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Optimal Collision Avoidance in Unconfined Waters

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ABSTRACT

COLLISION AVOIDANCE manoeuvring has been subject to studies for many years, especially since the introduction of radar and automated data handling. Most of these studies were done on a rather practical basis, meaning that practical maritime habits in this field were adopted without much criticism.

This paper presents a somewhat unconventional solution to collision avoidance problems, in which a quantitative knowledge of the own ship's dynamics can be fruitfully applied. As a slight premise, we only consider situations in which two ships are involved. Moreover the evading ship is assumed to have full navigational freedom, meaning that there are no positional restrictions, caused by shoals or bounds of depth channels.

The resulting evading manoeuvre is a loop with continuously changing course and speed. It is less time consuming than former designs with piecewise constant courses. Thus the evader's intentions become more clear to the other vessel, which is initially obliged to maintain course and speed.

In order to avoid "last-moment-hazardous" situations, the evading manoeuvre is proposed to be started at the earliest possible time.

INITIAL CONDITIONS FOR A COLLISION AVOIDING MANOEUVRE

In this study, a collision avoiding manoeuvre (c.a.m.) for a ship under way at sea will be understood to be a rudder manoeuvre with the engines running constantly full ahead.

Prior to the c.a.m. the own ship A and the observed ship B are assumed to have constant courses ψ_a and ψ_b and constant speeds v_a and v_b. The time origin is set at the moment when A has detected an approaching vessel B on the radar screen and A has been able to make a fair estimate of B's relative and true motion. In addition to B's course and speed, A has also determined B's distance of closest approach r_c.

The closest approach of B is assumed to be less than a preset value r_0—to be determined by A's shipmaster—so there is a danger of collision. We further assume that A is obliged to evade. A's first object is now to carry out an evading manoeuvre, resulting in an increased value of r_c, that has to be at least equal to r_0. B will be assumed to maintain course and speed.

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SHIP'S MOVEMENT AS A RESULT OF A RUDDER ANGLE

We shall now briefly describe and explain how the ship's position, velocity and course change with time, if a certain rudder angle δ is introduced.

Figure 2 explains that a starboard rudder angle δ, introduced on a full speed sailing ship, results in a lifting force L on the rudder plane. This force L has three effects, being

(i) an angular acceleration, caused by the moment of L about the ship's gravity center G,

(ii) a port directed acceleration, caused by L_2 and

(iii) a backward acceleration, caused by L_1.

These accelerations affect the ship's forward and beam velocities u_1 and u_2, as well as the ship's rate of turn ω and the ship's course ψ.
With respect to a rigid system of co-ordinate axes OX_1X_2 we can now evaluate the ship's velocity components $\dot{x}_1$ and $\dot{x}_2$ as functions of u_1, u_2 and ψ. (See figure 3.).

The change with time of the variables x_1, x_2,

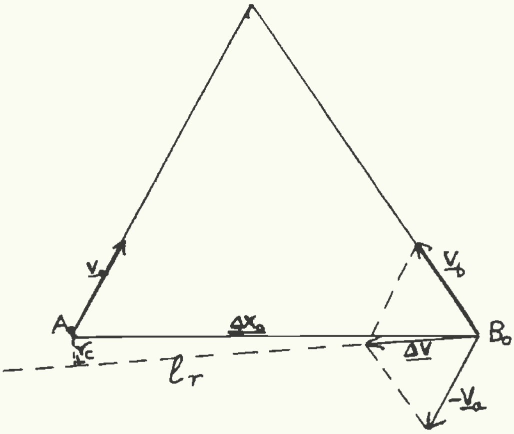

Fig. 1—A_0, B_0: initial positions of A and B. The dashed line 1_c is the relative path of B with respect to A, if both vessels would maintain their course and speed. The distance r_c from A_o to 1_r is the closest approach of B to A.

