A singular values approach in helicopter gas turbine engines flight testing analysis

Arush, Ilan; Pavel, Marilena D.; Mulder, Max

Citation (APA)
Arush, I., Pavel, M. D., & Mulder, M. (2020). A singular values approach in helicopter gas turbine engines flight testing analysis. *Proceedings of the Institution of Mechanical Engineers, Part G: Journal of Aerospace Engineering*, *234*(12), 1851-1865. https://doi.org/10.1177/0954410020920060

Important note
To cite this publication, please use the final published version (if applicable).
Please check the document version above.

A Singular Values Approach in Helicopter Gas Turbine Engines Flight Testing Analysis

Ilan Arush[1]
National Test Pilot School, Mojave, California, 93502. Email: iarush@ntps.edu

Marilena D. Pavel[2]
Faculty of Aerospace Engineering, Delft University of Technology, Delft, The Netherlands

Max Mulder[3]
Faculty of Aerospace Engineering, Delft University of Technology, Delft, The Netherlands

The process of empirical models evaluation is at the core business of experimental flight-testing data analysis. Accurate and convenient flight-testing of helicopter engine(s) available power is crucial for predicting the total helicopter performance. Common practice in estimation of in-flight helicopter gas turbine engine power consist of a reduction of flight-test data into simplistic single-variable analysis approach. While such an approach is convenient for practical use, it often results in unrealistic predictions of the available engine(s) power. A novel approach for the helicopter available power problem is the so-called Multivariable Polynomial Optimization under Constraints (MPOC) method. In this method eighteen regressors, constructed from the engine non-dimensional parameters, are used to define empirical polynomial models. This paper is intend to complement the MPOC method and answer the question of which multivariable-polynomial can be <u>generally</u> used in representing helicopter gas-turbine engine performance? In this sense, a variety of seven gas-turbine engines installed on different helicopters are analyzed, each one giving 512 possible polynomial models to be used for available-power calculations. While conventional statistical methods of hypothesis-testing failed in providing the answer to the question stated above of which the best general empirical model for representing engine performance is, an alternative approach based on the Singular-Value-Decomposition (SVD) theorem, was proven successful in providing the answer. Moreover,   this approach presented in the paper yielded a

---

[1] Corresponding Author. Chief Academic Officer, Department of Rotary Wing Performance and Flying Qualities, National Test Pilot School, Mojave, California.

[2] Associate Professor, Control and Simulation, Faculty of Aerospace Engineering, Delft University of Technology, Kluyverweg, 2629HS Delft, The Netherlands, AIAA Member.

[3] Professor, Department Head, Control and Simulation Group, Faculty of Aerospace Engineering, Delft University of Technology, Kluyverweg, 2629HS Delft, The Netherlands.

short list of 10 *simple* and *convenient* multivariable-polynomials, best representing the performance of *all* seven engines analyzed as a group.

Keywords

Gas-Turbine, Singular Values Decomposition, Helicopter-Performance, Flight-Testing.

Nomenclature

A	=	Matrix containing numerical regressors
α_j^i	=	Generic multivariable-polynomial coefficient
a_i, b_i, c_i =		Generic single-variable polynomial coefficients
Ng	=	Engine compressor-speed
TGT	=	Engine-Temperature (Turbine-Gas-Temperature)
SHP	=	Engine-Power (Shaft Horse-Power)
W_f	=	Engine Fuel-Flow
δ	=	Relative static air pressure
θ	=	Relative static air temperature
$\vec{E}_r\big|_j$	=	Engine power estimation error-vector, model (j).
$\vec{b}$	=	Column-vector to represent experimental CSHP
f_i	=	Engine multivariable regressors, i=1, 2,…, 9
s	=	Power estimation error standard-deviation
Z	=	Precision-measures matrix, models to engines (512,7)
$t_{i,j}$	=	Test-Statistics for Multivariable-Polynomial Model (i) wrt Engine (j)
$\zeta_{i,j} \Box \dfrac{1}{\left|t_{i,j}\right|}$		= Elements of Z, represent accuracy-measures between model (i) and engine (j)
$U, \left(\hat{U}\right)$ =		Left singular-vectors of matrix Z, (normalized)
$\Sigma, \left(\hat{\Sigma}\right)$ =		Singular-values of matrix Z, (normalized)
V^T	=	Right singular-vectors of matrix Z

σ_i =   Singular-values, elements along the main diagonal of matrix Σ

W   =   Matrix to represent the product of $\hat{U}, \hat{\Sigma}$

{S}   =   Column-vector to represent the combined-normalized scores of each model

μ_j =   Mean of prediction errors based on multivariable polynomial model 'j'

Non-Dimensional Parameters

$CN_g \triangleq \dfrac{N_g}{\sqrt{\theta}}$ = corrected engine compressor speed

$CTGT \triangleq \dfrac{TGT}{\theta}$ =   corrected engine temperature (Turbine Gas Temperature)

$CSHP \triangleq \dfrac{SHP}{\delta\sqrt{\theta}}$ = corrected engine output power (Shaft Horse Power)

$CW_f \triangleq \dfrac{W_f}{\delta\sqrt{\theta}}$ =   corrected engine fuel-flow

I. Introduction

FLIGHT test engineering is an interdisciplinary science that gathers data and develops methods with the objective of evaluating an aircraft or a system in its operational flight environment. This need for flight-testing means that the system or the vehicle under testing requires accurate assessment of its characteristics while operating in its flight environment rather than just relying on the results of ground-based verification methods such as wind tunnels, simulators, and software models [Ref. 1]. There are many disciplines involved in flight-testing based on the nature of the questions in search. Such ones include for example performance assessment, structural integrity testing, handling-qualities evaluation, etc. Regarding helicopter performance assessment, the useful performance of any helicopter is directly derived from the amount by which the engine power (the available-power) surpasses (or falls below) the power required by the main and tail rotor systems, the drag of the fuselage and all other consumers of power for the specific conditions [Ref. 2]. The 'off-the-shelf' engine available power as given by the manufacturer changes once installed in a particular type of helicopter. It usually reduces due to inlet loss. Moreover, the maximum output-power of the installed engine degrades as it matures. Therefore, the actual available-power of the installed engine during a

particular phase of its life is of high practicality to helicopter users. This paper relates to the methods used in flight-test engineering for measuring helicopter gas turbine engine performance and estimating the maximum available output power under a wide range of environmental conditions.

The current method widely used in flight testing for determining the maximum power of a helicopter gas turbine engine relies on empirical single-variable polynomials. The method which is detailed in Ref. 3, 4 and 5 is practicably demonstrated in Ref. 6 and 7. The method requires the collection of stabilized engine parameters while flying the helicopter throughout its operational envelope. The four main raw engine variables measured in flight (compressor speed, temperature, fuel-flow and the output power) are normalized (or 'corrected') using the surrounding atmospheric conditions. By applying linear regression methods, three 3rd order single-variable polynomials are defined, representing the empirical relation between the corrected engine power (CSHP) and each one of the three engine corrected variables; corrected compressor speed (CNg) – Eq. (1), corrected temperature (CTGT) – Eq. (2) and corrected fuel-flow (CWf)- Eq. (3).

$$CSHP = \tilde{f}_1\left(CN_g\right) \approx \sum_{i=0}^{n} a_i \left(CN_g\right)^i \therefore n = 3 \tag{1}$$

$$CSHP = \tilde{f}_2\left(CTGT\right) \approx \sum_{i=0}^{n} b_i \left(CTGT\right)^i \therefore n = 3 \tag{2}$$

$$CSHP = \tilde{f}_3\left(CW_f\right) \approx \sum_{i=0}^{n} c_i \left(CW_f\right)^i \therefore n = 3 \tag{3}$$

The maximum available power of the *installed* engine is next estimated by using these three empirical single-variable polynomials. The three calculated values of the engine output power are first compared with each other and then against the maximum transmission torque (transmission limitation). This comparison is performed through an iterative process executed for various atmospheric conditions. The maximum available power under various atmospheric condition is then chosen as the minimum value out of all four values compared. The main advantage of the single-variable method lies in its simplicity. The flight-tester does not need to be confused with which mathematical model to choose, since the method is based on 3rd order single-variable polynomials. However, this simplicity is also the method's biggest disadvantage since 1) it requires careful analysis of the data especially when the required flight

conditions are outside of the limitations of the helicopter; 2) it may not replicate performance limiting factors that depend on actual flight conditions although matching non-dimensional values has been targeted successfully; 3) it frequently yields poor estimations of the maximum engine output power, especially under atmospheric conditions outside of the actual tested range. A comprehensive demonstration of the poor estimation using the single-variable method was is presented in Ref. 8.

A novel method to estimate the maximum output power of a helicopter gas turbine engine more accurately and under a wider range of atmospheric conditions was proposed in Ref. 8. This method involves the analysis of the flight test data in terms of referred parameters [Ref. 9] through a so-called 'Multivariable Polynomial Optimization under Constraints' (MPOC). The main advantages to analyzing data using MPOC models over the single-variable method are: 1) it gives the ability to determine the relative influence of one or more predictor variables to the criterion value. 2) it has the ability to identify outliers, or anomalies; 3) it gives a superior estimation precision. As demonstrated in Ref. 8, MPOC could provide a more accurate engine power estimation (in some cases of more than 300% when compared to the single-variable method). However, the main weakness of the MPOC method is that it struggles with a large number of possible multivariable-polynomials (more exactly 512 polynomials) to choose from and there are no guidelines herein.

