact that the test statistic V_q for a certain alternative hypothesis has a larger value than any other alternative hypothesis doesn't mean that the alternative hypothesis concerned is the best one. De Heus et al. (1994b) proposed to use in such a situation the ratio of the test statistic with the critical value, while fixing the power γ at 50% (not 80% as before). The resulting new test is:

$$\text{If } \frac{F_{q,\infty}}{F_{\text{critical}}} = \frac{V_q}{q\sigma^2 F_{\text{critical}}} > 1 \quad \text{reject the null hypothesis.} \tag{3.49}$$

Test strategy The test strategy now becomes: try several different C-matrices and perform test (3.49). The C-matrix that gives the largest ratio that is larger than one indicates the best alternative hypothesis, assuming that no C-matrix exists that is not tested for and that would give an even larger ratio.

Because test (3.22) is S-system invariant also test (3.49) and the above test strategy are S-system invariant.

A justification for a test strategy using test ratios is given by de Heus et al. (1994a), de Heus et al. (1994b, appendix).

3.5.3 Several differently deformed partial point fields

In a geodetic network, consisting of many points, designed to detect deformations, it is plausible that one point may deform, but also that two, three, or even more points may deform. These points can deform in exactly the same way, for example shifting away with the same amount in the same direction. It is also possible that some deform in the same way, but others differently. To illustrate these possibilities three examples in 2D of C-matrices are given.

The first C-matrix says that the first two points have been deformed with the same amount in the same direction. The second C-matrix says also that the first two points have been deformed, but the second point has a deformation that differs from that of the first point. The third C-matrix says the size of the deformation of the second point is s times the size of the deformation of the first point. s may be, for example, the ratio of the distance of the second point to a certain fixed point and the distance of the first point to that point, reflecting the situation that the deformation of a point may depend on the distance to a certain fixed point.

The vertical dots indicate a not specified amount of rows, consisting merely of zeros.

$$C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \end{pmatrix} \quad C_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad C_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ s & 0 \\ 0 & s \\ \vdots & \vdots \end{pmatrix} \tag{3.50}$$

By using such specification matrices C one can test the deformation of a subset of points or even of several subsets of points. It is therefore a multi-point testing method, contrary to the often used point-by-point methods, described in section 3.2.

How points are deformed, depends on the physical processes that underlie the deformations. These processes may cause a simple shifting of (subsets of) points that can be described by simple C-matrices like the ones above. It is however also possible that more complex deformations result from the physical processes. If the parabolic form of a water storage dam is considered, its deformations might follow a pattern that can be described by a mathematical function. Linearisation of this function gives the C-matrix that can be used for testing. If a subset of points is subject to an elastic deformation, the points of this subset undergo an affine transformation. Linearisation of the equations of the affine transformation results in the following matrix C:

$$C = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline x_i^0 & y_i^0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_i^0 & y_i^0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{i+j}^0 & y_{i+j}^0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_{i+j}^0 & y_{i+j}^0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \tag{3.51}$$

with i until $i+j$ the points of the deformed subset.

The six columns of this matrix C correspond to the six parameters of the affine transformation that the points are subject to.

3.5.4 B-method of testing

It is assumed in the testing strategy that the B-method of testing is applied (Baarda, 1968b, p. 33). This means that first the overall model test is performed. If the null hypothesis is accepted, the testing stops. Only if the null hypothosesis is rejected

the search for the best fitting alternative hypothesis is started, beginning with data snooping. The overall model test and the one- and multidimensional tests are linked by the requirement that all tests have the same power if a deformation of the size of the minimal detectable deformation is present. The consequence is that the significance levels of the tests differ for different dimensions. The significance level for a certain dimension is chosen, from which the significance levels for all other dimensions are derived. Starting from the one-dimensional test or starting from the ρ-dimensional overall model test are two options given by Baarda (1968b, p. 25). Starting from the significance level of the one-dimensional test (α_0) may result in large values for the significance level α_ρ of the overall model test (over 50%), if ρ is large. Therefore for the validation in section 3.6.2 it is chosen to fix α_ρ and to derive α_0 from it. This may result in very small values for α_0.

3.6 Searching the best alternative hypothesis

3.6.1 Automating the process

The assumption in the test strategy of section 3.5.2 that no accidentally not tested C-matrix exists with a larger ratio, is tricky. The amount of possible alternative hypotheses, and thus of possible C-matrices, can be infinitely large. In practice not all are relevant, but extremely many can be plausible. It is possible to automate the process of formulating alternative hypotheses (and thus of designing C-matrices) and subsequently testing these hypotheses. Testing one alternative hypothesis can be done very fast, if a computer is used. The number of numerical computations depends on the size of matrix $C^* Q_{\hat{r}} C$ in eq. (3.18). The dimension of this matrix is equal to the amount of columns of matrix C, so equal to q. Computing the test statistic for small-dimensional tests can therefore be done extremely fast. Computing thousands or ten thousands of test statistics is just a matter of seconds.

An algorithm for the adjustment model of section 3.3 and its testing with q-dimensional tests, as described in sections 3.4 and 3.5, has been programmed for use in a computer.

The automated testing of large amounts of alternative hypotheses has been incorporated into the algorithm. The algorithm starts with testing all hypotheses that just one point has been deformed and all others haven't. Then it takes every combination of two points and tests if those two points have been deformed in the same way and that all other points have not been deformed. It also tests if the two points have been deformed in a different way.

Subsequently it takes every combination of three points and performs analogous tests. It goes on with testing combinations of four points, five points, etcetera.

Here the algorithm as for now stops in taking combinations of points to be tested. It could be possible to test, for example, if two points have been deformed in the same way and a certain third point in a different way. This the algorithm doesn't do, as so many other combinations are not tested.

Because point fields can have very large amounts of points, the amount of combinations can very rapidly attain incredibly large numbers, so the algorithm limits the total amount of alternative hypotheses to be calculated.

The algorithm calculates test ratios (see test (3.49)) for every alternative hypothesis and lists the 10 alternative hypotheses with the largest ratios.

Testing the algorithm showed that the deformation patterns of the 10 alternative hypotheses give more information about the deformations present than just data snooping or point data snooping can provide.

3.6.2 Validation of the method

To validate the proposed automated process a numerical test is presented. In section 13.5 of Niemeier (2008) a simulated network (figure 3.2), used before in a FIG working group (Welsch, 1983), is given. It is about a 2D geodetic network, where distances and directions have been measured in two epochs.

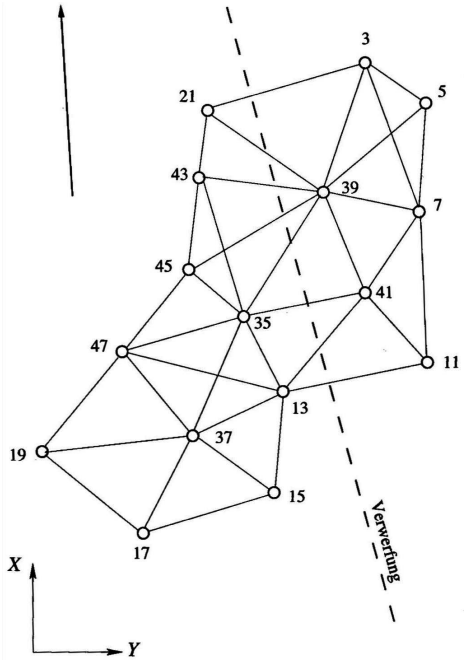

Figure 3.2: The simulated network from Niemeier (2008) ("Verwerfung" means "fault").

Niemeier's analysis Niemeier (2008) analyses the network in two ways. First he assumes that it is unknown if and where points are deformed and performs a one-step analysis, in which the forward and backward strategy, mentioned in the introduction, is performed. This strategy identifies significantly shifted points. Secondly a two-step strategy is described, in which the fault zone is assumed to be known. The points that

are supposed to be stable (reference points) are tested with an overall model test and a subsequent localisation of non-stable points is done. In the second step it is tested point-by-point, if a point should be attributed to the reference points. The results are given in table 3.1 together with the simulated shifts (deformations). Point 7 doesn't appear in the table, because it was not measured in two epochs and is therefore no connection point.

As can be seen in table 3.1 the one-step strategy doesn't succeed in giving satisfying results. The two-step strategy gives better results, helped, of course, by its knowing which points are deformed most probably.

	simul.		1-step		2-step		section 6.1	
Pointnr.	dx	dy	dx	dy	dx	dy	dx	dy
3	20	12	0	0	21.8	20.7	23.7	8.0
5	20	12	0	0	22.8	21.0	23.7	8.0
11	20	12	18.8	-8.3	23.3	5.8	23.7	8.0
21	0	0	-21.9	-5.2	0	0	0	0
35	0	0	-12.2	-4.4	0	0	0	0
39	20	12	0	0	22.4	9.2	23.7	8.0
41	20	12	10.2	-5.5	20.7	6.5	23.7	8.0
43	0	0	-19.4	-6.3	0	0	0	0
45	0	0	-21.4	-0.4	0	0	0	0
47	0	0	-17.2	2.8	0	0	0	0

Table 3.1: Simulated shifts in cm and results of one-step strategy, two-step strategy (from Niemeier (2008)) and algorithm of section 3.6.1.

Analysis with the algorithm To test the algorithm, described in section 3.6.1, first coordinates and their covariance matrices were computed for each epoch. The distance and direction observations and their standard deviations as given in Niemeier (2008) were used, except for the incorrect standard deviation of 1.5 cm for the distance observations, for which the correct standard deviation of 10.5 cm (personal communication of Prof. Niemeier) was used. The adjustment of each epoch was done as a free network adjustment. Both adjustments were tested with an overall model test and accepted. For the connection adjustment the model of section 3.3.2 was used with a similarity transformation. The full covariance matrices of the coordinates as computed in both epochs were used in the connection adjustment. The overall model test (eq. (3.36)) was not accepted with a square root of $F_{b,\infty}$ from eq. (3.22) equal to 2.33, using the B-method of testing with $\lambda_0 = 10.014$, $\gamma_0 = 50\%$ and $\alpha_\rho = 10\%$. The square root of the ratio according to test (3.49) was 1.98.

Because the overall model test was not accepted, the algorithm searched for the best alternative hypothesis. It generated 19,800 alternative hypotheses and tested each of them. The following three types of alternative hypotheses were generated:

1. one point is deformed, all others are not; to be tested for each of the 14 points that were measured in both epochs;
2. two, three, four and up to seven points are deformed; the deformation is the same for each point;
3. two, three, four and up to seven points are deformed; the deformation is different for each point.

The first type amounts to fourteen alternative hypotheses.

The second and third type amount each to

$$\sum_{i=2}^{7} \binom{14}{i} = 9,893 \text{ alternative hypotheses.} \tag{3.52}$$

Each alternative hypothesis with $i > 7$ gives an identical test result as one alternative hypothesis with $i \leq 7$, because testing that some specified points are deformed and the others not is equivalent to testing that those others are deformed and the specified aren't.

The three types of alternative hypotheses together give $14 + 9,893 + 9,893 = 19,800$ alternative hypotheses.

B-method of testing is used	
Tested alternative hypotheses	19,800
Level of significance overall model test	10%
Level of significance one-dimensional test	0.16%
Power	50%
Reference non-centrality parameter	10.014
Test ratio overall model test	1.98
Ten largest test ratios of alternative hypotheses	
Ratio Points affected:	
3.09 11 3 41 5 39	
2.74 11 15 3 41 5 39	
2.73 11 41 39	
2.72 11 3 41 5 21 39	
2.71 45 13 21 35 43 47	
2.63 11 15 37 41 17 39	
2.58 45 13 17 21 35 43 47	
2.58 11 15 3 37 41 5 39	
2.57 11 15 41 39	
2.56 11 41 5 39	

Table 3.2: Results of the algorithm.

For each alternative hypothesis the square root of the ratio according to test (3.49) was computed. The alternative hypotheses were sorted in order of this ratio. The largest

square root of the ratio was 3.09 (table 3.2) and belonged to the alternative hypothesis that the five points 3, 5, 11, 39 and 41 were deformed, all in the same way, and that all other points had not been deformed. This hypothesis gives exactly the points that are the deformed points according to Niemeier (2008), see table 3.1 and figure 3.2. It is clear from table 3.2 that the largest value of 3.09 was notably larger than the other ones.

For the best alternative hypothesis the deformations of the five deformed points were estimated according to equations (3.42) and (3.43). Estimated was a shift in the x-direction of 23.7 cm and in the y-direction of 8.0 cm. These results are shown in the last two columns of table 3.1. The root mean square (rms) of the differences between the estimated shifts and the simulated ones is 3.9 cm, where it is 5.1 cm for the two-step strategy.

Conclusion of the numerical test The numerical test shows that the method is effective in detecting the points affected by deformation and gives a smaller rms of the remaining coordinate differences than the conventional methods as described in Niemeier (2008, section 13). The advantage of this method is that it doesn't need to have prior information about which points might have been deformed (no information about the deformation zone is needed).

Not all possible alternative hypotheses were tested in the simulation. For example the hypothesis that two points were deformed in the same way and one other point in a different way, was not tested. Because the true hypothesis was among the tested ones, the method found it.

3.7 Considerations

3.7.1 Reference points and object points

It is common in the literature (de Heus et al., 1994b; Welsch and Heunecke, 2001; Rüeger, 2006) to distinguish between object points and reference points. In an absolute deformation analysis the stability of the reference points is checked first. Then the reference points are kept fixed and the object points are checked for deformation. Using the model of this chapter, the reference points are just one subset of points. The object points form another subset of points, or they may form several subsets. The search is for the functional relationship between the points and is equivalent to the search of the specification matrix C as described in section 3.6. The result can be that several subsets emerge that have changed their relative positions. The subset of reference points is one of them. If, for the analysis of the object points, the reference points are fixed, i.e. considered errorless, the stochasticity of the reference points is "pushed" to the object points and in this way disturbs the analysis. Specifying the right alternative hypothesis, i.e. specifying the right matrix C with the appropriate subsets of points and with stochastic reference points, is to be preferred.

3.7.2 S-transformation or implicit transformation

The model for the connection adjustment, as it is presented in this chapter, includes a transformation (eq. (3.2)). One could approach the connection problem without a transformation, in which case it is necessary that the coordinate vectors a and b are defined in the same reference system. Generally the geodetic datum is defined by some or all points and it is necessary to perform S-transformations (Baarda, 1973; Koch, 1985; Welsch et al., 2000) in the process of searching the best alternative hypothesis. To avoid this the connection model includes a transformation. This implicit transformation and the regularisation, described in section 3.3.3, take care of the datum problem, i.e. the transformation between S-systems. Because of the implicit transformation and the regularisation, it is possible to perform tests of points that are part of the datum definition in an S-system, and to estimate their deformations, without performing any S-transformation.

3.7.3 Testing with confidence ellipsoids

In this section the conditions are derived under which the more general test (3.22) is equal to the testing of confidence ellipsoids (Cederholm, 2003).

If a and b are defined in the same S-system, transformation f is estimated in eq. (3.10) as $\hat{f} = 0$. This means:

$$\hat{e}_d = d \tag{3.53}$$

$$Q_{\hat{e}_d} = Q_d \tag{3.54}$$

From this the test quantity of eq. (3.18) becomes:

$$V_q = d^* Q_d^{-1} C (C^* Q_d^{-1} C)^{-1} C^* Q_d^{-1} d \tag{3.55}$$

Divide Q_d^{-1} into four sub matrices and d into two sub vectors according to:

$$Q_d^{-1} = \begin{pmatrix} W_1 & W_3^* \\ W_3 & W_2 \end{pmatrix} \; ; \quad d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \tag{3.56}$$

with W_1 a (3×3)-matrix and d_1 a 3-vector.

In case of point data snooping in 3D and testing the first point, matrix C follows from eq. (3.41) and we get:

$$d^* Q_d^{-1} C = d_1^* W_1 + d_2^* W_3 \tag{3.57}$$

$$C^* Q_d^{-1} d = W_1 d_1 + W_3^* d_2 \tag{3.58}$$

$$C^* Q_d^{-1} C = W_1 \tag{3.59}$$

V_q becomes:

$$V_q = (d_1^* W_1 + d_2^* W_3) W_1^{-1} (W_1 d_1 + W_3^* d_2) \tag{3.60}$$

If $W_3 = 0$, i.e. there is no correlation between the coordinates of the first point and all other points for both a and b, we get:

$$V_q = d_1^* W_1 d_1 \tag{3.61}$$

Performing test (3.22) is now equal to testing coordinate differences with confidence ellipsoids (Cederholm, 2003), if the same level of statistical significance is chosen.

The choice of which point is the first point is arbitrary, which means that eq. (3.61) can be used for any point. The reasoning has been done for 3D, but is the same for 1D and 2D.

The conclusion is that testing using confidence ellipsoids (confidence regions in 1D, confidence ellipses in 2D) is the uniformly most powerful invariant test (in accordance with the definition of such a test in Teunissen (2006, p. 62)), if two conditions are fulfilled:

1. The coordinates of a and b are defined in the same S-system
2. The coordinate difference between a and b of any point is not correlated with any other coordinate difference.

If an S-transformation is performed on the coordinates of all points, they will all in general be correlated with each other *after* the transformation, even if they were not before. This means that testing with confidence ellipsoids is not the uniformly most powerful invariant test after an S-transformation. It also means that the results of a confidence ellipsoids test (Koch, 1985; Cederholm, 2003) is in general not invariant under a change of the S-system.

If Q_d is of full rank, it is not clear whether a and b are defined in the same S-system. In such a case using model (3.8), (3.9) is preferable for two reasons. First it has a transformation included, which solves the datum problem. Secondly it eliminates the uncertainty in the definition of the degrees of freedom of the S-system from the deformation analysis.

Model (3.8), (3.9) cannot be used, if the deformation is relative to points (objects, part of the earth's surface) that lie outside the point field under consideration, i.e. the reference points are not part of a_2 and b_2. This is the case, for instance, if the reference system itself is used as reference for the deformation analysis. In such a case the transformation should be omitted from model (3.5). This is not elaborated upon in this chapter.

3.7.4 Geometric and physical interpretation

A geometric deformation analysis can be improved significantly if physical causes of the deformation are taken into account. In section 3.4 it is shown how an alternative hypothesis is defined by a specification matrix C. A method is proposed to find the best C. The method applies in fact "brute force": it tries to test as many as possible alternative hypotheses to find the one with the largest test ratio. The search for the best alternative hypothesis could, however, be formulated in a general way:

Search Find the alternative hypothesis with a specification matrix C that maximises the test ratio of test (3.49).

A mathematical approach of this search means finding the derivative of the left hand side of the inequality of test (3.49) relative to the elements of matrix C and equating it to zero. It is a system of non-linear equations, whose solution is not straightforward.

Here we approach the search by trying different alternative hypotheses. The large amount of alternative hypotheses that has to be tested, can be reduced by omitting not plausible hypotheses. Such not plausible hypotheses can be found by geometric reasoning (e.g. points that are far away from each other probably don't undergo exactly the same deformation). Having knowledge, however, about the underlying physical processes that determine the deformations, helps considerably to reduce the amount of matrices C that has to be tested.

The method, proposed in section 3.4, can be used in the absence of knowledge of the underlying physical processes. If however knowledge about them is available, that knowledge should be used to reduce the amount of alternative hypotheses that has to be tested.

3.7.5 Outlook

Although the situation that two epochs of measurements of a limited amount of 1D-, 2D- or 3D-points are to be analysed, still happens often in professional practice, the trend is towards very frequent measuring, even continous monitoring, of objects and towards very dense coverage of an object with measured points (Niemeier, 2011). Extension of the presented method towards more epochs than just two and towards a large amount of points, is therefore desirable.

With an adjustment model that covers more than two epochs it is possible to specify alternative hypotheses that describe deformation processes during more epochs. The search for the best alternative hypothesis will be even more complicated. It is therefore important to find ways to reduce the amount of alternative hypotheses that have to be tested. Analysing underlying physical processes and finding functional relationships that describe the deformation processes is a way to do it.

3.8 Conclusions

A new approach to determine a multi-point deformation of a geodetic network, measured in two epochs, is presented. Monitoring an object or the earth's surface for deformations is possible by choosing a representing set of points and measuring them, and, if necessary, reference points outside the object, with geodetic means. If two epochs of measurements are available, it is shown how to test the measured points in such a way that subsets of points can be distinguished. Each subset can have its own deformation behaviour. The reference points are seen as just one subset, not to be treated differently from the other subsets.

The deformation analysis uses the null hypothesis that no deformation has occurred. This hypothesis is tested with an overall model test. If this test leads to rejection, a search starts for the best alternative hypothesis. An alternative hypothesis can concern the deformation of just one point, but in general it will affect more points (i.e. a subset of points). It can even distinguish more subsets of points. If no information is available on possible deformations, a method is given to test by "brute force" as many alternative hypotheses as possible.

The test method is invariant under changes of the S-systems in which the point coordinates are defined.

When the best alternative hypothesis has been detected, the least squares estimate of the deformation can be computed. The equations are given as well to compute the minimal detectable deformations that can be used in the design stage of a geodetic network for deformation analysis.

The method to find the best alternative hypothesis has been tested numerically in a 2D-network, where it succeeded to find the deformation.

The used adjustment model includes a transformation that makes it unnecessary to use S-transformations in the process of testing for deformations.

The relation between the proposed method and the method of testing confidence ellipsoids is shown.

Finally it is shown that the proposed method exceeds the boundaries of a purely geometric analysis. Application of the method yields improved results, if the underlying physical processes are taken into account.

4

Analysis model in 3D (2 epochs)[1]

4.1 Introduction

Geodetic deformation analysis is about analysing the geometric changes of objects on, above or under the earth's surface or changes of this surface itself. The objects are generally discretised by points, whose coordinates are registered at two or more epochs.

Affine, similarity and congruence transformations play an important role in geodetic deformation analyses, mainly for two reasons. First objects may undergo deformations that are well described by such transformations. This is the case, if the deformations comprise translations, rotations, shears and changes of size. The second reason is that objects are often described with geodetically determined coordinates that are defined by geodetic datums. To transform coordinates to a common datum, transformations are necessary. The three mentioned transformation types are often adequate for this purpose.

Deformations of objects may comprise much more complicated patterns than can be described by congruence, similarity or affine transformations. Therefore extended functional models have to be built to describe the relations between coordinate sets of two or more epochs of geodetic measurements. These extended models can often be described as extensions to the congruence, similarity or affine transformation or to combinations thereof. If necessary the models are complemented by covariance functions (collocation) or variograms (kriging) to capture the systematic effects that are not described by the functional model.

[1] This chapter has been published before in Survey Review (Velsink, 2015a). To fit the publication into this study minor changes have been made.

4.1.1 Problem definition

The problem addressed in this chapter is the problem of finding the relation between two coordinate sets in 3D, describing each object by means of the same points but at different moments in time. This relation describes the deformation that the object is subject to and a possible difference in geodetic datum. The coordinate sets are supposed to be Cartesian with full covariance matrices, reflecting the precision by which the coordinates have been acquired. Testing the correctness of the deformation model should be possible. If it is not correct, it should be possible to find least squares estimates of the deformations. These estimates can be used to extend the adjustment model.

4.1.2 Approach to solution

The problem is addressed by first finding the optimal affine transformation, or its special cases: the similarity and congruence transformation. Subsequently the transformation model is extended by additional parameters to find the best fitting solution.

The affine transformation in 3D involves a rotation. This rotation is often described by Euler angles. They have, however, the disadvantage that gimbal lock may occur, i.e. the impossibility to determine the three Euler angles, if a specific one of them is 0 or 90°. Which angle gives this problem and at what value, depends on the sequence in which Euler angles are applied to perform a rotation. If the problematic angle is close to the dangerous value, this may still cause problems to determine the other angles with sufficient precision. Therefore finding the deformation should avoid the use of Euler angles, because angles of 0 and 90° can occur in deformation problems.

The approach to solve the posed problem is setting up an adjustment model and solving the parameters of this model by means of the method of least squares. It is an adjustment model, because there are in general more coordinate differences than transformation parameters to be estimated.

The adjustment model describes a null hypothesis, in which it is supposed that no deformation occurred. It should be possible to extend the model with extra parameters to formulate alternative hypotheses, which describe possible deformations, and to test them against the null hypothesis. A linearised adjustment model makes it relatively easy to formulate many types of hypotheses. Linearisation of the model makes it easy as well to take account of full covariance matrices of coordinate sets. Such full covariance matrices are to be expected when the coordinate sets are the result of the adjustment of geodetic measurements (terrestrial, airborne or satellite).

4.1.3 Overview

The setup of this chapter is as follows.

First the general adjustment model for a transformation is given. Determining the transformation parameters between two or more coordinate sets is a problem that can be solved without linearisation of the functional model, if the transformation is one-,

two- or three-dimensional and an affine, similarity or congruence transformation. Direct solutions to the general adjustment model are referenced.

The chapter continues with the linearised adjustment model. If the deformations are more than just an affine, similarity or congruence transformation, it is hardly ever possible to find a direct solution for the transformation parameters. Linearisation of the functional model and solving the linearised model is necessary in that case. The linearised adjustment model is given as a model of observation equations with constraints on the parameters. The starting point is the affine transformation, whose parameters are constrained to get the parameters of the similarity or congruence transformation. In this way the use of Euler angles is avoided.

The least squares solution of a linearised adjustment model is reviewed. It is shown how to handle a singular cofactor matrix. Methods to solve an adjustment problem with constraints are given.

Then the model for the affine transformation is elaborated upon, followed by the congruence transformation. A linearised adjustment model needs approximate values for the unknown parameters. Their determination and the iteration to arrive at the final solution are treated. In each iteration step approximate parameters are necessary that fulfil the constraints. For the affine transformation it is easy to get approximate coordinates. To make subsequently the approximate coordinates comply to the constraints, use is made of the singular value decomposition of the rotation matrix.

Finally the similarity transformation is treated and an experimental validation is given.

Recent literature mentions total least squares as a method to find a least squares solution for transformation problems, where all coordinates are considered stochastic (Fang, 2011; Snow, 2012). In this chapter it is shown that it is very well possible to construct an adjustment model, where all coordinates are considered stochastic and the coefficient matrix does not contain stochastic elements. This makes application of the total least squares method unnecessary. Since the standard method of least squares is used, its extensive body of knowledge can be used.

4.2 General adjustment model for transformation

A general adjustment model for solving the transformation parameters between two sets of coordinates is treated in this section.

Let a point field be described by a set of Cartesian coordinates in reference system r_a, taken together in a vector $\underline{\mathbf{a}}$, and by another set of Cartesian coordinates in reference system r_b, taken together in a vector $\underline{\mathbf{b}}$. The vector names are underlined to indicate that they are random variables. It is supposed that vector $\underline{\mathbf{a}}$ differs from vector $\underline{\mathbf{b}}$ because of a deformation, a difference in geodetic datum and of stochastic noise in both vectors. Let the vector $\mathbf{c}$ contain the mathematical expectations of $\underline{\mathbf{a}}$, i.e.

$$\mathbf{c} = E\{\underline{\mathbf{a}}\}, \tag{4.1}$$

where $E\{\}$ indicates the mathematical expectation.

Let a transformation, represented by the vector function t, transform the vector $\underline{\mathbf{b}}$ into the vector $\underline{\mathbf{a}}$ by means of transformation parameters, taken together in vector $\mathbf{f}$. The transformation is supposed to describe the deformation and the difference in geodetic datum. The mathematical expectation of $\underline{\mathbf{a}}$ and of the transformed vector $\underline{\mathbf{b}}$ should then be equal

$$E\{\underline{\mathbf{a}}\} = E\{t(\underline{\mathbf{b}}, \mathbf{f})\}. \tag{4.2}$$

The transformation t is non-linear in the elements of vector $\underline{\mathbf{b}}$ and of vector $\mathbf{f}$ for the affine, congruence and similarity transformation. If, however, vector $\mathbf{b}$ is considered non-random, i.e. a constant vector, transformation t is linear for the affine transformation. Its direct least squares solution is given in section 'Step 1: affine transformation done simply'.

For the general case transformation t is non-linear, which means that a solution can be found by a direct solution, treated in the next section, or by solving iteratively a linearised model, treated subsequently. This chapter proposes a solution with a linearised model, which is elaborated upon in the section 'Linearised adjustment model'.

4.2.1 Direct solutions

A direct solution of the transformation parameters in (4.2) is, as mentioned above, straightforward in the case of an affine transformation if $\underline{\mathbf{b}}$ is considered a constant vector. The stochastic behaviour of $\underline{\mathbf{b}}$ cannot be taken account of, unless it is possible to have it included in the stochastic behaviour of vector $\underline{\mathbf{a}}$.

In the case of a congruence transformation the transformation consists of a rotation around a certain axis and a translation in a certain direction. It can be considered a special case of the similarity transformation, treated subsequently.

The similarity transformation in 3D consists of a rotation around a certain axis, a translation in a certain direction and a change of scale. It is also called a seven-parameter transformation or a Helmert transformation (Awange et al., 2004; Krarup, 2006).

Much literature is devoted to finding direct solutions for the parameters of these transformations, where finding the three rotation parameters is most difficult.

Menno Tienstra writes (Tienstra, 1969) that the first method published was by Thompson (1959) and that Schut (1961) gave a more elegant derivation of the same method. Tienstra himself gives a different method (Tienstra, 1969). At least eight solutions of more recent date (from 1981 up to 2006) can be found in literature. They are the solutions of Hanson and Norris (1981), Bakker et al. (1989, p. 55), Awange et al. (2004), Teunissen (1985a, p. 148), Arun et al. (1987), Hinsken (1987), Horn et al. (1988), and Krarup (2006).

Of the first three solutions a short description is given to illustrate the different approaches that are possible.

Hanson and Norris consider in their paper (Hanson and Norris, 1981) the estimation of the transformation parameters of the congruence and similarity transformations. Their

application is quality control of manufactured parts, where points on parts are matched with points on a drawing. They prove that the least squares estimation of the rotation can be performed independently from the translation, if the elements of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are mutually stochastically independent and the x, y and z-coordinate of each point i have the same weight γ_i. A procedure is given to compute the direct least squares estimates of the rotation parameters by means of singular value decomposition of a matrix that is computed from the elements of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$. Also a procedure is given to compute the direct least squares estimate of the change of scale of the similarity transformation. The direct least squares estimate of the translation is the difference between the centres of gravity of $\underline{\mathbf{a}}$ and the rotated and scaled $\underline{\mathbf{b}}$.

Bakker et al. (1989) consider the points of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ as a distribution of mass points with unit mass. They compute for both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ the centre of mass, the set of body-axes, the mean radius of gyration and the inertia tensor. After translating to the centre of mass, rotating with the inertia tensor and scaling both $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, the transformed vectors are equated and a formula for the transformation parameters derived. The stochastic characteristics of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are not taken into account, although it is mentioned by Bakker et al. (1989) that an adjustment is easy, especially if the coordinates have rotationally symmetric variances.

Awange et al. (2004) give a procedure to compute the least squares estimates of the parameters of the similarity transformation (seven-parameter transformation), based on finding the roots of univariate polynomials using a Groebner basis.

4.2.2 Solutions by linearisation

The transformation parameters in (4.2) can be solved by linearising the equation and using the standard least squares algorithm. The advantage of this approach is the possibility of solving adjustment models, for which direct solutions are not known. The disadvantages are the need to have approximate values for all unknown parameters and the need to iterate the computation with the risk of divergence, if the approximate values are not chosen well.

The linearisation of the affine transformation is simplest and is given first. For the linearisation of the congruence and similarity transformation, the rotation has to be parameterised. This is generally done, see e.g. Hofmann-Wellenhof et al. (2001, p. 294), by using three angles that describe the rotation around three coordinate axes, the so called Euler angles. As described in the introduction, the use of Euler angles may result in gimbal lock. This problem occurs for example for the rotation parameterisation of Hofmann-Wellenhof et al. (2001, p. 294), if the second rotation angle α_2 is a right angle ($\alpha_2 = 100$ gon), which yields the rotation matrix $\mathbf{R}$

$$\mathbf{R} = \begin{pmatrix} 0 & \sin(\alpha_1 + \alpha_3) & -\cos(\alpha_1 + \alpha_3) \\ 0 & \cos(\alpha_1 + \alpha_3) & \sin(\alpha_1 + \alpha_3) \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.3}$$

Only the sum $(\alpha_1 + \alpha_3)$ appears in the matrix, and angle α_1 nor angle α_3 can be determined separately, although the rotation matrix itself is well defined. This is called 'gimbal lock' after the equivalent problem in mechanical engineering, when two of three

gimbals become parallel to each other and one rotation possibility is lost. An additional disadvantage of Euler angles is that approximate values for them have to be determined in some way. In this chapter a different approach is chosen, where the rotations are not parameterised by three angles, but by the nine elements of the rotation matrix. By imposing six constraints on the nine elements, the matrix is forced to describe only rotations.

The use of the direct solutions of the previous section has the advantage that no approximate values and no iteration are needed. The advantage of a linearised adjustment model, however, is that extending the model with other parameters is easy. There are two reasons why the model should be extended:

(i) errors may be present in the coordinates, caused by errors in the measurements that were used to compute the coordinates, or caused by e.g. identification errors. To test for these errors the model is extended by parameters, describing these errors. The extended model is tested as an alternative hypothesis against the null hypothesis of no errors (Teunissen, 2006, p. 78).

(ii) the description by means of 6, 7 or 12 parameters of one congruence, similarity or affine transformation may be not adequate. Maybe not one but more such transformations are needed, for example each describing a subset of points. Or more complex transformations with more or other parameters are needed.

Extending or changing the model is easier for a linearised model than for a non-linear one.

4.3 Linearised adjustment model

4.3.1 Linearisation of general model

To solve the problem of finding the transformation parameters, a system of observation equations is set up. Constraints are added to this system. This constrained system is solved by means of the method of least squares. As the equations of this system are not linear, they are linearised.

First the system is given for an affine transformation. The affine parameters can be obtained without the addition of constraints. Then the system is augmented by constraints that force the parameters of the affine transformation to become the parameters of a congruence transformation. Finally it is shown how the parameters of a similarity transformation are obtained.

A linearised adjustment model is constructed by differentiating the function t of (4.2) relative to the elements of the vectors $\underline{\mathbf{b}}$ and $\mathbf{f}$ Teunissen (2000, p. 142). The resulting equations stay simpler, if vector $\underline{\mathbf{b}}$ is first transformed to coincide approximately with vector $\underline{\mathbf{a}}$. For this transformation approximate transformation parameters are needed. Their computation is treated later on.

The approximately transformed vector is

$$\underline{\mathbf{b}}' = t'(\underline{\mathbf{b}}, \mathbf{f}'). \tag{4.4}$$

The cofactor matrix of $\underline{\mathbf{b}}'$ is determined by applying the law of propagation of cofactors. To do this, equation (4.4) is linearised by a Taylor expansion, neglecting second and higher order terms. The expansion is relative to the elements of vector $\underline{\mathbf{b}}$.

Equation (4.2) now becomes

$$E\{\underline{\mathbf{a}}\} = E\{t(\underline{\mathbf{b}}', \mathbf{f})\}. \tag{4.5}$$

Define two matrices $\mathbf{B}$ and $\mathbf{F}$ of partial derivatives as follows

$$\mathbf{B} = \left(\frac{\partial t}{\partial \mathbf{b}'}\right)_0 ; \quad \mathbf{F} = \left(\frac{\partial t}{\partial \mathbf{f}}\right)_0 . \tag{4.6}$$

The sub index 0 indicates that approximate values of resp. $\underline{\mathbf{b}}'$ and $\mathbf{f}$ have to be entered into the matrices.

Assuming that $\mathbf{B}$ is square and invertible, which is the case in the situations treated in this chapter, the adjustment model can be constructed as

$$E\{\begin{pmatrix} \underline{\Delta\mathbf{a}} \\ \underline{\Delta\mathbf{b}}' \end{pmatrix}\} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{B}^{-1} & -\mathbf{B}^{-1}\mathbf{F} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{c} \\ \Delta\mathbf{f} \end{pmatrix}. \tag{4.7}$$

where $\underline{\Delta\mathbf{a}} = \underline{\mathbf{a}} - \mathbf{a}_0$; $\underline{\Delta\mathbf{b}}' = \underline{\mathbf{b}}' - \mathbf{b}'_0$; $\Delta\mathbf{c} = \mathbf{c} - \mathbf{c}_0$; $\mathbf{a}_0$, $\mathbf{b}'_0$, $\mathbf{c}_0$, $\mathbf{f}_0$ are approximate values; $\mathbf{I}$ is the unit matrix; $\mathbf{0}$ is the zero matrix; $\mathbf{c}_0 = \mathbf{a}_0$ and thus $E\{\underline{\Delta\mathbf{a}}\} = \Delta\mathbf{c}$.

On the left hand side of (4.7) the vector of observations can be found, which consists of the two vectors containing the coordinates. The right hand side contains the matrix of coefficients and the vector of unknown parameters. The parameters are the mathematical expectations of the coordinates in reference system r_a and the transformation parameters.

The vector of observations is assumed to have a normal distribution, described by a covariance matrix that is the product of a scalar variance factor and the cofactor matrix. The cofactor matrix is

$$\mathbf{Q}\{\begin{pmatrix} \underline{\mathbf{a}} \\ \underline{\mathbf{b}}' \end{pmatrix}\} = \begin{pmatrix} \mathbf{Q}_{\mathbf{a}} & \mathbf{Q}_{\mathbf{ab}'} \\ \mathbf{Q}_{\mathbf{b}'\mathbf{a}} & \mathbf{Q}_{\mathbf{b}'} \end{pmatrix} \tag{4.8}$$

where $\mathbf{Q}_{\mathbf{a}}$ and $\mathbf{Q}_{\mathbf{b}'}$ are the cofactor matrices of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}'$. $\mathbf{Q}_{\mathbf{ab}'} = (\mathbf{Q}_{\mathbf{b}'\mathbf{a}})^T$ gives the cofactors that describe the correlation between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}'$. It equals the zero-matrix if $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}'$ are supposed to be not correlated mutually. It is, however, possible to use non-zero matrices, for example to describe temporal correlation using a covariance function. The variance factor is chosen in such a way that the elements of the cofactor matrix have computationally convenient values. The situation that $\mathbf{Q}_{\mathbf{a}}$ or $\mathbf{Q}_{\mathbf{b}'}$ or both are not regular, but positive-semidefinite matrices, is treated later on.

4.3.2 Reduced general model

The system can be reduced to minimise the amount of unknown parameters, which is advantageous in case there is a large number of coordinates. The solution of the system of normal equations can be a computational burden with many coordinates, especially if

full covariance matrices are involved. The reduced system is acquired by premultiplying the second row of (4.7) with **B** and subtracting it from the first row

$$E\{\underline{\Delta\mathbf{a}} - \mathbf{B}\underline{\Delta\mathbf{b}}'\} = \mathbf{F}\Delta\mathbf{f}. \tag{4.9}$$

On the left hand side appears the vector of remaining coordinate differences ($\underline{\Delta\mathbf{a}} - \mathbf{B}\underline{\Delta\mathbf{b}}'$) as vector of observations. On the right hand side the only unknown parameters are the transformation parameters. The least squares residuals of the coordinates can be computed by means of the stochastic correlation of the adjusted coordinates with the estimated transformation parameters, because the adjusted coordinates are so called free variates Teunissen (2000, p. 75).

In this chapter use is made of the model as defined by (4.7), because it shows more clearly the structure, because constraints on the coordinates can be added, e.g. to describe more complex deformation behaviour, and because it can readily be extended to more than two epochs.

4.4 Least squares solution

The system of observation equations (4.7) is overdetermined if the number of parameters is less than the number of coordinates and can be solved by the method of least squares. A weighted least squares solution is obtained, if account is taken of cofactor matrix (4.8).

A system of linearised observation equations, like (4.7), has the following general structure

$$E\{\underline{\ell}\} = \mathbf{A}\mathbf{p}; \quad \mathbf{D}\{\underline{\ell}\} = \sigma^2\mathbf{Q}_\ell. \tag{4.10}$$

where $\underline{\ell}$ is the m-vector of observations, **A** is the $(m \times n)$-matrix of coefficients and **p** is the n-vector of unknown parameters. The equation behind the semicolon describes the stochastic model by giving the covariance matrix $\mathbf{D}\{\underline{\ell}\}$, the variance factor σ^2 and the cofactor matrix $\mathbf{Q}_\ell$. The least squares solution of system (4.10) is given in the appendix. There the equations for testing of the results are given as well.

For each system of linear observation equations an equivalent system of linear condition equations exists

$$\mathbf{K}^T E\{\underline{\ell}\} = \mathbf{0}; \quad \mathbf{D}\{\underline{\ell}\} = \sigma^2\mathbf{Q}_\ell. \tag{4.11}$$

where $\mathbf{K}^T$ is the $[(m-n)\times m]$-matrix of conditions, for which holds

$$\mathbf{K}^T\mathbf{A} = \mathbf{0}. \tag{4.12}$$

The least squares solution of system (4.11) is given in the appendix. The equations for testing are the same as those given for the model of observation equations.

4.4.1 Positive semidefinite cofactor matrix

The vector $\underline{\ell}$ of observations of system (4.7) contains coordinates, which may have resulted from geodetic measurements. In that case it can easily happen that the cofactor

matrix $\mathbf{Q}_\ell$ is positive semidefinite, because the coordinates are defined relative to a geodetic datum, defined by some of the points.

A solution for handling a positive semidefinite cofactor matrix of the observations is given for the reduced general model by Teunissen et al. (1987b, p. 231) for the similarity transformation. Matrix $\mathbf{Q}_\ell$ is regularised by adding a term to it, which does not change the least squares solution of the parameters $\mathbf{p}$. It is possible to generalise regularisation to any model of observation equations[2], if the coefficient matrix $\mathbf{A}$ is of full rank.

Matrix $\mathbf{Q}_\ell$ of (4.10) can be regularised by adding to it a matrix $\lambda\mathbf{A}\mathbf{A}^T$, with $\mathbf{A}$ from (4.10) and λ any real scalar with $\lambda > 0$

$$\mathbf{Q}'_\ell = \mathbf{Q}_\ell + \lambda\mathbf{A}\mathbf{A}^T. \tag{4.13}$$

Matrix $\mathbf{A}$ is a real matrix of full rank and therefore $\mathbf{A}\mathbf{A}^T$ is positive definite. Multiplying it with a positive factor λ does not change its positive definiteness. $\mathbf{Q}'_\ell$ is the sum of a positive semidefinite matrix and a positive definite matrix and is therefore positive definite. This means that its inverse $(\mathbf{Q}'_\ell)^{-1}$ exists.

Using $\mathbf{Q}'_\ell$ instead of $\mathbf{Q}_\ell$ for computing the least squares solution yields the same adjustment and testing results. This can be seen by switching to the model of condition equations (4.11). The equations in the appendix show that vector $\hat{\underline{\mathbf{r}}}$ and its cofactor matrix $\mathbf{Q}_{\hat{\mathbf{r}}}$ are used to get the adjustment and testing results. In the equations $\mathbf{Q}'_\ell$ appears only in the product $\mathbf{Q}'_\ell\mathbf{K}$ (or its transpose), for which we have

$$\mathbf{Q}'_\ell\mathbf{K} = (\mathbf{Q}_\ell + \lambda\mathbf{A}\mathbf{A}^T)\mathbf{K} = \mathbf{Q}_\ell\mathbf{K} + \lambda\mathbf{A}\mathbf{A}^T\mathbf{K}. \tag{4.14}$$

The second term is zero because of (4.12). Therefore

$$\mathbf{Q}'_\ell\mathbf{K} = \mathbf{Q}_\ell\mathbf{K}. \tag{4.15}$$

The conclusion is that the adjustment and testing results do not change by using $\mathbf{Q}'_\ell$ instead of $\mathbf{Q}_\ell$. Owing to the equivalence of using the model of observation equations and the model of condition equations, $\mathbf{Q}'_\ell$ can be used as well to get the least squares solution for the model of observation equations.

4.5 Constraining parameters

The parameters of an adjustment model can be constrained to satisfy certain linear or linearised relations

$$\mathbf{C}\mathbf{p} = \mathbf{0}, \tag{4.16}$$

where matrix $\mathbf{C}$ is a matrix of coefficients and $\mathbf{0}$ a zero vector. In the following sections constraints are used to force the affine transformation parameters to change into the parameters of a congruence or similarity transformation.