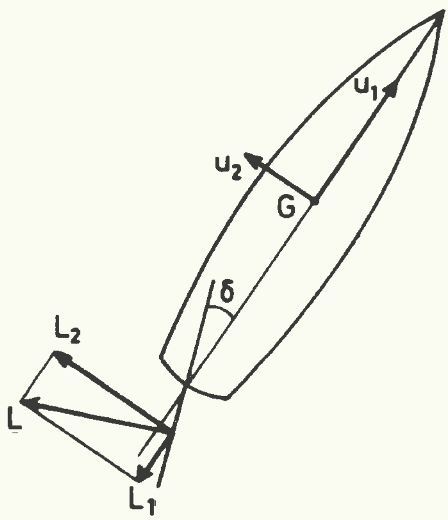

Fig. 2—A rudder angle δ generates a lifting force L with longships and thwartships components L_1 and L_2 and a moment L∗d about a vertical axis through the ship's gravity centre G.

u_1, u_2, ψ and ω can be described by the following set of equations.

$$\dot{x}_1 = u_1 \cos\psi - u_2 \sin\psi \qquad (3.1)$$

$$\dot{x}_2 = u_1 \sin\psi + u_2 \cos\psi \qquad (3.2)$$

$$\dot{\psi} = \omega \qquad (3.3)$$

$$\dot{\omega} = -a\omega - b\omega^3 + c\delta \qquad (3.4)$$

$$\dot{u}_1 = -fu_1 - W\omega^2 + S \qquad (3.5)$$

$$u_2 = -r_1\omega - r_3\omega^3 \qquad (3.6)$$

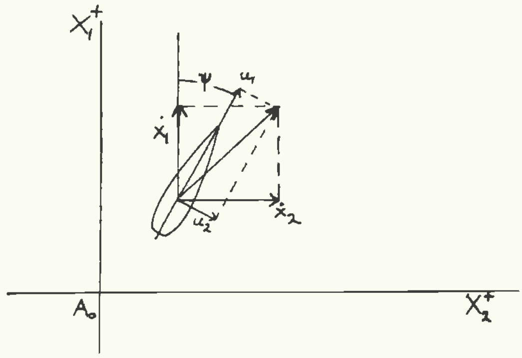

Fig. 3

Figure 3 serves to explain (3.1) and (3.2). Equation (3.3) is merely a definition; (3.4) is the simplified Nomoto-model, expressing that the ship's course is accelerated by the rudder angle δ, while the ship's reaction is described by the terms $-a\omega - b\omega^3$. For a stable ship, a and b are both positive, so the ship's rate of turn always counteracts the rudder action. For an unstable ship a is negative and b is positive. For a small rate of turn the ship "helps" the rudder action, but for larger values of ω the entire rate-of-turn reaction damps the angular acceleration. Equation (3.5) indicates how the propeller's thrust, resulting in a forward acceleration S, is counteracted by the ship's velocity as well as by the turning rate.

The relation between the beam velocity u_2 and the rate of turn, determined and affirmed by full scale and laboratory experiments, is given by equation (3.6). We finally remark, that the rudder angle is always bounded:

$$|\delta| \leq \delta_m, \; (\delta_m \doteq 0.6 \text{ rad} = 35°) \qquad (3.7)$$

For the purpose of this study, this set of equations gives a good description of what really happens for a large class of vessels, from well manoeuvrable coasters to large oil carriers.

Introducing the somewhat more systematic notations $x_3 = \psi$, $x_4 = \omega$, $x_5 = u_1$, $u = \delta$, the ship's manoeuvre can be described by the following set of differential equations:

$$\dot{x}_1 = x_5\cos(x_3) - (r_1x_4 + r_3x_4^3)\sin(x_3) \qquad (3.8.1)$$

$$\dot{x}_2 = x_5\sin(x_3) + (r_1x_4 + r_3x_4^3)\cos(x_3) \qquad (3.8.2)$$

$$\dot{x}_3 = x_4 \qquad (3.8.3)$$

$$\dot{x}_4 = -ax_4 - bx_4^3 + cu \qquad (3.8.4)$$

$$\dot{x}_5 = -fx_5 - Wx_4^2 + S \qquad (3.8.5)$$

This set can be written formally as

$$\dot{x} = F(x, u). \qquad (3.9)$$

THE COLLISION AVOIDANCE PROBLEM AS A TIME OPTIMAL CONTROL PROBLEM

The initial conditions of the state vector $x(t)$ are:

