On a single degree of freedom oscillator with a time-varying mass

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PROEFSCHRIFT

ter verkrijging van de graad van doctor
aan de Technische Universiteit Delft,
op gezag van de Rector Magnificus prof. ir. K.C.A.M. Luyben,
voorzitter van het College voor Promoties,
in het openbaar te verdedigen op

dinsdag 31 january 2012 om 15.00 uur

door

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geboren te Dnipropetrovsk region, Oekraïne.

Dit proefschrift is goedgekeurd door de promotor:
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Samenstelling promotiecommisie:

Rector Magnificus,	voorzitter
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ISBN/EAN: 978-94-6186-021-7

This research was carried out in the section of Mathematical Physics at the Department of Applied Mathematics, Delft Institute of Applied Mathematics, Delft University of Technology, Netherlands.

Cover by Alexandr Pischaskyy and Bau Winkel (Studio Bau Winkel).

Printed in the Netherlands by Optima Grafische Communicatie, Rotterdam.

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'There are some days when
I think I'm going to die from
an overdosis of satisfaction'

SALVADOR DALI
MARQUIS DE PUBOL

to my parents, Victor and Vera,
and to my beloved, Victoria

Contents

Chapter 1

Introduction

Systems with time-varying masses frequently occur in daily life. Many constructions and mechanisms such as for instance cable-stayed bridges and suspension bridges contain parts for which the masses change in time. These constructions play an important role in practice and must be reliable and safe. The vibrations of these systems are the subject for many studies.

A single degree of freedom oscillator model will be used in this thesis as an extremely simple model to study the stability of the vibrations of such systems with periodically time-varying masses.

1.1 Short historical overview and motivation

The periodic behaviour of most constructions is determined by the periodicity of the loading and by the periodic deformation of (parts of) supporting or connected structures. The rotation of the rotor of a turbine or the crankshaft of a combustion engine, the transverse movement of a beam or a string, and the rectilinear movement of a car on a rough surface, can cause unexpectable periodic motion of the systems in different directions. Such a periodic motion – *vibration* or *oscillation* – can also occur after an instantaneous disturbance of the system and without a visible influence further.

Vibrations can be desirable or even necessary. For example, the vibrations of a guitar string or the cone of a loudspeaker produce sounds. The transportation of heavy objects on short distances can be done by using vibrations. The vibrating ear drums transfer sound waves to the brain, and so on. But most of the vibrations are undesirable. Building any system, either static or dynamic, requires thorough study of the possible vibrations of this system.

The Tacoma Narrows suspension bridge in Washington State, USA, is a classical example of a structure which looses its stability due to an incomplete study of the dynamics of the system. This bridge was built in 1940 and after 4 months it collapsed in not extremely bad weather conditions. The most trustful theory to explain the collapse is that the profile of the deck of the bridge acted as a kind of airfoil, and due to a windflow the bridge deck experienced strong drag-and-lift forces.

Another interesting example is the Erasmusbrug – a cable-stayed bridge in Rotterdam in the Netherlands. Several months after its opening in 1996 a significant movement of the deck of the bridge was observed in windy and rainy weather. Numerical analysis and experiments showed high amplitudes for the vibrations of the cables of the bridge. The problem was studied in [7]. Hydraulic dampers were applied to the cables of the bridge, which reduced the amplitudes of the vibrations and solved the problem.

In case of the Erasmusbrug, the movement of the deck of the bridge was caused by the periodic movement of the cables. There were two major possibilities for such cables' vibrations discovered in [7]. These were axial air-flow and the formation of water rivulets on the cable's surface. Similar phenomena were also observed by engineers on many other bridges, see [12, 13] for additional information. The flow of water rivulets on the inclined surface has also been studied in [2, 3] experimentally.

Vibrations are a complicated dynamical phenomenon. In particular, the stability properties of the vibrations are usually difficult to determine, but are important to know. In engineering, numerical and experimental approaches are used to determine the system's behaviour under certain conditions. Different models based on for instance the Finite Element Methods are used to simulate the system's dynamics.

Not only engineers are interested in the study of stability of vibrations, but also theoretical studies of this interesting phenomenon are presented by many scientists. Many analytical and numerical approaches to the problem of stability of vibrations have been developed, see for example [16, 18, 21, 22], and many others.

The fast development of mechanics, mathematics and engineering in the XIX - XX centuries allowed to build complicated structures and mechanisms. It turned out to be not enough just to study the vibrations of those constructions, but it was necessary to take into account many other factors. One of those important factors is the changing masses of the constructions during operation. The Russian scientist I. Meshchersky considered problems with changing masses in his Doctoral thesis [15] published in 1897. His works on this subject became a basis for the development of rockets and space flights.

The process of change of mass of a body can be considered in general as either an addition of new particles to the body or a separation of particles from the body. The behaviour of the added and the separated particles is considered neither before the addition nor after the separation of the particles. Only the influence of the particles during the time-intervals when the particles are situated on the body is taken into account. It is usually assumed that any particle situated on the body is considered to be an essential part of the body, and has the same velocity and experiences the same forcing as the body itself. So, every time when a particle either joins to or separates from the body, the mass of the body changes and become constant again for some time until the next particle changes the mass of the body.

According to the previous assumption the problem of the motion of a body with a changing mass is usually studied first as a number of simpler sub-problems of the motion of a body with a constant mass. Although the differential equation of motion of a body with a changing mass is similar to the differential equation of motion of a body with a constant mass, it is always easier to solve the latter one. The initial conditions for this equation are usually taken from the solution of the initial value problem solved for the previous interval when the mass of the body was constant but different. The moment of change of the properties of the system (when a particle either joins to or separates from the body) is very short and is usually taken as an infinitesimally small time-interval. It is sometimes a challenge to determine the behaviour of the system on these small time-intervals, since the system is influenced mostly at those moments.

1.2 The model

The model studied in this thesis is the following (see Fig. 1.1). The cross section of a cable is represented by an elastically constrained symmetrical circular body. The model is assumed to have one degree of freedom, since the main vibration of the system is assumed to be in one (vertical) direction, and all the forces acting in the other directions are relatively small and/or can be neglected. Several cases are studied in this thesis, and include impulses, damping, harmonic excitations, and drag-and-lift forces.

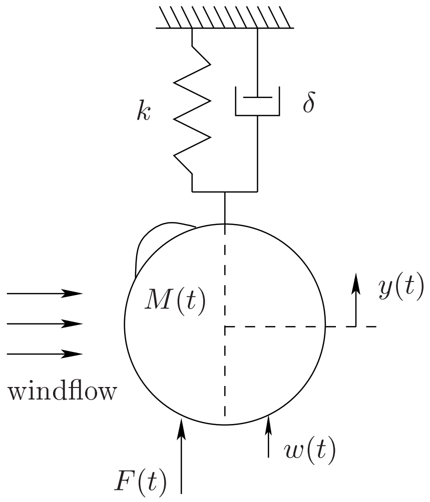

Figure 1.1: Single degree of freedom oscillator with a time varying mass.

It is assumed in all cases that the oscillator is influenced by the raindrops falling on and off the surface of the oscillator. The masses of the raindrops are much smaller than the mass of the oscillator itself. In some cases the raindrops' hits are considered to be impulses which add some energy to the system in the moments of adding to and separating from the oscillator.

It is rather difficult to obtain the equation of motion for a system with a changing mass. Such systems are usually described by a non-material volume inside a closed surface. Such a non-material volume is called a control volume in fluid mechanics and the surface of this volume is called a control surface. In case when the control surface coincides with the material surface, the general balance law can be used for the non-material volume, see [9] for details. Finally, the differential equation of motion of the oscillator with mass $M(t)$ can be determined, and is given by:

$$M(t)\ddot{y}(t) - \dot{M}(t)(w(t) - \dot{y}(t)) + \delta\dot{y}(t) + ky(t) = F(t, y(t), \dot{y}(t)), \qquad (1.1)$$

where $y(t)$ is the displacement of the centre of mass of the oscillator, $w(t)$ is the velocity at which a particle is added to or separated from the oscillator, δ is the damping coefficient, k is the stiffness coefficient, and $F(t, y(t), \dot{y}(t))$ is a function describing external excitations.

Eq. (1.1) was derived based on the principal of the conservation of energy and the balance of linear momentum of a non-material volume in [9] with references to Meshchersky [15], Levi-Civita, and others. According to [9], the particular case of this equation (the velocity of the added masses is equal to zero) was published by Levi-Civita in his work *Sul moto di un corpo di massa variabile* in 1928. The equation of Levi-Civita was obtained for the case of a planet for which the mass changes slowly due to the fall of meteorites on its surface. Some other scientists obtained similar equations by the balance of linear momentum for the non-material volume in other areas such as biology, chemistry, fluid mechanics etc, where a change of mass frequently occurs.

1.3 Methods

Theoretical studies of the problem of stability are mostly based on a combination of analytical and approximation methods depending on the considered problem. These methods are described for instance in [4, 6, 30], and used for many problems, such as in [1, 5, 20].

In this thesis the problem of stability of the vibrations of a single degree of freedom oscillator with a time-varying mass will be considered. This work is based on and employes the methods described in [25].

To determine the stability one first solves initial value problems for the differential equation (1.1). By using the solutions of the initial value problems maps are constructed. These maps, or equivalently systems of difference equations, are then studied to determine finally the stability properties of the solutions.

In case when the analytical solution is difficult to obtain explicitly, most of the researchers apply approximation methods. Such methods have been actively developed during the last few decades, see for example [18, 24]. A straight forward perturbation method will be used in this thesis.

In all cases of this research the solution of the initial value problem has been found in matrix form. Often, this solution has a form which is difficult to analyse directly, and transformations are required. There are many methods described in [4, 10, 17, 29], and others, which may be applied to the equations in order to change their form to perform the analysis. Several techniques, such as diagonalisation and Jordan-form-matrices have been used in this thesis. In the non-linear case the solution of the problem will lead to a system of two strongly non-linear algebraic equations. Numerical techniques have been applied to this system to obtain the stability characteristics of the solution. The search of equilibrium points of the system has been done by roots-finding algorithms for differential and difference equations. Then, the phase-space diagrams around those fixed points have been plotted and analysed in order to obtain the stability properties.

1.4 Reader's guide

This thesis is a collection of several modified journal papers and conference papers which describe research on the particular cases studied during the PhD project. The model used for each case is the same, except for the external excitations acting on the system, which differ from one case to another.

In chapter 2, the research results for the undamped linear case will be presented. This chapter is a continuation of the research done in [25], where a linear homogeneous model has been considered, and where the stability of this model has been studied. The oscillator studied in this chapter is influenced by two types of external excitations, such as impulses and harmonic forcing. The change of the mass of the oscillator is described by the periodic addition and separation of small masses. The influence of rain has been modelled as a number of impulses which are due to small masses hitting and leaving the surface of the oscillator. Wind acting on the system has been modelled mathematically as an harmonic function. For both of these forces the initial value problem has been formulated and solved. The obtained solution has been studied for its stability by using some analytical techniques. In case of harmonic forcing interesting resonance conditions have been found. These conditions relate the properties of the system to the frequency of the external excitations. Also the existence of periodic solutions has been investigated.

In chapter 3 a linear system with damping is studied. Three different cases with external excitations have been studied. The model is similar to the one considered in chapter 2, so the initial value problem and the methods are rather similar. The solution of this problem is more complicated, but the general analytical techniques still can be applied to find the stability properties of the system. The solutions have been studied in detail, and many interesting stability properties have been found. There are graphs in chapter 3 which describe the behaviour of the system for different values of the parameters. Also optimal damping rates have been computed for which the system is always stable.

In chapter 4 a non-linear case is investigated. The model is similar to the model from chapter 2 but the external excitation (is a non-linear function) due to a wind force. This non-linearity leads to non-linear sub-problems, and the application of numerical techniques, and perturbation methods are required. The instability regions in the parameter space and some phase-space figures for the non-linear problem will be computed numerically. Moreover, a lot of bifurcations will be presented.

Chapter 2

A linear case without damping

Abstract: In this chapter the forced vibrations of a linear, single degree of freedom oscillator (sdofo) with a time-varying mass will be studied. The forced vibrations are due to small masses which are periodically hitting and leaving the oscillator with different velocities. Since, these small masses stay for some time on the oscillator surface the effective mass of the oscillator will periodically vary in time. Additionally, an external harmonic force will be applied to the oscillator with a time-varying mass. Not only solutions of the oscillator equation will be constructed, but also stability properties for the forced vibrations will be presented for various parameter values.

2.1 Introduction.

Systems with time-varying masses frequently occur in practice. Examples of such systems can be found in robotics, rotating crankshafts, conveyor systems, excavators, cranes, biomechanics and in fluid-structure interaction problems [5, 9]. The oscillations of electric transmission lines and cables of cable-stayed bridges with water rivulets on the surface are also examples of time-varying dynamic systems [22]. For these mechanical constructions the 1-mode Galerkin approximation of the continuous model will lead to a sdofo-equation. These sdofos are considered to be representative models for testing numerical methods and for studying forces which are acting on the system [8].

In this chapter the forced oscillations of a linear sdofo with a (periodically and stepwise changing) time-varying mass will be studied. The free oscillations have been recently studied in [25].

Consider the oscillations of a sdofo with a linear restoring force and a mass which varies in time according to a periodic stepwise dependence. This model is perhaps the simplest model which describes the process of the vibrations of a cable with rainwater located on it. Part of the raindrops hitting the cylinder (i.e. the cable) will remain on the surface of the cylinder for some time, and will subsequently be blown or shaken off after some time. It will be assumed that when mass is added to

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This chapter is a slightly revised version of [27]: W.T. van Horssen, O.V. Pischanskyy, J.L.A Dubbeldam, On the forced vibrations of an oscillator with a periodically time-varying mass, *Journal of Sound and Vibration*, 329 (6): 721-732, 2010 or separated from the oscillator that the position of the center of the (total) mass of the oscillator is not influenced. The following equation of motion for the sdofo can now be derived (see for instance [9, p. 152]):

$$M\ddot{y} = \dot{M}(w - \dot{y}) - ky + F, \tag{2.1}$$

where $y = y(t)$ is the displacement of the oscillator (see Figure 2.1), $M = M(t)$ is the time-varying mass of the oscillator, $w = w(t)$ is the mean velocity at which masses (i.e. raindrops) are hitting or leaving the oscillator, k is the (positive) stiffness coefficient in the linear restoring force, $F = F(t)$ or $F = F(t, y, \dot{y})$ is an external force, and the dot denotes differentiation with respect to t. The force F and the velocity w are measured positive in positive y direction (see Figure 2.1).

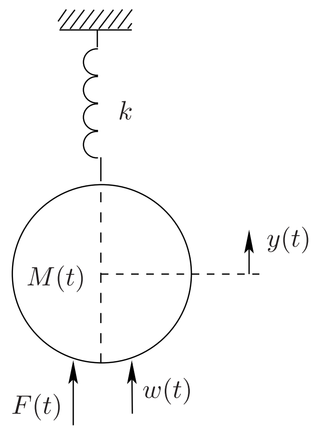

Figure 2.1: The single degree of freedom oscillator.

In [25] the free vibrations (i.e. $F \equiv 0$ and $w \equiv 0$ in (2.1)) of the sdofo have been studied, and in this chapter the forced vibrations will be studied. Following [25] it turns out to be convenient to separate the mass $M(t)$ into a time invariant part M_0 and into a time-varying part $m(t)$, that is,

$$M(t) = M_0 - m(t), \tag{2.2}$$

where M_0 is a positive constant, and $M_0 - m(t) > 0$. By substituting equation (2.2) into equation (2.1) it follows that (2.1) can be rewritten in:

$$\frac{d}{dt}\left((M_0 - m(t))\frac{dy}{dt}\right) + ky = \frac{-dm}{dt}w + F. \tag{2.3}$$

Then, by introducing the time-rescaling $t = \sqrt{M_0/k}\,\tau$ it follows that equation (2.3) becomes

$$\frac{d}{d\tau}\left(\left(1 - \frac{\tilde{m}(\tau)}{M_0}\right)\frac{d\tilde{y}(\tau)}{d\tau}\right) + \tilde{y}(\tau) = \frac{-\tilde{w}(\tau)}{\sqrt{M_0 k}}\frac{d\tilde{m}(\tau)}{d\tau} + \tilde{F}(\tau), \tag{2.4}$$

where $\tilde{y}(\tau) = y(\sqrt{M_0/k}\,\tau)$, $\tilde{m}(\tau) = m(\sqrt{M_0/k}\,\tau)$, $\tilde{w}(\tau) = w(\sqrt{M_0/k}\,\tau)$ and $\tilde{F}(\tau) = \frac{1}{k}F(\sqrt{M_0/k}\,\tau)$. In this chapter it will be assumed that $h(\tau) = \tilde{m}(\tau)/M_0$ with

$1 - h(\tau) > 0$ is a periodic step function, that is,

$$h(\tau) = \begin{cases} \varepsilon \text{ for } 0 < \tau < T_0, \\ 0 \text{ for } T_0 < \tau < T, \end{cases} \tag{2.5}$$

and $h(\tau + T) = h(\tau)$, and ε is a constant (in practice usually small) with $0 < \varepsilon < 1$. Also $\tilde{w}(\tau)$ is assumed to be T-periodic. It should be observed that in the analysis ε is defined to be the quotient m/M_0, where m is the mass which added at time T_0, and where M_0 is the mass of the oscillator. So, ε can be seen as a measure for the relative mass which is added at time T_0.

For the reasons of convenience the tildes in (2.4) will be dropped, and the prime will be introduced to denote differentiation with respect to τ, yielding

$$((1 - h(\tau))\, y'(\tau))' + y(\tau) = \frac{-w(\tau)\,\omega_0}{k}\, m'(\tau) + F(\tau), \tag{2.6}$$

where $\omega_0 = \sqrt{k/M_0}$ is the natural frequency of the oscillator. The initial displacement and the initial velocity of $y(\tau)$ are given by

$$y(0) = y_0 \text{ and } y'(0) = y_0' \tag{2.7}$$

respectively.

The chapter is organized as follows. In section 2 of this chapter the initial value problem (2.6) - (2.7) will be studied with $F(\tau) \equiv 0$. In this case the small masses which are periodically hitting and leaving the oscillator (with nonzero velocities) can be seen as an external force acting on the oscillator. The stability of the solution(s) of the initial value problem will be studied in detail, and the existence of periodic solutions will be investigated. In section 3 of this chapter it will be assumed that the external force $F(\tau)$ is a harmonic force, that is,

$$F(\tau) = A\cos(\alpha\tau + \beta),$$

where A and β are constants, and where α is the frequency of the external force. Then the following initial value problem for $y(\tau)$ is obtained

$$((1 - h(\tau))\, y'(\tau))' + y(\tau) = \frac{-w(\tau)\,\omega_0}{k}\, m'(\tau) + A\cos(\alpha\tau + \beta), \tag{2.8}$$

with initial conditions (2.7).

The initial value problem (2.7) - (2.8) will be studied in detail in section 3. The stability of the solutions will be studied as well as the existence of resonance frequencies (depending on α). Finally, in section 4 of this chapter some conclusions will be drawn, and remarks will be made about future research on this subject.

2.2 The case $F \equiv 0$.

In this section the initial value problem (2.6) - (2.7) with $F \equiv 0$ will be studied, or equivalently

$$((1 - h(\tau))\, y'(\tau))' + y(\tau) = \frac{-w(\tau)}{\omega_0}\, h'(\tau), \quad \tau > 0, \tag{2.9}$$

with $y(0) = y_0$, $y'(0) = y_0'$, $\omega_0 = \sqrt{k/M_0}$, and where $h(\tau)$ is given by (2.5). This section is organized as follows. In subsection 2.1 a representation for the solution $y(\tau)$ of the initial value problem will be given. The stability properties of the solution(s) will be discussed in the subsection 2.2, and in the subsection 2.3 the existence of periodic solutions will be investigated.

2.2.1 A representation of the solution.

It is obvious that the derivative of $h(\tau)$ with respect to τ for $0 < \tau < T_0$ and $T_0 < \tau < T$ is equal to 0. Thus, for $0 < \tau < T_0$ equation (2.9) becomes:

$$(1-\varepsilon)y'' + y = 0. \tag{2.10}$$

The initial value problem for (2.10) can easily be solved for $0 < \tau < T_0$, yielding:

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_1}(\tau) \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}, \tag{2.11}$$

where matrix $\mathbf{M_1}(\tau)$ is given by

$$\mathbf{M_1}(\tau) = \begin{pmatrix} \cos\left(\frac{\tau}{\sqrt{1-\varepsilon}}\right) & \sqrt{1-\varepsilon}\,\sin\left(\frac{\tau}{\sqrt{1-\varepsilon}}\right) \\ \frac{-1}{\sqrt{1-\varepsilon}}\sin\left(\frac{\tau}{\sqrt{1-\varepsilon}}\right) & \cos\left(\frac{\tau}{\sqrt{1-\varepsilon}}\right) \end{pmatrix}.$$

At $\tau = T_0$ the function $h(\tau)$ has a jump discontinuity. Consider the infinitesimal small time-interval ${T_0}^- \leq \tau \leq {T_0}^+$, where ${T_0}^- = T_0 - 0, {T_0}^+ = T_0 + 0$. For this interval the following conditions can be formulated: the displacement of the oscillator is continuous, and the impulse of the system at $\tau = T_0^+$ is equal to the impulse of the system at $\tau = T_0^-$ plus the impulse of the raindrop (which hits the oscillator). The continuity of the displacement at $\tau = T_0$ simply implies that $y({T_0}^-) = y({T_0}^+)$, and the impulse condition can be obtained by integrating (2.9) with respect to τ from $\tau = {T_0}^-$ to $\tau = {T_0}^+$, yielding

$$y'({T_0}^+) - (1-\varepsilon)y'({T_0}^-) = \frac{\varepsilon w(T_0)}{\omega_0}.$$

And so, it follows for $\tau = T_0^+$ that

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_2}(\tau) \begin{pmatrix} y({T_0}^-) \\ y'({T_0}^-) \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\varepsilon\, w(T_0)}{\omega_0} \end{pmatrix} =$$

$$= \mathbf{M_2}(\tau)\,\mathbf{M_1}(\mathbf{T_0}) \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{\varepsilon\, w(T_0)}{\omega_0} \end{pmatrix}, \tag{2.12}$$

where $\mathbf{M_2}(\tau)$ is given by

$$\mathbf{M_2}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}.$$

For $T_0 < \tau < T$ equation (2.9) has the following form:

$$y'' + y = 0 \tag{2.13}$$

and the solution of equation (2.13) is given by:

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_3}(\tau)\,\mathbf{M_2}(\mathbf{T_0})\,\mathbf{M_1}(\mathbf{T_0}) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{M_3}(\tau) \begin{pmatrix} 0 \\ \dfrac{\varepsilon\, w(T_0)}{\omega_0} \end{pmatrix}, \tag{2.14}$$

where matrix $\mathbf{M_3}(\tau)$ is given by

$$\mathbf{M_3}(\tau) = \begin{pmatrix} \cos(\tau - T_0) & \sin(\tau - T_0) \\ -\sin(\tau - T_0) & \cos(\tau - T_0) \end{pmatrix}.$$

At $\tau = T$ the function $h(\tau)$ has again a jump discontinuity. Consider the infinitesimal small time-interval $T^- \le \tau \le T^+$, where $T^- = T - 0, T^+ = T + 0$. For this interval the following conditions can be formulated: the displacement of the oscillator is continuous, and the impulse of the system at $\tau = T^-$ is equal to the impulse of the system at $\tau = T^+$ plus the impulse of the raindrop (which leaves the oscillator). The continuity of the displacement at $\tau = T$ simply implies that $y(T^-) = y(T^+)$, and the impulse condition can be obtained by integrating (2.9) with respect to τ from $\tau = T^-$ to $\tau = T^+$, yielding

$$(1-\varepsilon)y'(T^+) - y'(T^-) = \frac{-\varepsilon w(T)}{\omega_0}.$$

And so, it follows for $\tau = T^+$ that

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_4}(\tau) \begin{pmatrix} y(T^-) \\ y'(T^-) \end{pmatrix} + \begin{pmatrix} 0 \\ -\dfrac{\varepsilon w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix}, \tag{2.15}$$

where $\mathbf{M_4}(\tau)$ is given by

$$\mathbf{M_4}(\tau) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{1-\varepsilon} \end{pmatrix}.$$

So, the solution of equation (2.9) on the interval $0 < \tau \le T^+$ has been constructed, and at $\tau = T^+$ the solution is given by

$$\begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} = \mathbf{M_4}(\mathbf{T^+})\,\mathbf{M_3}(\mathbf{T^+})\,\mathbf{M_2}(\mathbf{T_0})\,\mathbf{M_1}(\mathbf{T_0}) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} +$$

$$+\,\mathbf{M_4}(\mathbf{T^+})\,\mathbf{M_3}(\mathbf{T^+}) \begin{pmatrix} 0 \\ \dfrac{\varepsilon w(T_0)}{\omega_0} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{-\varepsilon w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix}$$

or in a short form:

$$\begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{W}, \tag{2.16}$$

where

$$\begin{aligned}\mathbf{A} &= \mathbf{M_4}(\mathbf{T^+})\,\mathbf{M_3}(\mathbf{T^+})\,\mathbf{M_2}(\mathbf{T_0})\,\mathbf{M_1}(\mathbf{T_0}) = \\ &= \begin{pmatrix} a\,b - c\,d\sqrt{1-\varepsilon} & b\,c\sqrt{1-\varepsilon} + a\,d(1-\varepsilon) \\ a\,d\,\dfrac{-1}{1-\varepsilon} - b\,c\,\dfrac{1}{\sqrt{1-\varepsilon}} & c\,d\,\dfrac{-1}{\sqrt{1-\varepsilon}} + a\,b \end{pmatrix}, \end{aligned} \tag{2.17}$$

where

$$a = \cos\left(\frac{T_0}{\sqrt{1-\varepsilon}}\right),\ b = \cos(T - T_0),\ c = \sin\left(\frac{T_0}{\sqrt{1-\varepsilon}}\right),\ d = \sin(T - T_0), \tag{2.18}$$

and

$$\begin{aligned}\mathbf{W} &= \mathbf{M_4}(\mathbf{T^+})\,\mathbf{M_3}(\mathbf{T^+}) \begin{pmatrix} 0 \\ \dfrac{\varepsilon w(T_0)}{\omega_0} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{-\varepsilon w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix} = \\ &= \begin{pmatrix} \dfrac{\varepsilon w(T_0)}{\omega_0}\sin(T - T_0) \\ \dfrac{\varepsilon\left(w(T_0)\cos(T-T_0) - w(T)\right)}{\omega_0(1-\varepsilon)} \end{pmatrix}. \end{aligned} \tag{2.19}$$

To obtain the solution on the interval $0 < \tau \leq (n+1)T^+$, the same procedure should be applied to equation (2.9) n more times, yielding for $\tau = (n+1)T^+$:

$$\begin{pmatrix} y((n+1)T^+) \\ y'((n+1)T^+) \end{pmatrix} = \mathbf{A^{n+1}} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} + \sum_{r=0}^{n} \mathbf{A^r}\,\mathbf{W}. \tag{2.20}$$

The properties of matrix $\mathbf{A}$ are known from [25]. For $\mathbf{W} = (0\ \ 0)^T$ the oscillator is unstable when at least one of the eigenvalues λ_1 or λ_2 is such that $|\lambda_j| > 1$, or when $\lambda_1 = \lambda_2$ with $|\lambda_j| = 1$ and the dimension of the corresponding eigenspace is equal to one. In all other cases the oscillator is stable for $\mathbf{W} = (0\ \ 0)^T$. These results are summarized in Table 2.1, where $\lambda_{1,2} = \frac{1}{2}tr(\mathbf{A}) \pm \frac{1}{2}\sqrt{D}$ with $D = (tr(\mathbf{A}))^2 - 4$, and $tr(\mathbf{A})$ is the trace of matrix $\mathbf{A}$ (see also [25]).