The least squares solution for the parameters $\mathbf{p}$ from (4.7) has to be found under the condition that they fulfil the constraints (4.16). A method is the extension of the system

[2] in fact: not "any model"; see appendix C.6 for the restrictions.

of normal equations with the constraints as follows (Tienstra, 1956, section 7.3)

$$\begin{pmatrix} \mathbf{A}^T\mathbf{Q}_\ell^{-1}\mathbf{A} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{k} \end{pmatrix} = \begin{pmatrix} \mathbf{A}^T\mathbf{Q}_\ell^{-1}\underline{\ell} \\ \mathbf{c} \end{pmatrix}, \tag{4.17}$$

where $\mathbf{k}$ contains the and $\mathbf{0}$ is a zero matrix. Solving this system of normal equations delivers a solution for $\mathbf{p}$ and $\mathbf{k}$. Vector $\mathbf{k}$ plays no further role in the considerations of this chapter.

If $\mathbf{Q}_\ell$ is positive semidefinite, it has to be regularised. To see how this can be done, another method to solve (4.10), taking account of the constraints (4.16), is given. Take the null space of the space $\mathcal{R}(\mathbf{C})$, spanned by the columns of matrix $\mathbf{C}$ of (4.16). If $\mathbf{N_C}$ is a base matrix of this null space, we can write

$$\mathbf{p} = \mathbf{N_C}\lambda \tag{4.18}$$

with λ a vector of $(n - n_c)$ parameters (n_c is the amount of constraints). We can now write (4.10) as

$$E\{\underline{\ell}\} = \mathbf{A}_r\lambda; \quad \mathbf{D}\{\underline{\ell}\} = \sigma^2\mathbf{Q}_\ell, \tag{4.19}$$

with $\mathbf{A}_r = \mathbf{A}\mathbf{N_C}$. These are observation equations, for which a least squares solution is found in the normal way. It is the same solution as the one that follows from (4.17). The determination of $\mathbf{A}_r$ can be seen as the elimination of n_c parameters from (4.10), which can be done with the Gaussian algorithm or the Cholesky method (Wolf, 1982). If $\mathbf{A}_r$ is determined as $\mathbf{A}_r = \mathbf{A}\mathbf{N_C}$, the base matrix $\mathbf{N_C}$ can be determined e.g. by the Matlab-command 'null(C)'.

For the regularisation of $\mathbf{Q}_\ell$ we now use

$$\mathbf{Q}'_\ell = \mathbf{Q}_\ell + \lambda\mathbf{A}_r\mathbf{A}_r^T. \tag{4.20}$$

4.6 Model for affine transformation

If an object is subject to a force, the material and the force may be such that applying and releasing the force causes respectively an elastic deformation and the disappearance of the deformation. Such a deformation is often linear and can be described by an affine transformation. In this section the adjustment model for the affine transformation is given by defining the content of the matrices $\mathbf{B}$ and $\mathbf{F}$ of (4.6).

Let $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$, $\underline{\mathbf{z}}$ be the vectors with the x-, y- and z- coordinates of vector $\underline{\mathbf{a}}$, as described in the appendix. Let $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$, $\underline{\mathbf{w}}$ be the vectors with the x-, y- and z- coordinates of the same points in vector $\underline{\mathbf{b}}'$.

Let the vectors $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$, $\boldsymbol{\alpha}_3$ and the vector $\mathbf{t}$ be defined as

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} a_{11} \\ a_{12} \\ a_{13} \end{pmatrix}, \quad \boldsymbol{\alpha}_2 = \begin{pmatrix} a_{21} \\ a_{22} \\ a_{23} \end{pmatrix}, \quad \boldsymbol{\alpha}_3 = \begin{pmatrix} a_{31} \\ a_{32} \\ a_{33} \end{pmatrix}, \quad \mathbf{t} = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}. \tag{4.21}$$

Let the matrix $\mathbf{R}$ be defined as

$$\mathbf{R} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\alpha}_1^T \\ \boldsymbol{\alpha}_2^T \\ \boldsymbol{\alpha}_3^T \end{pmatrix}. \tag{4.22}$$

The equation for the affine transformation reads then

$$\begin{pmatrix} \underline{\mathbf{x}}^T \\ \underline{\mathbf{y}}^T \\ \underline{\mathbf{z}}^T \end{pmatrix} = \mathbf{R} \begin{pmatrix} \underline{\mathbf{u}}^T \\ \underline{\mathbf{v}}^T \\ \underline{\mathbf{w}}^T \end{pmatrix} + \mathbf{t}\boldsymbol{\varepsilon}^T, \tag{4.23}$$

where $\boldsymbol{\varepsilon} = (1, 1, \cdots, 1)^T$.

The parameters t_x, t_y, t_z describe a translation and the parameters $a_{11}, \ldots, a_{33}$ describe the rotation, scaling and shearing of the affine transformation.

To get the matrices $\mathbf{B}$ and $\mathbf{F}$ of (4.6) the coordinates $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$, $\underline{\mathbf{z}}$ have to be differentiated relative to the coordinates $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$, $\underline{\mathbf{w}}$ and to the parameters t_x, t_y, t_z and $a_{11}, \ldots, a_{33}$.

Matrix $\mathbf{B}$ has the following structure in the case of an affine transformation

$$\mathbf{B} = \begin{pmatrix} a_{11}^0\mathbf{I} & a_{12}^0\mathbf{I} & a_{13}^0\mathbf{I} \\ a_{21}^0\mathbf{I} & a_{22}^0\mathbf{I} & a_{23}^0\mathbf{I} \\ a_{31}^0\mathbf{I} & a_{32}^0\mathbf{I} & a_{33}^0\mathbf{I} \end{pmatrix}, \tag{4.24}$$

where a_{ij}^0, with $i, j = 1, 2, 3$, are the approximate values of the parameters of (4.21) and $\mathbf{I}$ is the $(n \times n)$ unit matrix.

Since with (4.4) vector $\underline{\mathbf{b}}'$ is already approximately equal to $\underline{\mathbf{a}}$, the approximate values of (4.24) can be chosen as follows

$$a_{ij}^0 = \delta_{ij},$$

with $i, j = 1, 2, 3$ and δ_{ij} the Kronecker delta. Then matrix $\mathbf{B}$ becomes a $(3n \times 3n)$ unit matrix.

During iteration of the least squares adjustment, the approximate values have to be adjusted. If this is done by adjusting transformation (4.4), matrix $\mathbf{B}$ does not need to be adjusted and can stay a unit matrix.

To give a simple structure for matrix $\mathbf{F}$ the vector of transformation parameters $\Delta\mathbf{f}$ is divided as follows

$$\Delta\mathbf{f} = \begin{pmatrix} \Delta\boldsymbol{\alpha}_1 \\ \Delta\boldsymbol{\alpha}_2 \\ \Delta\boldsymbol{\alpha}_3 \\ \Delta\mathbf{t} \end{pmatrix}, \tag{4.25}$$

with the Δ-quantities defined as described in the appendix (Conventions).

Matrix $\mathbf{F}$ has now the following structure

$$\mathbf{F} = \begin{pmatrix} \boldsymbol{\beta} & \mathbf{0} & \mathbf{0} & \boldsymbol{\varepsilon}_1 \\ \mathbf{0} & \boldsymbol{\beta} & \mathbf{0} & \boldsymbol{\varepsilon}_2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\beta} & \boldsymbol{\varepsilon}_3 \end{pmatrix}, \tag{4.26}$$

where $\boldsymbol{\beta}$, $\boldsymbol{\varepsilon}_1$, $\boldsymbol{\varepsilon}_2$, $\boldsymbol{\varepsilon}_3$ and $\mathbf{0}$ are all $(n \times 3)$ matrices, as follows

$\beta = (\mathbf{u}_0,\mathbf{v}_0,\mathbf{w}_0)$; $\mathbf{u}_0,\mathbf{v}_0,\mathbf{w}_0$: approximate values of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$.

$$\varepsilon_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \end{pmatrix}, \varepsilon_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \end{pmatrix}, \varepsilon_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix}.$$

$\mathbf{0}$ is the $(n \times 3)$ zero matrix.

The adjustment model for the affine transformation can be constructed with (4.7), with $\mathbf{B}$ and $\mathbf{F}$ according to (4.24) and (4.26). Solution of this model by the method of least squares follows by generating the normal equations and solving them.

4.7 Model for congruence transformation

One of the simplest deformations an object can undergo on, under or above the earth's surface is a movement that consists of a shift and a rotation, i.e. a congruence transformation (also called a rigid body transformation).

The congruence transformation is a special form of an affine transformation, in which no central dilatations (changes of scale) and no shears occur, only translations and rotations. It has less parameters than the affine transformation. It involves a translation, described by three parameters t_x, t_y, t_z, and a rotation, described by three Euler angles, or by one angle and a unit vector, around which the rotation occurs.

We can write down matrix $\mathbf{R}$ in (4.22) with only three parameters: the three Euler angles. Differentiating such a system results in matrices $\mathbf{B}$ and $\mathbf{F}$ that can be used to construct adjustment model (4.7). There are two disadvantages to such an approach:

1. determining the parameters to perform the approximate transformation (4.4) is no easy task (but it is easy for an affine transformation, as will be shown later on);
2. the Euler angles may cause problems, as it was discussed before.

The approach chosen in this chapter, is to use the model as it was derived for the affine transformation in the previous section, and constrain the parameters in such a way that they become the parameters of a congruence transformation.

4.7.1 Applying constraints to affine transformation

For a congruence transformation matrix $\mathbf{R}$ of (4.22) has to be an orthogonal matrix. This means that the rows of $\mathbf{R}$ are orthogonal and each row has length 1 (Strang, 1988, p. 166). So six conditions have to be satisfied, i.e. three orthogonality conditions (e.g.: row 1 is orthogonal to row 2 and to row 3, and row 2 is orthogonal to row 3) and three length conditions (three rows have length 1).

As an equation

$$\boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j = \delta_{ij}, \tag{4.27}$$

with i=1, 2, 3; j=1, 2, 3; $j \geq i$; $\boldsymbol{\alpha}_i$ defined as in (4.21) and δ_{ij} the Kronecker delta.

The six conditions can be linearised and added as constraints on the parameters to system (4.7). The six linearised constraints are

$$\begin{pmatrix} \boldsymbol{\alpha}_2^{0T} & \boldsymbol{\alpha}_1^{0T} & \mathbf{0} \\ \boldsymbol{\alpha}_3^{0T} & \mathbf{0} & \boldsymbol{\alpha}_1^{0T} \\ \mathbf{0} & \boldsymbol{\alpha}_3^{0T} & \boldsymbol{\alpha}_2^{0T} \\ \boldsymbol{\alpha}_1^{0T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\alpha}_2^{0T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\alpha}_3^{0T} \end{pmatrix} \begin{pmatrix} \underline{\Delta\boldsymbol{\alpha}}_1 \\ \underline{\Delta\boldsymbol{\alpha}}_2 \\ \underline{\Delta\boldsymbol{\alpha}}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{4.28}$$

where $\mathbf{0}$ is the (1×3) zero vector and $\boldsymbol{\alpha}_i^{0T}$ with i=1, 2, 3 is the transposed vector of approximate values of the parameters as defined in (4.21). As mentioned before, as approximate values can be chosen $a_{ij}^0 = \delta_{ij}$, with i, j=1, 2, 3, from which follows

$$\left(\boldsymbol{\alpha}_1^0, \boldsymbol{\alpha}_2^0, \boldsymbol{\alpha}_3^0\right) = \mathbf{I}_3, \tag{4.29}$$

with $\mathbf{I}_3$ the (3×3) unit matrix.

4.7.2 Determining approximate values

To determine the transformation parameters of the congruence transformation, adjustment model (4.7) is constructed, the matrices **B** and **F** are determined with (4.24) and (4.26), and the constraints are added with (4.28). Two methods to determine a solution of the adjustment model with constraints have been described before.

In the matrices **B** and **F**, in the constraints and in the Δ-quantities, however, approximate values have to be entered. The case of matrix **B** has been treated already when the adjustment model for the affine transformation was constructed: a unit matrix can be used. In matrix **F** the approximate values $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{w}_0$ are needed. Here the observed values of $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$, $\underline{\mathbf{w}}$ must be transformed with (4.4) and can then be used as $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{w}_0$.

For the transformation parameters of (4.4) approximate values are needed for all nine elements of matrix **R** and for the three translation parameters t_x, t_y, t_z.

The methods to get direct solutions of the transformation parameters in (4.2) have been treated before and can be used to get approximate values for the nine elements of matrix **R** and the three translation parameters.

Instead of using one of the treated direct methods a two-step procedure is described here to arrive at approximate transformation parameters. It is given to show its usability. The choice to use this method or one of the direct solutions depends on considerations like computational suitability.

In the two-step procedure approximate values for the translation parameters and the elements of matrix **R** are determined by first using a simplified version of the adjustment

model of the affine transformation and secondly using the singular value decomposition of matrix **R**.

Step 1: affine transformation done simply

If in (4.2) vector $\underline{\mathbf{b}}'$ is considered a non-random vector (i.e. $\mathbf{b}'$ without an underscore), and the equations of an affine transformation are used, the elements of vector $E\{\underline{\mathbf{a}}\}$ are a linear function of the transformation parameters

$$E\{\underline{\mathbf{a}}\} = \mathbf{F}\mathbf{f}, \tag{4.30}$$

where **F** is the matrix from (4.26) and **f** the vector of parameters

$$\mathbf{f} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_3 \\ \mathbf{t} \end{pmatrix}. \tag{4.31}$$

The least squares estimator of **f**, indicated as $\hat{\underline{\mathbf{f}}}$, with all observations considered as having the same variance and not being correlated, is

$$\hat{\underline{\mathbf{f}}} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\underline{\mathbf{a}}. \tag{4.32}$$

With (4.32) approximate values for all transformation parameters can be acquired.

Step 2: singular value decomposition of R

The approximate values, acquired with (4.32), are, however, not usable in transformation (4.4) and adjustment model (4.7), using the matrices **B** and **F** from (4.24) and (4.26), and the constraints from (4.28). The reason is that the approximate values must fulfil the constraints (4.27), which is a consequence of the linearisation process of the constraints by means of a first order Taylor expansion.

If the approximate values, computed with (4.32) are entered in matrix **R** from (4.22) the result should be an orthogonal matrix. In general this will not be the case. Changing **R** into an orthogonal matrix can be accomplished by performing a singular value decomposition of **R**. The result is three matrices, for which holds (Strang, 1988, p. 443)

$$\mathbf{R} = \mathbf{Q}_1\,\boldsymbol{\Sigma}\mathbf{Q}_2^T, \tag{4.33}$$

where $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are (3×3) orthogonal matrices and $\boldsymbol{\Sigma}$ is a (3×3) diagonal matrix that contains the three singular values on the main diagonal. How these matrices are computed can be found in textbooks on linear algebra, e.g. Strang (1988). Computation routines are available in mathematical software like Matlab.

Changing **R** into an orthogonal matrix is done by changing all singular values to a value of 1, i.e. by removing matrix $\boldsymbol{\Sigma}$ and computing the changed matrix $\mathbf{R}'$ as follows

$$\mathbf{R}' = \mathbf{Q}_1\mathbf{Q}_2. \tag{4.34}$$

Since $\mathbf{Q}_1$ and the transpose of $\mathbf{Q}_2$ are orthogonal matrices also their product $\mathbf{R}'$ is an orthogonal matrix. A proof that the elements of $\mathbf{R}'$ are as close as possible to the analogous elements of $\mathbf{R}$ is given by (Higham, 1989). Closeness is defined with the .

By using the results of (4.32) to construct matrix $\mathbf{R}$ and using singular value decomposition to arrive at the orthogonal rotation matrix of (4.34), approximate values for all transformation parameters of the congruence transformation can be computed.

4.8 Iteration

Since both adjustment model (4.7) and the added constraints (4.28) are linearised, solving the model by the method of least squares needs iteration. In each iteration step the estimates of the coordinate corrections $\Delta\mathbf{c}$ and the corrections to the transformation parameters $\Delta\mathbf{f}$ are used to compute new approximate values. The adjustment model with its added constraints is then solved again. The iteration continues until the difference between the newly computed approximate values and those from the previous step is less than a preset limit.

In each iteration step the new approximate transformation parameters should again fulfil the constraints (4.27). That means that in each iteration step the adaptation of matrix $\mathbf{R}$ by means of a singular value decomposition, as described in the previous section, has to be repeated.

As mentioned before, (4.4) is used in each iteration step to compute new coordinates $\underline{\mathbf{b}}'$. Their cofactor matrix is determined by applying the law of propagation of cofactors. This guarantees that matrices $\mathbf{B}$ and $\mathbf{F}$ and the linearised conditions (4.28) keep their simple structure.

4.9 Model for similarity transformation

A similarity transformation is a congruence transformation with an additional parameter to account for a change of scale (change in the unit of length). This additional scale parameter may be necessary for example if the two coordinate sets have been determined by measuring techniques that cannot be guaranteed to use exactly the same unit of length.

Because of the additional transformation parameter, equation (4.23) receives an additional parameter λ as follows

$$\begin{pmatrix} \underline{\mathbf{x}}^T \\ \underline{\mathbf{y}}^T \\ \underline{\mathbf{z}}^T \end{pmatrix} = \lambda\mathbf{R} \begin{pmatrix} \underline{\mathbf{u}}^T \\ \underline{\mathbf{v}}^T \\ \underline{\mathbf{w}}^T \end{pmatrix} + \mathbf{t}\boldsymbol{\varepsilon}^T. \tag{4.35}$$

Again matrix $\mathbf{R}$ has to be orthogonal.

Linearisation is done relative to the coordinates of $\underline{\mathbf{b}}'$ and to thirteen transformation parameters $\mathbf{f}$ (of which seven remain as independent parameters, because there are six constraints):

$$\mathbf{f} = \begin{pmatrix} \boldsymbol{\alpha}_1 \\ \boldsymbol{\alpha}_2 \\ \boldsymbol{\alpha}_3 \\ \mathbf{t} \\ \lambda \end{pmatrix} . \tag{4.36}$$

Linearisation gives matrices **B** and **F** that resemble very much the ones of the congruence transformation. Matrix **B** is the same as in (4.24), matrix **F** has one column more than matrix **F** in (4.26)

$$\mathbf{F} = \begin{pmatrix} \boldsymbol{\beta} & \mathbf{0} & \mathbf{0} & \varepsilon_1 & \boldsymbol{\beta}\boldsymbol{\alpha}_1 \\ \mathbf{0} & \boldsymbol{\beta} & \mathbf{0} & \varepsilon_2 & \boldsymbol{\beta}\boldsymbol{\alpha}_2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\beta} & \varepsilon_3 & \boldsymbol{\beta}\boldsymbol{\alpha}_3 \end{pmatrix} . \tag{4.37}$$

A slight disadvantage of this matrix **F** is that it makes the adjustment model singular without the constraints. Another approach is possible that does not have this disadvantage. To arrive at rotation matrix **R** six constraints are put on matrix **R** of (4.22). These six constraints were relaxed by allowing an extra parameter λ. It is also possible to put only five constraints on matrix **R**. The three constraints that the lengths of the three rows are equal to 1, are replaced by two constraints: the length of the first row equals the length of the second, and it equals the length of the third row. Matrix **F** now stays the one of (4.26). The linearised constraints are

$$\begin{pmatrix} \boldsymbol{\alpha}_2^{0T} & \boldsymbol{\alpha}_1^{0T} & \mathbf{0} \\ \boldsymbol{\alpha}_3^{0T} & \mathbf{0} & \boldsymbol{\alpha}_1^{0T} \\ \mathbf{0} & \boldsymbol{\alpha}_3^{0T} & \boldsymbol{\alpha}_2^{0T} \\ \boldsymbol{\alpha}_1^{0T} & -\boldsymbol{\alpha}_2^{0T} & \mathbf{0} \\ \boldsymbol{\alpha}_1^{0T} & \mathbf{0} & -\boldsymbol{\alpha}_3^{0T} \end{pmatrix} \begin{pmatrix} \Delta\underline{\boldsymbol{\alpha}}_1 \\ \Delta\underline{\boldsymbol{\alpha}}_2 \\ \Delta\underline{\boldsymbol{\alpha}}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} , \tag{4.38}$$

Approximate values are computed in the same way as for the congruence transformation. An approximate value for λ can be determined from matrix $\boldsymbol{\Sigma}$ of the singular value decomposition by taking the mean of the singular values, i.e. the mean of the three values on the main diagonal of $\boldsymbol{\Sigma}$.

4.10 Barycentric coordinates

The numeric values of the coordinates can be very large, for example if coordinates in a national grid are used. It is advisable to switch in such a case to barycentric coordinates $\underline{\mathbf{x}}_b$, $\underline{\mathbf{y}}_b$, $\underline{\mathbf{z}}_b$.

$$\begin{pmatrix} \underline{\mathbf{x}}_b^T \\ \underline{\mathbf{y}}_b^T \\ \underline{\mathbf{z}}_b^T \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{x}}^T \\ \underline{\mathbf{y}}^T \\ \underline{\mathbf{z}}^T \end{pmatrix} - \frac{1}{n} \begin{pmatrix} \underline{\mathbf{x}}^T\varepsilon \\ \underline{\mathbf{y}}^T\varepsilon \\ \underline{\mathbf{z}}^T\varepsilon \end{pmatrix} , \tag{4.39}$$

where $\varepsilon = (1, 1, \ldots, 1)^T$. Likewise barycentric coordinates $\underline{\mathbf{u}}_b, \underline{\mathbf{v}}_b, \underline{\mathbf{w}}_b$ are defined. The cofactor matrix should be adapted accordingly, but it is possible to consider the second term on the right hand side of (4.39) as a constant term (a non-stochastic shift). The cofactor matrix then remains unchanged. If barycentric coordinates are used, the

last column of (4.26) or the last-but-one column of (4.37) can be left out, because the pertinent parameters are zero, also after adjustment, as long as $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are not correlated mutually.

4.11 Use of adjustment model

Given two vectors of coordinates $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, which describe the same points in 3D, the relation between both is searched by estimating the parameters of a transformation between them. To solve this problem in case of a congruence or similarity transformation, the adjustment model (4.7) is set up and constrained by (4.28) or (4.38). The matrices $\mathbf{B}$ and $\mathbf{F}$ in this model have been defined for the affine, congruence and similarity transformation (resp. (4.24), (4.26) and (4.37). The adjustment model can be used in the following ways:

(i) Suppose that the two vectors of coordinates $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ are defined in different geodetic datums, i.e. reference system r_a is different from system r_b. Adjustment model (4.7) and the constraints (4.28) or (4.38) can be used to transform vector $\underline{\mathbf{b}}$ into system r_a. Coordinates in vector $\underline{\mathbf{b}}$ that have no analogous coordinates in vector $\underline{\mathbf{a}}$ can be transformed to reference system r_a as free variates (Teunissen, 2000, p. 75). Coordinates in vector $\underline{\mathbf{a}}$ that have no analogous coordinates in vector $\underline{\mathbf{b}}$ can be adjusted as free variates in the same way.

(ii) If the two vectors of coordinates $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ differ from each other because of a deformation that can be described by an affine, congruency or similarity transformation, adjustment model (4.7) and the constraints (4.28) or (4.38) can be used to estimate the deformation.

(iii) If a combination of both datum differences and a deformation describes the relation between $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, adjustment model (4.7) and the constraints (4.28) or (4.38) can be used to estimate both datum differences and the deformation.

Using adjustment model (4.7) in combination with the constraints (4.28) or (4.38) has the following characteristics:

(i) Positive-semidefinite, full covariance matrices of $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, and the correlation between both, can be taken account of, which is especially useful if $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$ stem from geodetic measurements, in which case such covariance matrices are to be expected.

(ii) Testing for biases in the coordinates can be easily done by adding parameters to the model that describe the biases (Teunissen, 2006, p. 71).

(iii) Testing for several simultaneous deformations can be easily done by extending the model with extra parameters and constraints that describe those deformations.

(iv) Suppose that a test shows that added parameters are significant. These parameters can easily be added to the adjustment model.

(v) Extending the model to more than two epochs can be readily accomplished by adding the coordinates of a new epoch to the vector of observations and adding

additional parameters to describe the datum of the new epoch and the deformation of this new epoch relative to the other epochs.

Model (4.7) contains a transformation. The model could be constructed without such a transformation, but the transformation fulfils a fundamental function: it takes care that the vectors $\underline{\mathbf{a}}$ and $\underline{\mathbf{b}}$, including their cofactor matrices, are compared with each other only by means of their intrinsic geometric information [the information that can be extracted from the underlying measurements with enough 'sharpness', a word used by Baarda (1995, p. 1)]. Information about the geodetic datum and the precision of its definition are eliminated from the deformation analysis. Tests of the deformation measurements are therefore more accurate. Also the description of the resulting precision and reliability is better. The necessity to eliminate the influence of the geodetic datum is closely related to the search of Baarda (1995, p. 1) for dimensionless quantities and the call of Xu et al. (2000) for invariant quantities.

4.12 Experimental validation

To show the effectiveness of the proposed adjustment model for deformation analysis, a deformation analysis task, as it is encountered in professional practice in the Netherlands, is used. For this task fictitious observations were generated (assuming a normal distribution) and two different deformations simulated.

Three monumental buildings are monitored by total station measurements. Because of underground works, movements of the buildings might occur. Fifteen points are monitored (Fig. 4.1) from an instrument point that is not monumented and varies from epoch to epoch. Two epochs are considered (99 and 100 are the instrument points). A state-of-the-art high precision total station is supposed to have been used. The standard deviation of horizontal and vertical direction measurements is 0.3 mgon, of distance measurements 1 mm. The precision with which a point is defined (idealisation precision) is supposed to be 0.5 mm. The generated observations are listed in the appendix. The second epoch has two different lists of observations. In the first one (called case 1 hereafter) a deformation of point 1 is intentionally introduced. In the second one (case 2) the first five points, belonging to one building, are deformed.

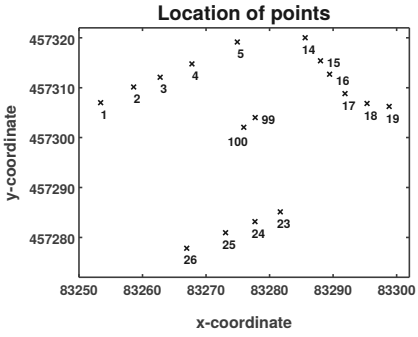

Figure 4.1: 15 object points, 2 instrument points.

The generated observations were processed by the commercial software package MOVE3 (2017), version 4.2.1(x64). 3D coordinates and their full covariance matrix were computed. The covariance matrices of both epochs were defined relative to the approximate x-, y-, z-coordinates of point 1 and 14 and z-coordinate of point 26. Note, however, that the testing results of the adjustment model of this chapter are invariant to a change of

base points. This was confirmed by computations with other base points. The approximate coordinates of the first epoch were in the national grid, those of the second epoch in a local system that was rotated on purpose over 100 gon relative to the national grid.

The adjusted coordinates and their covariance matrix, of both epochs, were transformed to the system of the first epoch and adjusted and tested with the adjustment model of this chapter. A similarity transformation was used, in accordance with the degrees of freedom of the adjustments of each epoch in MOVE3. To do the computations a specifically designed MATLAB programme was used. The results are shown in the appendix.

The covariance matrices of both epochs are rank deficient, because they are defined relative to a subset of the points. This results in a rank deficiency of 14 for the cofactor matrix of (4.8). Regularisation of the cofactor matrix has been used to handle the rank deficiency.

To estimate the parameters of the 3D similarity transformation and to adjust the coordinates, three computations (i.e. two iterations) were needed. In the last iteration the absolute value of the largest correction to the estimated parameters was less than 10^{-12}.

4.12.1 Testing of case 1

In case 1 a deformation of point 1 of 5.2 mm was induced by giving the x-, y-, z-coordinate each a bias of 3 mm.

The overall model test (F-test) of the adjustment yielded an F-value of 1.41, which was more than the critical value of 1.18 (computed with a one-dimensional significance level of 0.1%, a power of 80% and using the B-method of testing). Conventional w-tests were performed, using (B.21) and (B.22). None led to any rejection. Also point tests were performed, using (B.23). A point test is a three-dimensional test, where the alternative hypothesis is, that three independent biases are present for respectively the x-, y-, z-coordinate of a point. Point 1 was rejected with estimated errors (computed with (B.19)) of 4, 2 and 3 mm in resp. the x-, y-, z-direction of system r_a. This shows that using weighted least squares with full covariance matrices and applying a three-dimensional point test is capable of detecting deformed single points.

Point 1 is one of the base points that were held fixed on their approximate values in the epoch adjustments. Testing of this point, however, can be done with the model of this chapter like the testing of the non-base points, without any additional action. This is possible, because the model includes a transformation. The estimated least squares residuals from $\underline{\hat{\mathbf{e}}}$ of (B.7) are zero for the base points, but the reciprocal least squares residuals from $\underline{\hat{\mathbf{r}}}$ of (B.8) are not. The reciprocal residuals are used for testing ((B.17), (B.20) and (B.21)).

4.12.2 Testing of case 2

In case 2 the first five points, belonging to one building, are deformed, all with the same deformation: 3 mm along the x- and y-axis, 22 mm along the z-axis. The x-, y-, and z-axis are those of the local system of the second epoch.

The overall model test (F-test) of the adjustment yielded an F-value of 2.20, which was more than the critical value of 1.18. Both conventional w-tests and point tests were performed. Only the w-tests of the x- coordinate of point 5 and the y-coordinate of point 2 led to rejection. Points 2 and 5 were rejected by the point tests. If, however, the deformation of this one building was tested by formulating an alternative hypothesis that the five points of this building had undergone the same deformation, the pertinent test led to rejection with a test statistic that was 2.01 times larger than the critical value. This was larger than the same ratio for any other alternative hypothesis that was formulated. The estimated deformation was 3, 3 and 24 mm in the direction of respectively the x-, y-, and z-axis in system r_b. This shows that using weighted least squares with full covariance matrices and applying multidimensional tests gives the possibility to detect deformations that are below the noise level of individual points.

4.13 Conclusions

The problem of finding the relation between two Cartesian coordinate vectors that pertain to the same points of an object under deformation, is addressed. If the two vectors refer to two different epochs, the relation between them is determined by a possible difference in geodetic datum, by a possible deformation, which may have occurred between both epochs, and by measurement noise. In this chapter the relation is considered as describable by in principle one or more affine, congruence or similarity transformations, to be extended by other parameters.

An adjustment model is given to estimate the parameters of an affine, a congruence or a similarity transformation. The congruence and similarity transformation are formulated as an affine transformation with constraints. That makes it possible to avoid the use of Euler angles.

To compute approximate values for the parameters of a congruence and similarity transformation it is possible to compute first approximate values for an affine transformation and subsequently to change these values to those of a congruence or similarity transformation by applying a singular value decomposition of the rotation matrix. This computation of approximate values has to be repeated in each iteration step of solving the linearised adjustment model.

Applying the proposed linearised adjustment model with constraints makes it easy to extend the model as follows

1. Several transformations can be combined.
2. Different geodetic datums are taken account of.
3. Testing for biases in the coordinate vectors is made possible.
4. Testing alternative deformation models is possible.

The proposed linearised adjustment model can be solved to get a weighted least squares solution, where full covariance matrices of the coordinates of both epochs are taken account of. The covariance matrices may be singular positive semidefinite matrices.

The effectiveness of the proposed adjustment model is demonstrated in an experiment, where artificially added deformations are successfully detected.

Extending the proposed linearised adjustment model to more than two epochs and to more complex deformation models is straightforward.

5

Testing adjustment models with constraints[1]

5.1 Introduction

Tests for the misfit of one or more observations in a geodetic adjustment model are well-known, see e.g. Teunissen (2006, p. 71ff.). The set-up is to formulate a null-hypothesis and an alternative hypothesis and to derive a test statistic that can be used decide with a certain probability which one is better. In addition, complicated alternative hypotheses with one or more additional parameters, and concerning several or many observations, are tested in the same way. In this setup the test for a bias in an observation is approached as a test of the *complete* adjustment model, in which a bias for that observation is present (alternative hypothesis) or absent (null hypothesis).

The adjustment model is often formulated as a model of observation equations with additional constraints on the parameters. A test for a misfit between one or more constraints and the rest of the adjustment model and the observations, was derived by Rao (1945b). He formulated the constraints as estimable functions of the model parameters and tested the hypothesis that they are equal to some assigned values. According to Rao (1945a), the concept of an estimable function was introduced by Bose (1944).

Independently, it seems, a derivation was given by Aitchison and Silvey (1958) and Silvey (1959). They used the probability distribution of Lagrange multipliers to test the constraints.

Rao (1971) treats the general case of an adjustment model of observation equations with a possibly rank deficient coefficient matrix and a possibly singular covariance matrix of the observations. He defines the concept of a *consistent hypothesis* and shows how to test it. He also showed that an adjustment model with constraints is equivalent

[1] This chapter will be published in Journal of Surveying Engineering (Velsink, 2018). To fit the publication into this study minor changes have been made.

to an adjustment model without constraints, in which the constraints are added as nonstochastic observations. Nonstochastic observations are constants that are treated as if they were observations (pseudo-observations).

Teunissen (1985b) treated the test of Lagrange multipliers and, additionally, showed how to describe the reliability, both internal and external, of the test.

Lehmann and Neitzel (2013) showed how to test a subset of constraints and multiple individual constraints, and give an elaborate geodetic example.

If it is not clear which constraints are valid, a choice between alternative sets of constraints has to be made, either by using multiple outlier detection or by using an information criterion (Lehmann and Lösler, 2017).

In recent years the adjustment model with constraints on the parameters has been handled in the framework of the adjustment model of *total least squares*. It has been elaborated upon for the situation that the covariance matrix is the identity matrix (Schaffrin and Felus, 2009), for a full covariance matrix (Fang, 2014), and for multiple constraints and a full, singular covariance matrix (Jazaeri et al., 2014; Amiri-Simkooei, 2017). Parameters of an affine transformation, to be estimated in a total least squares adjustment, can be constrained to become the parameters of a similarity or congruence transformation (Tong et al., 2011; Zhang et al., 2016).

However, many adjustment problems that are given as examples for total least squares, can be handled equally well by the standard least-squares algorithm that uses a full, but possibly singular, covariance matrix, and a possibly rank deficient coefficient matrix. Such a standard algorithm is described in the "Adjustment with rank deficient cofactor matrix and coefficient matrix" section. In this chapter, it is shown that it can handle an adjustment model with constraints. Examples of its field of application are the two-dimensional (2D) similarity and congruence transformations (Velsink, 2015b), the 3D affine, similarity and congruence (rigid-body) transformations (Velsink, 2015a), and their extensions to a time series of transformations (Velsink, 2016b). The advantage of using the standard algorithm is that its complete body of knowledge can be applied. Moreover, all stochastic variables are elements of the observation vector and treated in the same way.

In this chapter an alternative approach is proposed for testing the adjustment model with constraints on the parameters. Here hypotheses about models with constraints are tested using nonstochastic observations. An advantage of this approach is that the same equations are used for testing hypotheses about stochastic observations as for hypotheses about constraints. Consequently hypotheses about simultaneous biases in both the stochastic observations and in (part of) the constraints can be tested. Test quality quantities, like minimal detectable biases, can be computed for such hypotheses. It enables the assessment of the test quality. A further advantage is that switching from hard constraints (standard deviation $= 0$) to soft constraints (standard deviation > 0) is accomplished by changing just that standard deviation, which makes a sensitivity analysis easy. The proposed approach gives a simple procedure to adjust and test both stochastic observations and constraints on the parameters, and quantify the test quality. An overview of six computation methods for the alternative approach is given.

The purpose of this chapter is to describe the alternative approach to test an adjustment model with constraints; to provide a still missing, systematic overview of six computation methods, as well as some not yet published derivations; and to show, in two examples, the applicability for geodetic deformation analysis. The setup of the chapter is as follows. The *adjustment model* of observation equations with constraints on the parameters, formulated as nonstochastic observations, is given in the next section. The introduction of nonstochastic observations in the adjustment model makes it inevitable that the covariance matrix of the observations is singular. The solution of the adjustment model with a singular covariance matrix is given. The *model testing* is treated in the "Testing" section. It needs special care because of the singular covariance matrix. For the computation of the test statistic, it is possible to use *reciprocal residuals*, which are defined in the "Overview of computation methods with rank deficient matrices" section where an overview is given of *six methods* to compute the test statistic. In the "Testing procedure and quality description" section the *procedure* for testing adjustment models with constraints is treated, including the test quality description. Finally two *examples* from geodetic deformation analysis are given to show the usability of the proposed test of an adjustment model with constraints.

5.2 Adjustment with rank deficient cofactor matrix and coefficient matrix

The adjustment model has the general structure:

$$\underline{\mathbf{y}} = \mathbf{A}\mathbf{x} + \mathbf{a}_0 + \underline{\mathbf{e}}, \quad E\{\underline{\mathbf{e}}\} = \mathbf{0}, \quad D\{\underline{\mathbf{y}}\} = \sigma^2\mathbf{Q}_y. \tag{5.1}$$

The m-vector $\underline{\mathbf{y}}$ contains m observations. The $\underline{\mathbf{y}}$ is underlined to indicate that it is *stochastic*: a probability density function, here assumed to be a normal distribution, is linked to it. The matrix $\mathbf{A}$ is the coefficient matrix (model matrix), which may be rank deficient, the n-vector $\mathbf{x}$ contains n parameters, $\mathbf{a}_0$ is an m-vector of constant terms and $\underline{\mathbf{e}}$ is the m-vector of random noise. The expectation of $\underline{\mathbf{e}}$, i.e. $E\{\underline{\mathbf{e}}\}$, is zero. $D\{\underline{\mathbf{y}}\}$ is the covariance matrix of $\underline{\mathbf{y}}$, split into a variance factor σ^2 and a symmetric positive semidefinite cofactor matrix $\mathbf{Q}_y$, so it can be rank deficient (singular). The variance factor is any positive real scalar.

A least-squares solution to model (5.1) with rank deficient coefficient matrix and singular cofactor matrix is as follows (Rao, 1971):

$$\overline{\mathbf{Q}}_y = \mathbf{Q}_y + \lambda\mathbf{A}\mathbf{A}^T, \tag{5.2}$$

$$\mathbf{A}_m^- = (\mathbf{A}^T\overline{\mathbf{Q}}_y^-\mathbf{A})^-\mathbf{A}^T\overline{\mathbf{Q}}_y^-, \tag{5.3}$$

$$\underline{\hat{\mathbf{x}}} = \mathbf{A}_m^-(\underline{\mathbf{y}} - \mathbf{a}_0), \tag{5.4}$$

with $\lambda > 0$ any positive real scalar. Let $\mathcal{R}(.)$ indicate the range space. If $\mathcal{R}(\mathbf{A}) \subset \mathcal{R}(\mathbf{Q}_y)$, the choice $\lambda = 0$ is possible.

The symbol $^-$ indicates a g-inverse (generalised inverse). For any matrix $\mathbf{X}$ the g-inverse $\mathbf{X}^-$ is defined by $\mathbf{X}\mathbf{X}^-\mathbf{X} = \mathbf{X}$. Matrix $\mathbf{A}_m^-$ is a minimum $\mathbf{Q}_y$-norm g-inverse of $\mathbf{A}$. It is a g-inverse with the additional property that any linear function $\mathbf{p}^T\underline{\hat{\mathbf{x}}}$, with $\underline{\hat{\mathbf{x}}}$ from

equation (5.4), that is an unbiased estimator of $\mathbf{p}^T\mathbf{x}$, has minimum variance among all linear unbiased estimators of $\mathbf{p}^T\mathbf{x}$. Expressions to compute this g-inverse are given by Rao and Mitra (1971, p. 148) and Rao (1971). The estimator $\mathbf{p}^T\underline{\hat{\mathbf{x}}}$ is called a best linear unbiased estimator (BLUE).

By adding the term $\lambda\mathbf{A}\mathbf{A}^T$ the space $\mathcal{R}(\mathbf{Q}_y)$ is *amplified* to the space $\mathcal{R}(\overline{\mathbf{Q}}_y)$ in such a way that $\mathcal{R}(\mathbf{A}) \subset \mathcal{R}(\overline{\mathbf{Q}}_y)$. The estimator of $\underline{\mathbf{e}}$ is $\underline{\hat{\mathbf{e}}} = \underline{\mathbf{y}} - (\mathbf{A}\underline{\hat{\mathbf{x}}} + \mathbf{a}_0)$. It minimises $\underline{\mathbf{e}}^T\overline{\mathbf{Q}}_y^{-}\underline{\mathbf{e}}$, but also $\underline{\mathbf{e}}^T\mathbf{Q}_y^{-}\underline{\mathbf{e}}$. It follows that $\underline{\hat{\mathbf{x}}}$ is the least-squares estimator we are looking for. Here it is proposed to call $\overline{\mathbf{Q}}_y$ an *amplified cofactor matrix*.

The cofactor matrix of $\underline{\hat{\mathbf{x}}}$, based on the original $\mathbf{Q}_y$, is:

$$\mathbf{Q}_{\hat{x}} = \overline{\mathbf{Q}}_{\hat{x}} - \lambda\mathbf{I}, \tag{5.5}$$

in which $\mathbf{I}$ is the unit matrix. $\overline{\mathbf{Q}}_{\hat{x}}$ is the cofactor matrix of $\underline{\hat{\mathbf{x}}}$ because it follows from applying the propagation law of cofactors to equation (5.4). The derivation is given in appendix C.5.

In the following sections, the constraints are formulated as nonstochastic observations. This makes it possible to test with the same method *all* constraints, or just a *subset* of the constraints, or even simultaneously a subset of the constraints *and* a subset of the stochastic observations. Therefore, the observations are split into the stochastic observations $\underline{\mathbf{y}}_s$ and the nonstochastic $\mathbf{y}_z$. The constraints are described by $\mathbf{y}_z$. The coefficient matrix, the vector of constants and the residual vector are split accordingly.

$$\begin{pmatrix}\underline{\mathbf{y}}_s\\ \mathbf{y}_z\end{pmatrix} = \begin{pmatrix}\mathbf{A}_s\\ \mathbf{A}_z\end{pmatrix}\mathbf{x} + \begin{pmatrix}\mathbf{a}_{0s}\\ \mathbf{a}_{0z}\end{pmatrix} + \begin{pmatrix}\underline{\mathbf{e}}_s\\ \mathbf{e}_z\end{pmatrix}, \quad E\{\begin{pmatrix}\underline{\mathbf{e}}_s\\ \mathbf{e}_z\end{pmatrix}\} = \begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}. \tag{5.6}$$
$$D\{\begin{pmatrix}\underline{\mathbf{y}}_s\\ \mathbf{y}_z\end{pmatrix}\} = \sigma^2\begin{pmatrix}\mathbf{Q}_{y_s} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}.$$

This model is a model of observation equations with constraints on the parameters. The constraints are described by the nonstochastic observations $\mathbf{y}_z$.

Model (5.1), and (5.6) likewise, is linear or linearised. If necessary, its least-squares solution is computed by iteration.

5.3 Testing

5.3.1 Null and Alternative Hypothesis

The standard method for testing a linear adjustment model uses a *nonsingular* cofactor matrix of the observations. It is given here in the formulation of Teunissen (2006, p. 71) and extended to handle a *singular* cofactor matrix. This is necessary, because the cofactor matrix $\mathbf{Q}_y$ in model (5.6) is singular. A null hypothesis H_0 and an alternative model H_a are formulated:

$$H_0 : E\{\underline{\mathbf{y}}\} = \mathbf{A}\mathbf{x} + \mathbf{a}_0 \tag{5.7}$$
$$H_a : E\{\underline{\mathbf{y}}\} = \mathbf{A}\mathbf{x} + \mathbf{a}_0 + \mathbf{C}\nabla \tag{5.8}$$

Equation (5.7) equals model (5.1). Equation (5.8) has an additional term $\mathbf{C}\nabla$, with $\mathbf{C}$ a known $(m \times q)$-coefficient matrix (model matrix), and an unknown q-vector ∇. The term $\mathbf{C}\nabla$ gives the bias in the functional model. Both hypotheses have the same stochastic model, described with a covariance matrix $\mathbf{D}\{\underline{\mathbf{y}}\}$, assumed to belong to a normal distribution.