$$x_1(0) = x_2(0) = x_3(0) = x_4(0) = 0,\ x_5(0) = S/f. \qquad (4.1)$$

The last condition means that the ship starts the c.a.m. in a steady state of full speed ahead. The viscous friction fx_5 then neutralizes the propelling force S. Furthermore, the control $u(t) = \delta(t)$ is taken to be zero right before the start. The c.a.m. now comes down to determinating a rudder angle control

$$u^* = \{u(t), 0 \leqq t < t_f, |u(t)| \leqq \delta_m\} \qquad (4.2)$$

that takes the vessel from the initial state to a final state x_f at some, unknown final time t_f. In this final state we want $x_2 = x_3 = x_4 = 0$, i.e. the ship must be back on her former track ($x_2 = 0$) and we want her to be steady on her old course ($x_3 = \psi = 0$, $x_4 = \omega = 0$). The final time is specified as the time at which the ship has regained 95% of her initial speed:

$$x_5(t_f) \geqq 0.95\, x_5(0). \qquad (4.3)$$

During this manoeuvre we want the ship's distance to the other vessel B to be not less than r_0:

$$r(t) = \Big(\sum_{i=1}^{2} (x_i(t) - y_i(t))^2\Big)^{1/2} \geqq r_0. \qquad (4.4)$$

In this expression $y_1(t)$ and $y_2(t)$ are B's co-ordinates. With the adopted co-ordinate system we have $y(0) = \Delta x_0$, so with B's course ψ_b and speed v_b assumed constant, we have

$$y_1(t) = y_1(0) + v_b t \cos(\psi_b - \psi_a(0)), \qquad (4.5.1)$$

$$y_2(t) = y_2(0) + v_b t \sin(\psi_b - \psi_a(0)). \qquad (4.5.2)$$

At the time of closest approach t_c the distance $r(t_c)$ is minimal, meaning that $\dot{r}(t_c) = 0$ and $\ddot{r}(t_c) > 0$. Naturally, t_c should not be later than t_f. This means that t_f, beside satisfying condition (4.3), also has to satisfy the condition

$$\dot{r}(t_f) \geqq 0. \qquad (4.6)$$

Having thus defined the conditions that have to be satisfied during the c.a.m., we can now pose a criterion for optimization by demanding that the manoeuvre must be carried out in the least possible time. This means that t_f must be minimized. This least time manoeuvre will also have the advantage of being very clear to the navigator of the other vessel B, who will observe a clear and fast change of bearing of the evader A.

SOLUTION OF THE TIME OPTIMAL CONTROL PROBLEM

From the theory on optimal control, it can be derived, that for the case under consideration, the control u^* is one of the bang-bang type. (See lit. ref. 1). In connection with the Rules of the Road at Sea, stating that the evading ship has to avoid to come to the port side of her original track, the c.a.m. will consist of the following sequence of rudder angle signals:

$$\delta(t) = 0 \quad \text{for} \quad 0 \leqq t < t_1, \qquad (5.1.1)$$

$$\delta(t) = +\delta_m \quad \text{for} \quad t_1 \leqq t < t_2, \qquad (5.1.2)$$

$$\delta(t) = -\delta_m \quad \text{for} \quad t_2 \leqq t < t_3, \qquad (5.1.3)$$

$$\delta(t) = +\delta_m \quad \text{for} \quad t_3 \leqq t < t_4, \qquad (5.1.4)$$

$$\delta(t) = -\delta_m \quad \text{for} \quad t_4 \leqq t < t_5, \qquad (5.1.5)$$

$$\delta(t) = 0 \quad \text{for} \quad t_5 \leqq t < t_f. \qquad (5.1.6)$$

The first switching time t_1 may coincide with the starting time $t = 0$. The graphs of $\delta(t)$, $\omega(t)$ and $\psi(t)$ and the corresponding ship's track are shown in Figures 4 and 5.