The stability of the oscillator when $\mathbf{W} \neq \mathbf{0}$ will be determined in the next subsection.

2.2.2 On the stability of the oscillator.

From the previous subsection (see (2.16) to (2.20)) it follows that the solution of equation (2.9) at $\tau = (n+1)T^+$ and at $\tau = nT^+$ can be linked by

$$\begin{pmatrix} y_{n+1} \\ y_{n+1}' \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n \\ y_n' \end{pmatrix} + \mathbf{W}, \tag{2.21}$$

stability properties / $tr(\mathbf{A})$	the oscillator for $\mathbf{W} = \mathbf{0}$ is
$-2 < tr(\mathbf{A}) < 2$ $(\lvert\lambda_{1,2}\rvert = 1)$	stable
$tr(\mathbf{A}) < -2$ or $tr(\mathbf{A}) > 2$ $(\lvert\lambda_j\rvert > 1$ for $j = 1$ or $j = 2)$	unstable
$tr(\mathbf{A}) = 2$ $(\lambda_1 = \lambda_2 = 1)$	only stable when $c = d = 0$ and $ab = 1$ in matrix $\mathbf{A}$, else unstable
$tr(\mathbf{A}) = -2$ $(\lambda_1 = \lambda_2 = -1)$	only stable when $c = d = 0$ and $ab = -1$ in matrix $\mathbf{A}$, else unstable

Table 2.1: Stability properties of the oscillator when $\mathbf{W} = \mathbf{0}$.

where $y_{n+1} = y((n+1)T^+)$, $y'_{n+1} = y'((n+1)T^+)$ and where $\mathbf{A}$ and $\mathbf{W}$ are given by (2.17) and (2.19) respectively. The solution of the system of difference equations (2.21) is given by (2.20). However, the representation (2.20) is not very convenient to determine the stability of the oscillator (due to an external force, that is, due to $\mathbf{W} \neq 0$). Also the use of a fundamental matrix for system (2.21) will lead to a representation from which it is not very convenient to determine the stability. In fact the following representation (see [6, p. 124] or [17]) will be obtained

$$\begin{pmatrix} y_n \\ y'_n \end{pmatrix} = \mathbf{\Phi}(n, n_0) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \sum_{r=n_0}^{n-1} \left[\mathbf{\Phi}(n, r+1)\mathbf{g}(r)\right],$$

where $\mathbf{g}(r)$ is a particular solution of (2.9), and where the fundamental matrix $\mathbf{\Phi}(n, n_o)$ is given by

$$\mathbf{\Phi}(n, n_o) = [v_1, v_2] \bullet diag\,[\lambda_1^n, \lambda_2^n],$$

in which v_1, v_2 are eigenvectors, and λ_1, λ_2 are eigenvalues of matrix $\mathbf{A}$.

Now a diagonalization method will be used to obtain a representation of the solution from which the stability of the oscillator can be determined immediately. From [17, p. 6] it follows that if the eigenvalues λ_1, λ_2 of a 2×2 matrix $\mathbf{A}$ are distinct or if the two eigenvalues are coinciding and the dimension of the corresponding eigenspace is 2, then from any set of linearly independent corresponding eigenvectors v_1, v_2 a matrix $\mathbf{P}$ can be formed, which is invertible and

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = diag[\lambda_1, \lambda_2]. \tag{2.22}$$

Let

$$\begin{pmatrix} y_n \\ y'_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} x_n \\ x'_n \end{pmatrix}, \tag{2.23}$$

and substitute the transformation (2.23) into (2.21). Then, multiply the left- and the right-hand sides of the so-obtained equation by the inverse matrix of $\mathbf{P}$. So, we can rewrite (2.21) in the following form:

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} x_n \\ x'_n \end{pmatrix} + \mathbf{G}, \tag{2.24}$$

where

$$\mathbf{G} = \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = \mathbf{P}^{-1}\mathbf{W}. \tag{2.25}$$

Divide the first equation in (2.24) by λ_1^{n+1}, and the second equation in (2.24) by λ_2^{n+1}, yielding:

$$\begin{cases} \dfrac{x_{n+1}}{\lambda_1^{n+1}} = \dfrac{x_n}{\lambda_1^n} + \dfrac{G_1}{\lambda_1^{n+1}}, \\ \dfrac{x'_{n+1}}{\lambda_2^{n+1}} = \dfrac{x'_n}{\lambda_2^n} + \dfrac{G_2}{\lambda_2^{n+1}}. \end{cases} \tag{2.26}$$

Then x_n and x'_n can be obtained, yielding:

$$\begin{pmatrix} x_n \\ x'_n \end{pmatrix} = \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix} + \sum_{r=0}^{n-1} \begin{pmatrix} \lambda_1^r & 0 \\ 0 & \lambda_2^r \end{pmatrix} \mathbf{G}. \tag{2.27}$$

Substitute (2.27) into (2.23) and multiply the result by matrix $\mathbf{P}$, to obtain for $\lambda_1 \neq 1$ and $\lambda_2 \neq 1$:

$$\begin{pmatrix} y_n \\ y'_n \end{pmatrix} = \mathbf{P} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \mathbf{P}^{-1} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{P} \begin{pmatrix} \dfrac{1-\lambda_1^n}{1-\lambda_1} & 0 \\ 0 & \dfrac{1-\lambda_2^n}{1-\lambda_2} \end{pmatrix} \mathbf{P}^{-1}\mathbf{W}. \tag{2.28}$$

For the eigenvalues $\lambda_{1,2} = 1$ and the dimension of the corresponding eigenspace is two, it is obvious from (2.27) that the solution (2.21) is unbounded, and that the oscillator is exponentially unstable for $\mathbf{W} \neq \mathbf{0}$. In [25] it has been shown that for $\mathbf{W} = \mathbf{0}$ the solution of (2.9) is bounded in this case. Remind that this method can be applied to 2×2 matrices which have two independent eigenvectors. From [25], eq. (20)-(22) it can be seen that the eigenvalues $\lambda_{1,2}$ of matrix $\mathbf{A}$ can be only coinciding for $\lambda_1 = \lambda_2 = 1$, or $\lambda_1 = \lambda_2 = -1$, and if one of the eigenvalues is equal to 1 (or -1) then the other eigenvalue is also equal to 1 (or -1). The case $\lambda_{1,2} = 1$ (and the dimension of the corresponding eigenspace is two) has just been considered, and for the case $\lambda_{1,2} = -1$ (and the dimension of the corresponding eigenspace is two) it easily follows from (2.28) that the solution is bounded, and so for $\lambda_1 = \lambda_2 = -1$ (and the dimension of the corresponding eigenspace is two) the oscillator is stable. For all other noncoinciding values of $\lambda_{1,2}$ the stability properties of the oscillator easily follow from (2.28).

Now the following case still has to be considered: matrix $\mathbf{A}$ has two coinciding eigenvalues and the dimension of the corresponding eigenspace is one (implying that matrix $\mathbf{A}$ cannot be diagonalized). For this case the Jordan-form matrix method can be used as for instance described in [6, 17]. It can be shown (see [6, 17]) that again an invertible matrix $\mathbf{P}$ exists such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{J} = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix}. \tag{2.29}$$

Instead of (2.24) the following system will be obtained:

$$\begin{pmatrix} x_{n+1} \\ x'_{n+1} \end{pmatrix} = \begin{pmatrix} \pm 1 & 1 \\ 0 & \pm 1 \end{pmatrix} \begin{pmatrix} x_n \\ x'_n \end{pmatrix} + \mathbf{G}, \tag{2.30}$$

For $\lambda_{1,2} = 1$ x_n and x'_n can be determined from (2.30), yielding:

$$\begin{cases} x_n = x_0 + n x'_0 + n G_1 + \dfrac{n(n-1)}{2} G_2, \\ x'_n = x'_0 + n G_2. \end{cases} \tag{2.31}$$

In (2.31) it can be seen that several terms are multiplied by n, so the vibrations of the oscillator will grow in time. For $\lambda_{1,2} = -1$, x_n and x'_n can be obtained similarly:

$$\begin{cases} x_n = (-1)^n x_0 - (-1)^{n+1} n \left(x'_0 - \dfrac{G_2}{2} \right) + \left(G_1 + \dfrac{G_2}{2} \right) \cos^2 \left(\dfrac{\pi(n+1)}{2} \right), \\ x'_n = (-1)^n x'_0 + G_2 \cos^2 \left(\dfrac{\pi(n+1)}{2} \right). \end{cases} \tag{2.32}$$

Again there are several unbounded terms in (2.32), so the vibrations of the oscillator will also grow in time. All of the stability properties of the oscillator (for $\mathbf{W} \neq \mathbf{0}$) are summarized in Table 2.2.

2.2.3 On the existence of periodic solutions.

In this subsection the existence of qT-periodic solutions (with $q \in \mathbf{Z}^+$) for equation (2.9) will be investigated. Since a small mass hits and leaves the oscillator with period T, it is natural to study the question whether qT-periodic solutions exist or not. In [16] a uniqueness result about the existence of T-periodic solutions for (2.9) has recently been presented. In this section the existence or non-existence, and the uniqueness or non-uniqueness of qT-periodic solutions for equation (2.9) will be discussed in detail. To study these properties the map (2.21) will be used, that is,

$$y_{n+1} = \mathbf{A} y_n + \mathbf{W}, \tag{2.33}$$

stability properties / $tr(\mathbf{A})$	the oscillator for $\mathbf{W} \neq \mathbf{0}$ is
$-2 < tr(\mathbf{A}) < 2$ $(\lvert\lambda_{1,2}\rvert = 1)$	stable
$tr(\mathbf{A}) < -2$ or $tr(\mathbf{A}) > 2$ $(\lvert\lambda_j\rvert > 1$ for $j = 1$ or $j = 2)$	unstable
$tr(\mathbf{A}) = 2$ $(\lambda_1 = \lambda_2 = 1)$	unstable
$tr(\mathbf{A}) = -2$ $(\lambda_1 = \lambda_2 = -1)$	only stable when $c = d = 0$ and $ab = -1$ in matrix $\mathbf{A}$, else unstable.

Table 2.2: Stability properties of the oscillator when $\mathbf{W} \neq \mathbf{0}$.

where $y_n = (y(nT^+), y'(nT^+))^T$, and where $\mathbf{A}$ and $\mathbf{W}$ are given by (2.17) and (2.19) respectively. For a T-periodic solution of (2.9) it follows from (2.33) that $y_{n+1} = y_n = y_{n-1} = \ldots = y$, and so y follows from (2.33):

$$y = \mathbf{A}y + \mathbf{W} \iff (\mathbf{I} - \mathbf{A})y = \mathbf{W}. \tag{2.34}$$

So, a unique, T-periodic solution of equation (2.9) exists when matrix $\mathbf{I} - \mathbf{A}$ is invertible, or equivalently $det(\mathbf{I} - \mathbf{A}) \neq 0$, or equivalently 1 is not an eigenvalue of matrix $\mathbf{A}$, or equivalently $tr(\mathbf{A}) \neq 2$. When $tr(\mathbf{A}) = 2$ or equivalently $\lambda = 1$ is an eigenvalue of matrix $\mathbf{A}$ then there are two possibilities: there are no T-periodic solutions of equation (2.9), or there are infinitely many T-periodic solutions. From (2.27) and (2.31) it is obvious that for $\mathbf{W} \neq (0,0)^T$ that there are no T-periodic solutions, and that for $\mathbf{W} \equiv (0,0)^T$ that there are infinitely many T-periodic solutions.

For a $2T$-periodic solutions of (2.9) it follows from (2.33) that $y_{n+2} = y_n = y_{n-2} = \ldots = y$, and so y follows from (2.33):

$$y_{n+2} = \mathbf{A}y_{n+1} + \mathbf{W} \;=\; \mathbf{A}(\mathbf{A}y_n + \mathbf{W}) + \mathbf{W} \implies$$

$$(\mathbf{I} - \mathbf{A}^2)y = (\mathbf{A} + \mathbf{I})\mathbf{W}. \tag{2.35}$$

So, a unique $2T$-periodic solution of equation (2.9) exists when matrix $\mathbf{I} - \mathbf{A}^2$ is invertible, or equivalently $det(\mathbf{I} - \mathbf{A}^2) \neq 0$, or equivalently 1 is not an eigenvalue of matrix $\mathbf{A}^2$, or equivalently those λ's with $\lambda^2 = 1$ (that is, 1 and -1) are not eigenvalues of matrix $\mathbf{A}$, or equivalently $tr(\mathbf{A}) \neq 2$ and $tr(\mathbf{A}) \neq -2$. When $\lambda = 1$ (or equivalently $tr(\mathbf{A}) = 2$) then the previous case of T-periodic solutions will be obtained. When $\lambda = -1$ (or equivalently $tr(\mathbf{A}) = -2$) it follows from (2.28) and (2.32) that there are infinitely many $2T$-periodic solutions of equation (2.9) for all vectors $\mathbf{W}$. This can also be seen from (2.35) in the following way. Rewrite (2.35) into

$$(\mathbf{A}+\mathbf{I})((\mathbf{I}-\mathbf{A})y-\mathbf{W}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.36}$$

Since $\lambda = -1$ is an eigenvalue of matrix $\mathbf{A}$ it follows that $\mathbf{I}+\mathbf{A}$ is not invertible, and that there is no eigenvalue equal to 1. So, $\mathbf{I}-\mathbf{A}$ is invertible, and equation (2.36) has at least one solution $y = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{W}$. Since $\mathbf{I}+\mathbf{A}$ is not invertible, and equation (2.36) has at least one solution, it follows that equation (2.36) has infinitely many solutions, that is, there are infinitely many $2T$-periodic solutions of equation (2.9) for all vectors $\mathbf{W}$.

For a qT-periodic solution of (2.9) with $q \in \mathbf{Z}^+$ and $q \geq 2$ it follows from (2.33) that $y_{n+q} = y_n = y_{n-q} = \ldots = y$, and so it follows from (2.33):

$$y_{n+q} = \mathbf{A}y_{n+q-1} + \mathbf{W} = \mathbf{A}(\mathbf{A}y_{n+q-2} + \mathbf{W}) + \mathbf{W} = \ldots \implies$$

$$y = \mathbf{A}^q y + (\mathbf{A}^{q-1} + \ldots + \mathbf{A} + \mathbf{I})\mathbf{W} \iff$$

$$(\mathbf{I}-\mathbf{A}^q)y = (\mathbf{A}^{q-1} + \ldots + \mathbf{A} + \mathbf{I})\mathbf{W} \iff \tag{2.37}$$

$$(\mathbf{A}^{q-1} + \ldots + \mathbf{A} + \mathbf{I})(\mathbf{I}-\mathbf{A})y = (\mathbf{A}^{q-1} + \ldots + \mathbf{A} + \mathbf{I})\mathbf{W} \iff$$

$$(\mathbf{A}^{q-1} + \ldots + \mathbf{A} + \mathbf{I})((\mathbf{I}-\mathbf{A})y - \mathbf{W}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.38}$$

So, a unique, qT-periodic solution of equation (2.9) exists (see (2.37)) when matrix $\mathbf{I}-\mathbf{A}^q$ is invertible, or equivalently $det(\mathbf{I}-\mathbf{A}^q) \neq 0$, or equivalently 1 is not an eigenvalue of matrix $\mathbf{A}^q$, or equivalently those λ's with $\lambda^q = 1$ are not eigenvalues of matrix $\mathbf{A}$. When λ is an eigenvalue of matrix $\mathbf{A}$, and $\lambda^q = 1$, and $\lambda \neq 1$ (the case of T-periodic solutions has already been studied) then $\mathbf{A}^{q-1} + \ldots + \mathbf{A} + \mathbf{I}$ is not invertible, and equation (2.38) has at least one solution $y = (\mathbf{I}-\mathbf{A})^{-1}\mathbf{W}$. And so, equation (2.38) has infinitely many solutions, that is, there are infinitely many qT-periodic solutions (with $q \geq 2$) of equation (2.9) for all vectors $\mathbf{W}$. It can be shown in an elementary way that $\lambda^q = 1$ and λ is an eigenvalue of matrix $\mathbf{A}$ is equivalent with $tr(\mathbf{A}) = 2\cos\left(\frac{2n\pi}{q}\right)$ for at least one n in the set $0, 1, 2, \ldots, q-1$. The results obtained so far about the existence (and uniqueness) of qT-periodic solutions of equation (2.9), can be summarized as follows. Let λ be an eigenvalue of matrix $\mathbf{A}$, and let q be an element in $\mathbf{Z}^+$. Then,

- If $\lambda = 1$ ($\Leftrightarrow tr(\mathbf{A}) = 2$) then there are only T-periodic solutions when $\mathbf{W} \equiv (0, 0)^T$.

- If $\lambda^q = 1$ and $\lambda \neq 1$ for a certain $q \geq 2$ ($\Leftrightarrow tr(\mathbf{A}) = 2\cos\left(\frac{2n\pi}{q}\right)$ for at least one n in the set $0, 1, 2, \ldots, q-1$) then there are infinitely many qT-periodic solutions of equation (2.9) for all vectors $\mathbf{W}$.

- If $\lambda^q \neq 1$ then there is a unique qT-periodic solution of equation (2.9) for all vectors $\mathbf{W}$.

2.3 The case with an external, harmonic force and $w(t) \equiv 0$.

In this section the initial value problem (2.7) - (2.8) with $w(t) \equiv 0$ will be studied, that is,

$$((1 - h(\tau))\, y'(\tau))' + y(\tau) = A\cos(\alpha\tau + \beta), \quad \tau > 0, \tag{2.39}$$

with $y(0) = y_0, y'(0) = y_0', \omega_0 = \sqrt{k/M_0}$, where $h(\tau)$ is given by (2.5), and where α, A and β are constants. This section is organized as follows. In subsection 2.3.1 a representation for the solution $y(\tau)$ of the initial value problem will be given. The amplitude increase after one period T will be discussed in subsection 2.3.2, and in subsection 2.3.3 the stability properties of the solution and the resonance cases will be investigated.

2.3.1 A representation of the solution

As in the previous section a map will be constructed which relates the solution at $\tau = (n+1)T + 0^+$ to the solution at $\tau = nT + 0^+$. For simplicity the following notation will be introduced: $y_n(0^+) = y(nT + 0^+)$, $y_{n+1}(0^+) = y((n+1)T + 0^+)$, $y_n(\tau^*) = y(nT + \tau^*)$ with $0 < \tau^* \leq T + 0^+$. Starting at $\tau = nT + 0^+$ the solution will now be constructed (leading to the solution at $\tau = (n+1)T + 0^+$). For $nT < \tau < nT + T_0$ or equivalently for $0 < \tau^* < T_0$ equation (2.39) becomes

$$(1 - \varepsilon)y'' + y = A\cos(\alpha\tau + \beta). \tag{2.40}$$

For $\alpha^2 \neq \dfrac{1}{1-\varepsilon}$ a particular solution of (2.40) is given by:

$$y_p(\tau) = y_{1p}\cos(\alpha\tau + \beta), \tag{2.41}$$

where

$$y_{1p} = \frac{A}{1 - \dfrac{\alpha^2}{\phi^2}}, \tag{2.42}$$

and $\phi = \dfrac{1}{\sqrt{1-\varepsilon}}$. The initial value problem (with $\alpha^2 \neq \phi^2$) can easily be solved for $0 < \tau^* < T_0$, yielding

$$\begin{pmatrix} y_n(\tau) \\ y_n'(\tau^*) \end{pmatrix} = \mathbf{M_1}(\tau^*) \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{N_1}(\tau^*) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{2.43}$$

where

$$\mathbf{M_1}(\tau^*) = \begin{pmatrix} a^* & \dfrac{c^*}{\phi} \\ -\phi c^* & a^* \end{pmatrix},$$

$$\mathbf{N_1}(\tau^*) = \begin{pmatrix} y_{1p}\left(c^* j - \dfrac{\alpha}{\phi} a^* l + f^*\right) & y_{1p}\left(a^* l + \dfrac{\alpha}{\phi} c^* j - g^*\right) \\ y_{1p}\left(\phi c^* j + \alpha a^* l - \alpha g^*\right) & y_{1p}\left(\alpha a^* j - \phi c^* l - \alpha f^*\right) \end{pmatrix},$$

and where a^*, c^*, j, l, f^*, g^* are given by

$$\begin{aligned} &a^* = \cos(\phi\tau^*),\ c^* = \sin(\phi\tau^*),\ j = \cos(\beta), \\ &l = \sin(\beta),\ f^* = \cos(\alpha\tau^* + \beta),\ g^* = \sin(\alpha\tau^* + \beta). \end{aligned} \tag{2.44}$$

For $\alpha^2 = \phi^2$ a particular solution of (2.40) on the time-interval $nT < \tau < nT + T_0$ is given by:

$$y_p(\tau) = \frac{A}{2}\phi\tau \sin(\phi\tau + \beta),$$

and an expression almost similar to (2.43) can be given. At $\tau^* = T_0$ the function $h(\tau)$ in (2.39) has a jump discontinuity. As in section 2.2 of this chapter it follows for $\tau^* = T_0^+$ that

$$\begin{pmatrix} y_n(\tau^*) \\ y_n'(\tau^*) \end{pmatrix} = \mathbf{M_2}\mathbf{M_1}(T_0) \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{M_2}\mathbf{N_1}(T_0) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{2.45}$$

where

$$\mathbf{M_2} = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\phi^2} \end{pmatrix}.$$

For $T_0 < \tau^* < T$ equation (2.39) is given by

$$y'' + y = A\cos(\alpha\tau + \beta), \tag{2.46}$$

and for $\alpha^2 \neq 1$ a particular solution of (2.46) can be written as:

$$y_p(\tau) = y_{2p} cos(\alpha\tau + \beta), \tag{2.47}$$

where

$$y_{2p} = \frac{A}{1 - \alpha^2}. \tag{2.48}$$

The initial value problem (with $\alpha^2 \neq 1$) can easily be solved for $T_0 < \tau^* < T$, yielding

$$\begin{aligned} \begin{pmatrix} y_n(\tau^*) \\ y_n'(\tau^*) \end{pmatrix} &= \mathbf{M_3}(\tau^*)\mathbf{M_2}\mathbf{M_1}(T_0) \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \\ &+ (\mathbf{M_3}(\tau^*)\mathbf{M_2}\mathbf{N_1}(T_0) + \mathbf{N_3}(\tau^*)) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \end{aligned} \tag{2.49}$$

where

$$\mathbf{M_3}(\tau^*) = \begin{pmatrix} b^* & d^* \\ -d^* & b^* \end{pmatrix},$$

$$\mathbf{N_3}(\tau^*) = \begin{pmatrix} y_{2p}(\alpha d^* g - b^* f + p^*) & y_{2p}(\alpha d^* f + b^* g - q^*) \\ y_{2p}(\alpha b^* g + d^* f - \alpha q^*) & y_{2p}(\alpha b^* f - d^* g - \alpha p^*) \end{pmatrix},$$

and where b^*, d^*, p^*, q^*, f, g are given by

$$\begin{aligned} &b^* = \cos(\tau^* - T_0),\ d^* = \sin(\tau^* - T_0),\ p^* = \cos(\alpha\tau^* + \beta), \\ &q^* = \sin(\alpha\tau^* + \beta),\ f = \cos(\alpha T_0 + \beta),\ g = \sin(\alpha T_0 + \beta). \end{aligned} \tag{2.50}$$

For $\alpha^2 = 1$ a particular solution of (2.46) on the time-interval $nT + T_0 < \tau < (n+1)T$ is given by

$$y_p(\tau) = \frac{A}{2}\tau \sin(\tau + \beta),$$

and an expression almost similar to (2.49) can be given. At $\tau^* = T$ the function $h(\tau)$ in (2.39) has again a jump discontinuity. As in section 2.2 of this chapter it follows for $\tau^* = T$ that

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \begin{pmatrix} y_n(T^+) \\ y'_n(T^+) \end{pmatrix} = \mathbf{M_4 M_3}(T)\mathbf{M_2 M_1}(T_0) \begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} +$$

$$+(\mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0) + \mathbf{M_4 N_3}(T)) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{2.51}$$

where

$$\mathbf{M_4} = \begin{pmatrix} 1 & 0 \\ 0 & \phi^2 \end{pmatrix}.$$

From (2.51) the following map can be obtained:

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} + \mathbf{W}_n, \tag{2.52}$$

where $\mathbf{A}$ is given by (2.17), and where $\mathbf{W}_n$ is given by

$$\mathbf{W}_n = (\mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0) + \mathbf{M_4 N_3}(T)) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}. \tag{2.53}$$

Comparing (2.53) to (2.21) it should be observed that the nonhomogeneous term now explicitly depends on n.