The difference between H_0 and H_a is the vector $\mathbf{C}\nabla$. Under H_0 ∇ is zero, under H_a it is not. Assume $\underline{\hat{\nabla}} \sim N_m(\nabla, \sigma^2\mathbf{Q}_{\hat{\nabla}})$, with $\underline{\hat{\nabla}}$ a least-squares estimate of ∇, and N_m denoting an m-variate normal distribution with mean ∇ and covariance matrix $\sigma^2\mathbf{Q}_{\hat{\nabla}}$. Let $(.)_{rs}^{-}$ indicate a reflexive symmetric g-inverse. Then, according to Rao and Mitra (1971, theorem 9.2.3), the quadratic form:

$$\underline{T}_q = \frac{1}{\sigma^2}\underline{\hat{\nabla}}^T (\mathbf{Q}_{\hat{\nabla}})_{rs}^{-} \underline{\hat{\nabla}} \tag{5.9}$$

has a $\chi^2(q, \delta)$-distribution, $q = \text{rank}(\mathbf{Q}_{\hat{\nabla}})$ and $\delta = \frac{1}{\sigma^2}\nabla^T(\mathbf{Q}_{\hat{\nabla}})_{rs}^{-}\nabla$.

The null hypothesis is tested against the alternative hypothesis by using $\underline{T}_q$ as the test statistic. The test is to choose a significance level α, to compute the critical value and to test whether the computed value of $\underline{T}_q$ exceeds the critical value. If this happens, the null hypothesis is rejected (Teunissen, 2006, p. 78). The test is a *generalised likelihood ratio test*, which is shown in appendix C.4.

5.3.2 Testable and Consistent Hypothesis, Invariance of $\underline{T}_q$

For the test statistic to be usable, it must be capable of distinguishing between the null and alternative hypothesis, that is, the alternative hypothesis must be testable. Matrix $\mathbf{C}$ describes a *nontestable hypothesis*, if it spans a space that is a subspace of the space spanned by matrix $\mathbf{A}$. ∇ is not estimable in this situation.

Another requirement for the test statistic is that it gives a unique value, independent of the choice of reflexive symmetric g-inverse in equation (5.9). Therefore, the concept of a *consistent hypothesis* is important. A hypothesis, described by a matrix $\mathbf{C}$, is consistent, if

$$\underline{\hat{\nabla}} \in \mathcal{R}(\mathbf{Q}_{\hat{\nabla}}). \tag{5.10}$$

$\underline{T}_q$ is invariant for the choice of reflexive symmetric g-inverse in equation (5.9), if the hypothesis is consistent. This follows from lemma 2.2.4 (ii) of Rao and Mitra (1971).

5.3.3 Comparison with Other Methods in the Literature

In the introduction of this chapter, several references were given that provide methods to test H_0 against H_a (Rao, 1945b; Aitchison and Silvey, 1958; Silvey, 1959; Rao, 1971; Teunissen, 1985b; Lehmann and Neitzel, 2013). In all these references H_0 and H_a are

written in a form similar to:

$$\begin{cases} H_0: & \underline{\mathbf{y}}_s = \mathbf{A}_s\mathbf{x} + \mathbf{a}_{0s} + \underline{\mathbf{e}}_s; \quad \mathbf{y}_z - \mathbf{a}_{0z} = \mathbf{A}_z\mathbf{x}, \\ H_a: & \underline{\mathbf{y}}_s = \mathbf{A}_s\mathbf{x} + \mathbf{a}_{0s} + \underline{\mathbf{e}}_s, \\ & E\{\underline{\mathbf{e}}_s\} = \mathbf{0};\ D\{\underline{\mathbf{y}}_s\} = \sigma^2\mathbf{Q}_{y_s}. \end{cases} \tag{5.11}$$

Here $\mathbf{y}_z - \mathbf{a}_{0z} = \mathbf{A}_z\mathbf{x}$ are constraints on the parameters $\mathbf{x}$.

Equation (5.11) states that the *null* hypothesis is equal to the *alternative* hypothesis, but with *additional constraints* on the parameters. It is tested whether these constraints are valid. The references give as test statistic $\underline{T}_q$:

$$\begin{aligned} &\underline{\mathbf{u}} = (\mathbf{y}_z - \mathbf{a}_{0z}) - \mathbf{A}_z\underline{\hat{\mathbf{x}}}, \quad \mathbf{Q}_u = \mathbf{A}_z\mathbf{Q}_{\hat{x}}\mathbf{A}_z^T, \\ &\underline{T}_q = \underline{\mathbf{u}}^T\mathbf{Q}_u^{-1}\underline{\mathbf{u}}, \end{aligned} \tag{5.12}$$

in which $\underline{\hat{\mathbf{x}}}$ is the least-squares solution of H_a (i.e. without constraints), and $\mathbf{Q}_u$ is assumed invertible.

This chapter treats testing for biases of an adjustment model with constraints, using equations (5.7) and (5.8). Here the *alternative* hypothesis is equal to the *null* hypothesis, but with *additional parameters* (the biases).

Suppose we want to test all constraints, just as it is done with equation (5.12), but now using equations (5.7) and (5.8). Here the constraints are incorporated into H_0 as nonstochastic observations. In H_a, equation (5.8), they are still present as nonstochastic observations, but they are neutralised, if we take $\mathbf{C} = (\mathbf{0}\ ,\ \mathbf{I})^T$, in which $\mathbf{0}$ matches $\mathbf{A}_s$ of equation (5.6), and the unit matrix $\mathbf{I}$ matches $\mathbf{A}_z$. It follows then that $\underline{\hat{\nabla}} = \underline{\mathbf{u}}$, and $\mathbf{Q}_{\hat{\nabla}} = \mathbf{Q}_u$, giving the same expression for $\underline{T}_q$.

However, $\mathbf{C} = (\mathbf{0}\ ,\ \mathbf{I})^T$ is only one possible choice for $\mathbf{C}$. Therefore, by using (5.7) and (5.8) to formulate H_0 and H_a, it is easy to test various types of alternative hypotheses by specifying appropriate matrices $\mathbf{C}$.

Thus the various special cases, distinguished by Lehmann and Neitzel (2013), as well as cases involving both constraints and stochastic observations, can be treated by one procedure. Methods to compute the appropriate test statistic $\underline{T}_q$ are elaborated upon in the next section.

5.4 Overview of computation methods with rank deficient matrices

5.4.1 Two Approaches, Resulting In Six Methods

Several methods are available to compute $\underline{T}_q$ when both the cofactor matrix $\mathbf{Q}_y$ and the coefficient matrix $\mathbf{A}$ have full rank (Teunissen, 2006, p. 71ff.). Here methods are proposed for the situation when both $\mathbf{Q}_y$ and $\mathbf{A}$ are rank deficient. Two approaches are given, from which the second one results in five methods. These six methods are not exhaustive, but should provide enough flexibility to solve most practical situations.

First approach Solve the *alternative* hypothesis using equation (5.8). It can be formulated as a model of observation equations:

$$E\{\underline{\mathbf{y}}\} = (\mathbf{A} \vdots \mathbf{C}) \begin{pmatrix} \mathbf{x} \\ \nabla \end{pmatrix} + \mathbf{a}_0. \tag{5.13}$$

Solving this model using equation (5.4), we get $\underline{\hat{\nabla}}$ and its cofactor matrix $\mathbf{Q}_{\hat{\nabla}}$, which can be used in equation (5.9) to compute the test statistic.

Second approach Solve the *null* hypothesis, and do not use matrix $\mathbf{C}$, until $\underline{T}_q$ is computed. This means that for testing different alternative hypotheses, characterised by different matrices $\mathbf{C}$, only one adjustment process is necessary.

Because $\underline{\mathbf{e}} \in \mathcal{R}(\mathbf{Q}_y)$ (proof in appendix C.2), it is possible to write, with $\underline{\mathbf{r}}$ some appropriate vector

$$\underline{\mathbf{e}} = \mathbf{Q}_y \underline{\mathbf{r}}. \tag{5.14}$$

Here the elements of vector $\underline{\mathbf{r}}$ are called *reciprocal residuals*. Using the least-squares estimates $\underline{\hat{\mathbf{r}}}$ and their cofactor matrix $\mathbf{Q}_{\hat{r}}$, the least-squares solution $\underline{\hat{\nabla}}$ and its cofactor matrix $\mathbf{Q}_{\hat{\nabla}}$ can be written (derivation in appendix C.2):

$$\underline{\hat{\nabla}} = (\mathbf{C}^T \mathbf{Q}_{\hat{r}} \mathbf{C})^- \ \mathbf{C}^T \underline{\hat{\mathbf{r}}}, \quad \mathbf{Q}_{\hat{\nabla}} = (\mathbf{C}^T \mathbf{Q}_{\hat{r}} \mathbf{C})_{rs}^-. \tag{5.15}$$

The least-squares estimates $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$ follow from the adjustment of the *null* hypothesis (see appendix C.2).

Test statistic $\underline{T}_q$ follows from inserting (5.15) in (5.9):

$$\underline{T}_q = \frac{1}{\sigma^2} \underline{\hat{\mathbf{r}}}^T \mathbf{C} (\mathbf{C}^T \mathbf{Q}_{\hat{r}} \mathbf{C})_{rs}^- \mathbf{C}^T \underline{\hat{\mathbf{r}}}. \tag{5.16}$$

Condition (5.10) is necessary for test statistic $\underline{T}_q$ to be unique. It can now be written:

$$\mathbf{C}^T \underline{\hat{\mathbf{r}}} \in \mathcal{R}(\mathbf{C}^T \mathbf{Q}_{\hat{r}} \mathbf{C}). \tag{5.17}$$

If $\mathbf{C}^T \mathbf{Q}_{\hat{r}} \mathbf{C}$ is an invertible matrix, the solution $\underline{\hat{\nabla}}$ is unique, and $\underline{T}_q$ is:

$$\underline{T}_q = \frac{1}{\sigma^2} \underline{\hat{\mathbf{r}}}^T \mathbf{C} (\mathbf{C}^T \mathbf{Q}_{\hat{r}} \mathbf{C})^{-1} \mathbf{C}^T \underline{\hat{\mathbf{r}}}. \tag{5.18}$$

If the cofactor matrix $\mathbf{Q}_y$ is regular, the reciprocal residuals can be computed as $\underline{\hat{\mathbf{r}}} = \mathbf{Q}_y^{-1} \underline{\hat{\mathbf{e}}}$. However, $\mathbf{Q}_y$ is singular in model (5.6) because of the nonstochastic observations. To compute $\underline{T}_q$ the two approaches are elaborated upon in six methods, treated in the next section.

5.4.2 Six Computation Methods

5.4.2.1 Observation equations of alternative hypothesis

To *test alternative hypotheses* concerning the model, two approaches have been given. The first approach uses equation (5.13), which is solved by using, for example, equations

(5.2)–(5.4). It gives $\hat{\underline{\nabla}}$ as part of the estimated parameter vector, and analogously its cofactor matrix. With them $\underline{T}_q$ can be computed.

The second approach uses equation (5.16) and needs, therefore, the reciprocal residuals $\hat{\underline{\mathbf{r}}}$ and their cofactor matrix. In the following an overview of five methods to compute them is given. These methods use the Pandora box matrix, condition equations, a regularised cofactor matrix, reduction after orthogonalisation, or standard deviations that are almost, not precisely, zero ("almost zero").

5.4.2.2 Pandora box

The first method to compute reciprocal residuals and their cofactor matrix uses the following system of equations with Pandora box matrix $\mathbf{P}$ (Kourouklis and Paige, 1981, eq. (2.8)):

$$\underbrace{\begin{pmatrix} \mathbf{Q}_y & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix}}_{\mathbf{P}} \begin{pmatrix} \hat{\underline{\mathbf{r}}} \\ \hat{\underline{\mathbf{x}}} \end{pmatrix} = \begin{pmatrix} \underline{\mathbf{y}} - \mathbf{a}_0 \\ \mathbf{0} \end{pmatrix}. \tag{5.19}$$

This system follows from the combination of equations (C.5) and (C.6) in appendix C.2. Any g-inverse of $\mathbf{P}$ is called by C.R. Rao a "Pandora box" (Rao, 1971), because it supplies all the ingredients needed for adjustment and testing. The g-inverse $\mathbf{P}^-$ gives the solution of system (5.19). Both $\mathbf{Q}_y$ and $\mathbf{A}$ may be rank deficient.

$$\begin{aligned} \begin{pmatrix} \hat{\underline{\mathbf{r}}} \\ \hat{\underline{\mathbf{x}}} \end{pmatrix} &= \underbrace{\begin{pmatrix} \mathbf{Q}_y & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0} \end{pmatrix}^-}_{\mathbf{P}^-} \begin{pmatrix} \underline{\mathbf{y}} - \mathbf{a}_0 \\ \mathbf{0} \end{pmatrix} = \\ &= \underbrace{\begin{pmatrix} \mathbf{C}_1 & \mathbf{C}_2 \\ \mathbf{C}_3 & -\mathbf{C}_4 \end{pmatrix}}_{\mathbf{P}^-} \begin{pmatrix} \underline{\mathbf{y}} - \mathbf{a}_0 \\ \mathbf{0} \end{pmatrix}. \end{aligned} \tag{5.20}$$

If $\mathbf{P}^-$ is computed explicitly, we get the cofactor matrices $\mathbf{Q}_{\hat{r}} = \mathbf{C}_1\mathbf{Q}_y\mathbf{C}_1$ and $\mathbf{Q}_{\hat{x}} = \mathbf{C}_4$. If $\mathbf{P}$ is nonsingular, $\mathbf{P}^-$ is the regular inverse $\mathbf{P}^{-1}$ of $\mathbf{P}$, and $\mathbf{Q}_{\hat{r}} = \mathbf{C}_1$. $\mathbf{P}$ is nonsingular, if the range space of $\mathbf{A}$, that is, $\mathcal{R}(\mathbf{A})$, contains the nullspace of $\mathbf{Q}_y$, and, simultaneously, $\mathbf{A}$ has full column rank. Even if $\mathbf{A}$ is rank deficient, but its nullspace does not intersect the nullspace of $\mathbf{Q}_y$, $\mathbf{P}$ is nonsingular. These conditions are fulfilled in many use cases.

5.4.2.3 Condition equations

The second method to compute $\hat{\underline{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$, is to switch from the model of observation equations to the model of condition equations. This model is dual to the model of observation equations, and is elaborated upon in appendix C.2. By using the model of condition equations, the necessity to invert the cofactor matrix $\mathbf{Q}_y$ is avoided.

The relation $\mathbf{B}^T\mathbf{A} = \mathbf{0}$ holds, with $\mathbf{A}$ being the coefficient matrix of the model of observation equations and $\mathbf{B}$ being the coefficient matrix of the model of condition

equations. This relation means that $\mathcal{R}(\mathbf{B})$, the range space of $\mathbf{B}$, is the null space of $\mathcal{R}(\mathbf{A}^T)$. The numerical determination of $\mathbf{B}$ can be accomplished by, for example, singular value decomposition.

The coefficient matrix $\mathbf{A}$ may be rank deficient. By switching to the model of condition equations, and determining $\mathbf{B}$ as a base matrix of the mentioned null space, the rank deficiency of $\mathbf{A}$ no longer poses a problem.

The reciprocal residuals and their cofactor matrix are computed with equation (C.7) in appendix C.2.

5.4.2.4 Regularised cofactor matrix

The third method to compute $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$ is to use equation (5.2). If the range space of $\mathbf{A}$, that is, $\mathcal{R}(\mathbf{A}) = \mathcal{R}(\mathbf{A}\mathbf{A}^T)$, contains the nullspace of $\mathbf{Q}_y$, $\overline{\mathbf{Q}}_y$ is a regular matrix and its regular (Cayley) inverse exists. We met this condition already in the treatment of the Pandora box method. It is equivalent to the condition that the normal matrix of the model of condition equations is invertible. It will be elaborated upon in appendix C.6. Equation (5.2) can now be called the *regularisation* of the cofactor matrix. $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$ are computed from:

$$\underline{\hat{\mathbf{r}}} = \overline{\mathbf{Q}}_y^{-1}\underline{\hat{\mathbf{e}}} \quad \text{and} \quad \mathbf{Q}_{\hat{r}} = \overline{\mathbf{Q}}_y^{-1}\mathbf{Q}_{\hat{e}}\overline{\mathbf{Q}}_y^{-1}. \tag{5.21}$$

They are unaffected by the regularisation (proof is given in appendix C.5), which means that testing is unaffected.

5.4.2.5 Reduction after orthogonalisation

The fourth method to compute reciprocal residuals and their cofactor matrix uses a reduced model in a first adjustment step, followed by a follow-up adjustment as second step. To solve model (5.1), the observations are first orthogonalised relative to $\mathbf{Q}_y$, resulting in stochastic observations $\underline{\bar{\mathbf{y}}}_1$ with the unit matrix as cofactor matrix, and nonstochastic observations $\underline{\bar{\mathbf{y}}}_2$ (Rao and Mitra, 1971, p. 149). The necessary matrices $\mathbf{F}$ and $\mathbf{N}$ are given in appendix C.2:

$$\underline{\bar{\mathbf{y}}}_1 = \mathbf{F}^T\underline{\mathbf{y}}; \quad \mathbf{Q}_{\bar{y}_1} = \mathbf{I}, \tag{5.22}$$

$$\underline{\bar{\mathbf{y}}}_2 = \mathbf{N}^T\underline{\mathbf{y}}; \quad \mathbf{Q}_{\bar{y}_2} = \mathbf{0}. \tag{5.23}$$

The nonstochastic observations are handled as constraints, and used to define an adapted vector of observations $\underline{\mathbf{y}}_R$ with the unit matrix as cofactor matrix and a reduced vector of parameters $\mathbf{x}_R$ (appendix C.2):

$$\underline{\mathbf{y}}_R = \mathbf{A}_R\mathbf{x}_R + \underline{\mathbf{e}}_R; \quad \mathbf{Q}_{y_R} = \mathbf{I}, \tag{5.24}$$

This reduced model has a coefficient matrix $\mathbf{A}_R$ of full rank. After obtaining the least-squares solution of this model, a follow-up model is formulated, which determines the reciprocal residuals of the original observations (appendix C.2).

Notice that the new nonstochastic observations $\underline{\bar{\mathbf{y}}}_2$ consist of those formulated in the original model, supplemented with nonstochastic observations that follow from a possible singularity of the cofactor submatrix of the stochastic observations $\underline{\mathbf{y}}_s$.

5.4.2.6 Almost zero

The fifth method to compute reciprocal residuals and their cofactor matrix is to change the zero standard deviations of the nonstochastic observations into very small values.

It may be that the cofactor matrix $\mathbf{Q}_y$ is still singular after this change because the submatrix $\mathbf{Q}_{y_s}$ of the stochastic observations $\underline{\mathbf{y}}_s$ is singular. It is proposed here to amplify $\mathbf{Q}_y$ in this situation into $\overline{\mathbf{Q}}_y$, using a base matrix $\mathbf{N}$ of the nullspace of $\mathbf{Q}_y$ and a very small value ε:

$$\overline{\mathbf{Q}}_y = \mathbf{Q}_y + \varepsilon \mathbf{N}\mathbf{N}^T. \tag{5.25}$$

$\mathcal{R}(\overline{\mathbf{Q}}_y)$ is the complete space $\mathbb{R}^m$ and $\overline{\mathbf{Q}}_y$ is a regular matrix. The regular (Cayley) inverse of $\overline{\mathbf{Q}}_y$ can now be computed and used in equation (5.21) to compute $\underline{\hat{\mathbf{r}}}$ and its cofactor matrix. The value, however, for the standard deviations or ε should, on the one hand, not be *too* small, in order to keep the inversion process stable. On the other hand, they should not be too large, in order to keep the results sufficiently close to the desired ones. In the first example at the end of this chapter, a standard deviation of 0.01 mm can be taken to give the same results as the other methods at submillimeter level.

The testing method of the first approach, and the four testing methods with reciprocal residuals and their cofactor matrix, all yield the same testing results. The testing results of this fifth method will approach those results, as ε approaches zero, cf. Rao and Mitra (1971, pp. 136, 149).

5.4.3 Independent Constraints

For four of the five methods to compute reciprocal residuals, a matrix exists whose nonsingularity indicates that there are only *independent constraints*, or, equivalently, independent nonstochastic observations. It is proposed here to call this matrix the *indicator matrix*. The independent constraints make it possible to apply equation (5.16). If the constraints are dependent, the computation of $\underline{T}_q$ is more involved. It may be necessary, for example, to amplify $\mathbf{Q}_t$, and $\underline{T}_q$ may not be unique. Therefore, it is assumed in this chapter that the indicator matrices are nonsingular. Table 5.1 lists the indicator matrices.

A comprehensive treatment of these indicator matrices is given in appendix C.6.

5.4.4 Comparison of Methods

Six methods have been shown to test an adjustment model with constraints, one for the first approach (which solves the alternative hypothesis), plus five for the second

Computation method	Indicator matrix
Pandora box	Pandora box matrix $\mathbf{P}$
Condition equations	Normal matrix $\mathbf{Q}_t$
Regularised cofactor matrix	Amplified cofactor matrix $\overline{\mathbf{Q}}_y$
Reduction after orthogonalisation	Constraints matrix $\mathbf{S}$
Almost zero	–

Table 5.1: Indicator matrix for each computation method.

approach (which solves the null hypothesis, and only uses the alternative hypothesis for the test statistic). Each method has its advantages and disadvantages. Computational time may be an important factor in deciding which method to use, as will the available software.

To give an impression of relative computation times, an example of the application of model (5.1) is considered. It concerns a geodetic deformation analysis with several epochs of 3D coordinate sets with their covariance matrices. The epochs are linked by transformations that are incorporated in the adjustment model (Velsink, 2016b). The model has been applied to adjust and test a deformation network of a large bridge in the Netherlands. It was a regular assignment by professional practice. Ten epochs of each more than 90 points with (x, y, z)-coordinates, had been measured. The adjustment model contained 5391 observations (2832 stochastic and 2559 nonstochastic), 2940 parameters and 2451 conditions. It was adjusted and tested with all six methods, described in this chapter. In the following, for each method it is indicated, which part of the computation takes the most time.

The model "almost zero" gave the fastest computation of the adjustment and testing results, followed by the method that applies condition equations. Slowest among the methods of the second approach was the method that uses a regularised cofactor matrix. The method of the first approach is slowest if more than one alternative hypothesis has to be tested.

1. **Observation equations of alternative hypothesis** In addition to $\underline{\hat{\nabla}}$, $\underline{\hat{\mathbf{x}}}$ is also computed under the alternative hypothesis. This means extra effort, which can become a disadvantage if a large number of alternative hypotheses has to be tested. It is possible to eliminate $\mathbf{x}$ as a parameter vector from the system, but then we arrive at the method that uses the model of condition equations.

   Any of the following methods can be used for the adjustment; however, only the estimated parameters and their cofactor matrix need to be computed, not the reciprocal residuals and their cofactor matrix. Yet, that does not give much reduction of computation time for any of the methods.

2. **Pandora box** The size of the Pandora box matrix is large: The sum of the number of observations and the number of parameters. This is a disadvantage when solving the system of equations. The advantage is, however, that the possible sparseness of $\mathbf{Q}_y$ and $\mathbf{A}$ is preserved. In the "bridge example" most time

was spent on the inversion of the Pandora box matrix.

A further advantage is that the solution gives both the adjustment results ($\hat{\underline{\mathbf{x}}}$ and its cofactor matrix) and the results needed for testing ($\hat{\underline{\mathbf{r}}}$ and its cofactor matrix).

3. **Condition equations**

   Switching to the model of condition equations has the advantage that the inverse of $\mathbf{Q}_y$ is not needed to compute a solution. This avoids computation time and the problem of inverting a singular matrix.

   The size of the normal equations in the model of condition equations is determined by the number of conditions, that is, the redundancy of the model. If it is small compared with the number of parameters in the model of observation equations, the model of condition equations may be preferred.

   In the previously discussed bridge example, the elimination of the parameters from matrix **A** to get matrix **B** was very fast. The most time was spent on the computation of the cofactor matrix of the reciprocal residuals.

   The misclosures of the conditions can be computed and tested separately, allowing for an extra possibility to detect biases in the model.

4. **Regularised cofactor matrix** The method to regularise the cofactor matrix uses the well-known and widely applied model of observation equations (the Gauss-Markov model). The size of the normal equations in the model of observation equations is determined by the amount of parameters. A disadvantage is the loss of sparseness of the cofactor matrix when it is amplified. The cofactor matrix has to be inverted to get the weight matrix. The inversion time was very large for the bridge example.

5. **Reduction after orthogonalisation** Depending on the number of stochastic observations, nonstochastic observations, and the redundancy, the model using a reduction of the observation vector and the coefficient matrix can be advantageous.

   In the bridge example, the most computation time was lost in the orthogonalisation.

6. **Almost zero** Existing software for least-squares adjustment can easily be accomodated to make it suitable for testing constraints, if the method "almost zero" is used, in which the zero standard deviations are replaced by very small values. The software needs to have an observation type that allows for the input of a 1-, 2- or 3D-vector, for which the observed values can be put to zero and the standard deviations to almost zero.

5.5 Testing Procedure and Quality Description

It has been shown in the previous sections how a null hypothesis and an alternative hypothesis regarding an adjustment model with constraints are tested against each other. In the following a testing procedure is given, which is based on the procedure

described by Baarda (1968b). The procedure starts with an overall model test of the null hypothesis. If it is rejected, all observations (stochastic observations and constraints) are tested one by one with so-called conventional w-tests. This should give indications how an alternative hypothesis can be formulated that may have biases in more than one observation, constraint or both, and that fits the data very well. A test of such an alternative hypothesis (a multidimensional test) is treated subsequently. It is also possible to use a more systematic way by testing a large number of alternative hypotheses (Velsink, 2015b): For each alternative hypothesis the test ratio is determined (see the "Multidimensional Test" section). The alternative hypothesis with the largest test ratio is considered the best one.

To judge the quality of tests, a quality description is necessary. It will be treated in a short section, after which two examples show the applicability of the proposed procedure.

5.5.1 Overall Model Test

The first step in testing adjustment results is performing an overall model test. Using the reciprocal residuals $\hat{\underline{\mathbf{r}}}$ and a generalised inverse $\mathbf{Q}_y^-$ of $\mathbf{Q}_y$, the test statistic of equation (5.16) is computed for $q = b$ as

$$\underline{T}_b = \frac{1}{\sigma^2} \hat{\underline{\mathbf{r}}}^T \mathbf{Q}_y \hat{\underline{\mathbf{r}}} = \frac{1}{\sigma^2} \hat{\underline{\mathbf{e}}}^T \mathbf{Q}_y^- \hat{\underline{\mathbf{e}}}. \tag{5.26}$$

This follows from equation (5.16) because $\mathbf{C}$ is now a matrix with the same dimensions $(m \times (m-n))$ as $\mathbf{B}$. To yield a testable alternative hypothesis, $\mathbf{B}^T\mathbf{C}$ has to be regular and invertible. Using this, and combining equation (5.16) and equation (C.7) in appendix C.1, equation (5.26) is obtained.

The test statistic is χ^2-distributed, if $\underline{\mathbf{e}}$ has a normal distribution. Its critical value can be computed by means of the B-method of testing (Baarda, 1968b, p. 33). That means that first a value for the significance level of a 1D test is chosen (in the elaborated example that follows, a value of 0.1% is chosen) and a value for the power (80% in the example). From these the value of the significance level of the b-dimensional test can be derived.

5.5.2 Conventional w-Tests

If the overall model test leads to rejection of the null hypothesis, an alternative hypothesis has to be found that describes the relations between the observations better. To this end, conventional w-tests are performed (Baarda, 1968b, p. 15). This means that a bias is assumed in only one observation, that is, we have for matrix $\mathbf{C}$ of equation (5.8)

$$\mathbf{C} = \begin{pmatrix} 0, \dots 0, 1, 0 \dots 0 \end{pmatrix}^T, \tag{5.27}$$

with 1 in the i^{th} row, if the i^{th} observation is tested. The test quantity $\underline{w}_i$ to test the i^{th} observation, is derived from test statistic $\underline{T}_q$ for $q = 1$. It is defined as:

$$\underline{w}_i = \frac{\mathbf{C}^T\underline{\hat{r}}}{\sigma\sqrt{\mathbf{C}^T\mathbf{Q}_{\hat{r}}\mathbf{C}}} = \frac{\underline{\hat{r}}_i}{\sigma\sqrt{(\mathbf{Q}_{\hat{r}})_{ii}}}, \tag{5.28}$$

with $\underline{\hat{r}}_i$ the i^{th} element of $\underline{\hat{r}}$ and $(\mathbf{Q}_{\hat{r}})_{ii}$ the element of $\mathbf{Q}_{\hat{r}}$ in the i^{th} row and i^{th} column. From this follows

$$\underline{w}_i^2 = \underline{T}_{q=1}. \tag{5.29}$$

If the observations have a normal distribution, $\underline{w}_i$ has a standard normal distribution.

5.5.3 Multidimensional Test

If there is reason to believe that more than one observation (stochastic or nonstochastic) is biased, a multidimensional test can be performed. A matrix $\mathbf{C}$ for the alternative hypothesis is formulated. Examples can be found in chapter 3. The test statistic $\mathbf{T}_q$ is computed and compared with a critical value, which is computed with the B-method of testing. It is not straightforward how to compare this result with the results of the 1D w-test or the overall model test because the dimensions, and so the probability density functions and the critical values, differ. Following the approach of de Heus et al. (1994b), we take the ratio between T_q and its critical value, and compare the ratio's, see also Chang (2015). The largest ratio indicates the best alternative hypothesis.

5.5.4 Quality Description

The quality of the test of a certain alternative hypothesis against the null hypothesis is described by the minimal detectable bias (Teunissen, 2006, p. 94ff.). For the i^{th} conventional w-test, it is computed as (Velsink, 2015b, eq. (46)):

$$\begin{aligned} \text{MDB}_i &= \sigma\sqrt{\frac{\lambda_0}{\mathbf{C}_i^T\mathbf{Q}_{\hat{r}}\mathbf{C}_i}}, \\ \mathbf{C}_i &= \left(0, \dots 0, 1, 0 \dots 0\right)^T, \ i = 1, \dots, m, \end{aligned} \tag{5.30}$$

with λ_0 the reference noncentrality parameter, which is 17.075 if the 1D level of significance is 0.1% and the reference power is 80%. The MDB gives the size of the bias that can be found with a probability of 80% (the reference power) if the observation is tested with the pertinent w-test.

5.6 Two examples

Two examples are given to show the application of testing an adjustment model with constraints in geodetic deformation analysis. The first example is a small one and is intended to show in detail how the adjustment model is built and used. The second

example is a larger one and is intended to compare the method with other methods to perform geodetic deformation analysis.

Both examples show that constraints can describe deterministic, unmeasured model elements that are often designated as a priori information. Here the deterministic elements are statements about deformation of points or the absence of deformation.

5.7 Example 1

5.7.1 Problem Description

To illustrate the testing of constraints, an example is taken from geodetic deformation analysis. Three bolts, named A, B and C, have been placed in three buildings to determine subsidence of each of the bolts relative to the other two. Three height differences ($\underline{\mathbf{y}}_1$, $\underline{\mathbf{y}}_2$, $\underline{\mathbf{y}}_3$) have been measured by levelling at time 1. At a later time 2, they are measured again, yielding $\underline{\mathbf{y}}_4$, $\underline{\mathbf{y}}_5$, $\underline{\mathbf{y}}_6$, see figure 5.1. How can whether relative subsidence has occurred be tested?

Here it is shown by regarding the three heights at time 1 as a network that is separate from the network of the three heights at time 2. Subsequently, both networks are connected by three constraints that state that the three heights are unchanged between time 1 and 2. Testing these constraints means testing the deformation hypothesis of stability.

This deformation analysis is conventionally done differently (Caspary, 2000).

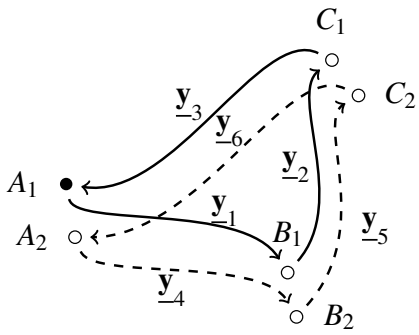

Figure 5.1: Three levelled height differences, measured twice to determine subsidence.

5.7.2 Adjustment Model

The purpose is to determine relative heights. The height of A at time 1, indicated as A_1, is fixed at an arbitrary height: 2 m. As unknown parameters, the heights of B and C at time 1, indicated by B_1 and C_1, and the heights of A, B and C at time 2, indicated as A_2, B_2 and C_2, are used.
The assumption is that no subsidence has taken place, and so we should have:

$$\begin{aligned} 0 &= A_2 - A_1, \\ 0 &= B_2 - B_1, \\ 0 &= C_2 - C_1. \end{aligned} \tag{5.31}$$

These are three constraints on the five parameters. We get the following elements of adjustment model (5.6):

$$
\begin{aligned}
&\underline{\mathbf{y}}_s = \left(\underline{y}_1, \cdots, \underline{y}_6\right)^T,\ \mathbf{y}_z = \left(y_7, y_8, y_9\right)^T = \left(0,0,0\right)^T,\\
&\mathbf{x} = \left(B_1, C_1, A_2, B_2, C_2\right)^T,\\
&\mathbf{A}_s = \begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & -1 & 1\\ 0 & 0 & 1 & 0 & -1 \end{pmatrix},\\
&\mathbf{A}_z = \begin{pmatrix} 0 & 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0 & 1 \end{pmatrix},\\
&\mathbf{a}_{0s} = \begin{pmatrix} -A_1 & 0 & A_1 & 0 & 0 & 0 \end{pmatrix}^T,\\
&\mathbf{a}_{0z} = \begin{pmatrix} -A_1 & 0 & 0 \end{pmatrix}^T.
\end{aligned}
\tag{5.32}
$$

With nine measurements (six levelled height differences, three nonstochastic observations) and five parameters, the model has a redundancy of four.

5.7.3 Observations and Adjustment Results

The observations are given in table 5.2. The standard deviation of all observations is 1 mm and no correlation between the observations is assumed, that is, the cofactor matrix $\mathbf{Q}_{\underline{y}_s}$ is the unit matrix if the variance of unit weight is chosen as 10^{-6}. For the six measured height differences exact fitting values are taken and are therefore without noise.

To present the capability of the proposed analysis method to test for both biases in the constraints and biases in the stochastic observations, two biases are added to these differences: point A subsides 4 mm, causing the height differences y_4 and y_6 to be each 4 mm larger than y_1 and y_3; and a bias of 4 mm in y_4, which is not a subsidence, but a measurement bias.

The observations y_7, y_8 and y_9 are, according to equation (5.32), the changes in height of A, B and C, respectively, and have values of zero.

Table 5.3 gives the estimated heights after least-squares adjustment.

5.7.4 Test Results and Quality Description

The value of T_b from equation (5.26) is 43.7 with 4 degrees of freedom and a critical value of 13.5. The overall model test leads thus to rejection of the null hypothesis, that is, the assumption of point stability and the simultaneous absence of measurement biases is not accepted.

Observation	From	To	Height difference
y_1	A_1	B_1	1.732 m
y_2	B_1	C_1	-0.634 m
y_3	C_1	A_1	-1.098 m
y_4	A_2	B_2	1.740 m
y_5	B_2	C_2	-0.634 m
y_6	C_2	A_2	-1.102 m

Table 5.2: Measured height differences.

Point	Height
$A_1 = A_2$	2.000 m
$B_1 = B_2$	3.735 m
$C_1 = C_2$	3.101 m

Table 5.3: Estimated heights (no subsidence assumed).

Subsequently, w-tests are performed. The absolute values of the w-quantities are given in table 5.4. With the previously chosen level of significance 0.1%, the critical value is 3.29. Four observations are rejected, two measured observations and two constraints:

Observ.	$\hat{e}_i$	$\hat{r}_i$	$\lvert w_i \rvert$	$\hat{\nabla}_i$	MDB_i
y_1	-3.33	-3.33	4.08	-5	5
y_2	0.67	0.67	0.82	1	5
y_3	2.67	2.67	3.27	4	5
y_4	4.67	4.67	5.72	7	5
y_5	0.67	0.67	0.82	1	5
y_6	-1.33	-1.33	1.63	-2	5
y_7	0	6	6	6	4
y_8	0	-4	4	-4	4
y_9	0	-2	2	-2	4

Table 5.4: Test results ($\hat{e}_i$, $\hat{\nabla}_i$ and MDB_i in mm, $\hat{r}_i$ in mm^{-1}, and w_i dimensionless).

y_1, y_4, y_7 and y_8 ($|w_i| > 3.29$). If only one observation is biased, it is most probably y_7 (the height change of A), but y_4 is not far off. The least-squares estimate of the bias of y_7, when this observation is biased and all others are not, is computed with equation (5.15) as 6 mm. The value is listed in table 5.4 under $\hat{\nabla}_i$.

Table 5.4 shows that a constraint (like y_7, y_8, y_9) is tested in the same way as an observation acquired by some measuring device.

In the observations both y_4 and y_7 are biased. The first bias is a measurement bias, the second bias a subsidence of point A. Table 5.4 gives the test results of 1D tests. Each alternative hypothesis states that only one observation is biased, and the rest is without bias. Two observations, however, are biased. None of the alternative hypotheses of table 5.4 is, therefore, a good hypothesis.

Let us test the hypothesis that both y_4 and y_7 are biased. The matrix $\mathbf{C}$ is:

$$\mathbf{C} = \begin{pmatrix} 0,0,0,1,0,0,0,0,0 \\ 0,0,0,0,0,0,1,0,0 \end{pmatrix}^T . \tag{5.33}$$

The test is 2D ($q = 2$). We have $T_q = 42.67$ and a critical value of 11.73, computed with the B-method of testing using the previously mentioned parameters. Here test ratio's are used to compare this result with the results in table 5.4. For the largest value of $|w_i|$ in table 5.4, we get the ratio $(6/3.2905)^2 = 3.32$. For the 2D test we get $42.67/11.73 = 3.64$. This indicates that, indeed, the hypothesis that two biases are present might be better.

To check this result, all 36 alternative hypotheses, in which biases in two observations are present, are tested against the null hypothesis. Indeed, the largest value of the test ratio encountered is 3.64, but it does also occur in other combinations of observations:

- y_4 and y_6 are biased simultaneously; and
- y_6 and y_7 are biased simultaneously.

It is easy to see that these biases cannot be distinguished by the test from the biases in y_4 and y_7 and therefore give the same test values. To overcome this problem of inseparability, additional observations would be necessary.

For the alternative hypothesis that y_4 and y_7 are biased, the least-squares estimates of the biases, computed with equation (5.15), are 4 mm for y_4 and 4 mm for y_7, as they should be.

Table 5.4 shows that the MDB for the subsidence of each of the points is 4 mm, if it is tested with a conventional w-test.

5.7.5 Conclusion of Example 1

This example shows that testing one or more constraints, or even testing constraints and measured observations simultaneously, can be done by using standard test statistics for testing alternative hypotheses. Least-squares estimates of biases in the constraints and the measured observations can be determined. Minimal detectable biases in the constraints and in the measured observations, which describe the sizes of the biases detectable with a predefined probability, can be determined for planning and standardisation purposes.

The computations have been done with a MATLAB programme made for this purpose. All six computation methods, derived before, have been used, yielding all the same results.

5.8 Example 2

To compare the method of geodetic deformation analysis of example 1 with other published methods, an example of a levelling network is taken, of which the analysis

with several methods is given by Wiśniewski and Zienkiewicz (2016). The network (figure 5.2) consists of nine points, that is, 4 reference points ($R_1, \cdots, R_4$) and 5 control points ($P_1, \cdots, P_5$), between which 16 height differences ($h_1, \cdots, h_{16}$) have been measured in two epochs α and β. The simulated measurements are given in table 5.5. The measurements of epoch β are also given in five variants, each variant with other biases, see table 5.6.

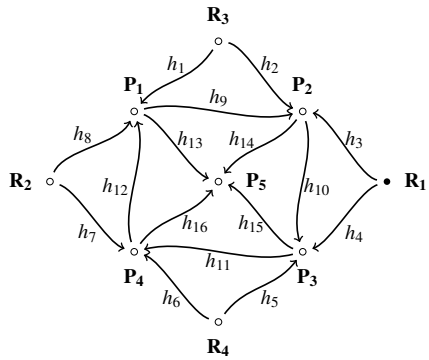

Figure 5.2: Levelling network, adapted from Wiśniewski and Zienkiewicz (2016).

Obs.	Epoch α		Epoch β	
h_1	1.0011	m	1.0037	m
h_2	0.9986	m	1.0017	m
h_3	1.0006	m	0.9974	m
h_4	0.9991	m	1.0007	m
h_5	1.0019	m	1.0030	m
h_6	0.9973	m	1.0011	m
h_7	1.0015	m	0.9999	m
h_8	1.0014	m	0.9996	m
h_9	-0.0002	m	0.0030	m
h_{10}	0.0028	m	0.0028	m
h_{11}	0.0013	m	-0.0024	m
h_{12}	0.0014	m	0.0033	m
h_{13}	0.0010	m	-0.0485	m
h_{14}	0.0015	m	-0.0506	m
h_{15}	0.0006	m	-0.0516	m
h_{16}	0.0018	m	-0.0478	m

Table 5.5: Measured height differences at epochs α and β (variant I). Simulated observations with a bias of -50 mm in point P_5 (= deformation). Data from Wiśniewski and Zienkiewicz (2016).

Variant	Biases	Changed observations
II	ΔP_5 = -0.050 m	h_{16} = -0.0228 m
	Δh_{16} = 0.025 m	
III	ΔP_5 = -0.050 m	h_{16} = 0.0022 m
	Δh_{16} = 0.050 m	
IV	ΔP_5 = -0.050 m	h_3 = 1.2474 m
	ΔR_1 = -0.250 m	h_4 = 1.2507 m
V	ΔP_5 = -0.050 m	h_3 = 1.2474 m
	ΔR_1 = -0.250 m	h_4 = 1.2507 m
	Δh_{16} = 0.050 m	h_{16} = 0.0022 m
VI	ΔP_5 = -0.050 m	h_3 = 1.2474 m
	ΔR_1 = -0.250 m	h_4 = 1.2507 m
	ΔR_2 = -0.250 m	h_7 = 1.2499 m
		h_8 = 1.2496 m

Table 5.6: Variants of simulated observations at epoch β. Data from Wiśniewski and Zienkiewicz (2016).

5.8.1 Adjustment Model

The adjustment model is built in the same way as in example 1. Because there is no fundamental difference in this example (example 2) between reference points and control points, they are treated in the same way. For point R_1, a fixed height of 9 m at epoch 1 is taken. Note that any height of any point at any epoch may be taken: It will give identical test results. Vector $\mathbf{y}_s$ consists of 32 elements: 16 height differences for two epochs. Vector $\mathbf{y}_z$ consists of 9 zeros, that is, the height of each of the 9 points at epoch α is assumed to be equal to the height of the same point at epoch β. Vector $\mathbf{x}$ consists of 17 elements: 8 heights in epoch α and 9 heights in epoch β (R_1 in epoch α is missing as a parameter because it is fixed at 9 m). Matrices and vectors $\mathbf{A}_s$, $\mathbf{A}_z$, $\mathbf{a}_{0s}$ and $\mathbf{a}_{0z}$ are constructed in the same way as in example 1. For the cofactor matrix $\mathbf{Q}_{y_s}$ the unit matrix is taken and the variance of unit weight is 10^{-6}, that is, all height differences have a standard deviation of 1 mm and are not correlated.

5.8.2 Adjustment and Test Results of Variant I

The adjustment of variant I (table 5.5) gives least-squares estimates for the heights of all points (table 5.7). Note that the estimated heights for both epochs α and β are the same, because of the constraints. These constraints are too strict, as is shown by the overall model test of the adjustment, which gives a value of 5257.2 for T_b of equation (5.26) with 24 degrees of freedom and a critical value of 31.8 (level of significance is 0.131).

Because of the rejection of the null hypothesis, subsequently conventional w-tests are done. The largest w-test value is for the constraint that the height of P_5 does not change: $w = 71.9$. The next largest w-test values are for biases in measured height

Point	Height	
R_1	9.0000	m
R_2	8.9985	m
R_3	8.9978	m
R_4	8.9986	m
P_1	9.9993	m
P_2	9.9989	m
P_3	10.0000	m
P_4	9.9988	m
P_5	9.9751	m

Table 5.7: Estimated heights, equal for epochs α and β; they are not used in the testing procedure. Point R_1 has been fixed at 9 meter.

differences: 31.3 ($h_{15-\beta}$), 30 ($h_{14-\beta}$), 30 ($h_{16-\alpha}$), 28.4 ($h_{14-\alpha}$), 28.4 ($h_{13-\alpha}$), and 27.3 ($h_{15-\alpha}$). The notably larger value of 71.9 clearly indicates that the height of P_5 has changed between epoch α and β.