If $t_1 = 0$, the c.a.m. is carried out at the earliest

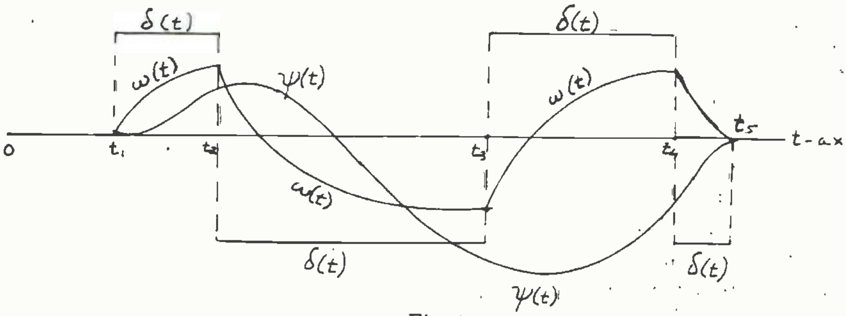

Fig. 4

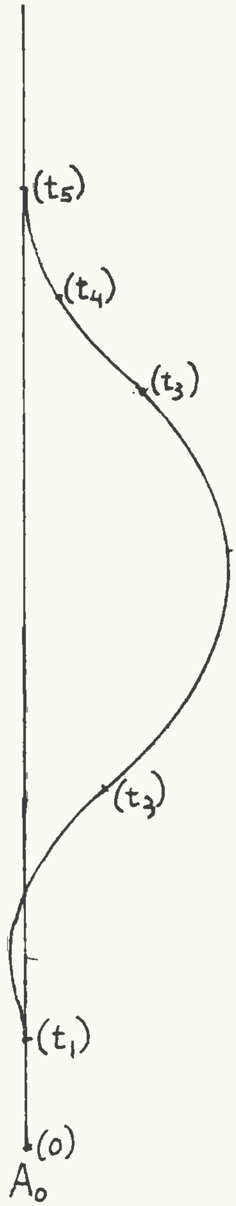

Fig. 5—Track of ship's gravity centre during a least time c.a.m.: $0 \leqq t < t_1$: Midships helm, $t_1 \leqq t < t_2$: Helm full astarboard, $t_2 \leqq t < t_3$: Helm full aport, $t_3 \leqq t < t_4$: Helm full astarboard, $t_4 \leqq t < t_5$: Helm full aport, $t_5 \leqq t < \cdots$: Course control by autopilot or helmsman.

possible moment. In that case the closest approach of A and B usually takes place at a moment t_c, which is later than t_5. In some cases it pays to postpone the c.a.m. until A has the possibility to make a turning loop around B's stern. In that event the moment of closest approach is earlier than t_5.

NUMERICAL SOLUTION OF THE CONTROL PROBLEM

With the object of obtaining a solution within some 10 seconds' computing time, the problem was somewhat simplified.

To start with, the set of simultaneous differential equations (3.8) was solved with Euler's predictor method with time steps Δt of 6 seconds.

Denoting the points of the trajectory at times $\tau_1 = \Delta t$, $\tau_2 = 2\,\Delta t$ etc. as $x(1)$, $x(2)$ etc., the Euler integration comes down to

(i) $x(0) = x_0$,

(ii) for $k = 1, 2$ etc.

$$x(k+1) = x(k) + F(x(k), u(k))\,\Delta t.$$

As to the evaluations of the switching times t_1, t_2 etc. we can make the following preconsiderations.

Once t_1 and t_2 have been selected, the values for t_3, t_4 and t_5 are determined implicitly by the three final conditions

$$x_2(t_5) = x_3(t_5) = x_4(t_5) = 0.$$

t_3, t_4 and t_5 can be determined numerically by minimization of the object function

$$J(t_3, t_4, t_5) = \sum_{i=2}^{4} (x_i(t_5))^2. \qquad (6.1)$$

This minimization was done with Hooke & Jeeve's method of Direct Search. We thus obtain good approximations of t_3, t_4 and t_5.

This minimization for each selection of t_1 and t_2 could lead to an unwanted accumulation of computing time. Therefore we first selected provisional values t_3^* and t_4^*, while introducing some simplifying modifications of the original dynamic system model (3.8).

