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ARBITRARY HIGH ORDER ACCURATE TIME INTEGRATION SCHEMES FOR LINEAR PROBLEMS

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Abstract. *Numerical schemes for wave propagation over long distances need good wave propagation properties with low dispersion and low dissipation errors. Suitable numerical methods are methods with high order of accuracy in space and time. For space discretization on structured grids, high order finite difference schemes are efficient, and, if a complicated computational domain requires an unstructured grid, discontinuous Galerkin methods are recently employed with success.*

The time integration is often performed by a Runge-Kutta scheme. These schemes need for the order of accuracy $\mathcal{O} > 4$ more than $\mathcal{O}$ stages, which reduces performance concerning CPU-time as well as storage requirements, because the numerical solution of more than one stage has to be stored. However, it is interesting to use schemes of accuracy order higher than 4, especially to capture wave propagation over long distances or if very accurate computations are needed. In this paper we consider a time integration approach for linear wave problems based on a Taylor expansion. Here we construct and analyze schemes of arbitrary high order accuracy in space and time using this time integration technique within the finite difference as well as the discontinuous Galerkin framework.

We present a stability analysis as well as performance comparisons with schemes relying on other time integration methods. A modification for the DG schemes is presented that accentuates the computational performance. Numerical experiments are realized for the system of linearized Euler Equations, but the formulation allows an application of the proposed schemes to any linear hyperbolic system.

1 INTRODUCTION

For the numerical simulation of wave propagation over long distances highly accurate numerical methods with minimal dissipation and dispersion have to be used. This was recognized in the field of computational aeroacoustics (CAA) as well as in computational

electromagnetics or in geophysics. Several approaches have been proposed that include explicit finite difference (FD) schemes, compact schemes and pseudo-spectral schemes, see e.g., the recent overview in.[8] One of the frequently used concepts in CAA is that of Tam and Webb proposed in.[9] They constructed a simple finite difference scheme of high order accuracy which is optimized not only with respect to the order of accuracy, but with respect to preserve wave propagation properties on coarse grids, too. They introduced a 7 point stencil for the difference formulae approximating the first order derivatives as in the case of a sixth order accurate finite difference approximation in space. They determined two of the coefficients in order to improve the wave propagation properties. Their criterion was to minimize the dispersion error, i.e., the difference between the effective and the exact wave number. Hence, this class of schemes are called the dispersion relation preserving schemes (DRP).

The DRP schemes are widely used in CAA. The original DRP scheme with the seven point stencil is fourth order accurate in space and is usually combined with a fourth order Runge-Kutta time discretization. Further development has been performed by several authors. Hu et al.[5] improved the Runge-Kutta time approximation and proposed low-dissipation and low-dispersion Runge-Kutta schemes. Bogey and Bailly[2] extended the spatial stencil and achieved further improvements in the wave propagation properties. They also presented Runge-Kutta schemes that are optimized with respect to these properties. Hu et al.[5] also described a time integration technique as considered in this paper and call it a low-storage Runge-Kutta scheme, despite the fact that it has important differences to usual Runge Kutta schemes. We will address this in the following. A DG-scheme for non-linear problems using a similar time integration technique was described by Shu and Dumbser in.[6]

The principal disadvantage of finite difference schemes is that it is difficult or even impossible to mesh complex geometries using smooth, structured grids. For this reason, there is an interest in numerical schemes that are able to simulate linear wave propagation problems on unstructured grids. Very promising results were obtained using discontinuous Galerkin (DG) space discretization often also combined with Runge-Kutta time integration. In,[4] Dumbser presented promising results using the DG space discretization in combination with the ADER time discretization.

In this paper we consider the alternative time discretization to the usual Runge-Kutta time discretization for the class of dispersion relation preserving schemes proposed by Tam in[7] as well as for DG schemes. We show how construct those schemes to obtain arbitrary order in space and time and show their stability.

2 DESCRIPTION OF THE TIME INTEGRATION TECHNIQUE

2.1 The Governing Equations

A general system of linear hyperbolic partial differential equations in two dimensions is given by

$$\underline{U}_t = -\underline{\underline{A}}\,\underline{U}_x - \underline{\underline{B}}\,\underline{U}_y + \underline{F}, \tag{1}$$

where $\underline{U} = \underline{U}(x, y, t)$ denotes the vector of the physical variables. The matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ as well as the right hand side $\underline{F} = \underline{F}(x, y, t)$ do not depend on $\underline{U}$.

2.2 The Discretization in Time

The numerical solution of equation (1) is represented in form of degrees of freedom (DOFs) of a space discretization. The DOFs are approximate values of the exact solution at some discrete points for a FD scheme and the coefficients of local polynomials for a DG scheme. The DOFs are a functions of the time t, and we use the abbreviation $\hat{\underline{U}}^n = \hat{\underline{U}}(t^n)$, with n being the index of the time step. In the following, expressions of the form $\frac{\partial}{\partial(\cdot)}\hat{\underline{U}}$ have to be understood as DOFs representing derivatives of a numerical solution represented by $\hat{\underline{U}}$.