5.8.3 Search for best alternative hypothesis

It may be that not only the height of P_5 has changed, but that also biases are present in measured height differences or in heights of other points (i.e., other points have moved). To test this, many alternative hypotheses are formulated systematically. Biases are assumed in 1, 2, 3 or 4 observations (stochastic observations or constraints) successively. One bias can be present in 41 observations. Two biases can be present in 820 combinations of the 41 observations. Similarly 3 and 4 biases can be present in a large number of combinations. The total for 1, 2, 3, or 4 biases is 112,791 combinations all together. To find the best alternative hypothesis, all these combinations are tested (as long as they are consistent and testable, see the "Testable And Consistent Hypothesis, Invariance Of $\underline{T}_q$" section) and compared with each other by using the test ratio (see chapter 3). The computation takes less than two minutes on a standard computer. The constraint on the height of P_5 (therefore, only one bias) gives the largest test ratio of 477.6. The next largest ones are 442.1 (P_5 and $h_{9-\beta}$ are both biased), 441,9 (P_5 and $h_{6-\alpha}$ are both biased) and 441.7 (P_5 and $h_{16-\beta}$ are both biased). The notably larger value of 477.6 is a strong indication that only the constraint on the height of P_5 is biased, that is, only this height has changed.

With equation (5.15), the least-squares estimate of the subsidence of P_5 can be determined as 50.9 mm, which is very close to the simulated value of 50 mm. The g-inverse in equation (5.15) is here a regular inverse because $\mathbf{C}^T\mathbf{Q}_{\hat{f}}\mathbf{C}$ has no rank defect.

5.8.4 Variants II-VI

After variant I, now variants II–VI are analysed. For all these variants the overall model test leads to rejection of the null hypothesis. The same search for the best alternative

Variant	Largest ratio	Biased observ.	Simul. biases	Estim. biases
II	386.6	P_5	50 mm	51.8 mm
		$h_{16-\beta}$	25 mm	28.8 mm
III	416.3	P_5	50 mm	51.8 mm
		$h_{16-\beta}$	50 mm	53.8 mm
IV	5735.4	P_5	50 mm	–
		R_1	250 mm	249.2 mm
V	5735.4	P_5	50 mm	–
		R_1	250 mm	249.2 mm
		$h_{16-\beta}$	50 mm	–
VI	5735.4	P_5	50 mm	–
		R_1	250 mm	249.2 mm
		R_2	250 mm	249.2 mm

Table 5.8: Variants II-VI: biased observations and estimated biases according to *largest test ratio*.
Note: Not all biases are detected (underfitting). Simulated biases for reference. Estim. = estimated; Obs. = observation; Simul. = simulated.

hypothesis as performed for variant I, is now performed for variants II-VI. This search includes also hypotheses that assume only one bias, the tests of which are equivalent to the w-test. The results are summarised in table 5.8.

Table 5.8 shows that the alternative hypotheses with the largest test ratios have biases that are indeed present. The least-squares estimates of the biases are very close to the simulated values. However, too few biases are detected. It means that the method using the largest test ratio has too large a preference for hypotheses with fewer parameters. It tends to *underfitting* of the model to the measurements, although it takes account of the number of parameters through the χ^2-distribution and its critical value.

The Akaike information criterion (AIC) is designed to take into account a difference in the number of parameters in the hypotheses that are compared (Akaike, 1974). For comparisons in which only a small number of observations are used, as in this example, a corrected AIC (AICc) has been proposed by Hurvich and Tsai (1989). For the comparison of geodetic point fields, the equation for the AICc can be written, omitting a constant term that is irrelevant in the comparisons, as follows (Lehmann and Lösler, 2017):

$$
\begin{aligned}
k &= n + q - d + 1, \\
\underline{\text{AICc}} &= 2k + \frac{2k(k+1)}{m-(k+1)} + m \ln \frac{\underline{T}_b - \underline{T}_q}{m},
\end{aligned}
\tag{5.34}
$$

in which d is the rank defect of model matrix $\mathbf{A}$. In this example $d = 0$. Table 5.9 shows the hypotheses that have the smallest AICc (which indicates the best hypothesis). All biases are detected without underfitting or overfitting (i.e., not too few, nor too many biases are detected). The least-squares estimates of the biases are close to the simulated ones.

The AICc performs better than the largest test ratio in this example. Just this example

Variant	Smallest AICc	Biased observ.	Simul. biases	Estim. biases
II	107.2	P_5	50 mm	51.8 mm
		$h_{16-\beta}$	25 mm	28.8 mm
III	107.2	P_5	50 mm	51.8 mm
		$h_{16-\beta}$	50 mm	53.8 mm
IV	112.0	P_5	50 mm	50.9 mm
		R_1	250 mm	249.2 mm
V	115.5	P_5	50 mm	51.8 mm
		R_1	250 mm	249.2 mm
		$h_{16-\beta}$	50 mm	53.8 mm
VI	119.2	P_5	50 mm	50.9 mm
		R_1	250 mm	249.2 mm
		R_2	250 mm	248.3 mm

Table 5.9: Variants II-VI: biased observations and estimated biases according to *AICc*. All biases are detected and no under- or overfitting occurs. Simulated biases for reference.

is, however, not enough to reach a general conclusion about the effectiveness of one or another information criterion. Here, the largest test ratio and the AICc are used to show how an adjustment model with constraints can be tested for the purpose of geodetic deformation analysis.

5.8.5 Conclusion of Example 2

This example shows that geodetic deformation analysis is possible by testing an adjustment model with constraints. It has the advantage relative to other methods of geodetic deformation analysis that points that define the geodetic datum are fixed in only one epoch. This means that datum points do not have to be stable points. Points are tested for movements by testing constraints, and, simultaneously, stochastic observations are tested for biases. This testing is invariant for a change of datum points (i.e., for S-transformations). Comparison with the results of the same example in Wiśniewski and Zienkiewicz (2016) shows that using the adjustment model with constraints, and using the AICc to select the best alternative hypothesis, is more effective in indicating which observations are biased and which points are unstable. The least-squares estimates of biases (biases in stochastic observations and deformations) are close to the simulated biases.

The extension of the analysis of two epochs to the analysis of more than two epochs is straightforward (see chapter 7).

5.9 Conclusions

It has been shown how constraints on parameters in an adjustment model of observation equations can be tested using standard generalised likelihood ratio tests. The constraints are formulated for this purpose as nonstochastic observations. This gives a simple procedure to adjust and test both stochastic observations and constraints on the parameters in an analogous way, and even simultaneously. The cofactor matrix of the observations is singular because of the nonstochastic observations. Adjustment of an adjustment model of observations with a singular, that is, positive semidefinite, cofactor matrix, is well-known. Testing of such a model needs, however, special consideration to compute the test statistic. Six methods have been treated to compute the test statistic. Two examples from geodetic deformation analysis have been given to illustrate testing of biases in both constraints and stochastic observations. These examples show that testing geodetic deformation hypotheses by testing constraints makes it possible to define a geodetic datum by unstable points. Detection of deformations with an information criterion has been demonstrated to be effective, and least-squares estimates of deformations to be adequate. Minimal detectable biases, usable for planning purposes, have been computed.

6

Time series analysis with geodetic observations[1]

6.1 Introduction

Geodetic measurements from e.g. levelling instruments, total stations and GPS receivers, acquired during many epochs, can be used for geodetic deformation analysis. In most standard applications Cartesian coordinates (1D, 2D, or 3D) are determined per epoch and transformed to a common geodetic datum. Subsequently coordinate differences are determined and analysed.

This chapter deals with the problem of finding the best mathematical description of a deformation. "Best description" is here defined by means of an information criterion, (e.g. the Akaike information criterion (Akaike, 1974)). Existing approaches to this problem suffer from one or more of the following three flaws. (1) They neglect stochastic information about the observations, especially correlation, where this can have a major influence (Borre and Tiberius, 2000; Beavan, 2005; Holst and Kuhlmann, 2016). (2) They cannot take into account the mathematical descriptions of driving forces, and focus on the analysis of displaced points (Heunecke et al., 2013; Caspary, 2000; Nowel, 2016). (3) Or they are not designed to handle a comparison between a multitude of alternative deformation descriptions (Xu et al., 2000; Boyd et al., 2015; Ng et al., 2015).

The existing approaches to find the best mathematical description of a deformation belong to one of three main approaches. Each approach suffers from one or more of the mentioned flaws.

The first approach, used in signal processing, tries to fit a deterministic *mathematical function in time* to a time series of coordinates. The time series is considered a random process and analysed accordingly (Papoulis, 1984). In general the stochastic nature of

[1] This chapter has been published before in Journal of Applied Geodesy (Velsink, 2017). To fit the publication into this study, minor changes have been made.

the observations (full covariance matrix) is not taken into account, and the deformations are analysed for each observing station separately (many publications apply this approach, we mention here (Chang and Hanssen, 2016; Ng et al., 2015)).

The second approach, traditionally used for geodetic deformation surveys, uses a *heuristic*, where the congruence of the same point field in different epochs is compared to find a satisfactory description (Heunecke et al., 2013; Caspary, 2000; Nowel, 2016). The heuristic uses the stochastic information of the observations, and it tests hypotheses that assume deformations per single point and per pair of two epochs. To be applicable, it requires some points to be stable through all epochs. If necessary, the heuristic successively removes points to arrive at an acceptable hypothesis. Then it tries to add previously rejected points. Many publications treat this and similar methods (Pelzer, 1971; Heunecke et al., 2013; Chen, 1983; Chrzanowski et al., 1986; Dong, 1993; Caspary, 2000). To what extent the goodness of fit, reached by such a heuristic, is the best one, is hard to determine.

The third approach, applied e.g. in geophysics, is based on knowledge of the *driving forces*, leading to a forward model, describing the deformations, and tests the model(s) subsequently. In many publications this approach is used; we mention Xu et al. (2000); Boyd et al. (2015). The number of models to be tested is generally limited.

This chapter proposes an alternative geodetic deformation analysis method to address the three flaws, mentioned earlier. This means, firstly, that it takes the stochastic information of the observations into account. Secondly, hypothesised deformations are included in the adjustment model as mathematical functions. The hypothesised deformations are confronted with the time series of observations, and tested. The mathematical functions provide the link between the driving forces and the observations. Thirdly, a statistical information criterion is used to select the best description of the deformations.

The proposed analysis method expresses the mathematical functions as constraints on the parameters of the adjustment model. The constraints are processed as observations without random noise (nonstochastic observations). In the appendix methods are given to solve an adjustment problem with nonstochastic observations rigorously (without approximations). The addition of nonstochastic observations enlarges the system of normal equations. This is a disadvantage relative to other solution methods of adjustment problems with constraints. The advantages are described in Section 6.3.1.4.

The original observations of all epochs are input for the analysis method. There is no intermediate step, where coordinates per epoch are determined. This means that there are no different geodetic datums for different epochs, and testing is directly related to the original observations. It may be a disadvantage, if the original observations of an epoch are not available.

Section 6.2 starts with an overview of the chosen method to solve the problem. Section 6.3 continues with the adjustment model for deformation analysis, and shows how a physical model of the deformation is included in the adjustment model. Testing and quality description are treated as well. It is shown that the testing method is invariant for a change of geodetic datum. Section 6.4 treats an elaborate example of geodetic deformation analysis.

6.2 Method overview

In Figure 6.1 an overview is given of the proposed method for geodetic deformation analysis.

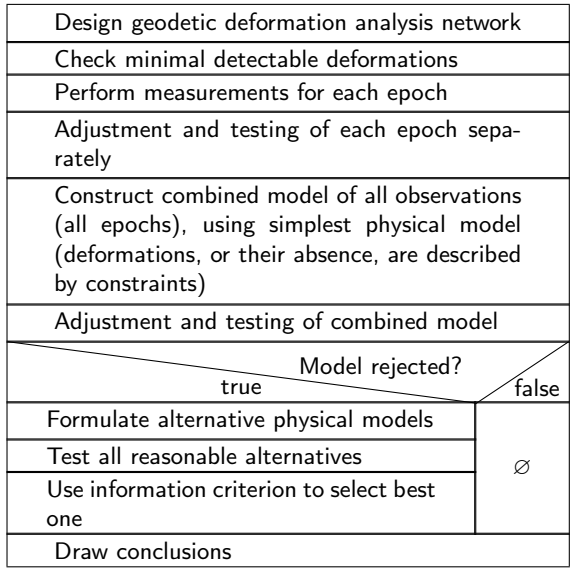

Figure 6.1: Overview of the proposed method to find the best mathematical description of a deformation.

To solve the problem of finding the best mathematical description of a deformation, we design a *geodetic deformation analysis network*. This network represents one or more objects. Such an object may, e.g., be a building, a tunnel, an area, a region. The subset of reference points is an object, which is considered stable. This object is not treated separately from the other objects. Each object possibly moves relative to the other objects or possibly experiences internal deformations, or both.

The point coordinates of the geodetic deformation analysis network are determined by measurements in each epoch. The measured points of any one epoch are considered to constitute a separate geodetic network, belonging solely to that epoch. Each point is linked to the equivalent point in another geodetic network, i.e. in another epoch, by constraints that define how the point is expected to move relative to the other points. It may be no deformation (the point is stable), or any kind of movement. It is illustrated for two epochs in Figure 6.2. Thus the constraints describe deformation hypotheses (which can be the absence of deformation).

To derive the point coordinates from the geodetic observations, a linear or linearised adjustment model is formulated, which is solved by least-squares. The adjustment model is a combined model, containing the observations of all epochs, the parameters of all epochs, and the constraints that link the parameters together into time series of parameters.

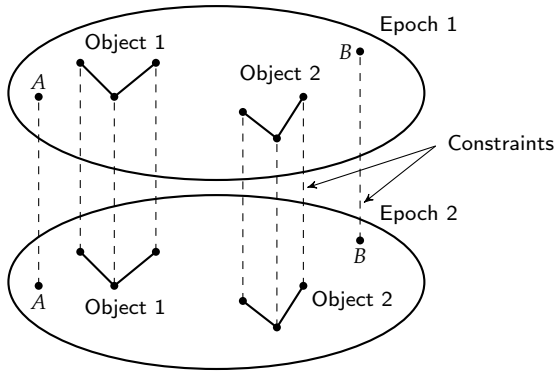

Figure 6.2: Two epochs linked by constraints, visualised by dashed lines. Constraints describe deformation behaviour, including no deformation. A, B: reference points, considered together a separate object.

The combined model must be capable of determining the deformations of interest. Therefore, a *check* is performed, whether the *minimal detectable deformations* (see chapter 3) comply with the specifications. If so, the measurements can start, and after collecting measurements of two or more epochs, adjustment and testing of the combined model can be carried out.

First, however, it is recommended to *adjust and test each epoch* separately. Only when the adjustment and testing of the observations of an epoch is accepted by its overall model test, the observations are included in the combined model. This makes it less likely that a bias in an epoch, caused by measurement noise or errors, is interpreted in the combined analysis as a deformation.

A *combined adjustment and testing* is performed, when all epochs have been included. Initially the assumed deformation is as simple as possible, often the absence of any deformation. If the overall model test of the combined adjustment model is rejected, alternative deformation models are added to the model, and tested. An alternative deformation model is derived from knowledge of the driving forces, i.e. from a *physical model*.

If the knowledge of the driving forces is limited, several, and often many deformation hypotheses are *reasonable alternatives*. Each of them can be tested, and the best among them is chosen. An *information criterion* is used to make a sensible choice.

To handle a time series of measured or derived 3D coordinates, for each epoch the coordinates and their covariance matrix can be taken, and the epochs can be linked by a chain of 3D transformations (see chapter 7).

In the present chapter, however, the focus is on a time series of geodetic measurements, of whatever measurement type they may be. The measurements of all epochs are direct input for the adjustment model. This avoids the intermediate step of determining coordinates and their covariance matrix for each epoch.

In the following section a time series analysis with the proposed method is described in more detail.

6.3 Time series analysis

The proposed method to find the best mathematical description of a deformation starts a time series analysis with a least-squares adjustment (Section 6.3.1) and testing, including a test quality description (Section 6.3.2). It is followed by the selection of the best deformation description with an information criterion (Section 6.3.3).

The complete process is illustrated in Figure 6.3, which is an elaborated version of Figure 6.1.

6.3.1 Adjustment model

6.3.1.1 Deformation analysis model

Our adjustment model for deformation analysis distinguishes three types of observations and two types of unknown parameters, which gives the following model:

$$\underbrace{\begin{pmatrix} \underline{\mathbf{y}}_s \\ \mathbf{z}_d \\ \mathbf{z}_g \end{pmatrix}}_{\underline{\mathbf{y}}} = \underbrace{\begin{pmatrix} \mathbf{A}_s & \mathbf{0} \\ \mathbf{Z}_d & \mathbf{Z}_\nabla \\ \mathbf{Z}_g & \mathbf{0} \end{pmatrix}}_{\mathbf{A}} \underbrace{\begin{pmatrix} \mathbf{x}_s \\ \nabla \end{pmatrix}}_{\mathbf{x}} + \underbrace{\begin{pmatrix} \underline{\mathbf{e}}_s \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}}_{\underline{\mathbf{e}}}, \tag{6.1}$$

$$\underline{\mathbf{y}}_s = \begin{pmatrix} \underline{\mathbf{y}}_1 \\ \vdots \\ \underline{\mathbf{y}}_p \end{pmatrix} ; \; \mathbf{x}_s = \begin{pmatrix} \mathbf{x}_c \\ \mathbf{x}_d \\ \mathbf{x}_n \end{pmatrix} ; \; \mathbf{x}_c = \begin{pmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_p \end{pmatrix},$$

$$\mathbf{D}\{\underline{\mathbf{y}}\} = \sigma^2 \mathbf{Q}_y = \sigma^2 \begin{pmatrix} \mathbf{Q}_{y_s} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

Underlined variables indicate that they are *stochastic*. Here we assume that the stochasticity can be described by a Gaussian probability density function. The m-vector $\underline{\mathbf{y}}$ contains m observations. The matrix $\mathbf{A}$ is the coefficient matrix, which may be rank deficient, the n-vector $\mathbf{x}$ contains n parameters, and $\underline{\mathbf{e}}$ is the m-vector of Gaussian noise. The expectation of $\underline{\mathbf{e}}$, i.e. $E\{\underline{\mathbf{e}}\}$, is zero.

$\mathbf{D}\{\underline{\mathbf{y}}\}$ is the covariance matrix of $\underline{\mathbf{y}}$, split into a variance factor σ^2 and a symmetric positive semidefinite cofactor matrix $\mathbf{Q}_y$, which is singular. The variance factor is any positive real scalar.

The observations, assembled in vector $\underline{\mathbf{y}}$, are split into:

- $\underline{\mathbf{y}}_s$: time series of observations that originate from measuring devices; clustered from epoch 1 to p in $\underline{\mathbf{y}}_1$ to $\underline{\mathbf{y}}_p$.

$\mathbf{z}_d$: nonstochastic observations (constraints) that stipulate which points in different epochs are the same points and how they move.

$\mathbf{z}_g$: nonstochastic observations (constraints) that define the geodetic datum. More generally: they define the minimal constraints, needed to deal with nonestimable parameters. We will come back on this in Section 6.3.1.3.

Parameters, assembled in vector $\mathbf{x}$, are split into:

$\mathbf{x}_c$: time series of heights or coordinates, clustered from epoch 1 to p in $\mathbf{x}_1$ to $\mathbf{x}_p$.

$\mathbf{x}_d$: deformation parameters that are part of the simplest physical model; in case the simplest physical model is "no deformation", this vector is empty.

$\mathbf{x}_n$: nuisance parameters, necessary to build the model, but not of primary interest (e.g. orientation parameters when directions are used as observations).

∇: deformation model parameters, describing the motion of points, for example the linear rate of a linear subsidence through the epochs.

Random noise, assembled in vector $\underline{\mathbf{e}}$, is split into:

$\underline{\mathbf{e}}_s$: the measurement noise of observations $\underline{\mathbf{y}}_s$.

$\mathbf{e}_d = \mathbf{0}$, because $\mathbf{z}_d$ is nonstochastic.

$\mathbf{e}_g = \mathbf{0}$, because $\mathbf{z}_g$ is nonstochastic.

Model (6.1) considers each epoch as a different geodetic network. The same point in another epoch is considered to be a different point. Only by linking corresponding points with nonstochastic observations, the geodetic networks of all epochs get connected, see Figure 6.2. This is reflected in matrix $\mathbf{Z}_d$ that contains in each row a 1 and a -1 in the columns of the pertaining points and epochs, in case the simplest physical model is "no deformation" (see the example in Section 6.4).

The nonstochastic observations $\mathbf{z}_d$ appear in two variants. If a point is not moving, the corresponding row in $\mathbf{Z}_\nabla$ contains zeros. In a stability analysis none of the points is expected to be moving. In that case, the column with $\mathbf{Z}_\nabla$ in matrix $\mathbf{A}$, as well as the parameter vector ∇ are missing. If one or more points are expected to be moving between two or more epochs, there are parameters ∇ and there is a matrix $\mathbf{Z}_\nabla$. These parameters are the rate of linear movement, or the coefficients of a power law function, or the coefficients of a periodic function, etc.

The nonstochastic observations $\mathbf{z}_g$ solve the datum problem. An essential difference between $\mathbf{z}_d$ and $\mathbf{z}_g$ is that $\mathbf{z}_d$ can be tested, which is not possible for $\mathbf{z}_g$.

Model (6.1) is linear or linearised. Its least-squares solution is treated in appendix D.1.

6.3.1.2 Physical model

To get a good description of a deformation, it is important to have information on the driving forces. This information is used to construct the *physical model* of the deformations, and to model the resulting deformations in the adjustment model. The

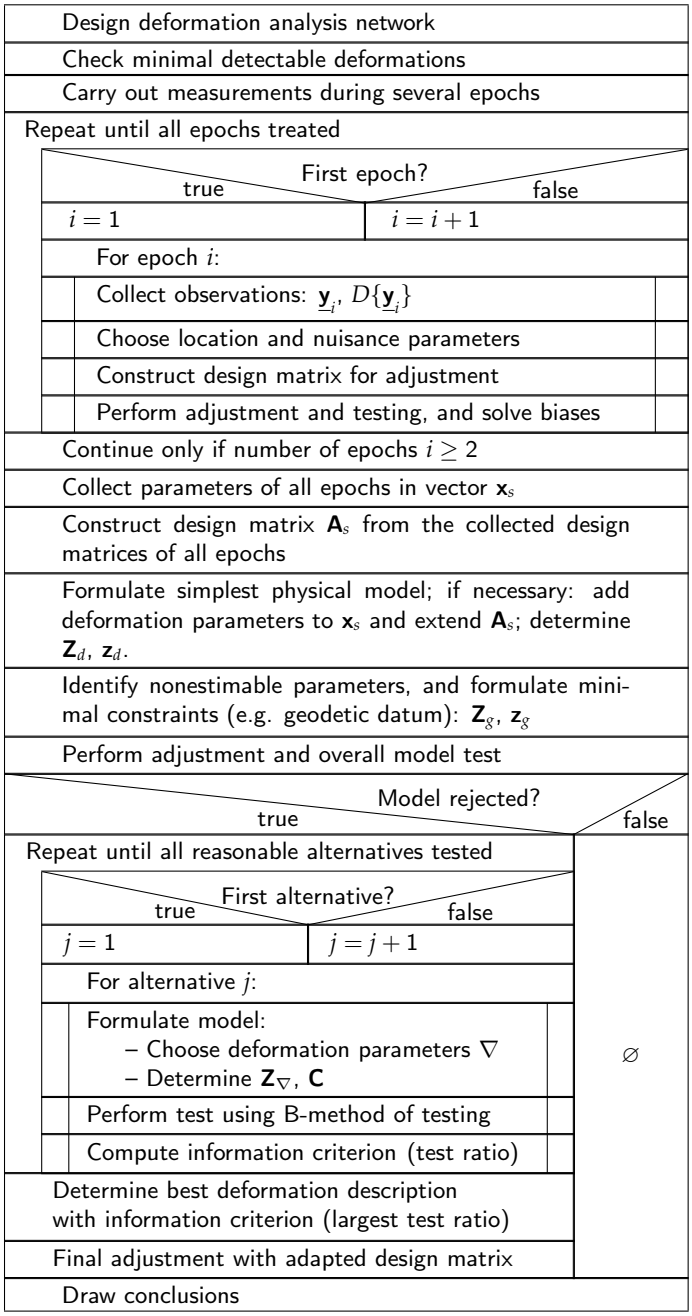

Figure 6.3: Proposed geodetic deformation analysis method.

physical model can have unknown parameters, like the unknown linear subsidence rate of a building, or the coefficients of a periodic movement. The physical model and its parameters are put into deformation analysis model (6.1) as nonstochastic observations $\mathbf{z}_d$, model matrices $\mathbf{Z}_d$ and $\mathbf{Z}_\nabla$, and parameters ∇.

Note that measurements can have been used to build the physical model. They can be put into model (6.1) as measurements $\underline{\mathbf{z}}_d$. They are, however, stochastic, and are treated in the same way as the geodetic measurements $\underline{\mathbf{y}}_s$. Standard deviations and correlations of the measurements have to be available, or an assumed standard deviation can be used. Variance component estimation can give information on the correctness of the used standard deviations.

6.3.1.3 Constraints

Model (6.1) contains both minimal and redundant constraints (Pope, 1971). $\mathbf{z}_g$ are minimal constraints, while $\mathbf{z}_d$ are in general redundant constraints.

Minimal constraints are added to solve a rank deficiency of matrix $\mathbf{A}$. In geodetic practice two major reasons exist, why rank deficiency occurs: (i) the definition of a geodetic datum, and (ii) the existence of a configuration defect, which means that some coordinates or other parameters are not or not sufficiently estimable, e.g. because of missing observations.

Redundant constraints are added as part of the functional model. In deformation analysis they can describe deformation hypotheses.

It may occur that without the constraints $\mathbf{z}_d$, the adjustment model is not solvable. In that case the elements of $\mathbf{z}_d$ also function as minimal constraints. In general they will be redundant as well (we have more of them than needed to solve the rank deficiency of matrix $\mathbf{A}$).

If standard deviations larger than zero are assigned to constraints, we speak of "weighted constraints" or "soft constraints". Soft constraints can be processed as observations. They differ from other observations in that they do not originate, directly or indirectly, from measuring devices. This distinction is, however, artificial. If a deformation analysis model states that a point is not moving, the human model designer, observing that "no motion" is a credible assumption, acts as a measuring device. The term "pseudo-observation" is, therefore, sometimes used to indicate a soft constraint.

If standard deviations of zero are assigned to constraints, we speak of "hard constraints". They can be treated as nonstochastic observations within model (6.1). Adjustment is possible with one of the three methods, described in appendix D.1.

If a generalised inverse is used, it is not necessary to explicitly formulate minimal constraints to solve a rank deficiency of matrix $\mathbf{A}$, because the rank deficiency is implicitly solved. Explicitly formulating minimal constraints makes it, however, clear, how the geodetic datum is defined, or how a configuration defect is solved.

If an extended system of normal equations is used, the constraints can be tested by testing the estimated Lagrange multipliers Lehmann and Neitzel (2013).

It is possible to test constraints by treating them as nonstochastic observations. This is described in appendix D.2. There it is shown that stochastic and nonstochastic observations are tested in the same way. The advantages are listed in the next section.

6.3.1.4 Discussion on nonstochastic observations

The advantage of handling hard constraints by means of nonstochastic observations, is that it yields a relatively simple adjustment model (6.1). This, in itself, may be a reason to use this option. But processing constraints by using nonstochastic observations has other advantages.

First, alternative hypotheses containing one or several additional bias parameters, having an influence on one or more observations (stochastic or nonstochastic ones, or both), are easily formulated and tested.

Secondly, the quality of the test can be described by minimal detectable biases (MDB's). They describe the minimal values of biases that can be detected with a certain reference probability by testing a model that contains these biases. A more detailed treatment will be given later (Section 6.3.2.3). If a constraint does not fit into the model, it is biased. The model can be tested for such a bias, and an MDB can be computed. Thus MDB's give information about detectable biases in both stochastic observations and constraints. One or more constraints in model (6.1) describe a deformation hypothesis. M.d.b.'s of these constraints describe the *minimal detectable deformations*. It is illustrated in an elaborated example in Section 6.4.5.

Thirdly, if a constraint is given a nonzero standard deviation and the adjustment is done again, it is relatively easy to see the effects of a less "hard" constraint. If a standard deviation is changed from zero to a nonzero value, it is called "relaxation" of the constraint. It enables a sensitivity analysis: the assessment of the effect of relaxing constraints on the estimated parameters and adjusted observations, and on the MDB's.

Relaxation of constraints makes it possible to estimate deformation effects that are unmodelled in the deterministic model matrix. Deformations that have a local influence can be modelled by covariance functions that determine the cofactors between constraints. The word "local" can have here a spatial and a temporal meaning. The option to estimate such a local deformation signal, comparable with collocation (Moritz, 1978) or Kriging (Krige, 1951), is left for future research.

6.3.1.5 Matrix C inside or outside model

Deformation parameters ∇ are included in model (6.1). They are *inside* the model, which is tested by means of an overall model test (see chapter 3). If the model is *not* rejected, the deformation is corroborated (confirmed).

However, we take in general as null hypothesis the simplest deformation description. Often this is the assumption of stability, i.e. no deformation is assumed to be present. It may be that the simplest deformation does include a deformation, described by the parameters $\mathbf{x}_d$, which are assumed to be undisputed. The parameters ∇ and the second

column of the coefficient matrix of model (6.1) do not appear in the null hypothesis. We now formulate the alternative hypothesis that a (more intricate) deformation has occurred. To test the alternative hypothesis against the null hypothesis, a test quantity $\underline{T}_q$ is computed. The equation for $\underline{T}_q$ is (appendix D.2):

$$\underline{T}_q = \frac{1}{\sigma^2}\hat{\underline{\mathbf{r}}}^T\mathbf{C}(\mathbf{C}^T\mathbf{Q}_{\hat{r}}\mathbf{C})^{-1}\mathbf{C}^T\hat{\underline{\mathbf{r}}}. \tag{6.2}$$

with $\hat{\underline{\mathbf{r}}}$ the reciprocal residuals, for which holds, using the least-squares residuals $\hat{\underline{\mathbf{e}}}$:

$$\hat{\underline{\mathbf{e}}} = \mathbf{Q}_y\hat{\underline{\mathbf{r}}}. \tag{6.3}$$

In equation (6.2) we take for matrix $\mathbf{C}$ the second column of the coefficient matrix of equation (6.1):

$$\underbrace{\begin{pmatrix}\mathbf{A}_s & \mathbf{0}\\ \mathbf{Z}_d & \mathbf{Z}_\nabla\\ \mathbf{Z}_g & \mathbf{0}\end{pmatrix}}_{\mathbf{A}'\quad\mathbf{C}} \tag{6.4}$$

Now we can write:

$$\underline{\mathbf{y}} = \mathbf{A}'\mathbf{x}_s + \mathbf{C}\nabla + \underline{\mathbf{e}} = \underbrace{(\mathbf{A}' \quad \mathbf{C})}_{\mathbf{A}}\underbrace{\begin{pmatrix}\mathbf{x}_s\\ \nabla\end{pmatrix}}_{\mathbf{x}} + \underline{\mathbf{e}}. \tag{6.5}$$

Matrix $\mathbf{C}$ is now *outside* the model that describes the null hypothesis. Model (6.5) describes the alternative hypothesis (appendix D.2), and the model:

$$\underline{\mathbf{y}} = \mathbf{A}'\mathbf{x}_s + \underline{\mathbf{e}} \tag{6.6}$$

is the null hypothesis.

Suppose that a certain alternative hypothesis is accepted as the best one to replace a rejected null hypothesis. It is easy to adapt the null hypothesis by adding $\mathbf{C}$ and to perform the final adjustment.

Before the best alternative hypothesis can be selected, the null hypothesis has been tested, and, if it was rejected, different alternatives have been tested. How this testing is done, is the subject of the next section.

6.3.2 Testing and test quality description

6.3.2.1 Testing

To test a time series of measurements, first a null hypothesis is formulated, often the absence of deformations. An overall model test of the null hypothesis is performed (appendix D.2.1). If the null hypothesis is rejected, an alternative hypothesis has to be searched for. Candidates are hypotheses, where only one observation is biased relative to the null hypothesis. Such hypotheses can be tested by conventional w-tests

(appendix D.2.2). In data snooping (Baarda, 1968b) all possible conventional w-tests are tested subsequently. For deformation analysis other types of systematic hypotheses are possible, e.g. the hypothesis that only one point is moving linearly through several epochs. If the driving forces are sufficiently known, more intricate hypotheses may be reasonable. They are tested by specific, multidimensional tests (appendix D.2.3).

From the test method follows what the quality of the tests is. This test quality, described by minimal detectable deformations, is needed already in the design stage of network analysis (Figure 6.3), and will be treated in Section 6.3.2.3. First, however, it will be shown that testing deformation hypotheses as described here, is invariant for S-transformations.

6.3.2.2 Deformations and S-transformation invariance

The coordinates and their covariance matrices contain information about the form and size of objects and also about the relative locations of these objects. They contain information about the geodetic datum as well. S-transformations of the coordinates and their covariance matrices change the geodetic datum, but not the form and size, nor the relative locations (Pope, 1971; Baarda, 1973).

The nonstochastic observations $\mathbf{z}_g$ define the geodetic datum, i.e. the S-system (Teunissen, 1985a, p. 41). The elements of the parameter vector that are constrained by $\mathbf{z}_g$ constitute the S-basis. A different $\mathbf{z}_g$ and a different $\mathbf{Z}_g$ mean a change of S-basis (an S-transformation), cf. Pope (1971); Baarda (1973). Different coordinates, and a different covariance matrix of the coordinates, will result. The choice of S-system is arbitrary, and should not influence the deformation analysis. This means, that the tests have to be invariant under an S-transformation, i.e. the test statistic $\underline{T}_q$ and the MDB's have to be S-transformation invariant. In appendix D.3 it is shown that this holds indeed.

It is worth noting that the S-system can be fixed by the height or coordinates of points *within one epoch*, although a time series of observations, pertaining to many epochs, is analysed. The relation to the other epochs is realised by the constraints (nonstochastic observations $\mathbf{z}_d$) that define the deformation hypothesis. The consequence is that deformations of S-base points can be tested without the need for an S-transformation. It is illustrated in the elaborated example in Section 6.4.6.

6.3.2.3 Test quality description

To assess the quality of a deformation analysis, the deformation size that can be detected, is important. A measure for it is the minimal detectable bias (MDB).

An MDB gives the bias in the functional model that can be detected by the associated test with a certain reference probability (Teunissen, 2006, p. 102). If the bias is a deformation, the MDB describes the *minimal detectable deformation* (MDD), and is an indication of the *sensitivity* of the network for deformations (Niemeier, 1982).

An MDB (indicated as ∇_0) can be determined as (cf. Teunissen (2006, p. 101)):

$$\sigma^2 \lambda_0 = \nabla_0^T \mathbf{C}^T \mathbf{Q}_{\hat{r}} \mathbf{C} \nabla_0 \,, \tag{6.7}$$

where $\mathbf{C}\nabla_0$ describes the bias in the observations that can be detected with a certain probability γ_0 (e.g. 80%), while using a certain significance level α_0 (e.g. 0,1%). λ is the non-centrality parameter of the χ^2-distribution, if a bias is present. The reference value λ_0 is computed for a test power γ_0 and a significance level α_0. The only unknowns in equation (6.7) are the elements of vector ∇_0.

If ∇_0 has only one element, and, as a consequence, matrix $\mathbf{C}$ has one column, ∇_0 is a scalar and follows from the equation:

$$\nabla_0 = \sigma \sqrt{\frac{\lambda_0}{\mathbf{c}^T \mathbf{Q}_{\hat{r}} \mathbf{c}}}, \tag{6.8}$$

where $\mathbf{c}$ is a lowercase letter, because it is a vector.

If ∇_0 has two elements, and $\mathbf{C}$ two columns, equation (6.7) describes an ellipse (because $\mathbf{Q}_{\hat{r}}$ is a semidefinite positive matrix). If ∇_0 has three elements the equation describes an ellipsoid, for more than three a hyperellipsoid.

The quality of a deformation analysis can be described by giving the tests that will be performed for the analysis, and listing the MDD's belonging to these tests.

An advantage of MDD's is that they can be computed in the design stage of a geodetic deformation network, when no measurements have been made yet (Figure 6.3). They allow to design an optimal network for the detection of those deformations that are most crucial (see chapter 3). Besides the criteria available for optimal design and sensitivity analysis of geodetic networks (Heunecke et al., 2013 (pp. 251ff.); Niemeier, 1982; Kuang, 1991), they yield the additional criteria to assess a network for its capability to detect deformations.

6.3.3 Best deformation description

6.3.3.1 Model identification and information criterion

To find the best mathematical description of a deformation, model identification is needed. Its purpose is to find the model that describes the relations between the observations most adequately. To start model identification the simplest deformation hypothesis is formulated. In geodetic deformation analysis this is often stability, which implies the absence of any deformation. The simplest deformation hypothesis may also describe something else than stability. For example a linear movement in time of some points may be assumed, if it is clear that these are not stable. This simple deformation is described by parameters $\mathbf{x}_d$ in model (6.1).

Testing a model, treated in appendix D.2, is done by opposing a null hypothesis to an alternative hypothesis. The alternative hypothesis has more parameters (parameters ∇ are supplementary). The null hypothesis is retained, unless it is rejected. This is in

accordance with *Occam's razor*, which means that the simplest hypothesis (translated as: with the fewest parameters; cf. Popper, 2002, p. 394ff.) is the preferred hypothesis.

Testing starts by performing an overall model test. If it does not reject the model, the simplest deformation hypothesis is corroborated, cf. Popper (2002, p. 248ff.). Otherwise, a better model needs to be identified.

Finding a better hypothesis implies comparing several models (i.e. several alternative hypotheses) with a different number of parameters. To choose between them, a criterion is needed (Lehmann and Lösler, 2016). Well-known are the Akaike Information Criterion (Akaike, 1974), and the Bayesian Information Criterion (Schwarz, 1978). In geodesy the B-method of testing (Baarda, 1968b, p. 33) is well-known. This method in combination with test ratios (cf. Chang (2015); de Heus et al. (1994b,a)) is used as information criterion in the example of Section 6.4.

To explain this information criterion, we first define the concepts "bias parameter", "valid model", and "reference minimal detectable bias". The alternative hypothesis deviates from the null hypothesis by its supplementary parameters, which are called here the *bias parameters*. If an alternative hypothesis is shown to give the best model among its competitors, the bias parameters of this hypothesis can be integrated into the model. The bias is in that case a vector of necessary additional parameters, and here: a deformation.

Suppose that a certain bias is indeed present, and that several models, each with a different vector of bias parameters, are valid descriptions of this bias. Here we call these models *valid models* for this bias. We take one of these models as reference and call its vector ∇_0 of minimal detectable biases the *reference minimal detectable bias* (reference MDB). Let the number of elements of ∇, i.e. its dimension, be q. We call the test of an alternative hypothesis with ∇ as vector of bias parameters a q-dimensional test. In general the MDB of a one-dimensional test is taken as reference MDB, i.e. the MDB of an alternative hypothesis, which is tested with a w-test (Baarda, 1968b, p. 18ff.).

The B-method states that the power of a test to find a bias that is equal to the reference MDB should be the same for all tests of valid models, whatever the dimension q of each of these tests may be. A consequence of the B-method is that tests of different dimension use different critical values (Baarda, 1968b, p. 25). Therefore, to evaluate the results of tests of different dimension, the values of test statistic T_q cannot be compared directly. Hence, the ratio of the test statistic and its critical value is used (Chang, 2015; de Heus et al., 1994b,a). A bias equal to ∇_0 will lead to rejection by tests of two valid models with an equal power. Therefore, the test ratio's of both tests will be larger than one with an equal power.

Hence, the basis for using the B-method of testing in combination with test ratio's as an information criterion, is the fact that the test ratio's of two hypotheses are larger than one with the same power, if both hypotheses are valid descriptions of the bias, and this bias is equal to the reference minimal detectable bias.

The use of this information criterion is shown in the elaborated example of Section 6.4.

6.3.3.2 Estimation of deformation parameters

If the best deformation description has been determined, the next step is drawing conclusions. This includes estimation of the deformation parameters ∇. They can be estimated by using model (6.1) ($\mathbf{C}$ is inside H_0) and one of the solution methods of appendix D.1. Another option is to use a model that assumes stability (no deformation; $\mathbf{C}$ is outside H_0). ∇ is estimated from the adjustment results of H_0 as:

$$\underline{\hat{\nabla}} = (\mathbf{C}^T\mathbf{Q}_{\hat{r}}\mathbf{C})^{-1}\mathbf{C}^T\underline{\hat{\mathbf{r}}}. \tag{6.9}$$

This result follows from the overview of testing equations in appendix D.2 (multiply equation (7.32) with $\mathbf{B}^T$ and solve for ∇). It is usable if $\mathbf{Q}_y$ is singular, which is the case for model (6.1). We notice that this equation is equal to equation (33) of (Teunissen, 2006), if $\mathbf{Q}_y$ is nonsingular (i.e. $\mathbf{Q}_y^{-1}$ exists):

$$\underline{\hat{\nabla}} = (\mathbf{C}^T\mathbf{Q}_y^{-1}\mathbf{Q}_{\hat{e}}\mathbf{Q}_y^{-1}\mathbf{C})^{-1}\mathbf{C}^T\mathbf{Q}_y^{-1}\underline{\hat{\mathbf{e}}}. \tag{6.10}$$

This is, because we have from equation (6.3), and the application of the propagation law of cofactors:

$$\underline{\hat{\mathbf{r}}} = \mathbf{Q}_y^{-1}\underline{\hat{\mathbf{e}}}\ ; \quad \mathbf{Q}_{\hat{r}} = \mathbf{Q}_y^{-1}\mathbf{Q}_{\hat{e}}\mathbf{Q}_y^{-1}, \tag{6.11}$$

with $\mathbf{Q}_{\hat{e}}$ the cofactor matrix of $\underline{\hat{\mathbf{e}}}$. If we insert this in equation (6.9), we get equation (6.10).

6.4 Experimental validation

The workability of the proposed method is experimentally validated by an example from deformation analysis, using spirit levelling measurements. The approach of this example can be applied to deformation description problems in e.g. the mining industry, where soil subsidence has to be monitored. An important advantage of the method is that no stable points are needed for the definition of an S-system. Professional practice in, for example, the mining industry in the Netherlands uses guidelines that demand such stable points, because it is assumed that they are necessary for the analysis (T.P.B., 2014).

6.4.1 Problem description

The problem solved in this experimental validation is the description of the deformation of five levelling bolts. The observations are height differences between the bolts, measured by spirit levelling. The bolts are indicated as A to E (Figure 6.4). The measurements are repeated four times, resulting in a time series of five epochs of measurements. The moments in time of these epochs are given in Table 6.1. Time is indicated in "units of time" (uot), which may be any unit (day, week, several months, etc). Both bolts B and C have been monumented in the same building, which is suspected to subside linearly with time. Bolts A, D and E are supposed to be stable and not influenced by the forces causing the subsidence. The suppositions are to be tested by adjusting and testing the measured height differences of all epochs.

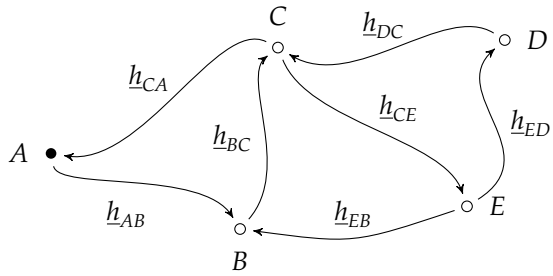

Figure 6.4: Levelling network.

Epoch	1	2	3	4	5
Time	1	2	4	7	10
h_{AB}	1.5730	1.5700	1.5690	1.5600	1.5550
h_{BC}	-0.1440	-0.1464	-0.1454	-0.1456	-0.1447
h_{CA}	-1.4290	-1.4270	-1.4220	-1.4160	-1.4090
h_{CE}	0.8682	0.8708	0.8754	0.8817	0.8877
h_{EB}	-0.7241	-0.7247	-0.7309	-0.7377	-0.7424
h_{ED}	0.6577	0.6572	0.6571	0.6577	0.6572
h_{DC}	-1.5280	-1.5290	-1.5320	-1.5380	-1.5440

Table 6.1: Time series of measurements of height differences in meter (time in any unit of time).