The control $\{\delta(t)\}$ is replaced by $\{\delta^*(t)\}$:

$$\delta^*(t) = +\delta_m \quad \text{for} \quad t_1 \leqq t < t_2, \qquad (6.2.1)$$

$$\delta^*(t) = -\delta_m \quad \text{for} \quad t_2 \leqq t < t_3^*, \qquad (6.2.2)$$

$$\delta^*(t) = +\delta_m \quad \text{for} \quad t_3^* \leqq t < t_4^*, \qquad (6.2.3)$$

$$\delta^*(t) = 0 \quad \text{for} \quad t_4^* \leqq t < t_f^*. \qquad (6.2.4)$$

The third switching time t_3 is estimated by the expression

$$t_3^* = t_2 + 2(t_2 - t_1)(1 + \eta). \qquad (6.3)$$

The last switching moment t_4^* is determined by the criterion that the course is back to zero again. In numerical language this means that, if for a certain value of k we have

$$\psi(k\Delta t) < 0 \quad \text{and} \quad \psi((k+1)\Delta t) \geqq 0$$

we take $t_4^* \doteq (k+1)\Delta t$.

For times later than t_4^* the ship's rate of turn is set back to zero, as well as the rudder angle. The dynamics then become

$$\begin{aligned} \dot{x}_1^* &= x_5^*, \\ \dot{x}_2^* &= \dot{x}_3^* = \dot{x}_4^* = 0, \qquad (6.4) \\ \dot{x}_5^* &= -fx_5^* + S. \end{aligned}$$

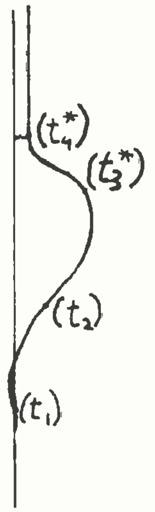

Fig. 6

With these modifications we get an approximated track $\{x^*(t)\}$, shown in Figure 6. For values of $t \geqq t_4^*$ this track can deviate from the original tract at the amount of $|x_2^*(t_4^*)|$. For the ship under consideration and using the linear estimation (6.3) for t_3^*, this final off-track-error was found to be minimal for $\eta = 0.15$.

For modified final time t_f^* can now be determined. For that purpose we proceed until time t_s: $x_5^*(t_s^*) = 0.95\, x_5(0)$. If the mutual distance is found to be non-decreasing, i.e. if

$$\dot{r}^*(t_s) \geqq 0,$$

then we can take $t_f^* = t_s$. In the other case we have to proceed until $r^*(t)$ has passed its minimal value.

We are now able to determine the optimal values of t_1 and t_2 for the modified system.

For a certain selected value of t_1 the corresponding value of t_2 can be determined. We do this by considering the minimal distance $r_{\min}$ of A and B as a function of t_1 and t_2

$$r_{\min} = R(t_1, t_2). \qquad (6.9)$$

The conditions to determine t_2 are:

$$R(t_1, t_2) \geqq r_0 \quad \text{and} \quad R(t_1, t_2 - \Delta t) < r_0. \qquad (6.10.\ 1\&2)$$

With these conditions we designed a searching procedure to determine t_2 for a given value of t_1. Once t_2 is known, the provisional final time t_f^* can be evaluated. Using this (t_1, t_2, t_f^*) proce-