Instead of using the ordinary differential (ODE) approach, e.g. a Runge-Kutta scheme, we base the time discretization on the Taylor expansion (2)

$$\hat{\underline{U}}(t^n + \Delta t) = \hat{\underline{U}}(t^n) + \sum_{k=1}^{\infty} \frac{\Delta t^k}{k!} \frac{\partial^k}{\partial t^k} \hat{\underline{U}}(t^n). \tag{2}$$

The expansion (2) about time t^n is truncated at the desired accuracy order $k = \mathcal{O}$. The time evolution of the discrete solution vector $\hat{\underline{U}}$ may then be expressed as

$$\hat{\underline{U}}^{n+1} = \hat{\underline{U}}^n + \sum_{k=1}^{\mathcal{O}} \frac{\Delta t^k}{k!} \frac{\partial^k}{\partial t^k} \hat{\underline{U}}^n, \tag{3}$$

where $\Delta t := t^{n+1} - t^n$ is the time step between the discrete time levels t^n and t^{n+1}. In this expansion all time derivatives $\frac{\partial^k}{\partial t^k}\hat{\underline{U}}^n$ have, for an explicit scheme, to be approximated in an appropriate way from the given discrete solution $\hat{\underline{U}}^n$.

In the case of the considered linear systems, we have Jacobian matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ which do not depend on $\underline{U}$ (and not on $\hat{\underline{U}}$). The application of the differential operator $\frac{\partial^k}{\partial t^k}$ on equation (1) directly yields:

$$\left(\frac{\partial^k}{\partial t^k}\underline{U}\right)_t = -\underline{\underline{A}}\left(\frac{\partial^k}{\partial t^k}\underline{U}\right)_x - \underline{\underline{B}}\left(\frac{\partial^k}{\partial t^k}\underline{U}\right)_y + \frac{\partial^k}{\partial t^k}\underline{F}. \tag{4}$$

Hence, the differential equation for $\underline{U}$ also holds for any of its time derivatives. On a discrete level, we can therefore write

$$\left(\frac{\partial^k}{\partial t^k}\hat{\underline{U}}\right)_t = -\underline{\underline{A}}\left(\frac{\partial^k}{\partial t^k}\hat{\underline{U}}\right)_x - \underline{\underline{B}}\left(\frac{\partial^k}{\partial t^k}\hat{\underline{U}}\right)_y + \frac{\partial^k}{\partial t^k}\underline{S}, \tag{5}$$

where $\underline{S}$ is the discretization of the source terms.

We define a discrete operator $\underline{\underline{\Theta}}$ such that it yields DOFs representing the right hand side of the equation (1) and hence the approximation of the time derivative when applied on $\underline{\hat{U}}$:

$$\frac{\partial}{\partial t}\underline{\hat{U}} = \underline{\underline{\Theta}}\underline{\hat{U}}. \tag{6}$$

Consequently, $\frac{\partial^k}{\partial t^k}\underline{\hat{U}}$ can be obtained from $\frac{\partial^{k-1}}{\partial t^{k-1}}\underline{\hat{U}}$ using equation (5). Thus, we can write

$$\frac{\partial^k}{\partial t^k}\underline{\hat{U}} = \underline{\underline{\Theta}}^k\underline{\hat{U}}. \tag{7}$$

Using equation (7) in equation (3), we can write

$$\underline{\hat{U}}^{n+1} = \underline{\hat{U}}^n + \sum_{k=1}^{\mathcal{O}} \frac{\Delta t^k}{k!}\underline{\underline{\Theta}}^k\underline{\hat{U}}. \tag{8}$$

So in order to evolve the numerical solution $\underline{\hat{U}}$ of a linear problem with arbitrary high order of accuracy in time, the only thing we still have to do is to define the discrete space operator $\underline{\underline{\Theta}}$ which has to perform a high order accurate differentiation, if applied on $\underline{\hat{U}}$.

For linear problems, the time integration scheme of order 4 has identical properties as a standard 4-stage Runge-Kutta time integration, that is, the modified differential equations which are solved by the 2 schemes are identical.

We note that we can even reduce the order of the spatial approximation in every step. For a scheme of designed order $\mathcal{O}$, only the first time derivative of $\underline{\hat{U}}$ has to be of accuracy order $\mathcal{O}$, the second order derivative $\frac{\partial^2}{\partial t^2}\underline{\hat{U}}$ may be of the order of accuracy $\mathcal{O}-1$ due to the fact that the time derivative is multiplied by Δt^2. So we could reduce the spatial accuracy of the operator $\underline{\underline{\Theta}}$ step for step in the Taylor expansion.