The primary objective of this paper is to face this disadvantage of MPOC method by developing a systematic and repeatable approach on which specific, pre-defined multivariable-polynomial models shall be used. For this goal the MPOC method is applied to a large set of flight-test data gathered from seven different types of helicopters as presented in Table 1. Using the Singular-Values-Decomposition (SVD) approach, the relative performance of 512 different potential models is compared towards the objective of identifying the best performing multivariable polynomial model to be generally used by the MPOC method.

The paper is structured as follows: after a short introduction, Section II reviews the MPOC method as applied to a set of flight-test data gathered from a BO-105 helicopter. This section also provides the formalization process of all possible multivariable polynomials fitted with the flight-test data. Section III screens between all 512 different polynomials used by the MPOC method, searching for the best-performing ones. This preliminary screening process is based on the conventional method of p-values and fails to provide the best-performing polynomials with respect to the *entire group of 7 helicopters* (Table 1). This unsuccessful attempt is then rectified in section IV which presents an alternative approach based on the Singular-Value-Decomposition (SVD). A short list of the 10 most accurate and

convenient multivariable-polynomial models to represent the *entire group* of all 7 engines is then formulated. Before concluding the paper, a short comparison is drawn between the results of this paper and similar studies. Final conclusions and recommendations complete the paper.

Table 1: List of gas-turbine engines and helicopters used for the analysis

Engine No.	Type of Engine	Type of Helicopter Installed in	Rated Power, [hp]	Installation Configuration
1	RR MTU250-C20B	MBB BO-105M	420	twin
2	Turbomeca Arriel 1E2	Eurocopter EC-145	740	twin
3	Allison T63-A-700	Bell OH-58C	420	single
4	Turbomeca Arriel 1C2	Aerospatiale SA365-N2	700	twin
5	PW-207E	MD-902 Explorer	710	twin
6	Turbomeca Arriel 1M1	Eurocopter AS565 Panther	780	twin
7	GE T700-GE-701A	Sikorsky UH-60A Blackhawk	1700	twin

II. Helicopter Gas-Turbine Engine Flight-Testing

Principles of Multivariable-Polynomial Optimization under Constraints (MPOC) Method

Unlike the single-variable polynomial method mentioned above, the MPOC method is seeking for a multivariable-polynomial model representing the engine power while capturing the interrelation between the engine variables. The maximum engine power can then be assessed as an optimization problem of a multivariable-function under constraints. Such a multivariable approach applied to engine analysis results in a more accurate and realistic available power prediction as it contains the intrinsic couplings between all engine variables. Reference 8 demonstrates that the empirical model for the engine output power (Eq. (4)) should rely on a basic model, superimposed with any possible combination of 9 regressors (f_1 to f_9) as listed in Table 2. The basic model, referred to as Model 1 and denoted hereinafter as $CSHP_{M(1)}$, is a 3rd order multivariable-polynomial in all engine variables given as Eq. 5. One should realize there are 511 different combinations of choosing from regressors f_1 to f_9 of Table 2 as demonstrated by Eq. (6). Adding the basic model with the 511 possible combinations sets the total number of candidate models to be as large as 512. Having a list of 512 candidate models is impractical as the flight-test engineer still needs to undertake a tedious task of evaluating the performance of each candidate model (Eq. (4)) against the actual flight-test data.

Table 2: List of Regressors to be superimposed on Model 1

$f_1 = (CN_g)(CTGT)$	$f_4 = (CN_g)^2(CTGT)$	$f_7 = (CN_g)(CTGT)^2$
$f_2 = (CN_g)(CW_f)$	$f_5 = (CN_g)^2(CW_f)$	$f_8 = (CN_g)(CW_f)^2$
$f_3 = (CTGT)(CW_f)$	$f_6 = (CTGT)^2(CW_f)$	$f_9 = (CTGT)(CW_f)^2$

$$CSHP_{M(i)} = CSHP_{M(1)} + f(CN_g, CTGT, CW_f)\big|_{(i-1)} \quad i = 1, 2, 3, \ldots, 512 \therefore Model\ number\ i \tag{4}$$

$$CSHP_{M(1)} = \alpha_1^1 \cdot CN_g^3 + \alpha_2^1 \cdot CN_g^2 + \alpha_3^1 \cdot CN_g + \alpha_4^1 \cdot CTGT^3 + \alpha_5^1 \cdot CTGT^2 + \alpha_6^1 \cdot CTGT + \tag{5}$$
$$+ \alpha_7^1 \cdot CW_f^3 + \alpha_8^1 \cdot CW_f^2 + \alpha_9^1 \cdot CW_f + \alpha_0^1 \qquad Model\ number\ 1$$

$$N = \binom{9}{1} + \binom{9}{2} + \ldots + \binom{9}{9} = \frac{9!}{1! \, 8!} + \frac{9!}{2! \, 7!} + \ldots + \frac{9!}{0! \, 9!} = 511 \tag{6}$$

Hypothesis Testing and P-Values

All 512 proposed polynomial models can be fitted with actual experimental flight-test, yielding the specific coefficients for the best-fit solution. A practical method to solve for the best-fit coefficient, based on linear concept known as projection onto subspaces, is thoroughly described in Ref. 10 and demonstrated in Ref. 8. The best-fit solution obtained for any candidate polynomial model can be used to evaluate how precisely this model predicts the actual measured flight-test data. The corrected engine power (CSHP) is estimated by substituting the *measured* independent variables in the model, i.e., corrected engine compressor speed (CNg), corrected engine temperature (CTGT) and corrected engine fuel-flow (CWf). The prediction errors of the arbitrary chosen model 122 for each measured data point are then calculated using Eq. (7) and presented graphically in Fig. 1. The prediction errors of model 122 are approximately normally distributed about a practically zero mean (actual mean is -4×10^{-10} hp).

$$\vec{E}_r\big|_{122} = \left\{ CSHP_i - (CSHP_{122})_i \right\} \therefore i = 1, \ldots 34 \tag{7}$$
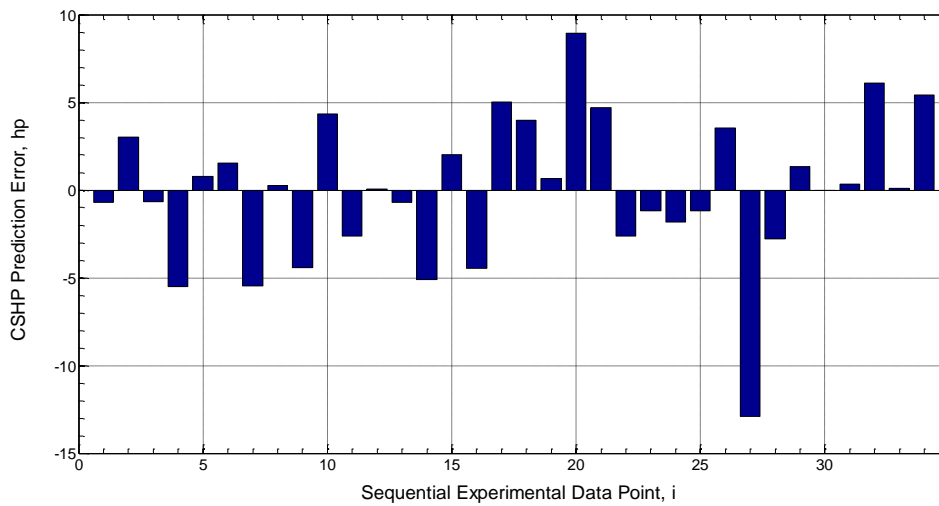
Figure 1. Prediction errors of the engine Corrected Output Power (CSHP), using the arbitrary model number 122

The conventional approach in flight-testing assessing prediction goodness is based on hypothesis testing and the associated p-values assigned. This approach follows from the Central Limit Theorem and is thoroughly discussed in literature [Ref. 11, 12]. In a nutshell, one can set up a hypothesis ('the null hypothesis') with regards to the mean value of the prediction errors and by using the actual measured data, the probability of falsely rejecting this hypothesis (making a 'type-I' error) is calculated. This probability numeral is known as the p-value and once it falls under a predefined value (the statistical significance level) it raises doubts about the statistical validity of the null hypothesis. Once again, the process is demonstrated by using the example BO-105 engine data and the arbitrary chosen, model 122. The hypothesis assigned claims all of model 122 prediction errors have a mean of zero. The test-statistic of this two-sided case is calculated as per Eq. (8) to be an extremely low value of -5.47×10^{-10}. In Eq. (8) the symbol 'n' represents the number of measured test-points and 'S' stands for the sample standard deviation with respect to the estimation errors of the engine power. One should realize for this particular case, low test-statistics values return large p-values and vice versa. This extremely low test-statistic value returns a *calculated* p-value of 1 (the maximum available due to software rounding errors). There is no statistical data to support rejection of the null hypothesis, meaning that model number 122 predicts the BO-105 engine performance with zero mean errors. Theoretically, this makes model 122 an excellent multivariable model for the available power prediction.

$$t_{122} = \frac{\overline{\left(\vec{E}_r\big|_{122}\right)}}{S/\sqrt{n}} \tag{8}$$

III. Prediction-Goodness Comparison between all Candidate Models

Assuming the arbitrary-chosen model number 122 is the 'perfect' multivariable-polynomial model to represent the output power of the BO-105 engine, how will all other 511 candidate models perform? Repeating the above-presented analysis for all other candidate models returned far too many *calculated* p-values of 1. The immediate conclusion one can draw is that the p-value by itself is not an effective screening tool. Since in our case the p-value and the *absolute-value* of the test-statistic are inversely proportional to each other, it is reasonable to use the test-statistic value itself as an indicator for prediction goodness. The screening process should be based then on minimum values of the test-statistics in lieu of a maximum p-values.