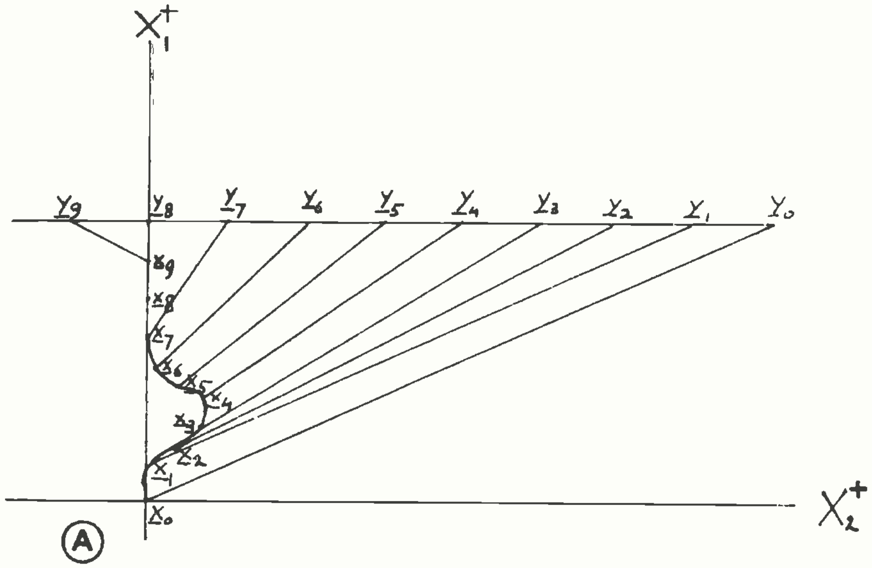

Fig. 7a—Time optimal collision avoiding manoeuvre of A: $\{x_k\}$, by keeping B: $\{y_k\}$ at a desired minimal distance.

dure we can minimize t_f by varying t_1. After having thus determined t_1 and t_2 for the provisional system, we can find the proper switching times t_3, t_4 and t_5 by minimizing (6.1).

COMPUTATIONAL RESULTS AND CONCLUSIONS

The program described above was run for a few encountering situations, in which the own ship was compelled to evade, while the other ship was primarily obliged to maintain course and speed.

For the own ship, the dynamic manoeuvring parameters of the m.s. Compass Island (See lit. ref. 2) were used. Adopting as units of time, length and angle respectively one minute, one nautical mile and one radian, the parameters in (3.4), (3.6), (3.7) and (3.8) were selected as follows:

(3.4): $a = 1.084$/min, $b = 0.62$ min, $c = 3.553$/rad/min,

(3.6): $\delta_m = 35° = 0.61$ rad,

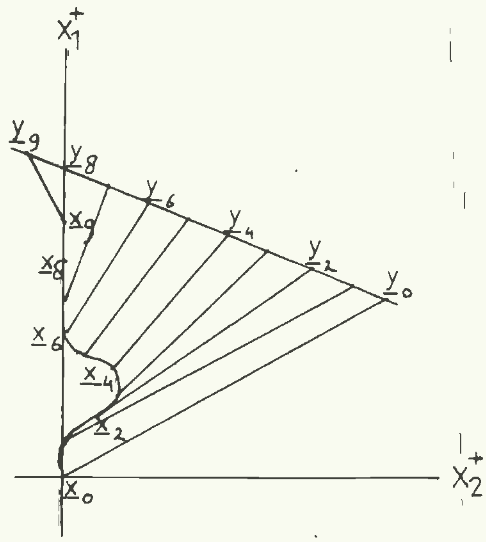

Fig. 8a—Collision avoiding manoeuvre for two ships with 70° initial heading difference. Evading ship A: x_k, evaded ship B: y_k.

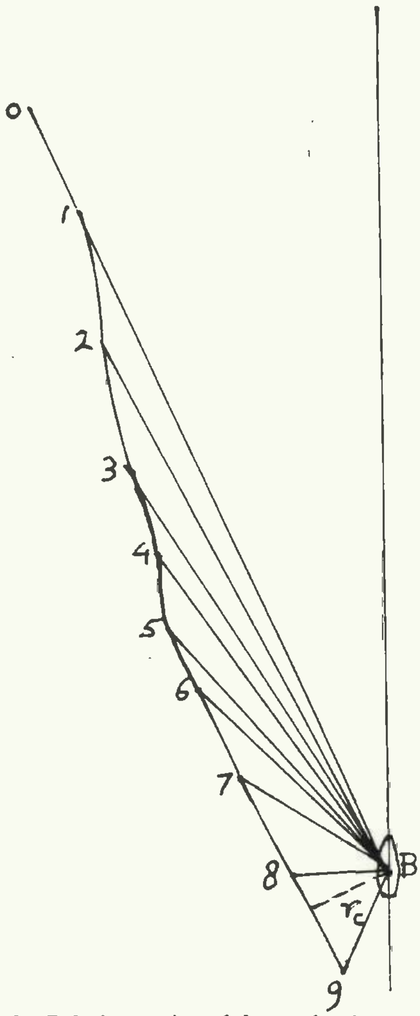

Fig. 7b—Relative motion of the evader A, seen from the evaded ship B.