3 ARBITRARY HIGH ORDER FINITE DIFFERENCE SCHEMES

Some general definitions concerning the finite difference approach are given in this section. We restrict ourselves to the two-dimensional case, but the extension to three space dimensions is straightforward. For simplicity we consider in this section equally spaced Cartesian grids in a rectangular domain only. Extension to curvilinear structured grids can easily be achieved when a smooth mapping from the physical to a Cartesian logical grid is given.

The computational domain $[a,b]\times[c,d]$ is covered with grid points

$$P_{i,j} \equiv (x_i, y_i), \quad \text{with } 1 \le i \le N_x,\ 1 \le j \le N_y, \tag{9}$$

where

$$a + \frac{\Delta x}{2} = x_1 < \ldots < x_N = b - \frac{\Delta x}{2} \tag{10}$$

and

$$c + \frac{\Delta y}{2} = y_1 < ... < y_N = d - \frac{\Delta y}{2}. \tag{11}$$

Furthermore, $\Delta x = x_i - x_{i-1} = constant$ and $\Delta y = y_j - y_{j-1} = constant$ denote the grid sizes.

The degrees of freedom in a FD scheme are point values $\underline{\hat{U}}_{i,j}(t)$ which approximate the exact solution $\underline{U}$ at point $P_{i,j}$ at discrete time levels.

An algorithm that yields a high order approximation of first the time derivative $\frac{\partial}{\partial t}\underline{\hat{U}}$ from a given $\underline{\hat{U}}$ is:

1. Approximate the first order space derivatives $\frac{\partial}{\partial x}\underline{\hat{U}}^n_{i,j}$ and $\frac{\partial}{\partial y}\underline{\hat{U}}^n_{i,j}$. This is done with arbitrary high order of accuracy using sufficiently large stencils into every space direction. Several authors (e.g.[7] or[2]) propose finite difference formulas that are not only optimized with respect to the order of consistency, but also with respect to wave propagation properties. Widely used is for example the DRP-scheme of Tam.[7]

2. Using equation (5) we can write:

$$\underline{\hat{U}}_t = -\underline{\underline{A}}\underline{\hat{U}}_x - \underline{\underline{B}}\underline{\hat{U}}_y + \underline{S}, \tag{12}$$

   where $\underline{S}$ is the discretization of the source terms. So the time derivatives are easily computed once the space derivatives have been approximated by a finite difference formula.

Formally, the steps 1 and 2 describe the operator $\underline{\underline{\Theta}}$, such that

$$\frac{\partial}{\partial t}\underline{\hat{U}} = \underline{\underline{\Theta}}\underline{\hat{U}}, \tag{13}$$

and we can immediately apply the previously proposed time evolution technique.

3.1 High order accurate approximation of space derivatives

An important part of the schemes described in the previous section is the computation of the space derivatives of a field $\underline{\hat{U}}$ into the different space directions.

The simplest way to derive a finite difference formula for the approximation of a space derivative is to use an interpolating polynomial of an order $\mathcal{O}$, and differentiate it with respect to the space coordinate. The interpolation necessitates $\mathcal{O}+1$ stencil points for symmetric stencils of even order $\mathcal{O}$. In the DRP-scheme a finite difference formula is used which minimizes the dispersion error, while it keeps a certain order of accuracy. We tested our time integration scheme with such dispersion-optimized finite difference formulas of arbitrary high order that have one minimized integral wavenumber error over one range of wavenumbers, see e.g.[7] for differentiation of such stencils. These stencils are constituted of $\mathcal{O}+3$ points, and the used coefficient sets are presented for some orders in Table 1.

Table 1: Wave number space optimized high order 1-D coefficient sets. The value η is the upper limit of the wave number range over which the integral error is minimized. The coefficients are antisymmetric to the interpolation point, so we present only one-sided coefficients.

order	η	Displacement	coefficient
4	1.1	+1	.7708823805182131
		+2	-.1667059044145704
		+3	.2084314277030928e-1
6	1.3	+1	.8204601252356066
		+2	-.2204601252356066
		+3	.4686386319621235e-1
		+4	-.5032866088257614e-2
8	1.5	+1	-.8543189294654197
		+2	.2620787765319082
		+3	-.7301454989443648e-1
		+4	.1391789132674661e-1
		+5	-.1293307844414755e-2
10	1.7	+1	.8795439639390204
		+2	-.2958585263523470
		+3	.9803266836188213e-1
		+4	-.2532417845586396e-1
		+5	.4294456142566486e-2
		+6	-.3500805348915692e-3
12	1.8	+1	.8958592609066766
		+2	-.3194790145422356
		+3	.1180814831288988
		+4	-.3662873256081292e-1
		+5	.8463524379799489e-2
		+6	-.1263351509433068e-2
		+7	.9024802742149065e-4