Figure 2 presents a wide perspective of the test-statistics, mean of prediction errors and errors standard deviations for *all* 512 candidate models. Figure 3 presents a closer look at the test-statistic of a group of only 84 candidate models, those involving the base model (Eq. 5) superimposed with any combination of 3 predictors out of the list of 9 (f_1 to f_9 as per Table 2). One should realize that each one of those 84 models returned a perfect *computed* p-value of 1, *including* models number 62, 88, 107 and 112 which seem to stand out from the group. The conclusion arising from Fig. 3 is that the conventional approach of screening models using the p-value is not practical for the specific task of finding the best empirical model to represent gas-turbine engine performance.
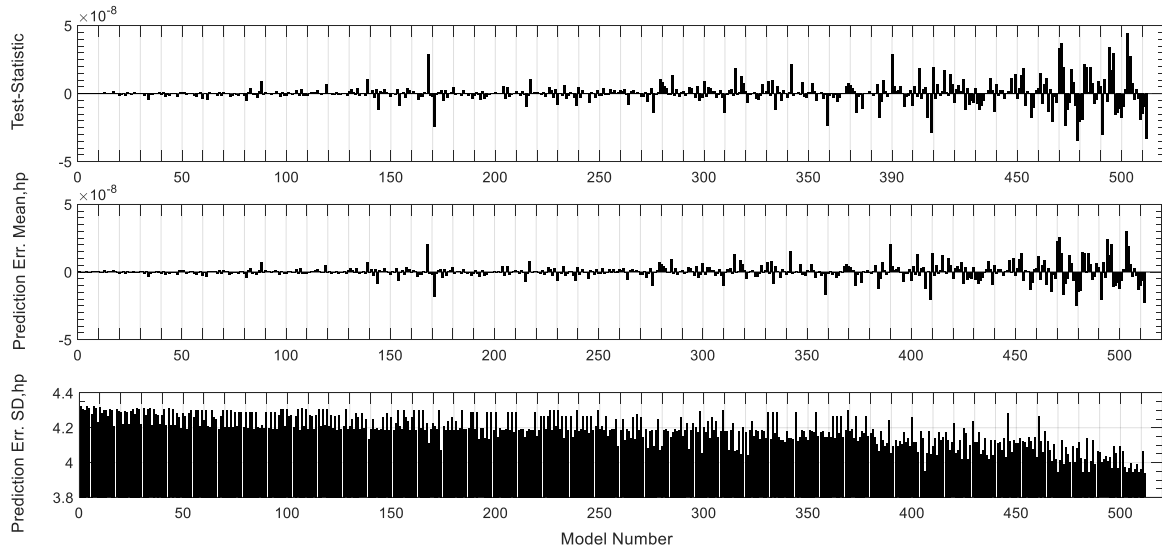
Figure 2. A wide perspective of all 512 candidate models prediction performance.
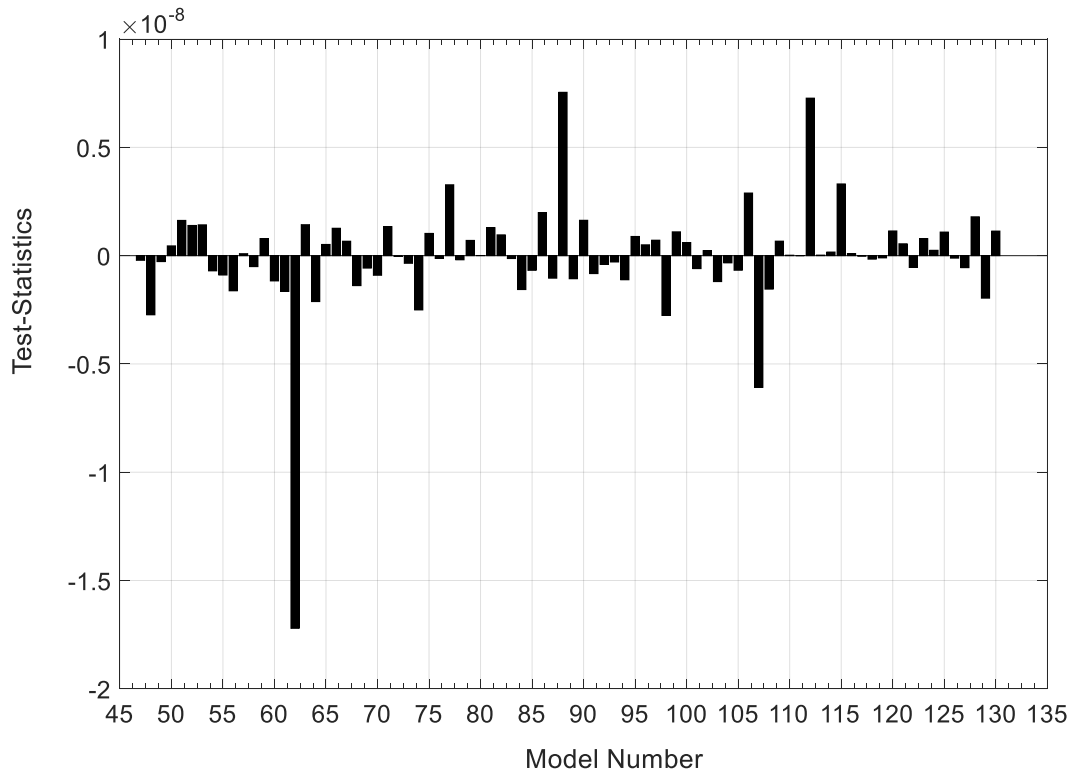

Figure 3. Test-Statistics of models number 47 to 130 which involve the base model (Eq. (5)) superimposed with any combination of 3 predictors from Table 2

The absolute-values of the test-statistics are then used instead of the p-values. Figure 4 presents the test-statistics (absolute-value) of the top 10 performing models for the BO105 engine. Table 3 describes these models in details. Examining Table 3, no pattern can be detected with respect to which predictors yield the best prediction performance. Nevertheless, the number of regressors used in the model has no clear influence on the prediction performance.

Within the set of 10 top-performing models there are models which involve additional one, two, three or four regressors to be superimposed over the basic model number 1. The obvious question to be asked next is Do these 10 top performing models also excel when applied to *different gas-turbine* engines? Can findings from the BO-105 engine be generalized to other types of helicopter gas-turbine engines? These questions are addressed hereinafter.
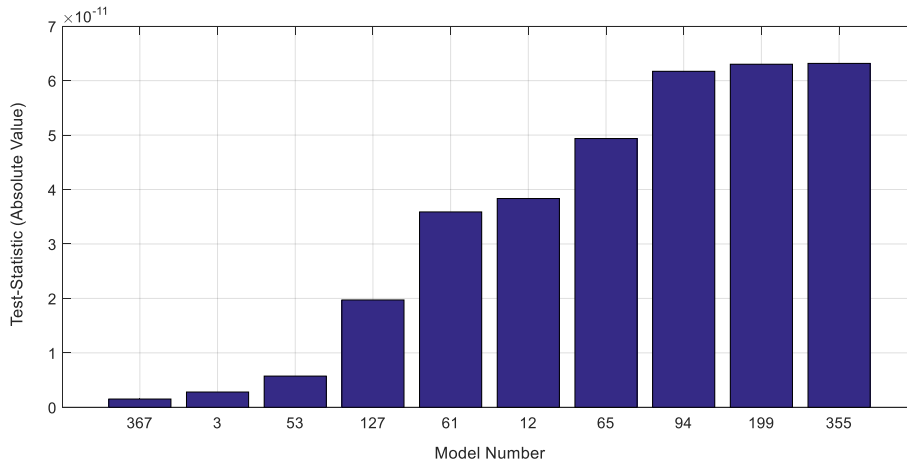

Figure 4. Test-Statistics (absolute-value) of the 10 top-performing models for the BO-105 engine

Table 3: List of 10 top-performing models for the BO-105 engine.