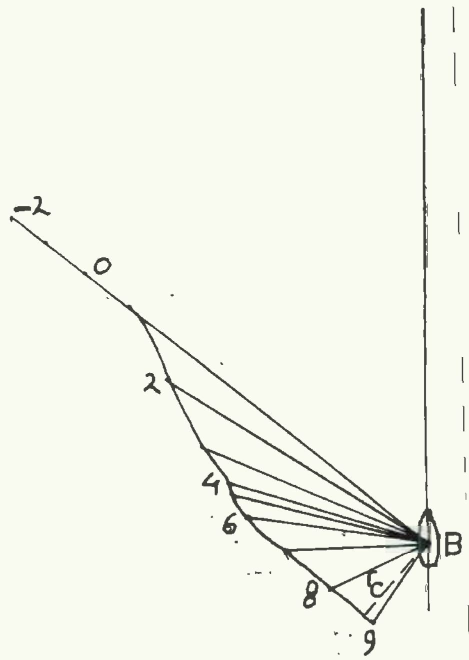

Fig. 8b—A's evading manoeuvre, as seen from the other ship B.

(3.7): $r_1 = -0.0375$ nm/rad, $r_3 = 0$,

(3.8): $f = 0.86$/min, $W = 0.067$ nm/rad^2, $S = 0.215$ nm/min^2.

With these parameters, the ship has a maximum speed of 0.25 nm/min = 15 knots. For a maximal rudder angle of 35°, the stationary rate of turn is 1°/sec.

Figures 7*a* and 7*b* show the fastest possible c.a.m. of this ship, when encountering a ship B with a speed of 30 knots and a course $\psi_b = \psi_a - 90°$. For this case the starting time t_1 of the c.a.m. (See (5.1.2)) is zero. A later time would not have generated a smaller final time t_f, so there is no point in postponing the instant of turning the wheel hard astarboard.

The same argument for selecting $t_1 = 0$ holds in the case, depicted in figures 8*a* and 8*b*. Here the other ship has a course $\psi_b = \psi_a - 70°$ and a speed of 15 knots.

Figures 8*b* and 9*b* show what the other ship B would see on its radar, when put in the relative motion mode. In both cases B observes a clear change of bearing of A at an early stage.

In Figures 9*a* and 9*b* we come closer to the case of almost opposite courses. Here we have

$$\psi_b = \psi_a - 123° \quad \text{and} \quad v_b = 22 \text{ knots.}$$

The time-optimal c.a.m. has a rather late start. Yet it is the best one, as the total manoeuvring time t_f decreases, as t_1 increases from zero upwards. A hazardous situation could be avoided in this case by setting a lower bound to the length of the time interval $[t_1, t_c^\circ)$. (t_c° is the time of closest approach or possibly the time of collision, if both ships would have proceeded in their old courses and speeds. See also (2.7))

In an attempt to promote the applicability of this time optimal collision avoidance manoeuvre, we should like to make the following remarks.

1. In order to reduce computing times to at most 10 seconds, it is recommended to work with a realistic, but not too sophisticated, model for the ship's dynamics. A simplified