3.2 Stability Analysis of the Finite Difference schemes

For stability analysis, we consider the scalar advection equation

$$u_t + a u_x = 0 \tag{14}$$

We expressing the discrete solution in form of point values $\underline{\hat{U}}$ and equation (8) defines our time stepping scheme. Note that the finite difference formulae for the approximation of the first order space derivatives are skew symmetric and therefore, if periodic boundary conditions are assumed, the matrix operator $\underline{\underline{\Theta}}$ is skew symmetric, that is, $\underline{\underline{\Theta}}^T = -\underline{\underline{\Theta}}$. So all eigenvalues of the matrix $\underline{\underline{\Theta}}$ are purely imaginary.

Assuming that the matrix $\underline{\underline{\Theta}}$ has full rank, any vector $\underline{\hat{U}}$ can be expressed as a linear combination of the eigenvectors $\underline{V_j}$ of $\underline{\underline{\Theta}}$. The scheme is therefore stable, if all the eigenvectors of $\underline{\underline{\Theta}}$ do not increase in the L2-norm, when the scheme is applied. For one eigenvector and the corresponding eigenvalue we can write equation (**??**) as

$$\underline{V_j}^{\star} = \sum_{k=0}^{\mathcal{O}} \frac{\Delta t^k}{k!} \lambda^k \underline{V_j}, \tag{15}$$

where $\underline{V_j}^{\star}$ denotes the eigenvector modified in one step of the scheme.

By replacing $\lambda \Delta t$ by α we get

$$\underline{V_j}^{\star} = \sum_{k=0}^{\mathcal{O}} \frac{\alpha^k}{k!} \underline{V_j}. \tag{16}$$

For a stable scheme, the expression

$$\left| \sum_{k=0}^{\mathcal{O}} \frac{\alpha^k}{k!} \right| \leq 1.0 \tag{17}$$

should hold.

In Figure 4.3, the stability regions with respect to α are shown for the orders 2, 4, 6, and 8 of the scheme. The stability region of the second order time integration scheme doesn't contain the imaginary axis, so the second order scheme would be unstable as $\underline{\underline{\Theta}}$ has imaginary eigenvalues. For the same reasons, the schemes of order 6, 10, 14, and so on are unstable. The stability region of the fourth order scheme contains a part of the imaginary axis, so we could expect a conditionally stable scheme for sufficiently small modulus of α, that is, for sufficiently small Δt. The stability regions of schemes of order 4, 8, 12, and so on, contain a part of the imaginary axis.

We carried out numerically the von Neumann stability analysis (for the system of linearized Euler Equations in two and three space dimensions) and found indeed that the schemes of order 2, 6, 10, and so on, are unstable, whereas the schemes of order 4, 8, 12, and so on, are stable under the usual CFL-condition (CFL=1.0).

3.3 Numerical Studies in Two and Three Dimensions

In order to get an idea of the performance of the high-order time integration DRP-schemes applied to the linearized Euler equations (LEE), we performed numerical computations with the convective transport of a density Gauss pulse in a domain with periodic boundaries. We chose the domain as $[0, 100]^3$ in three space dimensions and as the initial condition a pulse in density with $\rho_{max} = 1.0$, the halfwidth 5.0, and the center in the domain center. The other state variables were set to zero at $t = 0$. This Gauss pulse is transported diagonally across the domain by setting the velocity of the reference state of