6.4.2 Measurements

The simulated measurements of the height differences are listed in Table 6.1. They have a standard deviation of 1 mm and no correlation. The subsidence has been added to the heights of B and C by adding in each following epoch a height difference Δh defined by:

$$\Delta h = a\,(t_{(i+1)} - t_i), \tag{6.12}$$

with $a = -2$ mm/uot and t_i the time of epoch i and $i = 1, .., 4$. Both B and C subside at the same rate.

6.4.3 Null hypothesis: point field stability

We start with the null hypothesis that no subsidence has occurred. This is a simpler hypothesis with less parameters than the hypothesis that B and C have subsided, cf. Section 6.3.3.1. If this hypothesis is rejected, an alternative hypothesis with more parameters is formulated and tested.

To build model (6.1) for the null hypothesis, the heights of the five bolts are taken as the 25 parameters in vector **x**, five parameters for each epoch.

$$\mathbf{x} = \left(h_A^{(1)}, \dots, h_E^{(1)}, \dots, h_A^{(5)}, \dots, h_E^{(5)}\right)^T. \tag{6.13}$$

where by (1), (5) the epoch is indicated.

The observations of Table 6.1 give rise to 35 observation equations (vector $\underline{\mathbf{y}}_s$ in model (6.1)):

$$\begin{array}{lll}
\underline{y}_1 & = -h_A^{(1)} + h_B^{(1)} + \underline{e}_1 & = -x_1 + x_2 + \underline{e}_1 \\
\underline{y}_2 & = -h_B^{(1)} + h_C^{(1)} + \underline{e}_2 & = -x_2 + x_3 + \underline{e}_2 \\
\underline{y}_3 & = h_A^{(1)} - h_C^{(1)} + \underline{e}_3 & = x_1 - x_3 + \underline{e}_3 \\
\vdots & \vdots & \vdots \\
\underline{y}_8 & = -h_A^{(2)} + h_B^{(2)} + \underline{e}_8 & = -x_6 + x_7 + \underline{e}_8 \\
\vdots & \vdots & \vdots \\
\underline{y}_{35} & = h_C^{(5)} - h_D^{(5)} + \underline{e}_{35} & = x_{24} - x_{25} + \underline{e}_{35}
\end{array} \tag{6.14}$$

To these observations we add two groups of nonstochastic observations.

1. Twenty observations to establish the equality of heights of all points in all epochs (vector $\mathbf{z}_d$ in model (6.1)). The observation equations are:

$$\begin{array}{ll}
y_{36} = -h_A^{(1)} + h_A^{(2)} = -x_1 + x_6 \\
y_{37} = -h_B^{(1)} + h_B^{(2)} = -x_4 + x_9 \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
y_{41} = -h_A^{(2)} + h_A^{(3)} = -x_6 + x_{11} \\
y_{42} = -h_B^{(2)} + h_B^{(3)} = -x_7 + x_{12} \\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
y_{54} = -h_D^{(4)} + h_D^{(5)} = -x_{19} + x_{24} \\
y_{55} = -h_E^{(4)} + h_E^{(5)} = -x_{20} + x_{25}
\end{array} \tag{6.15}$$

   The values for all twenty observations are zero.

2. One observation to define the S-system (vector z_g in model (6.1)). Because from height differences no heights can be determined without having the height of at least one of the bolts, the height of one bolt in one epoch is fixed at some arbitrary height. We take $h_A^{(1)} = 2$ m, but any one height in any one epoch would do. This arbitrary height and its standard deviation of zero are the S-basis. The observation equation can be written as:

$$y_{56} = h_A^{(1)};\ \sigma_{y_{56}} = 0\,\text{mm}, \tag{6.16}$$

noticing that because the standard deviation is zero, the observation is non-stochastic and the residual $e_{56} = 0$. The value of $y_{56} = 2\,\text{m}$. Because the height of A is only fixed for the first epoch, the estimation of the height of A at other epochs can yield a different height, if under (1) not equality, but some movement is formulated. How a movement can be formulated is shown in the following sections. It means that A does not have to be a stable point to be usable as S-basis.

The 21 observations in these two groups are all taken as nonstochastic; their standard deviation is zero:

$$\sigma_{y_{36}} = ... = \sigma_{y_{56}} = 0 \tag{6.17}$$

There are 56 observations and 25 unknown parameters all together, which results in a redundancy of 31.

6.4.3.1 Adjustment and overall model test

The model, formulated in the previous section, is adjusted by means of least squares, cf. appendix D.1. In Table 6.2 the estimated heights are given. The heights of all points have the same values for all epochs. This is the consequence of the observations (constraints) under (1) in Section 6.4.3, which state that the heights do not change.

	h_A	h_B	h_C	h_D	h_E
Height	2.0000	3.5655	3.4205	4.9547	4.2974

Table 6.2: Estimated heights in meters for all epochs (no deformation assumed.)

The null hypothesis is tested by performing an overall model test. The significance level α of this test follows from the B-method of testing (Section 6.3.3.1). It is computed as α=0.172 after choosing the significance level α_0 of one-dimensional tests as $\alpha_0 = 0.001$. With α, the critical value is computed as $\chi^2_{crit.} = 38.3$.

The overall model test gives a value of test statistic $\underline{T}_q$ of 1195.6. The model is therefore clearly rejected. It means that the situation of Table 6.2, i.e. all heights stay the same, is not a valid hypothesis. Several alternative hypotheses are subsequently formulated and tested to find the best one. The following sections treat different possible alternative hypotheses.

6.4.3.2 Testing each observation

The search for the cause of the rejection starts with the conventional w-tests (appendix D.2.2). They test as many alternative hypotheses as there are observations. Each hypothesis has, with respect to the null hypothesis, only one additional parameter that affects only one observation. In tables 6.3 and 6.4 the absolute w-values for respectively the height differences and the point stability are shown. It is not tested, whether the nonstochastic observation of the S-basis (point A is fixed to a value of 2 m) is affected

by a bias. Its w-value is undefined, because this observation is not checked by any of the other observations.

Epoch	1	2	3	4	5
Time	1	2	4	7	10
h_{AB}	8.56	5.13	3.99	6.28	11.99
h_{BC}	1.11	1.54	0.44	0.66	0.34
h_{CA}	9.70	7.42	1.72	5.13	13.12
h_{CE}	9.57	6.69	1.61	5.36	11.99
h_{EB}	8.75	8.07	1.07	6.61	11.91
h_{ED}	0.38	0.19	0.30	0.38	0.19
h_{DC}	7.10	5.95	2.53	4.32	11.17

Table 6.3: absolute w-values for height differences (critical value is 3.29 with a significance level of 0.001).

Epoch interval	1-2	2-3	3-4	4-5
Δt	1	2	3	3
h_A	10.00	11.25	13.33	13.75
h_B	5.93	7.58	8.92	8.90
h_C	7.62	8.49	9.49	9.94
h_D	3.67	4.69	5.72	6.01
h_E	6.98	8.27	8.87	8.98

Table 6.4: absolute w-values for stability of points (critical value is 3.29 with a significance level of 0.001; Δt is time length of each epoch interval).

Most absolute w-values are much larger than the critical value of 3.29. That is an indication that biases in the observations are present. But which heights or height differences are biased? The w-test is a test that checks whether one observation has a bias, and the other observations are without bias. Even if all observations are tested by the respective w-tests, these tests cannot give the right conclusion, if the bias is in fact in more than one observation. This can be seen in the tables. Not B and C give the largest w-values, although we might expect the largest values for them. It seems as if the height of A between epochs 4 and 5 is biased, because it gives the largest w-value (Table 6.4).

6.4.3.3 One point subsiding, differently for each epoch interval

A more specific hypothesis states that one of the five points is subsiding, but without correlation between the epoch intervals. This hypothesis is formulated by specifying

matrix **C** of equation (6.4). Matrix **C** has four columns, pertaining to four extra parameters ∇. Each parameter describes the bias during one epoch interval. There are four epoch intervals (1–2, 2–3, 3–4, 4–5). Matrix C has the following structure (dots indicating zeros):

$$C = \begin{pmatrix} 0,\dots, & 1,\dots,0,\dots,0,\dots,0,\dots,0 \\ 0,\dots, & 0,\dots,1,\dots,0,\dots,0,\dots,0 \\ 0,\dots, & 0,\dots,0,\dots,1,\dots,0,\dots,0 \\ 0,\dots, & 0,\dots,0,\dots,0,\dots,1,\dots,0 \end{pmatrix}^T . \tag{6.18}$$

Test statistic $\underline{T}_q$ is computed and tested. The value is 493.0 with a critical value of 36.4. Because q=4, w-values cannot be computed.

Test statistic $\underline{T}_q$ is χ^2-distributed. The square of $\underline{w}$ is also χ^2-distributed, but the degrees of freedom and the critical values are different. We determine, as described in Section 6.3.3, the ratio of $\underline{T}_q$ and its critical value and likewise the ratio of $\underline{w}^2$ and its critical value to use the ratios as information criterion. The largest w-value encountered upto now is w=13.75, which gives a test ratio of 17.5. The test ratios of the above hypothesis of one point subsiding differently for each epoch interval are given in Table 6.5. Four of these test ratios exceed the largest previously computed w-test ratio. This

	h_A	h_B	h_C	h_D	h_E
Test ratio	36.4	23.8	38.5	6.6	25.3

Table 6.5: Test ratios of one point subsiding differently for each epoch interval.

indicates that points may be unstable during more epochs. However, bolt C and then A have the largest values. Is it possible to formulate still better hypotheses?

6.4.3.4 One point subsiding linearly

The next step can be to test whether one of the points is subsiding linearly in time with a constant rate. Performing such a test is done by constructing the matrix **C** that describes the alternative hypothesis. Matrix **C** has one column, pertaining to one extra parameter ∇, which is the linear subsidence rate. For example between the first and second epoch the subsidence for point A is expressed as:

$$h_A^{(2)} = h_A^{(1)} + t_{12}\,\nabla. \tag{6.19}$$

with $t_{12} = t_2 - t_1$ the time interval (epoch interval) between epoch 1 and 2 and ∇ the rate of change, constant for all epoch intervals.

Four nonstochastic observations are extended with this extra parameter to describe the linear subsidence of a point in each of the four epoch intervals 1–2, 2–3, 3–4 and 4–5. In matrix **C** in the pertinent rows appears the time difference t_{ij} of each epoch interval i–j. All other elements of matrix **C** are zero. We define three vectors $\mathbf{n}_s$, $\mathbf{n}_d$ and $\mathbf{n}_g$.

Vectors $\mathbf{n}_s$ and $\mathbf{n}_g$ have the same lengths as $\underline{\mathbf{y}}_s$ and $\underline{\mathbf{y}}_g$ respectively. $\mathbf{n}_d$ has five elements.

$$\mathbf{n}_s = (0, \ldots, 0)^T; \quad \mathbf{n}_g = 0; \quad \mathbf{n}_d = (1, 0, 0, 0, 0)^T. \tag{6.20}$$

The "1" corresponds to the point that is tested; here point A. Matrix $\mathbf{C}$ has one column as follows:

$$\mathbf{C} = \left(\mathbf{n}_s^T, \mathbf{n}_d^T, 2\mathbf{n}_d^T, 3\mathbf{n}_d^T, 3\mathbf{n}_d^T, \mathbf{n}_g^T\right)^T, \tag{6.21}$$

The test statistic $\underline{T}_q$ is computed and tested as described in appendix D.2. Because q=1, we can compute w-values as well. The results are given in Table 6.6. Two test ratios

	h_A	h_B	h_C	h_D	h_E
w-value	22.2	17.9	22.8	9.4	18.5
Test ratio	45.4	29.5	48.1	8.2	31.5

Table 6.6: absolute w-values and test ratios for testing linear subsidence of one point.

are notably larger than those of Table 6.5, so linear subsidence might have occurred. But it is not clear that only a linear subsidence of points B or C has occurred.

6.4.3.5 Points B and C subsiding linearly

Because it is known that bolts B and C are positioned in the same building, it is a logical step to test whether both bolts are subsiding in the same way. A matrix $\mathbf{C}$ is constructed to test this hypothesis.

We define two vectors $\mathbf{n}_s$ and $\mathbf{n}_g$ as in equation (6.20). Vector $\mathbf{n}_d$ is defined as:

$$\mathbf{n}_d = (0, 1, 1, 0, 0)^T. \tag{6.22}$$

Matrix $\mathbf{C}$ is taken as in equation (6.21). We compute a w-value of 34.3 and a test ratio of 108.4 (Table 6.7). These values are yet another step larger than the values computed in Table 6.6. Testing the same hypothesis for other combinations of two points gives smaller w-values (Table 6.7). Indeed the hypothesis that both B and C are linearly subsiding, is indicated as a very good one.

Finding the hypothesis that explains the available data best, is implemented here as finding matrix $\mathbf{C}$ that yields a large test ratio. To formulate the best hypothesis, it is important to have information about the causes of the deformation, as this example shows.

6.4.4 New null hypothesis

Based on the result of the previous section, a new null hypothesis can be formulated, cf. Section 6.3.1.5. Points B and C are now supposed to subside with a rate of ∇ mm per unit of time. This extra parameter is added to vector $\mathbf{x}$ of parameters as 26$^{\text{th}}$ parameter.

Point pair	w-value	Test ratio
A, B	0.2	0.0
A, C	7.2	4.7
A, D	22.3	46.0
A, E	28.3	74.1
B, C	34.3	108.4
B, D	7.9	5.8
B, E	0.5	0.0
C, D	16.2	24.2
C, E	6.1	3.5
D, E	26.1	63.1

Table 6.7: absolute w-values and test ratios for testing linear subsidence of two points.

Eight observation equations establish the linear subsidence of B and C. We use equations like equation (6.19) and replace eight observation equations by the following ones, using the additional parameter $\nabla = x_{26}$:

$$\begin{aligned} y_{37} &= x_2 - x_7 + t_{12}\, x_{26} \\ y_{38} &= x_3 - x_8 + t_{12}\, x_{26} \\ y_{42} &= x_7 - x_{12} + t_{23}\, x_{26} \\ y_{43} &= x_8 - x_{13} + t_{23}\, x_{26} \\ &\vdots \qquad\qquad \vdots \\ y_{53} &= x_{18} - x_{23} + t_{45}\, x_{26} \end{aligned} \tag{6.23}$$

The values of these eight nonstochastic observations are zero.

6.4.4.1 Adjustment results

The least-squares solution of the model, constructed in the preceding section, can be obtained by one of the methods of appendix D.1. The estimated heights are given in Table 6.8.

Epoch	1	2	3	4	5
h_A	2.0000	2.0000	2.0000	2.0000	2.0000
h_B	3.5734	3.5713	3.5672	3.5609	3.5547
h_C	3.4284	3.4263	3.4222	3.4159	3.4097
h_D	4.9547	4.9547	4.9547	4.9547	4.9547
h_E	4.2974	4.2974	4.2974	4.2974	4.2974

Table 6.8: Estimated heights in meters.

The table shows that the constraints, imposed by the nonstochastic observations, have been effective:

- Point A in epoch 1 has been fixed at 2 m.
- Points A, D and E have the same height in all five epochs.
- Points B and C have a linear subsidence with an estimated rate $\hat{\nabla}$ ($= \hat{x}_{26}$) of -2.07 mm/uot. It is in accordance with the value used for the simulated observations (Section 6.4.2). This rate is illustrated in Figure 6.5.

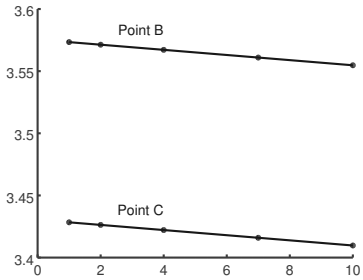

Figure 6.5: Subsidence of B (upper line) and C (lower line).

6.4.4.2 Test result

For the overall model test we have a value for $\underline{T}_q$ of 21.4, which means that the test doesn't reject the adjustment model. The assumed deformation (A, D and E are stable and B and C are subsiding linearly) is not rejected. And neither are the measured height differences rejected.

6.4.5 Quality description

To plan a deformation analysis before any measurement has yet been done, it is important to give an indication of the test quality. In this section for all tests of the previous section the quality is described by giving the minimal detectable biases, i.e. the minimum deformations that can be detected by each test.

6.4.5.1 All points stable, testing each observation

For the null hypothesis of Section 6.4.3.2 the MDB-values are:

1. 4.3 or 4.4 mm per epoch for all measured height differences;
2. listed in Table 6.9 for the point stability tests;

Table 6.9 shows that a lower MDB (= better detection of deformations) is achieved, if a height in a certain epoch is sandwiched in time (there are measurements before and after that epoch) and in space (there are measurements to surrounding points). The

Epoch interval	1-2	2-3	3-4	4-5
Δt	1	2	3	3
h_A	3.3	2.7	2.7	3.3
h_B	2.7	2.2	2.2	2.7
h_C	2.3	1.9	1.9	2.3
h_D	3.3	2.7	2.7	3.3
h_E	2.7	2.2	2.2	2.7

Table 6.9: MDB in mm for stability of points (significance level: 0.001, power: 80%).

lowest values are in epoch intervals 2-3 and 3-4 (sandwiched in time between epochs 1 and 4) for point C (sandwiched in space between four other points).

6.4.5.2 One point subsiding, differently in each epoch interval

For the null hypothesis of Section 6.4.3.3 (only one point subsiding, but differently in each epoch interval) four MDB-values for each point can be computed. They are determined by the intersections of the four-dimensional hyperellipsoid that follows from equation (6.7) with the coordinate axes, see Table 6.10.

Epoch interval	Δt	h_A	h_B	h_C	h_D	h_E
1-2	1	1.3	1.0	0.9	1.3	1.0
2-3	2	1.6	1.3	1.1	1.6	1.3
3-4	3	1.6	1.3	1.1	1.6	1.3
4-5	3	1.3	1.0	0.9	1.3	1.0

Table 6.10: MDB for each point in mm per epoch interval for subsidence, different for each interval.

6.4.5.3 One point subsiding linearly

For the null hypothesis of Section 6.4.3.4 (linear subsidence of only one point with a constant rate) the MDB-values are listed in Table 6.11.

	h_A	h_B	h_C	h_D	h_E
MDB	0.39	0.32	0.28	0.39	0.32

Table 6.11: MDB's in mm per uot for linear subsidence of one point with a constant rate (significance level: 0.001, power: 80%.)

Suppose we want to design a deformation analysis network that can detect a deformation of, say, 2 mm per epoch interval. From Table 6.9 it follows that the test of one point's

subsidence in only one epoch interval can detect a subsidence of 2 mm per epoch interval with a power of 80% in only two epoch intervals. Using, however, five epoch intervals (Table 6.11), one can easily detect a subsidence of 2 mm per epoch interval with a power of 80% (the epoch intervals are 1, 2 or 3 uot (units of time)).

6.4.5.4 Points B and C subsiding linearly

The MDB of testing the null hypothesis of Section 6.4.3.5 by a one-dimensional test is 0.25 mm/uot. This is better than the MDB's of Section 6.4.5.3, because more observations are used.

6.4.6 S-transformation invariance

The height of bolt A has been fixed for the analysis at a value of 2 m in epoch 1, to be able to determine the heights of the other bolts in all epochs. Point A in epoch 1 is the S-basis. The analysis doesn't change, if one of the other points is taken as S-basis. Let the S-basis be point B, not A. The height of point B in epoch 1 is fixed at 2 m. The only values that change are the estimated heights of the points. For the situation of Section 6.4.4.1 we get different estimated heights (Table 6.12), but all results regarding testing and quality description are the same.

Epoch	1	2	3	4	5
Time	1	2	4	7	10
h_A	0.4266	0.4266	0.4266	0.4266	0.4266
h_B	2.0000	1.9979	1.9940	1.9876	1.9814
h_C	1.8550	1.8529	1.8488	1.8426	1.8364
h_D	3.3813	3.3813	3.3813	3.3813	3.3813
h_E	2.7240	2.7240	2.7240	2.7240	2.7240

Table 6.12: Estimated heights in meters with h_B in epoch 1 fixed at 2 m.

This example makes it clear that there is no need for the S-system to be defined by one of the stable points.

6.4.7 Deformation analysis in 2D and 3D

A 2D or 3D deformation analysis can be effectuated with model (6.1) in the same way as a 1D problem. An S-system is defined by formulating the minimally necessary number of constraints on the coordinates with $\mathbf{z}_g$. In 2D an S-system is defined by 3 or 4 constraints (analysis of form and size, respectively only of form) (see chapter 3). In 3D it is defined by 6 or 7 constraints (see chapter 4). In appendix D.3 a proof is given of the invariance of the test results for an S-transformation, i.e. for the transition

to another S-system. Here the S-transformation is a 2D or 3D congruence or similarity transformation.

To define the equality (or deformation) of points in different epochs, vector $\mathbf{z}_d$ and matrices $\mathbf{Z}_d$ and $\mathbf{Z}_\nabla$ are used. Each point, for which stability or deformation is postulated, is represented in $\mathbf{z}_d$ by one, two or three elements (1D, 2D, 3D). These one, two or three elements within $\mathbf{z}_d$ can be seen as a vector that links the position of a point in one epoch to the position of the same point in another epoch. If the vector has elements with values zero, it is the null vector, and represents stability: the point does not move. If such a vector has elements different from zero, it represents a point that is moving.

Application of the method of this chapter to 2D and 3D problems, especially the testing of particular deformation problems (such as movements of a bridge or quay, torsion in a tunnel), will be the subject of future research.

Another approach to 2D and 3D problems is possible, which performs the analysis in two steps. In the first step the measurements of each separate epoch are processed to get adjusted coordinates and their (possibly singular) covariance matrix for each epoch. Succesively an adjustment model that has coordinates as input, and includes a transformation, can be used for the deformation analysis of two epochs (see chapter 4). This model can be extended to more than two epochs, i.e. to a time series of coordinates, for 2D and 3D deformation analysis problems (see chapter 7). Testing deformation hypotheses by testing constraints is possible for such a model in a similar way as described in this chapter.

6.5 Conclusions

In this study the problem of finding the best mathematical description of a deformation is addressed by testing physical deformation hypotheses by a time series of geodetic observations. The hypotheses are formulated as constraints on the parameters of an adjustment model. This approach allows for a flexible method to test deformation hypotheses, to find the best one, and to give a quantification of the test quality. The method uses the full stochastic information of the observations. Moreover, the tests are invariant for datum transformations, and the points that define the geodetic datum do not have to be stable points.

The constraints, describing the deformation hypotheses, can involve subsets of points and subsets of epochs. The constraints are formulated as nonstochastic observations (constants that are considered as observations). Thus the constraints can be tested as observations. Also minimal detectable deformations can be computed, by which the quality of the constraint tests can be assessed. In this chapter it has been shown how the best description of deformations is found with an information criterion. Here the B-method of testing combined with test ratio's is used as information criterion. The criterion is meant to have an optimal discriminatory power between alternative hypotheses. The MDD's are well suited to check geodetic deformation analysis models for the minimal detectable deformations, and the method is usable for 1D, 2D and 3D problems. The method's use has been demonstrated with an elaborate example.

Comparison of the performance of different information criteria; the testing of particular deformation problems, such as deformations of bridges, quays and tunnels; and also the option to estimate local deformation signals with covariance functions, is left for future research.

7

Time series analysis with coordinates[1]

In the previous chapter deformation analysis with a time series of geodetic observations has been treated. In geodetic practice it is a common situation that the original observations are not available any more, often because it is deemed convenient to store derived coordinates, and not the original observations, which are more numerous and difficult to interpret for non-specialists. Hence, in this chapter a time series of *coordinates* is used as input for the deformation analysis. The covariance matrix of the coordinates is known or unknown. It is *known*, when the coordinates stem from a previous adjustment of geodetic observations, and the covariance matrix has been estimated (and has also been stored together with the coordinates, and is available for further use). In many situations the covariance matrix is *unknown*, because it has not been estimated, it has not been stored, or it is not available. In the absence of the covariance matrix, it is replaced by a scaled unit matrix, or by a diagonal matrix. The scaling factor or the diagonal elements are known from the further unknown covariance matrix, or they are estimated from experience or requirements. With the adjustment model of this chapter it can easily be determined what the losses are, if a full covariance matrix is replaced by a scaled unit matrix, or a diagonal matrix.

This chapter treats the situation that 3D coordinates are used as input. The situations for 2D coordinates (x and y), and 1D coordinates (normally heights) are simpler, and the necessary results are easily derived from the 3D situation.

7.1 Introduction

Geodetic deformation analysis is about change of form and size of the earth's surface or of objects on, below or above it, and also of the relative position and orientation of the

[1] This chapter has been published before in Journal of Applied Geodesy (Velsink, 2016b). To fit the publication into this study, minor changes have been made.

objects. The objects to be analysed are represented by points that constitute a three-dimensional geodetic network. It is nowadays common practice to use total stations, GPS receivers and other devices for the analysis. If processing the measurements results in three-dimensional x, y, z coordinates, these can be presented in two-dimensional graphs, showing the displacements in time or in space or both. It is, however, generally difficult to come to statistically sound conclusions by analysing the graphs. Computational methods to test statistical hypotheses are desirable. For two epochs methods are available to perform an adjustment of the coordinates, taking into account their covariance matrix, and to perform hypothesis testing (Heunecke et al., 2013, p. 494ff.). The analysis is generally not invariant for the used S-basis (see chapter 3, section 3.7.3).

In this chapter an adjustment model is proposed that analyses a time series of 3D coordinates, taking account of the covariance matrices and analysing the deformations of all points and all epochs simultaneously, by computing statistics of deformation patterns and testing them. The model can be applied to any 3D geodetic network, observed quasi-continuously (i.e. with permanently installed sensors measuring frequently). Examples are the monitoring of the movement of a subset of points through all epochs, or the periodic oscillation of a subset of points.

In the next section the problem is defined. After describing existing approaches in section 7.3, section 7.4 describes the solution set-up. The adjustment model is treated in section 7.5. The adjustment itself, the deformation testing and the S-basis invariance are handled in section 7.6. Section 7.7 gives an experimental validation of the model.

7.2 Problem definition

The problem addressed in this chapter is the adjustment and testing for deformations of a time series of three-dimensional coordinates of a geodetic network, with a covariance matrix of the coordinates that is full and generally singular, because each epoch of the time series is adjusted as a free network.

The described problem will be handled by constructing a least squares adjustment model. As a practical application to show the usability of the model, the continuous monitoring by a total station of points, situated on built structures that are prone to deformations, is analysed.

7.3 Existing solutions

Heunecke et al. give an overview of existing approaches for deformation analysis of two time epochs of deformation measurements (Heunecke et al., 2013, p. 521) . The general approach is to compute displacement vectors between coordinates of two epochs and their covariance matrix. Different approaches exists to analyse the displacement vectors, e.g. by using 95%-confidence ellipsoids after a least squares adjustment (Kamiński and Nowel, 2013; Caspary, 2000) or a L1-norm adjustment (Chen, 1983; Caspary and Borutta, 1987), or using constraints on common points and analysing the quadratic form of the weighted estimated least squares residuals (Heunecke et al., 2013, p. 500ff.).

Heunecke et al. give also methods to analyse time series (Heunecke et al., 2013, p. 548). They do not take advantage of the covariance matrices of the coordinates and do not perform the analysis for all three dimensions (x, y and z) simultaneously. As a consequence the choice of datum definition and the solution's invariance for it, are not addressed.

A comprehensive 3D multi-epoch model is treated by Caspary (2000, p. 164ff.). It takes care of singular covariance matrices and incorporates deterministic deformation models. It assumes all epochs to be defined relative to the same S-basis, which has to be defined by points, measured in all epochs. Testing is treated for the sequential adjustment case. Quality description of the tests is not treated.

7.4 Solution set-up

7.4.1 Form and size, position and orientation

The subject of geodetic deformation analysis is the change in time of the form and size of objects, and also of the relative position and orientation of the objects. Form, size, relative position and orientation can be recorded by Euclidian x, y, z coordinates. It is assumed that the coordinates are normally distributed with a probability density function, which is fully described by a known covariance matrix, except for the first moments. This matrix may be singular, e.g. because it stems from a free network adjustment. If there are reference points, i.e. points that are considered not to be influenced by the deformation to be analysed, they are part of the geodetic network, and are analysed simultaneously with the object points.

The Euclidian coordinates describe the position and orientation of the network relative to the coordinate origin and axes as well. These, however, are *not* subject of the analysis. Their uncertaintity, as it is reflected in the covariance matrix, has to be eliminated from the analysis. This is realised in the adjustment model by a congruence or similarity transformation of the coordinates of each epoch to the coordinate system of the epoch!reference epoch. It is shown that after these transformations, testing of deformation hypotheses can be done independently from the S-bases chosen for the individual epochs. The first epoch is chosen in this chapter as reference epoch, but any other epoch as reference epoch would give the same analysis results.

The choice between a congruence and a similarity transformation depends on the question, whether the scale (unit of length) is considered stable between epochs and essential for the analysis.

The set-up of the adjustment model, with transformations between the epochs incorporated into it, not only removes the influence of origin, axes and scale of the reference system on the analysis. It also makes it possible to test, without additional S-transformations, for deformations of all kinds of subsets of points, independent of their being reference or object points, or being part of the S-basis or not. It is possible to include in one hypothesis that is to be tested, both reference and object points, and both points within and outside the S-basis.

7.4.2 Nonstochastic observations

The adjustment model is built as a model of observation equations with the coordinates as observations, arranged according to the epochs. The parameters are the expectations of the coordinates of all epochs. Each point has for each epoch different coordinates in the parameter vector. Also the transformation parameters of each epoch relative to the previous one appear in the parameter vector.

Constraints are stated concerning the coordinates of all epochs. In the case of stability analysis the constraints state that the expectations of coordinates of the same points in different epochs are equal. These constraints are added to the observation vector as nonstochastic observations, following the approach of chapter 5. If coordinates are assumed to be subject to some kind of deformation, for example a linear movement of one or more points, or a deformation pattern with a periodic character for a subset of points, the constraints add extra unknown parameters, for example the linear rate of movement, or the coefficients of the periodic pattern, to the parameter vector.

The advantage of using nonstochastic observations is that testing of deformation hypotheses is done in the same way as testing of one- or multidimensional hypotheses on biases in the other observations. Least squares estimates of the deformations are determined using standard formulas. Also minimal detectable biases can be computed with standard formulas, giving information on the deformation sizes that can be detected with the tests.

7.4.3 Full, singular covariance matrices

Observations, for example direction and distance observations of total stations, and their stochastic model are used for a deformation analysis, which is performed in two phases. In the first phase the direction and distance observations are adjusted for each epoch separately. The results are coordinates and their covariance matrices for all epochs. The second step is the subject of this chapter: the deformation analysis of the coordinates of many epochs. The covariance matrices of all epochs have to be used (Tienstra, 1956, p. 154). These matrices are generally full matrices (no or few zeros) and singular, because each epoch is adjusted as a free network, not connected to control points. The adjustment model of section 7.5 can handle full, singular covariance matrices.

7.4.4 Solution characteristics

An overview of the solution characteristics can now be given. The most relevant terms are listed below and the solution procedure is illustrated by a Nassi-Schneidermann diagram (figure 7.1).

A geodetic network per epoch is a set of points on, above, or under the earth's surface, in this chapter assumed to be represented by 3D Euclidian coordinates.

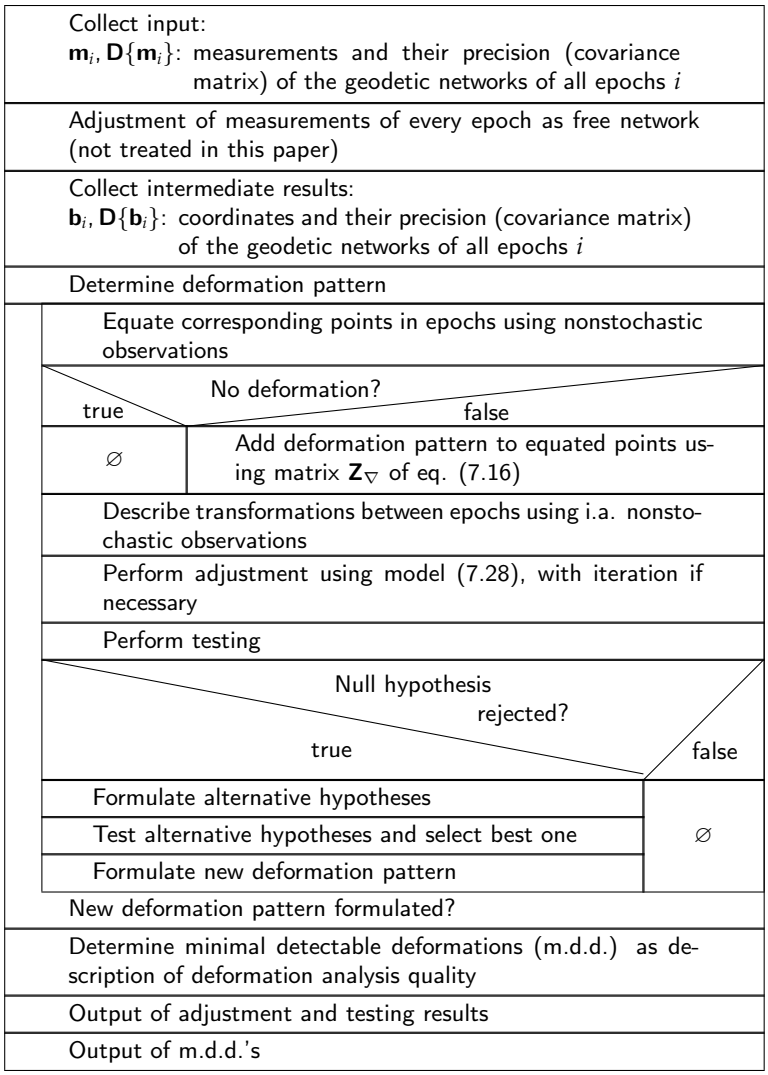

Figure 7.1: Solution procedure.

Form and size of a geodetic network (and their changes in time) are of interest, not the position and orientation relative to the reference system. Transformations are therefore included in the adjustment model.

Stability assumes the expectations of coordinates to be equal through all epochs, except for the above mentioned transformations.

A deformation pattern is the relation between the geodetic networks per epoch, formulated by giving the expectations of coordinates through the epochs using mathematical functions, described by nonstochastic observations, which depend on unknown deformation parameters, for example a linear movement rate, or the coefficients of a series expansion of a periodic function.

Inside or outside the adjustment model we put the description of the deformation. If it is *inside*, nonstochastic observations describe the deformation pattern, and extra deformation parameters are included in the parameter vector. If it is *outside*, the adjustment model assumes *stability* and no extra deformation parameters are in the parameter vector. The hypothesis of stability is tested against alternative hypotheses, describing deformation patterns, by appropiate test statistics, which make use of the nonstochastic observations to determine matrix $\mathbf{Z}_\nabla$ of eq. (7.16).

Singular, full covariance matrices result from the free network adjustments of each epoch, and are used in the adjustment model. If only coordinates are available for each epoch, a substitute matrix, for example a unit matrix, is used as covariance matrix, which yields sub optimal adjustment and testing results.

7.5 Adjustment model

7.5.1 Observations and parameters

The adjustment model is built taking as:

- observations:
  1. cartesian 3D point coordinates of a geodetic network and their covariance matrix, available for at least two epochs. For the first epoch they are assembled in vector $\underline{\mathbf{a}}_1$ (an underlined variable indicates a stochastic variable) with the covariance matrix $\mathbf{D}\{\underline{\mathbf{a}}_1\}$. For the second and later epochs they are assembled in vectors $\underline{\mathbf{b}}_i$, with i the epoch number, which runs from 2 to p with p the number of epochs. Each $\underline{\mathbf{b}}_i$ has a covariance matrix $\mathbf{D}\{\underline{\mathbf{b}}_i\}$;
  2. nonstochastic observations $\mathbf{z}_f$, describing constraints on the transformation parameters; their covariance matrix is the zero matrix;
  3. nonstochastic observations $\mathbf{z}_d$, describing the deformation pattern; their covariance matrix is the zero matrix.

- unknown parameters:
  1. expectations of cartesian 3D network point coordinates, for each epoch assembled in vector $\mathbf{c}_i$ of epoch i. Vector $\mathbf{c}$ takes all epochs together:
  $$\mathbf{c} = \left(\mathbf{c}_1, \cdots, \mathbf{c}_p\right)^T. \tag{7.1}$$
  2. vector of transformation parameters $\mathbf{f}$, subdivided in subvectors $\mathbf{f}_{i,i-1}$ for the transformation in each epoch interval between epoch i and $i-1$, with $i = 2, \ldots, p$.
  3. additonal parameters ∇ to describe the trend function of the deformation, see section 7.4.2.

7.5.2 Nonlinear adjustment model

In the adjustment model the expectations of all point coordinates are expressed in the reference system of the first epoch, and are *parameters* in vector $\mathbf{c}$. The *observed* coordinates in the first epoch are taken together in vector $\underline{\mathbf{a}}_1$. We have:

$$E\{\underline{\mathbf{a}}_1\} = \mathbf{c}_1 = \mathbf{P}_1\,\mathbf{c}, \tag{7.2}$$

with $E\{.\}$ the expectation operator, and $\mathbf{P}_1$ the matrix that selects the points of the first epoch from $\mathbf{c}$. $\mathbf{P}_1$ has only ones and zeros. The observed coordinates $\underline{\mathbf{b}}_i$ in a following epoch i ($i = 2, \cdots, p$) are assumed to be in a separate reference system, indicated by a superindex (i):

$$\underline{\mathbf{b}}_i = \underline{\mathbf{b}}_i^{(i)}. \tag{7.3}$$

These coordinates are transformed with a vector function $\varphi_{i,i-1}$ to the reference system of epoch $(i-1)$:

$$\underline{\mathbf{b}}_i^{(i-1)} = \varphi_{i,i-1}(\underline{\mathbf{b}}_i^{(i)}, \mathbf{f}_{i,i-1}), \tag{7.4}$$

then with $\varphi_{i-1,i-2}$ to the reference system of epoch $(i-2)$, and so on, and we get the transformed coordinates $\underline{\mathbf{a}}_i$:

$$\begin{aligned}\underline{\mathbf{a}}_i = \underline{\mathbf{b}}_i^{(1)} = \\ \varphi_{2,1}(\cdots(\varphi_{i,i-1}(\underline{\mathbf{b}}_i, \mathbf{f}_{i,i-1}), \ldots), \mathbf{f}_{2,1}),\end{aligned} \tag{7.5}$$

and:

$$E\{\underline{\mathbf{a}}_i\} = \mathbf{c}_i = \mathbf{P}_i\,\mathbf{c}. \tag{7.6}$$

$\mathbf{P}_i$ selects $\mathbf{c}_i$ from $\mathbf{c}$. It follows that:

$$E\{\varphi_{2,1}(\cdots(\varphi_{i,i-1}(\underline{\mathbf{b}}_i, \mathbf{f}_{i,i-1}), \ldots), \mathbf{f}_{2,1})\} = \mathbf{P}_i\,\mathbf{c}. \tag{7.7}$$

Hopping from epoch i through all intermediate epochs to the first one, is chosen, and not a direct transformation to the first epoch, because it is assumed that in general more common points are available for successive epochs.

Following the approach of chapter 4, the transformation $\varphi_{i,j}$ between epoch i and j, is of a general form, for example an affine transformation, which is changed to another type of transformation, for example a similarity transformation, by the use of constraints. These constraints are formulated as nonstochastic observations:

$$\mathbf{z}_f = \zeta_f(\mathbf{f}), \tag{7.8}$$

for which zeros are assumed as observed values.

The deformation pattern is described by a vector of nonstochastic observations $\mathbf{z}_d$ and a deformation function ζ_d, which gives a relation between the elements of $\mathbf{c}$ and the elements of a vector of deformation parameters ∇:

$$\mathbf{z}_d = \zeta_d(\mathbf{c}, \nabla). \tag{7.9}$$

For $\mathbf{z}_d$ we assume zeros as the observed values.

From equations (7.2), (7.7), (7.8) and (7.9) follows the following system for p epochs:

$$\begin{cases} E\{\underline{\mathbf{a}}_1\} & = \mathbf{P}_1\mathbf{c}, \\ E\{\varphi_{2,1}(\underline{\mathbf{b}}_2, \mathbf{f}_{2,1})\} & = \mathbf{P}_2\mathbf{c}, \\ & \vdots \\ E\{g(\underline{\mathbf{b}}_p, \mathbf{f})\} & = \mathbf{P}_p\mathbf{c}, \\ \mathbf{z}_f & = \zeta_f(\mathbf{f}), \\ \mathbf{z}_d & = \zeta_d(\mathbf{c}, \nabla), \end{cases} \tag{7.10}$$

with

$$g(\underline{\mathbf{b}}_p, \mathbf{f}) = \varphi_{2,1}(...(\varphi_{p,p-1}(\underline{\mathbf{b}}_p, \mathbf{f}_{p,p-1}), ...), \mathbf{f}_{2,1}).$$

A point may be present in an epoch, but missing in one or more other epochs. This is handled by matrix $\mathbf{P}_i$. The S-basis definition of an epoch is arbitrary (see section 7.6.3) and may be realised by only a few points, by many points, or by all. There can even be no S-basis, i.e. the covariance matrix is regular, and the S-basis can be considered to lie outside the geodetic network. The fact that a point is missing, be it in the first or in any other epoch, does therefore not pose any problem for the deformation analysis with model (7.10).

7.5.3 Transformations

7.5.3.1 Affine Transformation

As general form of transformation $\varphi_{i,j}$ the affine transformation is taken, written as:

$$\begin{aligned} &\begin{pmatrix} \mathbf{x}^T \\ \mathbf{y}^T \\ \mathbf{z}^T \end{pmatrix} = \mathbf{R} \begin{pmatrix} \mathbf{u}^T \\ \mathbf{v}^T \\ \mathbf{w}^T \end{pmatrix} + \mathbf{t}\boldsymbol{\epsilon}, \\ &\mathbf{R} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \ \mathbf{t} = \begin{pmatrix} t_x \\ t_y \\ t_z \end{pmatrix}, \\ &\boldsymbol{\epsilon} = (1, 1, \ldots, 1). \end{aligned} \tag{7.11}$$

The column vectors **u**, **v**, **w** contain resp. the x-, y-, z-coordinates of $\mathbf{b}_i$ before transformation. The vectors **x**, **y**, **z** contain the coordinates after transformation. **R** describes the rotation, shear and scale change of the affine transformation, **t** the translation.

7.5.3.2 Congruence Transformation

Equation (7.11) describes a congruence (or rigid body) transformation, if the nine coefficients of matrix **R** meet the following six constraints:

$$\begin{aligned} &\mathbf{a}_i^T\mathbf{a}_j = \delta_{ij},\ \mathbf{a}_i = \begin{pmatrix} a_{i1} \\ a_{i2} \\ a_{i3} \end{pmatrix},\ i,j = 1,2,3, \\ &j \geq i,\ \delta_{ij} = 1 \text{ if } i = j, \text{ otherwise } \delta_{ij} = 0. \end{aligned} \tag{7.12}$$

In the following sections a linearised adjustment model is derived. The linearised constraints are:

$$\begin{pmatrix} \mathbf{a}_2^{0T} & \mathbf{a}_1^{0T} & \mathbf{0} \\ \mathbf{a}_3^{0T} & \mathbf{0} & \mathbf{a}_1^{0T} \\ \mathbf{0} & \mathbf{a}_3^{0T} & \mathbf{a}_2^{0T} \\ \mathbf{a}_1^{0T} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_2^{0T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{a}_3^{0T} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{a}_1 \\ \Delta\mathbf{a}_2 \\ \Delta\mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{7.13}$$

where **0** is the (1×3) zero vector and $\mathbf{a}_i^0$ (i=1,2,3) is the vector of approximate values of $\mathbf{a}_i$. Δ indicates the difference of the quantity concerned and its approximate value.