The solution of the system of difference equations (2.52) is given by:

$$\begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} = \mathbf{A}^n \begin{pmatrix} y_0(0^+) \\ y'_0(0^+) \end{pmatrix} + \sum_{r=0}^{n-1} \mathbf{A}^r \mathbf{W}_r. \tag{2.54}$$

For $\alpha^2 \neq \phi^2$ and $\alpha^2 \neq 1$ the vector $\mathbf{W}_r$ is given by:

$$\mathbf{W}_r = \begin{pmatrix} w_{11}\cos(\alpha r T) + w_{12}\sin(\alpha r T) \\ w_{21}\cos(\alpha r T) + w_{22}\sin(\alpha r T) \end{pmatrix}, \tag{2.55}$$

where

$$\begin{aligned}
w_{11} &= y_{1p} b\left(\frac{\alpha}{\phi} cl - aj + f\right) + \\
&+ y_{1p}\frac{d}{\phi^2}(\phi cj + \alpha al - \alpha g) + y_{2p}(\alpha dg - bf + p), \\
w_{12} &= y_{1p} b\left(\frac{\alpha}{\phi} cj + al - g\right) + \\
&+ y_{1p}\frac{d}{\phi^2}(\alpha aj - \phi cl - \alpha f) + y_{2p}(\alpha df + bg - q), \\
w_{21} &= -y_{1p} d\phi^2\left(\frac{\alpha}{\phi} cl - aj + f\right) + \\
&+ y_{1p} b(\phi cj + \alpha al - \alpha g) + y_{2p}\phi^2(\alpha bg + df - \alpha q), \\
w_{22} &= -y_{1p} d\phi^2\left(\frac{\alpha}{\phi} cj + al - g\right) + \\
&+ y_{1p} b(\alpha aj - \phi cl - \alpha f) + y_{2p}\phi^2(\alpha bf - dg - \alpha p).
\end{aligned} \tag{2.56}$$

For $\alpha^2 = \phi^2$ the vector $\mathbf{W}_r$ is given by (2.55) with

$$\begin{aligned}
w_{11} &= y_{1p} 2b(\phi T_0 g - cl) + y_{1p}\frac{2d}{\phi}(\phi T_0 f + cj) + \\
&+ y_{2p}(\phi dg - bf + p), \\
w_{12} &= y_{1p} 2b(\phi T_0 f - cj) - y_{1p}\frac{2d}{\phi}(\phi T_0 g + cl) + \\
&+ y_{2p}(\phi df + bg - q), \\
w_{21} &= -y_{1p} 2\phi^2 d(\phi T_0 g - cl) + y_{1p} 2\phi b(\phi T_0 f + cj) + \\
&+ y_{2p}\phi^2(\phi bg + df - \phi q), \\
w_{22} &= -y_{1p} 2\phi^2 d(\phi T_0 f - cj) - y_{1p} 2\phi b(\phi T_0 g + cl) + \\
&+ y_{2p}\phi^2(\phi bf - dg - \phi p).
\end{aligned} \tag{2.57}$$

And for $\alpha^2 = 1$ the vector $\mathbf{W}_r$ is given by (2.55) with

$$\begin{aligned}
w_{11} &= y_{1p} b\left(\frac{cl}{\phi} - aj + f\right) + y_{1p}\frac{d}{\phi^2}(\phi cj + al - g) + \\
&+ y_{2p}(2(T - T_0)q + p - p_1),
\end{aligned}$$

$$
\begin{aligned}
w_{12} &= y_{1p} b \left(\frac{cj}{\phi} + al - g \right) + y_{1p} \frac{d}{\phi^2} (aj - \phi cl - f) + \\
&+ y_{2p} (2(T - T_0)p - q - q_1),
\end{aligned} \tag{2.58}
$$

$$
\begin{aligned}
w_{21} &= -y_{1p} \phi^2 d \left(\frac{cl}{\phi} - aj + f \right) + y_{1p} b (\phi cj + al - g) + \\
&+ y_{2p} \phi^2 (2(T - T_0)p + q + q_1),
\end{aligned}
$$

$$
\begin{aligned}
w_{22} &= -y_{1p} \phi^2 d \left(\frac{cj}{\phi} + al - g \right) + y_{1p} b (aj - \phi cl - f) - \\
&- y_{2p} \phi^2 (2(T - T_0)q + p + p_1).
\end{aligned}
$$

Coefficients a, b, c, d are given by (2.18), j, l are given by (2.44), and:
$p = \cos(\alpha T + \beta)$, $q = \sin(\alpha T + \beta)$, $p_1 = \cos(T - 2T_0 - \beta)$, $q_1 = \sin(T - 2T_0 - \beta)$.

2.3.2 The amplitude increase after one period T due to harmonic forcing.

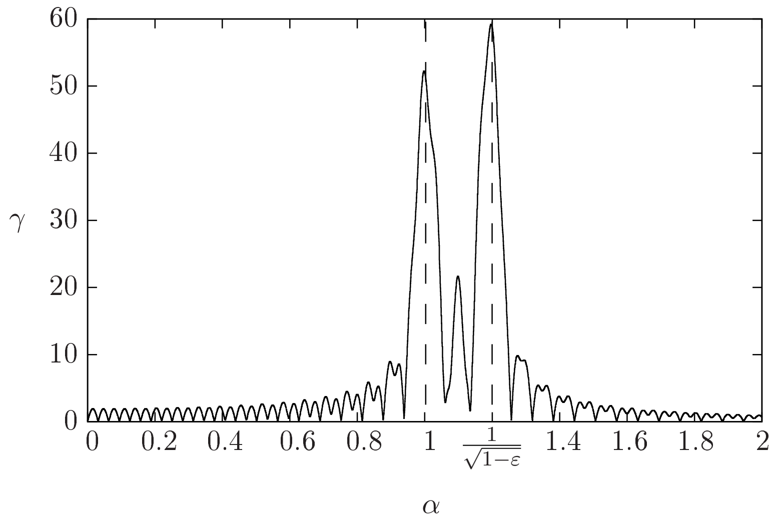

Figure 2.2: The maximum amplitude γ as function of α for $A = 1$, $T_0 = 100$, $T = 200$, $\delta = \pi/7$, and $\varepsilon = 0.3$.

In this section the possible amplitude increase of the displacement function $y(\tau)$ (after one period T) due to the external, harmonic force will be studied. From (2.54) and (2.55) it can easily be seen that this increase is completely determined by:

$$
w_{11} \cos(\alpha n T) + w_{12} \sin(\alpha n T) = \sqrt{w_{11}^2 + w_{12}^2} \sin(\alpha n T + \delta), \tag{2.59}
$$

where δ - is given by $\sin(\delta) = \dfrac{w_{11}}{\gamma}$ and $\cos(\delta) = \dfrac{w_{12}}{\gamma}$ in which:

$$
\gamma = \sqrt{w_{11}^2 + w_{12}^2}. \tag{2.60}
$$

The maximum amplitude response (in absolute value) is γ. Obviously, γ depends on $\alpha, A, T_0, T, \delta$, and ε. In Figure 2.2 γ as function of α is plotted for $A = 1$, $T_0 = 100$, $T = 200$, $\delta = \pi/7$ and $\varepsilon = 0.3$.

In Figure 2.2 it can be seen that there are two peaks. These two peaks are a consequence of the change of mass of the oscillator, and so the oscillator actually has two resonance frequencies (1 and $(1-\varepsilon)^{-1/2}$). Since only one period T for the amplitude response is considered these maximum amplitude responses are of course bounded. In Figure 2.3 an optimization program has been used to show the maximum amplitude responses when $A = 1$, T_0 and T are varied such that $0 < T_0 < T < 20$, $\delta = \pi/7$, and $\varepsilon = 0.3$. Similar results can be obtained for other values of A, T_0, T, δ, and ε. For instance, in Figure 2.4 the results have been shown for $A = 1$, $0 < T_0 < T < 100$, $\delta = \pi/7$, and $\varepsilon = 0.3$.

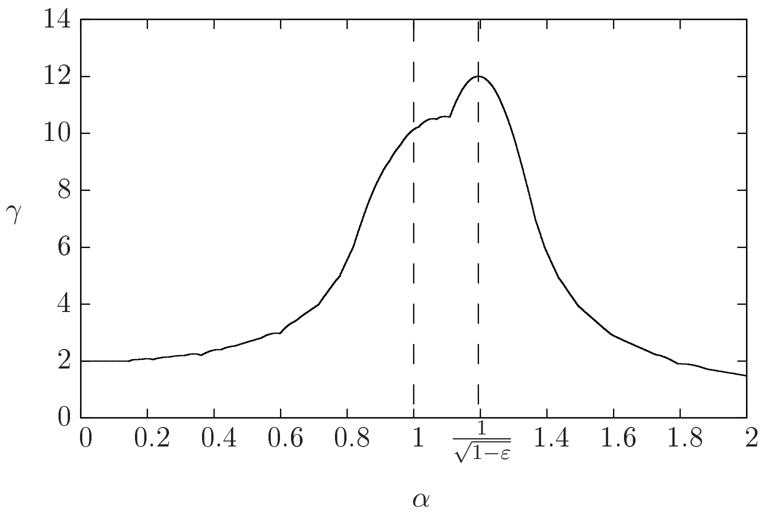

Figure 2.3: The maximum amplitude γ as function of α for $A = 1$, $0 < T_0 < T < 20$, $\delta = \pi/7$, and $\varepsilon = 0.3$.

2.3.3 Stability properties of the solution, and resonance.

In this subsection the stability properties and boundedness of the solution of (2.52) will be studied. In fact the solution has to satisfy

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} +$$

$$+(\mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0)\mathbf{M_4 N_3}(T)) \begin{pmatrix} \cos(\alpha n T) \\ \sin(\alpha n T) \end{pmatrix}, \qquad (2.61)$$

where $\mathbf{A}$ and $\mathbf{M_4}$, $\mathbf{M_3}(T)$, $\mathbf{M_2}$, $\mathbf{N_1}(T_0)$, and $\mathbf{N_3}$ are defined in subsection 2.3.1. It should be observed that in (2.61) the matrices $\mathbf{A}$ and $\mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0) + \mathbf{M_4 N_3}(T)$ are both n independent matrices. For simplicity $\mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0) + \mathbf{M_4 N_3}(T)$ will be denoted by $\mathbf{B}$, and the system of two first order ordinary difference

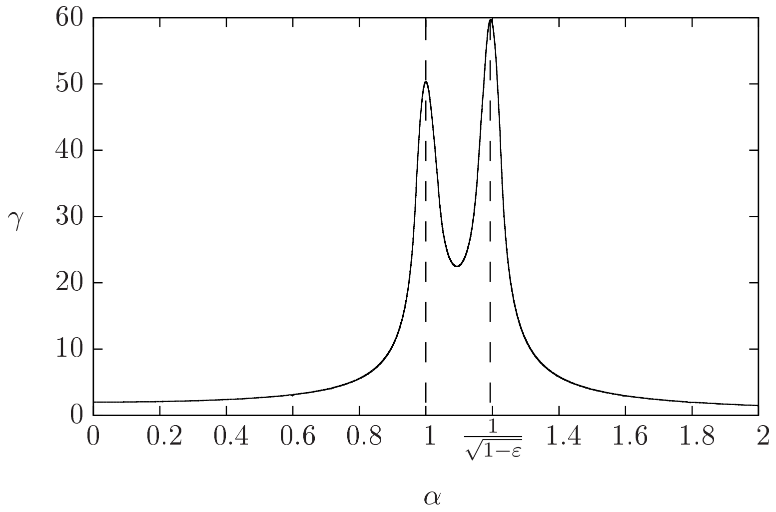

Figure 2.4: The maximum amplitude γ as function of α for $A = 1$, $0 < T_0 < T < 100$, $\delta = \pi/7$, and $\varepsilon = 0.3$.

equations (2.61) will be reduced to a single second order difference equation for $y_n(0^+) = y_n$, yielding:

$$\begin{aligned} y_{n+2} - (a_{11} + a_{22})y_{n+1} + (a_{11}a_{22} - a_{12}a_{21})y_n = \\ = c_0 \cos(\alpha n T) + s_0 \sin(\alpha n T) + \\ + c_1 \cos(\alpha(n+1)T) + s_1 \sin(\alpha(n+1)T), \end{aligned} \tag{2.62}$$

where $a_{ij}(i, j = 1, 2)$ are the components of matrix $\mathbf{A}$, and

$$c_0 = b_{21}a_{12} - b_{11}a_{22}, \;\; s_0 = b_{22}a_{12} - b_{12}a_{22},$$
$$c_1 = b_{11}, \;\; s_1 = b_{12},$$

and where $b_{ij}(i, j = 1, 2)$ are the components of matrix $\mathbf{B} = \mathbf{M_4 M_3}(T)\mathbf{M_2 N_1}(T_0) + \mathbf{M_4 N_3}(T)$ which are explicitly given by (2.56) - (2.58). In (2.62) $a_{11} + a_{22} = tr(\mathbf{A})$ is the trace of matrix $\mathbf{A}$, and $a_{11}a_{22} - a_{12}a_{21} = det(\mathbf{A})$ is the determinant of matrix $\mathbf{A}$ which is equal to 1 (see [25]). The solution y_n of (2.62) can be written as:

$$y_n = y_{h,n} + y_{p0,n} + y_{p1,n}, \tag{2.63}$$

where $y_{h,n}$ is the solution of the homogeneous equation (related to (2.62)):

$$y_{h,n+2} - tr(\mathbf{A})y_{h,n+1} + y_{h,n} = 0, \tag{2.64}$$

and where $y_{pm,n}$ (with $m = 0, 1$) are the particular solutions of (2.62) satisfying

$$y_{pm,n+2} - tr(\mathbf{A})y_{pm,n+1} + y_{pm,n} = c_m \cos(\alpha(n+m)T) + s_m \sin(\alpha(n+m)T). \tag{2.65}$$

The roots of the characteristic equation belonging to the homogeneous equation (2.64) are given by:

$$\lambda_{1,2} = \frac{1}{2}tr(\mathbf{A}) \pm \frac{1}{2}\sqrt{(tr(\mathbf{A}))^2 - 4},$$

and are, of course, coinciding with the eigenvalues of matrix $\mathbf{A}$. The corresponding stability properties of the homogeneous solution $y_{h,n}$ can be found in Table 2.1 or in [25].

The particular solutions $y_{pm,n}$ of (2.65) can be found in the following way. First one looks for a particular solution $y_{pm,n}$ in the form:

$$y_{pm,n} = C_{1m}\cos(\alpha(n+m)T) + C_{2m}\sin(\alpha(n+m)T), \tag{2.66}$$

where C_{1m} and C_{2m} are constants to be determined. By substituting (2.66) into (2.65), and then by collecting the coefficients of $\cos(\alpha(n+m)T)$ and of $\sin(\alpha(n+m)T)$ it follows that C_{1m} and C_{2m} have to satisfy

$$\begin{pmatrix} \cos(2\alpha T) - tr(\mathbf{A})\cos(\alpha T) + 1 & \sin(2\alpha T) - tr(\mathbf{A})\sin(\alpha T) \\ -\sin(2\alpha T) + tr(\mathbf{A})\sin(\alpha T) & \cos(2\alpha T) - tr(\mathbf{A})\cos(\alpha T) + 1 \end{pmatrix} \times \\ \times \begin{pmatrix} C_{1m} \\ C_{2m} \end{pmatrix} = \begin{pmatrix} c_m \\ s_m \end{pmatrix}. \tag{2.67}$$

The difference equation (2.62) has a unique solution when two initial conditions are given. And so, the particular solutions $y_{pm,n}$ can be determined uniquely. To have a unique particular solution $y_{pm,n}$ it follows from (2.67) that the determinant of the coefficient matrix in (2.67) should be nonzero. When the determinant is equal to zero then there are infinitely many solutions or there is no solution. This will occur when:

$$\begin{cases} \cos(2\alpha T) - tr(\mathbf{A})\cos(\alpha T) + 1 = 0, \quad \text{and} \\ \sin(2\alpha T) - tr(\mathbf{A})\sin(\alpha T) = 0, \end{cases} \tag{2.68}$$

or equivalently when:

$$tr(\mathbf{A}) = 2\cos(\alpha T). \tag{2.69}$$

So, the particular solutions $y_{pm,n}$ can be determined uniquely when $tr(\mathbf{A}) \neq 2\cos(\alpha T)$. When $tr(\mathbf{A}) = 2\cos(\alpha T)$ the particular solutions $y_{pm,n}$ will have the following form:

$$y_{pm,n} = n(\tilde{C}_{1m}\cos(\alpha(n+m)T) + \tilde{C}_{2m}\sin(\alpha(n+m)T)), \tag{2.70}$$

where $\tilde{C}_{1m}$ and $\tilde{C}_{2m}$ are constants to be determined. By substituting (2.70) into (2.65), and then by collecting the coefficients of $\cos(\alpha(n+m)T)$ and of $\sin(\alpha(n+m)T)$ it follows that $\tilde{C}_{1m}$ and $\tilde{C}_{2m}$ have to satisfy:

$$\begin{pmatrix} 2\cos(2\alpha T) - tr(\mathbf{A})\cos(\alpha T) & 2\sin(2\alpha T) - tr(\mathbf{A})\sin(\alpha T) \\ -2\sin(2\alpha T) + tr(\mathbf{A})\sin(\alpha T) & 2\cos(2\alpha T) - tr(\mathbf{A})\cos(\alpha T) \end{pmatrix} \times \\ \times \begin{pmatrix} \tilde{C}_{1m} \\ \tilde{C}_{2m} \end{pmatrix} = \begin{pmatrix} c_m \\ s_m \end{pmatrix}. \tag{2.71}$$

Again to have a unique particular solution $y_{pm,n}$ (in the form (2.70)) it follows from (2.71) that the determinant of the coefficient matrix in (2.71) should be nonzero.

stability properties / $tr(\mathbf{A})$	the oscillator for $\mathbf{W}_n \neq \mathbf{0}$ is
$-2 < tr(\mathbf{A}) < 2$ ($\lvert\lambda_{1,2}\rvert = 1$)	only unstable when $tr(\mathbf{A}) = 2\cos(\alpha T)$, else stable
$tr(\mathbf{A}) < -2$ or $tr(\mathbf{A}) > 2$ ($\lvert\lambda_j\rvert > 1$ for $j = 1$ or $j = 2$)	always unstable
$tr(\mathbf{A}) = 2$ ($\lambda_1 = \lambda_2 = 1$)	only stable when $c = d = 0$ and $ab = 1$ in matrix $\mathbf{A}$, and αT is not an even multiple of π, else unstable
$tr(\mathbf{A}) = -2$ ($\lambda_1 = \lambda_2 = -1$)	only stable when $c = d = 0$ and $ab = -1$ in matrix $\mathbf{A}$, and αT is not an odd multiple of π, else unstable.

Table 2.3: Stability properties of the oscillator with a harmonic external force when $\mathbf{W}_n \neq \mathbf{0}$.

When the determinant is equal to zero there are infinitely many solutions or there is no solution. This will occur when:

$$\begin{cases} 2\cos(2\alpha T) - tr(\mathbf{A})\cos(\alpha T) = 0, \quad \text{and} \\ 2\sin(2\alpha T) - tr(\mathbf{A})\sin(\alpha T) = 0, \end{cases} \tag{2.72}$$

or equivalently when:

$$tr(\mathbf{A}) = \pm 2 \quad \text{and} \quad \sin(\alpha T) = 0. \tag{2.73}$$

So, when $tr(\mathbf{A}) = 2\cos(\alpha T)$ and αT is not a multiple of π then the particular solution $y_{pm,n}$ will grow linearly in n (see (2.70)). The condition (2.69), that is $tr(\mathbf{A}) = 2\cos(\alpha T)$ will be called a resonance condition for that reason. The case $tr(\mathbf{A}) = 2\cos(\alpha T)$ and αT is a multiple of π still has to be studied. When αT is an even multiple of π the system of difference equations (2.61) becomes:

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} + \mathbf{B} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{2.74}$$

and $tr(\mathbf{A}) = 2\cos(\alpha T) = 2$. System (2.74) with $tr(\mathbf{A}) = 2$ already has been studied in section 2.2.2 of this chapter. From Table 2.2 it follows that the solution of (2.74) is unstable. Similarly, when αT is an odd multiple of π the system of difference equations (2.61) becomes:

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} + \mathbf{B} \begin{pmatrix} (-1)^n \\ 0 \end{pmatrix}, \tag{2.75}$$

and $tr(\mathbf{A}) = 2\cos(\alpha T) = -2$. Since the eigenvalues of matrix $\mathbf{A}$ are both equal to -1 it is not difficult to see that the particular solution of (2.75) will contain unbounded terms in n. So, also in this case the solution of (2.75) is unstable. All of the stability properties of the solution of the oscillator equation (2.39) with an external harmonic force are summarized in Table 2.3.

2.4 Conclusions and remarks

In this chapter the stability properties of the forced vibrations of a linear, single degree of freedom oscillator with a periodically and stepwise changing time-varying mass have been studied. Two types of forcing have been studied. First, a forcing has been investigated, due to a mass which hits the oscillator, stays for some time at the oscillator, and then leaves the oscillator. The stability properties of the oscillator, and the existence and (non) uniqueness of periodic vibrations have been studied in detail in section 2.2 of this chapter. Secondly, an external, harmonic forcing has been studied for an oscillator to which a mass (with zero velocity) is added for some time, and then is taken away (with zero velocity). For this case an interesting resonance condition, which relate the properties of the system to the frequency of the external excitations, has been found, and the stability properties of the oscillator problem have been presented in section 2.3 of this chapter. When both forcing types are applied to the oscillator the results as obtained in section 2.2 and in section 2.3 of this chapter can be combined, because the differential equation describing the problem is linear. It is also interesting to see in section 2.3 that due to the changing mass and due to the external harmonic forcing the instability region shows two peaks. For a similar oscillator equation with a constant mass and an external, harmonic forcing one usually has one peak in the instability region. This larger instability region might perhaps explain in part the instability mechanism for rain-wind induced oscillations of cables in windfields. Usually cables in windfields are stable, but due to rain these cables can become unstable. Water addition to the cables, water drop off, and water rivulets on these cables (and so, changing aerodynamic forcing acting on the cable), and changing eigenfrequencies of the cable system certainly enlarge the instability regions of these cables.

To obtain more realistic mathematical models for these rain-wind induced oscillations of cables in wind fields one might consider periodically and multi-stepwise changing time-varying masses. Other external forces (such as nonlinear drag-and-lift forces, damping forces, and so on) can also be included in the model equation. The aforementioned extensions to the model equation can be interesting subjects for future research.

Chapter 3

A linear case with damping

Abstract: In this chapter the vibrations of a damped, linear, single degree of freedom oscillator (sdofo) with a time-varying mass will be considered. Both the free and forced vibrations of the oscillator will be studied. For the free vibrations the minimal damping rates will be computed, for which the oscillator is always stable. The forced vibrations are partly due to small masses, which are periodically hitting and leaving the oscillator with different velocities. Since these small masses stay for some time on the oscillator surface the effective mass of the oscillator will periodically vary in time. Additionally, an external harmonic force will be applied to the oscillator. Not only solutions of the oscillator equations will be constructed, but also stability properties for the free, and for the forced vibrations will be presented for various parameter values. For the external, harmonic forcing case an interesting resonance condition will be derived.