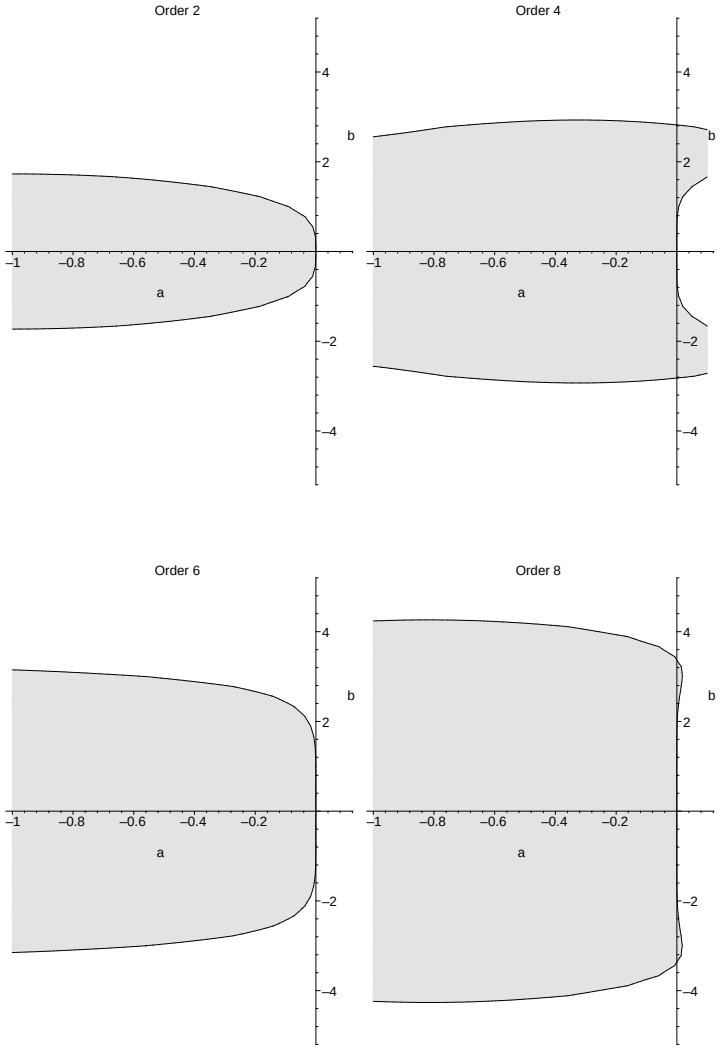
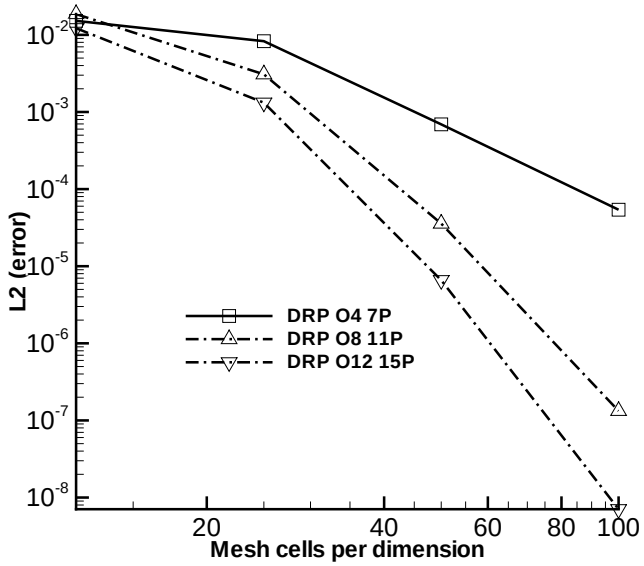


Figure 1: Stability regions according to equation (17) for $\alpha = a + bi$

Figure 2: L2 error over mesh size for 3-D Gausspuls convection



the LEE to $(v_x, v_y, v_z) = (1, 1, 1)$. At $t = 100$, the pulse should be transported to exact the same position as it was at $t = 0$, so we know the exact solution and can compute a norm of the error.

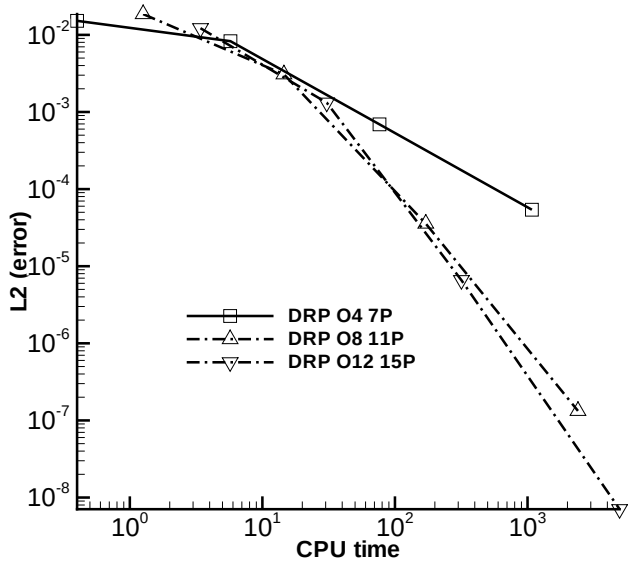
The described problem is effectively only a scalar transport, but in order to get an idea of the performance of the schemes in acoustics, the problem was computed using the complete linearized Euler equations. Figure 2 shows the L2-norm of the error over the mesh size for the schemes of order 4, 8, and 12. The results of the 4th order scheme is identical to those of standard Runge-Kutta $\mathcal{O}4$ time integration scheme. For space interpolation, we used dispersion optimized stencils as given in Figure 1. One can see that the higher order schemes are significantly more efficient, if highly accurate results are desired.

In terms of storage requirement note that unlike in standard Runge-Kutta time discretization it is not necessary to store information for all stages: it is sufficient to use one array for the successive time differentiation (on which the operator $\underline{\underline{\Theta}}$ is applied successively) and a second array for solution updating.