7.5.3.3 Similarity transformation

For the similarity transformation the affine transformation is constrained with five constraints. Three constraints state that the three rows of **R** are perpendicular to each other. The two remaining constraints state that the lengths of the first and second row, and those of the second and third row are equal. The linearised constraints are:

$$\begin{pmatrix} \mathbf{a}_2^{0T} & \mathbf{a}_1^{0T} & \mathbf{0} \\ \mathbf{a}_3^{0T} & \mathbf{0} & \mathbf{a}_1^{0T} \\ \mathbf{0} & \mathbf{a}_3^{0T} & \mathbf{a}_2^{0T} \\ \mathbf{a}_1^{0T} & -\mathbf{a}_2^{0T} & \mathbf{0} \\ \mathbf{a}_1^{0T} & \mathbf{0} & -\mathbf{a}_3^{0T} \end{pmatrix} \begin{pmatrix} \Delta\mathbf{a}_1 \\ \Delta\mathbf{a}_2 \\ \Delta\mathbf{a}_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{7.14}$$

7.5.3.4 Approximate transformation

Before the adjustment, $\underline{\mathbf{b}}_i$ is approximately transformed to $\underline{\mathbf{b}}_i'$ in the reference system of $\underline{\mathbf{a}}_1$, using equation (7.5). Likewise $\mathbf{D}\{\underline{\mathbf{b}}_i\}$ is transformed to $\mathbf{D}\{\underline{\mathbf{b}}_i'\}$ by applying the law of propagation of covariances. The approximate transformation parameters are

determined as affine parameters, and subsequently adapted to those of a congruence or similarity transformation using singular value decomposition (see chapter 4). The transformations of equation (7.10) are now differential transformations. In each iteration step of the adjustment this is repeated with adapted transformation parameters from the previous iteration step. Therefore as approximate values for a_{ij} in the constraints of the congruence or similarity transformation we can take $a_{ij}^0 = \delta_{ij}$.

In the following sections, if $\underline{\mathbf{b}}_i$ is written, $\underline{\mathbf{b}}_i'$ is meant.

7.5.4 Linearised adjustment model

A linearised adjustment model is built for the deformation analysis. Linearisation of all equations of system (7.10) is done with implicit differentiation relative to the observed vectors $\underline{\mathbf{a}}_1$, $\underline{\mathbf{b}}_i$ (i= 2,$\cdots$,p), $\mathbf{z}_d$, and $\mathbf{z}_f$, and the unknown parameter vectors $\mathbf{f}$, $\mathbf{c}$ and ∇.

The first and last two equations of system (7.10) are linearised as:

$$\begin{cases} E\{\underline{\Delta\mathbf{a}}_1\} & = \mathbf{P}_1\,\Delta\mathbf{c}, \\ \Delta\mathbf{z}_f & = (\dfrac{\partial\zeta_f}{\partial\mathbf{f}})_0\Delta\mathbf{f}, \\ \Delta\mathbf{z}_d & = (\dfrac{\partial\zeta_d}{\partial\mathbf{c}})_0\Delta\mathbf{c} + (\dfrac{\partial\zeta_d}{\partial\nabla})_0\Delta\nabla. \end{cases} \tag{7.15}$$

We define for later use:

$$\begin{aligned} \mathbf{Z}_f &= (\frac{\partial\zeta_f}{\partial\mathbf{f}})_0, \\ \mathbf{Z}_d &= (\frac{\partial\zeta_d}{\partial\mathbf{c}})_0,\ \mathbf{Z}_\nabla = (\frac{\partial\zeta_d}{\partial\nabla})_0. \end{aligned} \tag{7.16}$$

The partial derivatives of the vectors ζ_f and ζ_d with respect to the vectors $\mathbf{f}$, $\mathbf{c}$ and ∇ are matrices. The parentheses with zero $(.)_0$ indicate that approximate values of the parameters have to be used to get the values in the matrices.

For the equations with $\underline{\mathbf{b}}_i$ (i=2,$\cdots$,p) in system (7.10) the linearised equations are:

$$\mathbf{B}_i\,E\{\underline{\Delta\mathbf{b}}_i\} + \mathbf{F}_i\,\Delta\mathbf{f}_i = \mathbf{P}_i\,\Delta\mathbf{c}, \tag{7.17}$$

with the matrices $\mathbf{B}_i$ defined as follows:

$$\mathbf{B}_i = \mathbf{B}_{2,1}\mathbf{B}_{3,2}\cdots\mathbf{B}_{i,i-1}, \tag{7.18}$$

and with (j=2, ...,i−1):

$$\begin{aligned} \mathbf{B}_{j,j-1} &= (\frac{\partial\varphi_{j,j-1}}{\partial\varphi_{j+1,j}})_0 = (\frac{\partial\varphi_{j,j-1}}{\partial\mathbf{b}_i^{(j)}})_0, \\ \mathbf{B}_{i,i-1} &= (\frac{\partial\varphi_{i,i-1}}{\partial\mathbf{b}_i^{(i)}})_0. \end{aligned} \tag{7.19}$$

$\mathbf{F}_i$ is defined for $i = 2 \ldots p$ as follows:

$$\mathbf{F}_i = \left(\mathbf{F}_{2,1}, \cdots, \mathbf{F}_{i,i-1}, \mathbf{0}, \cdots, \mathbf{0}\right), \tag{7.20}$$

with (p-i) matrices $\mathbf{0}$ of zeros, which have the same number of rows as $\mathbf{F}_{2,1}$, and the partioning of $\mathbf{F}_i$ in columns in accordance with the partitioning of $\Delta\mathbf{f}$:

$$\Delta\mathbf{f} = \left(\Delta\mathbf{f}_{2,1}, \cdots, \Delta\mathbf{f}_{i,i-1}, \Delta\mathbf{f}_{i+1,i} \cdots, \Delta\mathbf{f}_{p,p-1}\right). \tag{7.21}$$

For $\mathbf{F}_{i,i-1}$ (i=2,...,p) we have:

$$\mathbf{F}_{i,i-1} = \mathbf{B}_{2,1}\, \mathbf{B}_{3,2} \cdots \mathbf{B}_{i-2,i-1} \left(\frac{\partial \varphi_{i,i-1}}{\partial \mathbf{f}_{i,i-1}}\right)_0. \tag{7.22}$$

Matrix $\mathbf{B}_{i,i-1}$ for an affine transformation is given in chapter 4 as follows:

$$\mathbf{B}_{i,i-1} = \begin{pmatrix} a^0_{11}\mathbf{I} & a^0_{12}\mathbf{I} & a^0_{13}\mathbf{I} \\ a^0_{21}\mathbf{I} & a^0_{22}\mathbf{I} & a^0_{23}\mathbf{I} \\ a^0_{31}\mathbf{I} & a^0_{32}\mathbf{I} & a^0_{33}\mathbf{I} \end{pmatrix}, \tag{7.23}$$

with a^0_{ij} (i, j=1, 2, 3) the approximate values of a_{ij} and $\mathbf{I}$ the $(n \times n)$ unit matrix and n the amount of points in $\underline{\mathbf{b}}_i$.

As explained in section 7.5.3.4, we can take $a^0_{ij} = \delta_{ij}$, which results in a unit matrix for $\mathbf{B}_{i,i-1}$, from which follows, see equations (7.18) and (7.22):

$$\begin{aligned} \mathbf{B}_i &= \mathbf{I}, \\ \mathbf{F}_{i,i-1} &= \left(\frac{\partial \varphi_{i,i-1}}{\partial \mathbf{f}_{i,i-1}}\right)_0. \end{aligned} \tag{7.24}$$

Matrix $\mathbf{F}_{i,i-1}$ for an affine transformation is given in chapter 4 as follows:

$$\mathbf{F}_{i,i-1} = \begin{pmatrix} \boldsymbol{\beta}_i & \mathbf{0} & \mathbf{0} & \boldsymbol{\epsilon}_1 \\ \mathbf{0} & \boldsymbol{\beta}_i & \mathbf{0} & \boldsymbol{\epsilon}_2 \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\beta}_i & \boldsymbol{\epsilon}_3 \end{pmatrix}, \tag{7.25}$$

where $\boldsymbol{\beta}_i$, $\boldsymbol{\epsilon}_1$, $\boldsymbol{\epsilon}_2$, $\boldsymbol{\epsilon}_3$ and $\mathbf{0}$ are all $(n \times 3)$ matrices, as follows:

$\boldsymbol{\beta}_i = (\mathbf{u}_0,\ \mathbf{v}_0,\ \mathbf{w}_0)$; $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{w}_0$ are approximate values of $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ (the x, y, z coordinates in $\underline{\mathbf{b}}_i$), which can be transformed to make the barycentre the origin.

$$\boldsymbol{\epsilon}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 0 \end{pmatrix},$$

$\boldsymbol{\epsilon}_2$ and $\boldsymbol{\epsilon}_3$ are analogous matrices as $\boldsymbol{\epsilon}_1$ with ones in the second, resp. third column,
$\mathbf{0}$ is the $(n \times 3)$ zero matrix.

We define $\mathbf{F}_1$ as the null matrix $\mathbf{0}$ and put it together with the $\mathbf{F}_i$, $i = 2 \ldots p$ of equation (7.20) into matrix $\mathbf{F}$. Analogously we take all $\mathbf{P}_i$ together in a matrix $\mathbf{P}$:

$$\begin{aligned} \mathbf{F} &= \left(\mathbf{F}_1, \ldots, \mathbf{F}_p\right)^T \\ \mathbf{P} &= \left(\mathbf{P}_1, \ldots, \mathbf{P}_p\right)^T \end{aligned} \tag{7.26}$$

We define vector $\Delta\mathbf{b}$ as:

$$\Delta\mathbf{b} = \left(\Delta\mathbf{a}_1, \Delta\mathbf{b}_2, \ldots, \Delta\mathbf{b}_p\right)^T. \tag{7.27}$$

We can now formulate the linearised equivalent of system (7.10):

$$E\{\begin{pmatrix} \underline{\Delta\mathbf{b}} \\ \Delta\mathbf{z}_f \\ \Delta\mathbf{z}_d \end{pmatrix}\} = \begin{pmatrix} \mathbf{P} & -\mathbf{F} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_f & \mathbf{0} \\ \mathbf{Z}_d & \mathbf{0} & \mathbf{Z}_\nabla \end{pmatrix} \begin{pmatrix} \Delta\mathbf{c} \\ \Delta\mathbf{f} \\ \Delta\nabla \end{pmatrix}. \tag{7.28}$$

The covariance matrix of the observation vector on the left-hand side consists of the covariance matrices of $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, i=2, $\cdots$, p, as described in section 7.5.1, approximately transformed as described in section 7.5.3.4, and zero matrices for the remainder if no correlation between the epochs is assumed (which is, however, not necessary to solve the model).

The model takes each epoch as a separate geodetic network: each point has a different point number for each epoch, for example point A is called A_1 in epoch 1, A_2 in epoch 2, etc. The hypothesis that no deformation has occured is formulated by stating that

$$\begin{cases} 0 = x_{A_2} - x_{A_1}, \\ 0 = y_{A_2} - y_{A_1}, \\ 0 = z_{A_2} - z_{A_1}, \\ \qquad \text{etc.} \end{cases} \tag{7.29}$$

The separate geodetic networks are linked together in this way. Equation (7.29) gives the nonstochastic observation equations (the zeros constitute together vector $\mathbf{z}_d$ and have a standard deviation of zero). The number of rows of matrix $\mathbf{Z}_d$ is three times the number of points. In each row there are zeros and one 1 and one -1 for respectively the coordinate of epoch 2 and epoch 1 (which are separate unknowns in the parameter vector). There are no parameters ∇ and no matrix $\mathbf{Z}_\nabla$.

Let us now assume that a deformation is present for point A. Let it be a linear movement for which we write:

$$\begin{aligned} 0 &= x_{A_2} - x_{A_1} + a_x t_{12}, \\ 0 &= y_{A_2} - y_{A_1} + a_y t_{12}, \\ 0 &= z_{A_2} - z_{A_1} + a_z t_{12} \end{aligned} \tag{7.30}$$

The a_x, a_y, a_z are unknown parameters, which enter the parameter vector ∇, and for which a least squares estimate is determined in the adjustment. t_{12} is the time interval between epoch 1 and 2. The matrix $\mathbf{Z}_\nabla$ is in this case a matrix with three columns and three elements t_{12} on the rows of the three nonstochastic observations mentioned, and with zeros on all other positions.

We can also leave a_x, a_y, a_z out of the adjustment. Then the last column of the coefficient matrix of equation (7.28) disappears. The null hypothesis states now that there is no deformation. We test for a linear movement by using $\mathbf{Z}_\nabla$ in the test statistic of equation (7.34).

Generally the transformation between epoch i and $i-1$ is a similarity or congruence, not an affine transformation. Matrix $\mathbf{F}_i$ is constructed according to equation (7.20) from matrices $\mathbf{F}_{i,i-1}$ as given in equation (7.22) for the affine transformation. Matrix $\mathbf{Z}_f$ is the matrix that describes the constraints for a congruence or similarity transformation. The coefficient matrix of equation (7.13) or (7.14) is used to construct matrix $\mathbf{Z}_f$.

7.6 Adjustment and testing

7.6.1 Adjustment

System (7.28) is a linear system of observation equations and can be solved by least squares. If sufficient points are available in all epochs to determine the transformation parameters, the coefficient matrix is of full rank.

Because of the nonstochastic observations, and because of possible singularities of the covariance matrices of $\underline{\mathbf{a}}_1$ and the vectors $\underline{\mathbf{b}}_i$, the covariance matrix of the observation vector of system (7.28) is singular. To get a least squares solution of the system, at least five methods are available that make it possible to test nonstochastic observations in the same way as stochastic observations (see chapter 5).

1. The adjustment model is split into two parts for the stochastic and the nonstochastic observations respectively, and a sequential adjustment is applied.
2. A switch is made from the model of observation equations to the model of condition equations.
3. The covariance matrix is regularised.
4. The standard deviations in the covariance matrix that are zero, are replaced by values that are very small.
5. The observations are orthogonalised and the nonstochastic observations eliminated. A follow-up adjustment determines the test quantities.

Because the system is linearised, iteration is needed to find the least squares solution. To start the iteration good approximate values for all observations and all parameters are needed, which have to satisfy the non-linear equations (7.10) and the non-linear constraints of section 7.5.3.2 or 7.5.3.3. As described in chapter 4, in each iteration step the approximate transformation parameters are updated, using singular value decomposition. Also, in each iteration step, all $\underline{\mathbf{b}}_i$ and their covariance matrices $\mathbf{D}\{\underline{\mathbf{b}}_i\}$ are transformed with the new approximate transformation parameters to new coordinates $\underline{\mathbf{b}}'_i$ and $\mathbf{D}\{\underline{\mathbf{b}}'_i\}$ that are (for the common points) almost equal to $\underline{\mathbf{a}}_1$ and in the reference system of $\underline{\mathbf{a}}_1$.

In each iteration step the approximate values of all observations, and of all parameters have to comply again with the non-linear equations (7.10) and the non-linear constraints of section 7.5.3.2 or 7.5.3.3.

7.6.2 Deformation testing

If one of the five methods, mentioned in the previous section, is used, standard methods for testing can be applied with the formulas given in chapter 5. Also the nonstochastic observations can be tested with the same formulas, which means that a method of testing deformation patterns is provided.

If it is not sure whether there is any deformation, or what type of deformation happens, a null hypothesis H_0 is formulated, where no deformation is assumed (∇ is missing in system (7.28)), and an alternative hypothesis H_a:

$$H_0: \; E\{\underline{\Delta \mathbf{y}}\} = \mathbf{A}\,\Delta\mathbf{x}, \tag{7.31}$$

$$H_a: \; E\{\underline{\Delta \mathbf{y}}\} = \mathbf{A}\,\Delta\mathbf{x} + \mathbf{Z}'_{\nabla}\Delta\nabla, \tag{7.32}$$

where $\underline{\Delta \mathbf{y}}$, $\mathbf{A}$ and $\Delta\mathbf{x}$ are respectively the observation vector, the coefficient matrix and the parameter vector of system (7.28). In $\mathbf{A}$ the last column of the coefficient matrix is missing and in $\Delta\mathbf{x}$ the parameters $\Delta\nabla$. $\mathbf{Z}'_{\nabla}$ is the last column of the coefficient matrix of equation (7.28):

$$\mathbf{Z}'_{\nabla} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{Z}_{\nabla} \end{pmatrix} \tag{7.33}$$

The alternative hypothesis is tested against the null hypothesis, without the need to perform a complete adjustment of (7.32), by using test statistic $\underline{T}_q$ (Teunissen, 2006, p. 77):

$$\underline{T}_q = \frac{1}{\sigma^2}\,\underline{\hat{\mathbf{r}}}^T\,\mathbf{Z}'_{\nabla}\,(\mathbf{Z}'^T_{\nabla}\,\mathbf{Q}_{\hat{r}}\,\mathbf{Z}'_{\nabla})^{-1}\,\mathbf{Z}'^T_{\nabla}\,\underline{\hat{\mathbf{r}}}. \tag{7.34}$$

q is the number of columns in $\mathbf{Z}'_{\nabla}$ and gives the degrees of freedom of the test. σ^2 is the variance factor of unit weight, and $\underline{\hat{\mathbf{r}}}$ are the reciprocal least squares residuals as they follow from a weighted least squares adjustment (see chapter 4) and for which holds, with $\underline{\hat{\mathbf{e}}}$ the usual least squares residuals and $\mathbf{Q}_y$ the cofactor matrix of the observations $\underline{\mathbf{y}}$:

$$\underline{\hat{\mathbf{e}}} = \mathbf{Q}_y\,\underline{\hat{\mathbf{r}}}. \tag{7.35}$$

$\mathbf{Q}_{\hat{r}}$ is the cofactor matrix of $\underline{\hat{\mathbf{r}}}$. $\mathbf{Z}'_{\nabla}$ describes a testable deformation pattern, if the product $\mathbf{Z}'^T_{\nabla}\,\mathbf{Q}_{\hat{r}}\,\mathbf{Z}'_{\nabla}$ is a regular matrix.

The probability density function of $\underline{T}_q$ is a χ^2-distribution with an expected value of q. The test is to choose a significance level α, to compute the critical value and to test, whether the computed value of $\underline{T}_q$ exceeds the critical value. If this happens, the null hypothesis is rejected (Teunissen, 2006, p. 78).

7.6.3 S-basis invariance

In chapters 3 and 5 it is shown that the test statistic of equation (7.34) is invariant for a change of S-basis of the parameter vector $\mathbf{x}$. It is evident from the fact that $\underline{\hat{\mathbf{r}}}$ can be computed from the model of condition equations, which is dual to the model of observation equations. In this dual model the parameter vector $\mathbf{x}$ has been eliminated, and therefore a change of S-basis of $\mathbf{x}$ doesn't influence $\underline{\hat{\mathbf{r}}}$.

The test statistic of equation (7.34) is also invariant for changes of S-bases of the observed coordinate vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, i=2, ... , p, if deformation patterns are tested. To see this, model (7.28) is simplified and reduced. To do this, we assume all observed vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, and also the parameter vectors $\mathbf{c}_i$ to contain coordinates of the same points in the same order, from which follows:

$$\mathbf{P} = \text{unit matrix.} \tag{7.36}$$

We also assume stability of all points, and therefore:

$$\Delta\mathbf{c}_i = \Delta\mathbf{c}_j, \text{ with } i,j = 1, \dots, p, \tag{7.37}$$

and we reduce $\Delta\mathbf{c}$ to a vector $\overline{\Delta\mathbf{c}}$ with the coordinate parameters of only one epoch. With matrix $\mathbf{I}_p$ defined with unit matrices $\mathbf{I}$ as:

$$\mathbf{I}_p = \left(\mathbf{I}, \dots, \mathbf{I}\right)^T, \tag{7.38}$$

we get:

$$\Delta\mathbf{c} = \mathbf{I}_p\,\overline{\Delta\mathbf{c}}. \tag{7.39}$$

This means that the nonstochastic observations $\mathbf{z}_d$ disappear. Furthermore we assume that the nonstochastic observations $\mathbf{z}_f$ are elimated. This can be done by noting that the equation:

$$\mathbf{0} = \mathbf{Z}_f\Delta\mathbf{f} \tag{7.40}$$

means that $\Delta\mathbf{f}$ lies in the nullspace of $\mathbf{Z}_f$. If $\mathbf{N}$ is a base matrix that spans this nullspace, we have:

$$\Delta\mathbf{f} = \mathbf{N}\overline{\Delta\mathbf{f}}, \tag{7.41}$$

with $\overline{\Delta\mathbf{f}}$ a vector of coefficients, which can be used as the new vector of unknown transformation parameters. If, for example, $\Delta\mathbf{f}$ contains 12 parameters of an affine transformation and there are 5 nonstochastic observations to constrain the transformation into a similarity transformation, $\overline{\Delta\mathbf{f}}$ contains 7 transformation parameters. With the definition $\overline{\mathbf{F}} = \mathbf{FN}$, it follows that:

$$\mathbf{F}\Delta\mathbf{f} = \overline{\mathbf{F}}\,\overline{\Delta\mathbf{f}}. \tag{7.42}$$

So if we use $\overline{\mathbf{F}}\,\overline{\Delta\mathbf{f}}$ instead of $\mathbf{F}\Delta\mathbf{f}$ in model (7.28), we can omit the nonstochastic observations $\mathbf{z}_f$.

Because of the stability assumption, no parameters ∇ exist and no matrix $\mathbf{Z}_\nabla$.

With (7.36), (7.39) and (7.42), model (7.28) is written as:

$$E\{\underline{\Delta\mathbf{b}}\} = \mathbf{I}_p\,\overline{\Delta\mathbf{c}} - \overline{\mathbf{F}}\,\overline{\Delta\mathbf{f}} \tag{7.43}$$

To eliminate $\overline{\Delta \mathbf{c}}$, we define matrix $\mathbf{H}$ as:

$$\mathbf{H} = \begin{pmatrix} -\mathbf{I} & \mathbf{I} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} & \mathbf{I} & \cdots & \mathbf{0} \\ \vdots & & \ddots & \ddots & \vdots \\ \mathbf{0} & \ldots & \ldots & -\mathbf{I} & \mathbf{I} \end{pmatrix}^T, \tag{7.44}$$

and the vector $\mathbf{d}$, containing the difference vectors of all epoch intervals, as:

$$\mathbf{d} = \mathbf{H}^T \mathbf{b}. \tag{7.45}$$

Premultiplying equation (7.43) with $\mathbf{H}^T$, we get:

$$E\{\underline{\Delta \mathbf{d}}\} = -\mathbf{H}^T \overline{\mathbf{F}}\, \overline{\Delta \mathbf{f}}. \tag{7.46}$$

This model has the same redundancy as model (7.43) and yields the same least squares solution.

Let the vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, $i = 2, \ldots, p$ all have been S-transformed to other S-bases. It means that we have new vectors $\underline{\mathbf{a}}'_1$ and $\underline{\mathbf{b}}'_i$, taken together in vector $\underline{\mathbf{b}}'$:

$$\underline{\mathbf{b}}' = \underline{\mathbf{b}} + \mathbf{S}\underline{\psi}, \tag{7.47}$$

where ψ is the vector of the differential transformations of the coordinate vectors of all epochs. These relate, however, to the same degrees of freedom as the transformations in $\overline{\Delta \mathbf{f}}$. This means that we can take $\mathbf{S} = \overline{\mathbf{F}}$.

A proof for two epochs that test statistic (7.34) is invariant for changes of S-bases of $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_2$, by proving that $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$ are invariant, is given in chapter 3. The extension to more than two epochs is possible by using reduced model (7.46). This model can be solved by switching to the model of condition equations with matrix $\mathbf{G}$, which is chosen to fulfil:

$$\mathbf{G}^T\ \mathbf{H}^T \overline{\mathbf{F}} = \mathbf{0}, \tag{7.48}$$

with $R(\mathbf{G})$ the complementary space of $R(\mathbf{H}^T \overline{\mathbf{F}})$. It follows with the same reasoning as given in chapter 3 that $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$, as they follow from solving model (7.46), are invariant for changes in S-bases of $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, $i = 2, \ldots, p$.

The conclusion is that if hypotheses concerning deformation patterns are formulated in terms of the original model (7.28), and they can be reformulated in terms of model (7.46), which is generally possible, test statistic (7.34) is invariant for changes to other S-bases of the coordinate vectors $\underline{\mathbf{a}}_1$ and $\underline{\mathbf{b}}_i$, $i = 2, \ldots, p$.

A deformation hypothesis may concern a point that is part of the S-basis definition and whose coordinates are fixed with a zero standard deviation. No S-transformation is needed to test such a point for deformation. It is demonstrated by the example of chapter 5.

7.7 Experimental validation

The proposed model can be applied to the 3D monitoring by GPS and total stations of deformations of buildings, harbour quays, bridges, tunnels, land slides, etc. The model gives the possibility to compute statistics and to test hypotheses that describe complex deformation patterns, like the abnormal movement of a subset of points through many epochs, or the periodic oscillation of a subset of points, for example caused by changes of temperature.

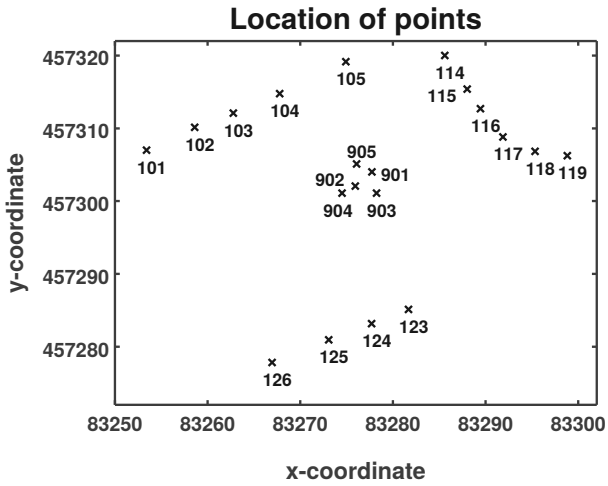

Figure 7.2: 15 object points, 5 instrument points.

To validate experimentally the model, the monitoring of some buildings is taken. To be able to judge effectively the performance of the model, observations have been generated with known standard deviations, to which artificially deformations have been added. Fifteen points have been measured with direction and distance observations from a total station during five epochs. The fifteen points are positioned on three buildings (figure 7.2), which are monitored because of construction works. The instrument point is not fixed (not monumented). The observations are adjusted using the software package MOVE3 (www.move3.com), resulting in x, y, z coordinates and their covariance matrix. The network is not attached to a control network.

A Matlab programme has been written to do the computations. The observations have been generated with the following standard deviations:

- directions: 0.3 mgon;
- distances: 1 mm;
- zenith angles: 0.3 mgon.

The precision with which a point is defined (idealisation precision) is supposed to be 0.5 mm, indicating the precision by which a removable prism can be put on a point.

First no deformation is put in the observations. The adjustment model to test stability of all points is created by adding for each epoch interval, for each point and for each

coordinate direction a nonstochastic observation, i.e. $4 \times 15 \times 3 = 180$ nonstochastic observations:

$$\begin{cases} \mathbf{0} = \Delta\mathbf{c}_1 - \Delta\mathbf{c}_2 \\ \mathbf{0} = \Delta\mathbf{c}_2 - \Delta\mathbf{c}_3 \\ \mathbf{0} = \Delta\mathbf{c}_3 - \Delta\mathbf{c}_4 \\ \mathbf{0} = \Delta\mathbf{c}_4 - \Delta\mathbf{c}_5 \end{cases} \tag{7.49}$$

From these equations follows matrix $\mathbf{Z}_d$ in equation (7.28).

The epochs are joined together with similarity transformations, which are realised for each epoch interval by five constraints on the affine transformation parameters.

The model contains 425 observations ($5 \times 15 \times 3 = 225$ coordinates, $4 \times 5 = 20$ transformation constraints, 180 point constraints), and 273 parameters (225 coordinates, $4 \times 12 = 48$ transformation parameters), which yields a redundancy of 152. Adjusting the model leads to an overall model test of 0.60. With a critical value of 1.004, based on the use of the B-method of testing with a significance level of a one-dimensional test of 0.1% and of 47% for the overall model test, the null hypothesis is accepted.

Then a movement of 1 mm in each epoch interval, in the direction of each coordinate axis of one point (point 101), is added and observations are generated. The same adjustment model as before is used and leads to rejection of the null hypothesis. The movement is then modelled with twelve nonstochastic observations. Assume vector $\mathbf{c}_{101}^{(i)}$ is the subvector of vector $\mathbf{c}$ that contains the x, y and z coordinates of point 101 in epoch i. Let vector $\mathbf{a}$ contain the movements in x, y and z direction between epoch i and j, and $\Delta\mathbf{a}$ the difference of $\mathbf{a}$ with its approximate value, necessary for the linearised model. The following 12 nonstochastic observations describe the deformation.

$$\begin{cases} \mathbf{0} = \Delta\mathbf{c}_{101}^{(1)} - \Delta\mathbf{c}_{101}^{(2)} + \Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101}^{(2)} - \Delta\mathbf{c}_{101}^{(3)} + \Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101}^{(3)} - \Delta\mathbf{c}_{101}^{(4)} + \Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}_{101}^{(4)} - \Delta\mathbf{c}_{101}^{(5)} + \Delta\mathbf{a} \end{cases} \tag{7.50}$$

From these equations the matrix $\mathbf{Z}'_\nabla$ of equation (7.32) is deduced, and the alternative hypothesis tested against the null hypothesis. The same test is used to test for linear movement of all other points. The deformed point shows the largest value of the test statistic (table 7.1 under "Stat."), with a critical value of 12.6 and a significance level of 0.6 %. The estimated deformation (equation (3.42)) in each epoch interval is given in the same table for point 101 and three other points with large test statistics. The estimated deformation of point 101 resembles closely the values that have been put intentionally into the coordinates, and the length of the deformation vector is even the same: 1.7 mm in each epoch interval.

Table 7.2 gives the minimal detectable deformations as the lengths of the semi-axes of the ellipsoid determined by equation (3.44):

$$\sigma^2\lambda_0 = \nabla_0^T \mathbf{Z}_\nabla'^T \mathbf{Q}_{\hat{l}} \mathbf{Z}'_\nabla \nabla_0, \tag{7.51}$$

	Stat.	Est. def. (mm)		
Pnt.		x	y	z
101	70.9	0.9	1.1	1.0
103	16.9	-0.4	-0.5	-0.3
102	7.0	-0.1	-0.1	-0.5
104	4.7	-0.2	-0.3	-0.1

Table 7.1: Test of linear point movement over 5 epochs. Values are per epoch interval.

with λ_0 the non-centrality parameter of the χ^2-distribution and ∇_0 describing the minimal detectable deformations. They give the deformations that can be detected with the three-dimensional point test of five epochs with a power of 80%.

	MDD (mm)		
Pnt.	axis 1	axis 2	axis 3
101	1.55	0.80	0.76
103	1.49	0.73	0.73
102	1.52	0.76	0.74
104	1.48	0.72	0.72

Table 7.2: Minimal detectable deformations (MDD). Values are per epoch interval.

Finally five points (101,...,105) are given a movement of 1 mm in both the x and y direction and -0.7 mm in the z direction in each epoch interval. It is modelled by 60 nonstochastic observations. Let vector $\mathbf{c}^{(i)}_{101-105}$ be the subvector of vector $\mathbf{c}$ that contains the x, y, z coordinates of the five points in epoch i. Let $\mathbf{k} = (1,1,1,1,1)^T$, $\mathbf{I}_3$ the (3×3)-unit matrix, and $\mathbf{E} = \mathbf{I}_3 \otimes \mathbf{k}$, with $\otimes$ denoting the kronecker product. From the following nonstochastic observations the matrix $\mathbf{Z}_\nabla$ is deduced.

$$\begin{cases} \mathbf{0} = \Delta\mathbf{c}^{(1)}_{101-105} - \Delta\mathbf{c}^{(2)}_{101-105} + \mathbf{E}\,\Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}^{(2)}_{101-105} - \Delta\mathbf{c}^{(3)}_{101-105} + \mathbf{E}\,\Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}^{(3)}_{101-105} - \Delta\mathbf{c}^{(4)}_{101-105} + \mathbf{E}\,\Delta\mathbf{a} \\ \mathbf{0} = \Delta\mathbf{c}^{(4)}_{101-105} - \Delta\mathbf{c}^{(5)}_{101-105} + \mathbf{E}\,\Delta\mathbf{a} \end{cases} \tag{7.52}$$

with $\Delta\mathbf{a}$ as defined before.

The null hypothesis is rejected again. The test of the hypothesis that the five points have shifted gives a very large test statistic (74.2 with a critical value of 12.6, if the significance level is 0.6 %), indicating that it is a very good hypothesis. The estimated deformation and the minimal detectable deformations are given in table 7.3. The length of the deformation vector is 1.6 mm, which is exactly the length of the vector that has been put intentionally into the coordinates.

	Est. def. (mm)			M.d.d. (mm)		
Pnt.	x	y	z	axis 1	axis 2	axis 3
101 - 105	1.0	0.7	-1.1	0.93	0.71	0.60

Table 7.3: Linear movement of points 101-105 over five epochs. Values are per epoch interval.

Point movements that are nonlinear in time are modelled by nonstochastic observations that are nonlinear functions of the deformation parameters. To be used in the model, the functions have to be linearised.

If the deformation pattern to be expected is not known, a search has to be performed for the best alternative hypothesis. A strategy is described in chapter 3 for two epochs. Extending it to more than two epochs, one could for example systematically test for a constant linear movement through all epochs of each point individually, of combinations of two points close together, of combinations of three points close together, etc. Because it is not needed to solve a complete adjustment model, only to compute test statistic (7.34), its degrees of freedom q determines the computational burden of testing many hypotheses.

7.8 Conclusions

A model has been built for the adjustment of a time series of 3D coordinates in a geodetic point field. The covariance matrices of the coordinates of all epochs of the time series are used and they may be full and singular. Deformation patterns, or their absence, are modelled as nonstochastic observations. To make the testing of the model invariant for S-transformations, transformations between all epochs are built into the model. The transformations can be similarity or congruence transformations, and are modelled as affine transformations, subject to constraints. The constraints are implemented as nonstochastic observations. The model is first built as a nonlinear one, and then linearised. The approximate parameter values and their updates in the iteration steps (needed because of the linearisation) have to comply with all nonstochastic observations. For the rotation parameters this is accomplished with singular value decomposition.

In many cases it is a sound deformation analysis procedure to formulate a null hypothesis that assumes no deformation. The nonstochastic observation equations state that the coordinate differences between the epochs are expected to be zero after the transformations. Alternative hypotheses are formulated that describe movements of one or many points over one or many epoch intervals. Standard hypothesis testing is used to test the alternative hypothesis against the null hypothesis. The quality of the tests is described by the sizes of the minimal detectable deformations.

The point movements are formulated as nonstochastic observation equations, which give the matrices to be used in the testing equations.

The model and its adjustment and testing have been verified experimentally with a geodetic network, where 15 points are measured by a total station during five epochs. The results show that 3D deformation analysis of time series of coordinates is possible with the model proposed.

8

Conclusions and recommendations

In this study a geodetic deformation analysis model has been developed and its essential elements have been clarified. The purpose at the outset of this study was to improve geodetic deformation analysis in professional practice by developing a model that is based on operational demands. The development made it necessary to approach geodetic deformation analysis in a novel way. This led to the development of a new geodetic deformation analysis model with at its centre an adjustment model, which has been developed in two variants. The applicability of the developed analysis model was tested for several use cases. In this chapter the conclusions of this study will be presented first, and subsequently an overview will be given of the contributions of this study to science. Finally, recommendations will be given for future research.

8.1 Conclusions

The research question for this study is (chapter 1, section 1.4):

> How can a generic mathematical-geodetic model be formulated that is:
> (i) usable for geodetic deformation analysis, (ii) enables standardisation of terminology, processes and presentation of results for geodetic deformation analysis, and (iii) is usable as a basis for communication about goals, possibilities and analysis results of geodetic deformation measurements?

The model is intended to use geodetic observables on the one hand and information on deformations in physical reality on the other hand, to test intricate deformation hypotheses. The standardisation and the communication have to be based on statistically valid methods.

The research question has been worked out in five subquestions. The answers that have been given in this study to the research question and its subquestions are summarised below. First the research question will be answered by describing the developed geodetic deformation analysis model. Then the five subquestions will be treated.

Geodetic deformation analysis model The generic mathematical-geodetic model of the research question has been developed as a *geodetic deformation analysis model*, of which a dedicated *adjustment model* constitutes the kernel. This geodetic deformation analysis model assumes a discretisation of monitored geo-objects[1] in discrete points. These points are measured by geodetic measuring techniques in discrete epochs[2]. Deformation analysis according to the proposed analysis model is divided in three stages: design, implementation and realisation. The *design stage* is concerned with the method that will be used to represent geo-objects, the geodetic observation techniques that will be used, and the deformation hypotheses that will be investigated. The *implementation stage* takes care of the construction of the adjustment model and gives a description of the quality of tests. The *realisation stage* is concerned with the acquisition of data (observations), the adjustment of data, and the testing of them; and finally with drawing conclusions on the best deformation hypothesis.

Central in the implementation stage is the *adjustment model*, which will be used in the realisation stage to adjust the observations according to least-squares theory. The adjustment model handles all epochs together. This means that the comparison of observations or coordinates of different epochs is not a seperate process, but an integrated part of the adjustment model. The adjustment is modelled as a model of observation equations using a model matrix with nonstochastic elements, and taking the covariance matrices of all observations (or coordinates, if they constitute the input) into account.

Two different adjustment models have been developed: the *measurements model* (in chapter 6) and the *coordinates model* (in chapter 7). Chapters 3 and 4 propose two simpler models that provide essential components, used in chapters 6 and 7. Chapter 5 provides essential algorithms and tools. The *measurements model* takes the measurements of all epochs as the observations of an adjustment model of observation equations (Gauss-Markov model), and the coordinates of all epochs as part of the parameters of the adjustment model. A point of a monitored geo-object has a different set of coordinates in each epoch. These sets are linked together by *constraints* (on the parameters of the adjustment model), which, in this way, define the deformation hypothesis.
The *coordinates model* divides the adjustment in two phases. In the first phase the measurements of each epoch are adjusted separately. The resulting adjusted coordinates per epoch are subsequently taken as input (i.e. pseudo-observations) for a second adjustment. In this second adjustment the coordinates of all epochs are taken as parameters in the adjustment model, just as in the measurements model; and the deformation hypothesis is again formulated by using constraints.

[1] A geo-object is an object on, above or under the earth's surface or a part of the surface itself.

[2] An epoch is a time period, in which a set of points is measured once. It is assumed that no movements or deformations occur during this time period, or that they are compensated for. For certain measuring techniques an epoch can be so short that it is considered a moment in time.

The seven essential elements of the proposed geodetic deformation analysis model (with its two adjustment models) are discussed in the answers below to the subquestions of the research question.

Least-squares and Delft school of geodesy
First subquestion (shortened):
How can a model be built in such a way that the full existing body of knowledge of least-squares theory is applied for deformation analysis? The focus is here on the achievements of the Delft school of geodesy.

Both the measurements model and the coordinates model compare the results of all epochs as part of the adjustment model. This enables the application of ordinary least-squares theory (with a full, possibly rank deficient covariance matrix of the observations, and with a model matrix without stochastic elements) for geodetic deformation analysis. It enables the application of the achievements of the Delft school of geodesy as well. Especially the results of the Delft school in the analysis of *size and form* (as opposed to absolute position, orientation and scale of a reference system); and of *reliability* have been used in the construction of the adjustment models (chapters 3, 4, 6 and 7). The preference of the Delft school of geodesy for the *adjustment model of condition equations* (as opposed to the model of observation equations, i.e. the Gauss-Markov model) has led to the successful application of this model for the derivation of algorithms of the proposed adjustment models (chapter 5).

The consideration that deformation (i.e. de-*form*-ation) analysis is about the analysis of *size and form* has led to the introduction of transformation parameters in the coordinates model (chapters 3, 4 and 7). It has also led to the definition of deformation hypotheses (in both measurements model and coordinates model) by means of *constraints* (chapters 5, 6 and 7), which makes it possible to use the coordinates of just one epoch to define the S-system[3]. These two characteristics of the adjustment model make the use of S-transformations superfluous, and makes it possible that the points that define the S-system are subject to deformation (chapter 6). This makes the conventional approach unnecessary, in which points that are stable (not moving) during all epochs, are searched for (to define the S-system), before the actual deformation analysis can start.

The emphasis of the Delft school on *reliability* with its concept of a minimal detectable bias has been at the basis of the description of *minimal detectable deformations*. They have been developed in this study as a valuable tool to express the power of a certain geodetic deformation analysis model to detect deformations (chapters 3 to 7).

Physical model, and time series of measurements
Second subquestion (shortened):
How can a physical model be incorporated in the geodetic deformation analysis model?

A physical model is a description of driving forces, from which a hypothesis can be deduced about the movements of one or more points, i.e. a deformation hypothesis. It is advantageous to combine the geodetic measurements and the deformation hypothesis

[3]An S-system is a coordinate reference system, of which absolute position, orientation and scale are defined by functions of the coordinates themselves, and not by external information.

in one adjustment model, because it allows for a simultaneous analysis. Both the measurements model and the coordinates model can accomplish this by formulating a deformation hypothesis by means of constraints on the coordinates in the parameter vector of the adjustment model. These constraints are added to the adjustment model as *nonstochastic observations* (constants that are treated as if they were observations; also known as pseudo-observations). This makes it possible to test both the geodetic measurements and the deformation hypothesis in the same way by standard statistical tests. The concept and algorithms of testing constraints on parameters are treated in chapter 5 and applied in the adjustment models of chapters 6 and 7.

A physical model may concern just two epochs, or a time series of measurements. In professional practice an analysis of just *two epochs* of measurements is used for the first deformation analysis, which is often crucial to determine whether more epochs of measurements will have to take place. And if so, how frequent they will have to be. An analysis of more than two epochs, i.e. a *time series*, of measurements, is the next step. The proposed adjustment models (the measurements model and the coordinates model) are capable of adjusting two epochs, but also a time series of measurements. A deformation hypothesis about a time series describes the movements of several points, or even several point sets, during several epochs. Each point set may show a different behaviour. Such an intricate deformation hypothesis is formulated as a set of constraints in the adjustment model. This set of constraints is tested, and minimal detectable deforamations are determined for the set. Thus intricate deformation hypotheses about time series of geodetic deformation measurements are tested and supllied with a description of the test quality (chapters 6 and 7).

Constraints and rank deficiency

Third subquestion (shortened):
How can constraints be used effectively to describe deformation hypotheses, and how can these constraints be tested and provided with a quality description, expressing minimal detectable deformations?

As described above, constraints can be used to describe deformation hypotheses. They are added to the adjustment model as nonstochastic observations. This causes the covariance matrix of the observations in the adjustment model to be necessarily a rank deficient (singular) matrix. The standard formulas for testing and description of test quality do not allow for singular covariance matrices. Therefore, in this study the formulas are developed that accept singular covariance matrices of the observations (chapter 5).

This study shows that (i) a singular covariance matrix of the observations, (ii) constraints on the parameters and a nonsingular covariance matrix of observations, and (iii) nonstochastic observations, are three ways to express the same phenomenon. The use of nonstochastic observations has the advantage that stochastic observations (e.g. geodetic measurements) and nonstochastic observations (e.g. deformation hypotheses) are handled in the same way. To arrive at the least-squares solution of the adjustment model, and, subsequently, to test the results and to determine the test quality, several algorithms are treated in this study. They make partially use of algorithms that were developed to solve (i) or (ii); and partially they use newly developed results (chapter 5).

Search for best hypothesis

Fourth subquestion:
What is a good search method to find the hypothesis that best describes the deformation?

If the task at hand is to test just one deformation hypothesis, this can be done with the tools described above. In general, however, several, or even many, deformation hypotheses exists, that might all be plausible hypotheses, considering the physical causes of the deformation, which are often not known well enough to reduce the number of plausible hypotheses to just one. If the number of plausible hypotheses is not utterly large, a search for the best hypothesis by systematically testing all hypotheses is possible (chapter 4). A deformation hypothesis may, however, concern several points, several epochs, and several deformation parameters[4]. Hence, it can be a quite intricate hypothesis. Several of such intricate hypotheses may relate to different amounts of deformation parameters. The task to determine by statistical tests which of the intricate hypotheses is the best one, is far from straightforward. Well-known methods to find the best hypothesis make use of an *information criterion*, such as the Akaike Information Criterion, and the Bayesian Information Criterion. This study uses the test quotient within the context of the B-method of testing as information criterion (chapter 6). Its use has been illustrated in several numerical examples of use cases (chapters 3, 4, 6 and 7).

Standardisation and communication

Fifth subquestion:
What are the requirements that a geodetic deformation analysis model has to fulfil to be usable for standardisation and, as a derivative, for effective and efficient communication.

To answer operational demands about deformations appropiately, a geodetic deformation analysis model has been developed in this study. This analysis model enables the definition of a *statistically significant deformation* and of *key performance indicators* of a geodetic deformation analysis. These concepts make it possible to define the requirements for standardisation and for effective and efficient communication of geodetic deformation analysis.