3.1 Introduction

In practice systems with time-varying masses frequently occur. These systems can be found in conveyor systems, robotics, cranes, in fluid-structure interaction problems, and in many other systems [5, 9, 22]. For these mechanical constructions the 1-mode Galerkin approximation of the continuous model will lead to a sdofo equation. These sdofo's are usually considered to be representative models for studying forces which are acting on the system. In this chapter the free and the forced oscillations of a damped, linear sdofo with a (periodically and step-wise changing) time-varying mass will be studied. The free, undamped oscillations, and the forced, undamped oscillations have been recently studied in [25] and [27] respectively. Some first results on nonlinear vibrations of these sdofos can be found in [8, 16]. In this chapter the oscillations of a sdofo with a linear restoring force and a mass which varies in time according to a periodic step-wise dependence will be considered. This model is perhaps the simplest model which describes the vibrations of a cable with rainwater located on it. Part of the raindrops hitting the cable will

This chapter is a slightly revised version of [26]: W.T. van Horssen, O.V. Pischanskyy, On the stability properties of a damped oscillator with a periodically time-varying mass, *Journal of Sound and Vibration*, 330 (13):3257-3269, 2011; and A. Pischanskyy, W.T. van Horssen, On a simple model for the rain-wind induced oscillations of a cable, *Proceedings of the 9th UK conference on Wind Engineering*, Bristol, 20-22 September 2010.

remain on the surface of the cable for some time, and will subsequently be blown or shaken off after some time. It will be assumed when mass is added to or separated from the oscillator that the position of the center of the (total) mass of the oscillator is not influenced. The following equation of motion for the sdofo can now be derived (see also for instance [9, p. 152]):

$$M\ddot{y} = \dot{M}(w - \dot{y}) - \delta\dot{y} - ky + F, \tag{3.1}$$

where $y = y(t)$ is the displacement of the oscillator (see Fig. 3.1), $M = M(t)$ is the time-varying mass of the oscillator, $w = w(t)$ is the mean velocity at which masses (i.e. raindrops) are hitting or leaving the oscillator, δ and k are the (positive) damping and stiffness coefficients, respectively, $F = F(t)$ or $F = F(t, y, \dot{y})$ is an external force (for instance, a windforce), and the dot denotes differentiation with respect to t. In the model it is assumed that the mass of for instance the raindrop which is hitting (or leaving) the oscillator, is the same mass which is added to (or taken away from) the oscillator. The force F and the velocity w are measured positive in positive y direction (see Fig. 3.1). Following [25, 27] it turns out to

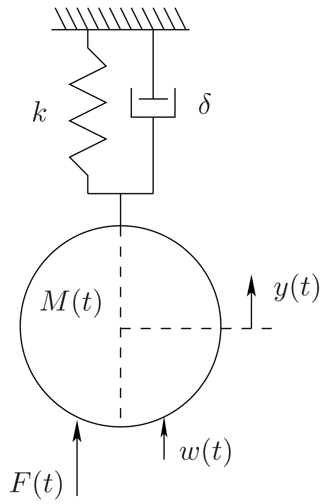

Figure 3.1: The single degree of freedom oscillator with damping.

be convenient to separate the mass $M(t)$ into a time invariant part M_0 and into a time-varying part $m(t)$, that is $M(t) = M_0 - m(t)$, where M_0 is a positive constant, and $M_0 - m(t) > 0$. Then it follows that Eq. (3.1) can be rewritten in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((M_0 - m(t))\frac{\mathrm{d}y}{\mathrm{d}t}\right) + \delta\frac{\mathrm{d}y}{\mathrm{d}t} + ky = \frac{-\mathrm{d}m}{\mathrm{d}t}w + F. \tag{3.2}$$

Then, by introducing the time-rescaling $t = \sqrt{M_0/k}\tau$, $2p = \delta/\sqrt{kM_0}$, $\tilde{y}(\tau) = y(\sqrt{M_0/k}\tau)$, $\tilde{m}(\tau) = m(\sqrt{M_0/k}\tau)$, $\tilde{w}(\tau) = w(\sqrt{M_0/k}\tau)$ and $\tilde{F}(\tau) = \frac{1}{k}F(\sqrt{M_0/k}\tau)$ Eq. (3.2) can be rewritten into

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left(1 - \frac{\tilde{m}(\tau)}{M_0}\right)\frac{\mathrm{d}\tilde{y}(\tau)}{\mathrm{d}\tau}\right) + 2p\frac{\mathrm{d}\tilde{y}(\tau)}{\mathrm{d}\tau} + \tilde{y}(\tau) = \frac{-\tilde{w}(\tau)}{\sqrt{M_0 k}}\frac{\mathrm{d}\tilde{m}(\tau)}{\mathrm{d}\tau} + \tilde{F}(\tau). \tag{3.3}$$

Now it will be assumed that $h(\tau) = \tilde{m}(\tau)/M_0$ is a periodic step function with $1 - h(\tau) > 0$, that is,

$$h(\tau) = \begin{cases} \varepsilon & \text{for } 0 < \tau < T_0, \\ 0 & \text{for } T_0 < \tau < T, \end{cases} \tag{3.4}$$

and $h(\tau + T) = h(\tau)$, and ε is a constant (in practice usually small) with $0 < \varepsilon < 1$. Also $\tilde{w}(\tau)$ is assumed to be T-periodic. It should be observed that in the analysis ε is defined to be the quotient m/M_0, where m is the mass which added at time T_0, and where M_0 is the mass of the oscillator. So, ε can be seen as a measure for the relative mass which is added at time T_0. For convenience the tildes in Eq. (3.3) will be dropped, and the prime will be introduced to denote differentiation with respect to τ, yielding

$$((1 - h(\tau))\, y'(\tau))' + 2py'(\tau) + y(\tau) = \frac{-w(\tau)\, \omega_0}{k}\, m'(\tau) + F(\tau), \tag{3.5}$$

where $\omega_0 = \sqrt{k/M_0}$ is the undamped natural frequency of the oscillator. The initial displacement and the initial velocity of $y(\tau)$ are given by

$$y(0) = y_0 \text{ and } y'(0) = y_0'. \tag{3.6}$$

The initial value problem (3.5) - (3.6) has (in our opinion) not been studied before in the literature. Only when the damping parameter p is equal to zero the stability properties of the oscillator have been presented in the chapter 2 and in [25, 27]. In the literature a related, homogeneous equation has been studied extensively $x''(\tau) + (1 - h(\tau)x(\tau) = 0$, where $h(\tau)$ is given by (3.4). This equation was introduced in 1918 by Meissner, and is nowadays known as Meissner's equation. The stability diagrams for this oscillator equation with a periodically and step-wise changing stiffness coefficient can for instance be found in [14, 23]. In this chapter a fairly complete treatment of the initial value problem (3.5) - (3.6) for the damped, and externally forced oscillator with a periodically and step-wise changing time-varying mass will be given.

This chapter is organised as follows. In section 2 the initial value problem (3.5) - (3.6) will be studied with $w(\tau) \equiv 0$ and $F(\tau) \equiv 0$, that is, the free vibrations of the oscillator will be studied. Depending on the value of the damping parameter p and mass ratio ε different cases have to be considered in the section. For the free vibrations the minimum value of the damping parameter p will also be determined in the section 2, such that for a given value of ε the oscillator is stable for all values of T_0 and T. In section 3 the initial value problem (3.5) - (3.6) will be studied with $F(\tau) \equiv 0$. In this case the small masses which are periodically hitting and leaving the oscillator (with nonzero velocities) can be seen as an external force acting on the oscillator. The stability of the solution(s) of the initial value problem will be studied. In section 4 it will be assumed that $w(\tau) \equiv 0$, and that the force $F(\tau)$ is a harmonic force, that is, $F(\tau) = A\cos(\alpha\tau + \beta)$, where A and β are constants, and where α is the frequency of the external force. The stability of the solutions will be studied as well as the existence of resonance frequencies (depending on α). Finally, in section 5 of this chapter some conclusions will be drawn, and remarks will be made about future research on this subject.

3.2 The free vibrations

In this section the initial value problem (3.5) - (3.6) with $w(\tau) \equiv 0$ and $F(\tau) \equiv 0$ will be studied, that is,

$$((1 - h(\tau))y'(\tau))' + 2py'(\tau) + y(\tau) = 0, \tag{3.7}$$

with $y(0) = y_0$, $y'(0) = y'_0$, $p > 0$, and $h(\tau)$ given by (3.4). To solve this initial value problem on, for instance, the interval $0 < \tau < T^+$ we have to split up this interval into $0 < \tau < T_0^-$, $T_0^- \leq \tau \leq T_0^+$, $T_0^+ < \tau < T^-$, $T^- \leq \tau \leq T^+$, where $T_0^- = T_0 - 0$, $T_0^+ = T_0 + 0$, $T^- = T - 0$, and $T^+ = T + 0$. On the first time-interval $0 < \tau < T_0^-$ Eq. (3.7) becomes

$$(1 - \varepsilon)y'' + 2py' + y = 0 \tag{3.8}$$

subject to the initial conditions $y(0) = y_0$ and $y'(0) = y'_0$. The solution of this initial value problem for Eq. (3.8) can readily be obtained, yielding on $0 < \tau < T_0^-$

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_1}(\tau) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{3.9}$$

where the fundamental matrix $\mathbf{M_1}(\tau)$ depends on p, that is on the roots $r_{1,2}$ of the characteristic equation of Eq. (3.8), where $r_{1,2}$ are given by

$$r_{1,2} = \frac{-p}{1 - \varepsilon} \pm \frac{\sqrt{p^2 - (1 - \varepsilon)}}{1 - \varepsilon}. \tag{3.10}$$

On the second, infinitesimal small time-interval $T_0^- \leq \tau \leq T_0^+$ we have to observe that the displacement of the oscillator is continuous, and that the impulse of the system at $\tau = T_0^+$ is equal to the impulse of the system at $\tau = T_0^-$ plus the impulse of the raindrop (which hits the oscillator). The continuity of the displacement at $\tau = T_0$ implies that $y(T_0^-) = y(T_0^+)$, and the impulse condition can be obtained by integrating Eq. (3.7) with respect to τ from $\tau = T_0^-$ to $\tau = T_0^+$, yielding $y'(T_0^+) - (1 - \varepsilon)y'(T_0^-) = 0$. And so,

$$\begin{pmatrix} y(T_0^+) \\ y'(T_0^+) \end{pmatrix} = \mathbf{M_2}(T_0) \begin{pmatrix} y(T_0^-)(\tau) \\ y'(T_0^-)(\tau) \end{pmatrix} = \mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{3.11}$$

where $\mathbf{M_2}(T_0)$ is given by $\mathbf{M_2}(T_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \varepsilon \end{pmatrix}$. On the time-interval $T_0^+ < \tau < T^-$ we have to solve

$$y'' + 2py' + y = 0 \tag{3.12}$$

subject to the initial conditions at $\tau = T_0^+$ and given by Eq. (3.11). The solution of this initial value problem for Eq. (3.12) can easily be obtained, yielding on $T_0^+ < \tau < T^-$

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_3}(\tau)\mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix}, \tag{3.13}$$

where the 'fundamental' matrix $\mathbf{M_3}(\tau)$ depends on p, that is, on the roots $r_{1,2}$ of the characteristic equation of Eq. (3.12), where $r_{1,2}$ are given by

$$r_{1,2} = -p \pm \sqrt{p^2 - 1}. \tag{3.14}$$

On the infinitesimal small time-interval $T^- \leq \tau \leq T^+$ we have to observe again that the displacement of the oscillator is continuous, and that the impulse of the system at $\tau = T^+$ is equal to the impulse of the system at $\tau = T^-$ plus the impulse of the raindrop (which leaves the oscillator). The continuity of the displacement at $\tau = T$ simply implies that $y(T^-) = y(T^+)$, and the impulse condition can be obtained by integrating Eq. (3.7) with respect to τ from $\tau = T^-$ to $\tau = T^+$, yielding $(1-\varepsilon)y'(T^+) - y'(T^-) = 0$. And so,

$$\begin{aligned} \begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} &= \mathbf{M_4}(T) \begin{pmatrix} y(T^-) \\ y'(T^-) \end{pmatrix} = \\ &= \mathbf{M_4}(T)\mathbf{M_3}(T)\mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}, \end{aligned} \tag{3.15}$$

where $\mathbf{M_4}(T)$ is given by $\mathbf{M_4}(T) = \begin{pmatrix} 1 & 0 \\ 0 & (1-\varepsilon)^{-1} \end{pmatrix}$. So, the solution of Eq. (3.7) on the interval $0 \leq \tau \leq T^+$ has been constructed, and at $\tau = T^+$ the solution is given by

$$\begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}, \tag{3.16}$$

where

$$\mathbf{A} = \mathbf{M_4}(T)\mathbf{M_3}(T)\mathbf{M_2}(T_0)\mathbf{M_1}(T_0). \tag{3.17}$$

To compute the solution at $2T^+, 3T^+, \ldots, (n+1)T^+$ the procedure can be repeated, giving the following system of difference equations

$$\begin{pmatrix} y((n+1)T^+) \\ y'((n+1)T^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y(nT^+) \\ y'(nT^+) \end{pmatrix} \tag{3.18}$$

for $n = 0, 1, 2, \ldots$. The stability properties of the oscillator are completely determined by the eigenvalues $\lambda_{1,2}$ of matrix $\mathbf{A}$. By putting

$$\begin{gathered} \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \mathrm{tr}(\mathbf{A}) = a_{11} + a_{22}, \\ \det(\mathbf{A}) = a_{11}a_{22} - a_{12}a_{21}, \quad \text{and} \quad \Delta = \mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A}) \end{gathered} \tag{3.19}$$

it follows that the eigenvalues $\lambda_{1,2}$ are given by

$$\lambda_{1,2} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) \pm \frac{1}{2}\sqrt{\mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) \pm \frac{1}{2}\sqrt{\Delta}. \tag{3.20}$$

Now we have to consider three cases: $\Delta < 0$, $\Delta = 0$, and $\Delta > 0$. When $\Delta < 0$ we have two complex-valued eigenvalues, and $|\lambda_{1,2}|^2 = \frac{1}{4}\mathrm{tr}^2(\mathbf{A}) + \frac{1}{4}(4\det(\mathbf{A}) - \mathrm{tr}^2(\mathbf{A})) =$

det($\mathbf{A}$). So, when $\Delta < 0$ we have stability if and only if $\det(\mathbf{A}) \leq 1$. When $\Delta > 0$ we have two real and distinct eigenvalues. To have stability in this case the eigenvalues have to satisfy $-1 \leq \lambda_1 \leq 1$ and $-1 \leq \lambda_2 \leq 1$, or equivalently $-2 \leq \mathrm{tr}(\mathbf{A}) + \sqrt{\Delta} \leq 2$ and $-2 \leq \mathrm{tr}(\mathbf{A}) - \sqrt{\Delta} \leq 2$. For $\Delta > 0$ it is obvious that when $\mathrm{tr}(\mathbf{A}) \geq 2$ or $\mathrm{tr}(\mathbf{A}) \leq -2$ then $|\lambda_1| \geq 1$ or $|\lambda_2| \geq 1$. So, for $|\mathrm{tr}(\mathbf{A})| \geq 2$ the system is unstable. Then, for $|\mathrm{tr}(\mathbf{A})| \leq 2$ we should have for stability: $-2-\mathrm{tr}(\mathbf{A}) \leq \sqrt{\Delta} \leq 2-\mathrm{tr}(\mathbf{A})$ and $-2+\mathrm{tr}(\mathbf{A}) \leq \sqrt{\Delta} \leq 2+\mathrm{tr}(\mathbf{A})$, or equivalently $\sqrt{\Delta} \leq 2-\mathrm{tr}(\mathbf{A})$ and $\sqrt{\Delta} \leq 2+\mathrm{tr}(\mathbf{A})$, or equivalently (after squaring and rearranging) $-1-\det(\mathbf{A}) \leq \mathrm{tr}(\mathbf{A}) \leq 1 + \det(\mathbf{A})$. When $\Delta = 0$ we have two real coinciding eigenvalues, that is, $\lambda_{1,2} = \frac{1}{2}\mathrm{tr}(\mathbf{A})$. Obviously, we will have (asymptotic) stability for $|\mathrm{tr}(\mathbf{A})| < 2$, and instability for $|\mathrm{tr}(\mathbf{A})| > 2$. When $|\mathrm{tr}(\mathbf{A})| = 2$, or equivalently when $\lambda_{1,2} = 1$ or $\lambda_{1,2} = -1$ we will have stability only when the dimension of the eigenspace of matrix $\mathbf{A}$ is two, else we will have instability. All of the stability properties of the oscillator

Stability properties for $\Delta = \mathrm{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})$	The oscillator for $w(\tau) \equiv 0$ and $F(\tau) \equiv 0$ is
$\Delta < 0$	stable for $\det(\mathbf{A}) \leq 1$, and unstable for $\det(\mathbf{A}) > 1$
$\Delta > 0$	stable for $|\mathrm{tr}(\mathbf{A})| < 2$ and $-1 - \det(\mathbf{A}) \leq \mathrm{tr}(\mathbf{A}) \leq 1 + \det(\mathbf{A})$, and unstable otherwise
$\Delta = 0$	stable for $|\mathrm{tr}(\mathbf{A})| < 2$, and for $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = \lambda$ with $\lambda = 1$ or $\lambda = -1$, and unstable otherwise

Table 3.1: Stability properties of the oscillator when $w(\tau) \equiv 0$ and $F(\tau) \equiv 0$.

for $w(\tau) \equiv 0$ and $F(\tau) \equiv 0$ are summarised in Table 3.1. It is also possible to depict these stability properties in the $(\mathrm{tr}(\mathbf{A}), \det(\mathbf{A}))$-plane (see Fig.3.2, and also [31] for a more extended figure). To determine matrix $\mathbf{A}$ completely we have to compute $\mathbf{M_1}(T_0)$ and $\mathbf{M_3}(T)$ in Eq. (3.17). These matrices will depend on the value of p (see also Eqs. (3.10) and (3.14)). In fact we have to distinguish five cases:

(*i*) $p^2 < 1 - \varepsilon$, the under-damped case,

(*ii*) $p^2 = 1 - \varepsilon$, the partly critically damped and partly under-damped case,

(*iii*) $1 - \varepsilon < p^2 < 1$, the partly over-damped and partly under-damped case,

(*iv*) $p^2 = 1$, the partly over-damped and partly critically damped case,

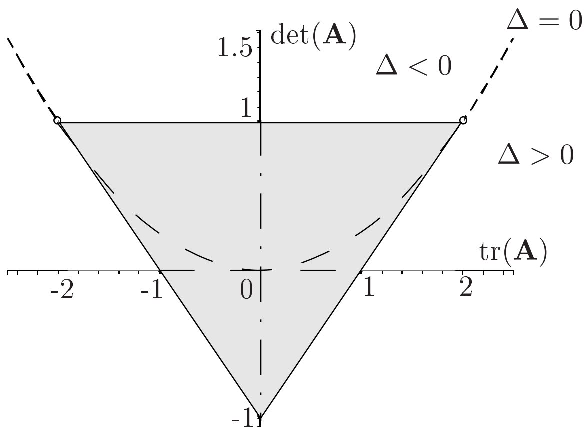

Figure 3.2: The stability diagram in the $(\text{tr}(\mathbf{A}), \det(\mathbf{A}))$-plane when $w(\tau) \equiv 0$ and $F(\tau) \equiv 0$ (grey colouring and —: stable, $\circ$: only stable in special cases (see Table 3.1), else unstable; dashed line indicates $\Delta = 0$).

(v) $p^2 > 1$, the over-damped case.

For given values of p, ε, T_0, and T the matrix $\mathbf{A}$ can be computed explicitly, and the stability properties of the oscillator can be found in Table 3.1 or in Fig. 3.2. To get more insight in the stability properties of the oscillator for different values of p, ε, T_0, and T we will now consider the aforementioned five cases in more detail. In the under-damped case $p^2 < 1 - \varepsilon$ the matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ with the components given by (see also Eq. (3.17)):

$$\begin{aligned} a_{11} &= \frac{e^{f_1+f_2}}{\psi_1\psi_2}\left((\psi_2 b - \phi_2 d)(\psi_1 a - \phi_1 c) - cd\right), \\ a_{12} &= \frac{e^{f_1+f_2}}{\psi_1\psi_2}\left(c(\psi_2 b - \phi_2 d) + d(1-\varepsilon)(\psi_1 a + \phi_1 c)\right), \\ a_{21} &= \frac{e^{f_1+f_2}}{\psi_1\psi_2}\left(\frac{-d(\psi_1 a - \phi_1 c) - c(\psi_2 b + \phi_2 d)}{1-\varepsilon}\right), \\ a_{22} &= \frac{e^{f_1+f_2}}{\psi_1\psi_2}\left((\psi_2 b + \phi_2 d)(\psi_1 a + \phi_1 c) - \frac{cd}{1-\varepsilon}\right), \end{aligned} \tag{3.21}$$

where

$$\begin{aligned} \phi_1 &= \frac{-p}{1-\varepsilon}, & \phi_2 &= -p, \\ \psi_1 &= \frac{\sqrt{(1-\varepsilon) - p^2}}{1-\varepsilon}, & \psi_2 &= \sqrt{1-p^2}, \\ a &= \cos(\psi_1 T_0), & c &= \sin(\psi_1 T_0), \\ b &= \cos(\psi_2(T-T_0)), & d &= \sin(\psi_2(T-T_0)), \\ f_1 &= \phi_1 T_0, & f_2 &= \phi_2(T-T_0). \end{aligned} \tag{3.22}$$

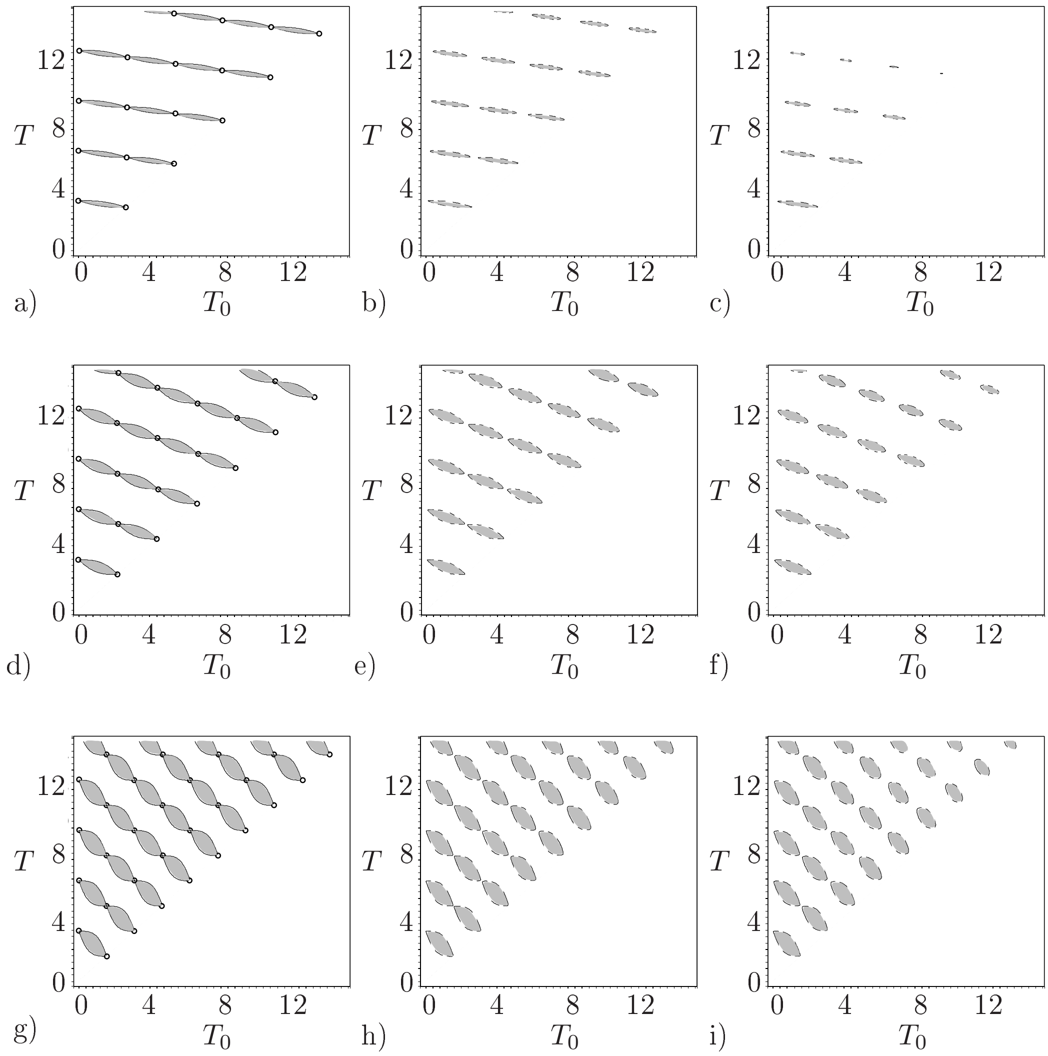

Figure 3.3: Stability and instability regions for the free vibrations of the oscillator when $p^2 < 1 - \varepsilon$ (grey colouring and solid lines: unstable; white colouring, dashed lines and $\circ$-points: stable). a) $\varepsilon = 0.25$, $p = 0$, b) $\varepsilon = 0.25$, $p = 0.005$, c) $\varepsilon = 0.25$, $p = 0.01$, d) $\varepsilon = 0.5$, $p = 0$, e) $\varepsilon = 0.5$, $p = 0.005$, f) $\varepsilon = 0.5$, $p = 0.01$, g) $\varepsilon = 0.75$, $p = 0$, h) $\varepsilon = 0.75$, $p = 0.005$, i) $\varepsilon = 0.75$, $p = 0.01$.