4 ARBITRARY HIGH ORDER DISCONTINUOUS GALERKIN SCHEMES

Again, we begin with some general definitions concerning the discretization in space. The computational domain is divided in conforming triangular elements Q_i being addressed by a unique index i. The numerical solution $\underline{U}$ of (1) is approximated inside each Q_i by a linear combination of time independent polynomial basis functions $\phi_l(x, y)$ of

Figure 3: L2 error over CPU-time for 3-D Gausspuls convection



degree N with support Q_i and with time dependent degrees of freedom $\hat{U}_{i,l}(t)$:

$$\underline{U}_i(t) = \sum_{l=0}^{N} \underline{\hat{U}}_{i,l}(t)\phi_l(x,y). \tag{18}$$

We use orthogonal polynomial basis functions ϕ_l on each Q_i.

4.1 Construction of the Time Differentiation Operator

Equation (1) is multiplied by a test function ϕ_m and is integrated over a triangle Q_i:

$$\int_{Q_i} \phi_m \frac{\partial}{\partial t}\underline{U} dV + \int_{Q_i} \phi_m (\underline{\underline{A}}\frac{\partial}{\partial x}\underline{U} + \underline{\underline{B}}\frac{\partial}{\partial y}\underline{U}) dV = 0. \tag{19}$$

Integration by parts yields

$$\int_{Q_i} \phi_m \frac{\partial}{\partial t}\underline{U} dV + \int_{\partial Q_i} \phi_m \underline{G} dS - \int_{Q_i} \frac{\partial \phi_m}{\partial x}\underline{\underline{A}}\underline{U} + \frac{\partial \phi_m}{\partial y}\underline{\underline{B}}\underline{U}) dV = 0, \tag{20}$$

where a numerical flux $\underline{G}$ has been introduced as integrand in the surface integral, since $\underline{U}$ may be discontinuous at an element boundary. We suppose rotational invariance of system (1), so the flux can be written in a coordinate system which is aligned with the outward pointing unit normal vector $\underline{n} = (n_x, n_y)$ on the boundary. The transformation

of the vector $\underline{U}$ from the global system to the vector $\underline{U}^\star$ in an edge-aligned coordinate system is given by

$$\underline{U} = \underline{\underline{T}}\underline{U}^\star. \tag{21}$$

For the linearized two-dimensional Euler equations the transformation matrix is

$$\underline{\underline{T}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & n_x & -n_y & 0 \\ 0 & n_y & n_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{22}$$

We use Godunov's exact flux as a numerical flux between two triangles Q_i and Q_j. For the global $x-y$ system, we finally obtain

$$\underline{G} = \frac{1}{2}\underline{\underline{T}}(\underline{\underline{A}}^\star + |\underline{\underline{A}}^\star|)(\underline{\underline{T}})^{-1}\underline{U}_i \tag{23}$$

$$\frac{1}{2}\underline{\underline{T}}(\underline{\underline{A}}^\star - |\underline{\underline{A}}^\star|)(\underline{\underline{T}}))^{-1}\underline{U}_j \tag{24}$$

in the linear case, where $\underline{U}_i$ and $\underline{U}_j$ are the boundary extrapolated values of the numerical solution from element Q_i and element Q_j which is adjacent to the considered side. $\underline{\underline{A}}^\star$ means the evaluation of matrix $\underline{\underline{A}}$ in the local edge-aligned system.

Defining the mass matrix

$$\underline{\underline{M}}[m,l] = \int_{Q_i} \phi_l \phi_m dQ, \quad l,m = 1,..,N+1, \tag{25}$$

we can rewrite equation(20) to

$$\frac{\partial}{\partial t}\hat{\underline{U}}_i = \underline{\underline{M}}^{-1}(-\int_{\partial Q_i} \phi_m \underline{G} dS + \int_{Q_i} \frac{\partial \phi_m}{\partial x}\underline{\underline{A}}\underline{U} + \frac{\partial \phi_m}{\partial y}\underline{\underline{B}}\underline{U})dV). \tag{26}$$

Expanding $\underline{U}$ in terms of DOFs (18) and comparing the resulting expression with equation (7), the operator $\underline{\underline{\Theta}}$ and therefore the numerical scheme are readily defined.

4.2 Reduction of Operations for the DG-scheme

So far, we use the same operator $\underline{\underline{\Theta}}$ in every substep of the scheme, that is, for any time derivative. As already mentioned in section 2.2, it would be sufficient, in order to get globally a scheme of order $\mathcal{O}$, to compute only the first time derivative of $\hat{\underline{U}}$ with the order of accuracy $\mathcal{O}$ and then to reduce successively the order with increasing order of the time derivative: the second time derivative is sufficiently accurate, when computed with order $\mathcal{O}-1$, and so on.

This order reduction can be used to increase the efficiency of the scheme, if we choose in our DG framework a hierarchically ordered basis. Then we simply can take the reduction

into account by ignoring all the terms representing the higher order test- or basis functions, which are not needed for the computation of the high order time derivatives.