Auxiliary Regressors Involved*	Model Number									
	367	*3*	*53*	*127*	*61*	*12*	*65*	*94*	*199*	*355*
$f_1 \equiv CNg \cdot CTGT$			x		x	x	x			
$f_2 \equiv CNg \cdot CWf$		x	x					x	x	x
$f_3 \equiv CTGT \cdot CWf$	x					x			x	
$f_4 \equiv CNg^2 \cdot CTGT$	x				x					x
$f_5 \equiv CNg^2 \cdot CWf$	x						x			
$f_6 \equiv CTGT^2 \cdot CWf$				x	x		x			x
$f_7 \equiv CTGT^2 \cdot CNg$				x				x	x	
$f_8 \equiv CWf^2 \cdot CNg$	x		x						x	x
$f_9 \equiv CWf^2 \cdot CTGT$	x		x					x		x

* Regressors to be superimposed to the basic model expressed as Eq. (5).

Models Performance with other Engines and Helicopters

Consider next a different type of gas-turbine engine (engine 2 as per Table 1) installed on a different type of helicopter and a new set of flight-test data. Performing similar analysis reveals completely different findings from the BO-105 case. Figure 5 presents test-statistic of the 10 top-performing models for engine number 2. Further analysis was undertaken to include flight-test data from 5 other types of gas-turbine engines installed on other helicopters (Table 1). Results merely confirmed the previously stated conclusion that the best performing model to describe helicopter gas-turbine power, if it exists, cannot be found using a conventional approach of screening between models using hypothesis testing, neither based on the p-value nor on the test-statistics. Concluding this section, an alternative *general* approach needs to be taken. This approach should relate to the Singular-Value-Decomposition (SVD) theorem and is discussed and demonstrated in the next section of the paper.
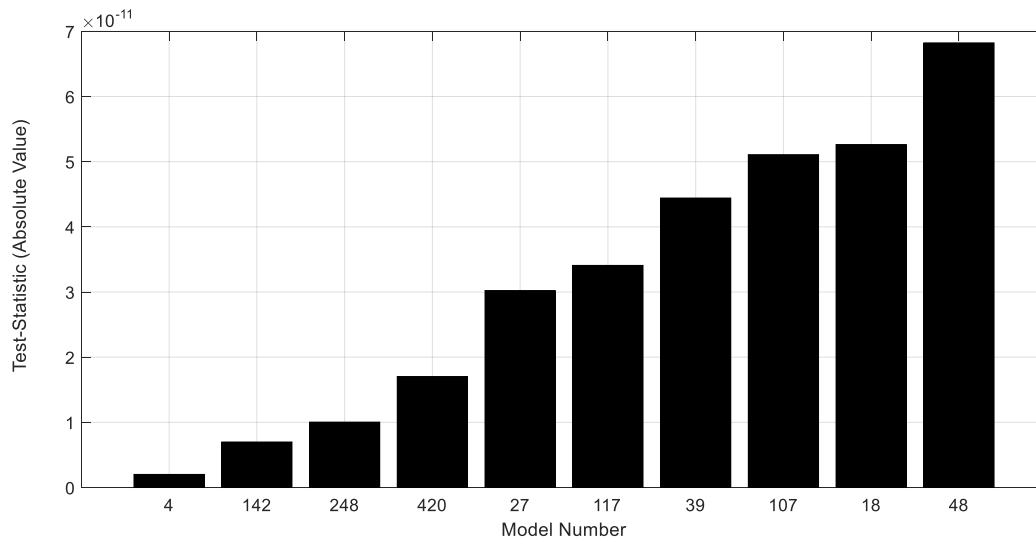

Figure 5. Test-Statistics (absolute-value) of the 10 top-rated models using engine number 2.

IV. The Singular-Value-Decomposition (SVD) Method

The SVD Theorem

The Singular-Value-Decomposition (SVD) theorem (see Ref. 13 for a thorough description) states that *any* matrix that holds real entries can be *uniquely* decomposed into a set of three special matrices as given in Eq. (9). Consider matrix Z to be of size m by n (notation (m,n)) and rank r. Matrix Z can be decomposed into three unique matrices:

- Matrix U called the 'left singular vectors' is an orthonormal matrix of size (m,r). The columns of U span a base in space.

- Matrix $\sum$ is a diagonal matrix which occupies all singular-values along its diagonal (the singular-values are arranged in a descending order).

- Matrix V^T called the 'right singular vectors' is an orthonormal matrix of size (r,n) with its rows spanning a base in space.

$$Z = U \cdot \Sigma \cdot V^T \tag{9}$$

SVD Implementation for Model Screening

The SVD theorem can be implemented to identify latent dimensions or concepts in the gas-turbine engine flight-test data. For this, matrix Z is defined with its elements to indicate measures of excellence (scores) for each *specific* multivariable-polynomial model in predicting performance of each s*pecific* engine tested (see Eq. (10)). This matrix Z is of size (512, 7) with its rows representing all 512 candidate multivariable-polynomial models and the columns representing the various engines/helicopters tested. Engine number 1 is represented by the most left column and engine number 7 by the most right column of matrix Z.

Next step is to assign scores as elements of matrix Z to quantify level of precision each model predicts a specific engine. As explained before, these scores are based on the absolute-values of the relevant test-statistics (Eq. (10)). Since prediction goodness and test-statistics (absolute-value) are inversely proportional to each other (that is the smaller the test-statistic absolute-value is the better the model represents the experimental data) the reciprocals of all test-statistics (absolute-value) are used as elements in matrix Z. The variable $t_{i,j}$ as appears in Eq. (10) represents the test-statistic calculated for model number (i) using the flight-test data of engine number (j). Note that matrix Z encapsulates the entire flight-test data base.

$$Z \equiv \begin{pmatrix} \zeta_{1,1} & \zeta_{1,2} & \zeta_{1,3} & \zeta_{1,4} & \zeta_{1,5} & \zeta_{1,6} & \zeta_{1,7} \\ \zeta_{2,1} & \zeta_{2,2} & \zeta_{2,3} & \zeta_{2,4} & \zeta_{2,5} & \zeta_{2,6} & \zeta_{2,7} \\ \zeta_{3,1} & \zeta_{3,2} & \zeta_{3,3} & \zeta_{3,4} & \zeta_{3,5} & \zeta_{3,6} & \zeta_{3,7} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \zeta_{511,1} & \zeta_{511,2} & \zeta_{511,3} & \zeta_{511,4} & \zeta_{511,5} & \zeta_{511,6} & \zeta_{511,7} \\ \zeta_{512,1} & \zeta_{512,2} & \zeta_{512,3} & \zeta_{512,4} & \zeta_{512,5} & \zeta_{512,6} & \zeta_{512,7} \end{pmatrix} \quad \therefore \zeta_{i,j} = \frac{1}{|t_{i,j}|} \qquad (10)$$

Equation 11 presents a SVD decomposition of matrix Z into its 3 unique matrices as defined above. The idea of linearly-independent vectors to span a base in space can be regarded as an exposure of hidden dimensions in the data. The conceptual interpretation of the SVD of matrix Z is illustrated in Figure 6 and further explained hereinafter.

The rank of matrix Z represents the number of independent hidden Principal Dimensions (PD's). The diagonal singular-values matrix ($\sum$) has all PD's represented by elements along its main diagonal. These elements are arranged in a descending order and indicate the relative 'strength' of appearance of each PD in the flight-test data.

The left singular-vectors matrix (U) has seven columns, each with 512 elements. These seven columns are orthonormal vectors which represent the level of correspondence between each one of the 512 models and a detected PD in the data. As illustrated in Figure 6 one can see that the first column vector indicates correspondence between each one of the 512 models to the first (*and the most significant*) PD. The second column vector specifies level of correspondence between all 512 models to the second PD and so on. Figure 6 specifies one element of the left-singular vector matrix (3rd row and 6th column) as an example to demonstrate this entry represent the level of correspondence between model number 3 and PD number 6.