In this under-damped case the determinant and the trace of matrix $\mathbf{A}$ are given by

$$\det(\mathbf{A}) = \det(\mathbf{M_4}(T))\det(\mathbf{M_3}(T))\det(\mathbf{M_2}(T_0))\det(\mathbf{M_1}(T_0)) = e^{2(f_1+f_2)}, \quad (3.23)$$

$$\mathrm{tr}(\mathbf{A}) = \frac{e^{f_1+f_2}}{\psi_1\psi_2}\left[2\psi_1\psi_2 ab + cd\left(2\phi_1\phi_2 - \frac{2-\varepsilon}{1-\varepsilon}\right)\right]. \quad (3.24)$$

From the Eqs. (3.22) and (3.23) it follows that $f_1 + f_2 = -p(T + T_0\frac{\varepsilon}{1-\varepsilon}) < 0$, and so $0 < \det(\mathbf{A}) < 1$. The stability boundaries follow from Table 3.1 or Fig. 3.2, and are given by $\mathrm{tr}(\mathbf{A}) = -1 - \det(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A}) = 1 + \det(\mathbf{A})$, where $\det(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A})$ are given by Eqs. (3.23) and (3.24) respectively. These stability boundaries are determined by four parameters: p, ε, T_0, and T. In Fig. 3.3 the instability regions (indicated by a grey colouring) and the stability region (indicated by a white colouring) in the (T_0, T)-plane are given for some fixed values of p and ε. From Fig. 3.3 it is clear that for larger values of ε the instability regions also become larger, and that for increasing value of p the instability regions become smaller. Moreover it can be seen in Fig. 3.3 that for fixed p and ε values and for increasing T_0, and T values the instability regions become smaller, because the damping can act longer. The stability / instability boundaries are plotted by using the 'implicitplot' command in the formula manipulation software package Maple.

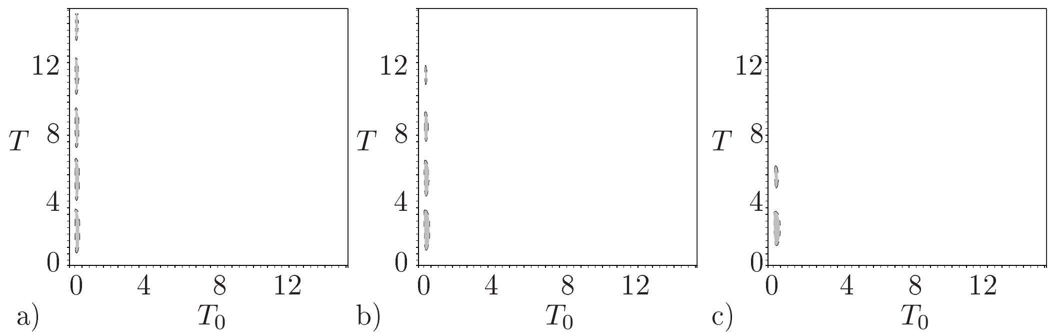

Figure 3.4: Stability and instability regions for the free vibrations of the oscillator when $p^2 = 1-\varepsilon$ (grey colouring: unstable; white colouring and dashed lines: stable). a) $\varepsilon = 0.995,\ 0.0707$, b) $\varepsilon = 0.99,\ 0.1$, c) $\varepsilon = 0.98,\ 0.1414$.

In the partly critically damped and partly under-damped case $p = \sqrt{1-\varepsilon}$ the components of matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ are given by:

$$\begin{aligned}
a_{11} &= \frac{e^{f_1+f_2}}{\psi_2}\left((\psi_2 b - \phi_2 d)(1 - \psi_1 T_0) - dT_0\right), \\
a_{12} &= \frac{e^{f_1+f_2}}{\psi_2}\left((\psi_2 b - \phi_2 d)T_0 + \frac{d(1+T_0)}{\phi_1}\right), \\
a_{21} &= \frac{e^{f_1+f_2}}{\psi_2}\left(-\phi_1^2(1 - \phi_1 T_0) - \phi_1^2 T_0(\psi_2 b + \phi_2 d)\right), \\
a_{22} &= \frac{e^{f_1+f_2}}{\psi_2}\left(-\phi_1^2 dT_0 + \phi_1(1+T_0)(\psi_2 b + \phi_2 d)\right),
\end{aligned} \quad (3.25)$$

where ϕ_1, ϕ_2, ψ_2, b, d, f_1, and f_2 are given by Eq. (3.22) with $p = \sqrt{1-\varepsilon}$. In this case the determinant and the trace of matrix $\mathbf{A}$ are given by

$$\det(\mathbf{A}) = e^{-2\sqrt{1-\varepsilon}\left(T + T_0\frac{\varepsilon}{1-\varepsilon}\right)}, \text{ and} \tag{3.26}$$

$$\operatorname{tr}(\mathbf{A}) = \frac{e^{f_1+f_2}}{\psi_2}\left[2\psi_2 b + dT_0\left(2\phi_1\phi_2 - \frac{2-\varepsilon}{1-\varepsilon}\right)\right]. \tag{3.27}$$

Again we have $0 < \det(\mathbf{A}) < 1$, and the stability boundaries are given by $\operatorname{tr}(\mathbf{A}) = -1 - \det(\mathbf{A})$ and $\operatorname{tr}(\mathbf{A}) = 1 + \det(\mathbf{A})$, where $\det(\mathbf{A})$ and $\operatorname{tr}(\mathbf{A})$ are given by Eqs. (3.26) and (3.27) respectively. In Fig. 3.4 the instability regions (indicated by a grey colouring) and the stability regions (indicated by a white colouring) in the (T_0, T)-plane are given for some fixed values of ε (and p). The same conclusions as for the case $p^2 < 1 - \varepsilon$ more or less hold.

In the partly over-damped and partly under-damped case $1 - \varepsilon < p^2 < 1$ the elements of matrix $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ are given by

$$\begin{aligned}
a_{11} &= g(\psi_2 b - \phi_2 d)\left((\psi_1 - \phi_1)e^{\psi_1 T_0} + (\psi_1 + \phi_1)e^{-\psi_1 T_0}\right) - \\
&- g\,d\left(e^{\psi_1 T_0} - e^{-\psi_1 T_0}\right), \\
a_{12} &= g(\psi_2 b - \phi_2 d)\left(e^{\psi_1 T_0} - e^{-\psi_1 T_0}\right) + \\
&+ g\,d(1-\varepsilon)\left((\psi_1 + \phi_1)e^{\psi_1 T_0} + (\psi_1 - \phi_1)e^{-\psi_1 T_0}\right), \qquad (3.28) \\
a_{21} &= \frac{-g\,d}{1-\varepsilon}\left((\psi_1 - \phi_1)e^{\psi_1 T_0} + (\psi_1 + \phi_1)e^{-\psi_1 T_0}\right) + \\
&+ \frac{-g}{1-\varepsilon}(\psi_2 b + \phi_2 d)\left(e^{\psi_1 T_0} - e^{-\psi_1 T_0}\right), \\
a_{22} &= \frac{-g\,d}{1-\varepsilon}\left(e^{\psi_1 T_0} - e^{-\psi_1 T_0}\right) + \\
&+ g(\psi_2 b + \phi_2 d)\left((\psi_1 + \phi_1)e^{\psi_1 T_0} + (\psi_1 - \phi_1)e^{-\psi_1 T_0}\right),
\end{aligned}$$

where $g = \frac{e^{f_1+f_2}}{2\psi_1\psi_2}$, $\psi_1 = \frac{\sqrt{p^2-(1-\varepsilon)}}{1-\varepsilon}$, and where ϕ_1, ϕ_2, ψ_2, b, d, f_1, and f_2 are given by Eq. (3.22). In this case the determinant and the trace of matrix $\mathbf{A}$ are given by

$$\det(\mathbf{A}) = e^{-2p\left(T + T_0\frac{\varepsilon}{1-\varepsilon}\right)}, \text{ and} \tag{3.29}$$

$$\begin{aligned}
\operatorname{tr}(\mathbf{A}) &= b\,e^{f_1+f_2}\left(e^{\psi_1 T_0} + e^{-\psi_1 T_0}\right) + \\
&+ \frac{d\,e^{f_1+f_2}}{2\psi_1\psi_2}\left(2\phi_1\phi_2 - \frac{2-\varepsilon}{1-\varepsilon}\right)\left(e^{\psi_1 T_0} - e^{-\psi_1 T_0}\right). \qquad (3.30)
\end{aligned}$$

As in the previous two cases we have $0 < \det(\mathbf{A}) < 1$, and the stability boundaries are given by $\mathrm{tr}(\mathbf{A}) = -1 - \det(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A}) = 1 + \det(\mathbf{A})$, where $\det(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A})$ are given by Eqs. (3.29) and (3.30) respectively. In Fig. 3.5 the instability regions (indicated by a grey colouring) and the stability regions (indicated by a white colouring) in the (T_0, T)-plane are given for some fixed values of ε and p. Again it can be seen that the instability regions decrease in size for increasing values of the damping parameter p.

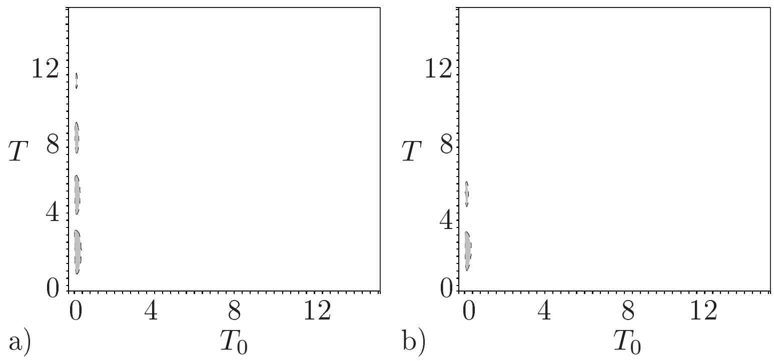

Figure 3.5: Stability and instability regions for the free vibrations of the oscillator when $1 - \varepsilon < p^2 < 1$ (grey colouring: unstable; white colouring and dashed lines: stable). a) $\varepsilon = 0.99$, $p = 0.105$, b) $\varepsilon = 0.99$, $p = 0.15$.

For the partly over-damped and partly critically damped case $p^2 = 1$, and for the over-damped case $p^2 > 1$ the matrices $\mathbf{A}$ also have been computed. Again it was found that $0 < \det(\mathbf{A}) < 1$, and that the stability boundaries are given by $\mathrm{tr}(\mathbf{A}) = -1 - \det(\mathbf{A})$ and $\mathrm{tr}(\mathbf{A}) = 1 + \det(\mathbf{A})$. For these two cases $p^2 = 1$ and $p^2 > 1$ no instability regions were detected by using the software package Maple.

In the analysis so far it can be seen that the instability regions shrink for increasing values of the damping parameter p. This raised the question whether it is possible (or not) to determine the smallest value of p (called the critical value $p_{cr}(\varepsilon)$) such that for all T_0 and T, and for a given value of ε with $(0 < \varepsilon < 1)$ the oscillator is stable for $p > p_{cr}(\varepsilon)$. In Fig. 3.6 the critical value p_{cr} is given for $0 < \varepsilon < 1$. The function $p_{cr}(\varepsilon)$ has been computed numerically in the following way. For a given value of ε, and for a given value of p, and for $0 < T_0 < T < 7$ the matrix $\mathbf{A}$, and its determinant and trace have been computed. By using Table 3.1 the (in)stability can be determined for all T_0 and T. When for some T_0 and T instability is detected, the value of p is increased, and the procedure is repeated until p should be decreased. When for all T_0 and T stability is detected, the value of p is decreased, and the procedure is repeated until p should be increased. From Fig. 3.6 it also follows that for $p^2 \geq 1$ the oscillator will always be stable.

3.3 External forcing: $F \equiv 0$ and $w \neq 0$.

In this section the initial value problem (3.5)-(3.6) with $F \equiv 0$ will be studied,

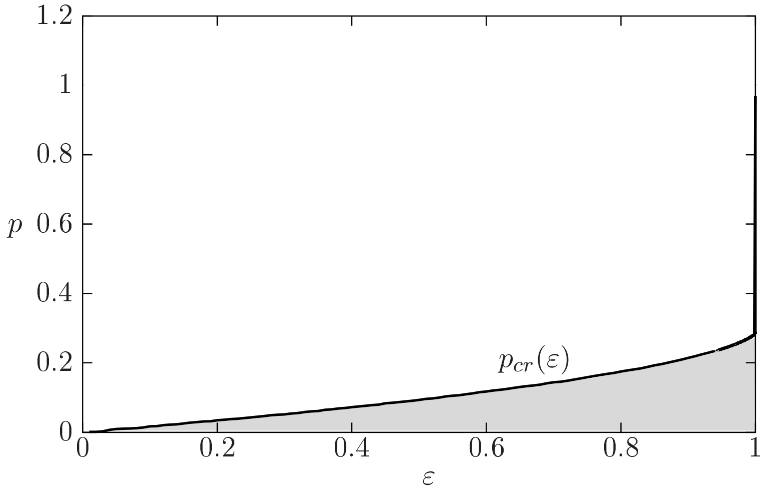

Figure 3.6: The curve $p_{cr}(\varepsilon)$. For $p > p_{cr}(\varepsilon)$ the oscillator is always stable, and for $p < p_{cr}(\varepsilon)$ the oscillator is unstable for some specific values of T_0 and T.

or equivalently

$$((1-h(\tau))\, y'(\tau))' + 2py'(\tau) + y(\tau) = \frac{-w(\tau)\,\omega_0}{k}\, h'(\tau), \;\; \tau > 0, \tag{3.31}$$

with $y(0) = y_0$, $y'(0) = y'_0$, $\omega_0 = \sqrt{k/M_0}$, and where $h(\tau)$ is given by Eq. (3.4). First we will compute the solution of the initial value problem for the five different cases: $p^2 < 1-\varepsilon$, $p^2 = 1-\varepsilon$, $1-\varepsilon < p^2 < 1$, $p^2 = 1$, and $p^2 > 1$. On the time-interval $0 < \tau < T_0^-$ it follows from Eq. (3.31) that we have to solve again Eq. (3.8) subject to the initial conditions $y(0) = y_0$, $y'(0) = y'_0$. The solution on $0 < \tau < T_0^-$ is given by Eq. (3.10). On the infinitesimal small time-interval $T_0^- < \tau < T_0^+$ we have to observe that the displacement of the oscillator is continuous, and that the impulse of the system at $\tau = T_0^+$ is equal to the impulse of the system at $\tau = T_0^-$ plus the impulse of the raindrop (which hits the oscillator). The continuity of the displacement at $\tau = T_0$ implies that $y(T_0^-) = y(T_0^+)$, and the impulse condition can be obtained by integrating Eq. (3.31) with respect to τ from $\tau = T_0^-$ to $\tau = T_0^+$, yielding $y'(T_0^+) - (1-\varepsilon)y'(T_0^-) = \varepsilon w(T_0)/\omega_0$. And so,

$$\begin{aligned} \begin{pmatrix} y(T_0^+) \\ y'(T_0^+) \end{pmatrix} &= \mathbf{M_2}(T_0) \begin{pmatrix} y(T_0^-) \\ y'(T_0^-) \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon w(T_0)/\omega_0 \end{pmatrix} = \\ &= \mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \begin{pmatrix} 0 \\ \varepsilon w(T_0)/\omega_0 \end{pmatrix}, \end{aligned} \tag{3.32}$$

where $\mathbf{M_2}(T_0)$ is given by $\mathbf{M_2}(T_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}$. On the time-interval $T_0^+ < \tau < T^-$ we have to solve again Eq. (3.12) subject to the initial conditions at $\tau = T_0^+$ and given by Eq. (3.32).The solution can easily be obtained , yielding on $T_0^+ < \tau < T^-$

$$\begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_3}(\tau)\mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{M_3}(\tau) \begin{pmatrix} 0 \\ \varepsilon w(T_0)/\omega_0 \end{pmatrix}, \tag{3.33}$$

where $\mathbf{M_3}(\tau)$ is the 'fundamental' matrix (see also Eq. (3.13)). On the infinitesimal small time-interval $T^- < \tau < T^+$ it should be observed that the displacement of the oscillator is continuous, and that the impulse of the system at $\tau = T^+$ is equal to the impulse of the system at $\tau = T^-$ plus the impulse of the raindrop (which leaves the oscillator). The continuity of the displacement at $\tau = T$ simply implies that $y(T^-) = y(T^+)$, and the impulse condition can be obtained by integrating Eq. (3.31) with respect to τ from $\tau = T^-$ to $\tau = T^+$, yielding $(1-\varepsilon)y'(T^+) - y'(T^-) = -\varepsilon w(T)/\omega_0$. And so,

$$\begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} = \mathbf{M_4}(T) \begin{pmatrix} y(T^-) \\ y'(T^-) \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{-\varepsilon w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix}, \tag{3.34}$$

where $\mathbf{M_4}(T)$ is given by $\mathbf{M_4}(T) = \begin{pmatrix} 1 & 0 \\ 0 & (1-\varepsilon)^{-1} \end{pmatrix}$. So, the solution of Eq. (3.31) on the interval $0 < \tau \le T^+$ has been constructed, and at $\tau = T^+$ the solution is given by

$$\begin{pmatrix} y(T^+) \\ y'(T^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_0 \\ y_0' \end{pmatrix} + \mathbf{W}, \tag{3.35}$$

where $\mathbf{A}$ is given by (3.17), and where $\mathbf{W}$ is a vector with constant elements and is given by

$$\mathbf{W} = \mathbf{M_4}(T)\mathbf{M_3}(T) \begin{pmatrix} 0 \\ \dfrac{\varepsilon w(T)}{\omega_0} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{-\varepsilon w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix}. \tag{3.36}$$

To compute the solution at $2T^+$, $3T^+$, ..., $(n+1)T^+$, the procedure can be repeated, giving the following system of difference equations

$$\begin{pmatrix} y((n+1)T^+) \\ y'((n+1)T^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y(nT^+) \\ y'(nT^+) \end{pmatrix} + \mathbf{W} \tag{3.37}$$

for $n = 0, 1, 2, \ldots$. The stability properties of the oscillator can be determined in the following way.First we try to compute the equilibrium point(s) of system (3.37) by assuming that such a point exists and is given by $\begin{pmatrix} y_{eq} \\ y_{eq}' \end{pmatrix}$. Then,

$$\begin{pmatrix} y_{eq} \\ y_{eq}' \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_{eq} \\ y_{eq}' \end{pmatrix} + \mathbf{W} \;\Rightarrow\; (\mathbf{I} - \mathbf{A}) \begin{pmatrix} y_{eq} \\ y_{eq}' \end{pmatrix} = \mathbf{W}.$$

When $\mathbf{I} - \mathbf{A}$ is invertible then the equilibrium is given by $\begin{pmatrix} y_{eq} \\ y_{eq}' \end{pmatrix} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{W}$, and we put

$$\begin{pmatrix} y(nT^+) \\ y'(nT^+) \end{pmatrix} = \begin{pmatrix} x_n \\ x_n' \end{pmatrix} + \begin{pmatrix} y_{eq} \\ y_{eq}' \end{pmatrix}. \tag{3.38}$$

By substituting Eq. (3.38) into Eq. (3.37) we obtain $\begin{pmatrix} x_{n+1} \\ x_{n+1}' \end{pmatrix} = \mathbf{A} \begin{pmatrix} x_n \\ x_n' \end{pmatrix}$. So,

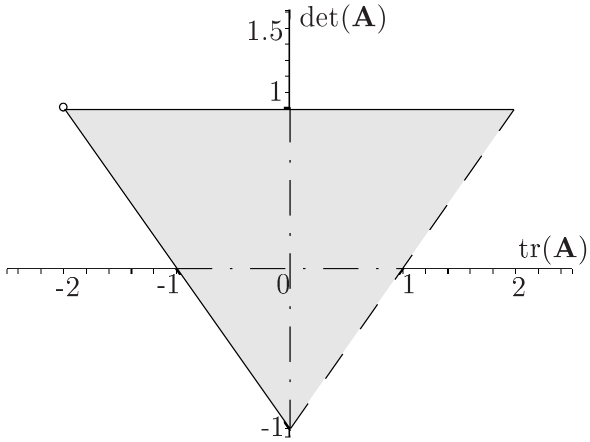

Figure 3.7: The stability diagram in the $(\text{tr}(\mathbf{A}), \det(\mathbf{A}))$-plane when $w(\tau) \neq 0$ and $F(\tau) \equiv 0$ (grey colouring and —: stable, white colouring and - - -: unstable, ∘: only stable in special cases (see Table 3.2), else unstable).

when $\mathbf{I} - \mathbf{A}$ is invertible the same stability properties for Eq. (3.37) hold as for the case $\mathbf{W} = \mathbf{0}$. The case when $\mathbf{I} - \mathbf{A}$ is not invertible will now be considered. It should be noted that when $\mathbf{I} - \mathbf{A}$ is not invertible then matrix $\mathbf{A}$ has at least one eigenvalue equal to 1. The characteristic equation for the eigenvalues belonging to matrix $\mathbf{A}$ is given by: $ch(\lambda) \equiv \lambda^2 - \text{tr}(\mathbf{A})\lambda + \det(\mathbf{A}) = 0$. And so, $ch(1) = 1 - \text{tr}(\mathbf{A}) + \det(\mathbf{A}) = 0 \Leftrightarrow \det(\mathbf{A}) = \text{tr}(\mathbf{A}) - 1$. From Eq. (3.20) it then follows that the eigenvalues are given by $\lambda_1 = 1$ and $\lambda_2 = \text{tr}(\mathbf{A}) - 1$. If the eigenvalues λ_1, λ_2 of a 2×2 matrix $\mathbf{A}$ are distinct or if the two eigenvalues are coinciding and the dimension of the corresponding eigenspace is two, then from any set of linearly independent corresponding eigenvectors v_1, v_2 a matrix $\mathbf{P}$ can be formed, which is invertible and $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \text{diag}[\lambda_1, \lambda_2]$. The solution of the system of difference equations (3.37) can then readily be obtained (see also [27]) and is given by

$$\begin{pmatrix} y(nT^+) \\ y'(nT^+) \end{pmatrix} = \mathbf{P} \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} \mathbf{P}^{-1} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{P} \sum_{r=0}^{n-1} \begin{pmatrix} \lambda_1^r & 0 \\ 0 & \lambda_2^r \end{pmatrix} \mathbf{P}^{-1}\mathbf{W}. \quad (3.39)$$

From Eq. (3.39) it is obvious that the oscillator is unstable when $\lambda_1 = 1$.

Now the following case still has to be considered: matrix $\mathbf{A}$ has two coinciding eigenvalues $\lambda_1 = \lambda_2 = 1$ and the dimension of the corresponding eigenspace is one (implying that matrix $\mathbf{A}$ cannot be diagonalised). For this case the Jordan-form matrix method can be used as for instance described in [6,17]. It can be shown that again an invertible matrix $\mathbf{P}$ exists such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The solution of Eq. (3.37) can again readily be obtained (see also [27]) and is given by

$$\begin{pmatrix} y(nT^+) \\ y'(nT^+) \end{pmatrix} = \mathbf{P} \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \mathbf{P}^{-1} \begin{pmatrix} y_0 \\ y'_0 \end{pmatrix} + \mathbf{P} \begin{pmatrix} n & \frac{n(n-1)}{2} \\ 0 & n \end{pmatrix} \mathbf{P}^{-1}\mathbf{W}. \quad (3.40)$$

Stability properties for $\Delta = \text{tr}^2(\mathbf{A}) - 4\det(\mathbf{A})$	The oscillator with $w(\tau) \neq 0$ and $F(\tau) \equiv 0$ is
$\Delta < 0$	stable for $\det(\mathbf{A}) \leq 1$, and unstable for $\det(\mathbf{A}) > 1$
$\Delta > 0$	stable for $\lvert\text{tr}(\mathbf{A})\rvert < 2$ and $-1 - \det(\mathbf{A}) \leq \text{tr}(\mathbf{A}) < 1 + \det(\mathbf{A})$, and unstable otherwise
$\Delta = 0$	stable for $\lvert\text{tr}(\mathbf{A})\rvert < 2$, and for $a_{12} = a_{21} = 0$ and $a_{11} = a_{22} = -1$, and unstable otherwise

Table 3.2: Stability properties of the oscillator when $w(\tau) \neq 0$ and $F(\tau) \equiv 0$.

From (3.40) it is also obvious that the oscillator is unstable when $\lambda_1 = \lambda_2 = 1$ and the dimension of the corresponding eigenspace is one. All of the stability properties of the oscillator for $\mathbf{W} \neq \mathbf{0}$ are summarised in Table 3.2, and in Fig. 3.7.