To illustrate the principle, we look at one part of the operator $\underline{\underline{\Theta}}$, say a local volume integral, which can be written as product of a matrix $\underline{\underline{K}}$ with the local vector of degrees of freedom (DOF) $\underline{\hat{U}}$. We assume to have a hierarchical ordered basis. For a fourth order scheme in 2D, we have 10 DOFs per element, so $\underline{\underline{K}}$ is a 10×10 matrix. For the first time derivative we have to use the whole matrix $\underline{\underline{K}}$, whereas it is sufficient for the second time derivative to be of order $\mathcal{O}-1$. Hence, we can ignore the lines of $\underline{\underline{K}}$ corresponding to fourth order DOFs as result. So this operation would be a matrix multiplication by a 6×10 matrix (a complete polynomial basis of order 3 consists of 6 basis functions in 2D). Similarly, in the next step, we would have a matrix multiplication by a 3×6 matrix.

This method reduces significantly the number of necessary operations of the DG operator. For a fourth order scheme we had 4 stages with 10×10 matrix operation, so 400 operations. Now we have $100 + 60 + 18 + 3 = 181$ operations. Hence, we reduced the number of operations by over 50%.

As described in the next subsection, the stability limit is slightly reduced by the reduction of operations as presented, but the error norms obtained by numerical experiments got even better. We call the original scheme Taylor-DG scheme and the scheme with reduced operations Taylor-DG-R scheme.

4.3 Stability Analysis

In this section we will give the maximal CFL numbers for the scheme applied to the one-dimensional linear scalar advection equation

$$u_t + a u_x = 0 \tag{27}$$

with $CFL = \frac{a\Delta t}{\Delta x}$. These stability limits hold, from our experience, for the system of linearized Euler Equations as well.

The stability of a periodic problem with a computational domain of 3 elements is analyzed. As the problem (27) is linear, one can construct a Matrix $\underline{\underline{W}}$ such that

$$\hat{U}^{n+1} = \underline{\underline{W}} \hat{U}^n, \tag{28}$$

where $\underline{\underline{W}} = \underline{\underline{W}}(CFL)$.

Assuming that the matrix $\underline{\underline{W}}$ has full rank, any vector $\underline{\hat{U}}$ can be expressed as a linear combination of the eigenvectors $\underline{V}_j$ of $\underline{\underline{W}}$. The scheme is therefore stable if all eigenvectors of $\underline{\underline{W}}$ do not increase in L2-norm when the scheme is applied. For one eigenvector $\underline{V}_j$ and the corresponding eigenvalue λ_j we can write Equation (28) to

$$\underline{V}_j^{n+1} = \lambda_j \underline{V}_j^n, \tag{29}$$

so the scheme is stable if the modulus of each eigenvalue λ_j of $\underline{\underline{W}}$ is lower or equal to 1.

scheme order	Taylor-DG	Taylor-DG-R	ADER-DG
1	1.00	1.00	1.00
2	0.333	0.333	0.3333
3	0.200	0.1875	0.171
4	0.145	0.115	0.104
5	0.118	0.10	0.076
6	0.095	0.075	0.050

Table 2: Stability limits of different DG schemes

Using a computer algebra system it is easy to compute, for any order, the largest modulus of the eigenvalue spectrum of $\underline{\underline{W}}$ for a given CFL. By varying CFL we can find the stability region. In Table 4.3, the maximal CFL-condition is given for schemes with and without operation reduction as described in the previous section.

In,[3] another arbitrary high order DG scheme for linear problems is described, the ADER-DG scheme. For comparison, we present the stability limit of the ADER-DG scheme as well.

4.4 Numerical Results

We present in this section results obtained for a numerical experiment with the same 2D problem than described in section 3.3.

The mesh consists now of regular triangles. In Table 3, error norms and CPU-times are presented for three different DG schemes. All computations have been done with the maximal possible CFL-number of the respective scheme and order. Especially for higher accuracy order, the Taylor time integration schemes with reduced operation number seem to be interesting in terms of performance.

Atkins and Shu[1] proposed a quadrature-free implementation for DG combined with standard Runge-Kutta time integration. As the scheme of order 4 (without operation reduction) has identical properties as a standard Runge-Kutta $\mathcal{O}4$ time integration (in terms of error norms and CPU cost), and as our implementations are quadrature-free, the scheme Taylor-$\mathcal{O}4$ corresponds to a fourth-order scheme as proposed by Atkins and Shu.[1]

5 Conclusions

We proposed a time integration technique of arbitrary high order which is especially efficient for the solution of linear hyperbolic systems. We applied this approach based on a Dimension-by-Dimension space discretization to finite difference schemes on structured grids as well as to discontinuous Galerkin schemes on unstructured grids. For the resulting finite difference schemes, the stability analysis showed that schemes of order 4, 8, 12, and so on are stable under the usual CFL-condition.