The right singular vectors matrix (V) has seven rows (the rank of matrix Z) with seven elements each (the 7 engines in the flight-test data base). As illustrated in Figure 6, these rows represent the level of correspondence between each specific engine (denoted by the column number) and a Principal Dimension (denoted by the row number). The first row vector indicates relative levels of correspondence between all 7 engines and the first PD. The second row specifies the relative strength between all engines and the second PD, and so on. Figure 6 specifies one element of the right-singular vector matrix (7th row and 2nd column) as an illustration of the level of correspondence between engine number 2 and PD number 7 (the least dominant PD exposed in the data).

$$
Z \equiv \begin{pmatrix}
u_{1,1} & u_{1,2} & u_{1,3} & u_{1,4} & u_{1,5} & u_{1,6} & u_{1,7} \\
u_{2,1} & u_{2,2} & u_{2,3} & u_{2,4} & u_{2,5} & u_{2,6} & u_{2,7} \\
u_{3,1} & u_{3,2} & u_{3,3} & u_{3,4} & u_{3,5} & u_{3,6} & u_{3,7} \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
u_{511,1} & u_{511,2} & u_{511,3} & u_{511,4} & u_{511,5} & u_{511,6} & u_{511,7} \\
u_{512,1} & u_{512,2} & u_{512,3} & u_{512,4} & u_{512,5} & u_{512,6} & u_{512,7}
\end{pmatrix}
\cdot
\begin{pmatrix}
\sigma_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sigma_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sigma_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \sigma_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \sigma_5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \sigma_6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \sigma_7
\end{pmatrix}
\cdot
\begin{pmatrix}
v_{1,1} & v_{1,2} & . & . & v_{1,7} \\
v_{2,1} & v_{2,2} & . & . & v_{2,7} \\
v_{3,1} & v_{3,2} & . & . & v_{3,7} \\
v_{4,1} & v_{4,2} & . & . & v_{4,7} \\
v_{5,1} & v_{5,2} & . & . & v_{5,7} \\
v_{6,1} & v_{6,2} & . & . & v_{6,7} \\
v_{7,1} & v_{7,2} & . & . & v_{7,7}
\end{pmatrix}
\tag{11}
$$

The relative strength of each PD which is demonstrated by the corresponding singular-value is then normalized as per Eq. (12). Figure 7 presents the normalized seven PD's singular-values. One can observe that the major PD detected in the data holds a relative strength of 36%, while the following two PD's share a similar relative strength of 23% and 22% ,respectively. The combination of the first 4 PD's encapsulates about 96% of PD presented in the data.

$$
\hat{\sigma}_i = \frac{\sigma_i}{\sum\limits_{k=1}^{r} \sigma_k} \quad \therefore r \, \square \, rank(Z)
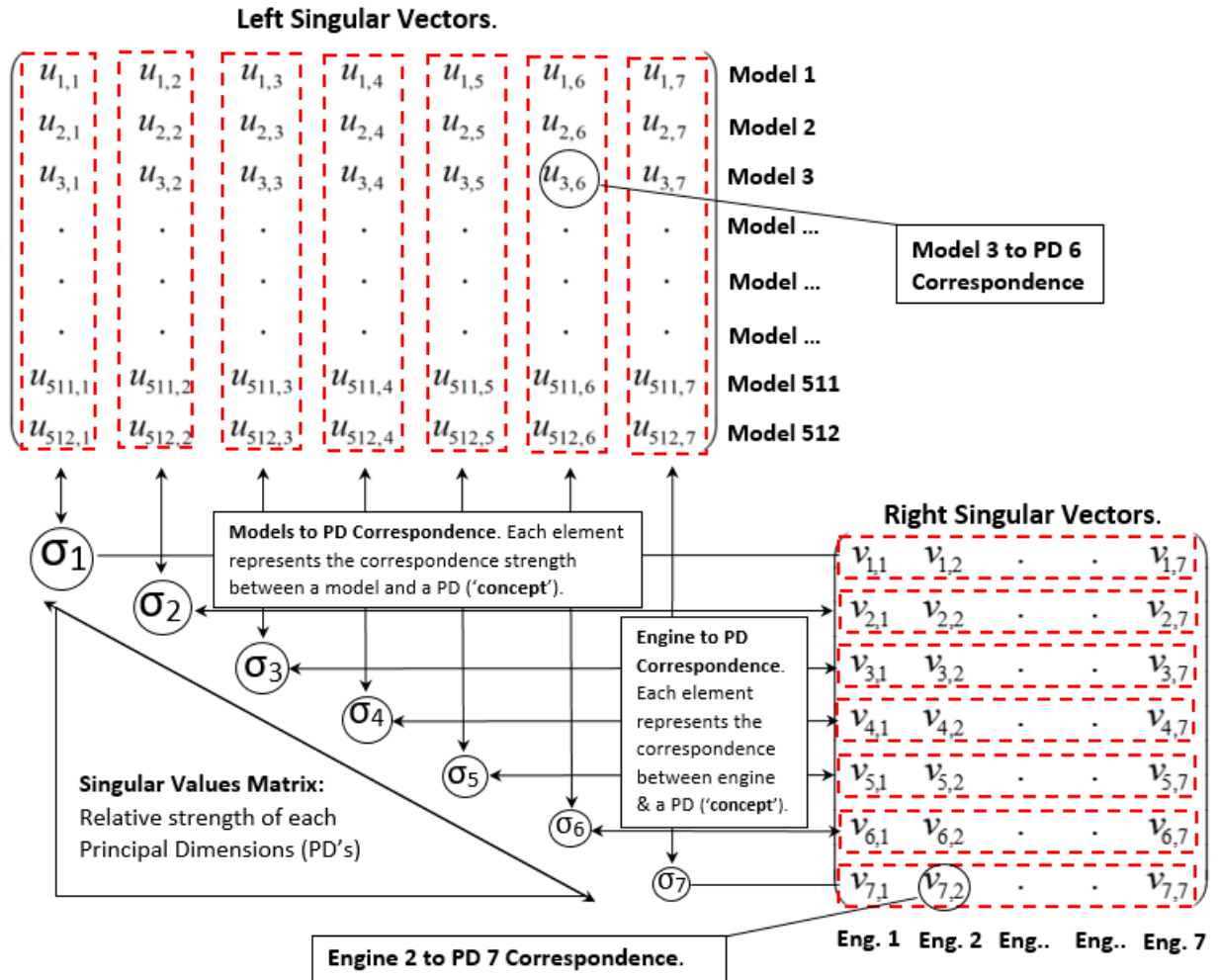\tag{12}
$$

Figure 6. A conceptual interpretation of the Singular-Value-Decomposition (SVD) of the matrix Z.
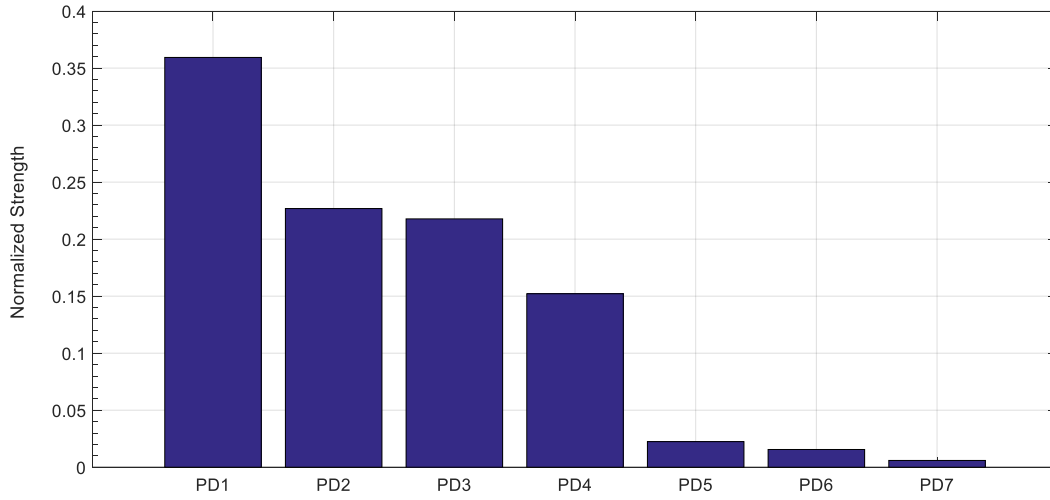
Figure 7. The normalized strength of the 7 PD's as demonstrated by the normalized singular-values of matrix Z

The Left Singular Vectors Matrix – Models to PD's Correspondence

The absolute-value of each element along a *column* vector indicates the level of correspondence between a specific model (row number of the vector) and the relevant PD. Each element along the column vectors is normalized as per Eq. (13). Figure 8 presents a collage of 7 plots to indicate the normalized elements along the 7 columns as level of correspondence between each of the 512 candidate models and the 7 Principal Dimensions. The first plot represent correspondence between each candidate model and the first and most dominant Principal Dimension (PD1).

One can observe that model number 320 demonstrates the strongest correspondence to PD1. The other plots on Fig. 8 are broadening the spectrum of models to PD's correspondence. Model 125 demonstrates strong correspondence to PD2, model 367 to PD3, model 4 to PD4, models 49 and 226 to PD5, model 7 to PD6 and model 282 to PD7.