3.4 External harmonic forcing: $w \equiv 0$ and $F(\tau) = A\cos(\alpha\tau + \beta)$.

In this section the initial value problem (3.5)-(3.6) with $w(\tau) \equiv 0$ will be studied, that is

$$((1 - h(\tau))y'(\tau))' + 2py'(\tau) + y(\tau) = A\cos(\alpha\tau + \beta), \quad \tau > 0, \tag{3.41}$$

with $y(0) = y_0$, $y'(0) = y_0'$, where $h(\tau)$ is given by Eq. (3.4), and where α, A, and β are constants. As in the previous section a map will be constructed which relates the solution at $\tau = (n+1)T + 0^+$ to the solution at $\tau = nT + 0^+$. For simplicity the following notation will be introduced: $y_n(0^+) = y(nT + 0^+)$, $y_{n+1}(0^+) = y((n+1)T + 0^+)$, $y_n(\tau^*) = y(nT + \tau^*)$ with $0 < \tau^* \leq T + 0^+$. Starting at $\tau = nT + 0^+$ the solution will now be constructed (leading to the solution at $\tau = (n+1)T + 0^+$). For $nT < \tau < nT + T_0$ or equivalently for $0 < \tau^* < T_0$ Eq. (3.41) becomes

$$(1 - \varepsilon)y'' + 2py' + y = A\cos(\alpha\tau + \beta). \tag{3.42}$$

The initial value problem for Eq. (3.42) can easily be solved on $0 < \tau^* < T_0$, and by separating the homogeneous and nonhomogeneous parts in the solution, we can rewrite the solution on this interval into

$$\begin{pmatrix} y_n(\tau^*) \\ y_n'(\tau^*) \end{pmatrix} = \mathbf{M_1}(\tau^*) \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{N_1}(\tau^*) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{3.43}$$

where the 'fundamental' matrix $\mathbf{M_1}(\tau^*)$ depends on p and ε, and where the matrix $\mathbf{N_1}(\tau^*)$ depends on p, ε, A, α, and β. These matrices can be computed explicitly for given p, ε, A, α, and β. At $\tau^* = T_0$ the function $h(\tau)$ in Eq. (3.41) has a jump discontinuity. As in the previous two sections of this chapter it follows for $\tau^* = T_0$ that

$$\begin{aligned} \begin{pmatrix} y_n(T_0^+) \\ y_n'(T_0^+) \end{pmatrix} &= \mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \\ &+ \mathbf{M_2}(T_0)\mathbf{N_1}(T_0) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \end{aligned} \tag{3.44}$$

where $\mathbf{M_2}(T_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}$. For $T_0 < \tau < T$ Eq. (3.31) is given by

$$y'' + 2py' + y = A\cos(\alpha\tau + \beta). \tag{3.45}$$

The initial value problem (3.44) - (3.45) can again easily be solved on $T_0 < \tau^* < T$, yielding

$$\begin{aligned} \begin{pmatrix} y_n(\tau^*) \\ y_n'(\tau^*) \end{pmatrix} &= \mathbf{M_3}(\tau^*)\mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \\ &+ (\mathbf{M_3}(\tau^*)\mathbf{M_2}(T_0)\mathbf{N_1}(T_0) + \mathbf{N_3}(\tau^*)) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \end{aligned} \tag{3.46}$$

where the 'fundamental' matrix $\mathbf{M_3}(\tau^*)$ and the matrix $\mathbf{N_3}(\tau^*)$ can be computed explicitly for given p, ε, A, α, and β. At $\tau^* = T$ the function $h(\tau)$ in Eq. (3.41) has a jump condition. As in the previous two section of this chapter it follows for $\tau^* = T^+$ that (observing that $y_{n+1}(0^+) = y_n(T^+)$ and $y_{n+1}'(0^+) = y_n'(T^+)$):

$$\begin{aligned} \begin{pmatrix} y_{n+1}(0^+) \\ y_{n+1}'(0^+) \end{pmatrix} &= \mathbf{M_4}(T)\mathbf{M_3}(T)\mathbf{M_2}(T_0)\mathbf{M_1}(T_0) \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \\ &+ (\mathbf{M_4}(T)\mathbf{M_3}(T)\mathbf{M_2}(T_0)\mathbf{N_1}(T_0) + \mathbf{M_4}(T)\mathbf{N_3}(T)) \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \end{aligned} \tag{3.47}$$

where $\mathbf{M_4}(T) = \begin{pmatrix} 1 & 0 \\ 0 & (1-\varepsilon)^{-1} \end{pmatrix}$. From Eq. (3.47) the following map can be obtained

$$\begin{pmatrix} y_{n+1}(0^+) \\ y_{n+1}'(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y_n'(0^+) \end{pmatrix} + \mathbf{B} \begin{pmatrix} \cos(\alpha nT) \\ \sin(\alpha nT) \end{pmatrix}, \tag{3.48}$$

where $\mathbf{A}$ is given by Eq. (3.17), and $\mathbf{B}$ is given by

$$\mathbf{B} = \mathbf{M_4}(T)\mathbf{M_3}(T)\mathbf{M_2}(T_0)\mathbf{N_1}(T_0) + \mathbf{M_4}(T)\mathbf{N_3}(T). \tag{3.49}$$

It should be observed that the elements of the matrices $\mathbf{A}$ and $\mathbf{B}$ are n-independent, and only depend on T_0, T, p, ε, A, α, and β. Now we will study the stability properties of the solution of the system of difference equations (3.48). First the system of two first-order ordinary difference equations (3.48) will be reduced to a single second-order difference equation for $y_n(0^+) = y_n$, yielding

$$\begin{aligned} y_{n+2} - (a_{11} + a_{22})y_{n+1} + (a_{11}a_{22} - a_{12}a_{21})y_n = \\ = c_0 \cos(\alpha nT) + s_0 \sin(\alpha nT) + \\ + c_1 \cos(\alpha(n+1)T) + s_1 \sin(\alpha(n+1)T), \end{aligned} \tag{3.50}$$

where $a_{ij}(i, j = 1, 2)$ are the components of matrix $\mathbf{A}$, and $c_0 = b_{21}a_{12} - b_{11}a_{22}$, $s_0 = b_{22}a_{12} - b_{12}a_{22}$, $c_1 = b_{11}$, $s_1 = b_{12}$, and where $b_{ij}(i, j = 1, 2)$ are the components of matrix $\mathbf{B}$. The solution y_n of Eq. (3.50) can be written as

$$y_n = y_{h,n} + y_{p0,n} + y_{p1,n}, \tag{3.51}$$

where $y_{h,n}$ is the solution of the homogeneous equation (related to Eq. (3.50)):

$$y_{h,n+2} - \mathrm{tr}(\mathbf{A})y_{h,n+1} + \det(\mathbf{A})y_{h,n} = 0, \tag{3.52}$$

and where $y_{pm,n}$ (with $m = 0, 1$) are the particular solutions of Eq. (3.50) satisfying

$$\begin{aligned} y_{pm,n+2} - \mathrm{tr}(\mathbf{A})y_{pm,n+1} + \det(\mathbf{A})y_{pm,n} = \\ = c_m \cos(\alpha(n+m)T) + s_m \sin(\alpha(n+m)T). \end{aligned} \tag{3.53}$$

The roots of the characteristic equation belonging to the homogeneous equation (3.52) are given by $\lambda_{1,2} = \frac{1}{2}\mathrm{tr}(\mathbf{A}) \pm \frac{1}{2}\sqrt{(\mathrm{tr}(\mathbf{A}))^2 - 4\det(\mathbf{A})}$, and are, of course, coinciding with the eigenvalues of matrix $\mathbf{A}$. The corresponding stability properties of the homogeneous solution $y_{h,n}$ can be found in Table 3.1. The particular solutions $y_{pm,n}$ of Eq. (3.53) can be found in the following way. First one looks for a particular solution $y_{pm,n}$ in the form:

$$y_{pm,n} = C_{1m} \cos(\alpha(n+m)T) + C_{2m} \sin(\alpha(n+m)T), \tag{3.54}$$

where C_{1m} and C_{2m} are constants to be determined. By substituting (3.54) into (3.53), and then by collecting the coefficients of $\cos(\alpha(n+m)T)$ and of $\sin(\alpha(n+m)T)$ it follows that C_{1m} and C_{2m} have to satisfy

$$\begin{pmatrix} \cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) + \det(\mathbf{A}) & \sin(2\alpha T) - \mathrm{tr}(\mathbf{A})\sin(\alpha T) \\ -\sin(2\alpha T) + \mathrm{tr}(\mathbf{A})\sin(\alpha T) & \cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) + \det(\mathbf{A}) \end{pmatrix} \times$$
$$\times \begin{pmatrix} C_{1m} \\ C_{2m} \end{pmatrix} = \begin{pmatrix} c_m \\ s_m \end{pmatrix}. \tag{3.55}$$

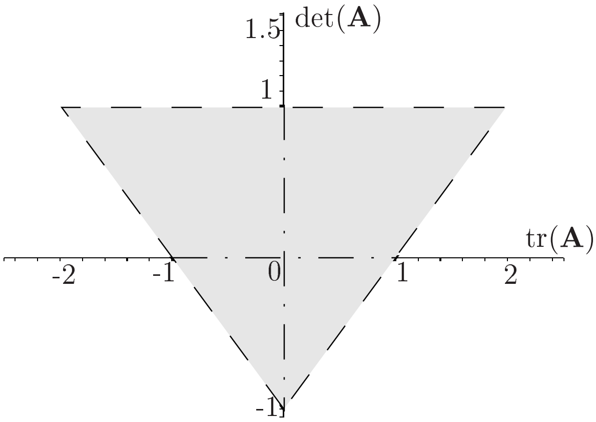

Figure 3.8: The stability diagram in the $(\mathrm{tr}(\mathbf{A}), \det(\mathbf{A}))$-plane when $w(\tau) \equiv 0$ and $F(\tau) = A\cos(\alpha\tau + \beta)$ (grey colouring: stable, white colouring: unstable, and - - -: only unstable in special cases (see Table 3.3)).

The difference equation (3.50) has a unique solution when two initial conditions are given. And so, the particular solutions $y_{pm,n}$ can be determined uniquely. To have a unique particular solution $y_{pm,n}$ it follows from (3.55) that the determinant of the coefficient matrix in (3.55) should be nonzero. When the determinant is equal to zero then there are infinitely many solutions or there is no solution. This will occur when:

$$\begin{cases} \cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) + \det(\mathbf{A}) = 0, \quad \text{and} \\ \sin(2\alpha T) - \mathrm{tr}(\mathbf{A})\sin(\alpha T) = 0, \end{cases} \quad \Leftrightarrow \tag{3.56}$$

$$\begin{cases} \cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) + \det(\mathbf{A}) = 0, \quad \text{and} \\ \sin(\alpha T)(2\cos(\alpha T) - \mathrm{tr}(\mathbf{A})) = 0, \end{cases} \quad \Leftrightarrow$$

$$\begin{cases} \cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) + \det(\mathbf{A}) = 0, \quad \text{and} \\ \alpha T = k\pi \text{ with } k \in \mathbf{Z}, \text{ or } \mathrm{tr}(\mathbf{A})) = 2\cos(\alpha T). \end{cases}$$

When $\alpha T = k\pi$ with $k \in \mathbf{Z}$ it follows from $\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) + \det(\mathbf{A}) = 0$ that $1 - (-1)^k \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = 0$. And when $\mathrm{tr}(\mathbf{A})) = 2\cos(\alpha T)$ it follows from $\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) + \det(\mathbf{A}) = 0$ that $\det(\mathbf{A}) = 1$. So, the particular solutions $y_{pm,n}$ as given by Eq. (3.54) can be determined uniquely when Eq. (3.56) is not satisfied. When Eq. (3.56), however, is satisfied the particular solutions $y_{pm,n}$ will have the following form:

$$y_{pm,n} = n(\tilde{C}_{1m}\cos(\alpha(n+m)T) + \tilde{C}_{2m}\sin(\alpha(n+m)T)), \tag{3.57}$$

where $\tilde{C}_{1m}$ and $\tilde{C}_{2m}$ are constants to be determined. By substituting Eq. (3.57) into Eq. (3.53), and then by collecting the coefficients of $\cos(\alpha(n+m)T)$ and of

Properties of matrix $\mathbf{A}$	The oscillator with $w(\tau) \equiv 0$ and $F(\tau) = A\cos(\alpha\tau + \beta)$ is
$\lvert \mathrm{tr}(\mathbf{A}) \rvert < 2$ and $\det(\mathbf{A}) < 1$ and $-1 - \det(\mathbf{A}) < \mathrm{tr}(\mathbf{A}) < 1 + \det(\mathbf{A})$	stable
$\lvert \mathrm{tr}(\mathbf{A}) \rvert < 2$ and $\det(\mathbf{A}) = 1$ and $\mathrm{tr}(\mathbf{A}) \neq 2\cos(\alpha T)$	stable
$-2 < \mathrm{tr}(\mathbf{A}) < 0$ and $\mathrm{tr}(\mathbf{A}) = -1 - \det(\mathbf{A})$ and αT is not an odd multiple of π	stable
$0 < \mathrm{tr}(\mathbf{A}) < 2$ and $\mathrm{tr}(\mathbf{A}) = 1 + \det(\mathbf{A})$ and αT is not an even multiple of π	stable
$a_{11} = a_{22} = 1$ and $a_{12} = a_{21} = 0$ and αT is not an even multiple of π	stable
$a_{11} = a_{22} = -1$ and $a_{12} = a_{21} = 0$ and αT is not an odd multiple of π	stable
$\mathrm{tr}(\mathbf{A}) = 0$ and $\det(\mathbf{A}) = -1$ and αT is not a multiple of π	stable
all other cases	unstable

Table 3.3: Stability properties of the oscillator when $w(\tau) \equiv 0$ and $F(\tau) = A\cos(\alpha\tau + \beta)$.

$\sin(\alpha(n+m)T)$ it follows that $\tilde{C}_{1m}$ and $\tilde{C}_{2m}$ have to satisfy:

$$\begin{pmatrix} 2\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) & 2\sin(2\alpha T) - \mathrm{tr}(\mathbf{A})\sin(\alpha T) \\ -2\sin(2\alpha T) + \mathrm{tr}(\mathbf{A})\sin(\alpha T) & 2\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) \end{pmatrix} \times$$
$$\times \begin{pmatrix} \tilde{C}_{1m} \\ \tilde{C}_{2m} \end{pmatrix} = \begin{pmatrix} c_m \\ s_m \end{pmatrix}. \tag{3.58}$$

Again, to have a unique particular solution $y_{pm,n}$ (in the form (3.57)) it follows from Eq. (3.58) that the determinant of the coefficient matrix in Eq. (3.58) should be nonzero. When the determinant is equal to zero there are infinitely many solutions or there is no solution. This will occur when:

$$\begin{cases} 2\cos(2\alpha T) - \mathrm{tr}(\mathbf{A})\cos(\alpha T) = 0, \quad \text{and} \\ 2\sin(2\alpha T) - \mathrm{tr}(\mathbf{A})\sin(\alpha T) = 0. \end{cases} \tag{3.59}$$

When the determinant in the coefficient matrix in Eq. (3.58) is nonzero, it is obvious from Eq. (3.57) that the solution of Eq. (3.48) is unbounded (and unstable). For that reason the condition $\mathrm{tr}(\mathbf{A})) = 2\cos(\alpha T)$ and $\det(\mathbf{A}) = 1$ or $\alpha T = k\pi$ with $k \in \mathbf{Z}$ and $1 - (-1)^k \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = 0$ is called the resonance condition. The case when this resonance condition and the condition (3.59) are satisfied still has to be studied. When $\mathrm{tr}(\mathbf{A})) = 2\cos(\alpha T)$ and $\det(\mathbf{A}) = 1$ it follows from Eq. (3.59) after some elementary calculations that αT is a multiple of π. And when $\alpha T = k\pi$ with

$k \in \mathbf{Z}$ and $1 - (-1)^k \mathrm{tr}(\mathbf{A}) + \det(\mathbf{A}) = 0$ it follows from Eq. (3.59) that $\det(\mathbf{A}) = 1$ and $\mathrm{tr}(\mathbf{A})) = 2(-1)^k$. So, we can conclude in this case that when αT is an even multiple of π the system of difference equations (3.48) becomes

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} + \mathbf{B} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{3.60}$$

where the eigenvalues of matrix $\mathbf{A}$ are coinciding and equal to one. System (3.60) with $\mathrm{tr}(\mathbf{A})) = 2$ and $\det(\mathbf{A}) = 1$ already has been studied in the previous section, and from Table 3.2 it follows that the solution of Eq. (3.60) is unstable. Similarly, when αT is an odd multiple of π the system of difference equations (3.48) becomes:

$$\begin{pmatrix} y_{n+1}(0^+) \\ y'_{n+1}(0^+) \end{pmatrix} = \mathbf{A} \begin{pmatrix} y_n(0^+) \\ y'_n(0^+) \end{pmatrix} + \mathbf{B} \begin{pmatrix} (-1)^n \\ 0 \end{pmatrix}, \tag{3.61}$$

and $\mathrm{tr}(\mathbf{A}) = 2\cos(\alpha T) = -2$. Since the eigenvalues of matrix $\mathbf{A}$ are both equal to -1 it is not difficult to see that the particular solution of Eq. (3.61) will contain unbounded terms in n. So, also in this case the solution of Eq. (3.61) is unstable. All of the stability properties of the solution of the oscillator equation (3.41) with an external harmonic force are summarised in Table 3.3 and in Fig. 3.8.

3.5 Conclusions.

In this chapter the vibrations of a damped, linear, single degree of freedom oscillator with a periodically and stepwise changing time-varying mass have been considered. Both the free and the forced vibrations of the oscillator have been studied. For the free vibrations the stability properties of the oscillator have been determined, and the minimal damping rates have been computed numerically for which the oscillator is always stable. Two types of forcing have been studied. First, a forcing has been studied due to a mass which hits the oscillator, stays for some time at the oscillator, and then leaves the oscillator. The stability properties for this forced oscillator problem have been determined in Section 3 of this chapter. A second forcing case has been considered in Section 4 of this chapter. In this section an external, harmonic forcing has been studied which is applied to an oscillator to which a mass (with zero velocity) is added for some time, and then is taken away (with zero velocity). For this case the stability properties of the oscillator have been presented, and an interesting resonance condition has been derived. When both forcing types are applied to the oscillator the results as obtained in Section 3 and in Section 4 of this chapter can be combined, because the differential equation describing the problem is linear.

The considered oscillator model is perhaps the simplest model which describes the vibrations of a cable with rainwater located on it. To obtain more realistic mathematical models for these rain-wind induced oscillations of cables in windfields one might consider periodically and multi-stepwise changing masses. For these rain-wind induced oscillations the formation of water rivulets on the cable surface should also be taken into account, leading to a (usually small) increase of the constant mass, because the constant component of water rivulet mass on the cable surface should be included. Other external forces (such as drag-and-lift forces, damping forces, and so on) can also be included in the model equation. Moreover, instead of an ordinary differential equation setting one can formulate the problem in a partial differential equation setting. Then by expanding the solution of the partial differential equation in a Fourier series, and by applying a Galerkin truncation method one obtains a finite system of ordinary differential equations. The ordinary differential equations will be of the same structure as the differential equation studied in this chapter. The aforementioned extensions to the model equation can be interesting subjects for further research, and some preliminary results in this direction can be found in [1, 28].

Chapter 4

A non-linear case

Abstract: In this chapter the forced vibrations of an undamped single degree of freedom oscillator with a time varying mass will be studied. An initial value problem for an oscillator equation with a Rayleigh type of non-linearity will be formulated, and by applying a straight-forward perturbation method the problem will be solved approximately. The approximations of the solutions will be used to construct a map. By using this map the stability properties of the solutions can be determined. The stability properties of the non-linear problem will be compared to those for the linear problem, which have been studied earlier in the literature. The instability regions in the parameter space and some phase-space figures for the non-linear problem will be computed numerically. It will also be shown how the behaviour of the solutions changes when the instability regions in the parameter space are crossed.

4.1 Introduction

Systems with time-varying masses frequently occur in practice. Examples of such systems are excavators, cranes, water rivulets on cables of bridges, and so on (see for instance [5, 9, 22]). The vibrations of such systems can usually be described by initial-boundary value problems for partial differential equations. The 1-mode Galerkin approximation of the solution for such problem will lead to a single degree of freedom oscillator (sdofo) equation. The analysis for some of these (mainly) linear oscillator equations has been presented in [16, 25–27]. In particular, the stability properties of the solutions have been determined.

In bridge engineering it has been observed [12, 13] that under rainy and windy conditions water rivulets are formed on the inclined cables of cable-stayed bridges near the points of wind flow separation. This results in a non-symmetrical profile of the cross section of the cable, and consequently, in drag and lift forces acting on the cable(s). Moreover, the wind flow approaching the cable divides into three flows: a cross flow, an axial flow from the upwind side of the cable, and an axial flow from the downwind side. It has been observed that the downwind axial flow results in an effect called in the literature as 'base bleed', which creates vacuum behind the

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This chapter is a slightly revised version of: O.V. Pischanskyy, W.T. van Horssen, On the non-linear dynamics of a single degree of freedom oscillator with a time-varying mass, *Journal of Sound and Vibration*, DOI: 10.1016/j.jsv.2011.12.009, 23-12-2011.

cable and an additional drag force, see for details, for example [12, 13]. It also has been observed in the literature that the water rivulets flowing along the cables have a periodic character.

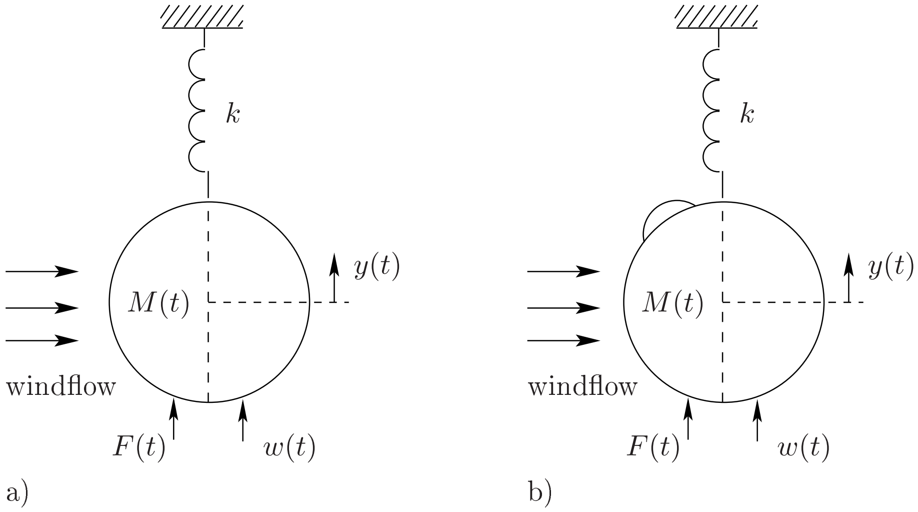

Figure 4.1: The single degree of freedom oscillator for time-interval a) $(0, T_0)$ and b) (T_0, T).

In this chapter an extremely simplified model is considered for the above discussed rain-wind induced oscillations of a cable. From time $t = 0$ to time $t = T_0$ a harmonic oscillator is considered, see Fig.4.1.a). At time $t = T_0$ a rain droplet instantaneously hits the oscillator with a velocity $w(T_0)$ and stays on the surface until time $t = T$, when the droplet moves off the oscillator with a velocity $w(T)$. The impacts of the droplets can excite the oscillator. It is assumed in the model that a wind flow (which is perpendicular to the y-axis) is acting on the oscillator. When no droplets are on the oscillator it is assumed that the profile of the oscillator is symmetric, and when droplets are on the surface of the oscillator it is assumed that a non-symmetric profile is obtained, see also Fig. 4.1.b).

The following equation of motion for the sdofo can now be derived ([9, p.152]):

$$M\ddot{y} - \dot{M}(w - \dot{y}) + ky = F, \tag{4.1}$$

where $y = y(t)$ is the displacement of the oscillator, $M = M(t)$ is the total time-varying mass of the oscillator, $w = w(t)$ is the mean velocity at which masses are hitting or leaving the oscillator, k is the positive stiffness coefficient, $F = F(t, y, \dot{y})$ is an external force, and the 'dot' denotes differentiation with respect to time t. The force F and the velocity w are measured positive in positive y direction (see Fig. 4.1). Following [25–27] it turns out to be convenient to separate the mass $M(t)$ into a time invariant part M_0 and into a time-varying part $m(t)$, that is $M(t) = M_0 - m(t)$, where M_0 is a positive constant, and $M_0 - m(t) > 0$. Then, it follows that Eq. (4.1) can be rewritten in

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((M_0 - m(t))\frac{\mathrm{d}y}{\mathrm{d}t}\right) + ky = \frac{-\mathrm{d}m}{\mathrm{d}t}w + F. \tag{4.2}$$

Then, by introducing the time-rescaling $t = \tau\sqrt{M_0/k}$, $\tilde{y}(\tau) = y(\tau\sqrt{M_0/k})$, $\tilde{m}(\tau) = m(\tau\sqrt{M_0/k})$, $\tilde{w}(\tau) = w(\tau\sqrt{M_0/k})$ and $\tilde{F}(\tau, \tilde{y}(\tau), \frac{\mathrm{d}\tilde{y}(\tau)}{\mathrm{d}\tau}) = \frac{1}{k}F(\tau, y(\tau), \dot{y}(\tau))(\tau\sqrt{M_0/k})$ Eq. (4.2) can be rewritten into

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\left(1 - \frac{\tilde{m}(\tau)}{M_0}\right)\frac{\mathrm{d}\tilde{y}(\tau)}{\mathrm{d}\tau}\right) + \tilde{y}(\tau) = \frac{-\tilde{w}(\tau)}{\sqrt{M_0 k}}\frac{\mathrm{d}\tilde{m}(\tau)}{\mathrm{d}\tau} + \tilde{F}(\tau, \tilde{y}(\tau), \frac{\mathrm{d}\tilde{y}(\tau)}{\mathrm{d}\tau}). \tag{4.3}$$

Now it will be assumed that $h(\tau) = \tilde{m}(\tau)/M_0$ is a periodic step function with $1 - h(\tau) > 0$, that is,

$$h(\tau) = \begin{cases} \varepsilon & \text{for } 0 < \tau < T_0, \\ 0 & \text{for } T_0 < \tau < T, \end{cases} \tag{4.4}$$

and $h(\tau + T) = h(\tau)$, and ε is a constant (in practice usually small) with $0 < \varepsilon < 1$. Also $\tilde{w}(\tau)$ is assumed to be T-periodic in τ. It should be observed that in the analysis ε is defined to be the quotient m/M_0, where m is the mass which added at time T_0, and where M_0 is the mass of the oscillator. So, ε can be seen as a measure for the relative mass which is added at time T_0. For convenience the tildes in Eq. (4.3) will be dropped, and the prime will be introduced to denote differentiation with respect to τ, yielding for $\tau > 0$

$$[(1 - h(\tau))\, y'(\tau)]' + y(\tau) = F(\tau, y(\tau), y'(\tau)) - \frac{w(\tau)}{\omega_0}h'(\tau), \tag{4.5}$$

where $\omega_0 = \sqrt{k/M_0}$ is the natural frequency of the oscillator.