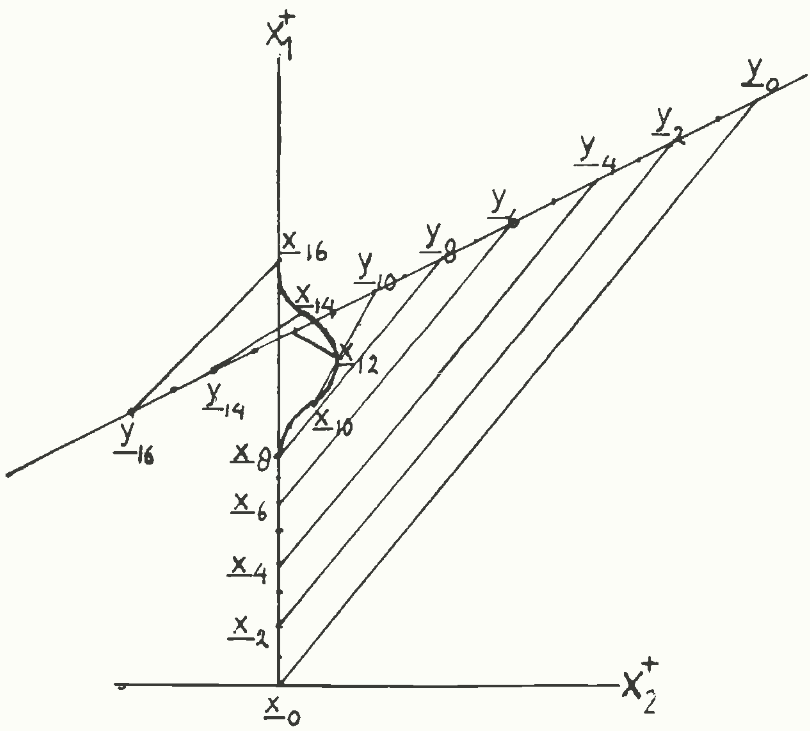

Fig. 9a—Collision avoidance for two ships, sailing at courses ψ_a and ψ_b with a difference $\Delta\psi$ = 123°. The evading manoeuvre is time optimal, but seems a bit hazardous with the rather large relative velocity.

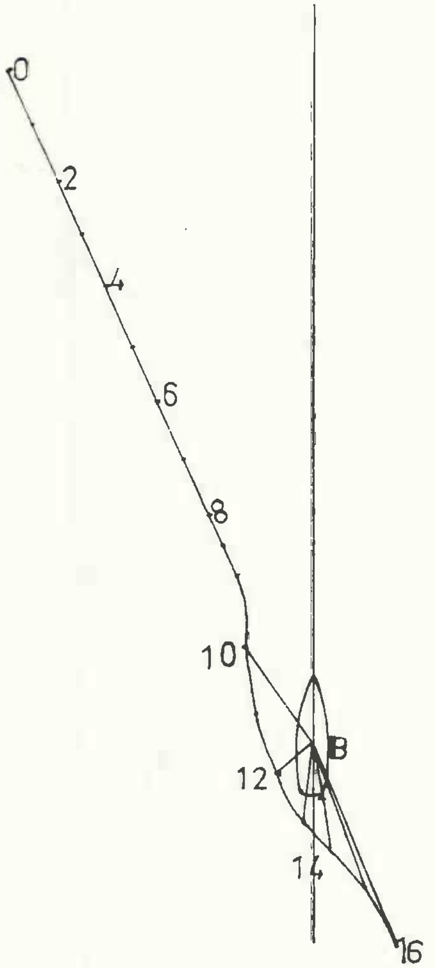

Fig. 9b—Relative motion of the evading ship A, as seen from the evaded ship B.

model, like the one adopted by Miloh and Sharma, seems too far from reality. One cannot do with a model that assumes a constant speed, regardless of the rudder angle and the ship's rate of turn. On the other hand, a complicated 9th order nonlinear model for the ship's course and speed dynamics would clearly be too much in view of the object.

2. We have no intention of introducing a rigorous set of prescriptions as to how to complete the c.a.m. In particular after time t_2—the moment of full port helm—it is left to the navigator's skill and judgment to bring the ship quietly back to its old course. A few cables off the original track won't hurt.
3. Beside setting a lower bound to $t_c^o - t_1$ with the intention to avoid nerve affecting situations, it seems practical to set an upper bound to the time interval (t_1, t_2). This is done to keep the ship from getting a course $\psi_a(t) \geqq \psi_a(0) + 90°$.
4. If the total manoeuvring time t_f turns out to be practically constant for t_1 increasing from 0 on upwards, t_1 should be selected as early as possible. In fact it seems worthwhile to take $t_1 = 0$ except for the case of almost opposite courses, where taking $t_1 = 0$ would merely postpone the moment of dangerous encounter, without producing a significant increase of the distance of closest approach.

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