$$\hat{U}(i, j) = \frac{|U(i, j)|}{\sum\limits_{i=1}^{512}|U(i, j)|} \therefore j = 1, 2, ..., r \therefore r = rank(Z) \tag{13}$$
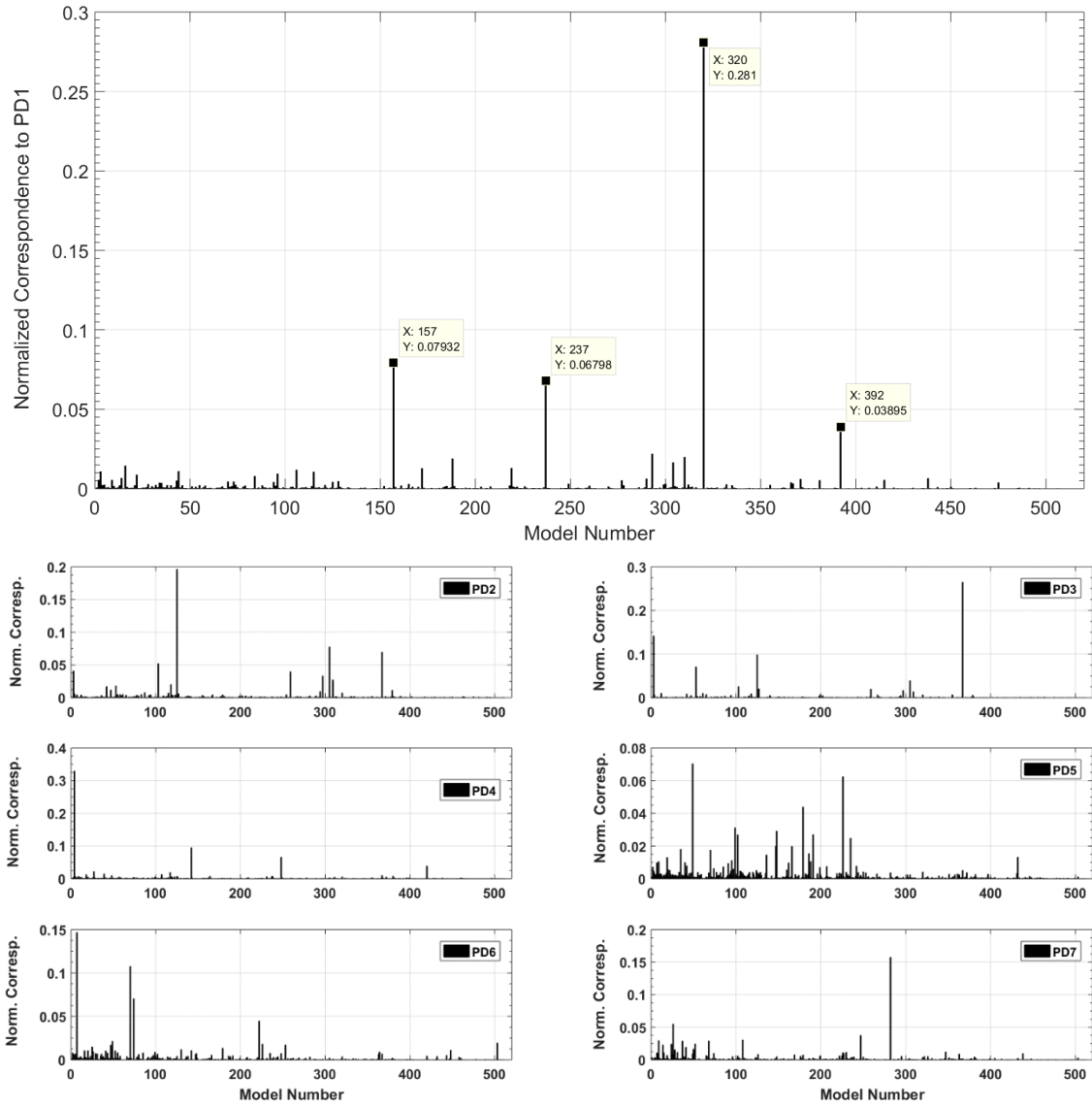
Figure 8. The normalized correspondence between all 512 candidate models and the 7 Principal Dimensions (PD's).

The Right Singular Vectors Matrix – Engines to PD's Correspondence

The absolute-value of each element along a row vector indicates the level of correspondence between a specific engine (column number of the row-vector) and the relevant PD. Each element along a row is normalized as per Eq. (14). Figure 9 presents a collage of 7 plots to indicate the normalized elements along the 7 row-vectors as level of correspondence between engines and Principal-Dimensions. It follows from the first plot that PD2 is mainly driven by two engines; engine number 1 and engine number 4. In a more general context, these two engines share a substantial

similarity with respect to performance models through the second most important Principal-Dimension (PD2). This demonstrates the capability of the SVD decomposition to detect latent dimensions in the data, hence to expose hidden similarities between different types of engines. The other 6 plots in Fig. 9 mostly expose the similarity shared between engines 1 and 4 through PD3.

$$\hat{V}(i,j) = \frac{\left|V(i,j)\right|}{\sum\limits_{j=1}^{7}\left|V(i,j)\right|} \therefore i = 1,2,...,r \therefore r = rank(Z) \tag{14}$$
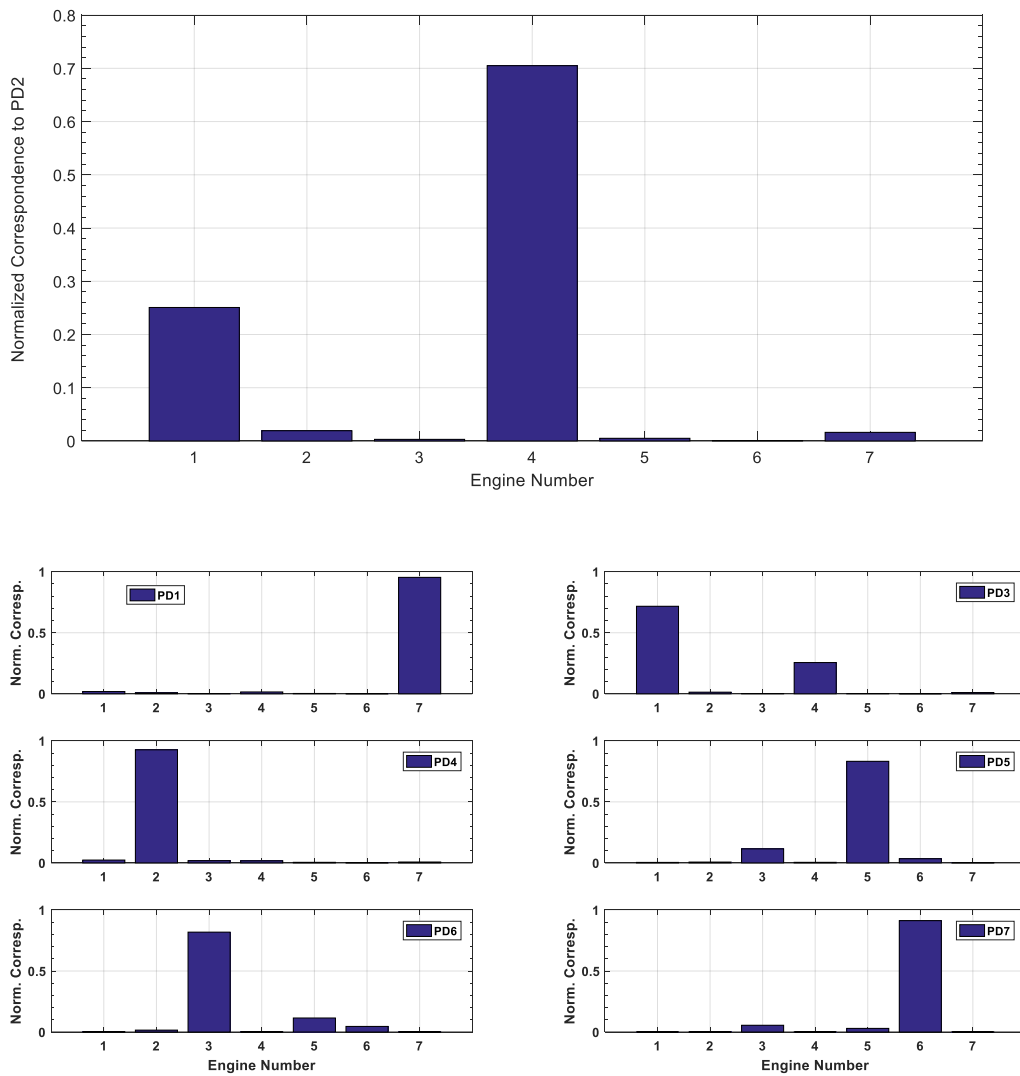

Figure 9. The normalized correspondence between all 7 engines and the Principal Dimensions (PD's).

Selecting the Best Multivariable Polynomial Model

Once the practical interpretation of the SVD for the flight-test data has been demonstrated, the central question raised in this paper can be readdressed , namely, is it practicable to find a general approach to the MPOC method for best prediction of the gas turbine engine available power? Can the flight-test data recommend a short list of multivariable polynomial models best describing the helicopter gas-turbine engine performance? The conventional method of hypothesis testing provided confusing results as was shown above. For this a new matrix (W) is defined (Eq. (15)).