To give a simple representation of the function F in (4.5) the following is observed. For $T_0 < \tau < T$ the droplet on the surface of the cable creates a non-symmetric cross section as shown in Fig. 4.1. b)). Due to the wind flow drag and lift forces in y direction are generated for $T_0 < \tau < T$. In [24] it has been shown that the total wind force in y direction can be modelled by a Rayleigh type of non-linearity, that is, by $ay' - b(y')^3$, where a and b are small, positive constants. In this chapter it will be assumed that $a = b = \varepsilon$. For $0 < \tau < T_0$ only a drag fore will act on the oscillator, and it will be assumed that the effect of the component of this force in y direction is so small that it can be neglected. Then, $F(\tau, y(\tau), y'(\tau))$ is T-periodic in τ and is given by

$$F = \begin{cases} 0, & \text{for } 0 < \tau < T_0, \\ \varepsilon(y' - (y')^3), & \text{for } T_0 < \tau < T. \end{cases} \tag{4.6}$$

The initial displacement and the initial velocity of $y(\tau)$ are given by

$$y(0) = y_0 \text{ and } y'(0) = y_0'. \tag{4.7}$$

The initial value problem (4.4) - (4.7) has (in our opinion) not been studied before in the literature. In this chapter approximations to the solutions of this problem will be constructed, and stability properties of the solution will be presented.

This chapter is organised as follows. In section 2 of this chapter the solution of the initial value problem (4.4) - (4.7) will be approximated. Since the problem is partially non-linear a straight-forward perturbation method will be applied to approximate the solution. By using these approximations the stability of the solutions will be investigated in section 3 of this chapter. Also phase-space figures for different values of the parameters will be presented. Finally, in section 4 some conclusions will be drawn, and remarks will be made about future research on this subject.

4.2 A representation of the solution

In this section the solution of the initial value problem (4.4) - (4.7) for $y(\tau)$ will be determined by using a similar approach as presented in [25–27]. Starting at $\tau = nT + 0^+$ the solution will be constructed (or approximated) on the interval $nT < \tau \leq (n+1)T + 0^+$ for all $n = 0, 1, 2, \ldots$, assuming that

$$y(nT + 0^+) = y_n(0^+), \text{ and } y'(nT + 0^+) = y_n'(0^+) \tag{4.8}$$

for all $n = 0, 1, 2, \ldots$. To solve this initial value problem the τ-interval has to be split up into four parts: $nT < \tau < nT + T_0^-$, $nT + T_0^- \leq \tau \leq nT + T_0^+$, $nT + T_0^+ < \tau < nT + T^-$, and $nT + T^- \leq \tau \leq nT + T^+$, where $T_0^- = T_0 - 0$, $T_0^+ = T_0 + 0$, $T^- = T - 0$, and $T^+ = T + 0$. On the first time-interval $nT < \tau < nT + T_0^-$ Eq (4.5) becomes $(1-\varepsilon)y'' + y = \frac{-w}{\omega_0}h'$. By using the initial conditions (4.8) the solution can be written in matrix form as follows:

$$\mathbf{Y}(\tau) = \begin{pmatrix} y(\tau) \\ y'(\tau) \end{pmatrix} = \mathbf{M_1} \begin{pmatrix} y_n \\ y_n' \end{pmatrix}, \tag{4.9}$$

where

$$\mathbf{M_1}(\tau) = \begin{pmatrix} a & \sqrt{1-\varepsilon}\, c \\ \dfrac{-1}{\sqrt{1-\varepsilon}} c & a \end{pmatrix}, \tag{4.10}$$

with $a = \cos\left(\frac{\tau - nT}{\sqrt{1-\varepsilon}}\right)$, $c = \sin\left(\frac{\tau - nT}{\sqrt{1-\varepsilon}}\right)$. For convenience, the following notation is introduced for $\mathbf{Y}(\tau)$:

$$\mathbf{Y}_n(T_0^-) = \mathbf{M_1}(nT + T_0^-)\, \mathbf{Y}_n(0^+), \tag{4.11}$$

where $\mathbf{Y}_n(0^+) = \mathbf{Y}(nT + 0^+) = (y_n, y_n')^T$ and $\mathbf{Y}_n(T_0^-) = \mathbf{Y}(nT + T_0^-)$ is the solution of the initial value problem at $\tau = nT + T_0^-$. On the second, infinitesimal small interval $nT + T_0^- \leq \tau \leq nT + T_0^+$ it should be observed that the displacement of the oscillator is continuous, and that the impulse of the system at $\tau = nT + T_0^-$ is equal to the impulse of the system at $\tau = nT + T_0^+$ plus the impulse of the raindrop (which hits the oscillator).

The continuity of the displacement at $\tau = nT + T_0$ implies that $y(nT + T_0^-) = y(nT + T_0^+)$. The impulse condition can directly be obtained by integrating Eq (4.5) with respect to τ from $\tau = nT + T_0^-$ to $\tau = nT + T_0^+$, yielding

$$y'(nT + T_0^+) = (1-\varepsilon)y'(nT + T_0^-) + \frac{\varepsilon w(nT + T_0)}{\omega_0},$$

where $w(nT + T_0) = w(T_0)$ is the velocity of a raindrop falling on the oscillator on time $\tau = nT + T_0$. The solution of the initial value problem for $\tau = nT + T_0^+$ is then given by:

$$\mathbf{Y}_n(T_0^+) = \mathbf{M_2}\mathbf{Y}_n(T_0^-) + \varepsilon \mathbf{N_2}, \tag{4.12}$$

where

$$\mathbf{M_2} = \begin{pmatrix} 1 & 0 \\ 0 & 1-\varepsilon \end{pmatrix}, \text{ and } \mathbf{N_2} = \begin{pmatrix} 0 \\ \frac{w(T_0)}{\omega_0} \end{pmatrix}. \tag{4.13}$$

From eqs. (4.9)-(4.13) is then follows that the solution of the initial value problem at $\tau = nT + T_0^+$ is given by:

$$\mathbf{Y}_n(T_0^+) = \mathbf{M_2}\,\mathbf{M_1}(nT + T_0)\,\mathbf{Y}_n + \varepsilon\mathbf{N_2}. \tag{4.14}$$

On the third time-interval ($nT + T_0^+ < \tau < (n+1)T^-$) the following Rayleigh differential equation has to be solved

$$y'' + y = \varepsilon(y' - (y')^3), \tag{4.15}$$

subject to the initial conditions $y(nT + T_0^+) = y_n(T_0^+)$ and $y'(nT + T_0^+) = y_n'(T_0^+)$. On the bounded time-interval $nT + T_0^+ < \tau < (n+1)T^-$ the solution of Eq. (4.15) can be written as an infinite series in powers of ε, that is,

$$\begin{aligned} y(\tau) &= f_0(\tau) + \varepsilon f_1(\tau) + O(\varepsilon^2), \\ y'(\tau) &= f_0'(\tau) + \varepsilon f_1'(\tau) + O(\varepsilon^2), \\ y''(\tau) &= f_0''(\tau) + \varepsilon f_1''(\tau) + O(\varepsilon^2). \end{aligned} \tag{4.16}$$

Then, by substituting eqs. (4.16) into Eq. (4.15) and by collecting terms of equal powers in ε, the following problems for f_0 and f_1 can be obtained:

$$f_0'' + f_0 = 0 \tag{4.17}$$

with as initial conditions:

$$f_0(nT + T_0^+) = y_n(T_0^+), \;\; f_0'(nT + T_0^+) = y_n'(T_0^+), \tag{4.18}$$

and

$$f_1'' + f_1 = f_0' - (f_0')^3, \tag{4.19}$$

with as initial conditions:

$$f_1(nT + T_0^+) = 0, \;\; f_1'(nT + T_0^+) = 0. \tag{4.20}$$

The solution of the initial value problem (4.17) - (4.18) can easily be found and written shortly as

$$\begin{pmatrix} f_0 \\ f_0' \end{pmatrix} = \mathbf{M_3}(\tau)\mathbf{Y}_n(T_0^+), \tag{4.21}$$

where

$$\mathbf{M_3}(\tau) = \begin{pmatrix} b & d \\ -d & b \end{pmatrix}, \tag{4.22}$$

with $b = \cos(\tau - nT - T_0),\; d = \sin(\tau - nT - T_0)$. By substituting Eq. (4.21) into Eq. (4.19) the solution of the initial value problem (4.19) - (4.20) for $f(\tau)$ can be determined, and at $\tau = nT + T^-$ this solution is given by

$$\begin{aligned} \begin{pmatrix} f_1 \\ f_1' \end{pmatrix} &= \begin{pmatrix} b_5 & b_6 \\ b_5' & b_6' \end{pmatrix} \begin{pmatrix} y_n(T_0^+) \\ y_n'(T_0^+) \end{pmatrix} + \begin{pmatrix} b_1 & b_4 \\ b_1' & b_4' \end{pmatrix} \begin{pmatrix} (y_n(T_0^+))^3 \\ (y_n'(T_0^+))^3 \end{pmatrix} + \\ &+ \begin{pmatrix} b_2 & b_3 \\ b_2' & b_3' \end{pmatrix} \begin{pmatrix} y_n(T_0^+)(y_n'(T_0^+))^2 \\ (y_n(T_0^+))^2 y_n'(T_0^+) \end{pmatrix}, \end{aligned} \tag{4.23}$$

where

$$
\begin{aligned}
b_1 &= \frac{9}{32}d - \frac{3}{8}(T^- - T_0^+)b + \frac{1}{32}l, & b_2 &= \frac{3}{32}b - \frac{3}{8}(T^- - T_0^+)d - \frac{3}{32}j, \\
b_3 &= \frac{21}{32}d - \frac{3}{8}(T^- - T_0^+)b - \frac{3}{32}l, & b_4 &= \frac{5}{32}b - \frac{3}{8}(T^- - T_0^+)d + \frac{1}{32}j, \\
b_5 &= \frac{-1}{2}(d - (T^- - T_0^+)b), & b_6 &= \frac{1}{2}(T^- - T_0^+)d, \\
b_1' &= \frac{-3}{32}b + \frac{3}{8}(T^- - T_0^+)d + \frac{3}{32}j, & b_2' &= \frac{-15}{32}d - \frac{3}{8}(T^- - T_0^+)b + \frac{9}{32}l, \\
b_3' &= \frac{9}{32}b + \frac{3}{8}(T^- - T_0^+)d - \frac{9}{32}j, & b_4' &= \frac{-17}{32}d - \frac{3}{8}(T^- - T_0^+)b - \frac{3}{32}l, \\
b_5' &= \frac{-1}{2}(T^- - T_0^+)d, & b_6' &= \frac{1}{2}(d + (T^- - T_0^+)b),
\end{aligned}
\tag{4.24}
$$

in which a and c are given by Eqs. (4.10) with $\tau = nT + T_0^-$, b and d are given by Eqs. (4.22) with $\tau = nT + T^-$, and in which $j = \cos(3(T - T_0))$, and $l = \sin(3(T - T_0))$. For convenience the right-hand side in Eq. (4.23) will be denoted by $\mathbf{N_3}$, that is,

$$
\begin{pmatrix} f_1(nT + T_0^-) \\ f_1'(nT + T_0^-) \end{pmatrix} = \mathbf{N_3}. \tag{4.25}
$$

The solution of the initial value problem for Eq. (4.15) at $\tau = nT + T^-$ is now given by:

$$
\mathbf{Y}_n(T^-) = \begin{pmatrix} y(nT + T^-) \\ y'(nT + T^-) \end{pmatrix} = \mathbf{M_3}(nT + T^-)\mathbf{Y}_n(T_0^+) + \varepsilon\mathbf{N_3} + O(\varepsilon^2). \tag{4.26}
$$

Since,the time-interval $nT + T_0^+ < \tau < nT + T^-$ is finite and independent of ε it is reasonable to neglect the $O(\varepsilon^2)$ corrections in the solution (4.26). By using eqs. (4.14) and (4.26) it now follows that the solution of the initial value problem (4.4) - (4.7) at $\tau = nT + T^-$ is given by:

$$
\begin{aligned}
\mathbf{Y}_n(T^-) &= \mathbf{M_3}(nT + T^-)\mathbf{M_2}\mathbf{M_1}(nT + T_0^-)\mathbf{Y}_n(0^+) + \\
&+ \varepsilon\left(\mathbf{M_3}(nT + T^-)\mathbf{N_2} + \mathbf{N_3}\right).
\end{aligned}
\tag{4.27}
$$

On the fourth, infinitesimal small time-interval $nT + T^- \leq \tau \leq nT + T^+$ it should again be observed that the displacement of the oscillator is continuous, and that the impulse of the system at $\tau = nT + T^+$ is equal to the impulse of the system at $\tau = nT + T^-$ plus the impulse of the raindrop (which leaves the oscillator). The continuity of the displacement at $\tau = nT + T$ implies that $y(nT + T^-) = y(nT + T^+)$. The impulse condition can directly be obtained by integrating Eq. (4.5) with respect to τ from $\tau = nT + T^-$ to $\tau = nT + T^+$, yielding

$$
y'(nT + T^+) = \frac{1}{1 - \varepsilon}y'(nT + T^-) + \frac{\varepsilon w(nT + T)}{\omega_0(1 - \varepsilon)},
$$

where $w(nT+T) = w(T)$ is the velocity of a raindrop leaving the oscillator at time $\tau = nT + T$. The solution of the initial value problem (4.4) - (4.7) at $\tau = nT + T^+$ is then given by:

$$\mathbf{Y}_{n+1}(0^+) = \mathbf{Y}_n(T^+) = \mathbf{M_4}\mathbf{Y}_n(T^-) + \varepsilon\mathbf{N_4}, \tag{4.28}$$

where

$$\mathbf{M_4} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{1-\varepsilon} \end{pmatrix}, \text{ and } \mathbf{N_4} = \begin{pmatrix} 0 \\ \frac{w(T)}{\omega_0(1-\varepsilon)} \end{pmatrix}. \tag{4.29}$$

So, the solution of the initial value problem (4.4) - (4.7) on the time-interval $nT < \tau \leq (n+1)T + 0^+$ has been constructed, and at $\tau = (n+1)T + 0^+$ the solution is given by

$$\begin{aligned} \mathbf{Y}_{n+1}(0^+) &= \mathbf{M_4}\mathbf{M_3}((n+1)T)\mathbf{M_2}\mathbf{M_1}(nT+T_0)\mathbf{Y}_n(0^+) + \\ &+ \varepsilon\left(\mathbf{M_4}\mathbf{M_3}((n+1)T)\mathbf{N_2} + \mathbf{M_4}\mathbf{N_3} + \mathbf{N_4}\right). \end{aligned} \tag{4.30}$$

Eq. (4.30) can be rewritten in the form of a map

$$\mathbf{Y}_{n+1} = \mathbf{A}\mathbf{Y}_n + \varepsilon\mathbf{W}(\mathbf{Y}_n), \tag{4.31}$$

where

$$\mathbf{A} = \mathbf{M_4}\mathbf{M_3}((n+1)T)\mathbf{M_2}\mathbf{M_1}(nT+T_0) \tag{4.32}$$

is a 2×2 matrix with constant elements, and

$$\mathbf{W}(\mathbf{Y}_n) = \mathbf{M_4}\mathbf{M_3}((n+1)T)\mathbf{N_2} + \mathbf{M_4}\mathbf{N_3} + \mathbf{N_4} \tag{4.33}$$

is a 2×1 vector which depends non-linearly on $\mathbf{Y}_n$ due to the component $\mathbf{N_3}$ which is given by:

$$\mathbf{N_3} = \begin{pmatrix} a_1 y_n + a_2 y'_n + a_3 y_n^3 + a_4 y_n^2 y'_n + a_5 y_n (y'_n)^2 + a_6 (y'_n)^3 \\ a'_1 y_n + a'_2 y'_n + a'_3 y_n^3 + a'_4 y_n^2 y'_n + a'_5 y_n (y'_n)^2 + a'_6 (y'_n)^3 \end{pmatrix}, \tag{4.34}$$

where

$$
\begin{aligned}
a_1 &= ab_5 - \frac{c}{\sqrt{1-\varepsilon}}b_6,\ \ a_2 = c\sqrt{1-\varepsilon}b_5 + ab_6,\\
a_3 &= a^3b_1 - \frac{c^3}{(1-\varepsilon)^{3/2}}b_4 - \frac{a^2c}{\sqrt{1-\varepsilon}}b_2 + \frac{ac^2}{1-\varepsilon}b_3,\\
a_4 &= 3a^2c\sqrt{1-\varepsilon}b_1 + \frac{3ac^2}{1-\varepsilon}b_4 + a(a^2-2c^2)b_2 + \frac{c}{\sqrt{1-\varepsilon}}(c^2-2a^2)b_3,\\
a_5 &= 3ac^2(1-\varepsilon)b_1 - \frac{3a^2c}{\sqrt{1-\varepsilon}}b_4 + c(2a^2-c^2)\sqrt{1-\varepsilon}b_2 + a(a^2-2c^2)b_3,\\
a_6 &= c^3(1-\varepsilon)^{3/2}b_1 + a^3b_4 + ac^2(1-\varepsilon)b_2 + a^2c\sqrt{1-\varepsilon}b_3,\\
a_1' &= ab_5' - \frac{c}{\sqrt{1-\varepsilon}}b_6',\ \ a_2' = c\sqrt{1-\varepsilon}b_5' + ab_6',\\
a_3' &= a^3b_1' - \frac{c^3}{(1-\varepsilon)^{3/2}}b_4' - \frac{a^2c}{\sqrt{1-\varepsilon}}b_2' + \frac{ac^2}{1-\varepsilon}b_3',\\
a_4' &= 3a^2c\sqrt{1-\varepsilon}b_1' + \frac{3ac^2}{1-\varepsilon}b_4' + a(a^2-2c^2)b_2' + \frac{c}{\sqrt{1-\varepsilon}}(c^2-2a^2)b_3',\\
a_5' &= 3ac^2(1-\varepsilon)b_1' - \frac{3a^2c}{\sqrt{1-\varepsilon}}b_4' + c\sqrt{1-\varepsilon}(2a^2-c^2)b_2' + a(a^2-2c^2)b_3',\\
a_6' &= c^3(1-\varepsilon)^{3/2}b_1' + a^3b_4' + ac^2(1-\varepsilon)b_2' + a^2c\sqrt{1-\varepsilon}b_3',
\end{aligned} \tag{4.35}
$$

where a and c are given by eqs. (4.10) with $\tau = nT + T_0$, and where b_i and b_i' are given by eqs. (4.24).

All properties of matrix $\mathbf{A}$ have been determined in [25, 27]. The parameters in matrix $\mathbf{A}$ can be such that the eigenvalues are both complex valued with moduli 1, or such that the eigenvalues are real with one of the moduli larger then one or both of the moduli equal to one. Vector $\mathbf{W}$ has a constant part $\mathbf{M_4M_3}((n+1)T)\mathbf{N_2} + \mathbf{N_4}$ (that is, a part which does not depend on y_n and y_n'), and a part $\mathbf{M_4N_3}$ which depends non-linearly on y_n and y_n'. The system of difference equations with $\mathbf{N_3} \equiv 0$ has been studied in [25, 27]. In the next section, the weakly non-linear system of difference equations (4.31) will be studied. Particularly, the stability of the solutions will be investigated and the behaviour of the solutions in the phase-space will be studied.

4.3 Stability analysis

In this section the system of difference equations (4.31) will be analysed numerically, because hardly any analytical methods are available (see [6, 10]). Since, the system is non-linear and contains many parameters the goal of the analysis presented in this section is to give an idea of the rich dynamics of the problem. By numerically computing the equilibrium points and the corresponding phase-space figures it will be shown that the system of difference equations (4.31) contains many interesting bifurcations and stability properties.

First the equilibrium points can be computed by letting $\mathbf{Y}_{n+1} = \mathbf{Y}_n = \mathbf{Y}$ in (4.31), yielding

$$\mathbf{AY} + \varepsilon\mathbf{W}(\mathbf{Y}) = \mathbf{Y}. \tag{4.36}$$

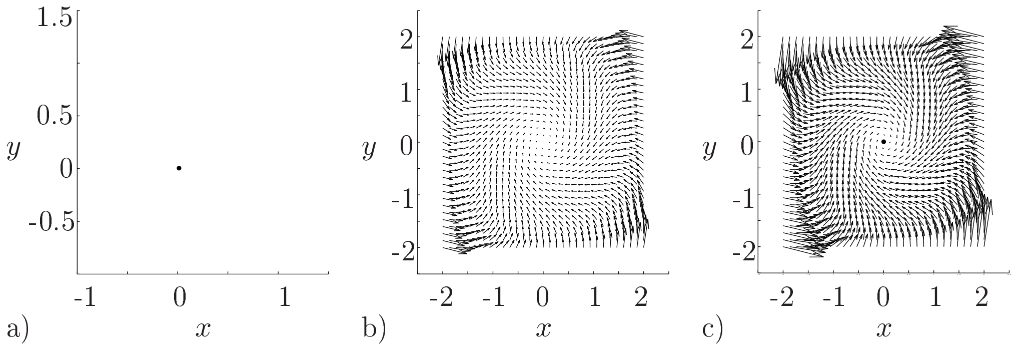

Figure 4.2: a) Fixed point near the origin, b) a map of system (4.31), c) fixed point and a map of system (4.31) overlay, for $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0.1$, $T_0 = 1.3$, $T = 13.9$.

The equations in system (4.36) are non-linear, and the equilibrium points $\mathbf{Y}$ can be found numerically. In this research the package Matlab has been used to compute the $\mathbf{Y}$'s.

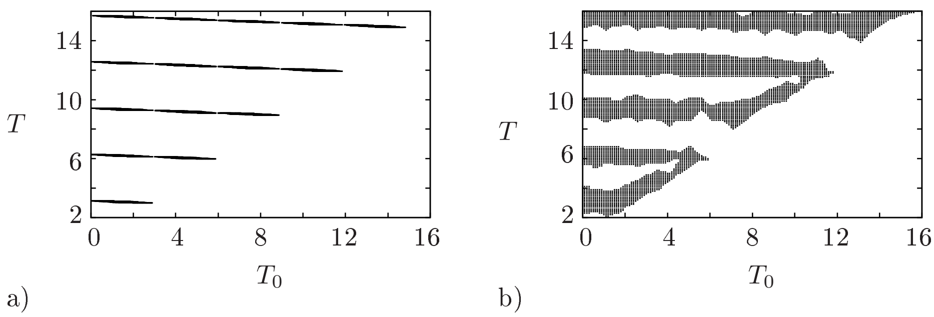

Figure 4.3: Instability regions (grey) for $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0$ of a) the linear homogeneous system (4.38), and b) the non-linear system (4.31).

For convenience, we let $\mathbf{Y}_n = (x_n,\ y_n)^T$ and $\mathbf{Y}_{n+1} = (x_{n+1},\ y_{n+1})^T$, and we rewrite system (4.31) into

$$\begin{cases} x_{n+1} = a_{11}x_n + a_{12}y_n + \\ \qquad +\varepsilon(w_1 + a_1x_n + a_2y_n + a_3x_n^3 + a_4x_n^2y_n + a_5x_ny_n^2 + a_6y_n^3), \\ y_{n+1} = a_{21}x_n + a_{22}y_n + \\ \qquad +\varepsilon(w_2 + a_1'x_n + a_2'y_n + a_3'x_n^3 + a_4'x_n^2y_n + a_5'x_ny_n^2 + a_6'y_n^3), \end{cases} \tag{4.37}$$

where w_1 and w_2 are the constant first and second componenents, respectively, in the vector $\mathbf{W}(\mathbf{Y}_n)$ as given by Eq. (4.33).

The search of the equilibrium points can be performed by solving numerically system (4.36) for the range of values of ε, ω_0, w, T_0, T by using Matlab functions. More programs have been written in Matlab to plot the phase-space diagrams for the same parameters values, and these graphs will be discussed further in this section.