A *statistically significant deformation* is defined in this study (chapter 2) as "a deformation that fits a deformation hypothesis, which is described by constraints on the parameters of an adjustment model of geodetic observations (or of coordinates, derived from them); the deformation hypothesis is described by physically interpretable parameters (they have been derived from a physical model); the hypothesis has been tested by means of a statistical test, which has a known probability of rejection, if the hypothesis is a valid one; and the deformation hypothesis has been shown to be the best among its competitors, where "best" is defined by an information criterion."

As *key performance indicators* the *test quantities of deformation hypotheses*, the *least squares estimates of deformation parameters*, and the *minimal detectable deformations* have been introduced in this study (chapter 2).

[4] A deformation parameter is a parameter that is used to describe a certain deformation behaviour, for example the linear rate of subsidence of some points, or the period and amplitude of some periodic movement of points.

The availability of the mentioned key performance indicators, based on the analysis model and its characteristic elements as described in this study, and the definition of a statistically significant deformation are the requirements that an analysis model has to fulfil to make a standardised procedure for geodetic deformation analysis possible. The geodetic deformation analysis model, developed in this study, fulfils these requirements. With the standardised procedure a tool is available for the improvement of communication about geodetic deformation analysis.

8.2 Contributions

The research for this study resulted in several contributions to the scientific field of geodesy, deformation analysis and adjacent fields. The contributions are summarised below in seven categories.

1. Application of achievements of the Delft school of geodesy
    - *Delft school of geodesy applied to deformation analysis*
      This study extends the application of the achievements of the Delft school of geodesy regarding the analysis of form and size, and the related achievements about S-systems and S-transformations, to the field of geodetic deformation analysis.
    - *Datum points do not have to be stable*
      This study uses test methods, in which points that are used in the definition of the S-system (the geodetic datum) can be points that are subject to deformation.
    - *Invariance for change of S-system*
      This study provides statistical tests and minimal detectable deformations (quality descriptors) that depend only on the form and size of the monitored geo-object(s), and are, therefore, invariant for the chosen S-system.
    - *S-transformation to other datum points is unnecessary*
      This study proves that testing successively many deformation hypotheses can be accomplished without any S-transformation to redefine datum points.
    - *Minimal detectable deformations based on constraints*
      This study defines minimal detectable deformations by defining deformation hypotheses with constraints on the parameter vector of the adjustment model, and computing minimal detectable biases for these constraints.
    - *Condition equations to solve and test a model with constraints*
      This study uses the preference of the Delft school of geodesy to use the adjustment model of condition equations to search and find a solution method for testing for biases in the constraints within an adjustment model of observation equations with constraints.

2. Estimation of transformation parameters; form and size

   - *Affine transformation with constraints*
     This study demonstrates how two sets of 3D stochastic coordinates of the same point field can be merged by using a 3D congruence or 3D similarity transformation by describing the merging process as an adjustment with affine transformation parameters in the vector of parameters, and with constraints on these transformation parameters to make them the parameters of a congruence or similarity transformation.

   - *Coordinates of all epochs are stochastic*
     This study shows that adjustment models can be constructed, in which the coordinates of all epochs are stochastic variables, and the model matrix contains only constant (nonstochastic) elements, thus avoiding the need to resort to a method like Total Least Squares.

   - *Overview of methods for 3D congruence and similarity transformation*
     This study provides an overview of methods, published in scientific literature, to determine the transformation parameters of the 3D congruence and similarity transformations between two sets of coordinates.

   - *Deformation analysis starting from size and form elements*
     This study analyses the amount of form elements and size elements, needed to describe the form and size of a geodetic point field in 3D, based on the same analysis for 2D. It introduces the concept of P-quantities to do the analysis. From this analysis, the geodetic deformation analysis in 1D, 2D and 3D can be derived.

3. Physical model

   - *Combination of geodetic and physical model*
     This study combines a geodetic and a physical model in one adjustment model, thus enabling the simultaneous analysis of geodetic measurements and deformation hypotheses.

   - *Separate analysis of reference points is unnecessary*
     This study shows that it is possible to consider reference and object points as subsets of points that can be analysed simultaneously and on an equal level, thus avoiding the necessity to first analyse the reference points.

   - *Displacements are not only relative to S-basis points*
     This study uses adjustment models that estimate deformation parameters and movements of points by using the method of least squares. The estimated movements of points are movements of one point, or of a subset of points, relative to the other points of the point field (which may or may not include reference points); they are not only relative to the points that constitute an S-basis (geodetic datum), fixed reference points, or similar references, used by conventional methods.

   - *Rank deficient covariance and model matrices*
     This study presents adjustment models that are capable to analyse time series of geodetic deformation measurements with consideration of full, possibly

rank deficient covariance matrices, and with possibly a rank deficient model matrix.

4. Algorithms

   - *Nonstochastic observations*
     This study uses nonstochastic observations as a means to handle an adjustment model with constraints on the parameters. It shows that they are an effective tool to incorporate deformation hypotheses and, thus, physical models in the adjustment model. Nonstochastic observations are also, as this study demonstrates, an effective tool to test constraints for biases, and, additionally to give a quality description with minimal detectable biases.

   - *Reciprocal residuals*
     This study introduces reciprocal residuals as an effective tool in testing theory. They allow for the testing of hypotheses regarding both stochastic and nonstochastic observations. Nonstochastic observations (also called "hard constraints") have a standard deviation of zero and no correlation to other observations. Hence, their estimated residuals in the least-squares adjustment are zero: they stay unchanged. But the reciprocal residuals of the nonstochastic observations are, in general, not zero and they can, therefore, be used to test hypotheses regarding nonstochastic observations.

   - *Amplification*
     This study introduces the term "amplification" for the adaptation of a rank deficient (singular) covariance matrix of the observations to solve an adjustment model of observation equations with such a covariance matrix. It shows how amplification can be applied for testing for biases in the constraints of an adjustment model of observation equations with constraints.

   - *Overview of methods to test a model with constraints*
     This study gives an overview of methods to test for biases in the constraints of an adjustment model of observation equations with constraints.

   - *Condition equations with singular covariance matrix*
     This study derives the formulas to solve an adjustment model of condition equations with a rank deficient (singular) covariance matrix of the observations.

5. Search for best deformation hypothesis

   - *Model identification*
     This study presents a search method for model identification, i.e. for finding the best deformation hypothesis. The search method uses a systematic search of all reasonable hypotheses, and applies the B-method of testing with test ratio's as information criterion to decide, which hypothesis is the best one. The search method is typically different from the search method as applied by data-snooping, or by the search heuristic to find moved points used in conventional congruence analysis of point fields.

6. Standardisation and communication

   - *Standardisation model*
     This study provides a foundation for a standardisation model.

   - *Four levels of stakeholders*
     This study distinguishes four levels of stakeholders in geodetic deformation analysis. For each level communication has to be different.

7. Professional practice

   - *Deformation analysis with existing software*
     This study provides an adjustment model (the measurements model) that is readily usable for deformation analysis with existing software for the adjustment of geodetic networks, especially if the method "almost zero" is applied.

   - *Comparison between measurements model and coordinates model*
     This study provides the theoretical tools to compare an analysis that uses the measurements model with an analysis that uses the coordinates model. The measurements model can be used by existing adjustment software for geodetic networks (see previous item). The coordinates model stongly resembles the methods predominantly used in professional practice, where coordinate sets, computed for separate epochs, are compared with each other.

   - *Invariance for changes in geodetic datum*
     This study provides tests of deformation hypotheses that are invariant for changes in geodetic datum.
     Conventional analysis methods used in professional practice compute coordinate sets per epoch relative to the geodetic datum (or S-basis, or computation basis). The choice of the best geodetic datum often poses a problem.

   - *Stable reference points not required*
     This study provides a geodetic deformation analysis model that does not require stable reference points.
     Conventional analysis methods use comparisons of coordinate sets relative to one geodetic datum. This requires a geodetic datum that consists of stable points. In geodetic practice, however, stable points are often hard to find, and the stability is generally difficult to be sure of.

   - *Key performance indicators*
     This study provides a geodetic deformation analysis model that delivers key performance indicators that are based on least-squares adjustment and statistical tests. They are *acceptable and credible* for both experts and non-experts.

   - *Minimal detectable biases and communication*
     This study shows that the concept of minimal detectable biases (MDB's) is suitable for judging the quality of geodetic deformation analysis. The MDB's provide a powerful tool for standardisation of the design of a geodetic deformation analysis. Standardisation, in its turn, enables better communication about the results of geodetic deformation analysis.

- *Physical model*
  This study presents an analysis model that provide a clear link between the geodetic and physical model, thus facilitating the translation of predictions from other sciences to deformations of geodetic networks.

8. Education

- *Transparent education and involvement of students*
  This study provides a geodetic analysis model that delivers key performance indicators that are acceptable and credible for experts and non-experts (as stated before), and also for students, thus stimulating transparent education and involvement of students in the subject.
- *Follow-up research*
  This study makes follow-up research necessary, both into further theoretical developments and in the practical implementations in professional practice (both for main stream applications and for specialised projects). This fundamental and applied research will stimulate education, internships and thesis work of students at academic universities and universities of applied sciences.
- *Cooperation between universities and professional practice*
  This study will stimulate cooperation between universities and professional practice to improve geodetic deformation analysis, thus bringing education and professional practice closer together.

8.3 Recommendations

Based on the results of this study, several recommendations can be given. They are grouped in seven categories

1. Physical model

   The incorporation of a physical model into the adjustment model for geodetic measurements can be improved:

   - The developed analysis model has to be applied to as many different use cases from professional practice as possible. The purpose of the research of these use cases is to acquire information to improve and show the weaknesses and strenghts of the developed analysis model. The use cases should provide intricate deformation hypotheses, such as: periodic movements of geo-objects, caused by, for example, sea tides, wind, sun, and human activities; specific deformation behaviour like twists in tunnels, and shifts of harbour quays; different deformation behaviour of various subsets of a geo-object, such as the abutments and the deck of a bridge.
   - Research is required into the applicability of machine learning for the search of the best deformation hypothesis.
   - The search for the best deformation hypotheses by trying all possible combinations of biases in points, leads to unacceptable large numbers of hypothe-

ses to be tested. Research into the reduction of the number of reasonable deformation hypotheses by physical considerations is desirable.

2. Information criterion

   This study uses the B-method of testing with test ratio's as an information criterion. Comparison of its performance with the performance of other information criteria (AKAIKE (AIC), Baysian (BIC)) is desirable. The theoretical study of these different information criteria may help in assessing the use and applicability of the criteria. Moreover, the consequences of the fact that multiple comparison influences the characteristics of the parameter estimator and the minimal detectable bias (Teunissen et al., 2017; Teunissen, 2018) has to be researched. Further, the error of type III, that is the error of a wrong identification of biases, and the related discernability of hypotheses (Förstner, 1990) needs a closer examination.

3. Pandora box

   Research into computation methods that employ the sparseness of the Pandora matrix is desirable to assess its properties for solving and testing the adjustment and testing models of this study.

4. Trend and signal

   It is possible to derive the formulas of collocation (Moritz, 1978) from an adjustment model with pseudo-observations (Strang van Hees, 1981), where collocation is used to separate the trend in stochastic observations from local signals. Consequently, it should be possible to change the adjustment models of this study in such a way that the nonstochastic observations become stochastic observations, where the stochastic information comes from a covariance function that describes a deformation signal. Introduction of a signal in the two adjustment models of this study is expected to improve the capability of the models to describe deformations.

5. Software

   For the numerical examples in this study research software has been written. This software can be extended to make it usable for testing more intricate deformation hypotheses. This will be necessary to use the software for new use cases from professional practice, as described above.

   The software has also to be made more robust for its use in professional practice.

6. Deformation analysis of point clouds

   Research is necessary to make the developed geodetic deformation analysis model suitable for deformation analysis by means of (extreme large) point clouds.

7. Governance, standardisation and communication

   A model has already been developed to make a taxonomy of the governance of geodetic deformation analysis (Velsink, 2012). Such a taxonomy will help in assessing the needs and possibilities for extending the use of the developed analysis model in professional practice.

A standardisation model has already been constructed in its first form (Velsink, 2016a). Elaboration of this model, and preparing it for application in national and international standards, is important to improve geodetic deformation analysis.

Appendices

A

Guidelines for Geodetic Deformation Monitoring

A.1 Research project DefoGuide

In 2014 the author of this study initiated a research project as part of the research activities for this study. The project was started under the leadership of Delft University of Technology (responsible professor: Prof. dr. ir. R.F. Hanssen) with participation of HU University of Applied Sciences Utrecht (responsible and acting researcher: ir. H. Velsink). Its title was "Guidelines for Geodetic Deformation Monitoring", short title: "DefoGuide". Partners in the project were Rijkswaterstaat (Ministry of Infrastructure and the Environment) and three Engineering Consultancies: Grontmij Nederland, Fugro Geoservices and Antea Group (Oranjewoud). Later others joined the project: Nederlandse Aardolie Maatschappij (Dutch Oil Company), and Engineering Consultancies Geomaat, RPS, and Brem Funderingsexpertise.

Funding was granted to the project by Research Programme *Maps4Society*, a programme of the government of the Netherlands in cooperation with several public and private companies.

The research question was formulated as: "How does a model look like hat describes the domains, the key players, the legislation, the measuring techniques and the products of geodetic deformation measurements, which model can be used as the basis of standards to be used in the management and the tendering of geodetic deformation measurements?"

The results of the research project have been published in the Dutch language (Velsink, 2016a). Some results, relevant to this study, are translated or summarised in the following sections, to make them accessible to those not familiar with the Dutch language.

A.2 Table of contents of the report

The table of contents of the report (Velsink, 2016a), translated from Dutch, is:

A.3 Domains[1]

Geodetic deformation measurements may differ from one another in terms of the applicable law, regulations, standardisation, method of assignment, mode of implementation and method of drawing conclusions. This appears to be dependent on the application domain. The following domains of activity are distinguished:

1. Residential and non-residential buidings
2. Hydraulic engineering (State)
3. Hydraulic engineering (Water boards)
4. Mining (gas, oil, salt, coal)
5. Infrastructure (state roads)
6. Infrastructure (provincial and municipal)
7. Rail infrastructure
8. Industrial installations
9. Scientific research / long-term movements (sea level and climate change)

A meeting of the partners of the project *DefoGuide* found that this categorisation properly covers the domains where the partners are active with geodetic deformation measurements.

A.4 Requirements for geometric accuracy in practice[2]

In the case of an assignment, which may take place within the organisation, outside the organisation by privately negotiated procedure, or outside the organisation through public procurement, standards must be formulated. The standards determine how the work has to be performed. They are explicitly and in detail written out, just vaguely defined, or implicitly known. This section focuses on formulations of standards for *precision and reliability of geometry*. The wording of standards comes from tenders delivered by partners to the project or found on the internet.

A.4.1 Rijkswaterstaat: Product Specifications Deformation Measurements of Infrastructure Works[3]

Merging of epochs

In performing deformation analyses with, for example, tacheometry measurements, objects are measured in more than one epoch. From each epoch the measurements are adjusted. This results in a description of form and size of the objects in each epoch. The analysis, whether deformations have taken place, and if so, which ones, is carried out by comparing the results of the epochs.

[1] This section is taken and translated from section 1.2.1 of Velsink (2016a).
[2] This section is taken and translated from section 1.4 of Velsink (2016a).
[3] This subsection is taken and translated from section 4.4.2 of Velsink (2016a).

Analysis with spreadsheet In current practice[4] the analysis often consists of putting the adjusted heights or x, y, and z coordinates in a spreadsheet, after which the differences between the epochs in the spreadsheet are calculated. If possible, graphs of the differences are made: on the x axis the epochs are presented, and on the y axis the height differences, the x coordinates, the y coordinates or the z coordinates. A verbal explanation of these time series concludes the analysis. This makes a multidimensional analysis difficult, because the presentation of the differences is only for two epochs, for only one point and for one dimension.

The *Product Specifications Deformation Measurements of Infrastructure Works* of Rijkswaterstaat (Rijkswaterstaat, 2014) can serve as an example of the way in which current requirements for geodetic deformation measurements are formulated. These product specifications are used not only by Rijkswaterstaat itself when outsourcing works, but also by many other public and private companies. Here are some examples:

- Outsourcing of deformation measurements by the Harbour Authority Rotterdam. Tender ICM-1011497 from the Harbour Authority, posted at www.tendernet.nl, published August 5, 2013, contains the document:
  "NVI 2 – BIJLAGE 1 – AANGEPASTE ANNEX 1", which states:
  "For the accuracy requirements of all objects, see document "Product Specifications Deformation Measurements of Infrastructure Works" version 1 of March 2012, by Rijkswaterstaat (Ministry of I&M)".

- Monitoring plan redirection N325, drafted by Fugro GeoServices B.V. on behalf of the Combination I-Lent, which works for the municipality of Nijmegen. Section 4.2 of the plan states:
  "The set up of the preliminary monitoring plan for the deformation measurements of the bridge and tunnel has been based, as much as possible, on the Product Specifications Deformation Measurements of Infrastructure Works of Rijkswaterstaat, dated 1 March 2012." (Municipality of Nijmegen, 2015).

- Master's thesis of Ivar Schols at TU Delft, based on research for research institute TNO: "Segment joint capacity of the Kiltunnel, Part 1: Immersed tunnels and settling". Section 3.3.4 states: "Information, guidelines and requirements regarding the deformation measurements can be found in the document used by Rijkswaterstaat "Product Specifications Deformation Measurements of Infrastructure Works".". The requirements of the document are taken over and not discussed. The report can be found at "repository.tudelft.nl".

The *Product Specifications Deformations of Infrastructure Works* thus play an important role in formulating requirements for geodetic deformation measurements in current practice in the Netherlands.

The Product Specifications assume that four types of measurements will be performed in each epoch:

[4] With "current practice" is meant the practice, as known to the author at the time of writing, February 2015.

1. Z measurements (measurement of height);
2. XY measurements (determination of x and y coordinates in the horizontal plane;
3. joint measurements;
4. tilt measurements.

For Z-measurements and XY measurements, the requirements are elaborated in Annex C and D of Rijkswaterstaat (2014). These annexes show that it is assumed that the measurements will be adjusted for each epoch separately. The least squares method is not mentioned explicitly, but the prescription of the MOVE3 network adjustment package (using the least squares method) in Appendix C.5, the mention of a preference for MOVE3 in Annex D.5.1 and the prescription of the B method of testing (based on the least-squared method (Baarda, 1968a)) clearly show that the least squares method is required for calculating the results per epoch. For the adjustment of each epoch, statistical parameters, such as the significance level and test power, are explicitly prescribed (Annex C.5.2, C.5.4, D.5.2).

However, the requirements for the analysis of the connection of the epochs are remarkably short. In Annex C.6 and D.7, only four and three paragraphs are devoted to this analysis. In both Annexes C.6 and D.7 it is stated: "The Client is provided with information on the nature and extent of the deformation and deformation process." It is not specified, how to determine and test the nature and extent of the deformation and deformation process.

Therefore, a least squares adjustment and subsequent statistical tests of the *connection* of the epochs are not treated.

A.4.2 Industrial Guidelines Mining Industry[5]

A.4.2.1 Guidelines

The *Industrial Guidelines Mining Industry* (T.P.B., 2014) do not describe a standard for deformation measurements, nor is it a manual for performing deformation measurements. It is somewhere in between. The measurement process of three measurement methods (spirit levelling, GNSS, InSAR) is described and statements are made about the achievable precision. Lidar is briefly discussed. The descriptions do not provide standards nor give they indications, how standards could be formulated. How the testing of measurement results against standards should be done is not dealt with at all.

Section 1.2 (page 6) states that in the guidelines "the establishment of geodetic networks, acquisition of data, mode of processing and reporting on soil movement by mineral extraction and/or storage, are formalized". Therefore, it is not the intention to formulate *product* standards, but *process* standards. This is special, because the general trend in the Netherlands is to move over to product standards, which makes functional procurement possible. In functional procurement, the *function* of the product to be delivered is specified and not how the product and its function are created. In deformation measurements, the product is a set of statements about the deformation pattern of a

[5]This section and the rest of section A.4 are taken and translated from section 1.4 of Velsink (2016a).

geo-object. The function of the statements is to provide support for statements about the risks of human actions regarding the geo-object under consideration.

The guidelines, therefore, do not have the purpose of making functional procurement possible.

The guidelines do not provide concrete standards for product features and products, and neither for the processes. It therefore appears that the guidelines are primarily intended to clarify which aspects of measurement and analysis methods should be addressed by the contractor during the work and in the report, without explicit requirements and without indications how testing with respect to the requirements should take place.

A.4.2.2 Precision, reliability and accuracy

The concepts of precision, reliability and accuracy are different concepts in the Dutch school of geodesy. The guidelines use the words sloppily.

They make statements about the precision of different measurement methods by giving numerical values. Usually it is not clear what the precise definition is of the concept, which the numerical values relates to. Generally, an indication of the S-basis, i.e. relative to what the value applies, is missing.

Typical is table 3.3 on page 58. The column "Precision deformation estimate" states for levelling: "1-5 mm" and "1 mm/$\sqrt{\text{km}}$". The standard deviation of levelling height differences is proportional to $\sqrt{\text{km}}$. Is the second value the standard deviation of a measured height difference? That is something different from a deformation estimate[6]! Do the first two numbers "1-5 mm" state something about the deformation estimate? Is it a standard deviation? And how is that estimate calculated? The guidelines provide two methods[7]:

1. pointwise multi-epoch deformation analysis;
2. continuous space-time deformation analysis.

Which method has been used to determine the mentioned values of 1-5 mm? If the answer is that the values indicate only an order of magnitude, the table hardly gives a clear lead how to draft standards and how to test them.

In Appendix B (page 76), one of the few formulations can be found, in which the standard deviation plays a role[8]: "Less than significant means that the subsidence rate of the neighboring point relative to the stable point is less than 3-sigma of the estimated velocity.". The word "significant" appears in the guidelines many times. It is not clear why the concept has been specified on page 76 only.

[6] If only one height difference is measured, it is immediately the deformation estimate. If more measurements are made, if there is an interpolation in time, if there are corrections for individual point movements, etc., this will no longer be the case.

[7] Scientific literature provides a lot more methods.

[8] The text speaks about "3-sigma". Sigma is likely to mean the standard deviation, but it is not defined.

A.4.2.3 Reference point

In 2009, an inquiry was commissioned on behalf of the Technical Committee on Soil Movement (Technische Commissie Bodembeweging (TCBB)) on "how we can achieve a consistent approach in the Netherlands for the clear determination of soil movements caused by mining". The final report "From Measurement to Subsidence" addresses i.a. the method recommended by SodM. It states:

(begin of quote)

1. The SodM method is based on the adjustment of levelling measurements per epoch (per measurement campaign), assuming one stable reference point.
2. The reference point should be carefully chosen so that it is expected to be stable throughout the period.
3. The SodM method is a break with the past when the levelling network was connected to multiple NAP[9]reference points, assumed to be stable. In fact, the absolute soil subsidence relative to NAP is no longer determined, only the relative movement of the benchmarks.

(end of quote)

The guidelines seem to use the SodM method as starting point. The "one reference point" is discussed in the guidelines many times. However, the above three statements are not defensible, combined together. It may be that one wants to know the change relative to a stable supposed environment, ór one wants to know the change of the geo-object relative to itself. In the *first case*, several points must be selected, which represent the stable supposed environment. Just one point is insufficient because there is no redundancy and the movement of this one reference point (relative to the stable environment) cannot be detected. The *second case* does not require any reference point that must be carefully chosen and stable throughout the entire period. One should carefully select *all* points representing the geo-object. Of course, calculations need to be relative to a reference point, but that can be any point, possibly a fictional point, such as the center of gravity. And it does not have to be stable.

A.4.2.4 Analysis with free network adjustment

On page 23 of the guidelines, two methods of deformation analysis are mentioned:

1. Analysis by Free network adjustment;
2. Analysis Space/Time.

The first method states: "With, for example, software package MOVE3 from Grontmij". Current version 4.3.0 of MOVE3 is a package that can perform deformation analyses with the observation type "shift vector" (MOVE3, 2017). But at the time of adoption of the guidelines, it was not yet possible. MOVE3 was able to perform a free network adjustment of a single epoch. MOVE3 could also merge the measurements of more

[9]Dutch national height reference network: *Normaal Amsterdams Peil*: Standardised Amsterdam Level.

epochs and perform an adjustment. However, the deformation analysis could not do more than deliver co-ordinates, between which co-ordinate differences were calculated (moreover, not by MOVE3). What is intended exactly by the "Analysis Free network adjustment"? What MOVE3 can do, is delivering coordinates including the covariance matrix. This allows for a good deformation analysis to be performed in a separate programme, if necessary equivalent to the "Analysis Space/Time".

A.4.3 Harbour Authority Rotterdam

A.4.3.1 European procurement of quay wall

From the Antea Group the standards have been obtained of a European tender by the municipality of Rotterdam for deformation measurements of quay walls. It is assumed that this tender is public. Delivered are annexes 5, 6 and 7 of the tender. Unfortunately, a further specification of the document is not available. The document is from before December 2014. The author received information in May 2016 showing that the tender has been withdrawn after some time, after which the tender has been restarted by the Port of Rotterdam, using the product specifications of Rijkswaterstaat (Rijkswaterstaat, 2014) as standards.

Annex 7 (Requirements Measurement and Processing of Quay Wall Deformations) states:

> The required standard quality is described in the section on horizontal and vertical deformation measurements. Quality class "Medium" is deemed sufficient for the quay walls. The result of the measurement has to possess the following quality features:
>
> | | Standard Deviation | Relative Precision | Change to be detected |
> |---|---|---|---|
> | XY (horizontal) | 4.0 mm | 2.0 mm | Abs. 8.0 mm, Rel. 4 mm |
> | Z (vertical) | 2.0 mm | 1.0 mm | Abs. 4.0 mm, Rel. 2 mm |
>
> The deformation points and fixed points must be measured redundantly in the horizontal measurements. The change to be detected is interpreted as follows: In re-measurement, the differences fall within the mentioned tolerance (tolerance = +/- 2 times the standard deviation) in 95% of the cases.

Comment:

1. "Standard Deviation" and "Relative Precision": What is meant by this? Presumably, the "Relative precision" value is a standard deviation, relative to something close to it. Presumably the "Standard deviation" is thought to be 'absolute', that is, in relation to the national reference system. But such an "absolute standard deviation" is always relative in the operational definition: the absolute standard deviation is relative to control points, whose co-ordinates are known in the national system. Then the question is: which S-basis (which reference points) is

taken for the relative and which for absolute precision? And how does one test whether the standard is met?

2. Why is there interest in the absolute standard deviation? In general the interest is only in the *approximate* absolute position of the measured object. Is a standard deviation of, for example, a meter already sufficient? Connection to a basis of three points, which was measured with GNSS on the mobile phone, might suffice.

   The deformation analysis requires a very good *relative* precision (small standard deviation). Its definition (S-basis, testing method) must be addressed, not the definition of absolute precision.

3. "Change to be detected": "Abs." is probably given a value (8.0 mm) that is twice the "Standard Deviation" and with "Rel." a value (4.0 mm) that is twice the "Relative Precision". The factor two comes from a two-sided critical area of the one-dimensional standard normal distribution, with a significance level of 5%, giving a critical value of 1.96 (rounded: 2). This follows from the sentence: "By re-measurement ...".

   But XY is not one, but two-dimensional. And change is determined by determining *twice* the XY and their difference. The discussion should be, then, about the standard deviation of that difference. Therefore, the numerical values in "Change to be detected" (both XY and Z) can not be set as standard simultaneously with the aforementioned standard deviations.

A.5 Software

Chapter 6 of Velsink (2016a) treats software used for geodetic deformation analysis, according to information acquired by the author in 2016. Several packages from Germany, the United States of America and Portugal are treated. From the Netherlands the package MOVE3 is reviewed and specialised software packages from the Antea Group (Engineering Consultancy), NAM (oil and gas company), Skygeo (InSAR-software) and TU Delft are described. Also from the Netherlands are a package for analysis of levelling measurements from Houtenbos Geodetic Consultancy, and the MATLAB-programmes of the author.

From the inventarisation and from questioning the Dutch partners in project DefoGuide it became clear that the software packages from abroad (Germany, United States of America, Portugal) are hardly used (maybe not at all) in professional practice in the Netherlands. The reason is unknown.

B

Appendices to chapter 4

B.1 Conventions

In chapter 4 use is made of the following conventions:

- T indicates the transpose of a matrix.
- Approximate values of a scalar or vector are indicated by a sub or super script 0, e.g. a scalar s^0 or a vector $\mathbf{v}_0$.
- If $\mathbf{v}$ is a certain vector, then $\Delta\mathbf{v}$ is the vector of differences of $\mathbf{v}$ with its vector $\mathbf{v}_0$ of approximate values.
$$\Delta\mathbf{v} = \mathbf{v} - \mathbf{v}_0.$$
For a scalar an analogous equation holds.
- a vector that contains the coordinates of n points contains first all x-coordinates from point 1 up to point n, then all y-coordinates and finally all z- coordinates
- the vector $\underline{\mathbf{a}}$ consists of three sub vectors $\underline{\mathbf{x}}$, $\underline{\mathbf{y}}$ and $\underline{\mathbf{z}}$, and the vector $\underline{\mathbf{b}}'$ of three sub vectors $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$ and $\underline{\mathbf{w}}$, such that
$$\underline{\mathbf{a}} = \begin{pmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{y}} \\ \underline{\mathbf{z}} \end{pmatrix} \text{ and } \underline{\mathbf{b}}' = \begin{pmatrix} \underline{\mathbf{x}} \\ \underline{\mathbf{v}} \\ \underline{\mathbf{w}} \end{pmatrix}. \tag{B.1}$$
- vector $\underline{\mathbf{x}}$ contains for each point the x-coordinates. If there are n points, then
$$\underline{\mathbf{x}} = \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \vdots \\ \underline{x}_n \end{pmatrix}. \tag{B.2}$$
In the same way $\underline{\mathbf{y}}$, $\underline{\mathbf{z}}$ and $\underline{\mathbf{u}}$, $\underline{\mathbf{v}}$, $\underline{\mathbf{w}}$ are defined.

B.2 Adjustment equations

Most of the following equations can be found in Teunissen (2000) and Teunissen (2006). To write the equations more concisely, and because of their central role in testing, the reciprocal least squares residuals $\underline{\hat{\mathbf{r}}}$ are introduced.

A system of linear or linearised observation equations has the following general structure

$$E\{\underline{\ell}\} = \mathbf{A}\mathbf{p}; \quad \mathbf{D}\{\underline{\ell}\} = \sigma^2\mathbf{Q}_\ell, \tag{B.3}$$

where $\underline{\ell}$ is the m-vector of observations, $\mathbf{A}$ is the $(m \times n)$ matrix of coefficients and $\mathbf{p}$ is the n-vector of unknown parameters. The equation behind the semicolon describes the stochastic model by giving the covariance matrix $\mathbf{D}\{\underline{\ell}\}$, the variance factor σ^2 and the cofactor matrix $\mathbf{Q}_\ell$. Both $\mathbf{A}$ and $\mathbf{Q}_\ell$, are considered in the following equations to be regular matrices. The situation that $\mathbf{Q}_\ell$ is singular is treated in the paper.

The least squares solution is given by

$$\underline{\hat{\mathbf{p}}} = (\mathbf{A}^T\mathbf{Q}_\ell^{-1}\mathbf{A})^{-1}\mathbf{A}^T\mathbf{Q}_\ell^{-1}, \tag{B.4}$$

$$\mathbf{Q}_{\hat{\mathbf{p}}} = (\mathbf{A}^T\mathbf{Q}_\ell^{-1}\mathbf{A})^{-1}, \tag{B.5}$$

with $\underline{\hat{\mathbf{p}}}$ the vector of estimated parameters and $\mathbf{Q}_{\hat{\mathbf{p}}}$ its cofactor matrix. We have also

$$\underline{\hat{\ell}} = \mathbf{A}\underline{\hat{\mathbf{x}}}; \quad \mathbf{Q}_{\hat{\ell}} = \mathbf{A}\mathbf{Q}_{\hat{\mathbf{x}}}\mathbf{A}^T, \tag{B.6}$$

$$\underline{\hat{\mathbf{e}}} = \underline{\ell} - \underline{\hat{\ell}}; \quad \mathbf{Q}_{\hat{e}} = \mathbf{Q}_\ell - \mathbf{Q}_{\hat{\ell}}, \tag{B.7}$$

$$\underline{\hat{\mathbf{r}}} = \mathbf{Q}_\ell^{-1}\underline{\hat{\mathbf{e}}}; \quad \mathbf{Q}_{\hat{r}} = \mathbf{Q}_\ell^{-1}\mathbf{Q}_{\hat{\mathbf{e}}}\mathbf{Q}_\ell^{-1} \tag{B.8}$$

with $\underline{\hat{\ell}}$ the adjusted observations, $\underline{\hat{\mathbf{e}}}$ the least squares residuals and $\underline{\hat{\mathbf{r}}}$ the reciprocal least squares residuals. The $\mathbf{Q}$-matrices are their cofactor matrices. The reciprocal least squares residuals are used in the testing equations.

For each system of linear observation equations an equivalent system of linear condition equations exists

$$\mathbf{K}^T E\{\underline{\ell}\} = \mathbf{0}; \quad \mathbf{D}\{\underline{\ell}\} = \sigma^2\mathbf{Q}_\ell. \tag{B.9}$$

where $\mathbf{K}^T$ is the $[(m-n)\times m]$-matrix of conditions, for which holds

$$\mathbf{K}^T\mathbf{A} = \mathbf{0}. \tag{B.10}$$

Matrix $\mathbf{K}$ is considered a regular matrix. $\mathbf{Q}_\ell$ is considered a positive semidefinite matrix, so it may be singular.

Define the vector of misclosures $\underline{\mathbf{t}}$ and its cofactor matrix $\mathbf{Q}_\mathbf{t}$ as

$$\underline{\mathbf{t}} = \mathbf{K}^T\underline{\ell}; \quad \mathbf{Q}_\mathbf{t} = \mathbf{K}^T\mathbf{Q}_\ell\mathbf{K}. \tag{B.11}$$

The least squares solution is

$$\underline{\hat{\mathbf{r}}} = \mathbf{K}\mathbf{Q}_\mathbf{t}^{-1}\underline{\mathbf{t}}; \quad \mathbf{Q}_{\hat{\mathbf{r}}} = \mathbf{K}\mathbf{Q}_\mathbf{t}^{-1}\mathbf{K}^T, \tag{B.12}$$

$$\underline{\hat{\mathbf{e}}} = \mathbf{Q}_\ell\underline{\hat{\mathbf{r}}}; \quad \mathbf{Q}_{\hat{e}} = \mathbf{Q}_\ell\mathbf{Q}_{\hat{\mathbf{r}}}\mathbf{Q}_\ell, \tag{B.13}$$

$$\underline{\hat{\ell}} = \underline{\ell} - \underline{\hat{\mathbf{e}}}; \quad \mathbf{Q}_{\hat{\ell}} = \mathbf{Q}_\ell - \mathbf{Q}_{\hat{\mathbf{e}}}. \tag{B.14}$$

B.3 Testing equations

Consider models (B.3) or (B.9) as the null hypothesis. Let an alternative hypothesis be defined as

$$E\{\underline{\ell}\} = \mathbf{A}\mathbf{p} + \mathbf{G}\nabla; \quad \mathbf{D}\{\underline{\ell}\} = \sigma^2\mathbf{Q}_\ell, \tag{B.15}$$

with $\mathbf{G}$ a $(m \times q)$ matrix of known coefficients and ∇ a q-vector of unknown bias parameters, with m the amount of observations and q the amount of bias parameters. The formulation for the model of condition equations is

$$\mathbf{K}^T E\{\underline{\ell}\} = \mathbf{K}^T\mathbf{G}\nabla; \quad \mathbf{D}\{\underline{\ell}\} = \sigma^2\mathbf{Q}_\ell. \tag{B.16}$$

To test this alternative hypothesis against the null hypothesis (model (B.3) or (B.9)) use is made of the following test statistic (Teunissen, 2006, p. 78)

$$\underline{V}_q = \underline{\hat{\mathbf{r}}}^T\mathbf{G}(\mathbf{G}^T\mathbf{Q}_{\hat{\mathbf{r}}}\mathbf{G})^{-1}\mathbf{G}^T\underline{\hat{\mathbf{r}}}. \tag{B.17}$$

$$\text{If } \underline{F}_{q,\infty} = \frac{\underline{V}_q}{q\sigma^2} > F_{\text{crit}} \tag{B.18}$$

with F_{crit} the critical value, the null hypothesis is rejected in favour of the alternative hypothesis.

If the null hypothesis is rejected, an estimate of the biases and its cofactor matrix can be determined as (Teunissen, 2006, p. 76)

$$\underline{\hat{\nabla}} = (\mathbf{G}^T\mathbf{Q}_{\hat{\mathbf{r}}}\mathbf{G})^{-1}\mathbf{G}^T\underline{\hat{\mathbf{r}}}; \quad \mathbf{Q}_{\hat{\nabla}} = (\mathbf{G}^T\mathbf{Q}_{\hat{\mathbf{r}}}\mathbf{G})^{-1} \tag{B.19}$$

For q we have $1 \leq q \leq m - n$. If $q > m - n$ the adjustment models (B.15) and (B.16) are underdetermined and cannot be solved. For the limiting case $q = m - n$ the test is equal to the overall model test and we have

$$\underline{V}_{m-n} = \underline{\hat{\mathbf{e}}}^T\mathbf{Q}_{\boldsymbol{\ell}}^{-1}\underline{\hat{\mathbf{e}}} = \underline{\hat{\mathbf{r}}}^T\mathbf{Q}_{\boldsymbol{\ell}}\underline{\hat{\mathbf{r}}}. \tag{B.20}$$

The rightmost expression can be used if the system of condition equations is used and $\mathbf{Q}_{\boldsymbol{\ell}}$ is singular.

The limiting case q=1 is the test of w-quantities (Baarda, 1968b, p. 13). It is called a w-test.

$$\text{If } \underline{w} = \frac{\mathbf{g}^T\underline{\hat{\mathbf{r}}}}{\sigma(\mathbf{g}^T\mathbf{Q}_{\hat{\mathbf{r}}}\mathbf{g})^{1/2}} > w_{\text{crit}} \tag{B.21}$$

reject the null hypothesis. Here w_{crit} is the critical value and $\mathbf{g}$ is the matrix $\mathbf{G}$, but written as a lower case letter, because it has only one column: it is a vector. The w-quantity has a normal distribution with an expectation of 0 and a standard deviation of 1. A conventional w-test (Baarda, 1968b, p. 15) is the test of one observation being biased and all other observations being without bias. The vector $\mathbf{g}$ is

$$\mathbf{g} = (0 \ldots 0\,1\,0 \ldots 0)^T, \tag{B.22}$$

with the 1 corresponding to the biased observation.

A point test is a three-dimensional test (q=3) of the x-, y- and z-coordinate of one point (used as observations). The matrix **G** is

$$\mathbf{G} = \begin{pmatrix} 0 & \dots & 0 & 1 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 & 0 & \dots & 0 \end{pmatrix}, \tag{B.23}$$

with the ones corresponding to the coordinates that are supposed to be biased.

B.4 Data of the experimental validation

Observations of the first epoch:

pnt	pnt	direction	distance	zenith angle
99	1	307.7820765	24.5060074	99.8055455
99	2	319.7344105	20.0876102	98.7070717
99	3	331.5296877	17.0625166	105.4327481
99	4	352.4563301	14.7326649	106.6349756
99	5	388.3056469	15.4070803	101.3380122
99	14	29.0918416	17.8292985	99.7871557
99	15	46.7749502	15.3559925	96.6129838
99	16	59.3812951	14.5848606	99.6001975
99	17	79.1592740	14.9923557	105.715829
99	18	89.8716178	17.9265803	106.5598073
99	19	93.2971810	21.2104291	98.4820515
99	23	186.8788155	19.3414323	103.9408223
99	24	200.1056248	20.9264019	105.3692816
99	25	212.6876307	23.5421679	99.0553208
99	26	224.8455986	28.3270008	98.4196188

Observations of the second epoch, point 1 is biased:

pnt	pnt	direction	distance	zenith angle
100	1	13,7372731	23,0672793	99,7859602
100	2	27,7594635	19,1262636	98,6398106
100	3	41,4401419	16,6097400	105,582282
100	4	63,6494039	15,1749445	106,4430463
100	5	96,2060486	17,1252428	101,2017896
100	14	131,4346394	20,3808279	99,8127807
100	15	146,8624786	18,0020637	97,1040642
100	16	157,5422153	17,1916389	99,6580831
100	17	174,4918375	17,3543574	104,9366243
100	18	184,6390687	20,0631725	105,8611633
100	19	188,5080762	23,2655969	98,6169971
100	23	279,1965362	17,9334770	104,2543189

100	24	294,0907282	19,0647202	105,8937977
100	25	308,5860882	21,3237391	98,9581652
100	26	322,5863556	25,8583270	98,2684568

Observations of the second epoch, points 1 up to 5 are biased:

pnt	pnt	direction	distance	zenith angle
100	1	13,7372731	23,0672631	99,7997607
100	2	27,7642973	19,1302064	98,6467483
100	3	41,4423295	16,6141034	105,5885051
100	4	63,6455986	15,1792709	106,4496345
100	5	96,1955807	17,1284537	101,2089988
100	14	131,4346394	20,3808279	99,8127807
100	15	146,8624786	18,0020637	97,1040642
100	16	157,5422153	17,1916389	99,6580831
100	17	174,4918375	17,3543574	104,9366243
100	18	184,6390687	20,0631725	105,8611633
100	19	188,5080762	23,2655969	98,6169971
100	23	279,1965362	17,9334770	104,2543189
100	24	294,0907282	19,0647202	105,8937977
100	25	308,5860882	21,3237391	98,9581652
100	26	322,5863556	25,8583270	98,2684568

Approximate coordinates of the first epoch:

pnt	x	y	z
099	83277.7360	457303.9790	2.0000
001	83253.4140	457306.9670	2.0740
002	83258.6090	457310.1060	2.4080
003	83262.7790	457312.0580	0.5460
004	83267.7820	457314.7330	0.4670
005	83274.9220	457319.1230	1.6760
014	83285.6020	457319.9780	2.0600
015	83288.0160	457315.3580	2.8170
016	83289.4500	457312.6660	2.0920
017	83291.8750	457308.7800	0.6560
018	83295.3420	457306.8040	0.1560
019	83298.8230	457306.2070	2.5050
023	83281.6860	457285.0840	0.8030
024	83277.7020	457283.1290	0.2370
025	83273.0750	457280.9050	2.3490
026	83266.9620	457277.7910	2.7030

Approximate coordinates of the second epoch:

pnt	x	y	z
100	0.000	0.000	0.000
1	4.936	22.528	0.074
2	80.750	173.330	0.408
3	100.270	131.630	-14.540
4	127.020	81.600	-15.330
5	170.920	10.200	-0.324
14	17.947	-9.660	0.060
15	133.270	-120.740	0.817
16	106.350	-135.080	0.092
17	67.490	-159.330	-13.440
18	47.730	-194.000	-18.440
19	41.760	-228.810	0.505
23	-169.470	-57.440	-11.970
24	-189.020	-17.600	-17.630
25	-211.260	28.670	0.349
26	-242.400	89.800	0.703
	.	.	