Matrix W is the product of the normalized *left singular vectors* matrix and the normalized *singular-values* matrix. This matrix is of the same size of matrix Z, i.e. 512 rows and 7 columns. Each column of W represents the relative correspondence of the 512 models to the relevant PD (the column number). Adding all column vectors of matrix W to each other results in a single column vector {S} with 512 elements (Eq. 16). Practically, each element of vector {S} holds a normalized value for the *overall/combined* performance of each model in predicting the performance of the 'generic' engine, a hypothetical engine that represents all engines tested. The elements of the column vector {S} can be regarded as the Combined Normalized Scores (CNS) of each one of the 512 models used in predicting the performance of a gas-turbine engine *in general.* Figure 10 presents the CNS for all 512 candidate multivariable polynomial models. Based on the highest Combined Normalized Score (CNS) achieved the best empirical model describing the gas-turbine engine performance is model number 320.

$$[W] = [\hat{U}][\hat{\Sigma}] \quad \therefore \quad [\hat{\Sigma}] \square \begin{pmatrix} \hat{\sigma}_1 & & 0 \\ & \ddots & \\ 0 & & \hat{\sigma}_r \end{pmatrix} \tag{15}$$

$$\{S\} = \sum_{i=1}^{r}(\{w_i\}) \therefore [W] \square \begin{bmatrix} \{w_1\} & \{w_2\} & . & . & \{w_r\} \end{bmatrix} \tag{16}$$

This result can be expanded to provide a short list of the 10 top-performing multivariable polynomial for the 7 engines tested (see Table 4). Looking at Table 4, one can find the similarities between this list and the one formulated for engine 1 (Fig. 4) and for engine 2 (Fig. 5). Although engines number 1 and 2 'sent' few of their top 10 performing models as 'representatives' to the final 10 top-performing models list, neither one nor the other shared the best final model proposed based on their level of correspondence to the major Principal Dimension (PD1). The engine with the maximum correspondence to PD1 was number 7 (Fig. 9) and its top-performing model came leading in the final list. As presented in Table 4, model number 320 involves the basic 10 predictors as given in Eq. (5) superimposed with 5

other predictors: f_1, f_4, f_6, f_8 and f_9. One should notice that adding more predictors to the basic model, Model number 1 (Eq. (5)), does not necessarily correlate with improving prediction performance.

The final top 10 list actually includes two models which are using only one extra predictor. These are models number 4 and number 3. Model number 4 (Eq. (17)) uses f_1 as the auxiliary predictor and model 3 (Eq. (18)) uses f_2. When analysis requires simple model to use, either one of the two is suitable.

$$CSHP_{M_{(4)}} = \alpha_1^4 (CN_g)^3 + \alpha_2^4 (CN_g)^2 + \alpha_3^4 (CN_g) + \alpha_4^4 (CTGT)^3 + \alpha_5^4 (CTGT)^2 + \alpha_6^4 (CTGT) + \alpha_7^4 (CW_f)^3 + \\ + \alpha_8^4 (CW_f)^2 + \alpha_9^4 (CW_f) + \alpha_{10}^4 (CTGT \cdot CW_f) + \alpha_0^4 \tag{17}$$

$$CSHP_{M_{(3)}} = \alpha_1^3 (CN_g)^3 + \alpha_2^3 (CN_g)^2 + \alpha_3^3 (CN_g) + \alpha_4^3 (CTGT)^3 + \alpha_5^3 (CTGT)^2 + \alpha_6^3 (CTGT) + \alpha_7^3 (CW_f)^3 + \\ + \alpha_8^3 (CW_f)^2 + \alpha_9^3 (CW_f) + \alpha_{10}^3 (CN_g \cdot CW_f) + \alpha_0^3 \tag{18}$$

Another point worth addressing is how well the basic model performs in the bigger scheme of all 7 engines? It appears that model number 1 was rated at the 173rd place, at the top of the second trimester of the pack.
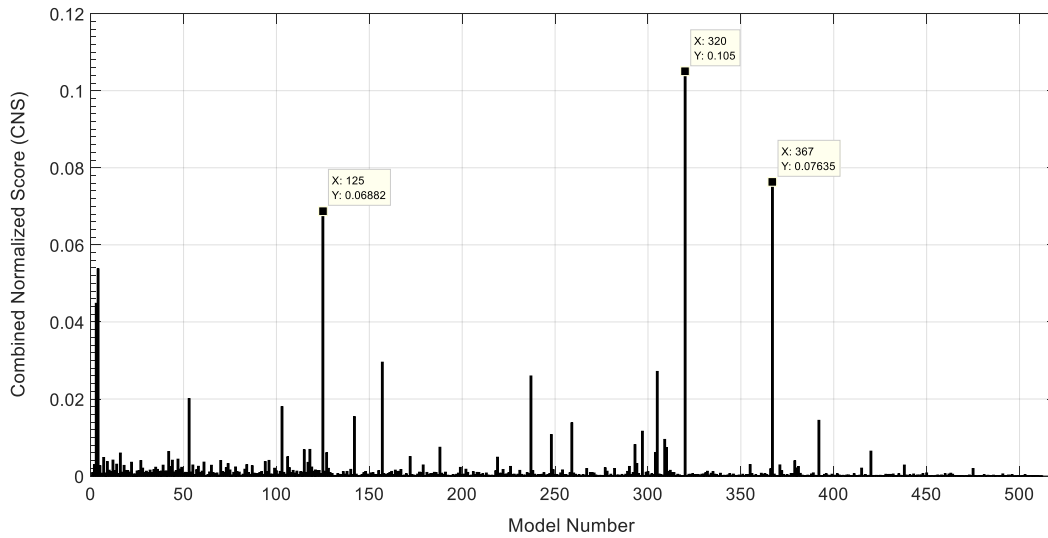

Figure 10. The Combined Normalized Score (CNS) returned for all 512 models evaluated based on all 7 engines tested

Table 4: List of 10 top-performing models for helicopter gas-turbine engine performance

Auxiliary Regressors Involved*	Model Number									
	320	*367*	*125*	*4*	*3*	*157*	*305*	*237*	*53*	*103*
$f_1 \equiv CNg \cdot CTGT$	x					x	x		x	
$f_2 \equiv CNg \cdot CWf$		x			x				x	
$f_3 \equiv CTGT \cdot CWf$		x		x		x	x	x		x
$f_4 \equiv CNg^2 \cdot CTGT$	x	x								
$f_5 \equiv CNg^2 \cdot CWf$		x	x			x	x	x		x
$f_6 \equiv CTGT^2 \cdot CWf$	x					x				
$f_7 \equiv CTGT^2 \cdot CNg$			x				x			
$f_8 \equiv CWf^2 \cdot CNg$	x	x					x	x		x
$f_9 \equiv CWf^2 \cdot CTGT$	x	x	x					x	x	

* Regressors to be superimposed to the basic-model expressed as Eq. (5)

V. Comparison to Conventional Methods and Applications

The conventional method for estimating helicopter *installed* gas turbine engine output power is based on single-variable analysis method. The innovative MPOC method, based on multivariable polynomial models, was shown to significantly improve prediction of maximum available power under a wider range of atmospheric conditions [Ref. 8]. The main drawback of the MPOC method is that it struggles with a large number of candidate multivariable polynomial models to choose from. Table 4 addresses this shortcoming by providing a brief list of 10 best-performing multivariable polynomial models to be used with the MPOC method. Figure 11 presents the mean of the prediction error of all these multivariable polynomial models using the seven engines of Table 1. The mean of the prediction errors presented in Fig 11 were calculated as per Eq. (19). In Eq. (19) the variable CSHPi represents the measured engine power for data point 'i' and CSHPj is the engine output power as estimated by sequential model number 'j'. The parameter 'n' represents the number of measured data points. Figure 11 also presents the prediction performance of the 7 engines of Table 1 using the conventional method based on the single-variable models (Eq. (1) - (3)). One can see that the multivariable polynomial models performed much better in predicting all 7 engine output-power. The maximum average prediction-error using a multivariable model was measured to be only 0.2%. This relatively low prediction error belongs to the two models 3 and 4 whilst predicting the output power of engine number 4. Comparing the prediction performance of the multivariable polynomial models to those achieved using single-variable models disclose a clear advantage for the multivariable models. The single-variable models returned much higher prediction errors for all seven engines tested. One can see in Fig. 11 these prediction errors reached up to 1.15% (for engine 3).

A different multivariable approach for helicopter engine performance determination is presented in Ref. 17. Although prediction accuracy achieved is not specifically discussed in Ref. 17, the method presented completely ignores the engine temperature as a predictor for the engine performance model. The engine temperature is essential for the determination of maximum available power, since the engine power is often limited by this variable reaching the maximum allowed. Discarding the engine temperature from the performance model, makes this approach useless for the MPOC method. Another fundamental difference between the MPOC method and the one presented in Ref. 17 is the order of the polynomials used to describe the engine output power. The MPOC method is based on 3rd orders while Ref. 17 uses 2nd orders. Limiting the engine performance model to a 2nd order only, prevents an inflection point and therefor fails from modeling an important physical characteristic of the engine that the rate of power increase with engine temperature increase must reduce while operating the engine close to its limitations. All multivariable models presented in Table 4 were found to estimate the output power of the seven different engines tested with an average accuracy of no more than 0.2% for each model tested. The absolute prediction error for a single measured point never exceeded 4.1% for all seven different engines. Similar analysis, based on conventional single-variable models, returned best estimation errors of only 8%. Putting the work presented in this paper in the larger context of gas turbine engine performance and comparing the prediction accuracy achieved using the multivariable polynomial with prediction accuracy of commonly used research simulation tools such as Turbomatch [Ref. 14] reveals similar or better results. Ref. 15 uses Turbomatch to predict helicopter gas-turbine performance for their work. Chapter 3 of Ref. 15 reports that the model has been matched at design point conditions with public domain data in terms of specific fuel consumption (SFC) with an accuracy of 0.3%. Ref. 15 presents a method of calculating gas-turbine engine output power based on flow-field simulation and aerodynamics modeling. The engine output-power estimation is based on the engine outlet temperature. Predictions for engine outlet temperature were validated against five measured steady-state engine operating data points (output power between 340 to 1394 hp) using five similar-type but different helicopter gas turbine engines tested on a bench. The reported temperature estimation errors ranged between 2.4% to 4.1% for all 25 data points measured.