Compared to standard Runge-Kutta time discretizations, it is much easier to obtain arbitrary high order. For higher order, no additional storage amount is required. If very

Nb cells	**Nb DOF**	$\|e\|_{L_2}$	$\mathcal{O}_{L_2}$	**CPU [s]**	$\|e\|_{L_2}$	$\mathcal{O}_{L_2}$	**CPU [s]**	$\|e\|_{L_2}$	$\mathcal{O}_{L_2}$	**CPU [s]**
		$\mathcal{O}2$ **Taylor**			$\mathcal{O}2$ **Taylor-R**			$\mathcal{O}2$ **ADER**		
100	**300**	6,07E+00		**0,48**	5,69E+00		**0,44**	5,88E+00		**0,32**
400	**1200**	3,24E+00	**0,9**	**3,82**	2,85E+00	**1,0**	**3,54**	3,13E+00	**0,9**	**2,46**
1600	**4800**	1,16E+00	**1,5**	**31,33**	9,40E-01	**1,6**	**29,25**	1,13E+00	**1,5**	**19, 33**
6400	**19200**	3,00E-01	**2,0**	**252,60**	2,40E-01	**2,0**	**235,00**	3,10E-01	**1,9**	**160,34**
		$\mathcal{O}3$ **Taylor**			$\mathcal{O}3$ **Taylor-R**			$\mathcal{O}3$ **ADER**		
100	**600**	3,01E+00		**1,42**	2,77E+00		**1,23**	3,15E+00		**1,03**
400	**2400**	7,20E-01	**2,1**	**11,54**	6,00E-01	**2,2**	**10,20**	8,20E-01	**1,9**	**8,3 2**
1600	**9600**	1,00E-01	**2,8**	**95,90**	8,10E-02	**2,9**	**87,60**	1,16E-01	**2,8**	**67, 26**
6400	**38400**	1,58E-02	**2,7**	**769,10**	1,20E-02	**2,8**	**691,00**	1,59E-02	**2,9**	**546,80**
		$\mathcal{O}4$ **Taylor**			$\mathcal{O}4$ **Taylor-R**			$\mathcal{O}4$ **ADER**		
100	**1000**	1,29E+00		**4,42**	1,17E+00		**3,09**	1,40E+00		**3,31**
400	**4000**	1,40E-01	**3,2**	**36,80**	1,20E-01	**3,3**	**26,60**	1,53E-01	**3,2**	**26, 63**
1600	**16000**	1,04E-02	**3,8**	**298,00**	8,65E-03	**3,8**	**226,60**	1,03E-02	**3,9**	**216,30**
6400	**64000**	7,69E-04	**3,8**	**2394,00**	6,06E-04	**3,8**	**1813,60**	7,92E-04	**3,7**	**1714,90**
		$\mathcal{O}5$ **Taylor**			$\mathcal{O}5$ **Taylor-R**			$\mathcal{O}5$ **ADER**		
100	**1500**	4,90E-01		**9,28**	4,40E-01		**5,92**	5,10E-01		**8,84**
400	**6000**	2,59E-02	**4,4**	**77,80**	2,14E-02	**4,4**	**51,12**	2,57E-02	**4,3**	**71, 10**
1600	**24000**	9,79E-04	**4,7**	**628,00**	7,98E-04	**4,7**	**420,24**	9,51E-04	**4,8**	**570,00**
6400	**96000**	3,79E-05	**4,8**	**5023,30**	2,85E-05	**4,8**	**3344,80**	3,53E-05	**4,8**	**4533,60**
		$\mathcal{O}6$ **Taylor**			$\mathcal{O}6$ **Taylor-R**			$\mathcal{O}6$ **ADER**		
100	**2100**	1,60E-01		**22,08**	1,50E-01		**12,56**	1,60E-01		**23,70**
400	**8400**	4,62E-03	**5,1**	**182,80**	3,97E-03	**5,2**	**109,00**	4,48E-03	**5,2**	**191,20**
1600	**33600**	8,88E-05	**5,7**	**1471,60**	7,31E-05	**5,8**	**904,80**	8,46E-05	**5,7**	**1529,00**
6400	**134400**	1,65E-06	**5,8**	**11709,70**	1,28E-06	**5,8**	**7177,30**	1,56E-06	**5,8**	**12070,70**

Table 3: Results of numerical experiment for different DG schemes and the Gausspuls transport problem

accurate results are required, the higher order schemes come out to be much more efficient than the usually used 4th order Runge Kutta schemes in storage as well as in terms of CPU-time.

For the scheme with DG space discretization, a technique was proposed how to reduce the computational cost which lead to a significant acceleration of the numerical scheme especially for high order.

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