B.5 Results case 1

```
Point 1 is biased
dx, dy, dz = 3 mm, -3 mm, -3 mm in x-system
Number_of_computations = 3
Max_criterion = 5.0e-013
Scale_factor = 1
Translation_x_y_z_in_m =
81475.939, 455202.030, 2.000
Alpha_Beta_Gamma_in_gon =
0, 0, -100
Overall model test:
Number_of_conditions = 38
Ftest = 1.41
Critical_value = 1.18
w-test: Critical value is 3.29
No rejections
Point test: Critical value = 4.21
Point F_q
001 6.2758
Est.def. x Est.def. y Est.def. z
3.6 mm -2.4 mm -2.5 mm
```

Differences between x and adj. coord. in mm

1	0.0	0.0	0.0
2	-1.1	0.1	0.9
3	-1.6	1.5	1.2
4	-1.1	-0.2	1.1
5	-0.7	0.2	0.4
14	0.0	0.0	0.0
15	-1.1	-1.5	-1.1
16	-0.3	-0.6	-0.5
17	-0.9	-0.8	-0.4
18	-1.2	-1.7	-0.4
19	-2.1	-1.4	-0.6
23	-2.5	-1.1	0.0
24	-3.2	-1.5	-0.6
25	-2.9	0.9	0.0
26	-3.2	-0.5	0.0

B.6 Results case 2

Points 1, 2, 3, 4, 5 are biased
dx, dy, dz = 3 mm, -3 mm, 2 mm
in x-system
Number_of_computations = 3
Max_criterion = 5.2e-012
Scale_factor = 1
Translation_x_y_z_in_m =
81475.940, 455202.029, 2.001
Alpha_Beta_Gamma_in_gon =
0, 0, -100
Overall model test
Number_of_conditions = 38
Ftest = 2.20
Critical_value = 1.18
w-test: Critical value is 3.29
x-coordinate
Point Ratio w Est.error
005 1.1582 3.8106 3.0011
y-coordinate
Point Ratio w Est.error
002 1.1816 3.8875 -3.2984
z-coordinate
no rejections
Point test: Critical value = 4.21
Point F_q
002 5.8141

Est.def. x Est.def. y Est.def. z
2.3 mm -3.9 mm -0.4 mm
More-dimensional test
q=3, points 001, 002, 003, 004, 005
F_q
5.8141
Est.def. x Est.def. y Est.def. z
2.8 mm -3.1 mm 3.7 mm
Differences between x and adj. coord. in mm

1	0.0	0.0	0.0
2	0.2	-1.2	-0.2
3	0.0	0.1	0.3
4	0.7	-1.7	0.6
5	1.0	-1.6	0.4
14	0.0	0.0	0.0
15	-1.2	-1.5	-0.6
16	-0.3	-0.7	0.2
17	-0.8	-0.9	0.7
18	-1.0	-1.9	1.1
19	-2.3	-1.4	1.2
23	-2.4	-1.2	1.1
24	-3.1	-1.7	0.2
25	-3.0	0.9	0.4
26	-3.4	-0.5	0.0

C

Appendices to chapter 5

C.1 Estimation of ∇

To derive the equations for the least-squares estimate $\underline{\hat{\nabla}}$ and its cofactor matrix $\mathbf{Q}_{\hat{\nabla}}$, the model of condition equations is used. This model is dual to the model of observation equations, model (5.1), yielding the same least-squares adjustment:

$$\mathbf{B}^T\underline{\mathbf{y}} - \mathbf{b}_0 = \mathbf{B}^T\underline{\mathbf{e}} = \underline{\mathbf{t}};\ E\{\underline{\mathbf{t}}\} = \mathbf{0}, \tag{C.1}$$

$$\text{with } \mathbf{B}^T\mathbf{A} = \mathbf{0} \text{ and } \mathbf{B}^T\mathbf{a}_0 = \mathbf{b}_0.$$

Matrix $\mathbf{B}$ is the $(m \times b)$ coefficient matrix and $\mathbf{b}_0$ a b-vector of constant terms. The number of conditions equals $b = m-n$. The vector $\underline{\mathbf{t}}$ contains the misclosures. The least-squares solution of model (C.1), using a positive semidefinite $\mathbf{Q}_y$, is derived in appendix C.3.

Under the alternative hypothesis H_a the model of condition equations takes the form:

$$\mathbf{B}^T\underline{\mathbf{y}} - \mathbf{b}_0 = \underline{\mathbf{t}};\ E\{\underline{\mathbf{t}}\} = \mathbf{B}^T\mathbf{C}\nabla, \tag{C.2}$$

This model can be solved as a model of observation equations:

$$E\{\underline{\mathbf{t}}\} = \mathbf{B}^T\mathbf{C}\nabla. \tag{C.3}$$

The cofactor matrix of $\underline{\mathbf{t}}$ is $\mathbf{Q}_t = \mathbf{B}^T\mathbf{Q}_y\mathbf{B}$. It is assumed that $\mathbf{Q}_t$ has full rank. Using equations (5.2)-(5.4) a least-squares solution $\underline{\hat{\nabla}}$ and its cofactor matrix can be computed.

This is rewritten in such a way that only the adjustment results of the *null* hypothesis are needed, until $\underline{T}_q$ is computed. Because $\underline{\mathbf{e}} \in \mathcal{R}(\mathbf{Q}_y)$ (proof in appendix C.2), it is possible to write, with $\underline{\mathbf{r}}$ some appropriate vector:

$$\underline{\mathbf{e}} = \mathbf{Q}_y\underline{\mathbf{r}}. \tag{C.4}$$

Here $\underline{\mathbf{r}}$ is called the vector of *reciprocal residuals*. Model (5.1), that is, the null hypothesis, is written

$$\underline{\mathbf{y}} = \mathbf{A}\mathbf{x} + \mathbf{a}_0 + \mathbf{Q}_y\underline{\mathbf{r}}, \quad E\{\underline{\mathbf{r}}\} = \mathbf{0}, \quad D\{\underline{\mathbf{y}}\} = \sigma^2\mathbf{Q}_y. \tag{C.5}$$

A least-squares estimate $\underline{\hat{\mathbf{r}}}$ has to fulfil the condition (Kourouklis and Paige, 1981, p. 621):

$$\mathbf{A}^T\underline{\hat{\mathbf{r}}} = \mathbf{0}. \tag{C.6}$$

We can write $\underline{\hat{\mathbf{r}}}$ and its cofactor matrix $\mathbf{Q}_{\hat{r}}$ in terms of the model of condition equations as

$$\underline{\hat{\mathbf{r}}} = \mathbf{B}\mathbf{Q}_t^{-1}\underline{\mathbf{t}}, \quad \mathbf{Q}_{\hat{r}} = \mathbf{B}\mathbf{Q}_t^{-1}\mathbf{B}^T. \tag{C.7}$$

It can be verified that $\mathbf{A}^T\underline{\hat{\mathbf{r}}} = \mathbf{0}$, and that $\underline{\hat{\mathbf{r}}}$ complies with the model equation $\mathbf{B}^T\underline{\hat{\mathbf{e}}} = \underline{\mathbf{t}}$.

A least-squares solution $\underline{\hat{\nabla}}$ and its cofactor matrix $\mathbf{Q}_{\hat{\nabla}}$ follow from solving equation (C.3):

$$\underline{\hat{\nabla}} = (\mathbf{C}^T\mathbf{Q}_{\hat{r}}\mathbf{C})^- \mathbf{C}^T\underline{\hat{\mathbf{r}}}, \quad \mathbf{Q}_{\hat{\nabla}} = (\mathbf{C}^T\mathbf{Q}_{\hat{r}}\mathbf{C})_{rs}^-. \tag{C.8}$$

Notice that $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$ follow from the adjustment of the null hypothesis.

C.2 Elaboration of Reduced Model after Orthogonalisation

This appendix describes the determination of the reciprocal residuals $\underline{\hat{\mathbf{r}}}$ and their cofactor matrix $\mathbf{Q}_{\hat{r}}$ from a model of observation equations with a rank deficient coefficient matrix $\mathbf{A}$ and singular cofactor matrix $\mathbf{Q}_y$. To accomplish this, the model of observation equations is changed into a model with a coefficient and cofactor matrix that are both nonsingular. First, the observations are orthogonalised relative to $\mathbf{Q}_y$, and subsequently the model is reduced (Rao and Mitra, 1971, p. 149). Then the least-squares solution is determined. This is used to determine with a follow-up adjustment the reciprocal residuals and their cofactor matrix.

Orthogonalisation According to Rao and Mitra (1971) the cofactor matrix $\mathbf{Q}_y$ can be split in $\mathbf{Q}_y = \mathbf{J}\mathbf{J}^T$ with $\text{rank}(\mathbf{Q}_y) = \text{rank}(\mathbf{J}) =$ number of columns in $\mathbf{J}$. Let $\mathbf{N}$ be an orthogonal complement of $\mathbf{Q}_y$, that is, $\mathbf{N}$ is a matrix of maximum rank such that $\mathbf{N}^T\mathbf{Q}_y = \mathbf{0}$. Let $\mathbf{F}^T$ be a left inverse of $\mathbf{J}$, that is $\mathbf{F}^T\mathbf{J} = \mathbf{I}$, with the additional condition $\mathbf{F}^T\mathbf{N} = \mathbf{0}$. For any positive semidefinite $\mathbf{Q}_y$ with rank ≥ 1 $\mathbf{F}$ exists. The observations of model (5.1) are transformed:

$$\underline{\bar{\mathbf{y}}}_1 = \mathbf{F}^T\underline{\mathbf{y}} = \mathbf{F}^T\mathbf{A}\,\mathbf{x} + \mathbf{F}^T\,\underline{\mathbf{e}}, \quad \mathbf{Q}_{\bar{y}_1} = \mathbf{I}, \tag{C.9}$$

$$\bar{\mathbf{y}}_2 = \mathbf{N}^T\underline{\mathbf{y}} = \mathbf{N}^T\mathbf{A}\,\mathbf{x}, \qquad \mathbf{Q}_{\bar{y}_2} = \mathbf{0}. \tag{C.10}$$

$\bar{\mathbf{y}}_2$ has the zero matrix as cofactor matrix and, therefore, defines constraints on the parameters, with $\mathbf{N}^T\mathbf{A}$ the *constraints matrix*, and $\bar{\mathbf{y}}_2$ the *vector of constants*. Equation (C.10) shows that $\mathbf{N}^T\underline{\mathbf{e}} = \mathbf{0}$. Therefore, $\underline{\mathbf{e}}$ has no component in the nullspace of $\mathbf{Q}_y$, or, equivalently, $\underline{\mathbf{e}} \in \mathcal{R}(\mathbf{Q}_y)$.

Reduction Define:

$$\begin{aligned}&\mathbf{B}^T = \mathbf{F}^T - \mathbf{F}^T\mathbf{A}(\mathbf{N}^T\mathbf{A})^-\mathbf{N}^T; \quad \mathbf{A}_R = \mathbf{F}^T\mathbf{A}\mathbf{U};\\ &\underline{\mathbf{y}}_R = \mathbf{B}^T(\underline{\mathbf{y}} - \mathbf{a}_0); \quad \underline{\mathbf{e}}_R = \mathbf{F}^T\underline{\mathbf{e}}.\end{aligned} \tag{C.11}$$

For $(.)^-$ any generalised inverse can be taken. $\mathbf{U}$ is a matrix for which holds $\mathbf{N}^T\mathbf{A}\,\mathbf{U} = \mathbf{0}$, and whose range space is complementary to the range space of $\mathbf{N}^T\mathbf{A}$. The reduced model is (Rao and Mitra, 1971, p. 144):

$$\underline{\mathbf{y}}_R = \mathbf{A}_R\mathbf{x}_R + \underline{\mathbf{e}}_R; \ \mathbf{Q}_{y_R} = \mathbf{I}, \tag{C.12}$$

with $\mathbf{I}$ being a unit matrix. Solution of the normal equations yields the least-squares solution $\hat{\underline{\mathbf{x}}}_R$.

Follow-up adjustment With least-squares solution $\hat{\underline{\mathbf{x}}}_R$, model (C.1) of condition equations is formulated for the original observations $\underline{\mathbf{y}}$, based on equations (C.11) and (C.12).

$$\mathbf{B}^T\underline{\mathbf{y}} - \underbrace{(\mathbf{B}^T\mathbf{a}_0 + \mathbf{A}_R\,\hat{\underline{\mathbf{x}}}_R)}_{\underline{\mathbf{b}}_0} = \underline{\mathbf{t}}; \ E\{\underline{\mathbf{t}}\} = \mathbf{0}. \tag{C.13}$$

The reciprocal residuals are given by equation (C.7):

$$\hat{\underline{\mathbf{r}}} = \mathbf{B}(\mathbf{B}^T\mathbf{Q}_y\mathbf{B})^{-1}(\mathbf{B}^T\underline{\mathbf{y}} - \underline{\mathbf{b}}_0). \tag{C.14}$$

Because $\underline{\mathbf{b}}_0$ is a stochastic vector, the computation of the cofactor matrix of $\hat{\underline{\mathbf{r}}}$ is lengthy.

$$\mathbf{H} = \mathbf{B}(\mathbf{B}^T\mathbf{Q}_y\mathbf{B})^{-1}(\mathbf{B}^T - \mathbf{A}_R(\mathbf{A}_R^T\mathbf{A}_R)^{-1}\mathbf{A}_R^T\mathbf{B}^T), \tag{C.15}$$

$$\hat{\underline{\mathbf{r}}} = \mathbf{H}(\underline{\mathbf{y}} - \mathbf{a}_0), \quad \mathbf{Q}_{\hat{r}} = \mathbf{H}\mathbf{Q}_y\mathbf{H}^T. \tag{C.16}$$

C.3 Condition Equations with Singular Cofactor Matrix

In this appendix the formulas are derived to perform a least-squares adjustment with a singular, that is, a semidefinite positive, cofactor matrix using the model of condition equations.

As described in appendix C.2, the observations in the model of observation equations can be orthogonalised with matrices $\mathbf{F}$ and $\mathbf{N}$, yielding $\bar{\underline{\mathbf{y}}}_1$ and $\bar{\underline{\mathbf{y}}}_2$. The BLUE of an estimable linear function of the parameters $\mathbf{x}$ is found by solving (C.9) for $\mathbf{x}$, while minimising (Rao and Mitra, 1971, p. 149)

$$\begin{aligned}&(\bar{\underline{\mathbf{y}}}_1 - \mathbf{F}^T\mathbf{A}\,\mathbf{x})^T(\bar{\underline{\mathbf{y}}}_1 - \mathbf{F}^T\mathbf{A}\,\mathbf{x}) \ =\\ &= \ (\underline{\mathbf{y}} - \mathbf{a}_0 - \mathbf{A}\,\mathbf{x})^T\mathbf{Q}_y^-(\underline{\mathbf{y}} - \mathbf{a}_0 - \mathbf{A}\,\mathbf{x}) \ =\\ &= \underline{\mathbf{e}}^T\mathbf{Q}_y^-\underline{\mathbf{e}},\end{aligned} \tag{C.17}$$

under the restriction that $\mathbf{N}^T\mathbf{A}\,\mathbf{x} = \bar{\mathbf{y}}_2$ and with $\mathbf{Q}_y^-$ any generalised inverse of $\mathbf{Q}_y$.

Equation (C.1) gives the model of condition equations. From the preceding condition for the BLUE of an estimable linear function of the parameters $\mathbf{x}$, it follows that the BLUE $\underline{\hat{\mathbf{e}}}$ is obtained by minimising $\underline{\mathbf{e}}^T\mathbf{Q}_y^-\underline{\mathbf{e}}$ under the restriction that (C.1) holds, that is, $\mathbf{B}^T\underline{\mathbf{e}} = \underline{\mathbf{t}}$, and under the restriction $\underline{\mathbf{e}} \in R(\mathbf{Q}_y)$. The solution can be found by using Lagrange multipliers. The optimum $\underline{\mathbf{e}} = \underline{\hat{\mathbf{e}}}$ of function Ω:

$$\Omega = \underline{\mathbf{e}}^T\mathbf{Q}_y^-\underline{\mathbf{e}} - 2\mathbf{k}^T(\mathbf{B}^T\underline{\mathbf{e}} - \underline{\mathbf{t}}), \tag{C.18}$$

is determined, under the restriction $\underline{\mathbf{e}} \in R(\mathbf{Q}_y)$, and with $\mathbf{k}$ the so-called *correlates*. Differentiating Ω relative to $\underline{\mathbf{e}}$, and equating the result to zero, we get:

$$2\mathbf{Q}_y^-\underline{\mathbf{e}} - 2\mathbf{B}\mathbf{k} = \mathbf{0}. \tag{C.19}$$

Premultiplication with $\mathbf{Q}_y$ and dividing by 2 gives:

$$\mathbf{Q}_y\mathbf{Q}_y^-\underline{\mathbf{e}} = \mathbf{Q}_y\mathbf{B}\mathbf{k} \tag{C.20}$$

Because $\underline{\mathbf{e}} \in R(\mathbf{Q}_y)$, we can write (Rao and Mitra, 1971, Lemma 2.2.4):

$$\mathbf{Q}_y\mathbf{Q}_y^-\underline{\mathbf{e}} = \underline{\mathbf{e}}. \tag{C.21}$$

And thus:

$$\underline{\hat{\mathbf{e}}} = \mathbf{Q}_y\mathbf{B}\underline{\hat{\mathbf{k}}}. \tag{C.22}$$

If we premultiply with $\mathbf{B}^T$ and use the conditions $\mathbf{B}^T\underline{\hat{\mathbf{e}}} = \underline{\mathbf{t}}$, we get the *normal equations* of the model of condition equations:

$$\mathbf{B}^T\mathbf{Q}_y\mathbf{B}\,\underline{\hat{\mathbf{k}}} = \underline{\mathbf{t}}. \tag{C.23}$$

The normal equations are identical to those used in case $\mathbf{Q}_y$ is regular. Solving the normal equations gives the least-squares solution $\underline{\hat{\mathbf{k}}}$. The least-squares residuals are determined with equation (C.22).

C.4 Generalised Likelihood Ratio Test

If $\mathbf{Q}_y$ is regular, the test of $\underline{T}_q$ of equation (5.16) is a generalised likelihood ratio test (Teunissen, 2006, p. 72ff). Therefore, if $\mathbf{Q}_y$ is not regular, $\mathbf{Q}_y$ is regularised (equation (5.2), and the conditions under which equation (5.2) is valid). This is possible, if the range space of $\mathbf{A}$ contains the nullspace of $\mathbf{Q}_y$. $\underline{T}_q$ is invariant for such a regularisation (proven in appendix C.5). If we make λ of equation (5.2) smaller, $\underline{T}_q$ stays the same and its test stays a generalised likelihood ratio test, and, therefore, also in the limit of $\lambda = 0$.

C.5 Consequences of Cofactor Matrix Amplification

The estimated residuals $\underline{\hat{\mathbf{e}}}$ and their cofactor matrix $\mathbf{Q}_{\hat{e}}$ are invariant for the amplification of the cofactor matrix in equation (5.2). It can be seen by using the model of condition

equations. This model gives identical results as the model of observation equations. Insert $\overline{\mathbf{Q}}_y$ in its equation according to the model of condition equations:

$$\underline{\hat{\mathbf{e}}} = \overline{\mathbf{Q}}_y \mathbf{B} (\mathbf{B}^T \overline{\mathbf{Q}}_y \mathbf{B})^{-1} \underline{\mathbf{t}}. \tag{C.24}$$

The matrix $\overline{\mathbf{Q}}_y$ appears in this equation only in the combination $\overline{\mathbf{Q}}_y \mathbf{B}$, for which we can write, taking into account that $\mathbf{A}^T\mathbf{B} = \mathbf{0}$ (eq. (C.1)):

$$\begin{aligned} \overline{\mathbf{Q}}_y \mathbf{B} &= (\mathbf{Q}_y + \lambda \mathbf{A}\mathbf{A}^T)\mathbf{B} = \\ &= \mathbf{Q}_y \mathbf{B} + \lambda \mathbf{A} \underbrace{\mathbf{A}^T \mathbf{B}}_{=\mathbf{0}} = \\ &= \mathbf{Q}_y \mathbf{B}. \end{aligned} \tag{C.25}$$

From this, it follows that $\underline{\hat{\mathbf{e}}}$ and, therefore, also $\underline{\hat{\mathbf{y}}}$ are the same, using $\mathbf{Q}_y$ or $\overline{\mathbf{Q}}_y$. In addition, the cofactor matrix $\mathbf{Q}_{\hat{e}}$ is unchanged by the amplification. It can be seen from the equation of $\mathbf{Q}_{\hat{e}}$:

$$\mathbf{Q}_{\hat{e}} = \overline{\mathbf{Q}}_y \mathbf{B} (\mathbf{B}^T \overline{\mathbf{Q}}_y \mathbf{B})^{-1} (\overline{\mathbf{Q}}_y \mathbf{B})^T, \tag{C.26}$$

in which again $\overline{\mathbf{Q}}_y$ only appears in the combination $(\overline{\mathbf{Q}}_y \mathbf{B})$.

Also $\underline{\hat{\mathbf{r}}}$ and $\mathbf{Q}_{\hat{r}}$ are invariant for the amplification. Their equations are the same, except for the leading $\overline{\mathbf{Q}}_y$.

$\mathbf{Q}_{\hat{x}}$ and $\mathbf{Q}_{\hat{y}}$ are not unaffected by the amplification of $\mathbf{Q}_y$. To determine $\mathbf{Q}_{\hat{y}}$, the fact is used that $\mathbf{Q}_{\hat{e}}$ is unaffected by the amplification:

$$\begin{aligned} \mathbf{Q}_{\hat{y}} &= \mathbf{Q}_y - \mathbf{Q}_{\hat{e}} = \\ &= \mathbf{Q}_y + \lambda \mathbf{A}\mathbf{A}^T - \lambda \mathbf{A}\mathbf{A}^T - \mathbf{Q}_{\hat{e}} = \\ &= \overline{\mathbf{Q}}_y - \mathbf{Q}_{\hat{e}} - \lambda \mathbf{A}\mathbf{A}^T = \\ &= \overline{\mathbf{Q}}_{\hat{y}} - \lambda \mathbf{A}\mathbf{A}^T. \end{aligned} \tag{C.27}$$

The cofactor matrix $\mathbf{Q}_{\hat{x}}$ can be computed from the amplified $\overline{\mathbf{Q}}_{\hat{x}}$. To see this, $\mathbf{Q}_{\hat{y}}$ is written as:

$$\begin{aligned} \mathbf{Q}_{\hat{y}} &= \overline{\mathbf{Q}}_{\hat{y}} - \lambda \mathbf{A}\mathbf{A}^T = \\ &= \mathbf{A} \overline{\mathbf{Q}}_{\hat{x}} \mathbf{A}^T - \lambda \mathbf{A}\mathbf{A}^T = \\ &= \mathbf{A} (\overline{\mathbf{Q}}_{\hat{x}} - \lambda \mathbf{I}) \mathbf{A}^T, \end{aligned} \tag{C.28}$$

with $\mathbf{I}$ being the unit matrix. From this, it follows that $\mathbf{Q}_{\hat{x}}$ can be determined from

$$\mathbf{Q}_{\hat{x}} = \overline{\mathbf{Q}}_{\hat{x}} - \lambda \mathbf{I}. \tag{C.29}$$

C.6 Indicator matrices

In this appendix a comprehensive treatment of the indicator matrix is given for each of the methods to compute the reciprocal residuals.

Let the constraints be formulated as $\mathbf{Sx} = \mathbf{s}$ with $\mathbf{S}$ the *constraints matrix* and $\mathbf{s}$ a *vector of constants*. Using the computation method "Reduction after orthogonalisation", $\mathbf{S}$ and $\mathbf{s}$ follow from the orthogonalisation (see appendix C.2). It is clear that the constraints matrix $\mathbf{S}$ has full rank, if the constraints are independent.

From the orthogonalisation, it is also clear that the rank defect of $\mathbf{Q}_y$ is equal to the number of constraints between the parameters (number of rows of $\mathbf{S}$).

Let us take the model of condition equations. In the derivations of computation methods in this chapter, the assumption was that $\mathbf{Q}_t$ is nonsingular (see text after equation (C.3)). This means that the conditions are independent. Because this model has translated the constraints on the parameters into conditions between the observations, it means that the constraints are independent.

The next method is the method that uses the regularised cofactor matrix.
If $\mathbf{Q}_t = \mathbf{B}^T\mathbf{Q}_y\mathbf{B}$ has full rank, $\mathbf{B}^T\mathbf{Q}_y$ must have full rank. To achieve this the direct sum of the range space of $\mathbf{Q}_y$ and the null space of $\mathbf{B}^T$ must fill the whole space $\mathbb{R}^m$. The null space of $\mathbf{B}^T$ is, however, the range space of $\mathbf{A}$. From this, it follows that the null space of $\mathbf{Q}_y$ must be filled by the range space of $\mathbf{A}$. This is the restriction we met in the method with a regularised cofactor matrix for the invertibility of the amplified cofactor matrix $\overline{\mathbf{Q}}_y$. Therefore, if the constraints are independent, $\overline{\mathbf{Q}}_y$ has full rank.

This is the same restriction we find in the Pandora box method. Hence, if the constraints are independent, and if coefficient matrix $\mathbf{A}$ fulfils the restrictions mentioned after equation (5.20) for the Pandora box method, Pandora box matrix $\mathbf{P}$ has full rank.

In the method "Almost zero", no constraints are present because they have received small standard deviations and have, therefore, become stochastic observations.

D

Appendices to chapter 6

D.1 Overview: adjustment

Model (6.1) has a rank deficient (singular) covariance matrix. It may have a rank deficient coefficient matrix, and therefore, elements in the parameter vector $\mathbf{x}$ may not be estimable. Let $\mathbf{p}^T\mathbf{x}$ be any unbiasedly estimable (u.e.) linear function of $\mathbf{x}$ under model (6.1), cf. (Rao and Mitra, 1971, p. 137). We mention below three methods to compute a best linear unbiased estimator (BLU-estimator) of $\mathbf{p}^T\mathbf{x}$. This estimator is also the least-squares estimator. It yields BLU-estimators for the adjusted observations (which are u.e. functions of $\mathbf{x}$) and also for those elements of $\mathbf{x}$ that are u.e. themselves.

The first method transforms model (6.1) into a model with full rank and a regular covariance matrix (Rao and Mitra, 1971, pp. 149, 144). In a first step the observations are orthogonalised relative to the covariance matrix. The result is a subvector of observations with a scaled unit matrix as covariance matrix, and a subvector of nonstochastic observations. The latter are considered constraints!on parameters, which implies that a model with less parameters is possible. In a second step such a model is derived. The result is a model with full rank and a regular covariance matrix, and no constraints.

The second method uses the fact that the BLU-estimator in model (6.1) for $\mathbf{p}^T\mathbf{x}$ is given by a minimum $\mathbf{Q}_y$-norm solution of $\mathbf{A}^T\mathbf{f} = \mathbf{p}$ as (Rao and Mitra, 1971, equation (7.4.2)):

$$\mathbf{f}^T\underline{\mathbf{y}} = \mathbf{p}^T[(\mathbf{A}^T)^-_{n(\mathbf{Q}_y)}]^T\underline{\mathbf{y}}. \tag{D.1}$$

where $(\mathbf{A}^T)^-_{n(\mathbf{Q}_y)}$ is a minimum $\mathbf{Q}_y$-norm generalised inverse (g-inverse) of $\mathbf{A}^T$. Expressions to compute this g-inverse are given in (Rao and Mitra, 1971, p. 148) and (Rao, 1971).

The third method is closely related to the use of a minimum $\mathbf{Q}_y$-norm g-inverse. This g-inverse is, in fact, a solution of an extended system of normal equations, in which

Lagrange multipliers are used (Rao and Mitra, 1971, eq. 7.4.10[1]). So formulating this system and solving it numerically, provides the desired solution of $\mathbf{x}$.

Model (6.1) is often formulated after linearisation. In that case iteration is needed to arrive at its solution.

D.2 Overview: testing

To test a model of observation equations like model (6.1), a null hypothesis H_0 and an alternative model H_a are formulated:

$$H_0 : \underline{\mathbf{y}} = \mathbf{A}\mathbf{x} + \underline{\mathbf{e}}, \tag{D.2}$$

$$H_a : \underline{\mathbf{y}} = \mathbf{A}\mathbf{x} + \mathbf{C}\nabla + \underline{\mathbf{e}}, \tag{D.3}$$

with $\mathbf{C}$ a known $(m \times q)$-coefficient matrix, and an unknown q-vector ∇. The product $\mathbf{C}\nabla$ describes the bias in the functional model. Both hypotheses have the same stochastic model.

The null hypothesis is tested against the alternative hypothesis with test statistic $\underline{T}_q$. If $\mathbf{Q}_y$ is an invertible matrix, $\underline{T}_q$ is (Teunissen, 2006, p. 78):

$$\underline{T}_q = \frac{1}{\sigma^2}\underline{\hat{\mathbf{e}}}^T\mathbf{Q}_y^{-1}\mathbf{C}(\mathbf{C}^T\mathbf{Q}_y^{-1}\mathbf{Q}_{\hat{e}}\mathbf{Q}_y^{-1}\mathbf{C})^{-1}\mathbf{C}^T\mathbf{Q}_y^{-1}\underline{\hat{\mathbf{e}}}. \tag{D.4}$$

$\underline{T}_q$ is the square of the norm of the difference between the vector of adjusted observations under H_0 and the same vector under H_a. The norm is determined by the metric of the vector space, which is defined by $\mathbf{Q}_y^{-1}$ (Teunissen, 2006, p. 85).

Let the *reciprocal residuals* $\underline{\hat{\mathbf{r}}}$ be defined by $\underline{\hat{\mathbf{r}}} = \mathbf{Q}_y^{-1}\underline{\hat{\mathbf{e}}}$, if $\mathbf{Q}_y$ is not singular. Let their cofactor matrix be called $\mathbf{Q}_{\hat{r}}$. Equation (D.4) becomes:

$$\underline{T}_q = \frac{1}{\sigma^2}\underline{\hat{\mathbf{r}}}^T\mathbf{C}(\mathbf{C}^T\mathbf{Q}_{\hat{r}}\mathbf{C})^{-1}\mathbf{C}^T\underline{\hat{\mathbf{r}}}. \tag{D.5}$$

Deformation analysis model (6.1) has a singular cofactor matrix $\mathbf{Q}_y$. Therefore, we need another, more general, definition of the reciprocal residuals. For this, we switch to the model of condition equations. It is defined as:

$$\mathbf{B}^T\underline{\mathbf{y}} = \underline{\mathbf{t}};\ E\{\underline{\mathbf{t}}\} = \mathbf{0}, \text{with } \mathbf{B}^T\mathbf{A} = \mathbf{0}. \tag{D.6}$$

Matrix $\mathbf{B}$ is the $(m \times b)$ - coefficient matrix. $\underline{\mathbf{t}}$ is the b-vector of misclosures. $b = m-n$ is the number of conditions. The model gives the same least-squares solution as model (D.2). Its solution, using a positive semidefinite $\mathbf{Q}_y$, is given by solving the normal equations. They have the same form as when using a positive definite (regular) $\mathbf{Q}_y$. The parameters to be solved are the *correlates* $\mathbf{k}$. They are used to define more generally the reciprocal residuals $\underline{\hat{\mathbf{r}}}$:

[1] The equation contains an error: the first $\mathbf{S}$ should be $\mathbf{S}^-$.

$$\mathbf{B}^T\mathbf{Q}_y\mathbf{B}\,\hat{\underline{\mathbf{k}}} = \underline{\mathbf{t}} \qquad \textit{normal equations}, \tag{D.7}$$

$$\hat{\underline{\mathbf{r}}} = \mathbf{B}\hat{\underline{\mathbf{k}}} \qquad \textit{reciprocal residuals}, \tag{D.8}$$

$$\hat{\underline{\mathbf{e}}} = \mathbf{Q}_y\hat{\underline{\mathbf{r}}} \qquad \textit{least-squares residuals}, \tag{D.9}$$

$$\mathbf{Q}_{\hat{e}} = \mathbf{Q}_y\mathbf{Q}_{\hat{r}}\mathbf{Q}_y \qquad \textit{cofactor matrix of } \hat{\underline{\mathbf{e}}}. \tag{D.10}$$

It is assumed that $(\mathbf{B}^T\mathbf{Q}_y\mathbf{B})$ is invertible (which implies that the conditions are linearly independent), but $\mathbf{Q}_y$ can be singular. The reciprocal residuals and their cofactor matrix are:

$$\hat{\underline{\mathbf{r}}} = \mathbf{B}(\mathbf{B}^T\mathbf{Q}_y\mathbf{B})^{-1}\underline{\mathbf{t}}; \quad \mathbf{Q}_{\hat{r}} = \mathbf{B}(\mathbf{B}^T\mathbf{Q}_y\mathbf{B})^{-1}\mathbf{B}^T. \tag{D.11}$$

Test statistic $\underline{T}_q$ is computed with equation (D.5).

To switch from model (D.2) to (D.6) $\mathbf{B}$ is computed as a base matrix of the nullspace of $\mathbf{A}^T$. It means solving the equation $\mathbf{A}^T\mathbf{B} = \mathbf{0}$ for $\mathbf{B}$, e.g. by singular value decomposition.

The probability density function of $\underline{T}_q$ is a χ^2-distribution, with $E\{\underline{T}_q\} = q$. After choosing a significance level α, the critical value is computed, and it is determined, whether the critical value is exceeded by the computed value of $\underline{T}_q$. In that case the null hypothesis is rejected.

D.2.1 Overall model test

Testing the adjustment results begins with the overall model test (Velsink, 2015b). It uses the reciprocal residuals $\hat{\underline{\mathbf{r}}}$ of equation (D.11) and the test statistic of equation (D.5) with q=m–n (m is the number of observation, n the number of parameters):

$$\underline{T}_{m-n} = \frac{1}{\sigma^2}\hat{\underline{\mathbf{r}}}^T\,\mathbf{Q}_y\,\hat{\underline{\mathbf{r}}}. \tag{D.12}$$

The test statistic is χ^2-distributed and its critical value is computed with the B-method of testing, after choosing the one-dimensional test significance level (often 0.1%) and the power (often 80%). The significance level of the $(m-n)$-dimensional test is then derived.

D.2.2 w-tests

If the overall model test doesn't lead to rejection of the adjustment model (the null hypothesis), more specific tests are not needed, if the B-method of testing is used. The B-method of testing uses the principle that if a hypothetical reference bias is present, the overall model will find it with the same statistical power as more specific tests, that have a smaller degree of freedom q (Baarda, 1968b, p. 33). A more specific test can be the one-dimensional test that the alternative hypothesis has only one additional parameter ∇ that affects only one observation (conventional w-test (Baarda, 1968b, p. 15)). As reference bias in general the minimal detectable bias of the conventional w-test is taken.

If the overall model test leads to rejection of the null hypothesis, more specific alternative hypotheses are formulated by specifying matrix $\mathbf{C}$ in equation (D.3). For the conventional w-tests we have $\mathbf{C} = \left(0, \ldots, 0, 1, 0, \ldots, 0\right)^T$.

D.2.3 Tests of specific deformation hypotheses

In deformation analysis one may expect certain deformations of subsets of points, like the gradual subsidence of a few points in the course of several epochs, or the temperature induced fluctuation of other points. Such expectations may be tested by formulating alternative hypotheses, using appropriate matrices $\mathbf{C}$. Examples are given in section 6.4.

D.3 S-transformation invariance

In this section it is shown that test statistic $\underline{T}_q$ and the m.d.b.'s are S-transformation invariant. Use the following definitions:

$$\underline{\mathbf{y}}_a = \begin{pmatrix} \underline{\mathbf{y}}_s \\ \mathbf{z}_d \end{pmatrix}; \; \mathbf{A}_a = \begin{pmatrix} \mathbf{A}_s & \mathbf{0} \\ \mathbf{Z}_d & \mathbf{Z}_\nabla \end{pmatrix}; \; \underline{\mathbf{e}}_a = \begin{pmatrix} \underline{\mathbf{e}}_s \\ \mathbf{0} \end{pmatrix}; \tag{D.13}$$

$$\mathbf{G} = \begin{pmatrix} \mathbf{Z}_g & \mathbf{0} \end{pmatrix}, \tag{D.14}$$

to write model (6.1) as:

$$\begin{pmatrix} \underline{\mathbf{y}}_a \\ \mathbf{z}_g \end{pmatrix} = \begin{pmatrix} \mathbf{A}_a \\ \mathbf{G} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \underline{\mathbf{e}}_a \\ \mathbf{0} \end{pmatrix}, \tag{D.15}$$

The S-system is defined by the second row: $\mathbf{z}_g = \mathbf{G}\mathbf{x}$. It solves the rank deficiency of $\mathbf{A}_a$. If $\mathbf{N}$ is a $(n \times (n - n_G))$ base matrix of the null space of $\mathbf{G}$, we can write (Rao and Mitra, 1971, p. 24):

$$\mathbf{x} = \mathbf{G}^- \mathbf{z}_g + \mathbf{N}\boldsymbol{\lambda}. \tag{D.16}$$

$\boldsymbol{\lambda}$ is a vector of $(n\text{-}n_G)$ parameters and $\mathbf{G}^-$ any generalised inverse of $\mathbf{G}$ (i.e. it is defined by: $\mathbf{G}\,\mathbf{G}^-\,\mathbf{G} = \mathbf{G}$). We can insert equation (D.16) into model (D.15). For $\mathbf{z}_g$ we get

$$\mathbf{z}_g = \mathbf{G}\mathbf{G}^-\mathbf{z}_g + \mathbf{G}\mathbf{N}\lambda = \mathbf{z}_g + \mathbf{0}\lambda, \tag{D.17}$$

which is valid for any λ. It can be left out of the model. Model (D.15) becomes therefore:

$$\underline{\mathbf{y}}_a - \mathbf{A}_a\mathbf{G}^-\mathbf{z}_g = \mathbf{A}_a\mathbf{N}\lambda + \underline{\mathbf{e}}_a, \tag{D.18}$$

Because $\mathbf{G}$ solves the rank deficiency of $\mathbf{A}_a$, the product $\mathbf{A}_a\mathbf{N}$ has full rank. The parameters λ are solved from equation (D.18) by least-squares. Then they are inserted into (D.16) to get the estimated parameters $\underline{\hat{\mathbf{x}}}$, relative to the S-basis, defined by the equation $\mathbf{z}_g = \mathbf{G}\mathbf{x}$.

Let the range space of any matrix $\mathbf{U}$ be written as $\mathcal{R}(\mathbf{U})$. Then $\mathcal{R}(\mathbf{A}_a\mathbf{N}) = \mathcal{R}(\mathbf{A}_a)$, because $\text{rank}(\mathbf{A}_a\mathbf{N}) = \text{rank}(\mathbf{A}_a)$.

If we switch to another S-basis, we have another $\overline{\mathbf{z}}_g$, $\overline{\mathbf{G}}$ and $\overline{\mathbf{N}}$. We have, however:

$$\mathcal{R}(\mathbf{A}_a\overline{\mathbf{N}}) = \mathcal{R}(\mathbf{A}_a\mathbf{N}) = \mathcal{R}(\mathbf{A}_a). \tag{D.19}$$

Let us switch to model (D.6), the model of condition equations. We have, with $\oplus$ indicating the direct sum of two vector spaces, and $\mathbb{R}^{m_a}$ the m_a-dimensional Euclidian space, where m_a is the number of elements in $\underline{\mathbf{y}}_a$:

$$\mathcal{R}(\mathbf{B}) \oplus \mathcal{R}(\mathbf{A}_a) = \mathbb{R}^{m_a}. \tag{D.20}$$

Because of (D.19) we have

$$\mathbf{B}^T\mathbf{A}_a\mathbf{N} = \mathbf{B}^T\mathbf{A}_a\overline{\mathbf{N}} = \mathbf{0}; \tag{D.21}$$

$$\mathcal{R}(\mathbf{B}) \oplus \mathcal{R}(\mathbf{A}_a\mathbf{N}) = \mathcal{R}(\mathbf{B}) \oplus \mathcal{R}(\mathbf{A}_a\overline{\mathbf{N}}) = \mathbb{R}^{m_a}. \tag{D.22}$$

So, if we use equation (D.18) using $\mathbf{N}$, and premultiply it with $\mathbf{B}^T$; or we use the same equation (D.18), but with another $\overline{\mathbf{N}}$, and premultiply it again with $\mathbf{B}^T$, we get the same model:

$$\mathbf{B}^T\underline{\mathbf{y}}_a = \mathbf{B}^T\underline{\mathbf{e}}_a, \;\; E\{\mathbf{B}^T\underline{\mathbf{e}}_a\} = \mathbf{0}. \tag{D.23}$$

This is a complete model to use for adjustment, irrespective of the choise of $\mathbf{N}$, i.e. irrespective of the S-system. Therefore $\hat{\underline{\mathbf{r}}}_a$ and $\mathbf{Q}_{\hat{r}_a}$ are S-transformation invariant. Then also test statistic $\underline{T}_q$ and the m.d.b.'s are S-transformation invariant, as long as matrix $\mathbf{C}$ contains zeros for the rows that pertain to S-basis elements (i.e. as long as the S-basis elements are not tested, which would be, by the way, meaningless). □

Curriculum vitae

Personal data

Name:	Hiddo Velsink
Date of Birth:	26 September 1958
Place of Birth:	Groningen, Netherlands
Place of Residence:	De Bilt, Netherlands
Marital Status:	Married to Marieke de Haas, two sons

Fields of Expertise

After a career at the Dutch Cadastre Hiddo Velsink lectured geodetic topics at Delft University of Technology and HU University of Applied Sciences Utrecht. In these functions he wrote ample lecture notes for the courses in geodesy. In 2001 he changed his activities to the management of education at HU University of Applied Sciences Utrecht and acquired broad experience in managing several courses up to the management of the Institute of Built Environment. After ten years of management, he decided to return to geodetic content and to start this PhD-research. Besides the PhD-research, he performs several tasks as programme manager and lecturer. In this connection, he gave several courses at the INES Institute of Applied Sciences, Ruhengeri, Rwanda.

Educational record

1970 – 1976: Secondary education (Gymnasium β)

1976 – 1983: Delft University of Technology, MSc (in Dutch: ingenieur (ir.))

Work experience

1983-1991: Dutch Cadastre (Head Office, Apeldoorn).

1983-1987: Geodetic Engineer; Main topics: cadastre and land registration, land surveying, cadastral cartography, land consolidation.

1987-1989: Research Fellow; Main topics: cadastre and land registration, geo-information infrastructure, design and implementation of Land Information Systems, efficient methods of land surveying and cartography, ICT-projects on land surveying and mathematical geodesy.

1989-1991: Head of the Department for Research and Development; Main topics: coordination of all geodetic research and development activities.

1991-1996: Lecturer at the Faculty of Geodetic Engineering, Delft University of Technology; Main topic: mathematical geodesy. Detachment at the HU University of Applied Sciences Utrecht; Main topics: see next item.

1996-2001: Lecturer at the Department of Geodesy/Geo-informatics of the HU University of Applied Sciences Utrecht; Main topics: mathematical geodesy, geodetic measurement techniques, geodetic positioning techniques, GPS, GIS, project management.

1993-2001: Course development and lecturing on cadastral and topographical surveying and mathematical geodesy and on GPS at the Institute for Post Tertiary Education of the Faculty of Science and Technology of HU University of Applied Sciences Utrecht.

2001-2002: Head of Department of Civil Engineering and Lecturer at the Department of Geodesy/Geo-informatics, HU University of Applied Sciences Utrecht.

2002-2005 Head of Departments of Civil Engineering and Geodesy/Geo-informatics.

2005-2009 Head of Departments of Construction Engineering and Construction Management.

2009-2011 Head of Institute of Built Environment.

2011-date Programme Manager & Research Fellow, Research Group Digital Smart Services, HU University of Applied Sciences Utrecht.

List of publications

For publications in the field of geodesy, see the References under 'Velsink'.
In addition to the publications that are referenced in this study, publications were produced:

- as student of Delft University of Technology (then: TH Delft) during internship at the German Geodetic Research Institute, Munich, Germany (subject: geodetic deformation analysis) (Boedecker and Velsink, 1982; Velsink, 1982)
- as student of Delft University of Technology (master's thesis; subject: three-dimensional geodesy) (Velsink, 1983)
- as employee of Dutch Cadastre (Velsink, 1987, 1990)
- as lecturer at HU University of Applied Sciences Utrecht (Velsink, 1991, 1997, 1998b, 1999)

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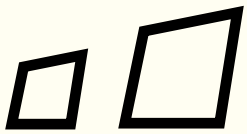

THE ELEMENTS OF DEFORMATION ANALYSIS

Blending Geodetic Observations and Deformation Hypotheses

This study provides a method to analyse time series of point positions. These points represent objects on the earth in consecutive moments in time. The point positions are determined by geodetic techniques, in order to describe land subsidence, land slides, continental drift, and deformations of buildings and civil engineering structures.

The method enables the incorporation of hypotheses about the movements and deformations in the analysis, thus combining geodetic measurements and deformation hypotheses.

The analysis is concerned with changes in form and size, and insensitive for changes in absolute position and orientation of the point fields. For each point a minimal detectable deformation can be computed. An important advantage is that reference points outside the measured objects do not have to be stable points.

Formulating standards by using the method supplies unambiguous key values for improving communication about geodetic deformation analysis. Specialists and stakeholders in the area of geodetic deformation analysis, and the interested or affected general public, will benefit from this novel method, as key performance indicators can provide a concise overview of deformations and movements.