Simple and accurate mathematical models that represent the available output power of the engine, such as presented in Table 4, can efficiently be used not only for the immediate prediction of a specific helicopter performance, but also in relevant adjacent research which requires a gas-turbine engine power model. Examples of such are improvement of existing gas-turbine engine technologies where current performance is needed for comparison

[Ref. 18]. Other examples can be improvement and validation of helicopter performance where the engine output power is needed as a module in the big scheme of total helicopter performance [Ref. 19, 20]

$$\mu_j = \frac{\sum_{i=1}^{n}\left(\frac{100\left(CSHP_i - CSHP_j\big|_i\right)}{CSHP_i}\right)}{n} \tag{19}$$
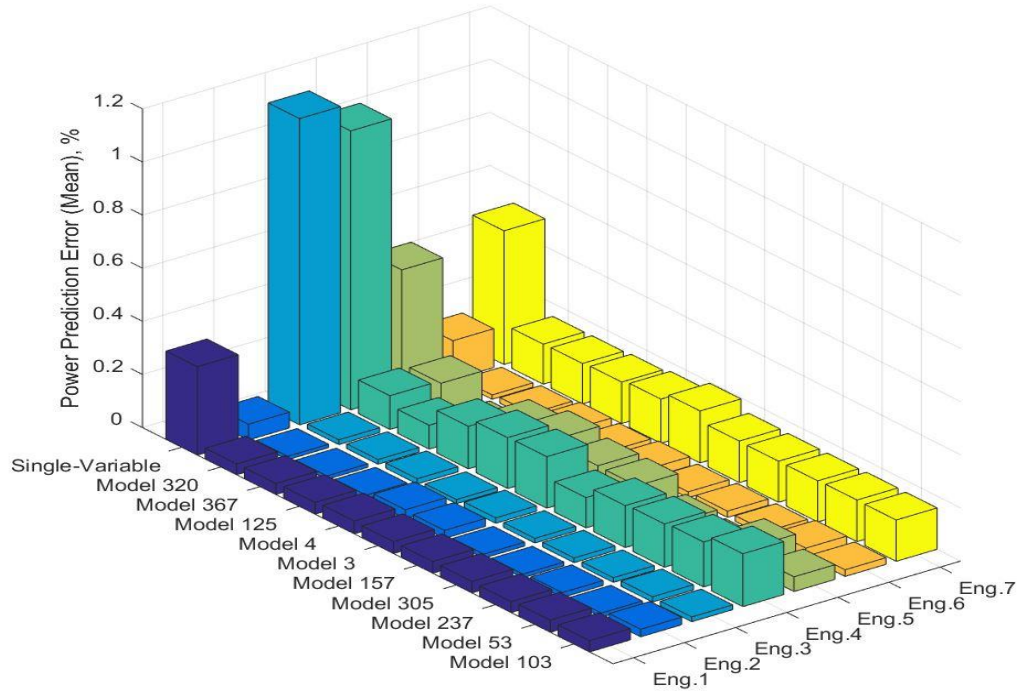

Figure 11. The mean output-power prediction error of all 7 engines based on multivariable models from Table 4 and the conventional single-variable model.

VI. Conclusions

The process of empirical models evaluation is at the core business of experimental flight-testing data analysis. A commonly used technique in experimental flight testing to sort between candidate models is the one based on hypothesis-testing and the associated P-values. The hypothesis-testing method is thoroughly demonstrated in this paper for the purpose of selecting the best empirical multivariable-polynomial model to represent the BO-105 engine. After applying minor adjustments, the hypothesis-testing method was implemented successful in ranking all 512

candidate multivariable-polynomial models based on their relative performance. The hypothesis-testing approach became completely ineffective once the experimental data-base was expanded to include 6 more engines. The method failed providing a clear answer to the question of which is the best-performing model when the *entire* experimental data from all 7 engines is analyzed as a whole.

The Singular-Value-Decomposition (SVD) approach was used successfully where the hypothesis-testing method failed and generated a list of top-performing multivariable polynomial models to describe gas turbine engine performance in *general*. The SVD approach was also successful in exposing latent similarities between different engines with respect to their performance models.

Analysis showed no correlation between the number of auxiliary predictors used in the multivariable-polynomial model and the prediction accuracy. The 4th and 5th best-performing models out of the 512 evaluated incorporated only 11 predictors, making either one of them a great choice if analysis simplicity is required. Although the SVD approach is demonstrated in this paper using engine data, it is not bounded only to gas-turbine engine testing. It can and should be used for any type of performance flight-testing analysis, where empirical models are being evaluated. Future research will focus on implementation of the SVD approach using other disciplines of helicopter performance flight-testing.

References

1. Pavlok, KM. Flight Test Engineering, Edwards California: NASA Dryden [Armstrong] Flight Research Center, NASA Report #DRFC-E-DAA-TN11035, September 19, 2013

2. Advisory Group for Aerospace Research & Development (AGARD), *Flight Test Techniques Series.* AGARD-AG-300, 1995

3. Knowles P. *The Application of Non-dimensional Methods to the Planning of Helicopter Performance Flight Trials and Analysis Results,* Aeronautical Research Council ARC CP 927, 1967

4. Cooke AK and Fitzpatrick EWH. *Helicopter Test and Evaluation*, 1st ed., AIAA Education Series, 2002, Chap. 3.

5. National Test Pilot School, *Professional Course Textbook Series, Vol. VII, Rotary Wing Performance Flight Testing,* Mojave, 2017, Chap. 5.

6. Benson TP, Buckanin RM, Mittag CF, et al. Evaluation of a OH-58A Helicopter with an Allison 250 C-20B Engine. Final Report, US Army Aviation Engineering Flight Activity, Edwards AFB, California, USA, April 1975.

7. Belte D, Stratton MV. Fuel Conservation Evaluation of US Army Helicopter, Part 4, OH-58C Flight Testing. Final Report, US Army Aviation Research and Development Command, St. Louis, MO, USA, August 1982.

8. Arush I and Pavel MD. Helicopter Gas Turbine Engine Performance Analysis: A Multivariable Approach, Proc IMechE Part G: Journal of Aerospace Engineering, Nov. 2017, pp 1-14, DOI: 10.1177/0954410017741329.

9. Engineering Science Data Unit, Non-dimensional methods for the measurement of level flight performance of turbine-engined helicopters, ESDU Item Number 74042

10. Strang G. *Introduction to Linear Algebra*, 5th ed., Wellesley-Cambridge Press, Wellesley MA, 2016, Chap. 4.

11. Guttman I, Wilks S. and Hunter J. *Introductory Engineering Statistics*, 2nd ed., John Wiley & Sons, Inc., New York, 1971, Chap. 10.

12. Kreyszig E. *Advanced Engineering Mathematics*, 3rd ed., John Wiley & Sons, Inc., New York, 1972, Chap. 19.

13. Strang G. *Linear Algebra and its Applications*, 4th ed., Thomson Brooks/Cole, Belmont CA, 2006, Chap. 6.

14. MacMillan WL. Development of a modular-type computer program for the calculation of gas-turbine of design performance, Ph.D. Dissertation, School of Engineering, Cranfield University, Cranfield, Bedford, 1974.

15. Goulos I, Giannakakis P, Pachidis V. et al. Mission Performance Simulation of Integrated Helicopter-Engine Systems using an Aeroelastic Rotor Model, Journal of Engineering for Gas Turbines and Power, Volume 135, September 2013, Vol. 135.

16. Heng W, Shufan Z, Jijun Z, et al. Gas Turbine Power Calculation Method of Turboshaft based on Simulation and Performance Model, MATEC Web of Conferences 189, 02003, 2018. DOI:10.1051/matecconf/201818902003.

17. Gomez A, Arantes J, De Andrade D, et al. Helicopter Engine Performance Determination using Analysis of Variance. In: AHS 71st Annual Forum, Virginia Beach, Virginia, USA, May 5-7 2015.

18. Zhang C, Gummer V. The potential of helicopter turboshaft engines incorporating highly effective recuperators under various flight conditions. Aerospace Science and Technology, 6 March 2019. http://doi.org/10.1016/j.ast.2019.03.008.

19. Savelle SA, Garrard GD. Application of transient and dynamic simulations to the US Army T55-L-712 helicopter engine. In: International Gas Turbine and Aeroengine Congress and Exhibition, Birmingham, UK, June 10-13 1996.

20. Yeo H, Bousman WG and Johnson W. Performance analysis of a utility helicopter with standard and advanced rotors. In: AHS Aerodynamics, Acoustics and Test and Evaluation Technical Specialist Meeting, San Francisco, California, USA, January 23 – 25 2002.