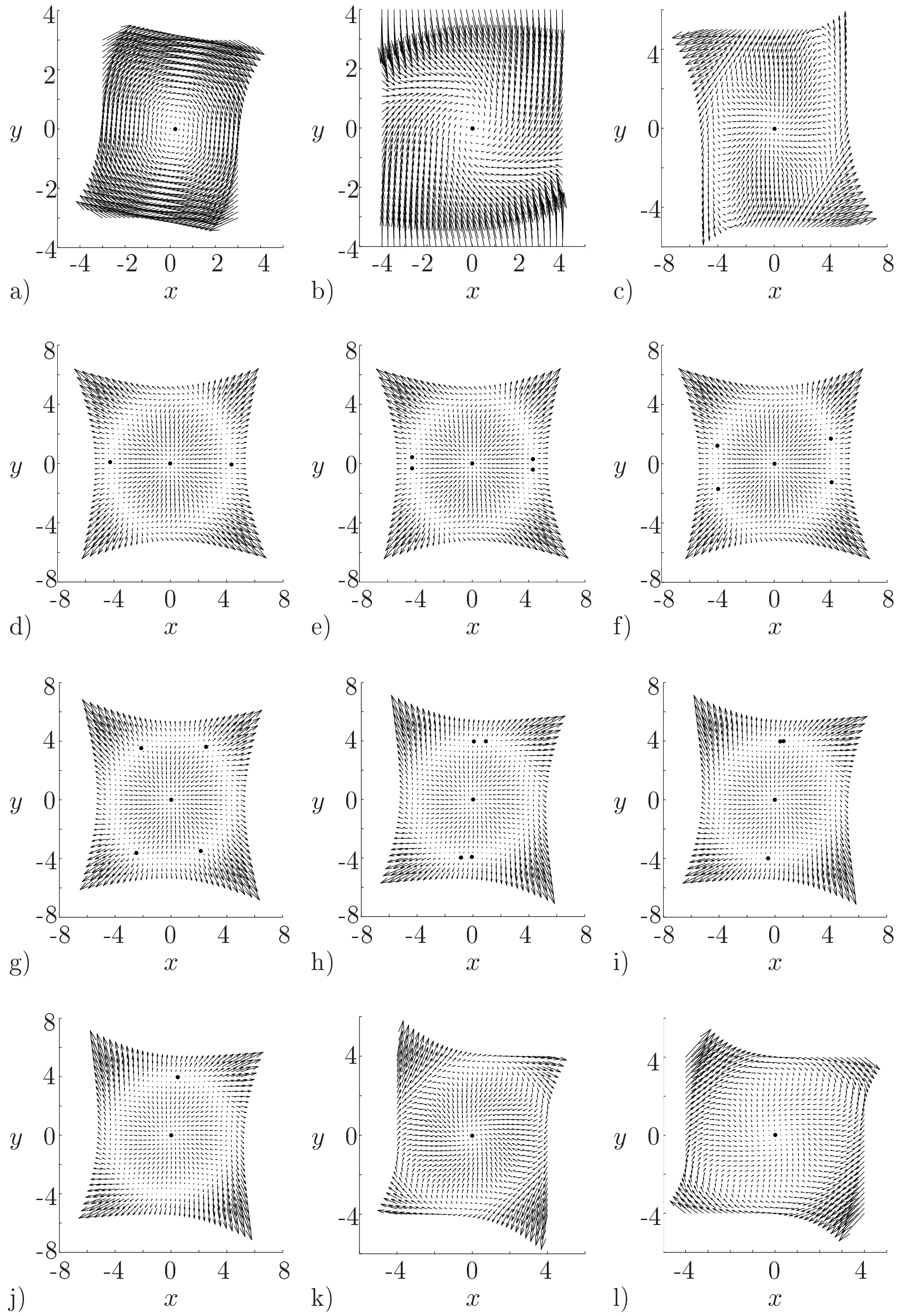

Figure 4.4: Phase-space diagram of Eq. (4.37) for $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0.1$, $T_0 = 0.1$, and a) $T = 0.1$, b) $T = 1.2$, c) $T = 2.5$, d) $T = 3.116$, e) $T = 3.117$, f) $T = 3.13$, g) $T = 3.3$, h) $T = 3.48$, i) $T = 3.49$, j) $T = 3.5$, k) $T = 3.9$, l) $T = 4.5$.

First of all the used algorithms will be discussed. The Matlab function 'fsolve' has been used to find the fixed point(s) of system (4.37). This function uses the ‘trust-region-dogleg’ algorithm for solving systems of non-linear algebraic equations like system (4.36). The ‘trust-region-dogleg’ algorithm is based on Newton’s finding-roots method but with a special Powell dogleg procedure, which allows one to avoid the failure of Newton’s method if the starting point is chosen too far from the solution, see for details [11]. The fixed points of system (4.37) can be plotted as shown, for example, in Fig. 4.2.a), where a single equilibrium point is found. Separately, a code using the Matlab function ‘quiver’ has been written to plot the Poincare maps. The latter code plots arrows which start from the points with the coordinates defined by the components of the vector $\mathbf{Y}_n$ and end at the points with the coordinates defined by the components of the vector $\mathbf{Y}_{n+1}$. There is a feature in Matlab which scales the arrows to fit the image while plotting, which gives a clear picture of the behaviour of the solutions of system (4.37). The map related to Fig. 4.2.a) is shown in Fig. 4.2.b). Overlaying these two graphs gives a coinciding result of the two Matlab codes as shown in Fig. 4.2.c). Around 40,000 graphs for different values of coefficients ε, ω_0, w, T_0, and T have been built up and have been analysed in this way.

The further analysis of the phase-space diagrams is performed for $0 < \varepsilon \ll 1$, such that the $O(\varepsilon)$ terms of Eq. (4.31) remain small. Then, the considered domain for x and y is taken not larger than 10×10 from -5 to 5 for each of the x and y. In this chapter any solution is considered to be ‘unstable’ for a certain set of parameters ε, ω_0, w, T_0, T if the phase-space diagram for this set of parameters has at least either one unstable equilibrium point or an unstable limit cycle. If only stable fixed points are found in the graph then the solution is considered as ‘stable’ for the given set of parameters ε, ω_0, w, T_0, T.

The instability regions of system (4.31) have been determined and shown in Fig. 4.3.b. In comparison, the instability regions of the linear homogeneous system considered in [25] and given by

$$\mathbf{Y}_{n+1} = \mathbf{A}\mathbf{Y}_n, \tag{4.38}$$

are shown in Fig. 4.3.a). The instabilities for both the linear and the non-linear systems occur along the lines $T = k\pi$, $k = 1, 2, 3, \ldots$. The angle of inclination and the width of the ‘instability lines’ depend on the value of ε. The instability regions for the non-linear system (4.31) are much wider and cover extra area along the line $T_0 = T$.

Following, for example, the line $T_0 = 0.1$ from $T = 0.1$ with step 0.1 the behaviour of the solution can be studied more detailed. The other parameters are assigned the following values: $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0.1$. It can be observed that for $T_0 = T = 0.1$ the phase-space diagram is similar to the diagram of a mathematical pendulum, that is, the single equilibrium point is a centre point, and it is shown in Fig. 4.4.a). Increasing T, the centre point changes smoothly into a stable node (see Fig. 4.4.b)) all the way up to around $T = 2.5$, where an unstable limit cycle with a stable equilibrium near the origin can be observed, see Fig. 4.4.c). The flow on the limit cycle is in the counter clockwise direction, and the maximum amplitude is around 4. The behaviour of the solutions changes significantly when approaching $T = \pi$. Two equilibrium points occur on the limit cycle around $T = 3.116$, as shown in Fig. 4.4.d). These points will be discussed further. Each of them separates into two points (Fig. 4.4.e)) which move along the limit cycle

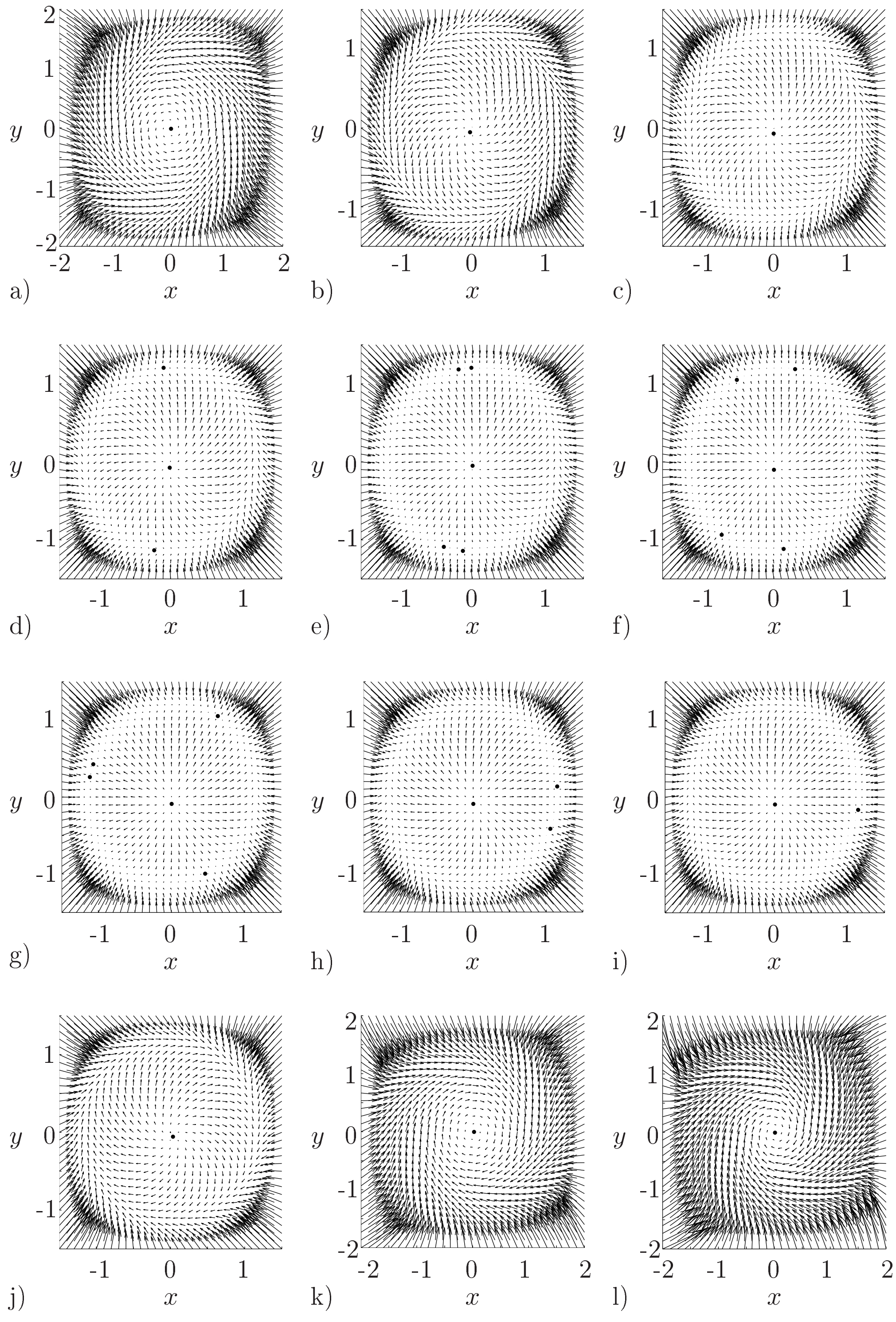

Figure 4.5: Phase-space diagram of Eq. (4.37) for $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0.1$, $T_0 = 0.1$, and a) $T = 5.7$, b) $T = 6.1$, c) $T = 6.2$, d) $T = 6.22$, e) $T = 6.243$, f) $T = 6.25$, g) $T = 6.268$, h) $T = 6.3$, i) $T = 6.3015$, j) $T = 6.4$, k) $T = 7$, l) $T = 7.4$.

away from each other (Fig. 4.4.f)) for increasing values of T. Increasing T further, it can be observed that the four points continue to move along the limit cycle and

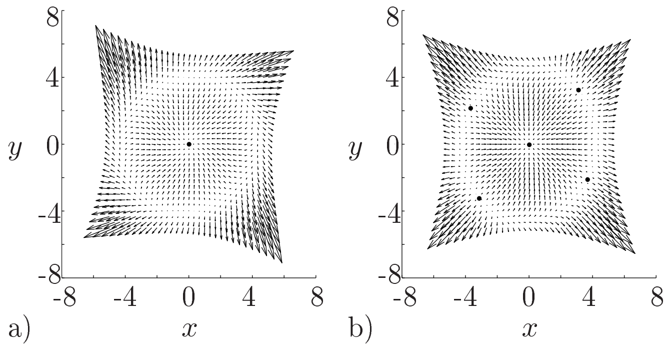

Figure 4.6: Phase-space diagrams of system (4.31) for $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0.1$, $T_0 = 0.2$, and a) $T = 3.5$, b) $T = 3.2$.

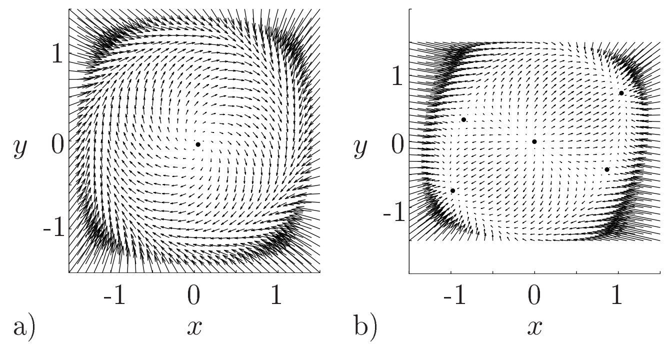

Figure 4.7: Phase-space diagrams of system (4.31) for $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0.1$, a) $T_0 = 0.2$, $T = 6.5$, and b) $T_0 = 3.6$, $T = 6.1$.

approach each other on the opposite sides of the limit cycle (Fig. 4.4.g) and h)). Finally, they merge into two points (Fig. 4.4.i)) and then disappear around $T = 3.5$, see Fig. 4.4.j). The unstable limit cycle smoothly dissapears and the flow will be directed to the equilibrium point near the origin forming a stable nodal point, see Fig. 4.4.k) and l)).

Moving up to $T = 2\pi$, it is observed that all the solutions are stable up to $T = 5.7$, where the stable limit cycle starts to expand around the fixed point near the origin, see Fig. 4.5.a) and b). The flow on the limit cycle is to the counter-clockwise direction as shown in Fig. 4.5.c). Two extra fixed points appear on the limit cycle around $T = 6.22$, where the maximum amplitude of approximately 1.2 is reached, Fig. 4.5.d). These extra points separate into two points each as is shown in Fig. 4.5.e), and move along the limit cycle away from each other for increasing values of T, Fig. 4.5.f). The speed of the movement is different for the different points, so two of them meet faster than the other two (see Fig. 4.5.g)) and disappear. The other two continue to move towards each other, merge, and finaly disappear (Figs. 4.5.h) and i)). The flow on the limit cycle changes to the clockwise direction and the limit cycle smoothly shrinks. Stable flow occurs after approximately $T = 7$ (Fig. 4.5.j) - l)), and all flow is directed towards a single equilibrium point near the origin.

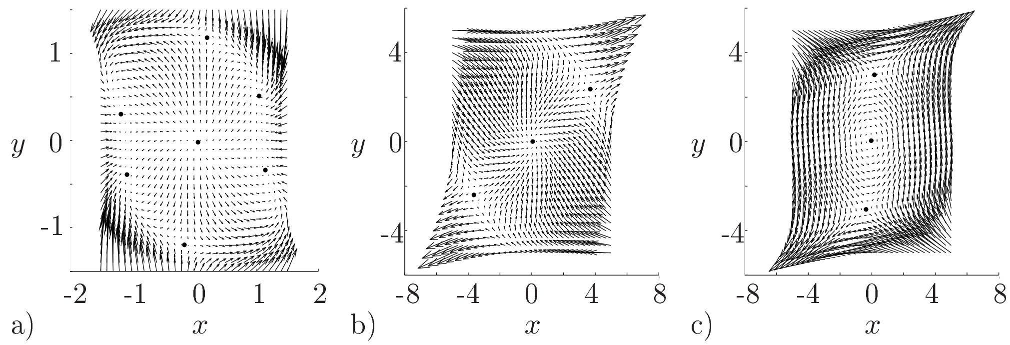

Figure 4.8: Phase-space diagrams of system (4.31) for $\varepsilon = 0.1$, $\omega_0 = 1$, $w(T_0) = w(T) = 0.1$, and a) $T_0 = 8.1$, $T = 12.1$, b) $T_0 = 7.6$, $T = 9.6$, c) $T_0 = 5.8$, $T = 5.8$.

During the analysis it has been observed that the formation of stable and unstable limit cycles with extra equilibrium points on them repeats with an interval of length 2π. The amplitude of the limit cycles and the number of extra fixed points on them differ from one instability region to another.

The nature of the extra fixed points in the diagrams for some values of the parameters ε, $w(T_0)$, $w(T)$, T_0, and T has not been studied yet. During the analysis of the phase-space diagrams it was observed that up to six such fixed points situated on the limit cycle can occur as shown for example in Fig 4.8.a). There may be even more of them, since the system of equations (4.37) is a system of two cubic equations.

The analysis of the phase-space diagrams of system (4.31) showed several types of instability of the solutions. Two types of unstable limit cycles are shown in Fig. 4.6. Fig. 4.6.a) presents a regular unstable limit cycle with one stable fixed point near the origin, typical for the solutions of Rayleigh-type equations. Fig. 4.6.b) presents a similar unstable limit cycle but with fixed points on it. These fixed points can be stable or unstable nodes, or saddle points. These types of limit cycles (unstable both with and without fixed points on them) with different amplitude occur in the instability regions along the lines $T = (2k - 1)\pi$, $k = 1, 2, 3, \ldots$, see Fig. 4.3.b). Stable limit cycles with different amplitudes, as for example shown in Fig. 4.7, occur in the instability regions along the lines $T = 2k\pi$, $k = 1, 2, 3, \ldots$, see Fig. 4.3.b). Similarly to the unstable limit cycles the stable ones have been observed having up to six fixed points of different types, like stable and unstable nodes, and saddle points.

There are also other types of instability such as shown in Fig. 4.8.b) and c) with different number of fixed points. The fixed point near the origin is usually a stable or unstable node, or a centre point, and other points are stable or unstable nodes, or saddle points.

4.4 Conclusions

In this chapter the vibrations of a single degree of freedom oscillator with a T-periodic and stepwise changing, time-varying mass have been considered. It is assumed that the T-periodic and stepwise changing mass, and the air flowing along the oscillator lead to T-periodic drag and lift forces acting on the oscillator. In this way a Rayleigh type of non-linearity is introduced in the initial value problem for the oscillator equation. The time-varying mass and the drag and lift forces can be both sources for instabilities. To investigate these instabilities the solution of the initial value problem is approximated analytically in section 2 of this chapter on the time-interval $nT \leq \tau \leq (n+1)T$. By doing this the solution at time $\tau = (n+1)T$ can be linked to the solution at time $\tau = nT$, and in this way a map is constructed from which the stability properties can be studied. The initial value problem, and so the solution of the problem, contain a lot of parameters (ε, T, T_0, ω_0, $w(T_0)$, $w(T)$). For certain choices of the parameters the rich dynamics of the problem has been shown in section 3 of this chapter by using some numerical methods. Stable and unstable equilibrium points and 'limit cycle' behaviour can be seen. Moreover, a lot of bifurcations can be noticed. To study these bifurcations in more detail (and completely) the weakly nonlinear map (or equivalently the system of weakly nonlinear difference equations) should be studied analytically, for instance, by using a multiple iteration-scales perturbation method. A preliminary research [19, Chapter 5] in this direction shows a good agreement with the numerical results as obtained in this chapter.

The considered oscillator model is perhaps one of the simplest models which describes rain-wind induced vibrations of a cable. To obtain more realistic mathematical models for these rain-wind induced oscillations of cables one might formulate the problem in a partial differential equation setting. Then by expanding the solution of the partial differential equation in a Fourier series, and by applying a Galerkin truncation method one obtains a finite system of ordinary differential equations. The ordinary differential equations will be of the same structure as the differential equation studied in this chapter. Some preliminary results in this direction can be found in [1, 28].

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Summary

In this thesis the free and forced vibrations of a single degree of freedom oscillator with a periodically time-varying mass have been studied. Linear and weakly non-linear oscillator equations have been considered. The forced vibrations of the oscillator are partly due to small masses which are T-periodically hitting and leaving the oscillator with T-periodic velocities. Since these small masses stay for some time on the oscillator surface the effective mass of the oscillator and the shape of the oscillator will periodically vary in time. The effect of a damping term (in the linear oscillator equation) on the solutions also has been considered. For the free vibrations the minimal damping rates have been computed for which the oscillator is always stable. Also cases with external, harmonic forcing have been investigated in detail for the linear oscillator equation, and interesting resonance conditions have been found.

As simple model to describe the rain-wind induced oscillations of a cable, an initial value problem for an oscillator equation with a Rayleigh type of non-linearity has been studied.By applying a straight-forward perturbation method the problem has been solved approximately on a time-interval of length T. In all cases studied in this thesis initial value problems for oscillator equations have been formulated. The constructed solutions on a time-interval of length T or the approximations of the solutions on the same time-interval have been used to construct maps. By using these maps (i.e. by using a system of difference equations) the stability properties of the solutions have been determined. The instability regions in the parameter space have been computed partly analytically and partly numerically. Some phase-space figures for the weakly non-linear problem have been computed numerically to show a number of interesting bifurcations, and to show the rich dynamics of the problem.

Samenvatting

In dit proefschrift zijn de vrije en gedwongen trillingen van een oscillator met één vrijheidsgraad bestudeerd. De massa van de oscillator varieert op een periodieke wijze in de tijd. Lineaire en zwak niet-lineaire oscillator zijn beschouwd. De gedwongen trillingen van de oscillator zijn gedeeltelijk een gevolg van kleine massa's die T-periodiek op en van de oscillator vallen met T-periodieke snelheden. Aangezien deze kleine massa's voor enige tijd op het oscillator oppervlak blijven, zullen de effectieve massa van de oscillator en de vorm van de oscillator periodiek in de tijd veranderen. Het effect van een dempings-term (in de lineaire oscillator vergelijking) op de oplossingen is ook beschouwd. Voor de vrije trillingen zijn de minimale dempings coëfficiënten berekend waarvoor de oscillator altijd stabiel is. Ook zijn gevallen met externe, harmonische krachten gedetailleerd onderzocht voor de lineaire oscillator vergelijking, en interessante resonantie voorwaarden zijn gevonden.

Als eenvoudig model ter beschrijving van de regen-wind geïnduceerde trillingen van een kabel, is een beginwaarde probleem voor een oscillator vergelijking met een Rayleigh niet-lineariteit bestudeerd. Een directe, eenvoudige storingsmethode is toegepast om de oplossing van het probleem te benaderen op een tijd-interval van lengte T. Alle gevallen, die bestudeerd zijn in dit proefschrift, zijn geformuleerd als beginwaarde problemen voor oscillator vergelijkingen. De geconstrueerde oplossingen op een tijd-interval ter lengte T (of de benaderingen van die oplossingen) zijn gebruikt om afbeeldingen te maken. Door gebruik te maken van deze afbeeldingen (dit zijn stelsels differentie vergelijkingen) zijn de stabiliteits - eigenschappen van de oplossingen bepaald. De instabiliteitsgebieden in de parameter ruimte zijn gedeeltelijk analitisch en gedeeltelijk numeriek berekend. Enkele fase-ruimte figuren voor het zwak niet-lineaire probleem zijn numeriek berekend om een aantal interessante bifurcaties te tonen, en om te laten dat het probleem een rijke dynamica heeft.

Acknowledgements

I started my research in the Mathematical Physics department of the Delft University of Technology in January 2007, where I met many pleasant and friendly people. Working in the department with its international community was exciting. I would like to mention here some people who made my work and life in Delft great.

First of all I would like to thank Prof. Igor Andrianov, Dr. Wim van Horssen, and Dr. Alexey Andrianov for giving me this opportunity. My thanks go specially to Wim van Horssen for his guidance, support, and interesting discussions. I also appreciate his help with the other issues in my daily life in Delft. Also I am grateful to Dr. Johan Dubbeldam who supervised me for some months and gave me many advices regarding my work.

I received much help regarding computers and programming from Kees Lemmens. He introduced Linux to me and I am thankful to him for many interesting discussions on this subject.

I would like to mention here also Evelyn Sharabi and Dorothee Engering from the Mathematical Physics department, and Theda Olsder, Veronique van der Varst, and other people from CICAT department for their help in different financial and organisational questions.

My work and life would not be complete without the other PhD students. We had many interesting discussions, fun, and parties with my colleagues and room mates: Timofey Gerasimov, Umer Altaf, Alina Barbu, Julius Sumihar, Slawomir Szklarz, Wiktoria Lawniczak, Dmitrii Penkin, Konstantin Vandyshev, and others.

I appreciate the time spent with my good friend Alexey Andrianov. He helped me much in arranging many important things in the beginning, and he supported me during all this time. We had many interesting talks, travels, and I learned much from him about life in The Netherlands and in Europe.

My special thanks go to my parents Victor and Vera. With anxiety they have been observing my life in another country, far away from them, and have been supporting me all the time. Their love and patience were helping me in sad moments, and their happiness made me also happy in good moments of our lives. Thank you!

Finally, I would like to emphasise a very special person for me. This is my girlfriend Victoria. We met in 2009 and from that moment my life has completely changed. She has turned everything over from head to feet. I have become the happiest man in the world. She helped me to survive the difficult moments and we were together 'on the top of the World' in times of happiness. She has opened for me a world of art and we spent many hours in fine and contemporary arts museums, around historical monuments, in photo exhibitions, in nature, and so on. I am appreciative to Victoria for our vacations in France, where we had very interesting

time and the most important moments in our lives. Victoria, thank you! I love you!

Curriculum Vitae

Alexandr Pischanskyy was born on 30 September, 1977 in Dnipropetrovsk region, Ukraine. He finished his secondary education in 1994 and in the same year started his master program in the department of Theoretical and Applied Mechanics of Dnipropetrovsk State University. He defended his MSc thesis entitled 'Boundary equilibrium of an anisotropic plate with a crack', and obtained the Diploma of Master of Science in Mechanics and Applied Mathematics in 1999. After the graduation Alexandr worked as a Mechanical Engineer in engineering consulting company DSB 'Pivdenne' in Dnipropetrovsk, Ukraine. Later he received a proposition from Prof. I. Andrianov and Dr. W.T. van Horssen for a PhD research in Delft University of Technology, The Netherlands. Alexandr started his research 'On the vibrations of a singe degree of freedom oscillator with a time-varying mass' in the Mathematical Physics department of Delft Institute of Applied Mathematics in Delft University of Technology in January 2007. This research has led to this thesis.